UNIVERSITI TUNKU ABDUL RAHMAN

ACADEMIC YEAR 2019/2020

JANUARY 2020 TRIMESTER

FINAL ASSESSMENT

ANSWER SCRIPT

Candidate is required to fill in ALL the information below:

Name : (as stated in Student Identity Card)	Ngu Yi Hui		
Faculty /Institute/ Centre:	FSc	Programme :	Statistical Computing And Operations Research
Index No. (in numbers) :	A00082DBSCF	Index No. (in words) :	A Zero Zero Zero Eight Two DBSCF
Course Code :	UDPS2013	Course Description :	Numerical Methods
Submission Date :	12th MAY 2020	Time :	9am - 11am

QUESTION NUMBER	**FOR EXAMINER'S USE ONLY**	
	MARKS	
	Internal	**External**
Q1		
Q2		
Q3		
Q4		
TOTAL MARKS		

JANUARY 2020 TRIMESTER FINAL ASSESSMENT ANSWER SHEETS

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I, Ngu Yi Hui (Name), Student ID No. 18ADB01438, hereby solemnly and fully declare and confirm that during my programme of study at Universiti Tunku Abdul Rahman, I shall abide and comply with all the rules, regulations and lawful instructions of Universiti Tunku Abdul Rahman and endeavour at all times to uphold the good name of the University.

I hereby declare that my submission for this Final Assessment is based on my original work, not plagiarised from any source(s) except for citations and quotations which have been duly acknowledged. I am fully aware that students who are suspected of violating this pledge are liable to be referred to the Examination Disciplinary Committee of the University.

Programme:	Statistical Computing And Operations Research
(Digital) Signature:	HUI
Student's I.C / Passport No.:	991110-14-6378
Index No:	A00082DBSCF
Date of Submission:	12th MAY 2020

Q1.

(a) $\frac{dy}{dx} = f(x,y) = x^2 + 2x\cos(xy)$, $y(0) = 0$ and $0 \le x \le 3$

Holding x as a constant and applying Mean Value Theorem to the function

$f(x,y) = x^2 + 2x\cos(xy)$ for $y_1 < y_2$ and $0 \le x \le 3$,

$$\frac{f(x,y_2) - f(x,y_1)}{y_2 - y_1} = \frac{\partial}{\partial y} f(x,y) = -2x^2 \sin(xy) \quad , y \in (y_1, y_2)$$

$$f(x,y_2) - f(x,y_1) = -2x^2 \sin(xy)\ (y_2 - y_1) \quad , y \in (y_1, y_2)$$

$$|f(x,y_2) - f(x,y_1)| = |2|\,|x^2|\,|\sin(xy)|\,|(y_2 - y_1)|$$

$$\le (2)(9)(1)\ |(y_2 - y_1)|$$

$$\le 18\ |(y_2 - y_1)| \quad , y \in (y_1, y_2)$$

f satisfies a Lipschitz condition in the variable y with Lipschitz constant $L = 18$.

As f continuous for when $0 \le x \le 3$ and $y \in (-\infty, \infty)$, the initial value problem $y' = f(x,y) = x^2 + 2x\cos(xy)$, $0 \le x \le 3$, $y(0) = 0$ has a unique solution.

Q1.

(b)

Cubic Spline Polynomial	Lagrange Polynomial
It forces the function f(x) to include all the data points over a close interval of x.	It forces the function f(x) to include all the data points over a close interval of x.
All the data points need to be sorted according to the x-coordinate.	It is flexible, the data points do not need to be sorted according to the x-coordinate.
It divides the whole interval into sub-divisions to perform the piecewise polynomial approximation where the cubic polynomial is used for each successive pair of nodes.	
It is more accurate than Lagrange Polynomial.	It is less accurate compared with Cubic Spline Polynomial.
It can retain the shape of the curve when it is well-fitted by the polynomial over each sub-intervals	It will be more accurate when higher degree of Lagrange Polynomial is performed.

Q1.

(c)

Newton Method	False Position Method
It is an open method as the iteration begins at an initial estimation (p_0) which is close to the root for the continuous $f(x)$.	It is a close method which requires the interval (a,b) to include the root of the continuous function $f(x)$.
It needs only one initial approximation which is close to the root (p).	It needs two initial approximation (p_0, p_1) which are close to the root (p), but the two initial approximation (p_0, p_1) cannot be too close to each other.
It uses the help of tangent line to approximate the solution, thus the assumption of $f'(x_n) \neq 0$ must be met. ($n = 0, 1, 2, \ldots$)	It make use of the secant line to find the solution, thus the condition $f(p_0) \neq f(p_1)$ must be met in the first iteration.
It usually converges faster than Bisection Method, but requires the model information providing the derivative exists	Its convergence rate usually faster than Bisection Method.
It converges faster when $f(x)$ is a straight line. Also, when the magnitude of the slope of the tangent line is larger, it converges faster.	It converges faster when the magnitude of the slope of the secant line is larger.

Q2.

(a) Accuracy of :

Euler's Method = $O(h^2)$

Taylor Series Method of Order p = $O(h^p)$

Runge Kutta Method of Order 4 = $O(h^4)$

The power of h is higher, the accuracy will be higher.

$\therefore$ Euler is least accurate.

Q2.

(b) (i) Using Trapezoidal Rule with $h = 1$,

$f(x) = \frac{1}{1+x^2}$

x	$f(x)$
0	1
1	0.5

$$\int_0^1 \frac{1}{1+x^2}\,dx \simeq \frac{1}{2}\left[f(0) + f(1)\right]$$

$$\simeq \frac{1}{2}(1 + 0.5)$$

$$\simeq 0.75$$

(ii) Using Composite Simpson's Rule with $h = \frac{1}{6}$,

$f(x) = \frac{1}{1+x^2}$

x	$f(x)$
0	1
1/6	0.9730
1/3	0.9
1/2	0.8
2/3	0.6923
5/6	0.5902
1	0.5

$$\int_0^1 \frac{1}{1+x^2}\,dx \simeq \frac{1}{6(3)}\left[1 + 0.5 + 4(0.9730 + 0.8 + 0.5902) + 2(0.9 + 0.6923)\right]$$

$$\simeq \frac{1}{18}(14.1374)$$

$$\simeq 0.7854$$

(iii) Actual $\int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1} x\right]_0^1 = \frac{\pi}{4}$

The accuracy of estimate in part (i):

$$\frac{\left|0.75 - \frac{\pi}{4}\right|}{\frac{\pi}{4}} = 0.04507$$

The accuracy of estimate in part (ii):

$$\frac{\left|0.7854 - \frac{\pi}{4}\right|}{\frac{\pi}{4}} = 0.0000002338$$

$\therefore$ Composite Simpson's Rule is more accurate than Trapezoidal Rule.

Q3.

(a) $y' = f(x,y) = 4(y-x)$, $y(1) = 2$

Using Heun's Method with $h = 0.1$,

$$y_{n+1}^{*} = y_n + h\, f(x_n, y_n)$$

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{*}) \right]$$

$$y_{n+1}^{*} = y_n + 0.1\left[4(y_n - x_n) \right]$$

$$y_{n+1} = y_n + 0.05\left[4(y_n - x_n) + 4(y_{n+1}^{*} - x_{n+1}) \right] \quad , n = 0, 1, \dots$$

x_n	y_n^{*}	y_n
1		2
1.1	2.4	2.46
1.2	3.004	3.0928
1.3	3.8499	3.9813

$\therefore\ y(1.3) \simeq 3.9813$ ※

Q3.

(b) (i) Expand function $f(x)$ in a 3rd Taylor polynomial about a point x_0 and evaluate at (x_0+h) and (x_0-h).

(1) — $f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2 f''(x_0)}{2!} + \frac{h^3 f'''(x_0)}{3!} + \frac{h^4 f''''(\xi_1)}{4!}$,

ξ_1 is between x_0 and x_0+h

(2) — $f(x_0-h) = f(x_0) - hf'(x_0) + \frac{h^2 f''(x_0)}{2!} - \frac{h^3 f'''(x_0)}{3!} + \frac{h^4 f''''(\xi_{-1})}{4!}$,

ξ_{-1} is between x_0-h and x_0

(1) + (2),

$$f(x_0+h) + f(x_0-h) = 2f(x_0) + h^2 f''(x_0) + \frac{h^4}{24}\left[f''''(\xi_1) + f''''(\xi_{-1})\right]$$

$$f''(x_0) = \frac{1}{h^2}\left[f(x_0-h) - 2f(x_0) + f(x_0+h)\right] - \frac{h^2}{12} f^{(4)}(\xi) ,$$

$\xi \in (x_0-h, x_0+h)$ and $f^{(4)}(\xi) = \frac{f^{(4)}(\xi_1) + f^{(4)}(\xi_{-1})}{2}$

$$f''(x_0) = \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} + O(h^2)$$

$$= \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} + K_1 h^2 + K_2 h^4 + \cdots ,$$

$|h| < 1$, and $K_1, K_2, \cdots$ are constant.

Q3.

(b)(ii) $f(x) = xe^x$

Using Richardson Extrapolation, with $h = 0.1$,

$$N_1(0.1) = \frac{f(1.1) - 2f(1) + f(0.9)}{0.1^2}$$

$$= \frac{3.3046 - 5.4366 + 2.2136}{0.1^2}$$

$$= 8.16$$

$$N_1(0.05) = \frac{f(1.05) - 2f(1) + f(0.95)}{0.05^2}$$

$$= \frac{3.0005 - 5.4366 + 2.4564}{0.05^2}$$

$$\doteq 8.12$$

$$N_1(0.025) = \frac{f(1.025) - 2f(1) + f(0.975)}{0.025^2}$$

$$= \frac{2.8568 - 5.4366 + 2.5849}{0.025^2}$$

$$= 8.16$$

$$N_2(0.1) = N_1(0.05) + \frac{N_1(0.05) - N_1(0.1)}{3}$$

$$= 8.1067$$

$$N_2(0.05) = N_1(0.025) + \frac{N_1(0.025) - N_1(0.05)}{3}$$

$$= 8.1733$$

$$N_3(0.1) = N_2(0.05) + \frac{N_2(0.05) - N_2(0.1)}{15}$$

$$= 8.1777$$

$$\therefore f''(1) \simeq N_3(0.1)$$

$$\simeq 8.1777$$ ※

Q4.

(a) $y'' - 4xy' + 4y = x + 1$, $y(1) = 1$, $y(1.4) = 2$

Using Finite Difference Method with $h = 0.1$,

Substitute $y'' \simeq \dfrac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$, $y' \simeq \dfrac{y_{i+1} - y_{i-1}}{2h}$, $x = x_i$ and $y = y_i$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - 4x_i \frac{y_{i+1} - y_{i-1}}{2h} + 4y_i = x_i + 1$$

$$100(y_{i+1} - 2y_i + y_{i-1}) - 20x_i(y_{i+1} - y_{i-1}) + 4y_i = x_i + 1 \quad , \quad i = 1, 2, 3$$

When $i = 1$,

$$100(y_2 - 2y_1 + 1) - 20(1.1)(y_2 - 1) + 4y_1 = 1.1 + 1$$

$$100y_2 - 200y_1 + 100 - 22y_2 + 22 + 4y_1 = 2.1$$

$$78y_2 - 196y_1 = -119.9$$

x	y
1	1
1.1	y_1
1.2	y_2
1.3	y_3
1.4	2

When $i = 2$,

$$100(y_3 - 2y_2 + y_1) - 20(1.2)(y_3 - y_1) + 4y_2 = 1.2 + 1$$

$$100y_3 - 200y_2 + 100y_1 - 24y_3 + 24y_1 + 4y_2 = 2.2$$

$$76y_3 - 196y_2 + 124y_1 = 2.2$$

When $i = 3$,

$$100(2 - 2y_3 + y_2) - 20(1.3)(2 - y_2) + 4y_3 = 1.3 + 1$$

$$200 - 200y_3 + 100y_2 - 52 + 26y_2 + 4y_3 = 2.3$$

$$-196y_3 + 126y_2 = -145.7$$

Using Cramer's Rule,

$$\begin{bmatrix} 0 & 78 & -196 \\ 76 & -196 & 124 \\ -196 & 126 & 0 \end{bmatrix} \begin{bmatrix} y_3 \\ y_2 \\ y_1 \end{bmatrix} = \begin{bmatrix} -119.9 \\ 2.2 \\ -145.7 \end{bmatrix}$$

(Continue next page).

Q4.

(a)

$$y_3 = \frac{\begin{vmatrix} -119.9 & 78 & -196 \\ 2.2 & -196 & 124 \\ -145.7 & 126 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 78 & -196 \\ 76 & -196 & 124 \\ -196 & 126 & 0 \end{vmatrix}} = \frac{6006987.2}{3756928} = 1.5989$$

$$y_2 = \frac{\begin{vmatrix} 0 & -119.9 & -196 \\ 76 & 2.2 & 124 \\ -196 & -145.7 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 78 & -196 \\ 76 & -196 & 124 \\ -196 & 126 & 0 \end{vmatrix}} = \frac{4999881.6}{3756928} = 1.3308$$

$$y_1 = \frac{\begin{vmatrix} 0 & 78 & -119.9 \\ 76 & -196 & 2.2 \\ -196 & 126 & -145.7 \end{vmatrix}}{\begin{vmatrix} 0 & 78 & -196 \\ 76 & -196 & 124 \\ -196 & 126 & 0 \end{vmatrix}} = \frac{4287992}{3756928} = 1.1414$$

$\therefore$ $y(1.1) \simeq 1.1414$

$y(1.2) \simeq 1.3308$

$y(1.3) \simeq 1.5989$

Q4.

(b) $\int_0^1 \frac{1}{1+x} dx$

Using Romberg Integration $R_{2,2}$,

$f(x) = \frac{1}{1+x}$

$R_{1,1} = \frac{h_1}{2}\left[f(0) + f(1)\right] = \frac{1}{2}(1+0.5) = 0.75$

$R_{2,1} = \frac{1}{2}\left[R_{1,1} + h_1 f(0.5)\right] = \frac{1}{2}\left[0.75 + 1(0.6667)\right] = 0.7083$

$R_{2,2} = R_{2,1} + \frac{R_{2,1} - R_{1,1}}{3} = 0.7083 + \frac{0.7083 - 0.75}{3} = 0.6944$

$\therefore \int_0^1 \frac{1}{1+x} dx \simeq 0.6944$

Q4.

(c) First, we need to check for the condition, Strictly Row Diagonally Dominant, For the following system of linear equations, they does not fulfil the condition,

eq1: $x + 10y + z = 18 \rightarrow 1 < 10 + 1$
eq2: $10x + y + z = 18 \rightarrow 1 < 10 + 1$ } Not fulfil
eq3: $x + y + 10z = 18 \rightarrow 10 > 1 + 1$ - Fulfil

So we need to interchange the row of the equations to make them follows Strictly Row Diagonally Dominant rule.
In this case, we interchange equation 1 and equation 2.

$$\left.\begin{aligned} 10x + y + z &= 18 \rightarrow 10 > 1+1 \\ x + 10y + z &= 18 \rightarrow 10 > 1+1 \\ x + y + 10z &= 18 \rightarrow 10 > 1+1 \end{aligned}\right\} \text{All fulfil}$$

The condition is fulfilled now.

Then, we can continue to perform Gauss - Seidel Method to solve it,

$$x^{(k)} = \frac{1}{10}\left[-y^{(k-1)} - z^{(k-1)} + 18\right]$$
$$y^{(k)} = \frac{1}{10}\left[-x^{(k)} - z^{(k-1)} + 18\right]$$
$$z^{(k)} = \frac{1}{10}\left[-x^{(k)} - y^{(k)} + 18\right], \quad k = 1, 2, \ldots$$

To obtain approximated solution for the system of linear equations up to n decimal places, $n \geq 1$, we stop the iterations when all the unknowns are having the absolute error smaller than 0.000005 ($n = 5$ decimal places, in this case).