Answers to problems.

Problem 1. (10 points) Answer the following questions regarding MLE.

(a) Explain what an unbiased estimator is. Is the MLE $\hat{\theta}$ always unbiased?

A point estimate $\hat{\theta}$ is unbiased if the expected value $E[\hat{\theta}]$ is exactly the parameter θ. The MLE is not always unbiased.

(b) Describe or formulate the Cramér-Rao lower bound for an unbiased estimator.

It is the reciprocal $1/I(\theta)$ of Fisher information $I(\theta)$.

(c) Explain what an efficient estimator is. Is MLE $\hat{\theta}$ always efficient?

A point estimate $\hat{\theta}$ is efficient if the variance $\mathrm{Var}(\hat{\theta})$ attains the Cramér-Rao lower bound. The MLE is not always efficient.

(d) Describe or formulate an asymptotic distribution of the MLE $\hat{\theta}$.

$\hat{\theta}$ is asymptotically normal with mean θ and variance $1/I(\theta)$; thus, MLE is "asymptotically" unbiased and efficient.

Problem 2. (25 points) Let $f(x;\theta) = \binom{k}{x}\theta^x(1-\theta)^{k-x}$, $x = 0, 1, \ldots, k$, be the binomial frequency function, and let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be a random sample from $f(x;\theta)$ with $0 < \theta < 1$, where k is known. Answer the following questions.

(a) Find the MLE $\hat{\theta}$ for θ.

Observe that

$$\ln f(\boldsymbol{x};\theta) = \sum_{i=1}^{n} \ln \binom{k}{x_i} + (\ln\theta - \ln(1-\theta)) \sum_{i=1}^{n} x_i + kn\ln(1-\theta)$$

$$\frac{\partial}{\partial\theta}\ln f(\boldsymbol{x};\theta) = \frac{1}{\theta(1-\theta)}\sum_{i=1}^{n} x_i - \frac{kn}{\theta(1-\theta)}$$

By solving $\frac{\partial}{\partial\theta}\ln f(\boldsymbol{x};\theta) = 0$, we obtain $\hat{\theta} = \sum_{i=1}^{n} x_i/nk$.

(b) Find the Fisher information $I(\theta)$ for the random sample $\boldsymbol{X}$.

$$I(\theta) = \mathrm{Var}\left(\frac{\partial}{\partial\theta}\ln f(\boldsymbol{X};\theta)\right) = \frac{1}{\theta^2(1-\theta)^2}\sum_{i=1}^{n}\mathrm{Var}(X_i) = \frac{nk}{\theta(1-\theta)}$$

(c) Is $\hat{\theta}$ unbiased?

Yes.

(d) Is $\hat{\theta}$ efficient?

Yes.

(e) Find an asymptotic distribution of $\sqrt{n}(\hat{\theta}-\theta)$ in terms of k and θ.

$N(\theta, \theta(1-\theta)/k)$.

Problem 3. (10 points) Answer the following questions regarding a natural sufficient statistic.

(a) Explain what a natural sufficient statistic is.

When a pdf for random sample is of exponential family, we always obtain a sufficent statistic of the form $\sum_{i=1}^{n} k(X_i)$, and call it "natural" sufficient statistic.

(b) Is a natural sufficient statistic always unbiased?

No.

(c) Is a natural sufficient statistic always complete?

Yes.

(d) Explain what Lehmann-Scheffé theorem does for complete and sufficient statistic.

Once a complete and sufficient statistic Y_1 is obtained, the MVUE can be calculated by $E(Y_2|Y_1)$ with any unbiased statistic Y_2.

Problem 4. (20 points) Continue **Problem 2**, and answer the following questions.

(a) Write the exponential family for the joint distribution $f(\boldsymbol{x};\theta)$ of $\boldsymbol{X}$.

$$f(\boldsymbol{x};\theta) = \exp\left[(\ln\theta - \ln(1-\theta))\sum_{i=1}^{n} x_i + kn\ln(1-\theta) + \sum_{i=1}^{n}\ln\binom{k}{x_i}\right]$$

(b) Find a natural sufficient statistic Y.

$Y = \sum_{i=1}^{n} X_i$

(c) Find the minimum variance unbiased estimator $\varphi(Y)$ for θ.

$\varphi(Y) = \sum_{i=1}^{n} X_i/nk$

(d) Find the variance of $\varphi(Y)$.

$\theta(1-\theta)/nk$

Problem 5. (10 points) Answer the following questions regarding hypothesis test.

(a) Describe a power function.

It is the function $K(\theta)$ of θ which provides the probability of rejecting the null hypothesis when the true parameter value is θ.

(b) Explain what the uniform most powerful (UMP) test is.

No other tests of the same size α has a larger value of power function when the alternative hypothesis is true.

(c) How can you find an optimal test statistic $T(\boldsymbol{X})$ when the joint density $f(\boldsymbol{x};\theta)$ is of exponential family and the alternative hypothesis is $H_A : \theta > \theta_0$?

See the lecture note #4.

(d) Continue from the previous question, how can you find the size α of the test in terms of power function.

$K(\theta_0)$

Problem 6. (25 points) Continue **Problem 4**. Here we wish to find evidence that $\theta > \frac{1}{2}$.

(a) Choose the null and the alternative hypothesis.

$H_0 : \theta \leq \frac{1}{2}$ versus $H_A : \theta > \frac{1}{2}$.

(b) Find an optimal test statistic $T(\boldsymbol{X})$.

$T(\boldsymbol{X}) = \sum_{i=1}^{n} X_i$

(c) Find the distribution for $T(\boldsymbol{X})$ given $\theta = \frac{1}{2}$, and express it in terms of n and k.

Binomial distribution with $(nk, \frac{1}{2})$.

(d) Suppose that $k = 1$ and $n = 7$. Construct the uniform most powerful (UMP) test of size $\alpha = \left(\frac{1}{2}\right)^7$.

$$\delta(\boldsymbol{X}) = \begin{cases} 1 & \sum_{i=1}^{n} X_i \geq 7; \\ 0 & \text{otherwise.} \end{cases}$$

(e) Continue from the previous question. Find the power function $K(\theta)$ in terms of θ. Is it an increasing function?

Yes, $K(\theta) = \theta^7$ is an increasing function.