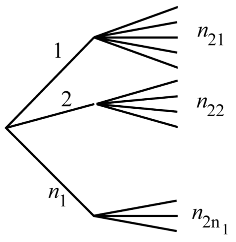
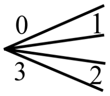
Chapter 1

1.1



(a) $\sum_{i=1}^{n_1} n_{2i}$

(b)



0 1 2 3
0 1 2 3
0 1 2
0 1

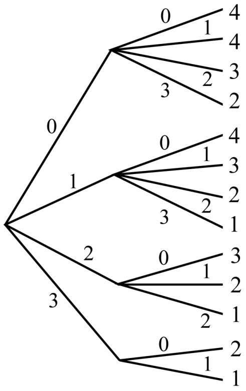
$\sum = 13$

1.2 $\sum_{i=1}^{n_1} n_2 i = \sum_{i=1}^{n_1} n_2 = n_1 n_2$

1.3 (a)

$n_{300} = 4$
$n_{301} = 4$
$n_{302} = 3$
$n_{303} = 2$
$n_{310} = 4$
$n_{311} = 3$
$n_{312} = 2$
$n_{313} = 1$

$n_{320} = 3$
$n_{321} = 2$
$n_{322} = 1$
$n_{330} = 2$
$n_{331} = 1$



(b) $\sum = 4 + 4 + 3 + ... + 2 + 1 = 32$

1.4 $\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} n_2 = n_1 n_2 n_3$

1.5 (b) 6, 20, and 70

"2 out of 3" $m = 2$ $2\left[\binom{1}{1}+\binom{2}{1}\right] = 2(1+2) = 6$

"3 out of 5" $m = 3$ $2\left[\binom{2}{2}+\binom{3}{2}+\binom{4}{2}\right] = 2(1+3+6) = 20$

"4 out of 7" $m = 4$ $2\left[\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\binom{6}{3}\right] = 2(1+4+10+20) = 70$

1.6 (a) $10! \approx \sqrt{20\pi}\left(\frac{10}{e}\right)^{10} = (7.92665)(3.678797)^{10} = (7.92665)(454{,}002.49) = 3{,}598{,}719$

$\%$ error $= \frac{3.6288 - 3.5987}{3.6288} \cdot 100 = 0.83\%$

$12! \approx \sqrt{24\pi}\left(\frac{12}{e}\right)^{12} = (8.683215)(4.41455)^{12} = 475{,}683{,}224$

$\%$ error $= \frac{4.7800 - 4.7568}{4.7900} \cdot 100 = 0.69\%$

(b) $$\binom{52}{13} = \frac{52!}{13!\ 39!} = \frac{\sqrt{104\pi}\left(\frac{52}{e}\right)^{52}}{\sqrt{26\pi}\sqrt{78\pi}\left(\frac{13}{e}\right)^{13}\left(\frac{39}{e}\right)^{39}}$$

$$= \frac{13^{52} \cdot 4^{52}}{\sqrt{19.5\pi}\ 13^{13} \cdot 13^{39} \cdot 3^{39}} = \frac{4^{52}}{\sqrt{19.5\pi}\ 3^{39}} = \text{ 639 billion}$$

1.7 Using Stirling's formula in $\binom{2n}{n} = \frac{2n!}{n!\ n!}$ yields

$$\frac{\binom{2n}{n}\sqrt{\pi n}}{2^{2n}} = \frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}{\left[\sqrt{2\pi n}\left(\frac{\pi}{e}\right)^{n}\right]^{2}} \cdot \frac{\sqrt{\pi n}}{2^{2n}} = 1$$

1.8 n^r and $12^3 = 1{,}728$

1.9 $\binom{r+n-1}{r}$ and $\binom{5+3-1}{5} = \binom{7}{5} = 21$

1.10 Substitute $r - n$ for r into result of 1.9

$$\binom{r-n+n-1}{r-n} = \binom{r-1}{r-n} \text{ and } \binom{5-1}{5-3} = \binom{4}{2} = 6$$

1.11 (b) Seventh row is 1, 6, 15, 20, 15, 6, 1

Eighth row is 1, 7, 21, 35, 35, 21, 7, 1

$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^2y^3 + 15x^2y^4 + 6xy^5 + y^6$$

$$(x+y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$$

1.14 (a) Set $x = 1$ and $y = 1$
(b) Set $x = 1$ and $y = -1$
(c) Set $x = 1$ and $y = a - 1$

1.19 (a) $$\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{24} = -\frac{15}{384} \text{ and } \frac{(-3)(-4)(-5)}{6} = -10$$

(b) $$\sqrt{5} = 2\left(1+\frac{1}{4}\right)^{\frac{1}{2}} = 2\left[1+\frac{1}{2}\left(\frac{1}{4}\right)+\frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{1}{4}\right)^2+\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{1}{4}\right)^3\right]$$
$$= 2\left[1+\frac{1}{8}-\frac{1}{64}+\frac{3}{512}\cdots\right] = 2\cdot\frac{512+64-8+3}{512}$$
$$= 2\cdot\frac{571}{512} = 2.23$$

$$\frac{1142}{512} = 2.230$$

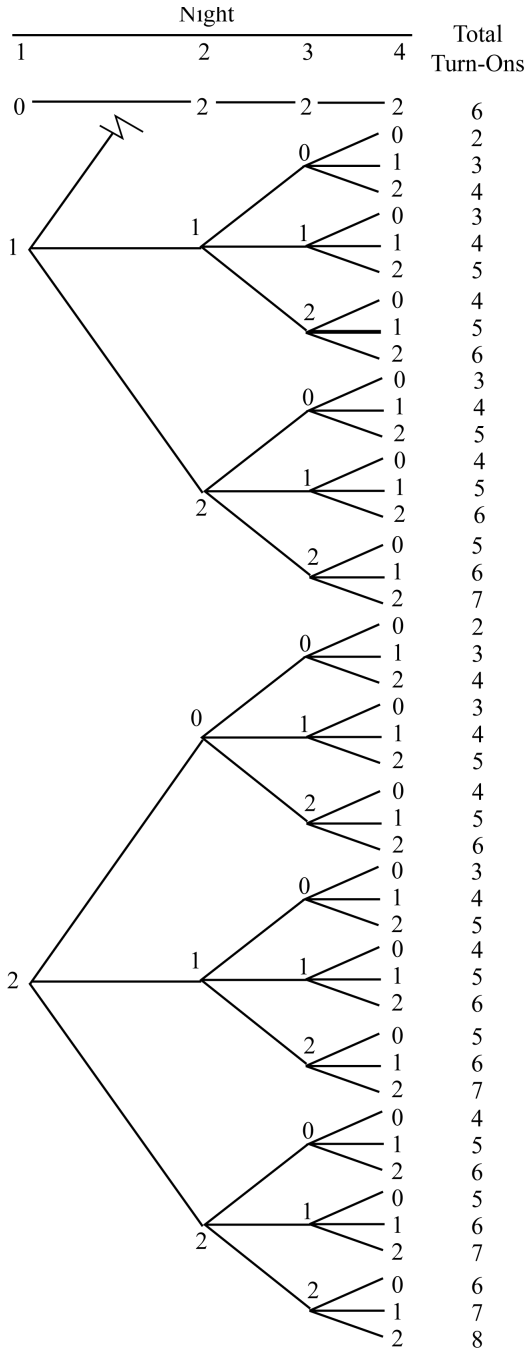
1.20 (a) $$\frac{(-1)(-2)\ldots(-r)}{r!} = (-1)^r$$

(b) $$\binom{-n}{r} = \frac{(-n)(-n-1)\ldots(-n-r+1)}{r!} = (-1)^r\,\frac{n(n+1)\ldots(n+r-1)}{r!}$$
$$= (-1)^r\,\frac{(n+r-1)\ldots(n+1)n}{r!} = (-1)^r\binom{n+r-1}{r}$$

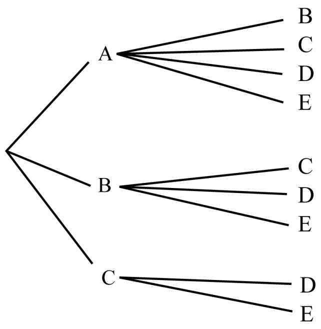
1.21 $$\frac{8!}{2!\;3!\;3!} = \frac{8\cdot7\cdot6\cdot5\cdot4}{2\cdot6} = 560$$

1.22 $$\frac{9!}{3!\;2!\;3!}\cdot2^3\cdot3^2\cdot(-4)^3 = \frac{9\cdot8\cdot7\cdot6\cdot5\cdot4}{12}\cdot8\cdot9\cdot64 = -23{,}224{,}320$$

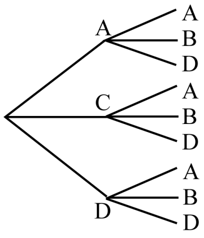
1.24 Note: If there are 0 turn-ons the first night, 6 turn-ons in four nights can only occur if there are 2 turn-ons on each of the subsequent three nights. Thus, we need to show only that part of the tree following this event.



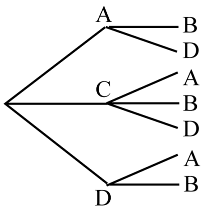
1.25



1.26 (a)



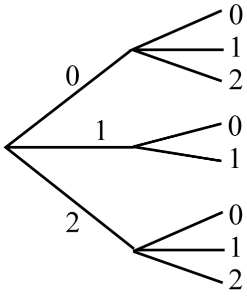
(b)



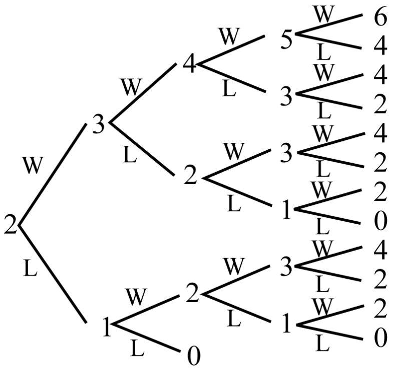
1.27 (a) 5

(b) 4

1.28



1.29



1.30 (a) $6 \cdot 5 = 30$; (b) $6 \cdot 6 = 36$;

1.31 (a) 6; (b) $6 \cdot 5 = 30$; (c) $5 \cdot 4 = 20$ first one fixed; (d) $6 + 30 + 20 = 56$

1.32 (a) $4 \cdot 5 \cdot 2 = 40$; (b) $5 \cdot 6 \cdot 3 = 90$

1.33 (a) $5 \cdot 4 = 20$; (b) $5 \cdot 4 \cdot 3 = 60$

1.34 $3^{15} = 14{,}348{,}907$

1.35 $\frac{15 \cdot 14}{2 \cdot 1} = 105$

1.36 (a) $10 \cdot 9 \cdot 8 \cdot 7 = 5040$; (b) $\frac{5040}{24} = 210$

1.37 (a) $\frac{14 \cdot 13}{2 \cdot 1} = 91$; (b) $\frac{14 \cdot 13 \cdot 12}{3 \cdot 2 \cdot 1} = 364$

1.38 $6! = 720$

1.39 $\frac{6!}{2!\,2!\,2!} = \frac{720}{8} = 90$

1.40 $5! = 120$ and $120 - 2 \cdot 4! = 72$

1.41 $7! = 5040$

1.42 (a) $5! = 120$; (b) $\frac{5!}{2!} = 60$

1.43 $\frac{10!}{3!\,3!\,2!} = \frac{3628800}{72} = 50{,}400$ and $\frac{8!}{3!\,2!} = \frac{40320}{12} = 3360$

1.44 $\frac{10!}{5!\,4!} = \frac{3628800}{120 \cdot 24} = 1{,}260$

1.45 $\frac{8!}{3!\,4!} = \frac{40320}{6 \cdot 24} = 280$

1.46 (a) $\binom{20}{7} = 77{,}520$; (b) $\binom{20}{10} = 184{,}755$

(c) $\binom{20}{17} + \binom{20}{18} + \binom{20}{19} + \binom{20}{20} = 1140 + 190 + 20 + 1 = 1351$

1.47 (a) $\binom{7}{2} = 21$; (b) $\binom{4}{2} = 6$; (c) $3 \cdot 4 = 12$

1.48 $\binom{3}{2}\binom{7}{2}+\binom{3}{3}\binom{7}{1}=3\cdot 21+1\cdot 7=63+7=70$

1.49 $\binom{4}{2}\binom{7}{3}\binom{3}{1}=6\cdot 35\cdot 3=630$

1.50 $\binom{13}{5}\binom{13}{3}\binom{13}{3}\binom{13}{2}=1287\cdot 286\cdot 286\cdot 78=8,211,173,256$

1.51 $\frac{7!}{3!\ 2!}=\frac{5040}{12}=420$

1.52 $3^{10}=59,049$

1.53 $5^{5}=15,625$

1.54 $\binom{12+6-1}{12}=\binom{17}{12}=\binom{17}{5}=6,188$

1.55 $\binom{12-1}{6}=\binom{11}{6}=462$

1.56 $\binom{14+3-1}{14}=\binom{16}{14}=120$

1.57 $\binom{r-2n+n-1}{n-1}=\binom{r-n-1}{n-1}$

$$\binom{r-n-1}{n-1}=\binom{10}{2}=45$$

Chapter 2

2.1 **(a)** $P[A] = P[(A \cap B) \cup (A \cup B')] = P(A \cap B) + P(A \cap B') \geq P(A \cap B)$

(b) $A \cup B = (A \cap B) \cup (A \cap B) \cup (A' \cap B) = A \cup (A' \cap B)$

2.6 $P(A) - P(A \cap B) = (a+b) - a = b = P(A \cap B')$

$P(A \cup B) = P(A) + P(A' \cap B) \geq P(A)$

2.7 $1 - P(A) - P(B) + P(A \cap B) = (a+b+c+d) - (a+b) - (a+c) + a = d$

$= P(A' \cap B')$

2.8 $P[(A \cap B') \cup (A' \cap B)] = b + c = (a+b) + a + c) - 2a$

$= P(A) + P(B) - 2P(A \cap B)$ Refer to Figure 2.6

2.9 **(a)** $P(A) + P(B) - P(A \cap B) \geq 0 \rightarrow P(A \cap B) \leq P(A) + P(B)$

(b) $P(A) + P(B) - P(A \cap B) \leq 1$ $P(A \cap B) \geq P(A) + P(B) - 1$

2.10 Refer to Figure 2.7 $P(A) = 1 \rightarrow e = c = f = 0$

$P(B) = 1 \rightarrow d = f = g = 0$

$P(C) = 1 \rightarrow b = e = g = 0$

Therefore $P(A) = a + b + d + g = a = 1$ QED

2.11 $P(A \cup B) = P(A) + P[A' \cap B)$

$= P(A) + P(A' \cap B) + P(A \cap B) - P(A \cap B)$

$= P(A) + P(B) - P(A \cap B)$ QED

2.12 $P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

$= (a + b + d + g + (a + b + c + e) + (a + c + d + f) - (a + b)$

$- (a + d) - (a + c) + a = a + b + c + d + e + f$

$= P(A \cup B \cup C \cup D)$

2.13 $$\begin{aligned}&P(A)+P(B)+P(C)+P(D)-P(A\cap B)-P(A\cap C)-P(A\cap D)-P(B\cap C)\\&-P(B\cap D)-P(C\cap D)+P(A\cap B\cap C)+P(A\cap B\cap D)\\&+P(A\cap C\cap D)+P(B\cap C\cap D)-P(A\cap B\cap C\cap D)\\&=(a+b+d+g+i+j+l+o)+(a+b+c+e+i+j+k+m)\\&\quad+(a+c+d+f+i+k+l+n)+(a+b+c+d+e+f+g+h)\\&\quad-(a+b+i+j)-(a+d+i+l)-(a+b+d+g)\\&\quad-(a+c+i+k)-(a+b+c+e)-(a+c+d+f)\\&\quad+(a+i)+(a+b)+(a+d)+(a+c)-a\\&=a+b+c+d+e+f+g+h+i+j+k+l+m+n+o\\&=P(A\cup B\cup C\cup D)\end{aligned}$$

2.14 For $n=2,\ P(E_1\cup E_2)=P(E_1)+P(E_2)-P(E_1\cap E_2)\le P(E_1)+P(E_2)$

Assume that for some n: $P(E_1\cup E_2\cup\ldots\cup E_n)=\sum_{j=1}^{n}P(E_j)$, then

$$\begin{aligned}&P\big((E_1\cup E_2\cup\ldots\cup E_n)\cup E_{n+1}\big)=P\big[(E_1\cup E_2\cup\ldots\cup E_n)\cup E_{n+1}\big]\\&\quad\le P(E_1\cup E_2\cup\ldots\cup E_n)+P(E_{n+1})\le\sum_{j=1}^{n+1}{}_{P(E_j)}\end{aligned}$$

where the first inequality follows from the first step of the induction, and the second inequality comes from the second step of the induction.

2.15 $\frac{p}{1-p}=\frac{A}{B}$, $pb=A-Ap$, $PA+pB=A$, $p(A+B)=A$, $p=\frac{A}{A+B}$

2.16 **(a)** P:ostulate 1 $P(A)=\frac{a}{a+b}\ge 0$

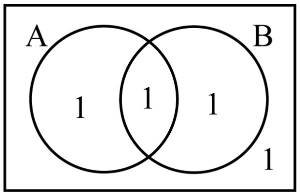
(B) Postulate 2 $P(A)=\frac{a}{a+b},\ P(A')=\frac{b}{a+b}$

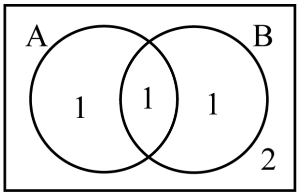
$$P(A)+P(A')=\frac{a}{a+b}+\frac{b}{a+b}=1=P(S)$$

2.17 **(a)** $P(A|B)=\frac{P(A\cap B)}{P(B)}\ge 0;$ **(b)** $P(B|B)=\frac{P(B\cap B)}{P(B)}=\frac{P(B)}{P(B)}=1$

(c) $$\begin{aligned}P(A_1\cup A_2\cup\ldots|B)&=\frac{P[A_1\cup A_2\cup\ldots)\cap B]}{P(B)}\\&=\frac{P(A_1\cap B)}{P(B)}+\frac{P(A_2\cap B)}{P(B)}+\cdots\\&=P(A_1|B)+P(A_2|B)+\cdots\end{aligned}$$

2.18





For example

(a) If $P(A \cap B) = P(A \cap B') = P(A' \cap B)$

$= P(A' \cap B') = \frac{1}{4}$ so that

$P(B|A) = \frac{1}{2}$, $P(B|A') = \frac{1}{2}$, and

$P(B|A) + P(B|A') = 1$

(b) If $P(A \cap B) = P(A \cap B') = P(A' \cap B) = \frac{1}{5}$

and $P(A' \cap B') = \frac{2}{5}$

$P(B|A) = \frac{1}{2}$, $P(B|A') = \frac{1}{3}$, and

$P(B|A) + P(B|A') = \frac{5}{6}$

2.19
$$\begin{aligned} P(A \cap B \cap C \cap D) &= P(A \cap B \cap C)P(D|A \cap B \cap C) \\ &= P(A \cap B)P(C|A \cap B)P(D|A \cap B \cap C) \\ &= P(A)P(B|A)P(C|A \cap B)P(D|A \cap B \cap C) \end{aligned}$$

2.20
$$P(C|A \cap B) = P(C|B) \rightarrow \frac{P(A \cap B \cap C)}{P(A \cap B)} = \frac{P(B \cap C)}{P(B)} \rightarrow \frac{P(A \cap B \cap C)}{P(B \cap C)} = \frac{P(A \cap B)}{P(B)}$$
$$\rightarrow P(A|B \cap C) = P(A|B)$$

2.21
$$P(B|A) = \frac{P(A \cap B)}{P(A)} = P(B) \rightarrow \frac{P(A \cap B)}{P(B)} = P(A) \rightarrow P(A|B) = P(A)$$

2.22 **(a)**
$$P(B) = P(A \cap B) + P(A' \cap B) = P(A)P(B) + P(A' \cap B)$$
$$P(A' \cap B) = P(B) - P(A)P(B) = P(B)[(1 - P(A)] = P(B)P(A') \quad \text{QED}$$

(b)
$$P(B') = P(A \cap B') + P(A' \cap B') = P(A' \cap B') + P(B')P(A)$$
$$P(A' \cap B') = P(B') - P(B')P(A\,|B') = P(B')[(1 - P(A)] = P(B')P(A') \quad \text{QED}$$

2.23 Assume that A and B' are independent and show that this leads to contradiction.

$$P(A) = P(A \cap B) + P(A \cap B') = P(A \cap B) + P(A)P(B')$$
$P(A \cap B) = P(A) - P(A)P(B') = P(A)[1 - P(B')] = P(A)P(B)$ and A and B are *independent*

2.24 $P(A) = 0.60$, $P(B) = 0.80$, $P(C) = 0.50$, $P(A \cap B) = 0.48$, $P(A \cap C) = 0.30$

$P(B \cap C) = 0.38$, $P(A \cap B \cap C) = 0.24$

$P(A \cap B \cap C) = 0.24$, $P(A)(B)(C) = (0.6)(0.8)(0.5) = 0.24$

$P(B \cap C) = 0.38$, $P(B)P(C) = (0.8)(0.5) = 0.40$ B and C *not independent*

2.25 Refer to 2.21

$P(A\cap B)=0.48,\ \ P(A)P(B)=(0.6)(0.8)=0.48$ A and B *independent*

$P(A\cap C)=0.30,\ \ P(A)P(C)=(0.6)(0.5)=0.30$ A and C *independent*

$P(B\cap C)=0.38,\ \ P(B)P(C)=(0.8)(0.5)=0.40$ B and C *not independent*

2.26 (Refer to 2.24 and 2.25) Already showed that A and B *independent*,.
A and C *independent*
$P[(A\cap(B\cap C)]=0.54,\ \ P(A)=0.60,\ \ P(B\cup C)=0.92,\ \ (0.6)(0.92)=0.552\neq 0.54$

2.27 **(a)** $P[(A\cap(B\cap C)]=P(A\cap B\cap C)=P(A)P(B)P(C)=P(A)P(B\cap C)$ QED

(b)
$$\begin{aligned}P[(A\cap(B\cup C)]&=P[(A\cap B)\cup(A\cap C)]\\&=P(A\cap B)+P(A\cap C)-P(A\cap B\cap C)\\&=P(A)P(B)+P(A)P(C)-P(A)P(B)P(C)\\&=P(A)[P(B)+P(C)-P(B\cap C)]\\&=P(A)P(B\cup C)\quad \text{QED}\end{aligned}$$

2.28 $P(A|B)\rightarrow \dfrac{P(A\cap B)}{P(A)}<P(B)\rightarrow P(B|A)<P(B)$

2.29 Proof by induction: If $n = 2$, then $P(A_1\cup A_2)=P(A_1)+P(A_2)-P(A_1)\cdot P(A_2)$
and $1-[1-P(A_1)]\cdot[1-P(A_2)]=1-1+P(A_1)+P(A_2)-P(A_1)P(A_2)$.
Assuming $P(A_1\cup A_2\cup\ldots\cup A_n)=1-[1-P(A_1)]\cdot[1-P(A_2)]\cdot\ldots\cdot[1-P(A_n)]$.
we can write
$$\begin{aligned}P(A_1\cup A_2\cup\ldots\cup A_n\cup A_{n+1})&=P(A_1\cup A_2\cup\ldots\cup A_n)+P(A_{n+1})\\&\quad-P(A_1\cup A_2\cup\ldots A_n)\cdot P(A_{n+1})\\&=P(A_1\cup A_2\cup\ldots\cup A_n)\cdot[1-P(A_{n+1})]+P(A_{n+1})\end{aligned}$$
$$=\{1-[P(A_1)]\cdot[1-P(A_2)]\cdot\ldots\cdot[1-P(A_n)]\}\cdot[1-P(A_{n+1})]+P(A_{n+1})$$
$$=1-[1-P(A_1)]\cdot[1-P(A_2)]\cdot\ldots\cdot[1-P(A_n)]\cdot[1-P(A_{n+1})]$$

2.30 Two at time $\begin{pmatrix}k\\2\end{pmatrix}$

Three at time $\begin{pmatrix}k\\3\end{pmatrix}$

k at time $\begin{pmatrix}k\\k\end{pmatrix}$

$$\begin{pmatrix}k\\2\end{pmatrix}+\begin{pmatrix}k\\3\end{pmatrix}+\ldots+\begin{pmatrix}k\\k\end{pmatrix}=2^k-\begin{pmatrix}k\\0\end{pmatrix}-\begin{pmatrix}k\\1\end{pmatrix}=2^k-1-k$$

2.31 $P(A\cap\varnothing)=P(A)\cdot P(\varnothing|A)=P(A)\cdot P(\varnothing),$ since $P(\varnothing|A)=P(\varnothing)=0.$

2.32 Since $B_1 \cup B_2 \cup \ldots \cup B_k = S$, $A \cap (B_1 \cup B_2 \cup \ldots \cup B_k) = A$. Thus, by the distributive property, $(A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_k) = A$, and
$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + \ldots + P(B_k)P(A|B_k)$ QED

2.33 The probability of no matches on any given trial is $\frac{n-1}{n}$; since the trials are independent, the probability of no match in n trials is $\left(\frac{n-1}{n}\right)^n = \left(1 - \frac{1}{n}\right)^n$.

2.34 $P(A \cup B) = P(A) + P(B) - P(A \cap B) = [1 - P(A')] + [1 - P(B')] - P(A \cap B)$
$= 1 - P(A') - P(B') + [1 - P(A \cap B)]$.
Since $1 - P(A \cap B) \geq 0$, $P(A \cup B) \leq 1 - P(A') - P(B')$ QED

2.35 **(a)** $\{6, 8, 9\}$; **(b)** **{8};** **(c)** $\{1, 2, 3, 4, 5, 8\}$; **(d)** $\{1, 5\}$;
(e) (2, 4, 8}; **(f)** $\varnothing$

2.36 **(a)** Los Angeles, Long Beach, Pasadena, Anaheim, Santa Maria, Westwood;
(b) San Diego, Long Beach, Pasadena, Anaheim, Santa Maria, Westwood;
(c) Santa Barbara; **(d)** $\varnothing$; **(e)** San Diego, Long Beach, Santa Barbara, Anaheim;
(f) San Diego, Santa Barbara, Long Beach; **(g)** Los Angeles, Santa Barbara, Anaheim;
(h) Los Angeles, Pasadena, Santa Maria, Westwood; **(i)** Los Angeles, Pasadena, Santa Maria, Westwood.

2.37 **(a)** $\{5, 6, 7, 8\}$; **(b)** $\{2, 4, 5, 7\}$; **(c)** $\{1, 8\}$ **(d)** (3, 4, 7, 8}

2.38 **(a)** He chooses a car with air conditioning.
(b) He chooses a car with bucket seats or no power steering.
(c) He chooses a car with bucket seats that is 2 or 3 years old.
(d) He chooses a car with bucket seats that is 2 or 3 years old.

2.39 **(a)** House has fewer than three baths;
(b) does not have fire place;
(c) does not cost more than $200,000
(d) is not new;
(e) has three or more baths and fire place;
(f) has three more baths and costs more than $200,000
(g) costs more than $200,000 but has no fire place;
(h) is new or costs more than $200,000
(i) is new or costs $200,000 or less
(j) has 3 or more baths and/or fire place;
(k) has 3 or more baths and/or costs more than $200,000;
(l) is new and costs more than $200,000

2.41 **(a)** (H,1), (H,2), (H,3), (H,4), (H,5), (H,6)
(T,H,H), (T,H,T), (T,T,H), (T,T,T)
(b) (H,1), (H,2), (H,3), (H,4), (H,5), (H,6), (T,H,T), (T,T,H)
(c) (H,5), (H,6), (T,H,T), (T,T,H), (T,T,T)

2.42 **(a)** S = {(0,0,0)…(1,1,1)}
A = {(1,0,1),(0,1,1),(1,1,1)}
B = {(0,1,1)}
C = {(1,0,1)}
(b) A & B *not* mutually exclusive, A & C *not* mutually exclusive, B & C are mutually exclusive.

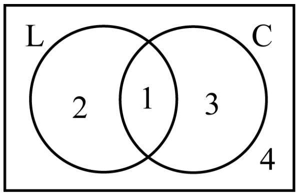
2.43 $3,\ x_1 3,\ x_1 x_2 3,\ x_1 x_2 x_3 3, \ldots$ where $x_1 = 1,2,4,5,6,$ for all i

(a) 5^{k-1}; **(b)** $1+5+\ldots 5^k = \dfrac{5^k-1}{4}$

2.44 $S = \{(x,y)\big|(x-2)^2+(y+3)^2 \le 9\}$

2.45 **(a)** $(x\big|3 < x < 10)$; **(b)** $(x\big|5 < x \le 8)$; **(c)** $(x\big|3 < x \le 5)$;
(d) $(x\big|0 < x \le 3 \text{ or } 5 < x < 10)$

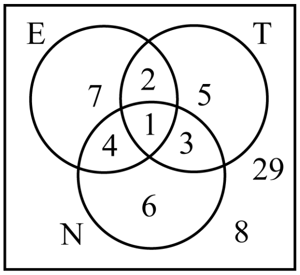
2.46



1 A driver has liability insurance and collision insurance.
2 A driver has liability insurance but not collision insurance.
3 A driver has collision insurance but not liability insurance.
4 A driver has neither liability insurance nor collision insurance.

2.47 **(a)** A driver has liability insurance.
(b) A driver does not have collision insurance.
(c) A driver has either liability or collision insurance, but not both.
(d) A driver does not have both kinds of insurance.

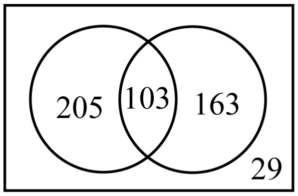
2.48



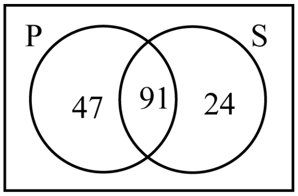
(a) A car brought to the garage needs engine overhaul, transmission repairs, and new tires.
(b) A car brought to the garage needs transmission repairs, new tires, but no engine overhaul.
(c) A car brought to the garage needs engine overhaul, but neither transmission repairs nor new tires.
(d) A car brought to the garage needs engine overhaul and new tires.
(e) A car brought to the garage needs transmission repairs, but no new tires.
(f) A car brought to the garage does not need engine overhaul.

2.49 **(a)** 5; **(b)** 1 and 2 together **(c)** 3, 5, and 6 together; **(d)** 1, 3, 4, and 6 together

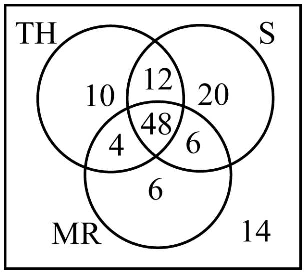
2.50 $500-(308+266)+103=29\neq 59$ results are *in*consistent



2.51 $200-(1.38+115)+91=38$



2.52 **(a)** 12; **(b)** 6; **(c)** 20



2.53 **(a)** permissible;
(b) not permissible because the sum of the probabilities exceeds 1;
(c) permissible;
(d) not permissible because $P(E)$ is negative
(e) not permissible because the sum of the probabilities is less than 1.

2.54 **(a)** $1-0.37=0.63$; **(b)** $1-0.44=0.56$; **(c)** $0.37+0.44=0.81$;
(d) 0; **(e)** 0.37, $P(A\cap B')=P(A)$ for mutually exclusive events;
(f) $1-0.81=0.19$

2.55 **(a)** Probability cannot be negative.
(b) $0.77+0.08=0.85\neq 0.95$
(c) $0.12+0.25+0.36+0.14+0.09+0.07=1.03>1$
(d) $0.08+0.21+0.29+0.40=0.98<1$

2.56 **(a)** $0.12+0.17=0.29$; **(b)** $0.17+0.34+0.29=0.80$
(c) $0.34+0.17+0.12=0.63$; **(d)** $0.34+0.29+0.08+0.71$

2.57 $(0,0), (1,0), (2,0), (3,0), (4,0), (5,0), (0,1), (1,1), (2,1), (3,1),$
$(4,1), (5,1), (0,2), (1,2), (2,2), (3,2), (4,2), (5,2), (0,3), (1,3),$
$(2,3), (3,3), (4,3), (5,3), (0,4), (1,4), (2,4), (3,4), (4,4), (5,4)$

(a) $\frac{10}{30}=\frac{1}{3}$; **(b)** $\frac{5}{30}=\frac{1}{6}$; **(c)** $\frac{15}{30}=\frac{1}{2}$; **(d)** $\frac{10}{30}=\frac{1}{3}$

2.58 **(a)** $\frac{20+10}{80}=\frac{3}{8}$; $\frac{4\cdot 5}{80}=\frac{1}{4}$; **(c)** $\frac{2\cdot 4}{80}=\frac{1}{10}$; **(d)** $\frac{4+2+1+1}{80}=\frac{1}{10}$;

(e) $\frac{8+14}{80}=\frac{22}{80}=\frac{11}{40}$

2.59 **(a)** $0.24+0.22=0.46$; **(b)** $0.15+0.03+0.22=0.40$
(c) $0.03+0.08=0.11$; **(d)** $0.15+0.03+0.28+0.22=0.68$

2.60 $$\frac{\binom{16}{2}}{\binom{52}{2}}=\frac{120}{1326}=\frac{20}{221}$$

2.61 Let $P(A)=4p,\ P(B)=2p, P(C)=2p,$ and $P(D)=p$.

Then $9p=1$ and $p=\frac{1}{9}$;

(a) $\frac{2}{9}$; **(b)** $1-\frac{4}{9}=\frac{5}{9}$

2.62 **(a)** $$\frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}44}{\binom{52}{5}}=\frac{13\cdot 12\cdot 6\cdot 6\cdot 44\cdot 120}{2\cdot 52\cdot 51\cdot 50\cdot 49\cdot 48}=\frac{198}{4165}=0.0475$$

(b) $$\frac{13\cdot 48}{\binom{52}{5}}=\frac{13\cdot 48\cdot 120}{51\cdot 51\cdot 50\cdot 49\cdot 48}=\frac{1}{4165}$$

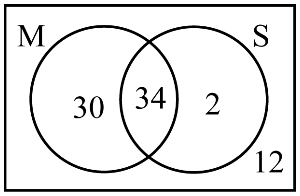
2.63 **(a)** $$\frac{\binom{6}{2}\binom{5}{2}\binom{3}{2}\cdot 4}{6^5}=\frac{15\cdot 10\cdot 3\cdot 4}{6\cdot 6\cdot 6\cdot 6\cdot 6}=\frac{25}{108}$$

(b) $$\frac{6\binom{5}{3}\cdot 5\cdot 4}{6^5}=\frac{6\cdot 10\cdot 5\cdot 4}{6\cdot 6\cdot 6\cdot 6\cdot 6}=\frac{25\cdot 4}{648}=\frac{25}{162}$$

(c) $$\frac{6\cdot 5\binom{5}{3}\binom{2}{2}}{6^5}=\frac{6\cdot 5\cdot 10}{6\cdot 6\cdot 6\cdot 6\cdot 6}=\frac{25}{648}$$

(d) $$\frac{6\binom{5}{4}\cdot 5}{6^5}=\frac{6\cdot 5\cdot 5}{6\cdot 6\cdot 6\cdot 6\cdot 6}=\frac{25}{1296}$$

2.65



$$\frac{78-[64+36-34]}{78}=\frac{12}{78}=\frac{2}{13}$$

2.64 **(a)** $P(A\cup B)$ is less than $P(A)$.
(b) $P(A\cap B)$ exceeds $P(A)$.
(c) $P(A\cup B)=0.72+0.84-0.52=1.04$ exceeds 1

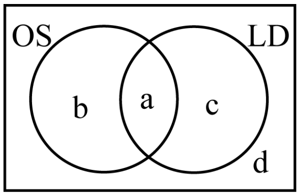
2.66 $\frac{2}{3}, 0$

2.67 The area of the triangle is $\frac{4\cdot 3}{2}=6$; If the point is a distance x from the vertex on the longer leg, then is will be $\frac{3x}{4}$ units from the vertex on the other leg. The area of the required triangle is $x\cdot\frac{3x}{4\cdot 2}=\frac{3x^2}{8}$. For this to be greater than 3, or half the area of the triangle, $x^2>8$, or $x>2\sqrt{2}$.
Thus, the probability of the line segment dividing the area in at least one-half is
$$\frac{4-2\sqrt{2}}{4}=1-\frac{\sqrt{2}}{2}$$

2.68 $0.21+0.28-0.15=0.34$

2.69 **(a)** $0.59+0.30-0.21=0.68$; **(b)** $0.59-0.21=0.38$
(c) $1-0.21=0.79$; **(d)** $1-0.68=0.32$

2.70



$$b+d=\frac{1}{3}$$

$$c+d=\frac{5}{9}$$

$$a+b+c=\frac{3}{4}; \text{ hence } d=\frac{1}{4}$$

$$b=\frac{1}{3}-\frac{1}{4}=\frac{1}{12},\ c=\frac{5}{9}-\frac{1}{4}=\frac{11}{36},\ a=1-\frac{1}{12}-\frac{11}{36}-\frac{1}{4}=\frac{13}{36}$$

$a = P$(out of state living on campus)
$b = P$(out of state not living on campus)
$c = P$(from Virginia living on campus)
$d = P$(from Virginia not living on campus)

2.71 **(a)** $(0.08)+0.05-0.02=0.11$; **(b)** $1-0.02=0.98$
(c) $0.08+0.05-2(0.02)=0.09$

2.72 $0.74+0.70+0.62-0.52-0.45-0.44+0.34=0.98$

2.73 $0.70+0.64+0.58+0.58-0.45-0.42-0.41-0.35-0.39-0.32$
$+0.23+0.26+0.21+0.20-0.12=0.94$

2.74 **(a)** The probability is $\frac{34}{34+21}=\frac{34}{55}$ that one of the eggs will be cracked.

(b) The probability is $\frac{11}{11+2}=\frac{11}{13}$ that they will not all be \$1 bills.

(c) The probability is $\frac{5}{5+1}=\frac{5}{6}$ that we will not get a meaningful word and $1-\frac{5}{6}=\frac{1}{6}$ that we will get a meaningful word.

2.75 **(a)** The odds are $\frac{6}{10}$ to $\frac{4}{10}$ or 3 to 2;

(b) The odds are $\frac{11}{16}$ to $\frac{5}{16}$ or 11 to 5;

(c) The odds are $\frac{7}{9}$ to $\frac{2}{9}$ or 7 to 2 against it.

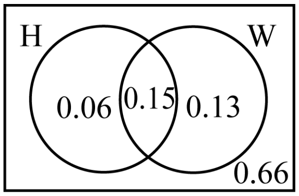
2.76 **(a)** $\frac{18+36}{90}=\frac{54}{90}=\frac{3}{5}$; **(b)** $\frac{36+27}{90}=\frac{63}{90}=\frac{7}{10}$; **(c)** $\frac{18}{90}=\frac{2}{10}=\frac{1}{5}$;

(d) $\frac{27}{90}=\frac{3}{10}$; **(e)** $\frac{18}{18+36}=\frac{18}{54}=\frac{1}{3}$; **(f)** $\frac{27}{27+36}=\frac{27}{63}=\frac{3}{7}$

2.77 **(a)** $\frac{1}{3} = \frac{1/5}{3/5}$ **(b)** $\frac{3}{7} = \frac{3/10}{7/10}$

2.78 $\frac{34}{34+2} = \frac{34}{36} = \frac{17}{18}$

2.79



$$\frac{0.15}{0.15+0.13} = \frac{0.15}{0.28} = \frac{15}{28}$$

2.80 $\frac{a}{a+b} = \frac{13/36}{13/36+1/12} = \frac{13/36}{13/36+3/36} = \frac{13}{16}$

2.81
$$P(R \cap W) = \frac{25 \cdot 40}{\binom{100}{2}}$$
$$= \frac{20}{99}$$

2.82 **(a)** $\frac{5}{12} + \frac{1}{12} = \frac{6}{12} = \frac{1}{2}$; consistent

(b) $\frac{1}{3} + \frac{1}{5} = \frac{8}{15} \neq \frac{7}{12}$; not consistent

2.83 **(a)**

Outcome	2	3	4	5	6	7	8	9	10	11	12
No.Combinations	1	2	3	4	5	6	5	4	3	2	1
Probability	1/36	1/18	1/12	1/9	5/36	1/6	5/36	1/9	1/12	1/18	1/36

(b) $(1+2+3+4+5+6+5+4+3+2+1)/36 = 1$

2.84 $\frac{1}{4} + \frac{3}{8} = \frac{5}{8}$; odds are 5 to 3 that either car will win.

2.85 Using MINITAB software, first we generate 1,000 uniformly distributed pseudo-random numbers, putting them in Column 1 (C1) as follows:

```
MTB> Random 1000 C1;
SUBC> Uniform 0.0 10.0.
```

Sorting these numbers facilitates counting the number that are less than 1. The sort is accomplished as follows:

```
MTB> Sort C1, C2;
SUBC> by C1.
```

When we did this, we obtained 111 numbers less than 1; thus, the required probability is estimated to be 111/1,000 = 0.111.

2.86 **(a)** Repeating the work of Exercise 2.59, we found the corresponding probability for the second set to be 99/1,000 = 0.099. Obtaining $P(A \cup B)$ is facilitated by using the LET command to add the two columns of random numbers and then sorting the resulting column. When we performed these operations, we noted that there were 22 cases in which the sum column contained a number less than 2. Thus, we estimated the required probability as 22/1,000 = 0.022.

(b) Using Theorem 2.7 with $P(A) = P(B) = 0.1$ we obtain $0.01 + 0.01 - 0.001 = 0.019$

2.87 $\frac{0.20}{0.20+0.30+0.10} = \frac{0.20}{0.60} = \frac{1}{3}$

2.88 **(a)** $\frac{0.52}{0.74} = \frac{25}{37}$; **(b)** $\frac{0.34}{0.52} = \frac{17}{26}$; **(c)** $\frac{0.18+0.16-0.10}{0.70+0.62-0.44} = \frac{0.24}{0.88} = \frac{3}{11}$

(d) $\frac{0.46-0.34}{0.30} = \frac{0.12}{0.30} = \frac{2}{5}$

2.89 $\frac{\binom{110}{3}}{\binom{120}{3}} = \frac{110 \cdot 109 \cdot 108}{120 \cdot 119 \cdot 118} = 0.7685$

2.90 $(0.55)(0.80) = 0.44$

2.91 **(a)** $(0.8)(0.2)(0.6) = 0.096$; **(b)** $(0.20)(0.40)(0.60) = 0.048$;

(c) $(0.8)(0.8)(0.2)(0.4) = 0.0512$; **(d)** $(0.8)(0.8) + (0.2)(0.6) = 0.76$

2.92 $\frac{15}{20} \cdot \frac{14}{19} \cdot \frac{13}{18} \cdot \frac{12}{17} = \frac{91}{323}$

2.93 **(a)** $\frac{3}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{3}{64}$; **(b)** $3 \cdot \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4} = \frac{27}{64}$

2.94 A even first, B even second, C same number both

$P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, $P(C) = \frac{1}{6}$, $P(A \cap B) = \frac{1}{4}$, $P(A \cap C) = \frac{3}{36} = \frac{1}{12}$

11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36,
41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66

$P(B \cap C) = \frac{1}{12}$, $P(A \cap B \cap C) = \frac{3}{36} = \frac{1}{12} \neq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{24}$

(a) Since $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, $\frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$, and $\frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$, events are pairwise independent.

(b) Since $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{6} \neq \frac{1}{12}$ the events are *not* independent.

2.95 **(a)** The required probability is approximately $(0.99)^4 = 0.9606$ (assuming independence). The exact probability is

$$\frac{990}{1{,}000}\cdot\frac{989}{999}\cdot\frac{988}{998}\cdot\frac{987}{997} = 0.9605$$

(b) The required probability is approximately $(0.99)^3(0.01) = 0.0097$ (assuming independence). The exact probability is

$$\frac{990}{1{,}000}\cdot\frac{989}{999}\cdot\frac{988}{998}\cdot\frac{10}{997} = 0.0097$$

2.96 **(a)** $(0.52)^3 = 0.1406$**;** **(b)** $(0.48)^2(0.52) = 0.1198$

2.97 $\frac{5}{10}\cdot\frac{4}{9}\cdot\frac{3}{8} = \frac{1}{12}$

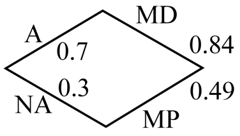
2.98 $1-(0.9)^{12} = 1-0.2824 = 0.7176$

2.99 $\frac{6}{15}\cdot\frac{5}{14}\cdot\frac{4}{13}\cdot\frac{3}{12} = \frac{1}{91}$

2.100 **(a)** $(0.9)(0.9)(0.9) = 0.729$
(b) $(0.6)(0.6)(0.4) = 0.144$

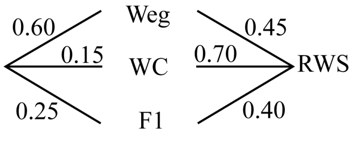
2.101 $P(A) = \frac{1}{2},\ P(B) = \frac{1}{2},\ P(C) = \frac{1}{3},\ P(D) = \frac{1}{3},\ P(A\cap B) = \frac{1}{4},\ P(A\cap C) = \frac{1}{6},$
$P(A\cap D) = \frac{1}{6},\ P(B\cap C) = \frac{1}{6},\ P(B\cap D) = \frac{1}{6},\ P(C\cap D) = \frac{1}{9},$
$P(A\cap B\cap C) = \frac{1}{12},\ P(A\cap B\cap D) = \frac{1}{12},\ P(A\cap C\cap D) = \frac{1}{18},$
$P(B\cap C\cap D) = \frac{1}{18},\ P(A\cap B\cap C\cap D) = \frac{1}{36}$**.** Substitution shows that all conditions for independence are satisfied.

2.102



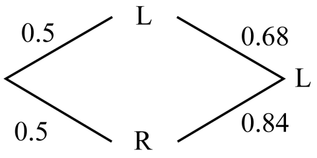
$(0.7)(0.84)+(0.3)(0.49) = 0.735$

2.103



$$(0.60)(0.45)+(0.15)(0.70)+(0.25)(0.40)$$
$$= 0.27+0.105+0.1 = 0.475$$

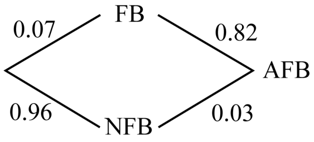
2.104



$$(0.5)(0.68)+(0.5)(0.84)=0.76$$

2.105 $\frac{0.27}{0.475}=0.5684$

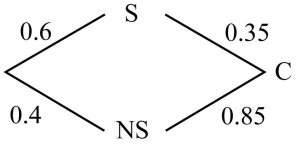
2.106



(a) $(0.04)(0.82)+(0.96)(0.03)$
$=0.0328+0.0288=0.0616$

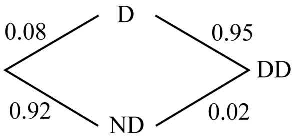
(b) $\frac{0.0328}{0.0616}=0.5325$

2.107



$$\frac{(0.6)(0.35)}{(0.6)(0.35)+(0.4)(0.85)}=\frac{0.21}{0.21+0.34}=\frac{0.21}{0.55}=0.3818$$

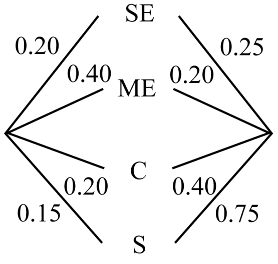
2.108



(a) $(0.08)(0.95)+(0.92)(0.02)$
$=0.076+0.0814=0.0944$

(b) $\frac{0.076}{0.0944}=0.8051$

2.109

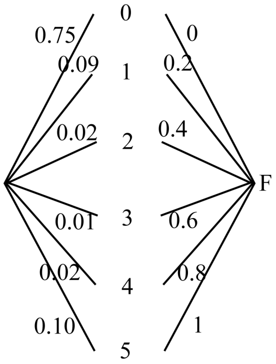


$0.05/0.3425=0.1460$
$0.08/0.3425=0.2336$
$0.10/0.3425=0.2920$
$0.1125/0.3425=0.3285$

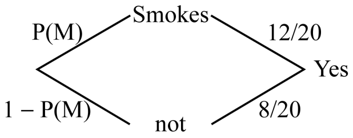
(a) Most likely cause is sabotage.
(b) Least likely cause is static electricity.

2.110 (a) 0.032; **(b)** 0.09375; **(c)** 0.625

2.111 $\dfrac{(0.10)1}{(0.09)(0.2)+(0.02)(0.4)+(0.01)(0.6)+(0.02)(0.8)+0.10}=\dfrac{0.10}{0.148}=0.6757$



2.112



$P(Y)=0.6\ P(M)+0.4[1-P(M)]$

(a) $P(Y)=0.4+0.2\ P(M)$

(b) $5P(Y)=2+P(M)$

$P(M)=5\cdot\dfrac{106}{250}-2=0.12$

2.113 $(0.95)^3(0.99)^3=0.832$

2.114 $(0.995)(0.990)(0.992)(0.995)(0.998)=0.970$

2.115 $R^6=0.95\therefore R=(0.95)^{1/6}=0.991$

2.116 $R^{10}=0.90\therefore R=(0.90)^{0.1}=0.990$

2.117 $1-(1-0.8)(1-0.7)(1-0.65)=0.979$

2.118 $1-(1-0.85)(1-0.80)(1-0.65)(1-0.60)(1-0.70)=0.999$

2.119 $(0.95)(0.90)\left[1-(1-0.60)^4\right]\left[1-(1-0.75)^2\right]=0.781$

2.120 $(0.98)(0.99)\left[1-(1-0.75)(1-0.60)(1-0.65)(1-0.70)(1-0.60)\right]=0.966$

Chapter 3

3.1 **(a)** No, because $f(4)$ is negative; **(b)** Yes; **(c)** No, because $f(1) + f(2) + f(3) + f(4) = \frac{18}{19}$ is less than 1.

3.2 **(a)** No, because $f(1)$ is negative; **(b)** Yes; **(c)** No, because $f(0) + f(1) + f(2) + f(3) + f(4) + f(5)$ is greater than 1.

3.3 $f(x) > 0$ for each value of x and

$$\sum_{x=1}^{k} f(x) = \frac{2}{k(k+1)}(1+2+\ldots+k) = \frac{2}{k(k+1)} \cdot \frac{k(k+1)}{2} = 1$$

3.4 **(a)** $c(1+2+3+\ldots 5) = 1$; thus $C = \frac{1}{15}$

(b) $c\left(5+\frac{5}{2}+\frac{5}{3}+\frac{5}{4}+1\right) = 1$**; thus,** $c = \frac{12}{137}$

(c) $$\sum_{x=1}^{k} f(x) = c\sum_{x=1}^{k} x^2 = cS(k,2)$$

From Theorem A.1 we obtain $S(k,2) = \frac{1}{6}k(k+1)(2k+1)$

Thus, for $f(x)$ to be a distribution function, $c = \frac{6}{k(k+1)(2k+1)}$, $k \neq 0$.

(d) $$\sum_{x=1}^{\infty} f(x) = c\sum_{x=1}^{\infty}\left(\frac{1}{4}\right)^x$$

The right-hand sum is a geometric progression with $a = 1$ and $r = 1/4$.
For $x = 1$ to n, this sum equals

$$S_n = \frac{1-\left(\frac{1}{4}\right)^n}{1-\frac{1}{4}} \to \frac{1/4}{3/4} = \frac{1}{3}$$ as $n \to \infty$. Therefore, $c = 3$.

3.5 For $f(x) = (1-k)k^x$ to converge to 1, $0 < k < 1$.

3.6 For $c > 0$, $f(x)$ diverges. For $c = 0$, $f(x) = 0$ for all x, and it cannot be a density function

3.9 **(a)** No, because $F(4) > 1$; **(b)** No, because $F(2) < F(1)$; **(c)** Yes.

3.10 $f(0)=\frac{4}{20}=\frac{1}{5}$; $f(1)=\frac{2\cdot 6}{20}=\frac{12}{20}=\frac{3}{5}$, $F(2)=\frac{4}{20}=\frac{1}{5}$

$$F(x)=\begin{cases} 0 & x<0 \\ 1/5 & 0\le x<1 \\ 4/5 & 1\le x<2 \\ 1 & 2\le x \end{cases}$$

3.11 **(a)** $\frac{5}{6}-\frac{1}{3}=\frac{1}{2}$; **(b)** $\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$; **(c)** $f(1)=\frac{1}{3}$, $f(4)=\frac{1}{6}$, $f(6)=\frac{1}{3}$ and $f(10)=\frac{1}{6}$, 0 elsewhere.

3.12

$$F(x)=\begin{cases} 0 & x<1 \\ 1/15 & 1\le x<2 \\ 3/15 & 2\le x<3 \\ 6/15 & 3\le x<4 \\ 10/15 & 4\le x<5 \\ 1 & 5\le x \end{cases}$$

3.13 **(a)** $\frac{3}{4}$ **(b)** $\frac{3}{4}-\frac{1}{2}=\frac{1}{4}$ **(c)** $\frac{1}{2}$ **(d)** $1-\frac{1}{4}=\frac{3}{4}$

(e) $\frac{3}{4}-\frac{1}{4}=\frac{1}{2}$ **(f)** $1-\frac{3}{4}=\frac{1}{4}$

3.14 $f(1)=\frac{3}{25}$, $f(2)=\frac{4}{25}$, $f(3)=\frac{5}{25}$, $f(4)=\frac{6}{25}$, $f(5)=\frac{7}{25}$

$$F(x)=\begin{cases} 0 & x<1 \\ 3/25 & 1\le x<2 \\ 7/25 & 2\le x<3 \\ 12/25 & 3\le x<4 \\ 18/25 & 4\le x<5 \\ 1 & 5\le x \end{cases}$$

$F(1)=\frac{6}{50}=\frac{3}{25}$, $F(2)=\frac{14}{50}=\frac{7}{25}$, $F(3)=\frac{24}{50}=\frac{12}{25}$, $F(4)=\frac{36}{50}=\frac{18}{25}$, $F(5)=\frac{50}{50}=1$, checks

3.15 **(a)** $P(x>x_1)=1-P(x\le x_1)=1-F(x_1)$ for $i = 1, 2, \ldots, n$

(b) $P(x>x_1)=1-P(x<x_i)=1-F(x_{i-1})$ for $i = 2, \ldots, n$ and

$P(x\ge x_1)=1$

3.16 $$F(x) = \begin{cases} 0 & x \le 2 \\ \frac{1}{5}(x-2) & 2 < x < 7 \\ 1 & 7 \le x \end{cases}$$

3.17 **(a)** $$\int_{-\infty}^{\infty} f(x)dx = \int_{2}^{7} \frac{1}{5} dx \frac{1}{5} \cdot x \Big|_{2}^{7} = \frac{1}{5}(7-2) = 1$$

(b) $$\int_{3}^{5} \frac{1}{5} dx = \frac{1}{5}(5-3) = \frac{2}{5}$$

3.18 **(a)** $f(x) \ge 0,\quad 0 < x < \infty,$ and $\int_{0}^{\infty} f(x)dx = \int_{0}^{\infty} e^{-x}dx = e^{0} = 1$

(c) $$P(x > 1) = \int_{1}^{\infty} e^{-x}dx = e^{-1}$$

3.19 **(a)** $f(x) \ge 0,\quad 0 < x < 1$ and $\int_{0}^{1} f(x)dx = 1$

(c) $$P(0.1 < x < 0.5) = \int_{0.1}^{0.5} 3x^{2}dx = 0.124$$

3.20 **(a)** $$\int_{2}^{3.2} \frac{1}{8}(y+1)dy = \frac{1}{8}\left(\frac{y^{z}}{2} + y\right)\Big|_{2}^{3.2} = \frac{1}{8}(8.32-4) = 0.54$$

(b) $$\int_{2.9}^{3.2} \frac{1}{8}(y+1)dy = \frac{1}{8}\left(\frac{y^{x}}{2} + y\right)\Big|_{2.9}^{3.2} = \frac{1}{8}(8.32-7.105) = 0.1519$$

3.21 $$\int_{2}^{y} \frac{1}{8}(t+1)dt = \frac{1}{8}\left(\frac{t^{2}}{2} + y\right)\Big|_{2}^{y} = \frac{1}{8}\left(\frac{y^{2}}{2} + y\right) - \frac{1}{8} \cdot 4 = \frac{1}{8}\left(\frac{y^{2}}{2} + y - 4\right)$$

$$F(y) = \begin{cases} 0 & y \le 2 \\ \frac{1}{8}\left(\frac{y^{2}}{2} + y - 4\right) & 2 < y < 4 \\ 1 & 4 \le y \end{cases}$$

(a) $$F(3.2) = \frac{1}{8}\left(\frac{3.2^{2}}{2} + 3.2 - 4\right) = 0.54$$

(b) $F(3.2)=F(2.9)=0.54-\frac{1}{8}\left(\frac{2.9^2}{2}+2.9-4\right)=0.54-0.3881=0.1519$

3.22 **(a)** $1=\int_0^4 \frac{c}{\sqrt{x}}dx=c\int_0^4 x^{-1/2}dx=c\frac{x^{1/2}}{1/2}\Big|_0^4=4c \qquad c=\frac{1}{4}$

(b) $P\left(x<\frac{1}{4}\right)=\int_0^{1/4}\frac{1}{4\sqrt{x}}dx=\frac{1}{4}\int x^{-1/2}dx=\frac{1}{4}\frac{\sqrt{x}}{1/2}\Big|_0^{1/4}=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$

$P(x>1)=1-\int_0^1\frac{1}{4\sqrt{x}}dx=1-\frac{1}{2}\sqrt{x}\Big|_0^1=\frac{1}{2}$

3.23 $F(x)=\frac{1}{2}\sqrt{x}$

$$F(x)=\begin{cases}0 & x\le 0\\ \frac{1}{2}\sqrt{x} & 0<x<4\\ 1 & 4\le x\end{cases}$$

$F\left(\frac{1}{4}\right)=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$ and $1=F(1)=1-\frac{1}{2}=\frac{1}{2}$

3.24 $F(z)=k\int_0^z ze^{-z^z}dz=k\int_0^z\frac{1}{2}e^{-u}du=\frac{k}{2}[1-e^{-u}]=\frac{k}{2}(1-e^{-z^z}) \qquad k=2$

3.25 $F(z)=\begin{cases}0 & z\le 0\\ 1-e^{-z^z} & z>0\end{cases}$

3.26 $P\left(x<\frac{1}{4}\right)=(3x^2-2x^3)\Big|_0^{1/4}=\frac{3}{16}-\frac{1}{32}=\frac{5}{32}$

$P\left(x>\frac{1}{2}\right)=\int_{1/2}^1 6x(1-x)dx=(3x^2-2x^3)\Big|_{1/2}^{1}=1-\left(\frac{3}{4}-\frac{1}{4}\right)=\frac{1}{2}$

3.27 $F(x)=\int_0^x 6x(1-x)dx=3x^2-2x^3 \qquad F(x)=\begin{cases}0 & x\le 0\\ 3x^2-2x^3 & 0<x<1\\ 1 & 1\le x\end{cases}$

$P\left(x<\frac{1}{4}\right)=\frac{3}{16}-\frac{2}{64}=\frac{5}{32}$ and $P\left(x>\frac{1}{2}\right)=1-\left(\frac{3}{4}-\frac{2}{8}\right)=\frac{1}{2}$

3.28 $F(x) = \int_0^x x\ dx = \frac{x^2}{2} \quad \text{0 to 1}$

$$F(x) = \frac{1}{2} + \int_1^x (2-x)dx = \frac{1}{2} + \left(2x - \frac{x^2}{2}\right)\Bigg|_1^x = \frac{1}{2} + 2x - \frac{x^2}{2} - \frac{3}{2}$$

$$= 2x - \frac{x^2}{2} - 1$$

$$F(x) = \begin{cases} 0 & x \le 0 \\ \frac{x^2}{2} & 0 < x < 1 \\ 2x - \frac{x^2}{2} - 1 & 1 \le x < 2 \\ 1 & 2 \le x \end{cases}$$

3.29 $F(x) = \int_0^x \frac{1}{3}dx = \frac{1}{3}x \qquad \text{0 to 1} \quad F(x) = \frac{1}{3} \qquad \text{1 to 2}$

$$F(x) = \frac{1}{3}(x-2) \quad \text{2 to 4} \qquad F(x) = \begin{cases} 0 & x \le 0 \\ \frac{1}{3}x & 0 < x < 1 \\ \frac{1}{3} & 1 \le x \le 2 \\ \frac{1}{3}(x-1) & 2 < x < 4 \\ 1 & 4 \le x \end{cases}$$

$$= \frac{1}{3}(x-1$$

3.30 **(a)** $\int_{0.8}^{1} x\ dx + \int_{1}^{1.2} (2-x)dx = \frac{x^2}{2}\Bigg|_{0.8}^{1} + \left(2x - \frac{x^2}{2}\right)\Bigg|_{1}^{1.2} = \left(\frac{1}{2} - 0.32)\right) + \left(2.4 - 0.72 - 2 + \frac{1}{2}\right) = 0.36$

(b) $F(1.2) - F(0.8) = 2(1.2) - \frac{(1.2)^2}{2} - 1 - \left(\frac{(0.8)^2}{2}\right)$

$$= 2.4 - 0.72 - 1 - 0.32 = 0.36$$

3.31

$$\begin{array}{lll} x \le 0 & F(x) = 0 & \\ 0 < x \le 1 & F(x) = \frac{x^2}{4} & F(1) = \frac{1}{4} \\ 1 < x \le 2 & F(x) = \frac{1}{2}x - \frac{1}{4} & F(2) = \frac{3}{4} \\ 2 < x < 3 & F(x) = \frac{3}{2}x - \frac{x^2}{4} - \frac{5}{4} & F(3) = 1 \\ 3 \le x & F(x) = 1 & \end{array}$$

3.32 **(a)** $F\left(\frac{1}{2}\right)-F\left(-\frac{1}{2}\right)=\frac{3}{4}-\frac{1}{4}=\frac{1}{2}; \quad F(3)-F(2)=1-1=0$

3.33 $\frac{\text{dF}}{\text{dx}}=\frac{1}{2}, \quad f(x)=\frac{1}{2}$ for $-1<x<1$; 0 elsewhere

$P\left(-\frac{1}{2}<x<\frac{1}{2}\right)=\frac{1}{2}\cdot 1=\frac{1}{2}; \quad P(2<x<3) \ =0$

3.34 **(a)** $F(5)=1-\frac{9}{25}=\frac{16}{25}$

(b) $1-F(8)=1-1+\frac{9}{64}=\frac{9}{64}$

3.35 $\frac{\text{dF}}{dy}=\frac{18}{y^2}$ for $y>0$; elsewhere

(a) $\int_3^5 \frac{18}{y^2}dy=-\frac{9}{y^2}\Big|_3^5=-\frac{9}{25}+1=\frac{16}{25}$; **(b)** $\int_8^\infty \frac{18}{y^2}dy=-\frac{9}{y^2}\Big|_8^\infty=0+\frac{9}{64}=\frac{9}{64}$

3.37 $P(x\le 2)=F(2)=1-3e^{-2}=1-3(0.1353)=1-0.4074=0.5926$

$P(1<x<3)=F(3)-F(1)=1-4e^{-2}-1+2e^{-1}-4e^{-2}$
$=2(0.3679)-4(0.0498)=0.7358-0.1992=0.5366$

$P(x>4)=1-F(4)=5e^4=5(0.0183)=0.0915$

3.38 $\frac{dF}{dx}=xe^{-x}$ for >0; 0 elsewhere

3.39 **(a)** for $x\le 0$ $\quad F(x)=0$
(b) for $0<x<0.5$ $\quad F(x)=\frac{1}{2}x$
(c) for $0.5\le x<1$ $\quad F(x)=\frac{1}{2}\left(x-\frac{1}{2}\right)+\frac{3}{4}=\frac{1}{2}(x+1)$
(d) for $x\ge 1$ $\quad f(x)=0$

3.40 **(a)** $f(x)=0$; **(b)** $f(x)=\frac{1}{2}$; **(c)** $f(x)=\frac{1}{2}$; **(d)** $f(x)=0$

3.41 $P(Z=-2)=\frac{-2+4}{8}=\frac{1}{4}, \ P(Z=2)=\frac{1}{4}, \ P(-2<Z<1)=\frac{5}{8}-\frac{1}{4}=\frac{3}{8}$

and $P(0\le z\le 2)=1-\frac{1}{2}=\frac{1}{2}$

3.42 **(a)** $\frac{1}{20}$; **(b)** $\frac{1}{4}+\frac{1}{8}=\frac{3}{8}$; **(c)** $\frac{1}{6}+\frac{1}{4}+\frac{1}{12}=\frac{1}{2}$; **(d)** $\frac{1}{6}+\frac{1}{24}+\frac{1}{40}=\frac{28}{120}=\frac{7}{30}$

3.43 **(a)** $\frac{1}{6}+\frac{1}{12}=\frac{1}{4}$; **(b)** 0; **(c)** $\frac{1}{12}+\frac{1}{6}+\frac{1}{24}=\frac{7}{24}$**;** **(d)** $1-\frac{1}{120}=\frac{119}{120}$

3.44 $c(2+5+10+1+4+9+2+5+10+10+13+18)=1$

$c=\frac{1}{89}$

3.45 **(a)** $\frac{1}{89}(10+9+10)=\frac{29}{89}$; **(b)** $\frac{1}{89}(1+4)=\frac{5}{89}$

(c) $\frac{1}{89}(9+5+10+13+18)=\frac{55}{89}$

3.46 **(a)** $k(0+2+8+0-1+2)=1$

f(3, 1) differs in sign from all other terms

3.47

		x			
		0	1	2	3
	0	0	$\frac{1}{30}$	$\frac{1}{15}$	$\frac{1}{10}$
y	1	$\frac{1}{30}$	$\frac{1}{15}$	$\frac{1}{10}$	$\frac{2}{15}$
	2	$\frac{1}{15}$	$\frac{1}{10}$	$\frac{2}{15}$	$\frac{1}{6}$

density

	y			
	0	1	2	3
0	0	$\frac{1}{30}$	$\frac{1}{10}$	$\frac{1}{5}$
1	$\frac{1}{30}$	$\frac{2}{15}$	$\frac{3}{10}$	$\frac{8}{15}$
2	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{3}{5}$	1

joint distribution function

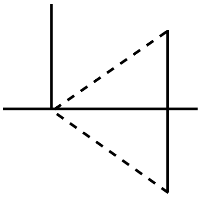
3.48 **(a)** $P(x\le-\infty, y\le-\infty)=0$

(b) $P(x\le\infty, y\le\infty)=1$

(c) $F(b,c)=F(a,c)+$ three probabilities

$F(b,c)\ge F(a,c)$

3.49



$$k\int_0^1\int_{-x}^{x} x(x-y)dy\ dx = k\int_0^1\left(x^2y-\frac{xy^2}{2}\right)\Bigg|_{-x}^{x} dx$$

$$k\int_0^1\left(x^3-\frac{x^3}{2}+x^3+\frac{x^3}{2}\right)dx = k\int_0^1 2x^3\ dx=\frac{k}{2}=1$$

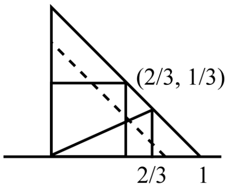
$k=2$

3.50 $$24\int_0^{1/2}\int_0^{1/2-x} xy\ dy\ dx = 24\int_0^{1/2} \frac{xy^2}{2}\bigg|_0^{1/2-x} dx = 12\int_0^{1/2} x\left(\frac{1}{2}-x\right)^2 dx$$

$$= 12\int_0^{1/2}\left(\frac{x}{4}-x^2+x^3\right)dx = 12\left[\frac{x^2}{8}-\frac{x^2}{3}+\frac{x^4}{4}\right]\Bigg|_0^{1/2} = 12\left[\frac{1}{32}-\frac{1}{24}+\frac{1}{64}\right]$$

$$= \frac{12}{64\cdot 3}(6-8+3) = \frac{12}{3\cdot 64} = \frac{1}{16}$$

3.51



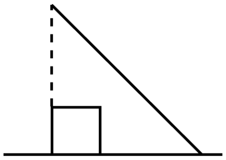
(a) $\frac{1}{2}$

(b) $1-2\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{2}{3} = 1-\frac{4}{9} = \frac{5}{9}$

(c) $2\left(\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{3}+\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{3}\right) = \frac{3}{9} = \frac{1}{3}$

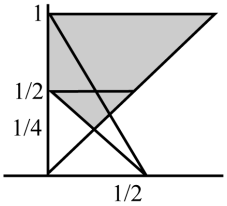
$F(x,y) = 2xy$ for $x>0,\ y>0,\ x+y<1$

3.52



(a) $2\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}$

3.53



$$\int_{1/4}^{1/2}\frac{1}{y}\int_{1/2-y}^{y} dx\, dy + \int_{1/2}^{1}\frac{1}{y}\int_0^{y} dx\, dy$$

$$= 1-\frac{1}{2}\ln\ 2\text{=1-0.3466=0.6534}$$

3.54 $$\frac{\partial F}{\partial y\,\partial x} = 2xe^{=x^2}\cdot 2ye^{-y^2} = 4xy^{-x^2}e^{-y^2} = 4xye^{-(x^2+y^2)} \qquad x>0,\ y>0$$

and $f(x,y)=0$ elsewhere

3.55 $$\int_1^2 2xe^{-x^2}dx\int_1^2 2ye^{-y^2}dy = \left[\int_1^4 e^{-u}du\right]^2 = \left(-e^{-u}\bigg|_1^4\right)^2 = (e^{-1}-e^{-4})^2$$

3.56 $$\frac{\partial F}{\partial x} = e^{-x}-e^{-x-y}\ \frac{\partial^2 F}{\partial x\partial y} = e^{-x-y} \qquad x>0,\ y>0$$

= 0 elsewhere

3.57 $$\int_2^3 e^{-X}dx\int_2^3 e^{-y}dy = \left[-e^{-x}\bigg|_2^3\right]^2 = (e^{-2} = e^{-3})^2$$

3.58 $F(b,d)-F(a,d)-F(b,c)+F(a,c)$

3.59 $a = 1, b = 3, c = 1, d = 2$

$$
\begin{aligned}
& F(3,2) - F(1,2) - F(3,1) + F(1,1) \\
& = (1-e^{-3})(1-e^{-2}) - (1-e^{-1})(1-e^{-2}) - (1-e^{-3})(1-e^{-1}) + (1-e^{-1})(1-e^{-1}) \\
& = (1-e^{-2})\left[(1-e^{-3}) - (1-e^{-1})\right] - (1-e^{-1})\left[(1-e^{-2}) - (1-e^{-1})\right] \\
& = \left[(1-e^{-2})(1-e^{-1})\right]\left[(1-e^{-3}) - (1-e^{-1})\right] \\
& = (e^{-1} - e^{-2})(e^{-1} - e^{-3}) = 0.074
\end{aligned}
$$

3.60

$$
\begin{aligned}
& F(2,2) - F(1,2) - F(2,1) + F(1,1) \\
& = (1-e^{-4})(1-e^{-4}) - (1-e^{-1})(1-e^{-4}) - (1-e^{-1})(1-e^{-4}) + (1-e^{-1})(1-e^{-1}) \\
& = (1-e^{-4})\left[(1-e^{-4}) - (1-e^{-1})\right] - (1-e^{-1})\left[(1-e^{-4}) - (1-e^{-1})\right] \\
& = (1-e^{-4})(e^{-1} - e^{-4}) - (1-e^{-1})(e^{-1} - e^{-4}) \\
& = (e^{-1} - e^{-4})(e^{-1} - e^{-4}) = (e^{-1} - e^{-4})^2
\end{aligned}
$$

3.61

$$
\begin{aligned}
& F(3,3) - F(2,3) - F(3,2) + F(2,2) \\
& = (1 - e^{-3} - e^{-3} + e^{-6}) \\
& -(1 - e^{-2} - e^{-3} + e^{-5}) - (1 - e^{-2} - e^{-2} + e^{-5} + (1 - e^{-2} - e^{-2} + e^{-4}) \\
& = e^{-4} - 2e^{-5} + e^{-6} = (e^{-2} - e^{-3})^2 \qquad \text{QED}
\end{aligned}
$$

3.62 $x = 1, 2$

$y = 1, 2, 3$

$z = 1, 2$

$$(1+2+2+4+3+6+2+4+4+8+6+12)k = 1$$

$$k = \frac{1}{54}$$

3.63 **(a)** $\frac{1}{54}(1+2) = \frac{1}{18}$

(b) $\frac{1}{54}(8+6) = \frac{14}{54} = \frac{7}{27}$

3.64 **(a)** $\frac{1}{54}(1+2+2+4) = \frac{9}{54} = \frac{1}{6}$**;** (b) 0; **(c)** 1

3.65

$$\int_0^1 \int_0^{1-z} \int_0^{1-y-z} xy(1-z)\,dx\,dy\,dz$$

$$\int_0^1 \int_0^{1-z} \frac{1}{2}(1-y-z)^2 y(1-z)\,dy\,dz$$

$$k\int_0^1 \int_0^{1-z} \int_0^{1-y-z} xy(1-z)\,dx\,dy\,dz = 1 \quad k = 144$$

3.66 $$\int_0^{1/2}\int_0^{1/2-x}\int_0^{1-x-y} 144\, xy(1-z)\, dz\, dy\, dx = 0.15625$$

3.68 **(a)** $$\frac{1}{3}\int_0^{1/2}\int_0^{1/2}\int_0^{1/2}(2x+3y+z)\, dz\, dy\, dx$$

$$= \frac{1}{3}\int_0^{1/2}\int_0^{1/2}\left[(2x+3y)z+\frac{z^2}{2}\right]\begin{matrix}1/2\\0\end{matrix} dy\, dx$$

$$= \frac{1}{3}\int_0^{1/2}\int_0^{1/2}\left(x+\frac{3}{2}y+\frac{1}{8}\right)dy\, dx$$

$$= \frac{1}{3}\int_0^{1/2}\left(xy+\frac{3}{4}y^2+\frac{1}{8}y\right)\Bigg|\begin{matrix}1/2\\0\end{matrix} dx = \frac{1}{3}\int_0^{1/2}\left(\frac{1}{2}x+\frac{3}{16}+\frac{1}{16}\right)dx$$

$$= \frac{1}{3}\left(\frac{1}{16}+\frac{3}{32}+\frac{1}{32}\right) = \frac{1}{3}\cdot\frac{6}{32} = \frac{1}{16}$$

3.69 **(a)** $g(-1)=\frac{1}{4},\ g(1)=\frac{3}{4}$

(b) $h(-1)=\frac{5}{8},\ h(0)=\frac{1}{4},\quad h(1)=\frac{1}{8}$

(c) $f(-1|-1)=\frac{1/8}{1/8+1/2}=\frac{1}{5};\quad f(1|-1)=\frac{1/2}{1/8+1/2}=\frac{4}{5}$

3.70 **(a)** $g(0)=\frac{1}{12}+\frac{1}{4}+\frac{1}{8}=\frac{1}{120}=\frac{7}{15};\quad g(1)=\frac{1}{6}+\frac{1}{4}+\frac{1}{20}=\frac{7}{15}$

$g(2)=\frac{1}{24}+\frac{1}{40}=\frac{1}{15}$

(b) $h(0)=\frac{1}{12}+\frac{1}{6}+\frac{1}{24}=\frac{7}{24}$**;** $\quad h(1)=\frac{1}{4}+\frac{1}{4}+\frac{1}{40}=\frac{21}{40}$

$h(2)=\frac{1}{8}+\frac{1}{20}=\frac{7}{40}$**;** $\quad h(3)=\frac{1}{120}$

(c) $f(0|1)=\frac{1/4}{21/40}=\frac{10}{21}$**;** $\quad f(1|1)=\frac{10}{21}$**;** $\quad f(2|1)=\frac{1/40}{21/20}=\frac{1}{21}$

(d) $w(0|0)=\frac{1/12}{56/120}=\frac{5}{28}$**;** $\quad w(1|0)=\frac{1/4}{56/120}=\frac{15}{28}$**;** $\quad w(2|0)=\frac{1/8}{56/120}=\frac{15}{56}$

$w(3|0)=\frac{1/120}{56/120}=\frac{1}{56}$

3.71 **(a)** $m(x,y)=\frac{xy}{108}(1+2)=\frac{xy}{36}$ for $x=1,2,3$; $y=1,2,3$

(b) $n(x,z)=\frac{xz}{108}(1+2+3)=\frac{xz}{18}$ for $x=1,2,3$; $z=1,2$

(c) $g(x)=\frac{x}{36}(1+2+3)=\frac{x}{6}$ for $x=1,2,3$

(d) $\phi(z|1,2)=\frac{z/64}{2/36}=\frac{z}{3}$ for $z=1,2$

(e) $\psi(y,z\Big|3)=\frac{yz/36}{1/2}=\frac{yz}{18}$ for $y=1,2,3$; $z=1,2$

3.72 **(a)** $g(0)=\frac{5}{12}$, $g(1)=\frac{1}{2}$; $g(2)=\frac{1}{12}$

$$G(x)=\begin{cases}0 & x<0\\ 5/12 & 0\le x<1\\ 11/12 & 1\le x<2\\ 1 & 2\le x\end{cases}$$

(b) $f(0|1)=\frac{2/9}{7/18}=\frac{4}{7}$

$f(1|1)=\frac{1/6}{7/18}=\frac{3}{7}$

$$F(x|1)=\begin{cases}0 & x<0\\ 4/7 & 0\le x<1\\ 1 & 1\le x\end{cases}$$

3.73 **(a)** $f(x)=\frac{1}{2}$ for $x=-1,\ 1$; $g(y)=\frac{1}{2}$ for $y=-1,\ 1$; $\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$, independent

(b) $f(0)=\frac{2}{3}$, $f(1)=\frac{1}{3}$, $g(0)=\frac{1}{3}$, $g(1)=\frac{2}{3}$

$f(0,0)=\frac{1}{3}\ne\frac{2}{3}\cdot\frac{1}{3}=\frac{2}{9}$ *not* independent

3.74 **(a)** $\frac{1}{4}\int_0^2(2x+y)\,dy=\frac{1}{4}\left[2xy+\frac{y^2}{2}\right]\Big|_0^2=\frac{1}{4}(4x+2)=\frac{1}{2}(2x+1)$ for $0<x<1$

$= 0$ elsewhere

(b) $f\left(y\Big|\frac{1}{4}\right)=\frac{\frac{1}{4}\left(\frac{1}{2}+y\right)}{\frac{1}{2}\cdot\frac{3}{2}}=\frac{1}{6}(2y+1)$ for $0<y<2$

$= 0$ elsewhere

3.75 **(a)** $\frac{1}{4}\int_0^1 (2x+y)\,dx = \frac{1}{4}(x^2+xy)\Big|_0^1 = \frac{1}{4}(1+y)$ for $0 < y < 2$

$= 0$ elsewhere

(b) $f(x|1) = \dfrac{\frac{1}{4}(2x+1)}{\frac{1}{4}(2)} = \frac{1}{2}(2x+1)$ for $0 < x < 1$

$= 0$ elsewhere

3.76 **(a)** $f(x) = 24\int_0^{1-x} (y - xy - y^2)dy = 24\left[\frac{y}{2} - \frac{xy^2}{2} - \frac{y^3}{3}\right]\Big|_0^{1-x}$

$$= 12(1-x)^2 - 12x(1-x)^2 - 8(1-x)^3$$
$$= 12(1-x)^3 - 8(1-x)^3 = 4(1-x)^3$$

$$f(x) = \begin{cases} 4(1-x)^3 & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

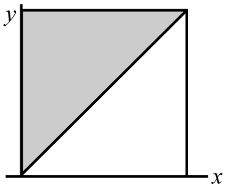
(b) $g(y) = 24\int_0^{1-y} (y - xy - y^2)dy = 24\left[y(1-y) - \frac{1}{2}y(1-y)^2 - y^2(1-y)\right]$

$$= 24(1-y)\left[1 - \frac{1}{2}(1-y) - y\right] = 24y\left(\frac{1}{2} - \frac{y}{2}\right)(1-y)$$

$$= \begin{cases} 12y(1-y)^2 & 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$f(x,y) \neq f(x)\cdot g(y)$ not independent

3.77



(a) $g(x) = \int_x^1 \frac{1}{y}\,dy = \ln y\Big|_x^1 = \ln 1 - \ln x = \begin{pmatrix} -\ln x & 0 < x < 1 \\ 0 & \text{elsewhere} \end{pmatrix}$

(b) $h(y) = \int_0^y \frac{1}{y}\,dx = \frac{1}{y}(y - 0) = \begin{pmatrix} 1 & 0 < x < 1 \\ 0 & \text{elsewhere} \end{pmatrix}$

$\frac{1}{y} \neq 1\cdot(-\ln x)$ *not* independent

3.78 **(a)** $$f(x_2|x_1,x_3) = \frac{(x_1+x_2)e^{-x_3}}{\left(x_2+\frac{1}{2}\right)e^{x_2}} = \frac{x_1+x_2}{x_1+\frac{1}{2}}$$

$$f\left(x_2\middle|\frac{1}{3},2\right) = \frac{\frac{1}{3}+x_2}{\frac{1}{3}+\frac{1}{2}} = \begin{cases} \frac{2+6x^2}{5} & 0 < x_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(b) $$g(x_2,x_3|x_1) = \frac{(x_1+x_2)e^{-x_3}}{x_2 + 2} = \begin{cases} \left(\frac{1}{2}+x_2\right)e^{-x_3} & 0 < x_2 < 1, x_3 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

3.79 $$g(x) = \int_{-\infty}^{\infty} f(x,y)\ dy \qquad G(x) = \int_0^x \int_{-\infty}^{\infty} f(x,y)\ dy = F(x,\infty)$$

$$G(x) = F(x,\infty) = \begin{cases} 1-e^{-x^2} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

3.80 $$M(x_1,x_3) = \int_{-\infty}^{\infty} f(x_1,x_2,x_3)\ dx_2 = F(x_1,\infty,x_3)$$

$$G(x_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1,x_2,x_3)\ dx_2\ dx_3 = F(x_1,\infty,\infty)$$

(a) $$M(x_2,x_3) = \begin{cases} 0 & x_1 \le 0 \text{ or } x_3 \le 0 \\ \frac{1}{2}x_1(x_1+1)(-1-e^{-x_3}) & 0 < x_1 < 1, x_3 > 0 \\ 1-e^{-x_3} & \mathrm{x}_1 \ge 1, x_3 > 0 \end{cases}$$

(b) $$G(x_1) = \begin{cases} 0 & x_1 \le 0 \\ \frac{1}{2}x_1(x_1+1) & 0 < x_1 < 1 \\ 1 & 1 \le x_1 \end{cases}$$

3.81 $g(x_1) = \begin{cases} x_1 + \ 2 & 0 < x_1 < 1 \\ 0 & \text{elsewhere} \end{cases}$ $\qquad h(x_2) = \begin{cases} x_2 + \ 2 & 0 < x_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$

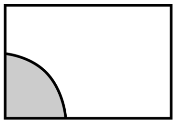
$\phi(x_3 = \begin{cases} e^{-x_3} & x_3 > 1 \\ 0 & \text{elsewhere} \end{cases}$

$f(x_{1,}x_2, x_3) \neq g(x_1) \cdot h(x_2) \cdot \phi(x_3)$ *not* independent

$m(x_1, x_3) = g(x_1)\phi(x_3)$ independent

$n(x_2, x_3) = h(x_2)\phi(x_3)$ independent

3.82



(a) $g(x, y) = \begin{cases} \frac{1}{6} & 0 < x < 2,\ 0 < y < 3 \\ 0 & \text{elsewhere} \end{cases}$

(b) $1 - \frac{\pi/4}{6} = 1 - \frac{\pi}{24}$

(a) $g(0) = \frac{5}{14}, \quad g(1) = \frac{5}{28}, \quad g(2) = \frac{3}{28}$

(b) $\phi(0|0) = \frac{3.28}{10/28} = \frac{3}{10}, \quad \phi(1|0) = \frac{6/28}{10/28} = \frac{6}{10}, \quad \phi(2|0) = \frac{1/28}{10/28} = \frac{1}{10}$

3.83

Heads	Tails	Probability	H-T
0	4	1/16	–4
1	3	4/16	–2
2	2	6/16	0
3	1	4/16	2
4	0	1/16	4

3.84

1	2	3
1	3	4
1	4	5
2	3	5
2	4	6
3	4	7

(a)

x	3	4	5	6	7
$f(x)$	1/6	1/6	2/6	1/6	1/6

$$F(x) = \begin{cases} 0 & x < 3 \\ 1/6 & 3 \leq x < 4 \\ 2/6 & 4 \leq x < 5 \\ 4/6 & 5 \leq x < 6 \\ 5/6 & 6 \leq x < 7 \\ 1 & 7 \leq x \end{cases}$$

3.85 $P(H)=\frac{2}{3}$

(a) $P(0)=\frac{1}{27}$, $P(1)=\frac{6}{27}$, $P(2)=3,\frac{1}{3},\frac{2}{3},\frac{2}{3}=\frac{12}{27}$, $P(3)=\frac{8}{27}$

(b) $\frac{1}{27}+\frac{6}{27}+\frac{12}{27}=\frac{19}{27}$

3.86 $$F(x)=\begin{cases} 0 & x<0 \\ 1/27 & 0\le x<1 \\ 7/27 & 1\le x<2 \\ 19/27 & 2\le x<3 \\ 1 & 3\le x \end{cases}$$

(a) $1-\frac{7}{27}=\frac{20}{27}$

(b) $1-\frac{19}{27}=\frac{8}{27}$

3.87 $$F(V)=\begin{cases} 0 & V<0 \\ 0.40 & 0\le V<1 \\ 0.70 & 1\le V<2 \\ 0.90 & 2\le V<3 \\ 1 & 3\le V \end{cases}$$

3.88 **(a)** $0.20+0.10=0.30$
(b) $1-0.70=0.30$

3.89 Yes; $f(x)\ge 0$ for $x=2,3,\ldots 12$ and $\sum_{x=2}^{12} f(x)=1$

3.90

S	2	3	4	5	6	7	8	9	10	11	12
	1/36	3/36	6/36	10/36	15/36	21/36	26/36	30/36	33/36	35/36	1

3.91 **(a)** $\frac{1}{5}(228.65-227.5)=0.23$; **(b)** $\frac{1}{5}(231.66-229.34)=0.464$;

(c) $\frac{1}{5}(232.5-229.85)=0.53$

3.92 $F(x)=\frac{1}{288}\int(36-x^2)dx+c=\frac{1}{288}\left(36x-\frac{x^3}{3}\right)+\frac{1}{2}$ so that $F(-6)=0$ and $F(6)=1$.

(a) $F(-2)=\frac{1}{288}(-72+\frac{8}{3})+\frac{1}{2}=\frac{1}{288}\cdot\frac{-208}{3}+\frac{1}{2}=\frac{7}{27}$

(b) $F(6)-F(1)=1-\frac{1}{288}(36-\frac{1}{3})-\frac{1}{2}=1-\frac{1}{288}\cdot\frac{107}{3}-\frac{1}{2}=\frac{757}{864}-\frac{1}{2}=\frac{325}{854}$

(c) $F(3)-F(1)=\frac{1}{288}(108-9)-\frac{1}{288}\left(36-\frac{1}{3}\right)=\frac{99}{288}-\frac{1}{288}\cdot\frac{107}{3}=\frac{190}{288\cdot 3}=\frac{95}{432}$

(d) 0

3.93 $F(x) = \int \frac{1}{30} e^{-x/30} dx + c = \frac{1}{30} \frac{e^{-x/30}}{-1/30} + c = c - e^{-x/30} = 1 - e^{-x/30}$

(a) $F(18) = 1 - e^{-18/30} = 1 - e^{-0.6} = 1 - 0.5488 = 0.4512$

(b) $F(36) - F(27) = e^{-27/30} - e^{-36/30} = e^{-0.9} - e^{-1.2} = 0.4066 - 0.3012 = 0.1054$

(c) $1 - F(48) = e^{-48/30} = e^{-1.6} = 0.2019$

3.94 $F(x) = \int \frac{20,000}{(x+100)^3} dx + c = \frac{20,000}{-2(x+100)^2} + 1 = -\frac{10,000}{(x+100)^2} + 1$

(a) $1 - F(200) = \frac{10,000}{300^2} = \frac{1}{9}$

(b) $f(100) = 1 - \frac{10,000}{40,000} = \frac{3}{4}$

3.95 **(a)** $1 - F(10) = \frac{25}{10^2} = 0.25 = \frac{1}{4}$

(b) $F(8) = 1 - \frac{25}{8^2} = \frac{39}{64}$

(c) $F(15) - F(12) = \frac{25}{12^2} - \frac{25}{15^2} = \frac{25(25-16)}{15^2 - 16} = \frac{1}{16}$

3.96 $F(x) = \frac{1}{9} \int_0^x x\, e^{-x/3} dx + c = \frac{1}{9} \frac{e^{-x/3}}{1/9} \left(-\frac{1}{3} x - 1 \right) + c = c - e^{-x/3} \left(\frac{1}{3} x + 1 \right)$

$c = 1$

(a) $F(6) = 1 - 3e^{-2} = 1 - 3e^{-2} = 1 - 3(0.1353) = 0.5491$

(b) $1 - F(9) = 4e^{-3} = 4(0.0498) = 0.1992$

3.97 $(0,0,2) = \binom{3}{0}\binom{2}{0}\binom{3}{2} = 3$ $\quad f(0,0) = \frac{3}{28},\ f(0,1) = \frac{6}{28},\ f(0,2) = \frac{1}{28}$

$(1,0,1) = \binom{3}{1}\binom{2}{0}\binom{3}{1} = 9$ $\quad f(1,0) = \frac{9}{28},\ f(2,0) = \frac{3}{28},\ f(1,1) = \frac{6}{28}$

$(0,1,1) = \binom{3}{0}\binom{2}{1}\binom{3}{1} = 6$

$(2,0,0) = \binom{3}{2}\binom{2}{0}\binom{3}{0} = 3$

$(1,1,0) = \binom{3}{1}\binom{2}{1}\binom{3}{0} = 6$

$(0,2,0) = \binom{3}{0}\binom{2}{2}\binom{3}{0} = 1$

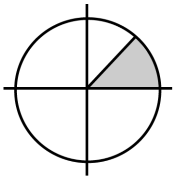
3.98 **(b)**

		x		
		0	1	2
	0	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{36}$
y	1	$\frac{1}{3}$	$\frac{1}{6}$	
	2	$\frac{1}{4}$		

3.99 $f(0,3)=\frac{1}{8},\ \ f(1,2)=\frac{3}{8},\ \ f(2,1)=\frac{3}{8},\ \ f(3,0)=\frac{1}{8}$

$g(0,-3)=\frac{1}{8},\ \ g(1,1)==\frac{3}{8},\ \ g(2,1)=\frac{3}{8},\, g(3,3)=\frac{1}{8}$

3.100 **(a)** Probability = 1/8



(b) $\frac{1}{\pi}\ \pi\cdot\ \frac{1}{4}=\frac{1}{4}$

3.101 **(a)**

$$\int_{0.2}^{0.3}\int_{2}^{\infty}5pe^{-ps}ds\ dp=\int_{0.2}^{0.3}-5e^{-ps}\Big|_{2}^{\infty}dp$$

$$=\int_{0.2}^{0.3}5e^{-2p}dp=\frac{5\cdot e^{-2p}}{-2}\Big|_{0.2}^{0.3}=\frac{5}{2}\left(e^{-0.4}-e^{-0.6}\right)=0.3038$$

(b)

$$\int_{0.25}^{0.30}\int_{0}^{1}5pe^{-ps}ds\ dp=\int_{0.25}^{0.30}-5e^{-ps}\Big|_{0}^{1}dp=\int_{0.25}^{0.30}5(1-e^{-p})\ dp$$

$$=5[p+e^{-p}]^{0.30}=5(0.30+e^{-0.30}-0.25-e^{-0.25}]=0.01202$$

3.102 **(a)**

$$\frac{2}{5}\int_{0}^{0.4}\int_{0}^{0.4}(2x+3y)\ dx\ dy=\frac{2}{5}\int_{0}^{0.4}(x^2+3xy)\Big|_{0}^{0.4}dy$$

$$=\frac{2}{5}\int_{0}^{0.4}\left((0.16+1.2y\right)dy$$

$$=\frac{2}{5}\left[(0.16)(0.4)+\frac{1.2(0.16)}{2}\right]=0.064$$

(b) $$\frac{2}{5}\int_0^{0.5}\int_{0.8}^{1}(2x+3y)\,dx\,dy=\frac{2}{5}\int_0^{0.5}(x^2+3xy)\Big|_{0.8}^{1}dy$$
$$=\frac{2}{5}\int_0^{0.5}\left[(1+3y)-(0.64+2.4y)\right]dy=\frac{2}{5}\int_0^{0.5}(0.6y+0.36)\,dy$$
$$=\frac{2}{5}(0.3y^2+0.36y)\Big|_0^{0.5}=\frac{2}{5}(0.075+0.18)=0.102$$

3.103 (a) $g(0)=\frac{5}{14},\ g(1)=\frac{15}{28}$ and $g(2)=\frac{3}{28}$

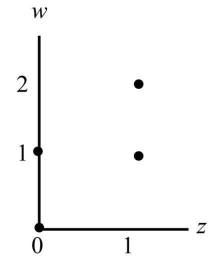
(b) $\phi(0|0)=\frac{3}{10},\ \phi(1|0)=\frac{6}{10}$, and $\phi(2|0)=\frac{1}{10}$

3.104 (a) $$\int_{0.3}^{1}\int_0^{1}\frac{2}{5}(x+4y)\,dy\,dx=\frac{2}{5}\int_{0.3}^{1}(xy+2y^2)\Big|_0^{1}dx=\frac{2}{5}\int_{0.3}^{1}(x+2)\,dx$$
$$=\frac{2}{5}\left[\frac{x^2}{2}+2x\right]\Big|_{0.3}^{1}$$
$$=\frac{2}{5}\left(\frac{1}{2}+2-\frac{0.09}{2}-0.6\right)=\frac{2}{5}(1.855)=0.742$$

(b) $$g(x)=\frac{2}{5}\int_0^{1}(x+4y)\,dy=\frac{2}{5}(x+2)$$
$$g(y|x)=\frac{(2/5)(x+4y)}{(2/5)(x+2)},\quad g(y|0.2)=\frac{4y+0.2}{2.2}$$
$$\frac{1}{2.2}\int_0^{0.5}(4y+0.2)\,dy=\frac{1}{2.2}(0.5+0.1)=\frac{0.6}{2.2}=0.273$$

3.105 (a) $$f(0,0)=\frac{48}{52}\cdot\frac{47}{51}=\frac{188}{221},\quad f(0,1)=\frac{48}{52}\cdot\frac{4}{51}=\frac{16}{221}$$
$$f(1,0)=\frac{4}{52}\cdot\frac{48}{51}=\frac{16}{221},\quad f(1,1)=\frac{48}{52}\cdot\frac{4}{51}=\frac{16}{221},\quad f(1,2)=\frac{4}{52}\cdot\frac{3}{51}=\frac{1}{221}$$

(b) $g(0)=\frac{188+16}{221}=\frac{204}{221},\quad g(1)=\frac{16+1}{221}=\frac{17}{221}$



(c) $\phi(0|1)=\frac{16/221}{17/221}=\frac{16}{17},\quad \phi(1|1)=\frac{1/221}{17/221}=\frac{1}{17}$

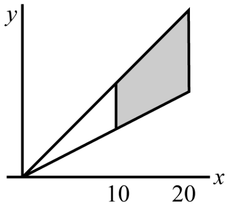
3.106 $f(p,s) = 5pe^{-ps} \quad 0.2 < p < 0.4 \quad s > 0$

(a) $$5p\int_0^{\infty} e^{-ps}\, ds = 5p\frac{e^{-ps}}{-p} = -5e^{-ps}\Big|_0^{\infty} = \begin{cases} 5 & 0.2 < p < 0.4 \\ 0 & \text{elsewhere} \end{cases}$$

(b) $$\frac{f(p,s)}{g(s)} = \frac{5pe^{-ps}}{5} = \begin{cases} pe^{-ps} & \text{for } s > 0 \\ 0 & \text{elsewhere} \end{cases}$$

(c) $$\int_0^3 \frac{1}{4}e^{-(1/4)s} ds = \left[e^{-s/4}\right]\Big|_0^3 = 1 - e^{-0.75}$$

3.107



(a) $$\frac{1}{25}\frac{20-x}{x}\int_{x/2}^{x} dy = \begin{cases} \dfrac{20-x}{50} & 10 < x < 20 \\ 0 & \text{elsewhere} \end{cases}$$

(b) $$\phi(y|x) = \frac{\frac{1}{25}\left(\frac{20-x}{x}\right)}{\frac{20-x}{50}} = \frac{2}{x}, \quad \phi(y|12) = \begin{cases} 1/6 & 6 < y < 12 \\ 0 & \text{elsewhere} \end{cases}$$

(c) $$\frac{1}{6}(12-8) = \frac{1}{6}\cdot 4 = \frac{2}{3}$$

3.108 $f(x,y) = \frac{2}{5}(2x+3y)$

$$g(x) = \frac{2}{5}\left[2xy + \frac{3y^2}{2}\right]\Big|_0^1 = \frac{2}{5}\left(2x + \frac{3}{2}\right)$$

$$= \begin{cases} \dfrac{4}{5}x + \dfrac{3}{5} & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$h(y) = \frac{2}{5}(x^2 + 3xy)\Big|_0^1$$

$$= \begin{cases} (2/5)(1+3y) & 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$f(x,y) \neq g(x)h(y)$$

3.109 **(a)**
$$f(x_1,x_2,x_3)=\begin{cases}\dfrac{(20{,}000)^3}{(x_2+100)^3(x_2+100)^3(x_3+100)^3} & x_1>0,\ x_2>0,\ x_3>0\\ 0 & \text{elsewhere}\end{cases}$$

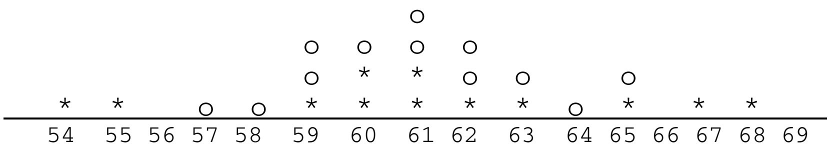
(b)
$$\int_0^{100}\frac{20{,}000}{(x_1+100)^3}dx_1\int_0^{100}\frac{20{,}000}{(x_2+100)^3}dx_2\int_{200}^{\infty}\frac{20{,}000}{(x_3+100)^3}dx_3$$
$$=\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{1}{9}=\frac{1}{16}$$

3.110 **(a)**
5 | 9 4 5 7 9 9 8
6 | 1 3 5 0 2 1 7 0 8 4 5 2 0 2 1 3 1

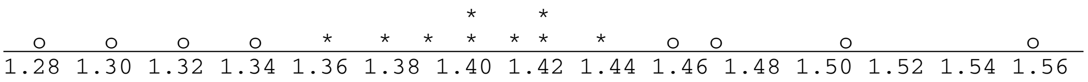
(b)
5f | 4
5s | 9 5 7 9 9 8
6f | 1 3 0 2 0 1 0 4 2 0 2 1 3 1
6s | 5 7 8 5

(c) The double-stem display is more informative.

3.111 *=Station 105 o = Station 107



3.112 *=Lathe A o = Lathe B



3.115

Class Limits	Frequency
40.0 – 44.9	5
45.0 – 49.9	7
50.0 – 54.9	15
55.0 – 59.9	23
60.0 – 64.9	29
65.0 – 69.9	12
70.0 – 74.9	8
75.0 – 79.9	1
	100

3.116

Class Limits	Frequency
3.0 – 4.9	15
5.0 – 6.9	25
7.0 – 8.9	17
9.0 – 10.9	11
11.0 – 12.9	8
13.0 – 14.9	4
	80

3.117 The class boundaries are: 39.95, 44.95, 49.95, 54.95, 59.95, 64.95, 69.95, 79.95;
the class interval is 5;
the class marks are: 42.45, 47.45, 52.45, 57.45, 62.45, 67.45, 72.45, 77.45.

3.118 The class boundaries are: 2.95, 4.95, 6.95, 8.95, 10.95, 12.95, 14.95;
the class interval is 2;
the class marks are: 3.95, 5.95, 7.95, 9.95, 9.95, 11.95, 13.95.

3.119

Class Limits	Frequency	Class Boundary	Class Mark
0 – 1	12	–0.5 – 1.5	0.5
2 – 3	7	1.5 – 3.5	2.5
4 – 5	4	3.5 – 5.5	4.5
6 – 7	5	5.5 – 7.5	6.5
8 – 9	1	7.5 – 9.5	8.5
10 – 11	0	9.5 – 11.5	10.5
12 – 13	1	11.5 – 13.5	12.5
	30		

3.120

Class Limits	Frequency	Percentage
3.0 – 4.9	15	18.75%
5.0 – 6.9	25	31.25
7.0 – 8.9	17	21.25
9.0 – 10.9	11	13.75
11.0 – 12.9	8	10.00
13.0 – 14.9	4	5.00
	80	100.00

3.121

Class Limits	Frequency	Percentage
40.0 – 44.9	5	5.0%
45.0 – 49.9	7	7.0
50.0 – 54.9	15	15.0
55.0 – 59.9	23	23.0
60.0 – 64.9	29	29.0
65.0 – 69.9	12	12.0
70.0 – 74.9	8	8.0
75.0 – 79.9	1	1.0
	100	100.0

3.122

Class Limits	Percentage Shipping Department	 Security Department
0 – 1	43.3%	45.0%
2 – 3	30.0	27.5
4 – 5	16.7	17.5
6 – 7	6.7	7.5
8 – 9	3.3	2.5
	100.0	100.0

The patterns seem comparable for the two departments.

3.123

Upper Class Boundary	Frequency	Cumulative Frequency
44.95	5	5
49.95	7	12
54.95	15	27
59.95	23	50
64.95	29	79
69.95	12	91
74.95	8	99
79.95	1	100
	100	

3.124

Upper Class Boundary	Frequency	Cumulative Frequency
4.95	15	15
6.95	25	40
8.95	17	57
10.95	11	68
12.95	8	76
14.95	4	80
	100	

3.125

Class Limits	Cumulative Percentage Shipping Department	 Security Department
1.5	43.3%	45.0%
3.5	73.3	72.5
5.5	90.0	90.0
7.5	96.7	97.5
9.5	100.0	100.0

3.126 **(a)**

Class Limits	Frequency
0 – 1	12
2 – 3	7
4 – 5	4
6 – 7	5
8 – 13	2
	30

(b) No. The class interval of the last class is greater than that of the others.

3.127 **(a)**

Class Limits	Frequency	Class Marks
0 – 99	4	49.5
100 – 199	3	149.5
200 – 299	4	249.5
300 – 324	7	312.0
325 – 349	14	337.0
350 – 399	6	374.5
	38	

(b) Yes, [see part (a)..

3.130 The class marks are found from the class boundaries by averaging them; thus, the first class mark is (2.95 + 4.95)/2 = 3.95, and so forth.

3.135 The MINITAB output is:

```
MIDDLE OF   NUMBER OF
INTERVAL    OBSERVATIONS
   6.0        2    **
   6.5        5    *****
   7.0        4    ****
   7.5        5    *****
   8.0        5    *****
   8.5        3    ***
   9.0        2    **
   9.5        2    **
  10.02       2    **
```

3.136 The MINITAB output is:

```
MIDDLE OF   NUMBER OF
INTERVAL    OBSERVATIONS
   40         1    *
   45         7    *******
   50        11    ***********
   55        21    *********************
   60        21    *********************
   65        23    ***********************
   70        10    **********
   75         6    ******
```

Chapter 4

4.1 **(a)** 0, 1, 4, 9

(b) $h(g_1) = f(0)$; $h(g_2) = f(-1) + f(1)$; $h(g_3) - f(-2) + f(2)$ **;** $h(g_4) = f(3)$

(c) $0 \cdot f(0) + 1[f(-1) + f(1)] + 4[f(-2) + f(2)] + 9 \cdot f(3)$

4.2 Replace $\int$ by $\sum$

4.3 Replace $\sum$ by $\int$

4.4 Replace $\int$ by $\sum$

4.5 (**a**) $E(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x, y)\, dy\, dx$; $E(x) = \int_{-\infty}^{\infty} x\ \mathrm{g}(x)\, dx$

4.6
$$E(x) = (-1)\left(\frac{3}{7}\right) + 0\left(\frac{2}{7}\right) + 1\left(\frac{1}{7}\right) + 3\left(\frac{1}{7}\right) = \frac{1}{7}$$

4.7
$$E(Y) = \frac{1}{8}\int_2^4 (y^2 + y)\, dy = \frac{1}{8}\left[\frac{y^3}{3} + \frac{y^2}{2}\right]_2^4 = \frac{1}{8}\left(\frac{64}{3} + 8 - \frac{8}{3} - 2\right)$$
$$= \frac{1}{8}\left(\frac{56}{3} + 6\right) = \frac{1}{8} \cdot \frac{74}{3} = \frac{37}{12}$$

4.8
$$E(x) = \int_0^1 x^2\ dx + \int_1^2 (2x - x^2)\ dx = \frac{1}{3} + \left[x^2 - \frac{x^2}{3}\right]\Bigg|_1^2 = \frac{1}{3} + 4 - \frac{8}{3} - 1 + \frac{1}{3}$$
$$= 3 - \frac{6}{3} = 1$$

4.9 **(a)**
$$E(x) = 0 \cdot \frac{1}{125} + 1 \cdot \frac{12}{125} + 2 \cdot \frac{48}{125} + 3 \cdot \frac{64}{125} = \frac{12 + 96 + 192}{125} = \frac{300}{125} = \frac{12}{5} = 2.4$$
$$E(x^2) = 0 \cdot \frac{1}{125} + 1 \cdot \frac{12}{125} + 4 \cdot \frac{48}{125} + 9 \cdot \frac{64}{125} = \frac{12 + 192 + 576}{125} = \frac{780}{125} = 6.24$$

(b) $E[(3x + 2)^2] = 9E(x^2) + 12E(x) + 4 = 56.16 + 28.8 + 4 = 88.96$

4.10 **(a)** $E(x) = \int_1^3 \frac{1}{\ln 3}\, dx = \frac{2}{\ln 3},\ E(x^2) = \int_1^3 \frac{x}{\ln 3}\, dx = \frac{4}{\ln 3},\ E(x^3) = \int_1^3 \frac{x^2}{\ln 3}\, dx = \frac{26}{3(\ln 3)}$

(b) $\frac{26}{3(\ln 3)} + \frac{8}{\ln 3} - \frac{6}{\ln 3} + 1 = \frac{32}{3(\ln 3)} + 1$

4.11 **(a)** $E(x) = \int_0^1 \frac{x^2}{2}\, dx + \int_1^2 \frac{x}{2}\, dx + \int_2^3 \frac{3x - x^2}{2}\, dx = \frac{3}{2}$

$E(x^2) = \int_0^1 \frac{x^3}{2}\, dx + \int_1^2 \frac{x^2}{2}\, dx + \int_2^3 \frac{3x^2 - x^3}{2}\, dx = \frac{8}{3}$

$E(x^2 - 5 + 3) = \frac{8}{3} - 5 \cdot \frac{3}{2} + 3 = -\frac{11}{6}$

4.12 $E(x) = 2, \quad E(Y) = \frac{19}{15}$, and $E(2x - Y) = \frac{60 - 19}{15} = \frac{41}{15} = 2\frac{11}{15}$

Marginal distributions

x	0	1	2	3
$g(x)$	1/10	1/5	3/10	2/5

y	0	1	2
h(y)	1/5	1/3	14/30

$E(x) = \frac{1}{5} + \frac{6}{10} + \frac{6}{5} = \frac{20}{10} = 2$

$E(Y) = \frac{1}{3} + \frac{28}{30} = \frac{38}{30} = \frac{19}{15}$

4.13 $E\left(\frac{x}{y}\right) = \int_0^1 \int_0^y \frac{x}{y^2}\, dx\, dy = \int \frac{1}{2}\, dy = \frac{1}{2}$

4.14 $k = \frac{1}{54}$

for x $g(1) = \frac{1}{54}(1 + 2 + 2 + 4 + 3 + 6) = \frac{18}{54} = \frac{1}{3}$

$g(2) = \frac{2}{3}$

for y $h(1) = \frac{1}{54}(1 + 2 + 2 + 4) = \frac{1}{6};\ h(2) = \frac{1}{3},\ h(3) = \frac{1}{2}$

for z $\phi(1) = \frac{1}{3};\ \phi(2) = \frac{2}{3}$

$E(x) = 1 \cdot \frac{1}{3} + 2 \cdot \frac{2}{3} = \frac{5}{3},\ E(Y) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{2} = \frac{1 + 4 + 9}{6} = \frac{14}{6} = \frac{7}{3}$

$E(z) = \frac{5}{3},\ E(u) = \frac{5}{3} + \frac{7}{3} + \frac{5}{3} = \frac{17}{3}$

4.15 $\int_0^1\int_0^1\int_0^1 \frac{1}{3}(2x+3y+z)(x^2-yz)\;dx\;dy\;dz = \frac{1}{12}$

4.16 $E(2^X) = 2^x\left(\frac{1}{2}\right)^x = 1+1+1+1+\ldots = \infty$

So $E(2^x)$ does not exist.

4.17 $\mu_0 = \int (x-\mu)^0 f(x)\;dx = \int f(x)\;dx = 1$

$\mu_1 = \int (x-\mu)^1 f(x)\;dx = \int xf(x)\;dx - \mu\int f(x)\;dx - \mu - \mu = 0$

4.18 $\mu = (-2)\frac{1}{2} + (2)\frac{1}{2} = 0,\;\; u_2' = (-2)^2\frac{1}{2} + (2)^2\frac{1}{2} = 4$

$\sigma^2 = 4 - 0^2 = 4$

4.19 $\mu = \int_0^2 \frac{x^2}{2}dx = \frac{x^3}{6}\Big|_0^2 = \frac{4}{3},\;\; \mu_2' = \int_0^2 \frac{x^3}{2}dx = \frac{x^4}{8}\Big|_0^2 = 2$

$\sigma^2 = 2 - \frac{16}{9} = \frac{2}{9}$

4.20 $\mu_r' = \frac{1}{\ln 3}\int_1^3 x^{r-1}dx = \frac{1}{\ln 3}\left[\frac{x^r}{r}\right]\Bigg|_1^3 = \frac{1}{r(\ln 3)}\cdot(3^r - 1) = \frac{3^r - 1}{r(\ln 3)}$

$\mu = \frac{2}{\ln 3},\;\; \mu_2' = \frac{8}{2(\ln 3)} = \frac{4}{\ln 3},\;\; \sigma^2 = \frac{4}{\ln 3} - \frac{4}{(\ln 3)^2} = \frac{4(\ln 3 - 1)}{(\ln 3)^2}$

4.21 $E[ax+b] = aE(x) + b$

$E[(ax+b)^2] = E[(a^2x^2 + 2abx + b^2) = a^2E(x^2) + 2abE(x) + b^2$

$\sigma^2 = a^2E(x^2) + 2abE(x) + b^2 - a^2[E(x)]^2 - 2abE(x) - b^2$

$= a^2\sigma^2$ QED

4.22 $\text{var}(2x+3) = 4\;\;\text{var}(x)$

$\mu = 1$ from Exercise 4.8

$\mu_2' = \int_0^1 x^3dx + \int_1^2 (2x^2 - x^3)dx = \frac{1}{4} + \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]\Bigg|_1^2$

$= \frac{1}{4} - \frac{16}{3} - 4 - \frac{2}{3} + \frac{1}{4} = \frac{7}{6},\;\; \sigma^2 = \frac{7}{6} - 1 = \frac{1}{6}$

$\text{var}(2x+3) = 4\cdot\frac{1}{6} = \frac{2}{3}$

4.23 $E(z) = E\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma}E(x-\mu) = \frac{1}{\sigma}(\mu-\mu) = 0$ exists

$$\text{var}(z) = E\left[\left(\frac{x-\mu}{\sigma}\right)^2\right] = \frac{1}{\sigma^2}E[(x-\mu)^2] = \frac{\sigma^2}{\sigma^2} = 1$$

4.24 $E(x) = \int_1^\infty 2x^{-2}dx = [-2x^{-1}] = \frac{2}{1} = 2$ exists

$\mu_2' = \int_1^\infty \frac{2}{x}dx = 2\ln x\Big|_1^\infty = \infty$ σ^2 does *not* exist

4.25 $\sum(x-\mu)^r f(x) = \sum x^r f(x) - \binom{r}{1}\mu\sum x^{r-1}f(x) + \binom{r}{2}\mu^2\sum x^{r-2}f(x)$

$$\ldots(-1)^{r-1}\mu^{r-1}\sum x\,f(x) + (-1)^r\mu^r\sum f(x)$$

$$= \sum x^r f(x) - \binom{r}{1}\mu\mu_{r-1}' + \binom{r}{2}\mu^2\mu_{r-2}^1 \ldots (-1)^{r-1}(r-1)\mu^r$$

$$\mu_3 = \mu_3' - 3\mu\mu_2' + 3\mu^2\cdot\mu - 1\mu^2 - \mu_3' - 2\mu\mu_2' + 2\mu^3$$

$$\mu_4 = \mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 4\mu^2\cdot\mu_2' + \mu^3 = \mu_4' - 4\mu\mu_2' + 6\mu^2\mu_2' - 3\mu^4$$

4.26 **(a)** $\mu = 1(0.05) + 2(0.15) + 3(0.30) + 4(0.30) + 5(0.15) + 6(0.05) = 3.50$

$$\mu_2' = 1^2(0.05) + 2^2(0.15) + 3^2(0.30) + 4^2(0.30) + 5^2(0.15) + 6^2(0.05) = 13.70$$

$$\mu_3' = 1^3(0.05) + 2^3(0.15) + 3^3(0.30) + 4^3(0.30) + 5^3(0.15) + 6^3(0.05) = 58.10$$

$\sigma^2 = 13.70 - 12.25 = 1.45$ $\mu_2 = 58.10 - 3(3.5)(13.7) + 2(3.5)^2 = 0$

$\alpha_3 = 0$

(b) $\mu = 3.5,\quad \mu_2' = 13.70,\quad \mu_3' = 1(0.05) + 2^3(0.20) + \ldots + 6^2(0.05) = 57.8$

$\mu_3 = 57.8 - 3(3.5)(13.7) + 2(3.5)^3 = -0.3$

$$\alpha_3 = \frac{-0.3}{\left(\sqrt{1.45}\right)^3} = \frac{-0.3}{1.746} = -0.172$$

4.27 **(a)** $\mu = 0$ by symmetry, $\mu_2' = 0$ by symmetry

$\mu_2' = 9(0.06) + 4(0.09) + 1(0.10) + 0(0.50) + 1(0.10) + 4(0.09) + 9(0.06) = 2$

$\mu_4' = 81(0.06) + 16(0.09) + 1(0.10) + 0(0.50) + 1(0.10) + 16(0.09) + 81(0.06) = 12.8$

$\sigma^2 = 2$ and $\mu_4 = 12.8$; $\alpha_4 = \frac{12.8}{4} = 3.2$

(b) $\mu = 0$ and $\mu_3' = 0$ by symmetry

$\mu_2' = 9(0.04) + 4(0.11) + 1(0.20) + \ldots = 2$

$\mu_4' = 81(0.04) + 16(0.11) + 1(0.20) + \ldots = 10.4$

$\sigma^2 = 2$ and $\mu_4 = 10.4$; $\alpha_4 = \frac{10.4}{4} = 2.6$

4.29 $\mu = \int_0^a xf(x)\,dx + \int_a^\infty xf(x)\,dx \geq a\int_a^\infty f(x)\,dx = aP(x \geq a)$

$$\frac{\mu}{a} \geq P(x \geq a) \qquad \text{QED}$$

4.30 $P[(x-\mu)^2 \geq a] \leq \frac{\sigma^2}{a} \qquad a = k^2\sigma^2$

$P[(x-\mu)^2 \geq k^2\sigma^2] \leq \frac{1}{k^2}$ or$P[|x = \mu| \geq k\sigma] \leq \frac{1}{k^2}$

$P[|x-\mu| < k\sigma] \geq 1 - \frac{1}{k^2}$

4.31 **(a)** $1 - \frac{1}{k^2} = 0.95$, $\frac{1}{k^2} = 0.05 = \frac{1}{20}$, $k = \sqrt{20} = 4.47$

(b) $1 - \frac{1}{k^2} = 0.99$, $\frac{1}{k^2} = 0.01 = \frac{1}{100}$, $k = 10$

4.32 $P\left(|x-\mu| < k\sigma\right) \geq 1 - \frac{1}{k^2}$ let $k\sigma = c$

$P\left(|x-\mu| < c\right) \geq 1 - \frac{\sigma^2}{c^2}$ Probability is at least $1 - \frac{\sigma^2}{c^2}$

4.33 $M_x(t) = \int_0^t \left(e^{tx}\,dx\right) = \frac{e^{tx}}{t}\Big|_0^1 = \frac{e^t - 1}{t}$

$\frac{e^t - 1}{t} = 1 + \frac{t}{2!} + \frac{t^2}{3!} + \ldots \mu_1' = \frac{1}{2}$ and $\mu_2' = \frac{1}{3}$

$$\sigma^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

4.34 $$M_x(t) = \sum 2\left(\frac{1}{3}\right)^x e^{tx} = \sum_1^{\infty} 2\left(\frac{e^t}{3}\right)^x = \frac{2\left(\frac{e^t}{3}\right)}{1-\left(\frac{e^t}{3}\right)} = \frac{2e^t}{3-e^t}$$

$$M'(t) = \frac{(3-e^t)2e^t - 2e^t(-e^t)}{(3-e^t)^2} = \frac{6e^t}{(3-e^t)^2}$$

$$M''(t) = \frac{(3-e^t)6e^t - 6e^t \cdot 2(3-e^t)(-e^t)}{(3-e^t)^4}$$

$$M\,'(0) = \frac{6}{4} = \frac{3}{2}\,, \quad M''(0) = \frac{24-12\cdot 2(-1)}{16} = 3$$

$$\mu_1' = \frac{3}{2} \text{ and } \mu_2' = 3$$

$$\sigma^2 = \mu_2' - (\mu_1')^2 = 3 - \frac{9}{4} = \frac{3}{4}$$

4.35 $$R_x(t) = \ln M_x(t),\ R_x'(t) = \frac{1}{M_x(t)} \cdot M_x'(t),\ R_x'(0) = \frac{M_x'(0)}{M_x(0)} = \frac{M}{1} = \mu$$

$$R''(t) = \frac{M_x(t)\cdot M_x''(t) - M_x'(t)M'(t)_x}{[M_x(t)]^2}$$

$$R''(0) = \frac{1\cdot \mu_2' - \mu^2}{1^2} = \sigma^2$$

$$R_x(t) = 4(e^t-1)\,,\ R_x'(t) = 4e^t \text{ and } R''(t) = 4e^t$$

$$\mu = 4 \text{ and } \sigma^2 = 4$$

4.36 $M_x(0) = 0 \neq 1$

4.37 $\frac{1}{2}\int_{-\infty}^{0} e^{tx}e^{x}\,dx + \frac{1}{2}\int_{0}^{\infty} e^{tx}e^{-x}\,dx$

$y = -x$

$cy = -dx$

$$\frac{1}{2}\int_{0}^{\infty} e^{-ty}e^{-y}\,dy + \frac{1}{2}\int_{0}^{\infty} e^{tx}e^{-x}\,dx = \frac{1}{2}\int_{0}^{\infty} e^{-(1+t)y}\,dy + \frac{1}{2}\int_{0}^{\infty} e^{-(1-t)x}\,dx$$

$$= \frac{\frac{1}{2}\left[e^{-(1+t)y}\right]\Big|_{0}^{\infty}}{-(1+t)} + \frac{\frac{1}{2}\left[e^{-(1-t)y}\right]\Big|_{0}^{\infty}}{-(1-t)} = \frac{1}{2}\left[\frac{1}{1+t} + \frac{1}{1-t}\right]$$

$$= \frac{1/2}{(1+t)(1-t)}[1-t+1+t] = \frac{1}{1-t^2}$$

4.38 $M_x(t) = 1 - t^2 + \frac{t^2}{2!} - \ldots$

(a) $\mu = 0$, $\sigma^2 = 2$

(b) $M'_x(t) = -(1-t^2)^{-2}(-2t) = \frac{2t}{(1-t^2)^2}$

$$M''_x(t) = \frac{(1-t^2)^2 2 - 2 + 2(1-t^2)(-2t)}{(1-t^2)^4} = \frac{2(1-t^2)^2 + 4t^2(1-t^2)}{(1-t^2)^4}$$

$M''_x(0) = 2$, $\sigma^2 = 2$

4.39 3. $M_{(x+a)/b}(t) = \int_{-\infty}^{\infty} e^{[(x+a)/b]t} f(x)\,dx = e^{at/b}\int_{-\infty}^{\infty} e^{xt/b} f(x)\,dx = e^{at/b}\cdot M_x\left(\frac{t}{b}\right)$ QED

1. Let $b = 1$;
2. Let $a = 0$ in above result.

4.40 $z = \frac{1}{4}(x-3)$, $a = -3$, $b = 4$

$M_z(t) = e^{-(3/4)t}\cdot e^{(3/4)t+(8/16)t^2} = e^{(1/2)t^2}$

$M_z(t) = 1 + \frac{1}{2}t^2 + \ldots$ $\mu = 0$ and $\sigma^2 = 1$

4.42 $(-3,-5)$, $(-1,1)$, $(1,1)$, $(3,5)$ probabilities are ¼

$E(X) = 0,\ E(Y) = 0,\ E(XY) = 15\cdot\frac{1}{4} + 1\cdot\frac{1}{4} + 1\cdot\frac{1}{4} + 15\cdot\frac{1}{4} = 8$

cov$(X,Y) = 8 - 0\cdot 0 = 8$

4.43 $$E(X) = 0\cdot\frac{56}{120} + 1\cdot\frac{56}{120} + 2\cdot\frac{8}{120} = \frac{72}{120} = 0.6$$
$$E(Y) = 0\cdot\frac{35}{120} + 1\cdot\frac{63}{120} + 2\cdot\frac{21}{120} + 3\cdot\frac{1}{120} = \frac{108}{120} = 0.9$$
$$E(XY) = 1\cdot 1\cdot\frac{1}{4} + 1\cdot 2\cdot\frac{1}{40} + 2\cdot 1\cdot\frac{1}{20} = \frac{16}{40} = 0.4$$
$$\operatorname{cov}(X,Y) = 0.4 - (0.6)(0.9) = 0.4 - 0.54 = -0.14$$

4.44 $$E(x_2) = \int_0^1\int_0^1\int_0^\infty x_2(x_1+x_2)e^{-x_3}\ dx_3\ dx_2\ dx_1 = \int_0^1\int_0^1 (x_1^2 + x_1x_2)dx_2\ dx_1$$
$$= \int_0^1 (x_1^2x_2 + x_1\frac{x_2^2}{2}\Big|_0^1 dx_1 = \int_0^1 (x_1^2 + \frac{1}{2}x_1)dx_1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

$$E(x_3) = \int_0^1\int_0^1\int_0^\infty x_3e^{-x_3}(x_1{+}x_2)\ dx_3\ dx_1\ dx_2 = \int_0^1\int_0^1 (x_1+x_2)dx_1\ dx_2$$
$$= \int_0^1\left(\frac{1}{2} + x_2\right)dx_2 = \frac{1}{2} + \frac{1}{2} = 1$$

$$E(x_2x_3) = \int_0^\infty x_3e^{-x_3}dx_3\int_0^1\int_0^1 (x_1^2 + x_1x_2)dx_2\ dx_1$$
$$= \int_0^1 (x_1^2 + \frac{x_1}{2})dx_2 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$
$$\operatorname{cov}(x_1, x_3) = \frac{7}{12} - \frac{7}{12}\cdot 1 = 0$$

4.45 $$E(X) = \frac{1}{4}\int_0^1\int_0^2 (2x^2 + xy)\ dy\ dx = \frac{1}{4}\int_0^1 (4x^2 + 2x)\ dx = \frac{1}{4}\left(\frac{4}{3} + 1\right) = \frac{7}{12}$$
$$E(Y) = \frac{1}{4}\int_0^2\int_0^1 (2xy + y^2)\ dx\ dy = \frac{1}{4}\int_0^2 (y + y^2)\ dy = \frac{1}{4}\left(2 + \frac{8}{3}\right) = \frac{14}{12}$$
$$E(XY) = \frac{1}{4}\int_0^1\int_0^2 (2x^2y + xy^2)\ dy\ dx = \frac{1}{4}\int_0^1 (4x^2 + \frac{8}{3}x)\ dx = \frac{1}{4}\left(\frac{4}{3} + \frac{4}{3}\right) = \frac{2}{3}$$
$$\operatorname{cov}(X,Y) = \frac{2}{3} - \frac{7}{12}\cdot\frac{14}{12} = \frac{2}{3} - \frac{49}{72} = -\frac{1}{72}$$

4.46 **(a)** $f(-1,1)=\frac{1}{4},\ f(0,0)=\frac{1}{6},\ f(0,1)=0,\ f(1,0)=\frac{1}{12}, f(1,1)=\frac{1}{2}$

$$E(X)=-1\left(\frac{1}{4}\right)+0\left(\frac{1}{6}\right)+1\left(\frac{7}{12}\right)=\frac{1}{3}$$

$$E(Y)=0\left(\frac{1}{4}\right)+1\left(\frac{3}{4}\right)=\frac{3}{4}$$

$$E(XY)=-\frac{1}{4}+\frac{1}{2}=\frac{1}{4}$$

$$\operatorname{cov}(X,Y)=\frac{1}{4}-\frac{1}{3}\cdot\frac{3}{4}=0$$

(b) $f(0,0)=\frac{1}{6},\ \mathrm{g}(0)h(0)=\frac{1}{6}\cdot\frac{1}{4}=\frac{1}{24},\ f(0,0)\neq g(0)h(0)$

4.47 **(a)** $E(U)=\int_{-1}^{0}(x+x^2)\ dx+\int_{0}^{1}(x-x^2)\ dx=-\frac{1}{2}+\frac{1}{3}+\frac{1}{2}-\frac{1}{3}=0$

$$E(V)=\int_{-1}^{0}(x^2+x^3)\ dx+\int_{0}^{1}(x^2-x^3)\ dx=-\frac{1}{3}-\frac{1}{4}+\frac{1}{3}-\frac{1}{4}=\frac{2}{3}-\frac{1}{2}=\frac{1}{6}$$

$$E(UV)=\int_{-1}^{0}(x^3+x^4)\ dx+\int_{0}^{1}(x^3-x^4)\ dx=-\frac{1}{4}+\frac{1}{5}+\frac{1}{4}-\frac{1}{5}=0$$

$$\operatorname{cov}(U,V)=0-0\cdot\frac{1}{6}=0$$

not independent; in fact $V=U^2$.

4.48 **(a)** $\dfrac{\partial\int\ldots\int e^{\Sigma t_i x_i}f(x_1\ldots x_k)dx_1\ldots dx_k}{\partial t_i}$

$$=\int\ldots\int x_i e^{\Sigma t_i x_i}f(x_1\ldots x_k)dx_1\ldots dx_k$$

at $t_i's=0$

$$=\int\ldots\int x_i f(x_1\ldots x_k)dx_1\ldots dx_k=\mu_i$$

(b) same

(c) $M_{XY}(t_1,t_2)=\int_0^\infty\int_0^\infty e^{xt_1-x}e^{yt_2-y}\ dx\ dy=\int_0^\infty\int_0^\infty e^{x(t_1-1)}e^{y(t_2-1)}dx\ dy$

$$=\frac{1}{t_1-1}\cdot\frac{1}{t_2-1}=\frac{1}{(1-t_1(1-t_2)}$$

$$\frac{\partial}{\partial t_1}=\frac{1}{(1-t_1)^2}\cdot\frac{1}{(1-t_2)}$$

$$\frac{\partial^2}{\partial t_1\ \partial t_2}=\frac{1}{(1-t_1)^2}\cdot\frac{1}{(1-t_2)^2}$$

$E(X)=1$

$E(Y)=1$

$E(XY)=1$

$\operatorname{cov}(X,Y)=0$

4.49 **(a)** $\mu_Y = 2(4) - 3(9) + 4(3) = -7$

$\sigma_Y^2 = 4(3) + 9(7) + 16(5) = 155$

(b) $\mu_Z = 1(4) + 2(9) - 1(3) = 19$

$\sigma_Z^2 = 1(3) + 4(7) + 1(5) = 36$

4.50 **(a)** $\mu_Y = -7, \ \sigma_Y^2 = 155 - 12 - 48 + 48 = 143$

(b) $\mu_Z = 19, \ \sigma_Z^2 = 36 + 4 + 6 + 8 = 54$

4.51 $$E(x) = \frac{1}{3}\int_0^1\int_0^2 (x^2 + xy)\, dy\, dx = \frac{1}{3}\int_0^1 (2x^2 + 2x)\, dx = \frac{1}{3}\left(\frac{2}{3} + 1\right) = \frac{5}{9}$$

$$E(x^2) = \frac{1}{3}\int_0^1\int_0^2 (x^3 + x^2 y)\, dy\, dx = \frac{1}{3}\int_0^1 (2x^3 + 2x^2)\, dx = \frac{1}{3}\left(\frac{1}{2} + \frac{2}{3}\right) = \frac{7}{18}$$

$$\sigma_X^2 = \frac{7}{18} - \frac{25}{81} = \frac{63 - 50}{162} = \frac{13}{162}$$

$$E(Y) = \frac{1}{3}\int_0^2\int_0^1 (xy + y^2)\, dx\, dy = \frac{1}{3}\int_0^2 \left(\frac{1}{2}y + y^2\right) dy = \frac{1}{3}\left(1 + \frac{8}{3}\right) = \frac{11}{9}$$

$$E(Y^2) = \frac{1}{3}\int_0^2\int_0^1 (xy^2 + y^2)\, dx\, dy = \frac{1}{3}\int_0^2 \left(\frac{1}{2}y^2 + y^3\right) dy = \frac{1}{3}\left(\frac{4}{3} + 4\right) = \frac{16}{9}$$

$$\sigma_Y^2 = \frac{16}{9} - \frac{121}{81} = \frac{144 - 121}{81} = \frac{23}{81}$$

$$E(XY) = \frac{1}{3}\int_0^1\int_0^2 (x^2 y + xy^2)\, dy\, dx = \frac{1}{3}\int_0^1 \left(2x^2 + \frac{8}{3}x\right) dx = \frac{1}{3}\left(\frac{2}{3} + \frac{4}{3}\right) = \frac{2}{3}$$

$$\text{cov}(X, Y) = \frac{2}{3} - \frac{5}{9}\cdot\frac{11}{9} = -\frac{1}{81}$$

$$\text{var}(w) = 9\cdot\frac{13}{162} + 16\cdot\frac{23}{81} + 24\cdot\left(-\frac{1}{81}\right) = \frac{177 + 736 - 48}{162} = \frac{805}{162}$$

4.53 $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y) + 2\,\text{cov}(X, Y) \qquad a_1 = 1, \ a_2 = 1$

$\text{var}(X - Y) = \text{var}(X) + \text{var}(Y) - 2\,\text{cov}(X, Y) \qquad b_1 = 1, \ \text{b}_2 = -1$

$\text{cov}(X + Y, X - Y) = \text{var}(X) - \text{var}(Y) + 0\cdot\text{cov}(X, Y) = \text{var}(X) - \text{var}(Y)$

4.54 $\text{cov}(Y_1, Y_2) = (-2)5 + (-6)(4) + 12(7) + 7(3) + (-2)(-2)$

$= -10 - 24 + 84 + 21 + 4 = 75$

$a_1 = 1, \ a_2 = -2, \ a_3 = 3$

$b_1 = -2, \ b_2 = 3, \ b_3 = 4$

4.55 $\text{cov}(Y, Z) = 2\cdot1\cdot3 - 3\cdot2\cdot7 - 4\cdot1\cdot5 = 6 - 42 - 20 = -56$

4.56 $F(-1\Big|-1)=\frac{1}{5},\ f(1\Big|-1)=\frac{4}{5}$

$$\mu_{x|-1}=(-1)\frac{1}{5}+1\cdot\frac{4}{5}=\frac{3}{5}$$

$$\mu_2'=1-\frac{1}{5}+1\cdot\frac{4}{5}=1 \qquad\qquad \sigma^2_{x|-1}=1-\left(\frac{3}{5}\right)^2=\frac{16}{25}$$

4.57 $f(z\big|1,2)=\phi(1\big|1,2)=\frac{1}{3},\ \phi(2\Big|1,2)=\frac{2}{3}$

$$E(z^2\big|1,2)=1^2\cdot\frac{1}{3}+2^2\cdot\frac{2}{3}=\frac{1}{3}+\frac{8}{3}=3$$

4.58 $f\left(y|\ \frac{1}{4}\right)=\frac{1}{6}(2y+1) \qquad 0<y<2)$

$$\mu_{Y|1/4}=\frac{1}{6}\int_0^2(2y^2+y)\ dy=\frac{1}{6}\left(\frac{16}{3}+2\right)=\frac{1}{6}\cdot\frac{22}{3}=\frac{11}{9}$$

$$\mu_2'=\frac{1}{6}\int_0^2(2y^3+y^2)\ dy=\frac{1}{6}\left(8+\frac{8}{3}\right)=\frac{1}{6}\cdot\frac{32}{3}=\frac{16}{9}$$

$$\sigma^2_{Y|1/4}=\frac{16}{9}-\frac{121}{81}=\frac{23}{81}$$

4.59 $f\left(x_2,x_3|\ \frac{1}{2}\right)=\left(x_2+\frac{1}{2}\right)e^{-x_3} \qquad 0<x_2<1 \text{ and } x_3>\sigma$

$$E\left(x_2^2x_3|\ \frac{1}{2}\right)=\int_0^1\left(x_2^3+\frac{x_2^2}{2}\right)dx_2\ {}_0^{\infty}\ x_2e^{-x_3}\ dx_3$$

$$=\left(\frac{1}{4}+\frac{1}{6}\right)\cdot 1=\frac{5}{12}$$

4.60 **(a)** $f(x\big|a)\le x\le b)=\frac{f(x)}{F(b)-F(a)} \qquad a\le x<b)$

$$f(x\big|a\le x\le b)=\int_a^x\frac{f(x)}{F(b)-F(a)}\ dx=\frac{1}{F(b)-F(a)}\cdot F(x)-F(a)$$

$$=\frac{F(x)-F(a)}{F(b)-F(a)} \qquad a<x<b$$

(b) $$f(x|a \le x \le b) = \frac{f(x)}{F(b)-F(a)}$$

$$E[u(x)|a \le x \le b] = \frac{\int_a^b u(x)f(x)\,dx}{F(b)-F(a)} = \frac{\int_a^b u(x)f(x)\,dx}{\int_a^b f(x)\,dx}$$

4.61 **(a)** $E(0) = N(0) = 10^6 - 0.0001 = 100$

(b) $E(p|0) = N(0)P(p|D) = 100 \cdot 0.98 = 98$

(c) $E(p|\bar{D}) = N(\bar{D})P(p|\bar{D}) = 999{,}900 \cdot 0.03 = 29{,}997$

4.62 $$3{,}000 \cdot \frac{3}{20} + 1{,}500 \cdot \frac{7}{20} + 0 \cdot \frac{7}{20} - 1{,}500 \cdot \frac{3}{20}$$
$$= \frac{1}{20}(9{,}000 + 10{,}500 - 4{,}500) = \frac{15{,}000}{20} = \$750$$

4.63 $10 \cdot \frac{1}{3} = A \cdot \frac{2}{3},\ A = \$5.00$

4.64 **(a)** $1 \cdot \frac{5}{6} - 0.4 \cdot \frac{1}{6} = \frac{4.6}{6} = \$0.77$

(b) $2 \cdot \frac{4}{6} + 0.6 \cdot \frac{1}{6} - 0.8 \cdot \frac{1}{6} = \frac{7.8}{6} = \$1.30$

(c) $-1.2 \cdot \frac{1}{6} + 0.2 \cdot \frac{1}{6} + 1.6 \cdot \frac{1}{6} + 3 \cdot \frac{3}{6} = \$1.60$

(d) $-1.6 \cdot \frac{1}{6} - 0.2 \cdot \frac{1}{6} + 1.2 \cdot \frac{1}{6} + 2.6 \cdot \frac{1}{6} + 4 \cdot \frac{2}{6} = \$1.67$

(e) $-2 \cdot \frac{1}{6} - 0.6 \cdot \frac{1}{6} + 0.8 \cdot \frac{1}{6} + 2.2 \cdot \frac{1}{6} + 3.6 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} = \$1.50$

Expected profit is greatest if he bakes *four* cakes.

4.65 $$E(x) = \int_{-1}^{5} \frac{x}{18}(x+1)\,dx = \frac{1}{18}\int_{-1}^{5}(x^2+x)\,dx = \frac{1}{18}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]\Bigg|_{-1}^{5}$$
$$= \frac{1}{18}\left[\frac{125}{3} + \frac{25}{2} + \frac{1}{3} - \frac{1}{2}\right] = \frac{1}{18}\left[\frac{126}{3} + 12\right] = 3 = \$3{,}000$$

4.66 $E(x) = \int_0^\infty \frac{x}{30} e^{-x/30} dx = 30$ or $30{,}0000$ kilometers

4.67 $$E(x)=\int_0^\infty \frac{x^2}{9}e^{-x/3}dx=\frac{1}{9}\left[-3x^2e^{-(1/3)x}-18xe^{-(1/3)x}-54e^{-(1/3)x}\right]\Bigg|_0^\infty$$
$$=\frac{1}{9}\cdot 54=6 \text{ million liters}$$

4.68 $$E(ps)=\int_{0.2}^{0.2}\int_0^\infty 5p^2s\ e^{-ps}ds\ dp=\int_{0.2}^{0.4}5p^2\cdot\frac{1}{p^2}\left[e^{-ps}(-ps-1)]\right]\Bigg|_0^\infty dp$$
$$=5\int_{0.2}^{0.4}dp=1=\$10{,}000$$

4.69 p = probability Adam will win
$$p\cdot b=(1-p)a,\ p(a+b)=a,\ p=\frac{a}{a+b}$$

4.70 $$\mu=0\cdot\frac{6}{11}+1\cdot\frac{9}{22}+2\cdot\frac{1}{22}=\frac{1}{2}$$
$$\mu_2'=1\cdot\frac{9}{22}+4\cdot\frac{1}{22}=\frac{13}{22}\qquad \sigma^2=\frac{13}{22}-\frac{1}{4}=\frac{26-11}{44}=\frac{15}{44}$$

4.71 $$\mu=\int_0^\infty\frac{1}{4}e^{-x/4}dx=\frac{1}{4}\cdot\frac{e^{-x/4}}{1/16}\left(-\frac{1}{4}x-1\right)\Bigg|_0^\infty=4$$
$$\mu_2'=\int_0^\infty\frac{1}{4}x^2e^{-x/4}dx=\frac{1}{4}\left[-\frac{2}{\left(-\frac{1}{4}\right)^2}\right]=32$$
$$\sigma^2=32-16=16$$

4.72 $$\mu=\frac{1}{288}\int_{-6}^{6}x(36-x^2)dx=\frac{1}{288}\left[18x^2-\frac{x^4}{4}\right]_{-6}^{6}=0$$
$$\mu_2'=\frac{1}{288}\int_{-6}^{6}x^2(36-x^2)dx=\frac{1}{288}\left(12x^3-\frac{x^5}{5}\right)_{-6}^{6}$$
$$=\frac{1}{288}\left[12\cdot 6^3-\frac{1}{5}6^5-12(-6)^3+\frac{1}{5}(-6)^5\right]=\frac{24\cdot 6^3}{288}-\frac{2\cdot 6^5}{288\cdot 5}=18-10.8=7.2$$
$$\sigma^2=7.2$$

4.73 $g(0)=0.4,.g(1)=0.3,\ g(2)=0.2,\ g(3)=0.1$
$$\mu=0(0.4)+1(0.3)+2(0.2)+3(0.1)=1\qquad \mu=1$$
$$\mu_2'=0^2(0.4)+1^2(0.3)+2^2(0.2)+3^2(0.1)=2\quad \sigma^2=2-1^2=1$$

4.74 **(a)** $P(x \geq 65) \leq \frac{41}{65} = 0.631$

(b) $P[(x-165) \leq 85)] \leq \frac{47}{85} = 0.553$

4.75 $\mu = 124,\ \sigma = 7.5,\ k(7.5) = 60,\ k = \frac{60}{7.5} = 8,\ p = 1 - \frac{1}{64} = \frac{63}{64}$, at least $\frac{63}{64}$

4.76 **(a)** $k = 6$ $\quad 0.260 \pm 6(0.005)$ between 0.230 and 0.290
0.030

(b) $k = 12$ $\quad 0.260 \pm 12(0.005)$ between 0.200 and 0.320
0.060

4.77 $\mu = 4,\ \sigma = 4$ at least $1 - \frac{1}{2.25} = \frac{1.25}{2.25} = \frac{5}{9}$

By Chebyshev's theorem probability $P(x < 10)$ is at least 59.

$$\int_0^{10} \frac{1}{4} e^{-(1/4)x} dx = -e^{-(1/4)x} \Big|_0^{10}$$

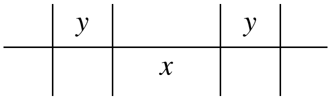
$$= 1 - e^{-2.5} = 1 - 0.0821 = 0.9179$$

4.78

	z	w	p	
(0, 0)	0	0	0.36	$E(0) = 0.60$
(1, 0)	1	1	0.24	$E(z) = 0(0.6) + 1(0.4) = 0.4$
(0, 1)	0	1	0.24	$E(w) = 0(0.36) + 1(0.48) + 2(0.16) = 0.8$
(1, 1)	1	2	0.16	$E(wz) = 0(0.36) + 1(0.24) + 0(0.24) + 2(0.16) = 0.56$

$\text{cov}(z, w) = 0.56 - 0.32 = 0.24$

4.79



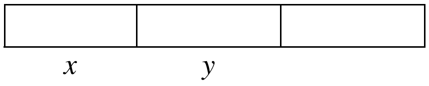
$\mu_x = 3 \qquad \sigma_x = 0.02$

$\mu_y = 0.3 \qquad \sigma_y = 0.005$ independent

$E(x + 2Y) = 3 + 2(0.3) = 3.6$

$\sigma^2_{x+2y} = (0.02)^2 + 4(0.005)^2 = 0.0005 \qquad \sigma = \sqrt{0.0005} = 0.0224$

4.80



… for x $\mu = 8 \qquad \sigma = 0.1$

… for y $\mu = 0.5 \qquad \sigma = 0.03$

$z = \sum_{i=1}^{50} x_i + \sum_{j=1}^{49} y_j \qquad E(z) = 50(8) + 49(0.5) = 424.5$ in.

$\text{var}(z) = 50(0.1)^2 + 49(0.03)^2 = 0.5441 \qquad \sigma_2 = 0.738$ in.

4.81 **(a)** X heads
Y getting 6
Z getting ace

$$E(X+Y+Z)=\frac{1}{2}+\frac{1}{6}+\frac{1}{13}=\frac{58}{78}=\frac{58}{78}=0.74$$

$$\operatorname{var}(X+Y+Z)=\frac{1}{4}+\frac{5}{36}+\frac{12}{169}=0.46 \qquad \sigma=0.68$$

(b) $3x+2y+z \qquad E=3\cdot\frac{1}{2}+2\cdot\frac{1}{6}+\frac{1}{13}=\frac{117+26+6}{78}=\frac{149}{78}=1.91$

$$\sigma^2=3\cdot\frac{1}{4}+2\cdot\frac{5}{36}+\frac{12}{169}=1.099 \qquad \sigma=1.05$$

4.82 $\mu=5(0.5)+5(0.45)=4.75$
$\sigma^2=5(0.5)(5)+5(0.45)(0.55)=1.25+1.2375=2.4875$
$\sigma=1.58$

4.83 $\phi(0|0)=3/10,\ \phi(1|0)=6/10,\ \phi(2|0)=1/10$
$E(Y)=0(0.3)+1(0.6)+2(0.10)=0.8$

4.84 $\phi(y|12)=\frac{1}{6}\ \ 6<y<12 \qquad \int_6^{12}\frac{y}{6}dy=\frac{1}{6}\left(\frac{y^2}{2}\right)\Big|_6^{12}=\frac{1}{6}(72-18)=\$9$

4.85 $$E=\frac{\int_1^\infty x f(x)\ dx}{\int_1^\infty f(x)\ dx}=\frac{N}{D}$$

$$N=\int_1^2\frac{x^2}{4}dx+\int_2^\infty\frac{4}{x^2}dx=\frac{x^3}{12}\Big|_1^2+\frac{-4}{x}\Big|_2^\infty=\frac{7}{12}+2=\frac{31}{12}$$

$$D=\int_1^2\frac{x}{4}dx+\int_2^\infty 4x^{-2}dx=\frac{x^2}{8}\Big|_1^2-\frac{2}{x^2}\Big|_2^\infty=\frac{1}{2}-\frac{1}{8}+\frac{1}{2}=\frac{7}{8}$$

$$E=\frac{31}{12}\cdot\frac{8}{7}=\frac{248}{84}=2.95\text{ min}$$

Chapter 5

5.2 $\mu = \lim_{t \to 0} \dfrac{e^t(1 - e^{kt} - ke^{kt} + ke^{t-kt})}{(e^t - 1)^2 k} = \dfrac{k+1}{2}$

5.3 $f(0) = 1 - \theta, \quad f(1) = \theta$

(a) $\sum_{x=0}^{1} x^r f(x) = 0^r(1-\theta) + 1^r \cdot \theta = \theta$

(b) $M_x(t) = \sum_{x=0}^{1} e^{tx} f(x) = (1-\theta) + e^t \cdot \theta = 1 + \theta(e^t - 1)$

$$= 1 + \theta\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \ldots\right)$$

$\mu_r' = \theta$

5.4 $\mu = \theta \qquad \mu_2' = \theta \qquad \sigma^2 = \theta - \theta^2 = \theta(1-\theta)$

(a) $\mu_3' = \theta \qquad \mu_3 = \theta - 3\theta \cdot \theta + 28^3 = \theta(1 - 3\theta + 2\theta^2) = \theta(1-2\theta)(1-\theta)$

$$\alpha_3 = \frac{\theta(1-\theta)(1-2\theta)}{\theta(1-\theta)\sqrt{\theta(1-\theta)}} = \frac{1-2\theta}{\sqrt{\theta(1-\theta)}}$$

$$\mu_4 = \theta - 4\theta^2 + 6\theta^3 - 3\theta^4 = \theta(1 - 4\theta + 6\theta^2 - 3\theta^2)$$

$$= \theta(1-\theta)[1 - 3\theta(1-\theta)]$$

(b) $\alpha_4 = \dfrac{\theta(1-\theta)[1-3\theta(1-\theta)]}{\theta^2(1-\theta)^2} = \dfrac{1-3\theta(1-\theta)}{\theta(1-\theta)}$

5.5 **(a)** $b(n-x;\ n,\ 1-\theta) = \binom{n}{n-x}(1-\theta)^{n-x}\theta^{n-(n-x)}$

$$= \binom{n}{x}\theta^x(1-\theta)^{n-x} = b(x;\ n,\ \theta)$$

(b) $B(x;\ n,\ \theta) - B(x-1;\ n,\ \theta) = \sum_{i=1}^{x} - \sum_{i=1}^{x-1} b = b(x;\ n, \theta)$

(c) $B(n-x;\ n,\ 1-\theta) = B(n-x-1;\ n,\ 1-\theta)$

$$= b(n-x;\ n,\ 1\text{-}\theta) == \binom{n}{n-x}(1-\theta)^{n-x}\theta^{n-(n-x)}$$

$$= \binom{n}{x}\theta^x(1-\theta)^{n-x} = b(x;\ n,\ \theta)$$

(d) $$1-B(n-x-1;\ n,\ 1-\theta)=1-\sum_{k=0}^{n-x-1} b(k;\ n,\ 1-\theta)$$

$$=\sum_{k=n=x}^{n} b(k;\ n,\ 1-\theta)$$

$$=\sum_{r=x}^{0} b(n-r;\ n,\ 1-\theta)=\sum_{\mathrm{r}=0}^{\mathrm{x}} b(r;\ n,\ \theta)=B(x;\ n,\ \theta) \qquad \text{QED}$$

5.6 **(a)** $$B(x;\ n,\ \theta)-B(x-1;\ n,\ \theta)=\sum_{i=1}^{x}-\sum_{i=1}^{x-1}=b(x;n,\theta)$$

(b) $$B(n-x;n,\ 1-\theta)-B(n-x-1;\ n,\ 1-\theta)$$

$$=b(n-x;\ n,\ 1-\theta)=\binom{n}{n-x}(1-\theta)^{n-x}\,\theta^{n-(n-x)}$$

$$=\binom{n}{x}\theta^{x}(1-\theta)^{n-x}=b(x;\ n,\ \theta)$$

(c) $$1-B(n-x-1;\ n,\ 1-\theta)=1-\sum_{k=0}^{n-x-1} b(k;\ n,\ 1-\theta)$$

$$=\sum_{k=n-x}^{n} b(k;\ n,\ 1-\theta)$$

$$=\sum_{r=x}^{0} b(n-r;\ n,\ 1-\theta)=\sum_{r=0}^{x} b(r;\ n,\ \theta)=B(x;\ n,\ \theta) \qquad \text{QED}$$

5.7 $$E(Y)=E\left(\frac{X}{n}\right)=\frac{1}{n}E(X)=\frac{n\theta}{n}=\theta$$

$$\mu_2'=E\left(\frac{X^2}{n^2}\right)=\frac{1}{n^2}[n\theta(1-\theta)+n^2\theta^2$$

$$\sigma_Y^2=\frac{1}{n^2}[n\theta-n\theta^2+n^2\theta^2-n^2\theta^2]=\frac{\theta(1-\theta)}{n}$$

5.8 $$b(x+1;\ n,\theta)=\binom{n}{x+1}\theta^{x+1}(1-\theta)^{n-x-1}$$

$$=\frac{n!}{(x+1)!(n-x-1)!}\theta^{x+1}(1-\theta)^{n-x-1}$$

$$=\frac{\theta}{1-\theta}\cdot\frac{n-x}{(x+1)}\cdot\binom{n}{x}\theta^{x}(1-\theta)^{n-x}=\frac{\theta(n-x)}{(x+1)(1-\theta)}b(x;\ n,\ \theta)$$

5.9 $\dfrac{b(x)}{b(x-1)} \geq 1 \qquad \dfrac{b(x+1)}{b(x)} \leq 1 \qquad \dfrac{\theta(n-x-1)}{x(1-\theta)} \geq 1 \qquad \dfrac{\theta(n-x)}{(x+1)(1-\theta)} \leq 1$

$\theta n - \theta x - \theta \geq x - \theta x \qquad \theta n - \theta x \leq x + 1 - \theta x - \theta$

$x \leq \theta(n-1) \qquad \theta n \leq x + 1 - \theta$

$x \leq \dfrac{n-1}{2} \qquad \theta(n+1) - 1 \leq x$

(b) odd maximum at $\dfrac{n-1}{2}$ $\qquad \dfrac{1}{2}n + \dfrac{1}{2} \leq x \qquad x \geq \dfrac{n+1}{2}$

(a) even maximum at $\dfrac{n-1}{2}$ and $\dfrac{n+1}{2}$

5.10 $b(x;\ n,\ \theta) = \dbinom{n}{x}\theta^x(1-\theta)^{n-x}$

$\ln b = \ln\dbinom{n}{x} + x\ln\theta + (n-x)\ln(1-\theta)$

$\dfrac{\partial b}{\partial\theta} = \dfrac{x}{\theta} - \dfrac{n-x}{1-\theta} = 0 \qquad x - \theta x = n\theta - \theta x \qquad x = n\theta$ and $\theta = \dfrac{x}{n}$

5.11 $\mu_2' = E(x^2) = E(x^2 - x + x) = \mu(_2') + \mu(_1')$ Since $x^2 = x(1-x) + x$

let $x^3 = x(x-1)(x-2) + ax(x-1) + bx$

$x = 1 \qquad 1 = b$

$b = 1 \qquad a = 3 \qquad \mu_3' = \mu(_3') + 3\mu(_2') + \mu(_1')$

$x = 2 \qquad 8 - 2a + 2$

$x^4 = x(x-1)(x-2)(x-3) + ax(x-1)(x-2) + bx(x-1) + cx$

$x = 1 \qquad 1 = c \qquad \mu_4' = \mu(_4') + 6\mu(_3') + 7\mu(_2') + \mu(_2')$

$x = 2 \qquad 16 = 2b + 2 \qquad b = 7$

$x = 3 \qquad 81 = 6a + 6b + 3c = 6a + 42 + 3$

$36 = 6a \qquad a = 6$

5.12 $F'(x) = \sum xt^{x-1}f(x) \qquad F'(1) = \sum xf(x) = \mu(_1')$

$F''(x) = \sum x(x-1)t^{x-2}f(x) \qquad F''(1) = \sum x(x-1)f(x) = \mu(_2')$

$F'''(x) = \sum x(x-1)(x-2)t^{x-3}f(x) \qquad F'''(1) = \sum x(x-1)(x-2)f(x) = \mu(_3')$

etc.

5.13 **(a)** $F_x(t) = t^\theta \cdot (1-\theta) + t\theta = 1 - \theta + \theta t \qquad F' = \theta \qquad F'' = 0$ etc.

$\mu_{(1)}' = \theta \qquad \mu_{(r)}' = 0$ for $r > 1$

(b) $$F_x(t) = \sum_x t^x \binom{n}{x} \theta^X (1-\theta)^{n-x} = \sum_x \binom{n}{x} (\theta t)^x (1-\theta)^{n-x}$$
$$= [\theta t + 1 - \theta]^n$$
$$= [1 + \theta(t-1)]^n$$

$$F' = n[1+\theta(t-1)]^{n-1}\theta \qquad F'(1) = n\theta$$
$$F'' = n(n-1)[1+\theta(t-1)]^{n-2}\theta^2 \qquad F''(1) = n(n-1)\theta^2$$
$$\mu = \mu'_{(1)} = n\theta \qquad \mu'_2 = \mu'_{(2)} + \mu'_{(1)} = n(n-1)\omega^2 + n\theta$$
$$\sigma^2 = n(n-1)\theta^2 + n\theta - n^2\theta^2 = n\theta - n\theta^2 = n\theta(1-\theta)$$

5.14 $M'_Y = e^{-\mu t} M'_X(t) + M_X(t)(-\mu)e^{-\mu t} = e^{-\mu t}[M'_X(t) - \mu M_X(t)]$

(a) Expand series.

(b) $$M_{X-\mu}(t) = e^{-n\theta t}[1+\theta(e^t-1)]^n$$
$$M'_{x-\mu}(t) = e^{-n\theta t} \cdot n[1+\theta(e^t-1)^{n-1} \cdot \theta e^t - n\theta e^{-n\theta t}[1+\theta(e^t-1)]^n$$
$$= n\theta e^{-n\theta t}[1+\theta(e^t-1)^{n-1}\left\{1-[1+\theta(e^t-1)\right\}$$
$$= n\theta e^{-n\theta t}[1+\theta(e^t-1)^{n-1}\left\{e^t(1-\theta)-(1-\theta)\right\} \qquad M'_{x-\mu}(0) = 0$$
$$= -n\theta^2 e^{-n\theta t}(e^t-1)[1+\theta(e^t-1)]^{n-1}$$

$$M''_{x-\mu}(t) = -n\theta^2 e^{-n\theta t}(e^t-1)(n-1)[1+\theta(e^t-1)]^{n-2}(e^t-1)$$
$$- n\theta^2[1+\theta(e^t-1)]^{n-1}\left\{e^{-n\theta t} \cdot e^t + (e^t-1)(-n\theta e^{-n\theta t})\right\}$$
$$= e^{-n\theta t}[1+\theta(e^t-1)]^n$$

5.15 **(a)** $\theta = \frac{1}{2},\ \alpha_3 = 0;$ **(b)** $\alpha_3 \to 0$ as $n \to \infty$

5.16 $f(y) = \binom{y+k+1}{k-1}\theta^k(1-\theta)^y$ $\qquad y = x-k$, $\quad y = 0,\ 1,\ 2,\ \ldots$

5.17 $$E(Y) = E(X) - k = \frac{k}{\theta} - k = k\left(\frac{1}{\theta}-1\right)$$
$$\sigma_Y^2 = \sigma_X^2 = \frac{k}{\theta}\left(\frac{1}{\theta}-1\right)$$

5.18 $b^*(x;k,\theta) = \binom{x-1}{k-1}\theta^k(1-\theta)^{x-k} = \frac{k}{x}\binom{x}{k}\theta^k(1-\theta)^{x-k} = \frac{k}{x}b(k;x,\theta)$ QED

5.19 $$E(x)=\sum_{x=k}^{\infty} x\binom{x-1}{k-1}\theta^{k}(1-\theta)^{x-k}=\frac{k}{\theta}\sum_{x=k}^{\infty}\binom{x}{k}\theta^{x+1}(1-\theta)^{x-k}\qquad \begin{matrix} y=x+1\\ m=k+1\end{matrix}$$

$$=\frac{k}{\theta}\sum_{y=m}^{\infty}\binom{y-1}{m-1}\theta^{y}(1-\theta)^{y-m}=\frac{k}{\theta}$$

$$E[x(x+1)]=\sum_{x=k}^{\infty}x(x+1)\binom{x-1}{k-1}\theta^{k}(1-\theta)^{x-k}$$

$$=\frac{k(k+1)}{\theta^2}\sum_{x=k}^{\infty}\binom{x+1}{k+1}\theta^{x+2}(1-\theta)^{x-k}\qquad \begin{matrix} y=x+2\\ m=k+2\end{matrix}$$

$$=\frac{k(k+1)}{\theta^2}\sum_{y=m}^{\infty}\binom{y-1}{m-1}\theta^{x+2}(1-\theta)^{y-k}=\frac{k(k+1)}{\theta^2}$$

$$\sigma^2=\frac{k(k+1)}{\theta^2}-\frac{k}{\theta}-\frac{k^2}{\theta^2}=\frac{k^2+k-k\theta-k^2}{\theta^2}=\frac{k(1-\theta)}{\theta^2}=\frac{k}{\theta}\left(\frac{1}{\theta}-1\right)$$

5.20 $g(x)=\theta(1-\theta)^{x-1}\qquad x=1,2,3,\ldots$

$$M=\sum_{x=1}^{\infty}e^{tx}\theta(1-\theta)^{x-1}=\sum_{x=1}^{\infty}\theta\frac{[e^{t}(1-\theta)]^{x}}{1-\theta}=\frac{\theta}{1-\theta}\sum_{x=1}^{\infty}[e^{t}(1-\theta)]^{x}$$

$$=\frac{\theta}{1-\theta}\cdot\frac{e^{t}(1-\theta)}{1-e^{t}(1-\theta)}=\frac{\theta e^{t}}{1-e^{t}(1-\theta)}\qquad \text{QED}$$

5.21 $$M'=\frac{[1-e^{t}(1-\theta)\theta e^{t}+\theta e^{t}(1-\theta)e^{t}}{[1-e^{t}(1-\theta)]^2}=\frac{\theta e^{t}-\theta e^{2t}(1-\theta)+\theta e^{2t}-\theta^2e^{2t}}{[1-e^{t}(1-\theta)]^2}$$

$$=\frac{\theta e^{t}}{[1-e^{t}(1-\theta)]^2}$$

$$M'(0)=\frac{\theta}{\theta^2}=\frac{1}{\theta}$$

$$M''=\frac{[1-e^{t}(1-\theta)]^2\theta e^{t}-\theta e^{t}\cdot 2[1-e^{t}(1-\theta)][-e^{t}(1-\theta)]}{[1-e^{t}(1-\theta)]^4}$$

$$M''(0)=\frac{\theta^2-2\theta\cdot\theta(1-\theta)}{\theta^4}-\frac{2-\theta}{\theta^2}\qquad \sigma^2=\frac{2-\theta}{\theta^2}-\frac{1}{\theta^2}=\frac{1-\theta}{\theta^2}$$

5.22 $$\sum_{x=1}^{\infty}\theta(1-\theta)^{x-1}=1$$

$$\theta+\sum_{x=2}^{\infty}\theta(1-\theta)^{x-1}=1 \qquad y=x-1$$

$$\sum_{y=1}^{\infty}\theta(1-\theta)^{y}=1-\theta$$

$$\sum_{y=1}^{\infty}[(1-\theta)^{y}+\theta y(1-\theta)^{y-1}(-\theta)]-1$$

$$\sum_{y=1}^{\infty}(1-\theta)^{y}-\sum_{y=1}^{\infty}\theta(1-\theta)^{y-1}=-1$$

$$\frac{1-\theta}{\theta}-\mu=-1$$

$$-\mu=-\frac{1}{\theta}\text{ and }\mu=\frac{1}{\theta}$$

$$\theta+\theta(1-\theta)+\sum_{x=3}^{\infty}\theta(1-\theta)^{x-1}=1 \qquad y=x-2$$

$$\theta+\theta(1-\theta)+\sum_{y=1}^{\infty}\theta(1-\theta)^{y+1}=1$$

$$\sum_{y=1}^{\infty}\theta(1-\theta)^{y+1}=1-\theta-\theta(1-\theta)=(1-\theta)^{2}$$

then differentiate *twice* with respect to θ.

5.23 $$P(X=x+n|x>n)=\frac{P(X=x+n)}{P(X>n)}=\frac{\theta(1-\theta)^{x+n}}{(1-\theta)^{n}}=\theta(1-\theta)^{x} \qquad \text{QED}$$

$$P(X>n)=\frac{\theta(1-\theta)^{n}}{1-(1-\theta)}=(1-\theta)^{n}$$

5.24 $$f(x)=\theta(1-\theta)^{x-1`} \quad F(x)=\sum_{t=1}^{x}\theta(1-\theta)^{tx-1}=\theta\cdot\frac{1-(1-\theta)^{x}}{1-(1-\theta)}=1-(1-\theta)^{x}$$

$$z(x)=\frac{\theta(1-\theta)^{x-1}}{(1-\theta)^{x-1}}=\theta$$

5.25 $X=X_1+X_2=\ldots X_n$

(a) $$E(X)=\sum E(X_i)=\sum\theta_i=n\frac{\sum\theta_i}{n}=n\theta$$

(b) $$\sigma_X^2=\sum\sigma_i^2=n\sum\theta_i(1-\theta_i)=n\sum\theta_i-\sum\theta_i^2$$
$$=n\theta-n\sigma_\theta^2+n\theta^2=n\theta(1-\theta)-n\sigma_\theta^2$$

5.26 $$h(x+1)=\frac{\binom{M}{x+1}\binom{n-M}{n-x-1}}{\binom{N}{n}}$$

$$=\frac{\dfrac{M!}{(x+1)!(M-x-1)!}\cdot\dfrac{(N-M)!}{(n-x-1)!(N-M-n+x+1)!}}{\binom{N}{n}}$$

$$=\frac{\dfrac{M-x}{x+1}\cdot\dfrac{M!}{x!(M-x)!}\cdot\dfrac{n-x}{N-M-n+x-1}\cdot\dfrac{(N-M)!}{(n-x)!(N-M-n+x)!}}{\binom{N}{n}}$$

$$=\frac{\dfrac{(M-x)(n-x)}{(x+1)(N-M-n+x+1)}\binom{M}{x}\binom{n-M}{n-x}}{\binom{N}{n}}$$

$$=\frac{(M-x)(n-x)}{(x+1)(N-M-n+x+1)}\cdot h(x)$$

$n=4,\ N=9,\ M=5$

$$h(0)=\frac{\binom{5}{0}\binom{4}{4}}{\binom{9}{4}}=\frac{1}{126},\quad h(1)=\frac{5\cdot 4}{1\cdot 1}\cdot\frac{1}{126}=\frac{20}{126}$$

$$h(2)=\frac{4\cdot 3}{2\cdot 2}\cdot\frac{20}{126}=\frac{60}{126},\quad h(3)=\frac{3\cdot 2}{3\cdot 3}\cdot\frac{60}{126}=\frac{40}{126}$$

$$h(4)=\frac{2\cdot 1}{4\cdot 4}\cdot\frac{40}{126}=\frac{5}{126}$$

5.27 $$E[X(X-1)] = \sum_{x=0}^{n} x(x-1)\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$$

$$= \sum_{x=2}^{n} M(M-1)\frac{\binom{M-2}{x-2}\binom{N-M}{n-x}}{\binom{N}{n}} \qquad \begin{array}{l} y = x-2 \\ m = n-2 \end{array}$$

$$= M(M-1)\sum_{y=0}^{m}\frac{\binom{M-2}{y}\binom{N-M}{m-y}}{\binom{N}{n}}$$

$$= \frac{M(M-1)n(n-1)}{N(N-1)}\sum_{y=0}^{m}\frac{\binom{M-2}{y}\binom{N-M}{m-y}}{\binom{N-2}{m}}$$

$$= \frac{M(M-1)n(n-1)}{N(N-1)} \qquad \text{QED}$$

5.28 $\theta = \frac{M}{N} \qquad \mu = n\frac{M}{N} = n\theta$

$$\sigma^2 = n\cdot\frac{M}{N}\cdot\left(1-\frac{M}{N}\right)\cdot\frac{N-n}{N-1} = n\theta(1-\theta)\cdot\frac{N-n}{N-1}$$

5.29 $$P(x+1;\ \lambda) = \frac{\lambda^{x+1}e^{-\lambda}}{(x+1)!} = \frac{\lambda}{x+1}\cdot\frac{\lambda^x e^{-\lambda}}{x!} = \frac{\lambda}{x+1}\cdot p(x;\ \lambda)$$

5.30 $$p(3;\ 10) = \frac{10^3 e^{-10}}{6} = \frac{1000(0.000045)}{6} = \frac{0.045}{6} = 0.0075$$

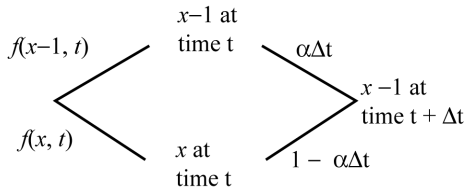
Table II yields 0.0076

(a) $$\binom{100}{3}(0.1)^3(0.90)^{97} = \frac{100!}{3!97!}(0.1)^2(0.9)^{97}$$

$$\begin{aligned}\log p &= 157.97000 - 0.77815 - 151.98314 + 3(-1) + 97(0.95424) - 1 \\ &= 5.20871 - 3 + 92.56128 - 97 \\ &= 0.77699 - 3,\ p = 0.0060\end{aligned}$$

(b) $p = 0.00598$

5.31



(a) $$f(x,\ t+\Delta t) = f(x,\ t)(1-\alpha\Delta t) + f(x-1,\ t)\alpha\Delta t$$

$$f(x,\ t+\Delta t) - f(x,\ t) = -\alpha\Delta t\, f(x,\ t) + \alpha\Delta t\, f(x-1,\ t)$$

$$\lim_{\Delta t\to 0}\frac{f(x,\ t+\Delta t)-f(x,\ t)}{\Delta t} = \alpha[f(x-1,\ t)-f(x,\ t)]$$

(b) $$f(x,\ \alpha t) = \frac{(\alpha t)^x e^{-\alpha t}}{x!}\quad \frac{\partial f}{\partial t} = \frac{\alpha^x x\ t^{x-1}\ e^{-at} + \alpha^x t^x(-\alpha\ e^{-\alpha t})}{x!}$$

$$= \frac{\alpha^x x\ t^{x-1}\ e^{-at} - \alpha^{x+1} t^x\ e^{-\alpha t}}{x!}$$

$$\alpha[f(x-1,\ t)-f(x,\ t)] = \frac{\alpha\cdot(\alpha t)^{x-1}e^{-\alpha t}}{(x-1)!} - \frac{\alpha(\alpha t)^x e^{-\alpha t}}{x!}$$

$$= \frac{\alpha^x\cdot x\ t^{x-1}e^{-\alpha t} - \alpha^{x+1}t^x e^{-\alpha t}}{x!}\qquad \text{QED}$$

5.32 $u = t^x dv = e^{-t}dt \qquad v = -e^{-t}du = x\ t^{x-1}dt$

$$\frac{1}{x!}\int_\lambda^\infty t^x e^{-t}dt = \frac{\lambda^x e^{-\lambda}}{x!} + \frac{1}{(x-1)!}\int_\lambda^\infty t^{x-1}e^{-t}dt$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} + \frac{\lambda^{x-1}e^{-\lambda}}{(x-1)!} + \frac{1}{(x-2)!}\int_\lambda^\infty t^{x-2}e^{-t}dt$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} + \ldots + \frac{\lambda^0 e^{-\lambda}}{0!} = \sum_{y=0}^{x}\frac{\lambda^x e^{-\lambda}}{y!}\qquad \text{QED}$$

5.33 $$E(X) = \sum_{x=0}^{\infty} x\cdot\frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=1}^{\infty}\lambda\cdot\frac{\lambda^{x-1}e^{-\lambda}}{(x-1)!} = \sum_{y=0}^{\infty}\lambda\cdot\frac{\lambda^y e^{-y}}{y!} = \lambda\cdot 1 = \lambda$$

$$E[X(X-1)] = \sum_{x=2}^{\infty} x(x-1)\frac{\lambda^x e^{-\lambda}}{x!}$$

$$= \sum_{x=2}^{\infty}\lambda^2\,\frac{\lambda^{x-2}e^{-\lambda}}{(x-2)!} = \sum_{y=0}^{\infty}\lambda^2\,\frac{\lambda^y e^{-y}}{y!} = \lambda^2$$

$\mu = \lambda,\ \sigma^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

5.34 $n \to \infty$, $\theta \to 0$, $n\theta = \lambda$

$$M_x = [1 + \lambda(e^t - 1)]^n$$

$$= \left[1 + \frac{n\lambda(e^t - 1)}{n}\right]^n = \left[1 + \frac{\lambda(e^t - 1)}{n}\right]^n$$

$$\lim_{n \to \infty} = e^{\lambda(e^{t-1})} \text{ QED}$$

5.35 $M = e^{\lambda(e^t - 1)}$

$$M' = \lambda\, e^t e^{\lambda(e^t-1)} \qquad M'(0) = \lambda$$

$$M'' = (\lambda e^t)^2 e^{\lambda(e^t-1)} + \lambda e^t e^{\lambda(e^t-1)} \quad M''(0) = \lambda^2 + \lambda$$

$$M''' = (\lambda e^t)^3 e^{\lambda(e^t-1)} + 2(\lambda e^t)^2 e^{\lambda(e^t-1)} \qquad M'''(0) = \lambda^3 + 3\lambda^2 + \lambda$$

$$+(\lambda e^t)^2 e^{\lambda(e^t-1)} + \lambda e^t e^{\lambda(e^t-1)}$$

$$\mu = \lambda,\ \sigma^2 = \lambda^2 + \lambda - \lambda^2 = \lambda,\ \mu^3 = \lambda^3 + 3\lambda^2 + \lambda - 3\lambda(\lambda^2 + \lambda) + 2\lambda^2 = \lambda$$

$$\alpha_3 = \frac{1}{\left(\sqrt{\lambda}\right)^3} = \frac{1}{\sqrt{\lambda}}$$

5.36 $$\frac{d\mu_r}{d\lambda} = \sum_{x=0}^{\infty} r(x-\lambda)^{r-1} \cdot \frac{\lambda^x e^{-x}}{x!} + \frac{(x-\lambda)^r}{x!}\left\{x\lambda^{x-1}e^{-\lambda} - \lambda^x e^{-\lambda}\right\}$$

$$= -r\mu_{r-1} + \sum_{x=0}^{\infty} \frac{(x-\lambda)^r}{x!}\lambda^{x-1}e^{-\lambda}(x-\lambda)$$

$$= -r\mu_{r-1} + \sum_{x=0}^{\infty} (x-\lambda)^{r+1} \frac{\lambda^{x-1}e^{-x}}{x!}$$

$$= -r\mu_{r-1} + {}_{\lambda}\mu_{r+1} \qquad \mu_{r+1} = \lambda\left[r\mu_{r-1} + \frac{d\mu_r}{d\lambda}\right]$$

$$\mu_0 = 1,\ \mu_1 = 0,\ r = 1,\ \mu_2 = \lambda\left[1 \cdot \mu_0 + \frac{d\mu_1}{d\lambda}\right] = \lambda$$

$$r = 2,\ \mu_3 = \lambda\left[2 \cdot \mu_1 + 1\right] = \lambda$$

$$r = 3,\ \mu_4 = \lambda\left[3 \cdot \lambda + 1\right] = \lambda + 3\lambda^2$$

5.57 $M_x = E(e^{xt}) = e^{\lambda(e^t-1)}$

$$M_Y = E[e^{(x-\lambda)t}] = e^{-\lambda t}E(e^{xt}) = e^{-\lambda t}e^{\lambda(e^t-1)} = e^{\lambda(e^t-t-1)}$$

$$M'_Y = \lambda(e^t - 1)e^{\lambda(e^t-t-1)}$$

$$M''_Y = \lambda^2(e^t-1)^2 e^{\lambda(e^t-t-1)} + \lambda e^t e^{\lambda(e^t-t-1)}$$

$$M'_Y(0) = \lambda$$

5.38 Marginal distribution of x_i is binomial distribution with parameter n and θ_i; therefore

$\mu_1 = n\theta_i$

5.39 $$E(x_i x_j) = \sum\sum x_i x_j \frac{n!}{x_i!x_j!(n-x_i-x_j)!}\theta_i^{x_i}\ \theta_j^{x_j}(1-\theta_i-\theta_j)^{n-x_i-x_j}$$
$$= n(n-1)\theta_i\theta_j \sum\sum \frac{(n-2)!}{(x_i-1)!(x_j-1)!(n-x_i-x_j)!}\ \theta_i^{x_i-1}\theta_j^{x_j-1}(1-\theta_i-\theta_j)^{n-x_i-x_j}$$
$$= n(n-1)(\theta_i)(\theta_j)$$

$$\operatorname{cov}(x_i,\ x_j) = n(n-1)\theta_i\theta_j - (n\theta_i)(n\theta_j)$$
$$= -n\theta_i\theta_j$$

5.40 $$\binom{8}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^4 = \frac{70\cdot 16}{6561} = 0.1707$$

5.41 $$\binom{5}{3}(0.1)^3(0.9)^2 + \binom{5}{4}(0.1)^4(0.9) + \binom{5}{5}(0.1)^5$$
$$= 10(0.001)(0.81) + 5(0.0001)(0.9) + (0.00001)$$
$$= 0.0081 + 0.00045 + 0.00001 = 0.0086$$

5.42 **(a)** $\binom{6}{5}(0.7)^5(0.3) = 0.3025$

(b) 0.3025

5.43 **(a)** $\binom{15}{6}(0.4)^6(0.6)^6 = 5005(0.004096)(0.01008) = 0.2066$

(b) 0.2066

5.44 **(a)** 0.1669

(b) $0.1669 + 0.1214 + 0.0708 + 0.0327 + 0.0117 + 0.0031 + 0.0006 + 0.0001 = 0.4073$

(c) $0.0000 + 0.0001 + \ldots + 0.1669 = 0.4073$

5.45 **(a)** $0.1529 + 0.0578 + 0.0098 = 0.2205$

(b) $1 - 0.7794 = 0.2206$

5.46 **(a)** $0.0285 + 0.0849 + 0.1734 = 0.2868$

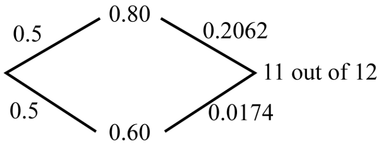
(b) $0.2939 - 0.0071 = 0.2868$

5.47 $p = 0.42,\ n = 15,\ x = 6,\ 0.2041$

5.48 $p = 0.51 \qquad n = 18$

(a) $x = 10$ 0.1731, **(b)** $1 - 0.5591 = 0.4409$, **(c)** 0.3742

5.49



$$\frac{2062}{2236} = 0.9222$$
$$1 - 0.9222 = 0.0778$$

5.50 **(a)** $\sigma_{\text{orig}} = \sqrt{np(1-p)}$. If $\sigma_{\text{new}} = \frac{1}{2}\sigma_{\text{orig}} = \frac{1}{2}\sqrt{np(1-p)} = \sqrt{\frac{n}{4}p(1-p)}$, then $n_{\text{new}} = \frac{1}{4}n_{\text{orig}}$

(b) $\sigma_{\text{orig}} = \sqrt{np(1-p)}$; $\sigma_{\text{new}} = \sqrt{nkp(1-p)} = \sqrt{k} \cdot \sqrt{np(1-p)} = \sqrt{k} \cdot \sigma_{\text{orig}}$

5.51 $P(x \geq 3) = 1 - b(0;20,0.05) - (b(1;20,0.05) - b(2;20,0.05)$
$= 1 - 0.3585 - 0.3774 - 0.1887 = 0.0754$

Thus, there is only a 0.0754 chance of obtaining 3 or more failures if the manufacturer's claim is correct.

5.52 Using MINITAB software we first enter 13 and 18 in C1 and then give the commands:

```
            MTB> CDF C1;
            SUBC> BINOMIAL 100 .16667.
obtaining   K  P(X LESS THAN OR = K)
            13  .2000
            10   .6964
```

(a) $P(x \leq 18) = 0.6984;\ P(x \leq 13) = 0.2000$
thus, $P(14 \leq x \leq 18) = 0.6954 - 0.2000 = 0.4964$

(b) No. The probability of obtaining more than 18 "sevens" is $1 - 0.6964 = 0.3036$

5.53 Using MINITAB with the number 12 entered into C1 and the commands:

```
            MTB> CDF C1;
            SUBC> BINOMIAL 80 .10.
we get      K  P(X LESS THAN OR = K)
            12  .9462
```

(a) $P(x \leq 12) = 0.9462$; thus $P(x > 12) = 1 - 0.9462 = 0.0538$

(b) With a probability of only 0.0538 the assumption is borderline questionable.

5.54 $k = 6$

(a) $\mu = 450;\ \sigma = 15$ $\quad \frac{450 \pm 90}{900}$ or 0.40 to 0.60

(b) $\mu = 5{,}000;\ \sigma = 50$ $\quad \frac{5{,}000 \pm 300}{10{,}000}$ or 0.47 to 0.53

(c) $\mu = 500{,}000;\ \sigma = 500$ $\quad \frac{500{,}000 \pm 3{,}000}{100{,}000}$ or 0.497 to 0.503

5.57 **(a)** $\theta = 0.5,\ x = 4,\ k = 1$

$$b^* = \binom{3}{0}(0.5)^1(0.25)^3 = 1 \cdot (0.5)(0.125) = 0.0625$$

(b) $\theta = 0.5,\ x = 7,\ k = 2$

$$b^* = \binom{6}{1}(0.5)^1(0.5)^5 = 6(0.25)(0.003125) = 0.0469$$

(c) $\theta = 0.5,\ x = 10,\ k = 4$ and 5

$$b^* = \binom{9}{3}(0.5)^4(0.5)^6 = \binom{9}{4}(0.5)^5(0.5)^5$$

$$= (84 + 126)(0.5)^{10} = 210(0.0009765) = 0.2051$$

5.58 **(a)** $\theta = 0.75,\ x = 8,\ k = 5$

$$b^* = \binom{7}{4}(0.75)^5(0.25)^3 = 35(0.2373)(0.015625) = 0.1298$$

(b) $\theta = 0.75,\ x = 15,\ k = 10$

$$b^* = \binom{14}{9}(0.75)^{10}(0.25)^5 = 2002(0.05631)(0.0009765) = 0.1101$$

5.59 $b^* = \binom{6-1}{1-1}(0.3)^1(0.7)^5 = 1 - (0.3)(0.16807) = 0.0504$

5.60 $\theta = 0.05,\ x = 15,\ k = 2$

(a) $b^* = \binom{14}{1}(0.5)^2(0.95)^{13} = 14(0.0025)(0.51334) = 0.0180$

(b) $b^* = \frac{2}{15} \cdot (2;\ 15,\ 0.05) = \frac{2}{15}(0.1348) = 0.0180$

5.61 $g(x;\ 1,\ \theta) = \frac{1}{x} b(x;\ 1,\ \theta)$

(a) $x = 4,\ \theta = 0.75$

$$g = \frac{1}{4} b(1;\ 4,\ 0.75)$$

$$= \frac{1}{4}\binom{4}{1}(0.75)^1(0.25)^3 = 0.0117$$

(b) $x = 6,\ \theta = 0.30$

$$g = \frac{1}{6} b(1;\ 6,\ 0.30)$$

$$= \frac{1}{6}\binom{6}{1}(0.3)(0.70)^5 = 0.0504$$

5.62 $g = (0.999)^{800}$

$$\begin{aligned}\log g &= 800(\log 0.999)\\ &= 800(0.99957 - 1)\\ &= 799.656 - 800 = 0.656 - 1\end{aligned}$$

$g = 0.4529$ (depends on rounding)

5.63 **(a)** $$\frac{\binom{14}{2}\binom{4}{0}}{\binom{18}{2}} = \frac{91}{153} = 0.5948$$

(b) $$\frac{\binom{10}{2}\binom{8}{0}}{\binom{18}{2}} = \frac{45}{153} = 0.2941$$

(c) $$\frac{\binom{6}{2}\binom{12}{0}}{\binom{18}{2}} = \frac{15}{153} = 0.980$$

5.64 **(a)** $$\frac{\binom{10}{0}\binom{6}{3}}{\binom{16}{3}} = \frac{1 \cdot 20}{560} = \frac{2}{56} = \frac{1}{28}$$

(b) $$\frac{\binom{10}{1}\binom{6}{2}}{\binom{16}{3}} = \frac{10 \cdot 15}{560} = \frac{15}{56}$$

(c) $$\frac{\binom{10}{2}\binom{6}{1}}{\binom{16}{3}} = \frac{45 \cdot 6}{560} = \frac{27}{56}$$

(d) $$\frac{\binom{10}{3}\binom{6}{0}}{\binom{16}{3}} = \frac{120}{560} = \frac{3}{14}$$

5.65 **(a)**

$$\mu = 0 \cdot \frac{2}{56} + 1 \cdot \frac{15}{56} + 2 \cdot \frac{27}{56} + 3 \cdot \frac{12}{56} = \frac{105}{56} = \frac{15}{8}$$

$$\mu_2' = 0^2 \cdot \frac{2}{56} + 1^2 \cdot \frac{15}{56} + 2^2 \cdot \frac{27}{56} + 3^2 \cdot \frac{12}{56} = \frac{231}{56}$$

$$\sigma^2 = \frac{231}{50} - \left(\frac{15}{8}\right)^2 = \frac{1848 - 1575}{448} = \frac{273}{448} = \frac{39}{64}$$

(b)

$$\mu = \frac{3 \cdot 10}{16} = \frac{15}{8}$$

$$\sigma^2 = \frac{3 \cdot 10 \cdot 6 \cdot 13}{16 \cdot 16 \cdot 15} = \frac{39}{64}$$

5.66

$$\frac{\binom{9}{2}\binom{6}{3}}{\binom{15}{5}} = \frac{36 \cdot 20}{3003} = 0.2398$$

5.67 **(a)** $12 > 0.05(200) = 10$; condition *not* satisfied

(b) $20 < 0.05(500) = 25$; condition satisfied

(c) $30 < 0.05(640) = 32$; condition satisfied

5.68 **(a)**

$$\frac{\binom{4}{1}\binom{76}{2}}{\binom{80}{3}} = \frac{4 \cdot 76 \cdot 75}{2} = \frac{6}{80 \cdot 79 \cdot 78} = \frac{285}{2054} = 0.1388$$

(b)

$$\binom{3}{1}(0.05)(0.95)^2 = 0.1354$$

5.69 $n = 300,\ M = 240,\ n = 6,\ x = 4$

(a)

$$\frac{\binom{240}{4}\binom{60}{2}}{\binom{300}{6}} = \frac{240 \cdot 239 \cdot 238 \cdot 237 \cdot 60 \cdot 59 \cdot 720}{24 \cdot 2 \cdot 300 \cdot 299 \cdot 298 \cdot 297 \cdot 296 \cdot 295} = 0.2478$$

(b)

$$\binom{6}{4}(0.80)^4(0.2)^2 = 15(0.4096)(0.04) = 0.2458$$

5.70 $\dfrac{\binom{30}{1}\binom{270}{11}}{\binom{300}{12}} \div \dfrac{\binom{30}{0}\binom{270}{12}}{\binom{300}{12}} = \dfrac{360}{259} = 1.39$, and hence, less than 3 to 2

5.71 Good $n \geq 20$ and $\theta \leq 0.05$ excellent $n \geq 100$ and $n\theta < 10$

(a) $125 \geq 20$ and $0.10 > 0.05$, also $n\theta = 12.5 > 10$; neither rule is satisfied

(b) $25 > 20,\ 0.04 \leq 0.05$; good approximation

(c) $120 > 100,\ n\theta = 6 < 10$; excellent approximation

(d) $0.06 > 0.05,\ 40 < 100$; neither rule is satisfied

5.72 $\lambda = 150(0.014) = 2.1$ from Table II

$p(2;\ 2.1) = 0.2700$

5.73

5 $\dfrac{0.1904 - 0.1088}{0.1088} \cdot 100 = 0.55\%$

6 $\dfrac{0.1367 - 0.1384}{0.1384} \cdot 100 = -1.23\%$

7 $\dfrac{0.1465 - 0.1499}{0.1499} \cdot 100 = -2.27\%$

8 $\dfrac{0.1373 - 0.1410}{0.1410} \cdot 100 = -2.62\%$

9 $\dfrac{0.1144 - 0.1171}{0.1171} \cdot 100 = -2.31\%$

10 $\dfrac{0.0858 - 0.0869}{0.0869} \cdot 100 = -1.27\%$

11 $\dfrac{0.0585 - 0.0582}{0.0582} \cdot 100 = 0.52\%$

12 $\dfrac{0.0366 - 0.0355}{0.0355} \cdot 100 = 3.10\%$

13 $\dfrac{0.0211 - 0.0198}{0.0198} \cdot 100 = 6.57\%$

14 $\dfrac{0.0113 - 0.0102}{0.0102} \cdot 100 = 10.78\%$

15 $\dfrac{0.0057 - 0.0049}{0.0049} \cdot 100 = 16.33\%$

$x = 15$

5.74 $\lambda = 150(0.04) = 6$ from Table II

(a) 0.1606

(b) $0.0025 + 0.0149 + 0.0446 + 0.892 = 0.1512$

5.75 $\lambda = 1000(0.0012) = 1.2$ from Table II

$p(0) + p(1) + p(2) = 0.3012 + 0.3614 + 0.2169 = 0.8795$

5.76 **(a)** $0.1373 + 0.1144 + 0.0858 + 0.0585 + 0.0366 = 0.4326$

(b) $0.9573 - 0.5246 = 0.4327$

5.77 $f(2;\ 3.3) = \dfrac{3.3^2 e^{-3.3}}{2!} = (5.445)(0.037) = 0.201$

5.78 **(a)** $f(0;\ 1.8)=\dfrac{(1.8)^0 e^{-1.8}}{0!}=0.165$

(b) $f(1;\ 1.8)=\dfrac{1.8\, e^{-1.8}}{1}=0.297$

5.79 **(a)** 0.1653; **(b)** 0.2975

5.80 **(a)** $\lambda=0.5 \qquad 0.6065+0.3033=0.9098$

(b) $\dfrac{(0.5)^0 e^{-0.5}}{0!}+\dfrac{(0.5)e^{-0.5}}{1!}=1.5(0.607)=0.9105$

5.81 **(a)** $f(3;\ 5.2)=0.1293$

(b) $0.0220+0.0104+0.0045+0.0018+0.0007+0.0002+0.0001=0.0397$

(c) $0.1681+0.1748+0.1515=0.4944$

5.82 **(a)**

$$h(0;\ 100,\ 100,\ 6)=\frac{\binom{6}{0}\binom{994}{100}}{\binom{1000}{100}}$$

Calculation of such large binomial coefficients is not possible with MINITAB. However, other statistical (e.g., MICROSTAT) yield 3.3876×10^{139} for the large coefficient in the numerator and 6.3850×10^{139} for denominator. Thus, the required probability is given by

$$1-h(0;\ 100,\ 1000,\ 6)=1-\frac{1\cdot 3.3876}{6.3850}=0.4695$$

(b) Using MINITAB software we enter 1 in C1 and give commands:

```
MTB> CDF C1;
SUBC? Binomial 100 .006.
```

obtaining

```
K P(X LESS THAN OR = K)
1.5478
```

Thus, the approximate probability is $1-0.5478=0.4522$

(c) Using the Poisson distribution having the mean $100\times 0.006=0.6$, we obtain the probability $1-0.5478=0.4522$ from Table II.

5.83 $\dfrac{10!}{3!\ 6!\ 1!}(0.40)^3(0.50)^6(0.10)=840(0.064)(0.015625)(0.10)=0.0840$

5.84 $\dfrac{12!}{5!\ 4!\ 2!\ 1!}(0.6)^5(0.2)^4(0.1)^2(0.1)=83160(0.07776)(0.0016)(0.001)=0.0103$

5.85 $\dfrac{9!}{4!\ 3!\ 2!\ 0!}\left(\dfrac{9}{16}\right)^4\left(\dfrac{3}{16}\right)^3\left(\dfrac{3}{16}\right)^2=1260(0.1001128)(0.0002317)=0.0292$

5.86 **(a)** $$\frac{\binom{15}{4}\binom{7}{1}\binom{3}{0}}{\binom{25}{5}} = \frac{1365 \cdot 7 \cdot 24 \cdot 5}{25 \cdot 24 \cdot 23 \cdot 22 \cdot 21} = 0.1798$$

(b) $$\frac{\binom{15}{3}\binom{7}{1}\binom{3}{1}}{\binom{25}{5}} = \frac{455 \cdot 7 \cdot 3 \cdot 120}{25 \cdot 24 \cdot 23 \cdot 22 \cdot 21} = 0.1798$$

5.87 $$\frac{\binom{10}{3}\binom{5}{1}\binom{3}{2}}{\binom{18}{6}} = \frac{120 \cdot 5 \cdot 3}{18564} = 0.0970$$

5.88 $P(\text{rejection} | \% \text{ defective} = 0.01) = 0.10,$ thus the producer's risk is 0.10.

$P(\text{rejection} | \% \text{ defective} = 0.03) = 0.95,$ thus the consumer's risk is $1 - 0.95 = 0.05$.

5.89 **(a)** Since producer's risk = 0.05 with an AQL of 0.03, the probability is $1 - 0.95 = 0.05$.
(b) Since the consumer's risk is 0.10 with an LTPD of 0.07, the probability is 0.10.

5.90 If $c = 2$, we get the following from Table I.

p	0	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
$L(p)$	1	0.9245	0.6769	0.4049	0.2061	0.0913	0.0355	0.0121	0.0036	0.0009	0.0002

Sketching the OC curve and finding values of p for $L(p) = 1 - 0.05 = 0.95$ and 0.10, we obtain: AQL = 0.03 and LTPD = 0.26.

5.91 **(a)** Producer's risk = 1 value of $L(p)$ when $p = 0.10$, or 0.17.
(b) LTPD = value of p for which $L(p) = 0.05$

5.92 If $n = 10$ and $c = 1$, we get the following from Table I.

p	0	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
$L(p)$	1	0.9139	0.7361	0.5443	0.3758	0.2440	0.1493	0.0860	0.0464	0.0009	0.0002

5.93 If $n = 15$ and $c = 2$, we get the following from Table I.

p	0	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
$L(p)$	1	0.9638	0.8159	0.6042	0.3980	0.2361	0.1268	0.0617	0.0271	0.0107	0.0037

5.94 If $n = 8$ and $c = 0$, we get the following from Table I.

p	0	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$L(p)$	1	0.6634	0.4305	0.2725	0.1678	0.1001	0.0576	0.0319	0.0168	0.0084

5.95 The AQL is the value of p for which $L(p) = 1 - 0.10 = 0.90$, or 0.07.
The LTPD is the value of p for which $L(p) = 0.10$ or 0.33.

5.96 The producer's risk is $1-$ value of $L(p)$ for which $p = 0.10$, or $1 - 0.74 = 0.26$.
The consumer's risk is the value of $L(p) = 0.25$, or 0.24.

5.97 **(a)** If $n = 10$ and $c = 0$, we get the following from Table I.

p	0	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$L(p)$	1	0.5987	0.3487	0.1969	0.1074	0.0563	0.0282	0.0135	0.0060	0.0010

(b) For plan 1 ($n = 10$, $c = 1$, see Exc. 5.93), the producer's risk = $1 - 0.9139 = 0.0861$ and the consumer's risk = 0.1493.

(c) For plan 2 ($n = 10$, $c = 0$, see preceding table), the producer's risk = $1 - 0.5987 = 0.4013$ and the consumer's risk = 0.0282.

Chapter 6

6.1 $$\int_{\alpha}^{\alpha+p(\beta-\alpha)} \frac{1}{\beta-a}\,dx = \frac{1}{\beta-\alpha}[\alpha+p(\beta-\alpha)-\alpha] = p$$

6.2 $$\mu = \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{1}{\beta-a}\,x\,dx = \frac{1}{\beta-\alpha}\left(\frac{\beta^2}{2}-\frac{\alpha^2}{2}\right) = \frac{1}{2(\beta-\alpha)}\cdot(\beta-\alpha)(\beta+\alpha) = \frac{\alpha+\beta}{2}$$

$$\mu_2' = \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} x^2\,dx = \frac{1}{3(\beta-\alpha)}(\beta^3-\alpha^3) = \frac{1}{3}(\beta^2+\alpha\beta+\alpha^2)$$

$$\sigma^2 = \frac{1}{3}(\beta^2+\alpha\beta+\alpha^2) - \frac{(\alpha+\beta)^2}{4} = \frac{1}{12}[4\beta^2+4\alpha\beta+4\alpha^2-3\alpha^2-6\alpha\beta-3\beta^2)$$
$$= \frac{1}{12}(\beta^2-2\alpha\beta+\alpha^2) = \frac{1}{12}(\beta-\alpha)^2$$

6.3 $$F(x) = \frac{1}{\beta-\alpha}\int_{\alpha}^{x} dx = \frac{x-\alpha}{\beta-\alpha} \qquad f(x) = \begin{cases} 0 & x \le \alpha \\ \dfrac{x-\alpha}{\beta-\alpha} & \alpha < x < \beta \\ 1 & \beta \le x \end{cases}$$

6.4 $$\mu_r = \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\left[x-\frac{\alpha-\beta}{2}\right]^r dx = \frac{1}{(\beta-\alpha)2^r}\int_{\alpha}^{\beta}[2x-(\alpha+\beta)]^r\,dx$$
$$= \frac{1}{(\beta-\alpha)2^r}\left[\frac{[2x-(\alpha+\beta)]^{r+1}}{2(r+1)}\right]\Bigg|_{\alpha}^{\beta}$$
$$= \frac{1}{(\beta-\alpha)2^r}\cdot\frac{(\beta-\alpha)^{r+1}-(-1)^{r+1}(\beta-\alpha)^{r+1}}{2(r+1)}$$

(a) $= 0$ when r is odd

(b) $= \dfrac{1}{(\beta-\alpha)2^{r+3}(r+1)}2(\beta-\alpha)^{r+1} = \dfrac{1}{r+1}\left(\dfrac{\beta-\alpha}{2}\right)^r$ when r is even

6.5 $$\mu_1 = 0,\ \mu_2 = \frac{1}{3}\frac{(\beta-\alpha)^2}{4} = \frac{(\beta-\alpha)^2}{12},\ \mu_3 = 0,\ \mu_4 = \frac{1}{5}\left(\frac{\beta-\alpha)}{2}\right)^4 = \frac{1}{80}(\beta-\alpha)^4$$

$$\alpha_3 = 0 \text{ and } \alpha_4 = \frac{\frac{1}{80}(\beta-\alpha)^4}{\frac{(\beta-\alpha)^4}{144}} = \frac{9}{5}$$

6.6 Intergals do not exist.

6.7 $$\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}dx$$

$$=-x^{\alpha-1}e^{-x}\Big|_0^\infty+(\alpha-1)\int_0^\infty x^{\alpha-2}e^{-x}\ dx$$

$$=(\alpha-1)\Gamma(\alpha-1)\qquad \text{QED}$$

$u=x^{\alpha-1}$

$dv=e^{-x}\ dx$

$du=(\alpha-1)x^{\alpha-2}\ dx$

$v=-e^{-x}$

6.8 $y=\frac{1}{2}z^2$

$dy=z\ dz$

$$\Gamma(\alpha)=\int_0^\infty y^{\alpha-1}e^{-y}dy=\int_0^\infty\left(\frac{z^2}{2}\right)^{\alpha-1}e^{-(1/2)z^2}z\ dz$$

$$=2^{1-\alpha}\int_0^\infty z^{2\alpha-1}er^{-(1/2)z^2}\ dz$$

6.9 $x=r\cos\theta \quad y=r\sin\theta \quad dx\ dy=r\ dr\ d\theta$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2=2\int_0^{\pi/2}\int_0^\infty re^{-(1/2)r^2}\ dr\ d\theta=\pi\int_0^\infty re^{-(1/2)r^2}\ dr$$

$$=\pi\int_0^\infty -e^u\ du=-\pi[e^u]\Big|_0^\infty=\pi\qquad \text{QED}$$

$u=-\frac{1}{2}r^2$

$du=-r\ dr$

6.10 **(a)** $$\alpha=2,\ \beta=3,\ x>4, p=\int_4^\infty\frac{1}{9\cdot 1}x\ e^{-x/3}\ dx=\frac{1}{9}\int_4^\infty x\ e^{-x/3}\ dx$$

$$=\frac{1}{9}\left[\frac{e^{-x/3}}{1/9}\left(-\frac{1}{3}x-1\right)\right]=e^{-4/3}\left(\frac{7}{3}\right)=\frac{7}{3}e^{-4/3}=\frac{7}{3}(0.2645)=0.6171$$

(b) $$\alpha=3,\ \beta=4,\ p=\int_4^\infty\frac{1}{64\cdot 2}x^2e^{-x/4}\ dx=\frac{1}{128}\int_4^\infty x^2e^{-x/4}\ dx=0.7818$$

6.11 $$\frac{\partial}{\partial x}=x^{\alpha-1}\left(-\frac{1}{\beta}e^{-x/\beta}\right)+e^{-x/\beta}(\alpha-1)x^{\alpha-2}$$

$$=x^{\alpha-2}e^{-x/\beta}\left(-\frac{x}{\beta}+\alpha-1\right)=0\qquad x=\beta(\alpha-1))$$

$0<\alpha<1$ function $\rightarrow\infty$ when $x\rightarrow 0$

$\alpha=1$ function has absolute maximum at $x=0$.

6.13 $$M = (1-\beta t)^{-\alpha} = 1-\alpha(-\beta t)+\alpha(\alpha+1)\frac{(-\beta t)^2}{2}-\alpha(a+1)(\alpha+2)\frac{(-\beta t)^2}{3!}$$
$$= 1+\alpha\beta t+\alpha(\alpha+1)\frac{\beta^2 t^2}{2!}+\frac{\alpha(\alpha+1)(\alpha+2)\beta^2 t^2}{3!}+\alpha(\alpha+1)(\alpha+2)(a+3)\frac{\beta^4 t^4}{4!}+\ldots$$

$$\mu_1' = \alpha\beta,\ \mu_2' = \alpha(\alpha+1)\beta^2,\ \mu_3' = \alpha(\alpha+1)(\alpha+2)\beta^3$$
$$\mu_4' = \alpha(\alpha+1)(\alpha+2)(\alpha+3)\beta^4$$

6.14 $$\mu_3 = (\alpha+1)(\alpha+2)\beta^3-3\alpha(\alpha+1)\beta^2\alpha\beta+2\alpha^3\beta^3$$
$$= \alpha\beta^3[(\alpha+1)(\alpha+2)-3\alpha(\alpha+1)+2\alpha^2] = \alpha\beta^3[2] = 2\alpha\beta^3$$

$$\alpha_3 = \frac{2\alpha\beta^2}{(\alpha\beta^2)^{3/2}} = \frac{2}{\sqrt{\alpha}}$$

$$\mu_4 = \alpha(\alpha+1)(\alpha+2)(\alpha+3)\beta^4-4\alpha(\alpha+1)(\alpha+2)\beta^3\cdot\alpha\beta+6\alpha(\alpha+1)\beta^2\cdot\alpha^2\beta^2-3\alpha^4\beta^4$$
$$= 2\beta^4[(\alpha+1)(\alpha+2)(\alpha+3)-4\alpha(\alpha+1)(\alpha+2)+6\alpha^2(\alpha+1)-3\alpha^3] = \alpha\beta^4$$

$$\alpha^4 = \frac{\alpha\beta^4(3\alpha+6)}{\alpha^2\beta^4} = 3+\frac{6}{\alpha}$$

6.15 $$f(x) = \frac{1}{\theta}e^{-x/\theta} \qquad p = \int_0^{-\theta\ln(1-p)}\frac{1}{\theta}e^{-x/\theta}d\theta = [-e^{-x/\theta}]\Big|_0^{-\theta\ln(1-p)}$$
$$= 1-e^{\ln(1-p)} = 1-(1-p) = p$$

6.16 $$\frac{p(x\geq t+T)}{P(x\geq T)} = \frac{e^{-(t+T)/\theta}}{e^{-T/\theta}} = e^{-t/\theta} = p(x\geq t)$$

6.17 $$M_x = (1-\theta t)^{-1} \qquad M_{x-\theta} = e^{-\theta t}(1-\theta t)^{-1} = \frac{e^{-\theta t}}{1-\theta t}$$

6.18 $$\left(1-\theta t+\frac{\theta^2t^2}{2!}-\frac{\theta^3t^3}{3!}+\frac{\theta^4t^4}{4!}\ldots\right)(1+\theta t+\theta^2t^2+\theta^3t^3+\theta^4t^4\ldots)$$
$$1+\left(1+\frac{1}{2}-1\right)\theta^2t^2+\left(-\frac{1}{6}+\frac{1}{2}-1+1\right)\theta^3t^3+\left(\frac{1}{24}-\frac{1}{6}+\frac{1}{2}-1+1\right)\theta^4t^4\ldots$$

$$1+\frac{\theta^2t^2}{2!}+2\cdot\frac{\theta^3t^3}{3!}+\frac{9\theta^4t^4}{4!}+\ldots$$

$$\alpha_3 = \frac{2\theta^3}{\theta^3} = 2$$
$$\alpha_4 = \frac{9\theta^4}{\theta^4} = 9$$

6.19 $\alpha = \frac{\nu}{2},\ \beta = 2$ See 6.11

From 6.11 $x = \beta(\alpha - 1) = 2\left(\frac{\nu}{2} - 1\right) = \nu - 2$

$0 < \nu < 2$ function $\rightarrow \infty$ when $x \rightarrow 0$

$\nu = 2$ function has absolute maximum at $x = 0$

6.20 $\mu = 2\alpha\int_0^\infty x^2\ e^{-\alpha x^2}\ dx \qquad u = \alpha x^2 \qquad du = 2\alpha x\ dx$

$$= \frac{1}{\sqrt{\alpha}}\int_0^\infty u^{1/2} e^{-u}\ du = \frac{1}{\sqrt{a}}\Gamma\left(\frac{3}{2}\right) = \frac{1}{\sqrt{\alpha}} \cdot \frac{1}{2} \cdot \sqrt{\pi} = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\mu_2' = 2\alpha\int_0^\infty x^3\ e^{-\alpha x^2\ dx} = \frac{1}{\alpha} \qquad \sigma^2 = \frac{1}{\alpha} - \frac{1}{4}\cdot\frac{\pi}{\alpha} - \frac{1}{\alpha}\left(1 - \frac{\pi}{4}\right)$$

6.21 $\mu_r' = \alpha\int_1^\infty x^{r-\alpha-1}\ dx$ exists only if $r - \alpha - 1 < 1$

$r < \alpha - 2$

6.22 $\mu_1' = \alpha\int_1^\infty x^{-\alpha}\ dx = \alpha\left.\frac{x^{1-\alpha}}{1-\alpha}\right|_1^\infty = \frac{\alpha}{\alpha - 1}$

6.23 **(a)** $k\int_0^\infty x^{\beta-1} e^{-\alpha x^\beta}\ dx = 1$ let $u = \alpha x^\beta$ $du = \alpha\beta x^{\beta-1}\ dx$

$$= k\int_0^\infty \frac{1}{\alpha\beta} e^{-u}\ du = \frac{k}{\alpha\beta} = 1 \qquad k = \alpha\beta$$

(b) $\mu = \alpha\beta\int_0^\infty x^\beta\ e^{-\alpha x^\beta}\ dx$

$$= \alpha^{-1/\beta}\int u^{1/\beta}\ e^{-u}\ du = \alpha^{-1/\beta}\Gamma\left(1 + \frac{1}{\beta}\right)$$

6.24 **(a)** $f(x) = \frac{1}{\theta}e^{-x/\theta} \qquad F(t) = \int_0^{t/\theta} e^{-u}\ du = 1 - e^{-t/\theta} = 1 - e^{-t/\theta}$

$$\frac{f(t)}{1 - F(t)} = \frac{\frac{1}{\theta}e^{-t/\theta}}{e^{-t/\theta}} = \frac{1}{\theta}$$

(b) $$F(t) = \alpha\beta\int_0^t x^{\beta-1}\ e^{-\alpha x^{\beta}}\ dx = 1 - e^{-\alpha t^{\beta}}$$

$$\frac{f(t)}{1-F(t)} = \frac{\alpha\beta t^{\beta-1}\ e^{-\alpha t^{\beta}}}{e^{-\alpha t^{\beta}}} = \alpha\beta t^{\beta-1}$$

6.25 **(a)** $$\frac{\Gamma(6)}{\Gamma(2)\Gamma(4)}\int_0^1 x(1-x)^3\ dx = 20\left[\frac{x^2}{2} - x^3 + \frac{3x^4}{4} - \frac{x^5}{5}\right]\Bigg|_0^1$$

$$= 20\left(\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5}\right) = 20\cdot\frac{1}{20} = 1$$

(b) $$\frac{\Gamma(6)}{\Gamma(3)\Gamma(3)}\int_0^1 x^2(1-x)^2\ dx = 30\left[\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right] = 30\cdot\frac{1}{30} = 1$$

6.26 $$f(x) = kx^{\alpha-1}(1-x)^{\beta-1}$$

$$\frac{df}{dx} = k\ x^{\alpha-1}(\beta-1)(1-x)^{\beta-2}(-1) + k(1-x)^{\beta-1}(\alpha-1)x^{\alpha-2}$$

$$= k\ x^{\alpha-2}(1-x)^{\beta-2}[-x(\beta-1) + (\alpha-1)(1-x)]$$

$$x(2-\alpha-\beta) = 1-\alpha \text{ and } x = \frac{\alpha-1}{\alpha+\beta-2}$$

6.28 $$\mu_2' = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 x^{\alpha+1}(1-x)^{\beta-1}\ dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot\frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} = \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)}$$

6.29 $$\mu = \frac{}{\alpha+\beta} \qquad \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$\alpha+\beta = \frac{\alpha}{\mu} \qquad \sigma^2 = \mu(1-\mu)\frac{1}{\alpha+\beta+1}$$

$$\alpha+\beta+1 = \frac{\mu(1-\mu)}{\sigma^2},\ \frac{\alpha}{\mu} = \frac{\mu(1-\mu)}{\sigma^2} - 1,\ \alpha = \mu\left[\frac{\mu(1-\mu)}{\sigma^2} - 1\right]$$

$$\beta = \frac{\alpha}{\mu} - \alpha = \alpha\left(\frac{1}{\mu} - 1\right) = \frac{\alpha(1-\mu)}{\mu}$$

$$= (1-\mu)\left[\frac{\mu(1-\mu)}{\sigma^2} - 1\right]$$

6.30 **(a)**
$$\frac{1}{f(x)}\frac{df(x)}{dx}=\frac{d-x}{bx}=\frac{d}{bx}-\frac{1}{5}$$
$$\ln f(x)-\frac{d}{b}\ln x=-\frac{1}{b}x+c$$
$$\ln\frac{f(x)}{x^{b/d}}=-\frac{1}{b}x,\ f(x)=kx^{b/d}e^{-(1/b)x}$$

(b)
$$\frac{1}{f(x)}\frac{df(x)}{dx}=-\frac{1}{b}\ln f(x)=-\frac{1}{b}x+c \qquad f(x)=ke^{-(1/b)x}$$

(c)
$$\frac{1}{f(x)}\frac{df(x)}{dx}=-\frac{d-x}{cx(1-x)}=\frac{-d/c}{x(1-x)}+\frac{1/c}{(1-x)}$$
$$\frac{-d/c}{x(1-x)}=\frac{A}{x}+\frac{B}{1-x}=\frac{A(1-x\ +Bx}{x(1-x)} \qquad A=-d/c=B$$
$$\frac{1}{f(x)}\frac{df(x)}{dx}=\frac{-d/c}{x}-\frac{d/c}{1-x}+\frac{1/c}{1-x}=\frac{-d/c}{x}-\frac{(d-1)/c}{1-x}$$
$$\ln f(x)=-\frac{d}{c}\ln x+\frac{(d-1)}{c}\ln(1-x)$$
$$f(x)=k\ x^{-d/c}(1-x)^{(d-1)/c}$$

6.31
$$f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-(1/2)[(x-\mu)/\sigma]^2} \qquad \ln f(x)=-\ln\sqrt{2\pi}\sigma-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$$

(a)
$$\ln f(x)=k-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$$
$$\frac{1}{f(x)}\frac{df(x)}{dx}=-\frac{1}{\sigma}\left(\frac{x-\mu}{\sigma}\right) \qquad -\frac{1}{\sigma}\left(\frac{x-\mu}{\sigma}\right)=0 \qquad x=\mu$$

(b)
$$\frac{df(x)}{dx}=-\left(\frac{x-\mu}{\sigma^2}\right)f(x)$$
$$\frac{d^2f(x)}{dx}=-\frac{1}{\sigma^2}f(x)-\left(\frac{x-\mu}{\sigma^2}\right)\cdot\left[-\left(\frac{x-\mu}{\sigma^2}\right)f(x)\right]$$
$$=-\frac{f(x)}{\sigma^2}\left[1-\left(\frac{x-\mu}{\sigma}\right)^2\right]=0$$
$$\left(\frac{x-\mu}{\sigma}\right)^2=1 \qquad \frac{x-\mu}{\sigma}=\pm1 \qquad x=\mu\pm\sigma$$

6.32
$$\frac{1}{f(x)}\frac{df(x)}{dx}=\frac{d-x}{a} \qquad \ln f(x)=-\frac{(d-x)^2}{2a}+c$$
$$f(x)=ke^{-1/2a}(x-d)^2 \qquad \text{QED}$$

6.33 $M''' = [(\mu+\sigma^2 t)^2+\sigma^2](\mu+\sigma^2 t)M + M[2(\mu+\sigma^2 t)\sigma^2]$

$= M(\mu+\sigma^2 t)[(\mu+\sigma^2 t)^2+3\sigma^2]$

$M'''(0) = \mu(\mu^2+3\sigma^2) = \mu^3+3\mu\sigma^2]$

$M'''' = M(\mu+\sigma^2 t)[2\sigma^2(\mu+\sigma^2 t)] + M[(\mu+\sigma^2 t)^2+3\sigma^2]\sigma^2$

$+(\mu+\sigma^2 t)[(\mu+\sigma^2 t)^2+3\sigma^2](\mu+\sigma^2 t)M$

$M''''(0) = 3\sigma^4+6\sigma^2\mu^2+\mu^4$

$\mu_3 = \mu^3+3\mu\sigma^2-3(\mu^2+\sigma^2)\mu+2\mu^3 = 0$

$\mu_4 = 3\sigma^4+6\sigma^2\mu^2+\mu^4-4(\mu^3+3\mu\sigma^2)\mu+6\mu^2(\mu^2+\sigma^2)-3\mu^4 = 3\sigma^4$

6.35 $\alpha_3 = 0$ and $\alpha_4 = \dfrac{3\sigma^4}{\sigma^4} = 3$

6.36 $M_x(t) = e^{\mu t+(1/2)\sigma^2 t^2}$

$M_{(x-\mu)/\sigma} = e^{-(\mu/\sigma)t}\cdot e^{\mu(t/\sigma)+(1/2)\sigma^2(t/\sigma)^2} = e^{(1/2)t^2}$

6.37 $E(x)=\mu,\ E(x^2)=\sigma^2+\mu^2,\ E(x^3)=\mu^3+3\mu\sigma^2$

$\operatorname{cov}(x,x^2) = (\mu^3+3\mu\sigma^2)-\mu(\sigma^2+\mu^2) = 2\mu\sigma^2$

for standard normal distribution $\mu = 0 \rightarrow \operatorname{cov}(x,x^2) = 0$

6.38 $$M = e^{(1/2)t^2} = 1+\frac{\left(\frac{1}{2}t^2\right)}{1!}+\frac{\left(\frac{1}{2}t^2\right)^2}{2!}+\ldots\frac{\left(\frac{1}{2}t^2\right)^{r/2}}{(r/2)!}$$

$$\downarrow$$

$$\frac{t^r}{2^{r/2}(r/2)!} = \frac{r!}{2^{r/2}(r/2)!}\cdot\frac{t^r}{r!}$$

(a) $\mu_r = 0$ since coefficient of t with r odd is zero.

(b) $\mu_r = \dfrac{r!}{(r/2)!}\dfrac{1}{2^{r/2}}$ read off for r even.

6.39 $M_{x-\mu} = e^{-\mu t} M_x(t) \qquad K_x(t) = -\mu t + \ln M_x(t)$

$$M_x(t) = 1 + \mu_1' t + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \mu_4' \frac{t^4}{4!}$$

$$\ln M_x(t) = \ln\left[1 + \left(\mu_1' t + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \mu_4' \frac{t^4}{4!} + \dots\right)\right]$$

$$\ln(1+z) = z - \frac{1}{2} z^2 + \frac{1}{3} z^3 - \frac{1}{4} z^4 + \dots$$

$$\begin{aligned} K_x(t) = 1 - \mu t + &\left[\mu_1' t + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \mu_4' \frac{t^4}{4!} + \dots\right] \\ &- \frac{1}{2}\left\{\mu_1' t + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \dots\right\}^2 \\ &+ \frac{1}{3}\left\{\mu_1' t + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \dots\right\}^3 \\ &- \frac{1}{4}\left\{\mu_1' t + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \dots\right\}^4 \end{aligned}$$

$$= \frac{t^2}{2!}\left[\mu_2' - (\mu_1')\right]^2 + \frac{t^2}{3!}\left[\mu_2'\mu_1' + 2(\mu_1')^2\right] + \frac{t^2}{4!}\left[\mu_4' - 3(\mu_2')^2 - 4\mu_1'\mu_3' + 12(\mu_1')^2 \mu_2' - 6(\mu_1')^4\right] + \dots$$

(a) $K_2 = \mu_2$, **(b)** $K_3 = \mu_3$, **(c)** $K_4 = \mu_4 - 3\mu_2^2$

6.40 $M_{x-\mu} = e^{-\mu t} M_x(t) = e^{-\mu t + \mu t + (1/2)t^2\sigma^2}$

$$\ln M_{x-\mu}(t) = \frac{1}{2} t^2 \sigma^2$$

$$K_x(t) = \frac{1}{2} t^2 \sigma^2$$

$K_1 = 0,\ K_2 = \sigma^2;\ K_r = 0$ for $r > 2$

6.41 $M_x(t) = e^{\lambda(e^t - 1)} \qquad \mu = \lambda,\ \sigma = \sqrt{\lambda}$

$$M_{(x-\mu)/\sigma}(t) = e^{-(\mu/\sigma)t} M_x\left(\frac{t}{\sigma}\right) = e^{-\sqrt{\lambda} t} e^{\lambda(e^{t/\sigma} - 1)}$$

$$\begin{aligned} \ln M_{(x-\mu)/\sigma}(t) &= -\sqrt{\lambda} t + \lambda(e^{t/\sigma} - 1) \\ &= -\sqrt{\lambda} t + \lambda(e^{t/\sqrt{x}} - 1) \\ &= -\sqrt{\lambda} t + \lambda\left[\frac{t}{\sqrt{\lambda}} + \frac{t^2}{2\lambda} + \frac{t^3}{3\lambda\sqrt{\lambda}} + \dots\right] \\ &= -\sqrt{\lambda} t + \sqrt{\lambda} t + \frac{t^2}{2} + \frac{t^3}{3\sqrt{\lambda}} + \dots \end{aligned}$$

$$\lambda \to \infty \qquad = \frac{1}{2} t^2$$

6.42 $M_x(t) = (1-\beta t)^{-\alpha}$ $\qquad \mu = \alpha\beta,\ \sigma = \beta\sqrt{\alpha}$

$$M_{(x-\mu)/\sigma} = e^{-\sqrt{\alpha}t}\left(1-\frac{t}{\sqrt{\alpha}}\right)^{-\alpha}$$

$$\ln M_{(x-\mu)/\alpha} = -\sqrt{\alpha}t - \alpha\ln\left(1-\frac{t}{\sqrt{\alpha}}\right) \qquad \ln(1+z) = +z+\frac{z^2}{2}+\frac{z^3}{3}+\ldots$$

$$= -\sqrt{\alpha}t + \alpha\left[\frac{t}{\sqrt{a}} - \frac{t^2}{2\alpha} + \frac{t^3}{3\alpha\sqrt{\alpha}}\ldots\right] = +\frac{t^2}{2} \text{ when } \alpha \to \infty$$

6.43 **(a)** Constant terms of $g(x)$ and $h(y)$ are $\frac{1}{\sigma_1\sqrt{2\pi}}$ and $\frac{1}{\sigma_2\sqrt{2\pi}}$

Constant term of $f(x,y) = \frac{1}{2\pi\sigma_2\sigma_2\sqrt{1-p^2}}$

If independent then $\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-p^2}} = \frac{1}{\sigma_1\sqrt{2\pi}}\cdot\frac{1}{\sigma_2\sqrt{2\pi}}\ \sqrt{1-p^2} = 1,\ p = 0$

(b) Substitute $p = 0$ into $f(x, y)$ and it becomes product of $g(x)$ and $h(y)$.

6.44 Substitute $y = a + bx$ into $f(x, y)$

6.45 **(a)** $\mu_1 = -2,\ \mu_2 = 1$; Let k^2 be suitable constant.

$$\frac{k^2}{\sigma_1^1} = 1,\ \frac{k^2}{\sigma_2^2} = 4,\ \frac{2pk^2}{\sigma_1\sigma_2} = 2.8, \text{ so that } \sigma_1 = k,\ \sigma_2 = \frac{k}{2} \text{ and } \frac{2pk^2}{k^2/2} = 2.8,$$

$$4p = 2.8,\ p = 0.7$$

$$-\frac{1}{2(1-p^2)} = \frac{-1}{2(0.51)} = \frac{-1}{1.02}$$

$$-\frac{1}{102}\left[\left(\frac{x+2}{10}\right)^2 - 2.8\left(\frac{x+2}{10}\right)\left(\frac{y-1}{10}\right) + \left(\frac{y-1}{5}\right)^2\right]$$

so that $\sigma_1 = 10$ and $\sigma_2 = 5$

6.46 Equating coefficients of x^2, xy, and y^2 with those of bivariate normal density

$27 = (1-\rho^2)\sigma_1^2$ $\qquad$ multiply first and third and divide by square of second

$$-27 = \frac{(1-\rho^2)\sigma_1\sigma_2}{\rho}$$

$$27 = 4(1-\rho^2)\sigma_2^2 \qquad \frac{27\cdot 27}{(-27)^2} = \frac{4(1-\rho^2)^2\sigma_1^2\sigma_2^2}{(1-\rho^2)^2\sigma_1^2\sigma_2^2}\cdot\rho^2$$

$$\rho^2 = \frac{1}{4} \qquad \rho = \pm\frac{1}{2}$$

from second equation must be $\rho = -\frac{1}{2}$

$$\sigma_1^2 = \frac{27}{0.75} = 36,\ \sigma_1 = 6$$

$$\sigma_2^2 = \frac{27}{4(0.75)} = 9,\ \sigma_2 = 3$$

6.47 $\mu_1 = 2,\ \mu_2 = 5,\ \sigma_1 = 3,\ \sigma_2 = 6,\ p = \frac{2}{3}$

$$\mu_{Y|1} = 5 + \frac{2}{3}\cdot\frac{6}{3}(1-2) = 5 - \frac{4}{3} = \frac{11}{3}$$

$$\sigma_{Y|1}^2 = 36\left(1 - \frac{4}{9}\right) = \frac{36\cdot 5}{9} = 20 \qquad \sigma_{Y|1} = \sqrt{20} = 4.47$$

6.48 $U = X + Y,\ V = X - Y$

$E(U) = \mu_1 + \mu_2,\ E(V) = \mu_1 = \mu_2$

$\sigma_U^2 = \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2$

$\sigma_V^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$

$E(UV) = E[(X+Y)(X-Y)] = E(X^2 - Y^2) = \sigma_1^2 + \mu_1^2 - \sigma_2^2 - \mu_2^2$

$\text{cov}(UV) = \sigma_1^2 + \mu_1^2 - \sigma_2^2 - \mu_2^2 - (\mu_1 + \mu_2)(\mu_1 - \mu_2) = \sigma_1^2 - \sigma_2^2$

$$\rho = \frac{\sigma_1^2 - \sigma_2^2}{\sqrt{\left(\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2\right)\left(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2\right)}}$$

$$\rho = \frac{\sigma_1^2 - \sigma_2^2}{\sqrt{\left(\sigma_1^2 - \sigma_2^2\right)^2 - 4\rho^2\sigma_1^2\sigma_2^2}}$$

6.49 **(a)** $M(t_1, t_2) = e^{t_1\mu_1 + t_2\mu_2 + (1/2)[\sigma_1^2 t_1^2 + 2p\sigma_1\sigma_2 t_1 t_2 + \sigma_2^2 t_2^2]} = e^Q$

$$\frac{\partial}{\partial t_1} = (\mu_1 + \sigma_1^2 t_1 + \rho\sigma_1\sigma_2 t_2)e^Q = \mu_1 \text{ at } t_1 = t_2 = 0$$

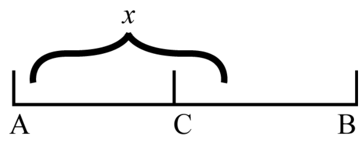
(b) $\frac{\partial^2}{\partial t_1^2} = (\mu_1 + \sigma_1^2 t_1 + \rho\sigma_1\sigma_2 t_2)^2 e^Q + \sigma_1^2 e^Q = (\mu_1^2 + \sigma_1^2) = \sigma_1^2 + \mu_1^2 \text{ at } t_1 = t_2 = 0$

(c) $\frac{\partial^2}{\partial t_1 \partial t_2} = (\mu_1 + \sigma_1^2 t_1 + \rho\sigma_1\sigma_2 t_2)e^Q(\mu_2 + \sigma_2^2 t_2 + \rho\sigma_1\sigma_2 t_2) + \rho\sigma_1\sigma_2 \cdot e^Q$

$$= \mu_1\mu_2 + \rho\sigma_1\sigma_2 \text{ at } t_1 = t_2 = 0$$

6.50 **(a)** $\frac{0.003 - (0.002)}{0.03} = \frac{0.005}{0.030} = \frac{1}{6}$; **(b)** $\frac{2(0.1)}{0.03} = \frac{2}{3}$

6.51



$$x+(a-x)>\frac{a}{2} \qquad a>\frac{a}{2}$$

$$x+\frac{a}{2}>a-x \qquad x>\frac{a}{4}$$

$$(a-x)+\frac{a}{2}>x \qquad x<\frac{3}{4}a$$

Probability is $\frac{1}{2}$

$\alpha=-0.015$ and $\beta=0.015$, $\beta-\alpha=0.03$

6.52 $\alpha=3,\ \beta=2$

$$\rho=\frac{1}{8\cdot 2}\int_{12}^{\infty} x^2 e^{-x/2}\,dx=\frac{1}{16}\left[\frac{x^2 3^{-(1/2)x}}{-1/2}-\frac{2}{-1/2}\cdot\frac{e^{(-1/2)x}}{1/4}\left(-\frac{1}{2}x-1\right)\right]\Bigg|_{12}^{\infty}$$

$$=\frac{1}{16}\left[-2x^2e^{-(1/2)x}+16e^{-(1/2)x}\left(\frac{1}{2}x+1\right)\right]\Bigg|_{12}^{\infty}$$

$$=\frac{1}{16}\left[288e^{-6}+16e^{-6}0.7\right]=25e^{-6}=25(0.002479)=0.062$$

6.53 $\mu=\alpha\beta=80\cdot 2\sqrt{n}=160\sqrt{n}$

$$E=160\sqrt{n}-8n \qquad \frac{dE}{dn}=\frac{160}{2\sqrt{n}}-8=0 \qquad n=100$$

6.54 **(a)** $$\int_0^{24}\frac{1}{120}e^{-(1/120)x}dx=-e^{-x/120}\Bigg|_0^{24}=1-e^{-0.2}=1-0.8187=0.1813$$

(b) $$\int_{180}^{\infty}\frac{1}{120}e^{-1/120}dx=-e^{-x/120}\Bigg|_{180}^{\infty}=e^{-1.5}=0.2231$$

6.55 **(a)** $$\int_{20}^{\infty}\frac{1}{40}e^{-(1/40)x}dx=-e^{-x/40}\Bigg|_{20}^{\infty}=e^{-1/2}=0.6065$$

(b) $$\int_0^{30}\frac{1}{40}e^{-(1/40)x}dx=-e^{-x/40}\Bigg|_0^{30}=1-e^{-3/4}=1-0.4724=0.5276$$

6.56 $\lambda=0.4$ per hour $\int_2^{\infty}0.4e^{-0.4t}dt=-e^{-0.4t}\Big|_2^{\infty}=e^{-0.8}=0.4493$

6.57 $\lambda=1.2$ per hour $\int_1^{\infty}1.2e^{-1.2t}dt=-e^{-1.2t}\Big|_1^{\infty}=e^{-1.2}=0.1827$

6.58 $\alpha = 2,\ \beta = 9$

$$90\int_0^{0.1} x(1-x)^8\ dx \qquad\qquad y = 1-x \qquad\qquad dy = -dx$$

$$= 90\int_{0.9}^{1} y^8(1-y)\ dy = 90\left[\frac{1}{9}-\frac{1}{10}-\frac{(0.9)^9}{9}+\frac{(0.9)^{10}}{10}\right] = 0.2463$$

6.59 $\lambda = 0.5 \int_3^{\infty} e^{-0.5t}dt = -e^{-0.5t}\Big|_3^B = e^{-1.5} = 0.2231$

6.60 $\alpha = 1,\ \beta = 4$

(a) $\mu = \frac{1}{1+4} = \frac{1}{5}$

(b) $\frac{\Gamma(5)}{\Gamma(1)\Gamma(4)}\int_{0.25}^{1}(1-x)^3dx = 4\int_0^{0.75} y^2dy$ $\qquad y = 1-x \qquad dy = -dx$

$$= 4\cdot\frac{y^4}{4}\Big|_0^{0.75} = (0.75)^4 = \left(\frac{3}{4}\right)^4 = \frac{81}{256} = 0.3164$$

6.61 $\alpha = 0.025,\ \beta = 0.5$

(a) $\mu = (0.025)^{-2}\Gamma(3) = \frac{2}{(0.025)^2} = 3200$ hours

(b) $\alpha\beta\int_{4000}^{\infty} x^{\beta-1}e^{-\alpha x^{\beta}}dx$ $\qquad y = \alpha x^{\beta} \qquad y = 0.025\cdot\sqrt{4000} = 1.58$

$dy = \alpha\beta x^{\beta-1}dx$

$$= \int_{1.58}^{\infty} e^{-y}dy = e^{-1.58} = 0.2060$$

6.62 **(a)** $0.5 + 0.4082 = 0.9082$
(b) $0.5 + 0.2852 = 0.7852$
(c) $0.3888 - 0.2088 = 0.1800$
(d) $0.4713 + 0.1700 = 0.6413$

6.63 **(a)** $0.5 - 0.3729 = 0.1271$
(b) $0.5 + 0.1406 = 0.6406$
(c) $0.1772 - 0.359 = 0.1413$
(d) $0.2190 + 0.3686 = 0.5876$

6.64 **(a)** $z_1 = 1.48$
(b) $z_2 = -0.74$
(c) $z_3 = 0.55$
(d) $z_4 = 2.17 \qquad 0.4850$

6.65 **(a)** $z = 1.92$
(b) $z = 2.22$
(c) $z = 1.12$ 0.3686
(d) $z = \pm 1.44$ 0.4251

6.66 **(a)** $2(0.3413) = 0.6826$
(b) $2(0.4772) = 0.9544$
(c) $2(0.4987) = 0.9974$
(d) $2(0.49997) = 0.99994$

6.67 **(a)** $z_{0.05} = 1.645$ 0.4500
(b) $z_{0.025} = 1.96$ 0.475
(c) $z_{0.01} = 2.33$ 0.49
(d) $z_{0.005} = 2.575$ 0.495

6.68 **(a)** Using MINITAB and entering $= -2.159$ and 0.5670 into C1, then giving the commands

```
MTB> CDF C1;
SUBC> Normal 1.786 1.0416
```

we get

```
K          P(X LESS THAN OR = K)
-2.1590    0.3601
0.5670     0.9881
```

Thus the required probability is $0.9881 - 0.3601 = 0.6280$

(b) $z_1 = \dfrac{-2.159 + 1.786}{1.0416} = -0.958$ $z_2 = \dfrac{0.5670 + 1.786}{1.0416} = 2.25$

The corresponding cumulative probabilities are obtained from Table II (with interpolation) to be 0.3602 and 0.9881. Thus the required probability is $0.9881 - 0.3602 = 0.6279$

6.69 **(a)** Using MINITAB and entering 8.626 into C1,

```
MTB> CDF C1;
SUBC> Normal 5.853 1.361
   K    P(X LESS THAN OR = K)
8.626   .9792
```

Thus, the required probability is $1 - 0.9792 = 0.0208$.

(b) $z = \dfrac{8.625 - 5.853}{1.361} = 2.0367; \quad \therefore p = 0.5 - 0.47915 = 0.02085$

6.70 **(a)** $z = \dfrac{44.5 - 37.6}{4.6} = 1.5$ $0.5 - 0.4332 = 0.0668$

(b) $z = \dfrac{35 - 37.6}{4.6} = -.565$ $0.5 - 0.214 = 0.2860$

(c) $z_1 = \dfrac{30 - 37.6}{4.6} = -1.65$ $0.4505 + 0.1985 = 0.6490$

$z_2 = \dfrac{40 - 37.6}{4.6} = 0.52$

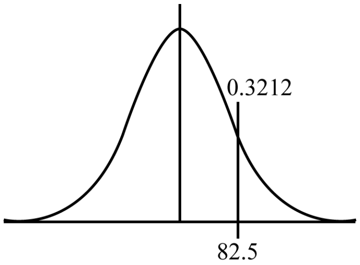
6.71 **(a)** $z = \dfrac{16 - 15.40}{0.48} = 1.25$ $\quad 0.5 - 0.3944 = 0.1056$

(b) $z = \dfrac{14.2 - 15.4}{0.48} = -2.5$ $\quad 0.5 - 0.4938 = 0.0062$

(c) $z_1 = \dfrac{15 - 15.4}{0.48} = -0.83$ $\quad 2(0.2967) = 0.5934$

$z_2 = 0.83$

6.72



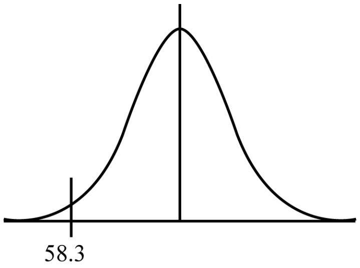
$$\frac{82.5 - \mu}{10} = 0.92$$

$$82.5 - \mu = 9.2$$

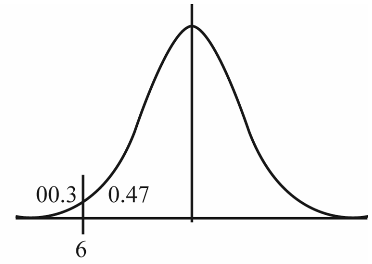
$$\mu - 73.3$$

$$z = \frac{58.3 - 73.3}{10} = -1.5$$

$$0.5 + 0.4332 = 0.9332$$



6.73



$z = -1.88$ $\quad \dfrac{6 - \mu}{0.05} = -1.88$

$6 - \mu = 0.094$

$\mu = 6.094$ ounces

6.74 **(a)** $n\theta = 3.2,\ n(1-\theta) = 15.68$, No

(b) $n\theta = 6.5,\ n(1-\theta) = 58.5$, Yes

(c) $n\theta = 117.6,\ n(1-\theta) = 2.4$, No

6.75 **(a)** $n\theta = 7.5,\ n(1-\theta) = 142.5$, Yes

(b) $\mu = 7.5,\ \sigma^2 = 150(0.05)(0.95) = 7.125,\ \sigma = 2.6693$

$z_1 = \dfrac{0.5 - 7.5}{2.6693} = -2.6224,$ $\quad z_2 = \dfrac{1.5 - 7.5}{2.6693} = -2.2478$

Probability $= 0.4956 - 0.4877 = 0.0079$

(c) $\dfrac{0.0079 - 0.0036}{0.0036} \cdot 100 = 119\%$

6.76 $n = 14,\ x = 7,\ \theta = \frac{1}{2},\ z_1 = \frac{6.5 - 7}{1.871} = -0.27,\ z_2 = \frac{7.5 - 7}{1.871} = 0.27$

$\rho = 2(0.1064) = 0.2128$ Table yields 0.2095

6.77 $\lambda = 7.5,\ p(1;\ 7.5) = \frac{7.5^1 e^{-7.5}}{1!} = 7.5(0.00055) = 0.0041$

6.78 $n = 120,\ \theta = -0.23$

$\mu = 27.6,\ \sigma = \sqrt{21.25} = 4.61$

$z = \frac{32.5 - 27.6}{4.61} = 1.06$

$0.5 - 0.3554 = 0.1446$

6.79 $n = 225,\ \theta = 0.2,\ \mu = 45,\ \sigma = 6$

$z = \frac{40.5 - 45}{6} = -0.75$

$0.5 - 0.2734 = 0.2266$

6.80 **(a)** $\mu = 50,\ \sigma = 5,\ z = \frac{51.5 - 50}{5} = 0.3$

49 to 51 $2(0.1179) = 0.2358 = 0.24$

(b) $\mu = 500,\ \sigma = 15.81,\ z = \frac{510.5 - 500}{15.81} = 0.664$

490 to 510 $2(0.2454) = 0.49$

(c) $\mu = 5000,\ \sigma = 50,\ z = \frac{5100.5 - 5000}{50} = 2.01$

4900 to 5100 $2(0.4778) = 0.96$

Chapter 7

7.1 $G(y) = P(Y \le y) = P(\ln X \le y) = P(X \le e^y)$

$$= \int_0^{e^y} \frac{1}{8} e^{-x/\theta} dx = -e^{-x/\theta} \Big|_0^{e^y} = 1 - e^{-(1/\theta)e^y}$$

$g(y) = \frac{1}{8} e^y e^{-(1/\theta)e^y}$ for $-\infty < y < \infty$

7.2 $G(y) = P(Y \le y) = P(X^2 \le y) = P(X \le \sqrt{y})$

$$= \int_0^{\sqrt{y}} 2xe^{-x^2} dx \qquad u = x^2 \qquad du = 2x\ dx$$

$$= \int_0^{y} e^{-u} du = -e^{-u} \Big|_0^{y} = 1 - e^{-y}$$

(a) $G(y) = \begin{cases} 1 - e^{-y} & y > 0 \\ 0 & \text{elsewhere} \end{cases}$

(b) $g(y) = \frac{dG(y)}{dy} = e^{-y}$ for $y > 0$ and 0 elsewhere

7.3 $G(y) = P(Y \le y) = P(\sqrt{X} \le y) = P(X \le y^2)$

$$= \int_0^{y^2} dx = y^2 \text{ for } 0 < y < 1$$

$g(y) = 2y$ for $0 < y < 1$ and 0 elsewhere

7.4 $G(z) = P(Z \le z) = P(X^2 + Y^2 + z^2)$

$$= \int_0^{z} \int_0^{\sqrt{z^2 - y^2}} 4xye^{-(x^2+y^2)} dx\ dy \qquad \text{let } u = x^2 \text{ and } v = y^2$$

$= 1 - (1 + z^2)e^{-z^2}$ for $z > 0$ and $G(z) = 0$ elsewhere

$g(z) = -(1 + z^2)e^{-z^2}(-2z) - 2z\ e^{-z^2}$

$= 2z^3 e^{-z^2}$ for $z > 0$ and elsewhere

7.5 $G(y) = P(Y \le y) = P(X_1 + X_2 \le y)$

$$= \int_0^y \int_0^{y-x_2} \frac{1}{\theta_1} e^{-x_1/\theta_1} \frac{1}{\theta_2} e^{-x_2/\theta_2}\, dx_2\, dx_2$$

$$= \int_0^y \left[\frac{1}{\theta_2} e^{-x_2/\theta_2} - \frac{1}{\theta_2} e^{-x_2/\theta_2} e^{-(y-x_2)/\theta_1} \right] dx_2$$

(a) $\theta_1 \ne \theta_2$

$$g(y) = \frac{1}{\theta_1 - \theta_2}\left[e^{-y/\theta_1} - e^{y/\theta_2} \right] \qquad y > 0$$

(b) $\theta_1 = \theta_2 = \theta$

$$G(y) = \int_0^y \left[\frac{1}{\theta_2} e^{-x_2/\theta_2} - \frac{1}{\theta_2} e^{-y/\theta_2} \right] dx_2$$

$$= 1 - e^{-y/\theta} - y\frac{1}{\theta} e^{-y/\theta}$$

$$g(y) = \frac{1}{\theta^2} y e^{-y/\theta} \qquad y > 0$$

7.6 **(a)** $F(y) = 0,$ **(b)** $F(y) = \frac{1}{2} y^2,$ **(c)** $F(y) = 1 - \frac{1}{2}(2 - y)^2,$ **(d)** $F(y) = 1$

$f(y) = 0, f(y) = y, f(y) = 2 - y, f(y) = 0$

7.7 $G(Z) = P(Z \le z) = P\left(\frac{X_1}{X_1 + X_2} \le z \right)$

$$x = xz + yz$$

$$yz = x(1-z)$$

$$y = \frac{x(1-z)}{z}$$

$$= \int_0^\infty \int_0^{x(1-z)/z} e^{-x} e^{-y} dy\, dx$$

$$= \int_0^\infty e^{-x} \int_0^{x(1-z)/z} e^{-y} dy\, dx$$

$$= \int_0^\infty \int_{x(1-z)/z}^\infty e^{-x} e^{-y} dy\, dx = \int_0^\infty e^{-x} \int_{x(1-z)/z}^\infty e^{-y} dy\, dx$$

$$= \int_0^\infty e^{-x} \left[e^{-x(1-z)/z} \right] dx \int_0^\infty e^{-x/z}\, dx = z$$

$g(z) = 1$ QED

7.8 $P(Z \le z) = P\left(\dfrac{X+Y}{2} \le z\right)$

$$= \int_0^{2z}\int_0^{2z-x} e^{-x}e^{-y}dy\ dx = \int_0^{2z} e^{-x}\left[-e^{-y}\right]\Bigg|_0^{2z-x} dx$$

$$= \int_0^{2z} e^{-x}[1-e^{x-2z}]dx = \int_0^{2z}(e^{-x}-e^{-2z})dx$$

$$= [-e^{-x}-xe^{-2z}]\Bigg|_0^{2z} = -e^{-2z}-2z\ e^{-2z}+1$$

$$g(z) = 2e^{-2z}-2e^{-2z}+4ze^{-2z} = 4ze^{-2z}$$

7.9 $h(0) = \dfrac{\binom{3}{0}\binom{3}{2}}{\binom{6}{2}} = \dfrac{3}{15} = \dfrac{1}{5}$, $h(1) = \dfrac{\binom{3}{1}\binom{3}{1}}{15} = \dfrac{9}{15} = \dfrac{3}{5}$

$$h(2) = \frac{\binom{3}{2}\binom{3}{0}}{\binom{6}{2}} = \frac{3}{15} = \frac{1}{5}$$

$x-(2-x)$

$2x-2$

X	0	1	2
Y	−2	0	2

Y	−2	0	2
$h(y)$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$

7.10

X	0	1	2
Z	1	0	1

Z	0	1
$h(z)$	$\frac{3}{5}$	$\frac{2}{5}$

7.11 $f(0) = 1\cdot\dfrac{8}{27} = \dfrac{8}{27}$, $f(1) = 3\cdot\dfrac{1}{3}\cdot\dfrac{4}{9} = \dfrac{12}{27}$, $f(2) = 3\cdot\dfrac{1}{9}\cdot\dfrac{2}{3} = \dfrac{6}{27}$, $f(3) = 1\cdot\dfrac{1}{27} = \dfrac{1}{27}$

(a)

$x = 0$	1	2	3
$y = 0$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$

y	0	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$
$g(y)$	$\frac{8}{27}$	$\frac{12}{27}$	$\frac{6}{27}$	$\frac{1}{27}$

(b)

$x = 0$	1	2	3
$y = 1$	0	1	16

y	0	1	16
$g(y)$	$\frac{12}{27}$	$\frac{14}{27}$	$\frac{1}{27}$

7.12 $f(x) = \theta(1-\theta)^{x-1}, \qquad x = 1,\ 2,\ 3,\ \ldots \qquad x - 1 = \dfrac{-1-y}{5}$

$y = 4 - 5x \qquad x = \dfrac{4-y}{5} \qquad x - 1 = \dfrac{-(1+y)}{5}$

$g(y) = \theta(1-\theta)^{-(1+y)/5}$ for $y = -1,\ -6,\ -11,\ -16,\ \ldots$

7.13

X	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$g(0) = \frac{2}{36} + \frac{5}{36} + \frac{4}{36} + \frac{1}{36} = \frac{12}{36} = \frac{1}{3}$$

$$g(1) = \frac{3}{36} + \frac{6}{36} + \frac{3}{36} = \frac{12}{36} = \frac{1}{3}$$

$$g(2) = \frac{1}{36} + \frac{4}{36} + \frac{5}{36} + \frac{2}{36} = \frac{12}{36} = \frac{1}{3}$$

7.14 $g(z) = \dfrac{dx}{dz} \cdot f(x) \qquad x - \mu = \sigma z \qquad x - \sigma z + \mu \qquad \dfrac{dx}{dz} = \sigma$

$g(z) = \sigma \cdot \dfrac{1}{\sqrt{2\pi}\sigma} e^{-(1/2)[(x-\mu)/\sigma]^2} = \dfrac{1}{\sqrt{2\pi}} e^{-(1/2)z^2}$ QED

7.15 $f(x) = 2xe^{-x^2} \qquad y = x^2 \qquad 1 = 2x\dfrac{dx}{dy}$

$$g(y) = \frac{1}{2x} \cdot 2xe^{-x^2} = \begin{cases} e^{-y} & \text{for } y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

7.16 $y = \dfrac{2x}{1+2x},\ y(1+2x) = 2x \qquad 1 + 2x = \dfrac{1}{1-y}$

$y = 2x(1-y) \qquad 2x = \dfrac{y}{1-y} \qquad x = \dfrac{y}{2(1-y)^2}$

$g(y) = \dfrac{dx}{dy} f(x) \qquad 2\dfrac{dx}{dy} = \dfrac{(1-y)+y}{(1-y)^2} = \dfrac{1}{(1-y)^2} \qquad \dfrac{dx}{dy} = \dfrac{1}{2(1-y)^2}$

$$g(y) = \frac{k\,y^3(1-y)^2}{8(1-y)^3} \cdot \frac{1}{2(1-y)^2} = \frac{k}{16} y^3(1-y)$$

Beta distribution with $\alpha = 4$ and $\beta = 2$

$$\frac{k}{16} = \frac{\Gamma(6)}{\Gamma(2)\Gamma(4)} = \frac{5!}{1!\ 3!} = 20,\ k = 320$$

7.17 $f(x) = \frac{x}{2} \qquad 0 < x < 2$

$y = x^3 \qquad 1 = 3x^2 \frac{dx}{dy}$

$$g(y) = \frac{1}{3x^2} \cdot \frac{x}{2} = \frac{1}{6y^{1/3}}$$

$$g(y) = \begin{cases} \frac{1}{6} y^{-2/3} & \text{for } 0 < y < 8 \\ 0 & \text{elsewhere} \end{cases}$$

7.18 $f(x) = 1 \qquad 0 < x < 1 \qquad y = -2\ln x \qquad 1 = \frac{-2}{x\frac{dx}{dy}}$

$g(y) = e^{-(1/2)y} \qquad 0 < y < \infty \qquad \frac{dx}{dy} = -\frac{x}{2}$

$\alpha = 1$ and $\beta = 2 \qquad -\frac{1}{2} y = \ln x \qquad x = e^{-(1/2)y}$

7.19 $f(x) = 1 \qquad 0 < x < 1$

$y = x^{-1/\alpha}, \ x = y^{-\alpha}, \ \frac{dx}{dy} = -\alpha y^{-(1+a)}$

$g(y) = 1 \cdot \alpha y^{-(1+\alpha)} = \frac{\alpha}{y^{1+\alpha}}$ for $x > 1$

7.20 **(a)** $Y = |x| \qquad g(y) = f(y) + f(-y)$

$$= \begin{cases} 3y^2 & 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(b) $z = y^2 \qquad 1 = 2y \cdot \frac{dy}{dz}$

$$h(z) = \frac{1}{\sqrt{z}} \cdot 3z = \begin{cases} \frac{3}{2\sqrt{z}} & \text{for } 0 < z < 1 \\ 0 & \text{elsewhere} \end{cases}$$

7.21 $f(x) = \frac{1}{4} \qquad \alpha = 1 \qquad \beta = 3$

(a) $y = |x|$

$$g(y) = \begin{cases} \frac{1}{2} & \text{for } 0 < y < 1 \\ \frac{1}{4} & \text{for } 1 < y < 3 \end{cases}$$

(b) $z = y^4 \qquad 1 = 4y^3\dfrac{dy}{dz}$

$$g(z) = \begin{cases} \dfrac{1}{4z^{3/4}}\cdot\dfrac{1}{2} = 8\, z^{-3/4} & 0 < z \le 1 \\[2ex] \dfrac{1}{4z^{3/4}}\cdot\dfrac{1}{4} = \dfrac{1}{16} z^{-3/4} & 1 < z < 81 \end{cases}$$

7.22

$$\begin{array}{cc|ccc|} & & & x_1 & \\ & & 1 & 2 & 3 \\ \hline & 1 & \frac{1}{36} & \frac{2}{36} & \frac{3}{36} \\ x_2 & 2 & \frac{2}{36} & \frac{4}{36} & \frac{6}{36} \\ & 3 & \frac{3}{36} & \frac{6}{36} & \frac{9}{36} \\ \hline \end{array}$$

(a)

x_1x_2	1	2	3	4	6	9
$g(x_1x_2)$	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{6}{36}$	$\frac{9}{36}$

(b)

x_1 / x_2	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	1	$\frac{3}{2}$	2	3
$h(x_1 / x_2)$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{6}{36}$	$\frac{14}{36}$	$\frac{6}{36}$	$\frac{2}{36}$	$\frac{3}{36}$

7.23 **(a)**

$$\begin{array}{cc|cccccc|} & & & & & y_1 & & \\ & & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline & -2 & & & & \frac{3}{36} & & \\ & -1 & & & \frac{2}{36} & & \frac{6}{36} & \\ y_2 & 0 & & \frac{1}{36} & & \frac{4}{36} & & \frac{9}{36} \\ & 1 & & & \frac{2}{36} & & \frac{6}{36} & \\ & 2 & & & & \frac{3}{36} & & \\ \hline \end{array}$$

(b)

y_1	2	3	4	5	6
$g(y_1)$	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$

7.24 $f(x, y) = \dfrac{(x-y)^2}{7}$ $\qquad x = 1, 2 \qquad y = 1, 2, 3$

$x \backslash y$	1	2	3
1	0	$\frac{1}{7}$	$\frac{4}{7}$
2	$\frac{1}{7}$	0	$\frac{1}{7}$

(a)

$u = x + y$
$v = -x - y$

$v \backslash u$	2	3	4	5
−2			$\frac{4}{7}$	
−1		$\frac{1}{7}$		$\frac{1}{7}$
0	0		0	
1		$\frac{1}{7}$		

(b)

u	2	3	4	5
$g(u)$	0	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$

7.25

x_1	x_2	x_3		y_1	y_2	y_3	
2	0	0	1/16	2	2	0	$g(0,\ 0,\ 2) = \frac{25}{144}$
0	2	0	1/9	2	−2	0	$g(1,\ -1,\ 1) = \frac{5}{18}$
0	0	2	25/144	0	0	2	$g(1,\ 1,\ 1) = \frac{5}{24}$
1	1	0	1/6	2	0	0	$g(2,\ -2,\ 0) = \frac{1}{9}$
1	0	1	5/24	1	1	1	$g(2,\ 0,\ 0) = \frac{1}{6}$
0	1	1	5/18	0	−1	1	$g(2,\ 2,\ 0) = \frac{1}{16}$

7.26

		X		
		0	1	2
Y	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$
	1	$\frac{2}{9}$	$\frac{1}{6}$	
	2	$\frac{1}{36}$		

(a)

u	0	1	2
	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$
		$\frac{2}{9}$	$\frac{1}{6}$
			$\frac{1}{36}$
$f(u)$	$\frac{1}{6}$	$\frac{5}{9}$	$\frac{5}{18}$

(b)

v	0	1
$g(v)$	$\frac{5}{6}$	$\frac{1}{6}$

w	-2	-1	0	1	2
	$\frac{1}{36}$	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$
		$\frac{1}{6}$			
$h(w)$	$\frac{1}{36}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{12}$

7.27 $f(x_1, x_2) = \binom{n_1}{x_1}\binom{n_2}{x_2}\theta^{x_1+x_2}(1-\theta)^{n_1+n_2-(x_1+x_2)}$

$x_1 + x_2 = y \qquad g(y) = \sum_{x_1=0}^{y}\binom{n_1}{x_1}\binom{n_2}{y-x_1}\theta^{y}(1-\theta)^{n_1+n_2-y}$

$= \binom{n_1+n_2}{y}\theta^{y}(1-\theta)^{n_1+n_2-1-y}$

7.28 $f(x_1, x_2) = \theta(1-\theta)^{x_1-1}\theta(1-\theta)^{x_2-1} \qquad x_1 + x_2 = y$

$g(y) = k\theta^2(1-\theta)^{y-2} \qquad b^*(y;\ 2,\ \theta) = (y-1)\cdot\theta^2(1-\theta)^{y-2}$

k is number of ways in which $x_1 + x_2 = y$ (with y fixed)

which is $y-1 \qquad g(y) = (y-1)\theta^2(1-\theta)^{y-2} = \binom{y-1}{1}\theta^2(1-\theta)^{y-2}$

7.29 $\frac{1}{2\pi}e^{-(1/2)(x^2+y^2)} \qquad z = x + y$

$$\frac{1}{2\pi}e^{-(1/2)[x^2+(z-x)^2]}$$

$$\frac{1}{2\pi}e^{-(1/2)[(x-z)^2/(1/2)]}\cdot e^{-(1/2)(z^2/2)}$$

$$\frac{\sqrt{2}}{\sqrt{2\pi}}e^{-(1/2)[(x-z/2)/(1/\sqrt{2})]^2}\cdot\frac{\sqrt{2}}{\sqrt{2\pi}}e^{-(1/2)(z/\sqrt{2})^2}$$

$$\frac{1}{\sqrt{2\pi}}e^{-(1/2)(z/\sqrt{2})^2}$$

normal $\mu = 0 \qquad \sigma^2 = 2$

7.30 $f(x, y) = 12xy(1-y) \qquad z = xy^2 \qquad 1 = \frac{dx}{dz}y^2$

$$g(z, y) = 12\cdot\frac{z}{y^2}(1-y)\cdot\frac{1}{y^2}$$

$$= 12(y^{-3} - y^2) \qquad \text{bounded by } z = 0,\ u = 1,\ \ z = u^2$$

$$h(z) = 12z\int_{\sqrt{z}}^{1}(y^{-3} - y^{-2})dy = 12z\left[\frac{y^{-2}}{-2} - \frac{y^{-1}}{-1}\right]\Bigg|_{\sqrt{z}}^{1}$$

$$= 12z\left[-\frac{1}{2} + 1 + \frac{1}{2z} - \frac{1}{\sqrt{z}}\right]$$

$$= 6z + 6 - 12\sqrt{z} \qquad 0 < z < 1$$

$$0 \qquad \text{elsewhere}$$

7.31 $z = xy^2 \qquad x = \frac{z}{u^2} \qquad \frac{\partial x}{\partial u} = \frac{-2z}{u^2} \qquad \frac{\partial y}{\partial u} = 1$

$u = y \qquad y = u \qquad \frac{\partial x}{\partial z} = \frac{1}{u^2} \qquad \frac{\partial y}{\partial z} = 0$

$$J = \begin{vmatrix} \frac{-2z}{u^2} & \frac{1}{u^2} \\ 1 & 0 \end{vmatrix} = \frac{1}{u^2}$$

$$g(z, u) = 12\frac{z}{u^2}u(1-u)\cdot\frac{1}{u^2} = 12z(u^{-3} - u^{-2})$$

from here same as in 7.30

7.32 $f(x_1, x_2) = \dfrac{1}{\pi^2(1+x_1^2)(1+x_2^2)}$ $\qquad y = x_1 + x_2$

$g(x_1, y_2) = \dfrac{1}{\pi^2(1+x_1^2)[1+(y_1-x_2)^2]}$

Use partial fractions to perform necessary integration

Result is $g(y) = \dfrac{1}{\pi}\dfrac{2}{4+y_1^2}$

$-\infty < y_1 < \infty$ Cauchy distribution

7.34 $g(u, y) = \begin{matrix} \dfrac{1}{2} & \text{over region bounded by } y = 0,\ u = y, \text{ and } 2y - u = 0 \\ 0 & \text{elsewhere} \end{matrix}$

$-2 < u < 0 \quad h(u) = \displaystyle\int_0^{(1/2)(u+2)} \frac{1}{2}\,dy = \frac{1}{4}(u+2)$

$0 < u < 2 \quad h(u) = \displaystyle\int_u^{(1/2)(u+2)} \frac{1}{2}\,dy = \frac{1}{4}(2-u)$

elsewhere it is 0

7.35 $u = y - x,\ v = x \qquad \dfrac{\partial u}{\partial x} = -1 \qquad \dfrac{\partial u}{\partial y} = 1 \qquad \dfrac{\partial v}{\partial x} = 1 \qquad \dfrac{\partial v}{\partial y} = 0 \qquad \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = -1$

$$f(u,v) = \begin{cases} \dfrac{1}{2} & \text{over the region bounded by } v = 0,\ u = -v, \text{ and } 2v + u = 2 \\ 0 & \text{elsewhere} \end{cases}$$

$g(u) = \displaystyle\int_0^{(1/2)(2-u)} \frac{1}{2}\,dv = \frac{1}{4}(2-u) \quad \text{for } 0 < u < 2$

$g(u) = \displaystyle\int_{-u}^{(1/2)(2-u)} \frac{1}{2}\,dv = \frac{1}{2}\left[\frac{1}{2}(2-u) + u\right]$

$= \dfrac{1}{4}(2+u) \quad \text{for } -2 < u < 0$

7.36 $f(x_1, x_2) = 4x_1x_2 \qquad y_1 = x_1^2 \qquad y_2 = x_1x_2$

$x_1 = \sqrt{y} \qquad \dfrac{\partial x_1}{\partial y_1} = \dfrac{1}{2\sqrt{y_1}} \qquad \dfrac{\partial x_1}{\partial y_2} = 0$

$x_2 = y_2 / \sqrt{y_1} \qquad \dfrac{\partial x_2}{\partial y_1} = -\dfrac{1}{2}y_2y_1^{-3/2} \qquad \dfrac{\partial x_2}{\partial y_2} = \dfrac{1}{\sqrt{y_1}}$

$$g(y_1, y_2) = 4\sqrt{y_1}\frac{y_2}{\sqrt{y_1}} \cdot \frac{1}{2y_1} \qquad J = \begin{vmatrix} \frac{1}{2\sqrt{y_1}} & 0 \\ -\frac{1}{2}y_2 y_1^{-3/2} & \frac{1}{\sqrt{y_1}} \end{vmatrix} = \frac{1}{2y_1}$$

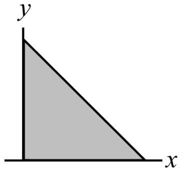
$$= \frac{2y_2}{y_1}$$

over region bounded by $y = 1,\ y_2 = 0,$ and $y_1 = y_2^2$

7.37 $f(x, y) = 24xy$

$z = x + y \quad w = x \to x = w$

and $y = z - w$



$$\frac{\partial x}{\partial w} = 1 \qquad \frac{\partial x}{\partial z} = 0$$

$$\frac{\partial y}{\partial \omega} = -1 \qquad \frac{\partial y}{\partial z} = 1 \qquad J = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

$$g(w, z) = \begin{cases} 24w(z - w) & \text{over region bounded by } w = 0,\ z = 1, \text{ and } z = w \\ 0 & \text{elsewhere} \end{cases}$$

7.38 **(a)** $u = \dfrac{x}{x + y}$ and $v = x + y$

$$x = uv \qquad \frac{\partial x}{\partial u} = v \qquad \frac{\partial x}{\partial v} = u$$

$$y = v(1 - u) \qquad \frac{\partial y}{\partial u} = -v \qquad \frac{\partial y}{\partial v} = 1 - u$$

$$J = \begin{vmatrix} v & u \\ -v & (1 - u) \end{vmatrix} = v(1 - u) + uv = v$$

$$f(x, y) = \frac{1}{[\beta^\alpha \Gamma(\alpha)]^2} x^{\alpha-1} y^{\alpha-1} e^{-(1/\beta)(x+y)}$$

$$g(u, v) = \frac{1}{\beta^{2\alpha}\Gamma(\alpha)]^2} [u(1 - u)]^{\alpha-1} v^{2\alpha-1} e^{-(1/\beta)v}$$

for $0 < u < 1,\ 0 < v < \infty$

(b) $$h(u) = \frac{1}{\beta^{2\alpha}\Gamma(\alpha)]^2}[u(1 - u)]^{\alpha-1}\int_0^\infty v^{2\alpha-1} e^{-(1/\beta)v} dv$$

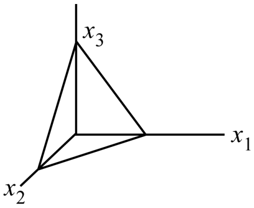
$$= \frac{1}{\beta^{2\alpha}\Gamma(\alpha)]^2} \cdot \beta^{2\alpha}\Gamma(2\alpha) \cdot [u(1 - u)]^{\alpha-1}$$

$$= \frac{\Gamma(2\alpha)}{\Gamma(\alpha)\Gamma(\alpha)} u^{\alpha-1}(1 - u)^{\alpha-1} \quad \text{for } 0 < u < 1$$

U has beta distribution with $\beta = \alpha$

7.39 $y = x_1 + x_2 + x_3$

$g(x_1, x_2, y) = e^{-y} \qquad x_1 > 0,\ x_2 > 0,\ y > 0$



$$h(y) = \int_0^y \int_0^{y-x_2} e^{-y}\,dx_1 dx_2 = \begin{cases} \frac{1}{2} y^2 e^{-y} & \text{for } y > 0 \quad x_1 + x_2 \le y \\ 0 & \text{elsewhere} \end{cases}$$

7.40 $g(y, x_3) = h(y)$ as given in Example 7.13

(a) $$g(y, u) = h(y)\cdot 1 = \begin{cases} y & \text{I} + \text{II} \\ 2 - y & \text{III} + \text{IV} \\ 0 & \text{elsewhere} \end{cases}$$

(b) $$h(u) = \int_0^u g(y, u)\,dy = \int_0^u y\,dy = \frac{u^2}{2} \qquad \text{for } 0 < u < 1$$

$$h(u) = \int_{u-1}^{1} y\,dy + \int_1^u (2 - y)\,dy = \frac{1}{2}u^2 - \frac{3}{2}(u-1)^2 \qquad 1 < u < 2$$

$$h(u) = \int_{u-1}^{2} (2 - y)\,dy = \frac{1}{2}u^2 - \frac{3}{2}(u-1)^2 + \frac{3}{2}(u-2)^2 \qquad 2 < u < 3$$

$h(u) = 0$ elsewhere; $h(1) = \frac{1}{2}$, $h(2) = \frac{1}{2}$ will make it continuous

7.41 $M_Y = [1 + \theta(e^t - 1)^{n_1}[1 + \theta(e^t - 1)]^{n_2}$

$= [1 + \theta(e^t - 1)]^{n_1 + n_2}$

Y is random variable having binomial distribution with the parameter θ and $n_1 + n_2$.

7.42 $$M_Y = \left[\frac{\theta e^t}{1 - e^t(1-\theta)}\right]^k = \frac{\theta^k e^{kt}}{[1 - e^t(1-\theta)]^k}$$

7.43 $M_X = (1 - \beta t)^{-\alpha}$

$M_Y = (1 - \beta)^{-\alpha n}$

Y is a random variable having gamma distribution with the parameter α and β.

7.44 $M_X = e^{\mu t + (1/2)t^2\sigma^2}$

$M_Y = \prod e^{\mu_i t + (1/2)t^2\sigma_i^2} = e^{t\left(\sum \mu_i\right) + (1/2)t^2\left(\sum \sigma_i^2\right)}$

Y is a random variable having normal distribution with $\mu = \sum \mu_i$ and $\sigma^2 = \sum \sigma_i^2$

7.45 Let $Z_i = a_i X_i$

$M_{Z_i} = M_{x_i}(a_i t)$

since $Y = \sum Z_i$

$M_Y = \prod M_{x_i}(a_i t)$ QED

7.46 $M_{x_i} = e^{\mu_i t + (1/2)t^2 \alpha_1^2}$ $\qquad Y = \sum a_i X_i$

$M_Y \prod e^{\mu_i a_i t + (1/2)t^2 a_i^2 \sigma_i^2}$

This is normal distribution with $\mu = \sum a_i \mu_i$ and variance $\sigma^2 = \sum a_i^2 \sigma_i^2$

7.47 $G(v) = P(V \le v) = P(SP \le v)$

$$= \int_{0.2}^{0.4} 5p \int_{0}^{v/p} e^{-sp} ds\ dp = \int_{0.2}^{0.4} 5p \left[-\frac{1}{p} e^{-sp} \right] \Bigg|_{0}^{v/p} dp$$

$$= \int_{0.2}^{0.4} 5[1 - e^{-v}] dp = 1 - e^{-v}$$

$g(v) = e^{-v}$ for $v > 0$ and 0 elsewhere

7.48 $x + y = 2u$

$$G(u) = \int_{0}^{2u} \int_{0}^{2u-x} \left[-\frac{1}{30} e^{-x/30} \right] \left[-\frac{1}{30} e^{-y/30} \right] dy\ dx$$

$$= 1 - e^{-u/15} - \frac{u}{15} e^{-u/15} \qquad y > 0$$

$g(u) = \dfrac{u}{255} e^{-u/15}$ for $y > 0$ and 0 elsewhere

7.49 $z = x - y$

for $0 < z < 5$ $\qquad G(z) = \displaystyle\int_{10}^{20} \int_{x-z}^{x} \frac{1}{25} \left(\frac{20 - x}{x} \right) dy\ dx$

$$= \frac{1}{25} z(20 \ln 2 - 10)$$

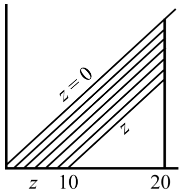
$g(z) = \dfrac{1}{25}(20 \ln 2 - 10)$ and 0 elsewhere

for $5 < z < 10$

$G(z) = 1 - \displaystyle\int_{2z}^{20} \int_{x/2}^{x-z} \frac{1}{25} \left(\frac{20 - x}{x} \right) dy\ dx$ leads to

$g(z) = \dfrac{1}{25} \left(2z - 20 - 20 \ln \dfrac{z}{10} \right)$ for $5 < z < 10$

7.50 $f(x,\ y) = \dfrac{1}{200}$ $\qquad 0 < y < x \qquad z = x - y$



$$G(z) = 1 - \frac{(20-z)^2}{z} \cdot \frac{1}{200} \qquad 0 < z < 20$$

$$g(z) = -\frac{2(20-z)(-1)}{2} \cdot \frac{1}{200} = \frac{20-z}{200}$$

for $0 < z < 20$
0 elsewhere

7.51 for $0 < y < 1$

$$G(y) = \int_0^y \int_0^{y-x_1} \frac{3}{11}(5x_1 + x_2)dx_2\ dx_1 = \frac{3}{11}y^3$$

$$g(y) = \frac{9}{11}y^2$$

for $1 < y < 2$

$$G(y) = 1 - \int_0^{2-y} \int_{y-x_2}^{2(1-x_2)} \frac{3}{11}(5x_1 + x_2)dx_1\ dx_2$$

$$= 1 - \frac{1}{11}(1+7y)(2-y)^2$$

$$g(y) = \frac{3(2-y)(7y-4)}{11}$$

7.52 $f(v) = kv^2e^{-\beta v^2} \qquad v > 0$

$$E = \frac{1}{2}mv^2 \qquad 1 = \frac{1}{2}m \cdot 2v\frac{dv}{dE} = mv\frac{dv}{dE} \qquad v = \sqrt{\frac{2}{m}E}$$

$g(E) = \dfrac{k}{m}v\ e^{-\beta 2E/m} = KE^{1/2}e^{-cE}$ which is a gamma distribution

7.53 $f(x,\ y) = \dfrac{1}{\pi} \qquad 0 < x^2 + y^2 < 1 \qquad r^2 = x^2 + y^2$

$$g(r, y) = \frac{4}{\pi}\frac{dx}{dr} \qquad 2r = \frac{dx}{dr} \qquad \frac{dx}{dr} = \frac{r}{x}$$

$$= \frac{4}{\pi} \cdot \frac{r}{x} = \frac{1}{\pi} \cdot \frac{r}{\sqrt{r^2 - y^2}}$$

$$h(r) = \frac{4}{\pi}\int_0^r \frac{r\ dy}{\sqrt{r^2 - y^2}} = \frac{4}{\pi}\int_0^r \frac{dy}{\sqrt{r^2-y^2}} = \frac{4r}{\pi} \cdot \sin^{-1}\frac{y}{r}\bigg|_0^r$$

$$= \frac{4r}{\pi} \cdot (\sin^{-1}1 - \sin^{-1}0) = \frac{4r}{\pi}\left[\frac{\pi}{2} - 0\right]$$

$= 2r$ for $0 < r < 1$

7.54 $f(x, y) = \frac{2}{5}(2x+3y)$ $\quad 0<x<1$, $0<y<1$ $\quad z = \frac{x+y}{z}$

$g(z, y) = \frac{2}{5}[4z+y]\cdot 2$ $\quad 2z = x+y$

$z = \frac{dx}{dz}$

$$= \begin{cases} \frac{4}{5}(4z+y) \text{ over } y=0,\ y=1,\ 2z=y, \text{ and } 2z=y+1 \\ 0 \text{ elsewhere} \end{cases}$$

$h(z) = \frac{4}{5}\int_0^{2z}(4z+y)dy = 8z^2$ $\quad$ for $0<z<\frac{1}{2}$

$h(z) = \frac{4}{5}\int_{2z-1}^{1}(4z+y)dy = 8z(1-z)$ $\quad$ for $\frac{1}{2}<z<1$

$h(z) = 0$ elsewhere

Also, let $h\left(\frac{1}{2}\right) = 2$

7.55 $f(p, s) = 5pe^{-ps}$ $\quad 0.2<p<0.4$ and $s>0$

$v = sp$ $\quad s = \frac{v}{w}$ $\quad \frac{\partial s}{\partial v} = \frac{1}{w},\ \frac{\partial s}{\partial w} = -\frac{v}{w^2},\ \frac{\partial p}{\partial v} = 0,\ \frac{\partial p}{\partial w} = 1$

$w = p$ $\quad p = \text{w}$ $\quad J = \begin{vmatrix} \frac{1}{w} & -\frac{v}{w^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{w}$

$g(v, w) = 5we^{-v}\cdot\frac{1}{w} = 5e^{-v}$ $\quad$ for $0.2<w<0.4$ and $v>0$

$h(v) = 5e^{-v}\int_{0.2}^{0.4}dw = e^{-v}$ $\quad$ for $v>0$

7.56 Using MINITAB, we generate 10 "pseudo-random" numbers in C1 having the standard normal distribution with the following commands:

```
MTB> Random 10 C1
SUBC> Normal 0.0 1.0.
```

7.57 First the computer generates 10 "pseudo-random" numbers on the interval (0, 1). For example, for numbers to two decimal places, the interval (0, 1) is regarded as the union of the subintervals (-0.0050, 0.0049), (0.0050, 0.0149), ..., (0.9950, 1.049), corresponding to the numbers 0.00, ..., 0.01, ..., 1.00, respectively. Since there are 101 such intervals (numbers) each one is chosen with probability 1/101.. Then, the required numbers are generated with the inverse of the probability integral transformation.

7.58 Total number of calls per hour is random variable having Poisson distribution with parameter $\lambda = 2.1+10.9 = 13$. From Table II

(a) 0.1021

(b) 0.0002 + 0.0008 + 0.0027 + 0.0070 + 0.0152 = 0.0259

7.59 Total number of inquiries is a random variable having Poisson distribution with $\lambda = 3.6 + 5.8 + 4.6 = 14$. From Table II

(a) 0.0001 + 0.0004 + 0.0013 + ... + 0.0473 = 0.1093
(b) 0.0989 + 0.0866 + ... + 0.0286 = 0.3817
(c) 0.0554 + 0.0409 + ... + 0.0001 = 0.1728

7.60 Six inquiries with $\lambda_2 = 5.8$ $p(6;\ 5.8) = 0.1601$ Table ii
Eight inquiries with $\lambda = 8.2$ $p(8;\ 8.2) = 0.1392$

$$(0.1601)(0.1392) = 0.0222$$

7.61 **(a)** $p(2;\ 3.3) = 0.2008$
(b) $p(5;\ 6.6) = 0.1420$
(c) p(at least 12; 9.9) = 0.0928 + 0.0707 + ... + 0.0001 = 0.2919

7.62 **(a)** $p(4;\ 3.2) = 0.1781$
(b) p(at least 2; 4.8) = $1 - (0.0082 + 0.0395) = 0.9523$
(c) p(at most 3; 6.4) = $0.0017 + 0.0106 + 0.0340 + 0.0726 = 0.1189$

7.63 **(a)** Gamma with $\alpha = 2$ and $\beta = 5$

$$\frac{1}{5^2 \cdot 1} t \int_0^8 x\ e^{-x/5} dx = 0.475$$

(b) Gamma with $\alpha = 3$ and $\beta = 5$

$$\frac{1}{5^3 \cdot 2!} \int_{12}^{\infty} x^2\ e^{-x/5} dx = 0.570$$

7.64 **(a)** $\frac{1}{9} \int_{20}^{\infty} e^{-x/9} dx = e^{-20/9} = e^{-2.22} = 0.1086$

(b) Gamma with $\alpha = 2$ and $\beta = 9$

$$\frac{1}{81 \cdot 1} \int_{20}^{\infty} x\ e^{-x/9} dx = 0.3492$$

(c) Gamma with $\alpha = 3$ and $\beta = 9$

$$\frac{1}{9^3 \cdot 2} \int_{20}^{\infty} x^2\ e^{-x/9} dx = 0.6168$$

7.65 $f(x) = \binom{3}{x}\left(\frac{1}{6}\right)^x \cdot \left(\frac{5}{6}\right)^{3-x}$, $x = 0,\ 1,\ 2,\ 3$. For $x^2 > 2,\ x > 1$. The probability that $x > 1$ is given by $3 \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right) + \left(\frac{1}{6}\right)^3 = \frac{16}{216} = \left(\frac{2}{27}\right)$

7.66 $P(x>1)=\int_1^\infty 0.5\cdot e^{-0.5x}ex=e^{-0.5}$.

7.67 **(a)** $\frac{1}{k}=\int_0^6\left(1-\frac{d}{5}\right)dd=2.5,\ \therefore k=\frac{2}{5}$.

(b) $A=\pi\frac{d^2}{4}\ \therefore d=\frac{2\sqrt{A}}{\sqrt{\pi}}$. Thus, $dA=\frac{\pi}{2}d\cdot dd;\ dd=\frac{dA}{d}\frac{2}{\pi}=\frac{1}{\sqrt{\pi}}A^{-1/2}dA$.

Substituting for d in $\int\left(1-\frac{d}{5}\right)dd$, we obtain

$\int\left(1-\frac{2\sqrt{A}}{5\pi}\right)\cdot\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{A}}dA=\int\left(\frac{1}{\sqrt{\pi A}}-\frac{2}{5\pi^{3/2}}\right)dA$ so that the integrand is

$g(A)=\pi^{-1/2}A^{-1/2}-\frac{2}{5}\pi^{-3/2}$ for $0<A<25\pi/4$, and $g(A)=0$ elsewhere.

7.69 $f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/2\sigma^2}$. Substituting $y=\ln x$, with $x=e^y$ and $dx=e^y dy$, we obtain

$g(y)=\frac{1}{\sqrt{2\pi}\sigma}\cdot y^{-1}e^{(\ln y-\mu)^2/2\sigma^2}$ for $y>0$, and $g(y)=0$ elsewhere.

7.70 Since $G=\log\frac{I_o}{I_i}$, and G is normally distributed with the mean 1.8 and the standard deviation 0.05, we calculate $z=\frac{6-1.8}{0.05}=84$ and conclude that the probability of the gain exceeding 6 is negligible.

Chapter 8

8.1 $a_1 = -\frac{1}{n},\ \ldots,\ a_r = 1 - \frac{1}{n}\ \ldots\ a_n = -\frac{1}{n}$

$$b_1 = \frac{1}{n},\ \ldots,\ \mathrm{b}_r = \frac{1}{n}\ \ldots\ \mathrm{b}_n = \frac{1}{n}$$

$$\mathrm{cov} = \left(-\frac{1}{n^2} + \ldots + \frac{1}{n} - \frac{1}{n^2} + \ldots - \frac{1}{n^2}\right)\sigma^2$$

$$= \left[\frac{1}{n} + n\left(-\frac{1}{n^2}\right)\right]\sigma^2 = \left(\frac{1}{n} - \frac{1}{n}\right)\sigma^2 = 0$$

8.2 $Y = \bar{x}_1 - \bar{x}_2$

(a) $E(Y) = E(\bar{x}_1 - \bar{x}_2) = \frac{1}{n_1}\sum E(x_{1i}) - \frac{1}{n_2}\sum E(x_{2i})$

$$= \frac{n_1}{n_1}\mu_1 - \frac{n_2}{n_2}\mu_2 = \mu_1 - \mu_2$$

(b) $\mathrm{var}(Y) = \sum \frac{1}{n_1^2}\mathrm{var}(x_{1i}) + \sum \frac{1}{n_2^2}\mathrm{var}(x_{2i}) = \frac{1}{n_1^2}\cdot n_1\sigma_1^2 + \frac{1}{n_2^2}\cdot n_2\sigma_2^2 = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$

8.3 $M_Y(t) = \prod_{i=1}^{n_1} M_{X_{1i}}\left(\frac{t}{n_1}\right)\cdot\prod_{j=1}^{n_2} M_{X_{2j}}\left(\frac{-t}{n_2}\right)$

$$= \prod_{i=1}^{n_1} e^{\mu_1}(t/n_1) + (1/2)\sigma_1^2)(t/n_1)^2 \cdot \prod_{j=1}^{n_2} e^{\mu_2(-t/n_1)+(1/2)\sigma_2^2(-t/n_2)^2}$$

$$= e^{\mu_2 t(1/2)(\sigma_1^2/n_1)t^2}\cdot e^{\mu_2 t+(1/2)(\sigma_2^2/n_2)t^2}$$

$$= e^{(\mu_1-\mu_2)t+(1/2)[(\sigma_1^2/n_1)+(\sigma_2^2/n_2)]t^2}$$

$$\mu = \mu_1 - \mu_2$$

$$\sigma^2 = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

8.4 $M_x = [1 + \theta(e^t - 1)]$

$$M_{\bar{x}} = [1 + \theta(e^{t/n} - 1)]^n$$

$$M' = n[1 + \theta(e^{t/n} - 1)]^{n-1}\frac{\theta}{n}e^{t/n} = \theta[1 + \theta(e^{t/n} - 1)]^{n-1}e^{t/n}$$

$$M'(0) = \theta$$

$$M'' = \theta[1 + \theta(e^{t/n} - 1)]^{n-1}\cdot\frac{1}{n}e^{t/n} + \theta e^{t/n}(n-1)[1 + \theta(e^{t/n} - 1)]^{n-2}\cdot\frac{\theta}{n}e^{t/n}$$

$$M''(0) = \frac{\theta}{n} + \frac{\theta^2(n-1)}{n}$$

$$\sigma^2 = \frac{\theta}{n} + \frac{\theta^2(n-1)}{n} - \theta^2 = \frac{\theta(n-1)}{n}$$

8.5 $E(Y) = \mu_1 - \mu_2 = \theta_1 - \theta_2$

$$\text{var}(Y) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} = \frac{\theta_1(1-\theta_1)}{n_1} + \frac{\theta_2(1-\theta_2)}{n_2}$$

Follows directly by substitution.

8.6 $M_{\bar{x}} = [1+\theta(e^t - 1)]^n \qquad \mu = \theta \qquad \sigma = \sqrt{\dfrac{\theta(1-\theta)}{n}}$

$$M_{(\bar{x}-\mu)/\sigma} = e^{-\mu/\sigma} \cdot M_{\bar{x}}\left(\frac{t}{\sigma}\right) = e^{-\sqrt{[\theta/n(1-\theta)]}t} \cdot \left[1+\theta\left(e^{t/\sqrt{n\theta(1-\theta)}} - 1\right)\right]^n$$

Use series expansion to show that as $n \to \infty$

$M_{(\bar{x}-\mu)/\sigma} \to e^{(1/2)t^2}$

8.7 (1) independent

(2) information bounded with $k = \frac{1}{2}$

(3) $$E(x_i) = \frac{1}{2}\left[1-\left(\frac{1}{2}\right)^i\right] + \frac{1}{2}\left[\left(\frac{1}{2}\right)^i - 1\right] = 0$$

$$E(x_i)^2 = \frac{1}{2}\left[1-\left(\frac{1}{2}\right)^i\right]^2 + \frac{1}{2}\left[\left(\frac{1}{2}\right)^i - 1\right]^2 = \left[1-\left(\frac{1}{2}\right)^i\right]^2$$

$$= 1 - \left(\frac{1}{2}\right)^{i-1} + \left(\frac{1}{4}\right)^i$$

$$E(Y_n) = n - \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} + \frac{1}{4}\frac{1-\left(\frac{1}{4}\right)^n}{1-\frac{1}{4}}$$

$$\lim_{n\to\infty} E(Y_n) = \lim_{n\to\infty}\left(n - 2 + \frac{1}{3}\right) \to \infty \qquad \text{QED}$$

8.8 (1) independent

(2) uniformly bounded $k = 2$

(3) $$E(x_i) = \frac{1}{2-\frac{1}{i}}\int_0^{2-(1/i)} x\,dx = \frac{1}{2-\frac{1}{i}}\frac{\left(2-\frac{1}{i}\right)^2}{2} = 1 - \frac{1}{2i}$$

$$E\left(x_i^2\right) = \frac{1}{2-\frac{1}{i}}\int_0^{2-(1/i)} x^2\,dx$$

$$E\left(x_i^2\right) = \frac{1}{2-\frac{1}{i}} \cdot \frac{\left(2-\frac{1}{i}\right)^3}{3} = \frac{\left(2-\frac{1}{i}\right)^2}{3} \qquad \sigma^2 = \frac{\left(2-\frac{1}{i}\right)^2}{3} - \frac{\left(2-\frac{1}{i}\right)^2}{4} = \frac{\left(2-\frac{1}{i}\right)^2}{12}$$

$$\begin{aligned}\sigma^2_{Y_n} &= n-\left(\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)+\frac{1}{4}\left(1+\frac{1}{4}+\frac{1}{9}\ldots+\frac{1}{n^2}\right)\\ &> n-\left(\frac{1}{2}+\frac{1}{2}+\ldots\frac{1}{2}\right)+\frac{1}{4}\left(1+\frac{1}{4}+\ldots\frac{1}{n^2}\right)\\ &> \frac{n}{2}+\frac{1}{4}\left(1+\frac{1}{4}+\ldots\frac{1}{n^2}\right)\to\infty\end{aligned}$$

8.9 $C_i = E\left(|x_i|^2\right)=\left[1-\left(\frac{1}{2}\right)^i\right]^3$

$$\sigma_i^2=\left[1-\left(\frac{1}{2}\right)^i\right]^2$$

$$\operatorname{var}(Y_n)=\sum_{i=1}^{n}\left[1-\left(\frac{1}{2}\right)^i\right]^2$$

Let $Q=[\operatorname{var}(Y_n)]^{-3/2}\cdot\sum_{i=1}^{n}C_i$

$$=\frac{\sum_{i=1}^{n}\left[1-\left(\frac{1}{2}\right)^i\right]^3}{\sum_{i=1}^{n}\left[1-\left(\frac{1}{2}\right)^i\right]^{2\ 3/2}}$$

$$=\frac{n+\ldots}{\{n+\ldots\}^{3/2}}=\frac{n+\ldots}{n\sqrt{n}+\ldots}$$

$$\lim_{n\to\infty}Q=0$$

8.10 $E(x_i)=0$ $\qquad\sigma^2=\dfrac{\left(2-\frac{1}{i}\right)^2}{4}$

$$\operatorname{var}(Y_n)=n-\left(\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)+\frac{1}{4}\left(1+\frac{1}{4}\ldots+\frac{1}{n^2}\right)$$

$$C_i=\int_0^{2-(1/i)}\frac{1}{2-\frac{1}{i}}x^2\,dx=\frac{1}{4}\left(2-\frac{1}{i}\right)^2=2-\frac{1}{i}+\frac{3}{2}\cdot\frac{1}{i^2}-\frac{1}{4}\cdot\frac{1}{i^2}$$

$$\sum_{i=1}^{n}C_i=2n-\left(1+\frac{1}{2}+\frac{1}{3}\ldots\frac{1}{n}\right)+\frac{3}{2}\left(1+\frac{1}{4}+\ldots\frac{1}{n^2}\right)-\frac{1}{4}\left(1+\frac{1}{8}+\frac{1}{27}+\ldots\frac{1}{n^2}\right)$$

$$\frac{\sum C_i}{\left[(\text{var}(Y_n)^{3/2}\right]} = \frac{n - \left(\frac{1}{2} + \frac{1}{3} \ldots \frac{1}{4}\right) + \frac{1}{4}\left(1 + \frac{1}{4} \ldots \frac{1}{n^2}\right)}{\left\{2n - \left(1 + \frac{1}{2} \ldots\right) + \frac{3}{2}\left(1 + \frac{1}{4} \ldots\right) - \frac{1}{4}\left(1 + \frac{1}{8} \ldots\right)\right\}^{3/2}}$$

$$= \frac{n + \ldots}{k\sqrt{n}n + \ldots} \to 0 \qquad \text{when } n \to \infty$$

8.11 When we sample with replacement from a finite population we satisfy all the conditions for random sampling from an infinite population. The random variables $x_1, x_2, \ldots, x_n$ are independent and identically distributed.

8.12 Hypergeometric distribution applies to sampling without replacement from a finite population

$$\mu = \frac{k}{N}$$

Consider population of k 1's and $N - k$ 0's.

$$\mu = \frac{k}{N} \text{ and } \sigma^2 = \frac{k}{N} - \frac{k^2}{N^2} = \frac{k(N-k)}{N^2}$$

by theorem 8.6 $E(\overline{x}) = \frac{k}{N}$ and $\text{var}(\overline{x}) = \frac{k(N-k)}{nN^2} \cdot \frac{N-n}{N-1}$

and for $Y = n\overline{x}$ $\quad E(Y) = \frac{nk}{N}$ and $\text{var}(Y) = \frac{k(N-k)}{N^2} \cdot \frac{N-n}{N-1}$

Then let $\theta = \frac{k}{N}$ $\quad E(Y) = \theta$ and $\text{var}(Y) = n\theta(1-\theta)\frac{N-n}{N-1}$

Y is a random variable having the hypergeometric distribution.

8.15 **(a)** $\mu = \frac{1+2+3\ldots+N}{N} = \frac{N(N+1)}{2N} = \frac{N+1}{2} \qquad \mu_{\overline{x}} = \frac{N+1}{2}$

(b) $\sigma^2 = \frac{1^2 + 2^2 + \ldots N^2}{N} - \frac{(N+1)^2}{4} = \frac{(N+1)(2N+1)}{6} - \frac{(N+1)^2}{4} = \frac{N^2-1}{12}$

$$\text{var}(\overline{x}) = \frac{N^2-1}{12n} \cdot \frac{N-n}{N-1} = \frac{(N+1)(N-n)}{12n}$$

(c) $\mu_Y = \frac{n(N+1)}{2}$ and $\text{var}(Y) = \frac{n^2(N+1)(N-n)}{12n} = \frac{n(N+1)(N-n)}{12}$

8.16 $\sum c = 130 \qquad \mu = 13 \qquad \sum (C-13)^2 = 256$

$$\sigma^2 = \frac{256}{10} = 25.6$$

8.17 $$\sigma^2 = \sum_{i=1}^{N}(c_i = \mu)^2 \cdot \frac{1}{N}$$
$$= \frac{1}{N}\left(\sum_{i=1}^{N} c_i^2 - 2\mu\sum_{i=1}^{N} c_i + N\mu^2\right)$$
$$= \frac{1}{N}\left(\sum_{i=1}^{N} c_i^2 - 2N\mu^2 + N\mu^2\right)$$
$$= \frac{\sum_{i=1}^{N} c_i^2}{N} - \mu^2$$

In Exercise 8.14 we have

$$\mu = (15+13+\ldots+9)\cdot\frac{1}{10} = 13.0;\ \sigma^2 = \frac{15^2+13^2+\ldots+9^2}{10} - (13.0)^2 = 25.8$$

8.18 $$S^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$$
$$= \frac{1}{n-1}\left(\sum_{i=1}^{n} X_i^2 - 2\bar{X}\sum_{i=1}^{n} X_i + n\bar{X})^2\right)$$
$$= \frac{1}{n-1}\left(\sum_{i=1}^{n} X_i^2 - 2n\bar{X}^2 + n\bar{X}^2\right)$$
$$= \frac{\sum_{i=1}^{n} X_i^2}{n-1} - \frac{n\bar{X}^2}{n-1}$$

From the given data we calculate

$$\sum_{i=1}^{8} X_i = 108; \qquad \sum_{i=1}^{8} X_i^2 = 1{,}486$$

$$S^2 = \frac{1{,}486}{7} - \frac{8\cdot\left(\frac{108}{8}\right)^2}{7} = 4$$

8.19 Multiplying both sides of the last equation in Exercise 8.18 by n we have

$$nS^2 = \frac{n\sum_{i=1}^{n} X_i^2}{n-1} - \frac{(n\bar{X})^2}{n-1}$$
$$\therefore S^2 = \frac{n\sum_{i=1}^{n} X_i^2 - \left(\sum_{i=1}^{n} X_i\right)^2}{n(n-1)}$$

Substituting the data of Exercise 8.18 we obtain

$$S^2 = \frac{8(1{,}486)-(108)^2}{8(7)} = 4$$

8.20 $M_{x_i}(t) = (1-2t)^{-(1/2)v_i} \qquad Y = \sum x_i$

$$M_Y(t) = \prod_{i=1}^{n} (1-2t)^{-(1/2)v_i} = (1-2t)^{-(1/2)\sum v_i}$$

chi square with $\sum v_i$ degrees of freedom

8.21 $M_{x_i}(t) \cdot M_{x_2}(t) = M_{x_i+x_2}(t)$

$(1-2t)^{-(1/2)v_2} \cdot M_{x_2}(t) = (1-2t)^{-(1/2)(v_1+v_2)}$

$M_{x_2}(t) = (1-2t)^{(1/2)v_2}$ QED

chi square with v_2 degrees of freedom

8.22 $\sum_{i=1}^{n}(x_i-\mu)^2 = \sum_{i=1}^{n}\left[(x_i-\overline{x})+(\overline{x}-\mu)\right]^2$

$$= \sum_{i=1}^{n}(x_i-\overline{x})^2 + 2\sum_{i=1}^{n}(x_i-\overline{x})(\overline{x}-\mu) + \sum_{i=1}^{n}(\overline{x}-\mu)^2$$

$$= \sum_{i=1}^{n}(x_i-\overline{x})^2 + 2(\overline{x}-\mu)\sum_{i=1}^{n}(x_i-\overline{x}) + n(\overline{x}-\mu)^2$$

$$= \sum_{i=1}^{n}(x_i-\overline{x})^2 + n(\overline{x}-\mu)^2 \qquad \text{QED}$$

8.23 $E\left[\frac{(n-1)S^2}{\sigma^2}\right] = n-1 \qquad E(S^2) = \frac{\sigma^2(n-1)}{n-1} = \sigma^2$

$\text{var}\left[\frac{(n-1)S^2}{\sigma^2}\right] = 2(n-1) \qquad \text{var}(S^2) = \frac{\sigma^4}{(n-1)^2}\cdot 2(n-1) = \frac{2\sigma^4}{n-1}$

8.24 Follows *directly* from central limit theorem

x_i has chi square distribution with 1 degree of freedom

$\mu = 1$ and $\sigma = \sqrt{2}$

8.25 From 8.24 with $z = \frac{Y_n - n}{\sqrt{2n}} \to N(0,1)$

Here Y_n is a Chi-Square random variable with *n* degrees of freedom.

8.26 $\mu = 50$ and $\sigma = \sqrt{2}\cdot 50 = 10 \qquad z = \frac{68-50}{50} = 1.8$

Probability is $0.5000 - 0.4641 = 0.0359$

8.27 $\sqrt{2x} - \sqrt{2v} < k$

$$\sqrt{2x} < k + \sqrt{2v}$$
$$2x < k^2 + 2k\sqrt{2v} + 2v$$
$$2x - 2v < k^2 + 2k\sqrt{2v}$$
$$\frac{x-v}{\sqrt{2v}} < \frac{k^2}{2\sqrt{2v}} + k$$

8.28 From 8.27 $P\left[\frac{x-v}{\sqrt{2v}} < k + \frac{k^2}{2\sqrt{2v}}\right] \to P\left[\frac{x-v}{\sqrt{2v}} < k\right] = P\left[\sqrt{2x} - \sqrt{2v} < k\right]$

Since $\frac{x-v}{\sqrt{2v}} \to N(0,\ 1)$ for $n \to \infty$, also $P\left[\sqrt{2x} - \sqrt{2v} < k\right] \to N(0,\ 1)$

Also, $z = \sqrt{2 \cdot 68} - \sqrt{2 \cdot 50} = 11.66 - 10 = 1.66$

$0.5000 - 0.4515 = 0.0485$

8.29 From 8.26 probability is 0.0359; % error = $\frac{0.0359 - 0.04596}{0.04596} \cdot 100 = -21.9\%$

From 8.27 probability is 0.0485; % error = $\frac{0.0485 - 0.04596}{0.04596} \cdot 100 = 5.53\%$

8.35 $$f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\,\Gamma\left(\frac{n}{2}\right)}\left(1 + \frac{t^2}{n}\right)^{-(n+1)/2}$$

$$\to \frac{\sqrt{2\pi\left(\frac{n-1}{2}\right)}\left(\frac{n-1}{2e}\right)^{(n-1)/2}}{\sqrt{\pi n}\sqrt{2\pi\left(\frac{n-2}{2}\right)}\left(\frac{n-2}{2e}\right)^{(n-2)/n}}\left(1 + \frac{t^2}{n}\right)^{-(n+1)/2} \qquad \mu = \frac{t^2}{n}$$

$$= \frac{k(n-1)^{n/2}}{\sqrt{n}(n-2)^{(n-1)/2}}(1+u)^{-[(t^2/2u)-(1/2)]}$$

$$f(x) = \frac{k(n-1)^{n/2}}{\sqrt{n}(n-2)^{(n-1)/2}}\left[(1+u)^{1/u}\right]^{-t^2/2}(1+u)^{-1/2}$$

$$= k\sqrt{\frac{(n-1)^n}{n(n-2)^{n-1}}}\left[(1+u)^{1/u}\right]^{-t^2/2}(1+u)^{-1/2}$$

$$\qquad\quad \downarrow \qquad\qquad\quad \downarrow \qquad\qquad\quad \downarrow$$

$$\qquad\quad 1 \qquad\qquad\quad e^{-t^2/1} \qquad\qquad\quad 1$$

$$= ke^{-t^2/2} \qquad \text{QED}$$

8.36 The Cauchy distribution

8.37 $F = \frac{u / v_1}{v / v_2} \qquad w = v \qquad u = Fw\frac{v_1}{v_2} \qquad v = w$

$$\frac{\partial u}{\partial F} = w\frac{v_1}{v_2}, \quad \frac{\partial u}{\partial w} = F\frac{v_1}{v_2}, \quad \frac{\partial v}{\partial F} = 0, \quad \frac{\partial v}{\partial w} = 1$$

$$J = \begin{vmatrix} w\frac{v_1}{v_2} & F\frac{v_1}{v_2} \\ 0 & 1 \end{vmatrix} = w\frac{v_1}{v_2}$$

$$f(u,\ v) = k\ u^{(v_1-2)/2} v^{(v_2-2)/2} e^{-(1/2)(u+v)}$$

$$g(F,\ w) = k\left(Fw\frac{v_1}{v_2}\right)^{(v_1-2)/2} w^{(v_2-2)/2} e^{-(1/2)w[F(v_1/v_2)+1]} \cdot w\frac{v_1}{v_2}$$

$$= k'\ F^{(v_1-2)/2} w^{(v_1+v_2-2)/2} e^{-(1/2)w[F(v_1/v_2)+1]}$$

$$h(F) = k''\ F^{(v_1-2)/2} \int_0^\infty w^{[(v_1+v_2)/2]-1} e^{-(1/2)[F(v_1/v_2)+1]} dw$$

Gamma distribution with $\alpha = \frac{v_1 + v_2}{2}$

$$\beta = \frac{2}{\left(F\frac{v_1}{v_2} + 1\right)}$$

$$= CF^{(v_1-2)/2}\left(F\frac{v_1}{v_2} + 1\right)^{-(1/2)(v_1+v_2)} \qquad \text{QED}$$

8.38 Make use of the fact that $F = \frac{u / v_1}{v / v_2}$ where u and v are independent chi square random variables, so that $E(F) = \frac{v_2}{v_1}E(u)E\left(\frac{1}{v}\right) = \frac{v_2}{v_1} \cdot v_1 \cdot \frac{1}{v_2 - 2} = \frac{v_2}{v_2 - 2}$ QED

8.39 $$\left(1 + \frac{v_1}{v_2}F\right)^{-(1/2)(v_1+v_2)} = \left(1 + \frac{v_1}{v_2}F\right)^{[(v_2/v_1F)(-v_1F/2)-(1/2)v_1]}$$

$$\to e^{-v_1F/2} \therefore g(F) \to k\ F^{[(v_1/2)-1]} e^{-v_1F/2}$$

$$f(v_1F) = kF^{[(v_1/2)-1]}\ e^{-(1/2)F} \to x^2(v_1)$$

8.40 T defined as $T = \dfrac{Z}{\sqrt{Y/v}}$ in Theorem 8.12 where $Z + Y$ are independent.

$T^2 = \dfrac{Z^2}{Y/v}$ where $Z^2 = \chi^2(1)$ by Theorem 8.7 $\quad Y = \chi^2(v) \quad$ QED

8.41 $F = \dfrac{u/v_1}{v/v_2}$ in Theorem 8.14 $\quad \left.\begin{array}{l} U \text{ is } \chi^2(v_1) \\ V \text{ is } \chi^2(v_1) \end{array}\right\}$ independent

$\dfrac{1}{F} = \dfrac{v(v_1)}{u(v)}$ is ratio of 2 chi square random variables with v_2 and v_1 degrees of freedom

So $\dfrac{1}{F}$ has F distribution with v_2 and v_1 degrees of freedom.

8.42 $x \rightarrow F(v_1, v_2)$

$y \rightarrow F(v_1, v_2)$ by Exercise 8.41

$$P(x \geq F_{\alpha,v_1,v_2}) = \alpha$$

$$P(\frac{1}{Y} \geq F_{\alpha,v_1,v_2}) = \alpha$$

$$P\left(Y \leq \frac{1}{F_{\alpha,v_1,v_2}}\right) = \alpha$$

$$P\left(Y \leq F_{1-\alpha,v_2,v_1}\right) = \alpha \qquad \therefore F_{1-\alpha,v_2,v_1} = \frac{1}{F_{\alpha,v_1,v_2}}$$

8.43
$$f(y) = \frac{\Gamma\left(\frac{v_1}{2} + \frac{v_2}{2}\right)}{\Gamma\left(\frac{v_1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)} y^{(v_1/2)-1}(1-y)^{(v_2/2)-1}$$

$$x = \frac{v_2 y}{v_1(1-y)} \rightarrow y = \frac{v_1 x}{v_2 + v_1 x} \rightarrow \frac{dy}{dx} = \frac{v_2 v_1}{(v_2 + x v_1)^2}$$

$$g(x) = \frac{\Gamma\left(\frac{v_1}{2} + \frac{v_2}{2}\right)}{\Gamma\left(\frac{v_1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)} \cdot \left(\frac{v_1 x}{v_2 + v_1 x}\right)^{(v_1/2)} - 1\left(\frac{v_2}{v_2 + v_1 x}\right)^{(v_1/2)-1} \cdot \left(\frac{v_2 v_1}{x v_2 + x v_1}\right)^2$$

$$= \frac{\Gamma\left(\frac{v_1 + v_2}{2}\right)}{\Gamma\left(\frac{v_1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)} \left(v_1^{v_1/2} v_2^{v_2/2} x^{(v_1/2)-1}\right) \cdot \frac{1}{(v_2 + v_1 x)^{(1/2)(v_1+v_2)}}$$

$$g(x) = \frac{\Gamma\left(\frac{v_1 + v_2}{2}\right)}{\Gamma\left(\frac{v_1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)} \left(\frac{v_1}{v_2}\right)^{v_1/2} x^{(v_1/2)-1}\left(1 + \frac{v_1}{v_2} x\right)^{-(1/2)(v_1+v_2)} \qquad \text{QED}$$

8.44 Substituting into formula of Theorem 8.14 yields

$$g(F) = \frac{\Gamma(4)}{\Gamma(2)\Gamma(2)} \cdot F(1+F)^{-4} \cdot \frac{6F}{(1+F)^4}$$

Since $\frac{1}{F}$ has same distribution as F by Ex. 8.41

$$\text{probability} = 2\int_2^\infty \frac{6F}{(1+F)^4} dF \quad \text{let } u = 1+F \qquad du = dF$$

$$= 2\int_3^\infty \frac{6(u-1)}{u^4} du = \frac{14}{27}$$

8.45 $$g_1(y_1) = n\frac{1}{8}e^{-y_1/\theta}\left[\int_{y_1}^\infty \frac{1}{\theta}e^{-x/\theta}dx\right]^{n-1} = \frac{n}{\theta}e^{-y_1/\theta}\left[e^{-y_1/\theta}\right]^{n-1}$$

$$= \frac{n}{\theta}e^{-y_1 n/\theta} \qquad \text{for } y_1 > 0 \text{ and } g_1(y_1) = 0 \text{ elsewhere}$$

$$g_1(y_n) = n\frac{1}{\theta}e^{-y_n/\theta}\left[\int_0^{y_1} \frac{1}{\theta}e^{-x/\theta}dx\right]^{n-1} = \frac{n}{\theta}e^{-y_n/\theta}\left[1-e^{y_n/\theta}\right]^{n-1}$$

for $y_n > 0$ and $g_n(y_n) = 0$ elsewhere

$$h(\bar{x}) = \frac{(2m+1)!}{m!\ m!}\left[\int_0^{\bar{x}} \frac{1}{\theta}e^{-x/\theta}dx\right]^m \frac{1}{\theta}e^{-\bar{x}/\theta}\left[\int_{\bar{x}}^\infty \frac{1}{\theta}e^{-x/\theta}dx\right]^m$$

$$= \frac{(2m+1)!}{m!\ m!}\left[1-e^{-\bar{x}/\theta}\right]^m \frac{1}{\theta}e^{-\bar{x}/\theta}\left[e^{-\bar{x}/\theta}\right]^m$$

$$= \frac{(2m+1)!}{m!\ m!\ \theta}e^{-\bar{x}(m+1)/\theta}\left[1-e^{-x/\bar{\theta}}\right]^m \quad \text{for } \bar{x} > 0 \text{ and } h(\bar{x}) = 0 \text{ elsewhere}$$

8.46 $$g_1(y_1) = n \cdot 1 \cdot \left[\int_{y_1}^1 dx\right]^{n-1} = n(1-y_1)^{n-1} \text{ for } 0 < y_1 < 1 \quad g_1(y_1) = 0 \text{ elsewhere}$$

$$g_n(y_n) = n \cdot 1 \cdot \left[\int_0^{y_n} dx\right]^{n-1} = ny_n^{n-1} \text{ for } 0 < y_n < 1 \quad g_n(y_n) = 0 \text{ elsewhere}$$

8.47 $$h(\bar{x}) = \frac{(2m+1)!}{m!\ m!}\left[\int_0^{\bar{x}} dx\right]^m \cdot 1 \cdot \left[\int_{\bar{x}}^1 dx\right]^m = \frac{(2m+1)!}{m!\ m!}\bar{x}(1-\bar{x})^m$$

for $0 < \bar{x} < 1 \quad h(\bar{x}) = 0$ elsewhere

8.48 $E(y_1) = n\int_0^1 y_1(1-y_1)^{n-1}dy_1 \qquad \text{let } u = 1 - y_1$

$$= n\int_0^1 (1-u)u^{n-1}du = n\left[\frac{1}{n} - \frac{1}{n+1}\right] = 1 - \frac{n}{n+1} = \frac{1}{n+1}$$

$$E(y_1^2) = n\int_0^1 y_1^2(1-y_2)^{n-1}dy_1 \qquad u = 1 - y$$

$$= n\int_0^1 (1-u)^2u^{n-1}du = n\left[\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right] = \left[\frac{2}{(n+1)(n+2)}\right]$$

$$\text{var}(y_2) = \frac{2}{(n+1)(n+2)} - \left(\frac{1}{n+1}\right)^2 = \frac{n}{(n+1)(n+2)}$$

8.49 $g_1(y_1) = n\ 12y_1^2(1-y_1)\left[12\int_{y_1}^1 x^2(1-x)\ dx\right]^{n-1}$

$$= 12\ ny_1^2(1-y_1)\left[1-4y_1^3+3y_1^4\right]^{n-1} \quad \text{for } 0 < y_1 < 1 \quad g_1(y_1) = 0 \text{ elsewhere}$$

$$g_n(y_n) = n\ 12y_n^2(1-y_n)\left[12\int_0^{y_n} x^2(1-x)\ dx\right]^{n-1}$$

$$= 12\ ny_n^2(1-y_n)y_n^{3(n-1)}(4-3y_n)^{n-1}$$

$$= 12\ ny_n^{3n-1}(1-y_n)(4-3y_n)^{n-1} \quad \text{for } 0 < y_n < 1 \quad g_n(y_n) = 0 \text{ elsewhere}$$

8.50 $h(\bar{x}) = \frac{(2m+1)!}{m!\ m!}\left[12\int_0^{\bar{x}} x^2(1-x)\ dx\right]^m 12\bar{x}^2(1-\bar{x})\left[12\int_{\bar{x}}^1 x^2(1-x)\ dx\right]^m$

$$= \frac{12(2m+1)!}{m!\ m!}\bar{x}^{3m+2}(1-\bar{x})[4-3\bar{x}]^m[1-4\bar{x}^3+3\bar{x}^4]^m$$

$h(\bar{x}) = 0$ elsewhere

8.51 **(a)**

1 and 2
1 and 3
1 and 4
1 and 5
2 and 3
2 and 4
2 and 5
3 and 4
3 and 5
4 and 5

y_2	$g_1(y_1)$
1	4/10
2	3/10
3	2/10
4	1/10

(b)

11	31	51
12	32	52
13	33	53
14	34	54
15	35	55
21	41	
22	42	
23	43	
24	44	
25	45	

y_1	$g_1(y_1)$
1	9/25
2	7/25
3	5/25
4	3/25
5	1/25

8.52 **(a)**
$$g(y_1,\, y_n) = n(n-1)\frac{1}{\theta^2}e^{-y/\theta}e^{-y_n/\theta}\left[\int_{y_1}^{y_n}\frac{1}{\theta}e^{-x/\theta}\,dx\right]^{n-2}$$
$$= \frac{n(n-1)}{\theta^2}e^{-(1/\theta)(y_1+y_n)}\left[e^{-y_1/\theta} - e^{-y_n/\theta}\right]^{n-2} \quad \text{for } 0 < y_1 < y_n < \infty$$
$$g(y_1,\, y_n) = 0 \text{ elsewhere}$$

(b)
$$g(y_1,\, y_n) = n(n-1)\left[\int_{y_1}^{y_n} dx\right]^{n-2}$$
$$= n(n-1)(y_n - y_1)^{n-2} \quad \text{for } 0 < y_2 < y_n < 1$$
$$g(y_1,\, y_n) = 0 \text{ elsewhere}$$

8.53 From 8.48 $E(y_1) = n\int_0^1 y_1(1-y_1)^{n-1}dy_1 = \dfrac{1}{n+1}$

and $E(Y_n) = n\int_0^1 y_n^n\; dy_n = \dfrac{n}{n+1}$

$$E(Y_1,\, Y_n) = n(n-1)\int_0^1\int_0^{y_n} y_1 y_n (y_n - y_1)^{n-2}dy_1\; dy_n = \frac{1}{n+2}$$

$$\text{cov}(Y_1,\, Y_2) = \frac{1}{n+2} - \frac{1}{n+1}\frac{n}{n+1} = \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)^2} = \frac{1}{(n+1)^2(n+2)}$$

8.54 $h(y_1,\, R) = n(n-1)f(y_1)f(y_1+R)\left[\displaystyle\int_{y_1}^{y_1+R} f(x)\; dx\right]^{n-2}$

Let $y_n = y_1 + R$

and transform holding y_1 fixed. $\quad \dfrac{dR}{dy_n} = 1$

8.55
$$h(y_1,\, R) = n(n-1)\frac{1}{\theta^2}e^{-y_1/\theta}e^{-(y_1+R)/\theta}\left[\int_{y_1}^{y_1+R}\frac{1}{\theta}\mathrm{e}^{-x/\theta}\; dx\right]^{n-2}$$
$$= \frac{n(n-1)}{\theta^2}e^{-y_1(n-1)/\theta}e^{-(y_1+R)/\theta}\left[1 - e^{-R/\theta}\right]^{n-2}$$
$$= \frac{n(n-1)}{\theta^2}e^{-y_1 n/\theta}e^{-R/\theta}\left[1 - e^{-R/\theta}\right]^{n-2}$$
$$= \underbrace{\frac{n}{\theta}e^{-y_2 n/\theta}}_{g(y_1)} \cdot \underbrace{\frac{n-1}{\theta}e^{-R/\theta}\left[1 - e^{-R/\theta}\right]^{n-2}}_{f(R)} \qquad \text{independent}$$

$$f(R) = \frac{n-1}{\theta}e^{-R/\theta}\left[1 - e^{-R/\theta}\right]^{n-2} \quad \text{for } R > 0$$
$$g(R) = 0 \text{ elsewhere}$$

8.56 $h(y_1,\,R) = n(n-1)\left[\int_{y_1}^{y_1+R} dx\right]^{n-2} = n(n-1)R^{n-2} \qquad 0 < R < 1 - y_1 < 1$

and 0 elsewhere

$$g(R) = n(n-1)R^{n-2}\int_0^{1-R} dy = n(n-1)R^{n-2}(1-R) \qquad 0 < R < 1; \ = 0 \text{ elsewhere}$$

8.57 $E(R) = n(n-1)\int_0^1 R^{n-1}(1-R)\,dR = n(n-1)\cdot\dfrac{1}{n(n+1)} = \dfrac{n-1}{n+1}$

$$E(R^2) = n(n-1)\int_0^1 R^{n}(1-R)\,dR = n(n-1)\cdot\frac{1}{(n+1)(n+2)} = \frac{n(n-1)}{(n+1)(n+2)}$$

$$\sigma^2 = \frac{n(n-1)}{(n+1)(n+2)} - \frac{(n-1)^2}{(n+1)^2} = \frac{n(n-1)(n+1)-(n+2)(n-1)^2}{(n+1)^2(n+2)} = \frac{2(n-}{(n+1)^2(n+2)}$$

8.58 **(a)** $p = \int_{y_1}^{y_n} f(x)\,dx \qquad \dfrac{dp}{dy_n} = f(y_n)$

$$h(y_1,\,p) = n(n-1)f(y_1)f(y_n)p^{n-2}\frac{1}{f(y_1)} = n(n-1)f(y_2)p^{n-2}$$

(b) $w = \int_{-\infty}^{y_1} f(x)\,dx \qquad \dfrac{dw}{dy_1} = f(y_1)$

$$\phi(w,\,p) = n(n-1)f(y_2)p^{n-2}\frac{1}{f(y_2)} = n(n-1)p^{n-2}$$

$$w > 0,\ p > 0,\ w + p < 1$$

$$\phi(w,\,p) = 0 \text{ elsewhere}$$

(c) $g(p) = \int_0^{1-p} n(n-1)p^{n-2}\,dw = n(n-1)p^{n-2}(1-p) \qquad 0 < p < 1 \qquad g(p) = 0$ elsewhere

8.59 Density of P is same density as R obtained in Exercise 8.56, so the formula for the mean and the variance are the same as those obtained in Exercise 8.57. When n is large $E(p) \to 1$ and $\text{var}(p) \to 0$.

8.60 **(a)** $\binom{12}{3} = \frac{12 \cdot 11 \cdot 10}{6} = 220$

(b) $\binom{20}{3} = \frac{20 \cdot 19 \cdot 18}{6} = 1140$

(c) $\binom{50}{3} = \frac{50 \cdot 49 \cdot 48}{6} = 19{,}600$

8.61 **(a)** $\frac{1}{\binom{12}{4}} = \frac{1}{495}$ **(b)** $\frac{1}{\binom{22}{5}} = \frac{120}{12 \cdot 21 \cdot 20} = \frac{1}{77}$

8.62 $\frac{\binom{49}{2}}{\binom{50}{3}} = \frac{49!}{2!\ 47!} \cdot \frac{47!\ 3!}{50!} = \frac{3}{50} = 0.06$

8.63 **(a)** It is divided by 2 $\sqrt{120/30} = 2$
(b) It is divided by 1.5 $\sqrt{180/80} = 1.5$
(c) It is multiplied by 3 $\sqrt{450/50} = 3$
(d) It is multiplied by 2.5 $\sqrt{250/40} = 2.5$

8.64 **(a)** $\frac{200-5}{200-1} = 0.9799$; **(b)** $\frac{300-50}{300-1} = 0.8361$; **(c)** $\frac{800-200}{800-1} = 0.7509$

8.65 **(a)** $n = 100,\ \mu = 75,\ \sigma = 16,\ \therefore \sigma_{\bar{x}} = \frac{16}{\sqrt{100}} = 1.6$

$$P\left(\left|\bar{X} - 75\right| < 5 \cdot 1.6\right) \geq 1 - \frac{1}{5^2} = \frac{24}{25} = 0.96$$

(b $Z_1 = \frac{67-75}{\frac{16}{\sqrt{100}}} = -5$; $Z_2 = \frac{83-75}{\frac{16}{\sqrt{100}}} = 5$

From Table III, $P(67 < \bar{x} < 83) = 2 \cdot 0.4999997 = 0.9999994$

8.66 $\sigma_{\bar{x}} = \frac{6.3}{9} = 0.7$ $\frac{129.4-128}{0.7} = 2$

(a) Probability is at most $\frac{1}{4}$

(b) $1 - 2(0.4772) = 1 - 0.9544 = 0.0455$

8.67 $\sigma_{\bar{x}} = 0.7\sqrt{\frac{400-81}{400-1}} = 0.7(0.8941) = 0.626$ $z = \frac{1.4}{0.626} = 2.24$

$1 - 2(0.4875) = 0.025$

8.70 $\sigma_{\bar{x}} = \dfrac{6.8}{8} = 0.85$

(a) $z = \dfrac{52.9 - 51.4}{0.85} = 1.765$

$0.5 - 0.4612 = 0.0388$

(b) $\dfrac{52.3 - 51.4}{0.85} = 1.06$

$\dfrac{50.5 - 51.4}{0.85} = -1.05$

$2(0.3554) = 0.7108$

(c) $\dfrac{50.6 - 51.4}{0.85} = -0.94$

$0.5 - 0.3264 = 0.1736$

8.71 $\sigma_{\bar{x}} = \dfrac{25}{\sqrt{100}} = 2.5$ $\qquad z = \dfrac{3}{2.5} = 1.2$

$1 - 2(0.3849) = 1 - 0.7698 = 0.2302$

8.72 $\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\dfrac{20^2}{400} + \dfrac{30^2}{400}} = \sqrt{1 + 2.25} = 1.803$ $\qquad k = 10$

$k\sigma = 18.03$ The value of $\bar{x}_1 - \bar{x}_2$ will fall between -18.03 and 18.03.

8.73 $z = 2.57$ $\qquad k = 2.57(1.803) = 4.63$

8.74 $\mu_{\bar{x}_1 - \bar{x}_2} = 78 - 75 = 3$ $\qquad \sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\dfrac{150}{30} + \dfrac{200}{50}} = 3$

$z = \dfrac{4.8 - 3}{3} = 0.6$ $\qquad 0.5 - 0.2257 = 0.2743$

8.75 $E(\hat{\theta}) = 0.70$ $\qquad \text{var}(\hat{\theta}) = \dfrac{0.70(0.30)}{84} = 0.0025$ $\qquad \sigma = 0.05$

(a) $k = \dfrac{0.06}{0.05} = 1.2$

Probability is at least $1 - \dfrac{1}{1.2^2} = 0.3056$

(b) $2(0.3849) = 0.7698$

8.76 $1-\frac{1}{k^2}=1-0.9375=0.0625,\ k=4$

$$\sigma=\sqrt{\frac{(0.4)(0.6)}{500}+\frac{(0.25)(0.75)}{400}}=\sqrt{0.00048+0.00047}=0.0308$$

It will fail between $0.40-0.25\pm k\sigma=0.15\pm4(0.0308)=0.15\pm0.1232$
0.0268 and 0.2732

8.77 $n=5 \qquad \sigma^2=25 \qquad y=\frac{4s^2}{25}\to\chi^2(4)$

$f(y)=\frac{1}{4}y\,e^{-y/2} \qquad s^2=20 \qquad y=\frac{80}{25}=\frac{16}{5}=3.2$

$s^2=30 \qquad y=\frac{120}{25}=\frac{24}{5}=4.8$

$$\text{probability }=\frac{1}{4}\int_{3.2}^{4.8}y\,e^{-y/2}\,dy=\left[-e^{-y/2}\left(\frac{1}{2}y+1\right)\right]\Bigg|_{3.2}^{4.8}$$

$$=-3.4e^{-2.4}+2.6e^{-1.6}=-3.4(0.091)+2.6(0.202)=0.216$$

8.78 $n=16 \qquad \sigma^2=25 \qquad y=\frac{15s^2}{25}=0.6s^2$

has chi-square distribution with 15 degrees of freedom
probability $\left[y\ge0.6(54.668)\right]=P(y\ge32.801)=0.005$
probability $\left[y\le0.6(12.102)\right]=P(y\le7.2612)=0.05$
total probability = 0.055

8.79 $\sigma^2=4 \qquad n=9 \qquad y=\frac{8s^2}{4}=2s^2$

probability $\left[y\ge2(7.7535)\right]=P(y\ge15.507)$ 8 degrees of freedom
$=0.5$ (Table V)

8.80 $t=\frac{47-42}{7/\sqrt{25}}=3.57$ Since 3.57 exceeds $t_{0.005,\,24}=2.797$ for $v=24$,
result is highly unlikely and conjecture is probably false.

8.81 $t=\frac{27.8-28.5}{1.8/\sqrt{12}}=-\frac{0.7}{1.8/3.464}=-1.347$

Since this value is fairly small (close to $-t_{0.10,\,11}$) the data tend to support the claim.

8.82 $F = \dfrac{s_1^2/12}{s_2^2/18} = 1.5\dfrac{s_1^2}{s_2^2}$

$$P\left(\frac{s_1^2}{s_2^2} > 1.16\right) = P\left[1.5\frac{s_1^2}{s_2^2} > (1.16)(1.5)\right] = P(F > 1.74)$$

for 60, 30 degrees of freedom

From Table V $F_{0.05,\ 60,\ 30} = 1.74$ So probability is 0.05.

8.83 $F = \dfrac{s_1^2}{s_2^2}$

$P\left(\dfrac{s_1^2}{s_2^2} < 4.03\right) = 1 - P(F > 4.03)$ with 9 and 14 degrees of freedom

From Table VI $F_{0.01,\ 9,\ 14} = 4.03$

So probability = $1 - 0.01 = 0.99$

8.84 Giving the MINITAB commands

```
MTB> CDF 1.363;
SUBC> T 11.
```

We obtain 0.8999, which verifies that $t_{1,\ 11} = 1 - 0.8999 = 0.1001$ The remaining four values also can be verified to within an error of at most 0.0001.

8.85 Following the procedure of Exercise 8.84, but using 21 in place of 11, we verify all five table entries to four decimal places.

8.86 Using the MINITAB commands

```
MTB> CDF 4.21;
SUBC> F 7 6.
```

We obtain 0.9501, verifying the entry in Table V to within an error of 0.0001. The remaining entries are similarly verified to within an error of at most 0.2

8.87 Following the procedure of Exercise 8.86, but using

```
SUBC> F 12 17
```

We obtain 0.9900. The remaining entries are similarly verified

8.88 From 8.46 $g_1(y_1) = n(1 - y_1)^{n-1}$ $y < y_1 < 1$

probability = $n\displaystyle\int_{0.2}^{1} (1 - y_1)^{n-1} dy_1 = (1 - y_1)^n \Big|_{1}^{0.2}$

$= (0.8)^4 = 0.4096$

8.89 $g(y_n) = 36y_n^2(1-y_n)(4-y_n^3-3y_n)^2$ for $n = 3$

$$= 36[16y^8 - 40y^9 + 33y^{10} - 9y^{11}]$$

$$\text{probability} = \int_0^{0.9} g(y_n)dy_n = 0.851$$

8.90 $g(R) = 20R^2(1-R)$ for $0 < R < 1$

$$\text{probability} = 20\int_{0.75}^{1} (R^3 - R^4)\, dR = (5R^4 - 4R^5)\Big|_{0.75}^{1} = 0.3672$$

8.91 $g(p) = n(n-1)p^2(1-p)$ $0 < p < 1$

$$= 90p^3(1-p)$$

$$\text{probability} = 90\int_{0.8}^{1} p^8(1-p)\, dp = (10p^9 - 9p^{10})\Big|_{0.8}^{1}$$

$$= 1 - 1.3422 + 0.9664 = 0.6242$$

8.92 $g(p) = n(n-1)p^{n-2}(1-p)$

$$\alpha = n(n-1)\int_0^p p^{n-2}(1-p)\, dp = np^{n-1} - (n-1)p^n = p^{n-1}\left[n-(n-1)p\right]$$

$$p^{n-1} = \frac{\alpha}{n-(n-1)p}$$

for $\alpha = 0.05$ and $p = 0.90$ $\qquad (0.90)^{n-1} = \dfrac{0.05}{n-(n-1)0.9} = \dfrac{1}{2n+18}$

$$n = \frac{1}{2} + \frac{1}{4}\cdot\frac{1.90}{0.10}\cdot 9.488 = 0.5 + 45.068 = 45.568 \text{ rounded up to } n = 46$$

8.93 The top cans have less pressure on them and may be less prone to damage.

8.94 **(a)** The sample, without the "bad" parts, will make the lathe seem better than it is.
(b) The sample is representative of product produced by the lather after inspection.

8.95 The sample is more likely to include longer sections than shorter ones; they take more time to pass the inspection station.

8.96 A systematic sample (e.g. every so many millimeters) may produce results always near the top or bottom of a wave, over- or understating the oxide thickness. To avoid this kind of problem, it is best to choose the locations to sample at random.

Chapter 9

9.2 Let a_{ij} be element in *i*th row and *j*th column. Since saddle point is minimum of row and maximum of column

	j	l
i	a_{ij}	a_{il}
k	a_{kj}	a_{kl}

$a_{ij} \geq a_{kj} \geq a_{kl} \geq a_{il} \geq a_{ij}$

$\therefore$ must all be equal signs

$a_{ij} = a_{kj} = a_{kl} = a_{il}$ and both parts are proved

9.3 If we let $x = 0$ for *n* heads, $x = 1$ at least one tail
Only changes in risk functions are that
$R(d_1,\ \theta_2) = \dfrac{1}{2^n}$ and $R(d_4,\ \theta_2) = 1 - \dfrac{1}{2^n}$

dominance same as before

resulting risk functions given by

$$\begin{array}{c|c|c|}
 & d_1 & d_2 \\
\hline
\theta_1 & 0 & 1 \\
\hline
\theta_2 & 1/2^n & 0 \\
\hline
\end{array}$$

9.4

$$R(d_1,\ \theta) = \int_0^\theta c(kx-\theta)^2 1 \frac{1}{\theta} d\theta$$

$$= \frac{c}{\theta}\left[\frac{(kx-\theta)^3}{3k}\right]\Bigg|_0^c$$

$$= \frac{c}{\theta}\left[\frac{(k\theta-\theta)^3}{3k} - \frac{\theta^3}{3k}\right] = \frac{c\theta^2}{3}(k^3 - 3k + 3)$$

9.5 $p(x<k) = \int_0^k \dfrac{2x}{\theta^2} dx = \dfrac{k^2}{\theta_2}$

$$\begin{array}{c|c|c|}
 & \theta_1 & \theta_2 \\
\hline
\theta_1 & 0 & C \\
\hline
\theta_2 & C & 0 \\
\hline
\end{array}$$

Probability	
$\dfrac{k^2}{\theta_1^2}$	$1-\dfrac{k^2}{\theta_1^2}$
$\dfrac{k^2}{\theta_2^2}$	$1-\dfrac{k^2}{\theta_2^2}$

$$R(d,\ \theta_1) = C\left(1-\frac{k^2}{\theta_1^2}\right),\ R\left(d,\ \theta_2\right) = C\cdot\frac{k^2}{\theta_2^2}$$

For mimimax solution $C\left(1-\dfrac{k^2}{\theta_1^2}\right) = C\cdot\dfrac{k^2}{\theta_2^2}$ $\qquad k = \dfrac{\theta_1\theta_2}{\sqrt{\theta_1^2+\theta_2^2}}$

9.6 Maximizing $R(d,\ \theta)$ with respect to θ yields

$$\theta = \frac{2ab-n}{2(b^2-n)}$$

Substituting this value into $R(d,\ \theta)$ and differentiating partially with respect to a and b yields $a = \frac{1}{2}\sqrt{n}$ and $b = \sqrt{n}$.

9.7 $E(\Theta) = \int_0^1 x\ dx = \frac{1}{2},\ E(\Theta^2) = \int_0^1 x^2\ dx = \frac{1}{3}$

Substituting into $R(d,\ \Theta)$ yields

Bayes Risk = $\frac{c}{(n+b)^2}\left[\frac{1}{3}(b^2-n)+\frac{1}{2}(n-2ab)+a^2\right]$

Differentiating partially with respect to a and equating to 0 yields $a = \frac{b}{2}$. Substituting $a = \frac{b}{2}$ into Bayes risk and differentiating with respect to b yields b= 2. So a = 1 and $d(x) = \frac{x+1}{n+2}$.

9.8 $g(x) = \int_x^{\infty} e^{-\theta} d\,\theta = e^{-x}$ for $x > 0$

$g(x) = 0$ elsewhere

$\phi(\theta|x) = \frac{f(x,\ \theta)}{g(x)} = \frac{e^{-\theta}}{e^{-x}} = e^{x-\theta}$ for $\theta > x$

$\phi(\theta|x) = 0$ elsewhere

9.9 **(a)** $g(x,\ \theta) = \theta(1-\theta)^{x-1}$ $x = 1,\ 2,\ 3,\ ...$

$f(x,\ \theta) = \theta(1-\theta)^{x-1}\cdot 1$ $x = 1,\ 2,\ 3,\ ...$ $0 < \theta < 1$

Beta distribution with $a = 2,\ \beta = x$

$$g(x) = \int_0^1 \theta(1-\theta)^{x-1} d\theta = \frac{\Gamma(2)\Gamma(x)}{\Gamma(x+2)} = \frac{1}{x(x+1)}$$

$\phi(\theta|x) = \frac{\theta(1-\theta)^{x-1}}{1/x(x+1)} = x(x+1)\theta(1-\theta)^{x-1}$ $0 < \theta < 1$

$\phi(\theta|x) = 0$ elsewhere

(b) $$\sum_{x=1}^{\infty}\left\{\int_0^1 c\left[d(x)-\theta\right]^2 \theta(1-\theta)^{x-1} x(x+1)\, d\theta\right\}$$

$$c\int_0^1 2\left[d(x)-\theta\right]\theta(1-\theta)^{x-1} x(x+1) d\theta$$

$$2cx(x+1)\int_0^1\left[d(x)-\theta\right]\theta(1-\theta)^{x-1} d\theta = 0$$

$$d(x)\int_0^1 \theta(1-\theta)^{x-1} d\theta = \int_0^1 \theta^2(1-\theta)^{x-1} d\theta$$

$$d(x)\cdot\frac{1}{x(x+1)} = \frac{\Gamma(3)\Gamma(x)}{\Gamma(x+3)} = \frac{2(x+1)!}{(x+2)!} = \frac{2}{(x+2)(x+1)x}$$

$$d(x) = \frac{2}{x+2}$$

9.10

	expand	wait		
Good times	−164,000	−80,000	0.4	4/11
Recession	40,000	−8,000	0.6	7/11

(a) $E = (0.4)(-164,000) + (0.6)(40,000) = -41,600$

$E = (0.4)(-80,000) + (0.6)(-8,000) = -36,800$

Manufacturer should expand now.

(b) $E = \frac{4}{11}(-164,000) + \frac{7}{11}(40,000) = -34,182$

$E = \frac{4}{11}(-80,000) + \frac{7}{11}(-8,000) = -34,182$

Does not matter.

9.11 **(a)**

	expand	wait	
Good times	−200,000	−80,000	1/3
Recession	40,000	−8,000	2/3

$E = \frac{1}{3}(-200,000) + \frac{2}{3}(40,000) = -40,000$

$E = \frac{1}{3}(-80,000) + \frac{2}{3}(-8,000) = -32,000$

Manufacturer should expand now. Decision reversed.

(b)

	expand	wait	
good times	−164,000	−80,000	2/5
recession	60,000	−8,000	3/5

$$E = \frac{2}{5}(-164{,}000) + \frac{3}{5}(60{,}000) = -29{,}600$$

$$E = \frac{2}{5}(-80{,}000) + \frac{3}{5}(-8{,}000) = -36{,}800$$

Manufacturer should expand now. Decision reversed.

9.12

	Reservation at			
	x	Y	(a)	(b)
x	65	68.40	3/4	5/6
Y	72	62.40	1/4	1/6

(a) $EC = \frac{3}{4}(66) + \frac{1}{4}(72) = 67.50$

$EC = \frac{3}{4}(68.40) + \frac{1}{4}(62.40) = 66.90$　Make reservation at Hotel Y.

(b) $EC = \frac{5}{6}(66) + \frac{1}{6}(72) = 67$

$EC = \frac{5}{6}(68.40) + \frac{1}{6}(62.40) = 67.40$　Make reservation at Hotel x

9.13

	go to				
27	27	33	(a)	(b)	(c)
should go to	27	45	1/6	1/3	1/4
33	39	33	5/6	2/3	3/4

(a) $ED = \frac{1}{6}(27) + \frac{5}{6}(39) = 37$

$ED = \frac{1}{6}(45) + \frac{5}{6}(33) = 35$　Should go to site 33 miles from lumberyard.

(b) $ED = \frac{1}{3}(27) + \frac{2}{3}(39) = 35$

$ED = \frac{1}{3}(45) + \frac{2}{3}(33) = 37$　Should go to site 27 miles from lumberyard.

(c) $ED = \frac{1}{4}(27) + \frac{3}{4}(39) = 36$　Does not matter.

$ED = \frac{1}{4}(45) + \frac{3}{4}(33) = 36$

9.14 **(a)** If he goes to x worst cost is 72.00; if he goes to Y worst cost is 68.40. Worst cost is minimized if he chooses Y.

(b) If he goes to (27) worst distance is 39; if he goes to (33) worst distance is 45; worst distance is least if he goes to site 27 miles from lumberyard.

9.15 **(a)** If he expands now, maximum gain is 164,000; if he waits maximum gain is 80,000. Maximum gain is maximized if he expands now.

(b) If she chooses x, minimum cost is 66; if she chooses Y minimum cost is 62.40; minimum cost is minimized if she chooses Y.

(c) If he goes to (27), minimum distance is 27; if he goes to (33) minimum distance is 33; minimum distance is minimized if he goes to site 27 miles from lumberyard.

9.16 **(a)** opportunity losses are

0	84,000
48,000	0

(b) Maximum opportunity losses are 48,000 and 84,000; these are minimized if he expands now.

9.17 **(a)** opportunity losses are

0	2.40
9.60	0

Maximum opportunity losses are 9.60 and 2.40; they are minimized if she chooses Hotel Y.

(b) opportunity losses are

0	18
6	0

Maximum opportunity losses are 6 and 18; they are minimized if he chooses to go to site 27 miles from lumberyard.

9.18 Expected losses with perfect information = $\frac{1}{3}(-164{,}000)+\frac{2}{3}(-8{,}000)=-60{,}000$

60,000 exceeds 28,000 and 32,000 by more than 15,000
Hiring the forecaster is worthwhile.

9.19 **(a)** Cross out first row, cross out second column, optimum strategies I and 2; value = 5
(b) Cross out first column, cross out second row, optimum strategies II and 1; value = 11
(c) Cross out third column, cross out second row, cross out second column, cross out third row, optimum strategies I and 1; value = -5.
(d) Cross out third column, cross out third row, cross out second column, cross out first row, optimum strategies I and 2; value = 8.

9.20 **(a)** Mimima of rows are $-2,\ 0,\ -4$; only second is largest of its column. Saddle point corresponds to I and 2; value = 0.

(b) Mimima of rows are 2, 3, 5, and 5; first two are not maxima of their columns; others are. Saddle point corresponds to I and 3; I and 4, III and 3, III and 4; value = 5 in each case.

9.21 **(a)**

	no glasses	glasses
no knives	0	−6
knives	8	3

(b) Minimum of second row is maximum of second column saddle point. Optimum strategies are for Station A to give away glasses and Station B to give away knives.

9.22

p	8	−5
$1-p$	2	6

$$8p+2(1-p)=-5p+6(1-p)$$

$$8p+2-2p=-5p+6-6p$$

$$17p=4 \qquad p=\frac{4}{17}$$

probabilities are $\frac{4}{17}$ and $\frac{13}{17}$

9.23

	x	$1-x$
y	3	−4
$1-y$	−3	1

(a) $3x-4(1-x)=-3x+(1-x)$

$$11x=5 \qquad x=\frac{5}{11}$$

probabilities are $\frac{5}{11}$ and $\frac{6}{11}$

(b) $3y-3(1-y)=-4y+(1-y)$

$11y-4$ probabilities are $\frac{4}{11}$ and $\frac{7}{11}$

(c) $3\cdot\frac{4}{11}-3\cdot\frac{7}{11}=-\frac{9}{11}$

9.24

x	$1-x$
66	68.40
72	62.40

$$66x+68.40(1-x)=72x+62.40(1-x)$$

$$6(1-x)=6x \qquad 1-x=x \qquad x=\frac{1}{2}$$

probabilities are $\frac{1}{2}$ and $\frac{1}{2}$

9.25

		enemy attacks	
		y 2	$1-y$ 10
country defends	x 2	12	2
	$1-x$ 10	10	12

$12x+10(1-x)=2x+12(1-x)$

$12x=2 \qquad x=\frac{1}{6}$

for defends $\frac{1}{6}$ and $\frac{5}{6}$

$12y+2(1-y)=10y+12(1-y)$

$12y=10 \qquad y=\frac{5}{6} \qquad$ for enemy $\frac{5}{6}$ and $\frac{1}{6}$

value is $12\cdot\frac{5}{6}+2\cdot\frac{1}{6}=10\frac{1}{3}$ which is \$10,333,333

9.26 **(a)**

		first person	
		1	4
second	0	−1	2
	3	2	−7

(b) $-x+2(1-x)=2x-7(1-x)$

$12x=9 \qquad x=\frac{3}{4} \qquad \frac{3}{4}$ and $\frac{1}{4}$

(c) $-y+2(1-y)=2y-7(1-y)$

$12y=9 \qquad y=\frac{3}{4} \qquad \frac{3}{4}$ and $\frac{1}{4}$

9.27

		first	
		lowers	not
second	lowers	\$80	\$70
	not	\$140	\$100

(a) Minima are \$80 and \$70. Maximized if he lowers prices.

(b) They might lower prices on alternate days.

$$\frac{140+70}{2}=105$$

9.28 **(a)**

	0	1/2	1
0	0	50	100
1/2	50	0	50
1	100	50	0

(b) $d_1(0)=0,\ d_1(1)=0;\ d_2(0)=0,\ d_2(1)=\frac{1}{2};\ d_3(0)=0,\ d_3(1)=1;$

$d_4(0)=\frac{1}{2},\ d_4(1)=0;\ d_5(0)=\frac{1}{2},\ d_5(1)=\frac{1}{2};\ d_6(0)=\frac{1}{2},\ d_6(1)=1;$

$d_7(0)=1,\ d_7(1)=0;\ d_8(0)=1,\ d_8(1)=\frac{1}{2};\ d_9(0)=1,\ d_9(1)=1$

(c) The risk functions are

	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9
0	0	0	0	50	50	50	100	100	100
1/2	50	25	50	25	0	25	50	25	50
1	100	50	0	100	50	0	100	50	0

d_1, d_4, d_7, and d_8 are eliminated by dominances; only d_2, d_3, d_5, d_6 are admissible and by inspection the maximum is 50 in each case. Accordingly by minimax criterion they area all equally good.

(d) Bayes risks are

$$d_2 \quad 0\cdot\frac{1}{3}+25\cdot\frac{1}{3}+50\cdot\frac{1}{3}=25$$

$$d_3 \quad 0\cdot\frac{1}{3}+50\cdot\frac{1}{3}+0\cdot\frac{1}{3}=16\frac{2}{3}$$

$$d_5 \quad 50\cdot\frac{1}{3}+0\cdot\frac{1}{3}+50\cdot\frac{1}{3}=33\frac{1}{3}$$

$$d_6 \quad 50\cdot\frac{1}{3}+25\cdot\frac{1}{3}+0\cdot\frac{1}{3}=25$$

Bayes risk is minimum for d_3.

9.29 **(a)**

	1/4	1/2
1/4	0	160
1/2	160	0

(b)

$$d_1(0)=\frac{1}{4},\ d_1(1)=\frac{1}{4};\ d_1(2)=\frac{1}{4},\ d_2(0)=\frac{1}{4};\ d_2(1)=\frac{1}{4},\ d_2(2)=\frac{1}{2};$$

$$d_3(0)=\frac{1}{4},\ d_3(1)=\frac{1}{2};\ d_3(2)=\frac{1}{4},\ d_4(0)=\frac{1}{4};\ d_4(1)=\frac{1}{2},\ d_4(2)=\frac{1}{2};$$

$$d_5(0)=\frac{1}{2},\ d_5(1)=\frac{1}{4};\ d_5(2)=\frac{1}{4},\ d_6(0)=\frac{1}{2};\ d_6(1)=\frac{1}{4},\ d_6(2)=\frac{1}{2};$$

$$d_7(0)=\frac{1}{2},\ d_7(1)=\frac{1}{2};\ d_7(2)=\frac{1}{4},\ d_8(0)=\frac{1}{2};\ d_8(1)=\frac{1}{2},\ d_8(2)=\frac{1}{2}$$

(c) The risk functions are

	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8
1/4	0	10	60	70	60	100	150	160
1/2	160	120	80	40	120	80	40	0

probabilities for $\theta=\frac{1}{4}$ are $\frac{9}{16}, \frac{6}{16}, \frac{1}{16}$

$\theta=\frac{1}{2}$ are $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$

d_7 is dominated by d_4, d_6 is dominated by d_3, d_5 is dominated by d_2 and d_3.

(d) The maxima corresponds to d_1, d_2, d_3, d_4, and d_8 are 160, 120, 80, 70, and 160. So the minimax criterion yields d_4.

(e) The five Bayes risks are $\frac{160}{3}, \frac{140}{3}, \frac{200}{3}, \frac{180}{3}, \frac{320}{3}$ so that d_2 is best.

9.30 **(a)**

		no inspection	inspect 1	inspect 2
	0	0	β	2β
repeat	1	$\alpha+2\beta+\phi$	$\frac{\alpha}{2}+2\beta+\phi$	$2\beta+\phi$
	2	$\alpha+2\beta+2\phi$	$2\beta+2\phi$	$2\beta+2\phi$

(b)

0	β	2β
$35+2\beta$	$22.50+2\beta$	$10+2\beta$
$45+2\beta$	$20+2\beta$	$20+2\beta$

maxima

$45+2\beta$ $\quad$ $22.50+2\beta$ $\quad$ $20+2\beta$ $\quad$ should inspect both

$\uparrow$

min

(c)

0	12	24
64	59	54
94	84	85

Bayes risks are $0(0.70)+64(0.20)+94(0.10)=22.2 \leftarrow$

$12(0.70)+59(0.20)+84(0.10)=28.6$

$24(0.70)+54(0.20)+84(0.10)=36.0$

Minimized if shipped without inspection

9.31 $\delta(\theta)=R(d_1,\theta)-R(d_2,\theta)=(1{,}000\theta-2{,}000)[B(1;10,\theta)-B(0;10,\theta)]$

As in the example, the first term always negative, and the second term is always positive; thus, $\delta(\theta)$ is always negative. As before, d_1 dominates d_2 and it is preferred.

9.32 $\delta(\theta)=(C_w \cdot n\theta-C_d)[B(2;n,\theta)-B(1;n;\theta)]$.

Since the second term of this product is always positive, d_2 will dominate d_1 when the first term is positive, that is, when $C_w n\theta > C_d$, as long as there is a value of $\theta \le 1$ that satisfies this inequality. Thus, strategy 2 will be preferable to strategy 1 whenever $\frac{C_4}{nC_w} < \theta \le 1$

Chapter 10

10.1 $E\left[\sum a_i x_i\right] = \sum a_i E(x_i) = \sum a_i \mu = \mu \sum a_i$

$$\therefore \sum_{i=1}^{n} a_i = 1$$

10.2 $E[k_1\hat{\theta}_1 + k_2\hat{\theta}_2] = k_1\theta + k_2\theta = \theta,\ \ k_1 + k_2 = 1$

10.3 $h(\tilde{x}) = \dfrac{(2m+1)!}{m!\ m!}\left[\displaystyle\int_{-\infty}^{\tilde{x}} f(x)\ dx\right]^m f(\tilde{x})\left[\displaystyle\int_{\bar{x}}^{\infty} f(x)\ dx\right]^m$

$$h(\tilde{x}) = \frac{(2m+1)!}{m!\ m!}\left[\int_{\theta-(1/2)}^{\tilde{x}} dx\right]^m \cdot 1 \cdot \left[\int_{\tilde{x}}^{\theta+(1/2)} dx\right]^m$$

$$= \frac{(2m+1)!}{m!m!}\left(\tilde{x} - \theta + \frac{1}{2}\right)^m \left(\theta + {}_{2} - \tilde{x}\right)^m \qquad\qquad m = 1$$

$$h(\tilde{x}) = 6(\left(\tilde{x} - \theta + \frac{1}{2}\right)\left(\theta + \frac{1}{2} - \tilde{x}\right)$$

$$E(\tilde{x}) = 6\int_{\theta-(1/2)}^{\theta+(1/2)} \tilde{x}\left(\tilde{x} - \theta + \frac{1}{2}\right)\left(\theta + \frac{1}{2} - \tilde{x}\right) d\tilde{x}$$

$$\text{let } u = \tilde{x} - \theta + \frac{1}{2}$$

$$= 6\int_0^1 \left(u + \theta - \frac{1}{2}\right) u(1-u)du = \theta$$

10.4 $h(\bar{x}) = \dfrac{6}{8} e^{-2\bar{x}/\theta}\left[1 - e^{-\bar{x}/\theta}\right]$

$$E[\bar{x}] = \frac{6}{\theta}\int_0^{\infty} \tilde{x}\ e^{-2\tilde{x}/\theta}\left[1 - e^{-\tilde{x}/\theta}\right]d\tilde{x}$$

$$= \frac{6}{\theta}\int_0^{\infty} \tilde{x}\ e^{-2\tilde{x}/\theta}d\tilde{x} - \frac{6}{\theta}\int_0^{\infty} \tilde{x}\ e^{-3\tilde{x}/\theta}d\tilde{x}$$

$$= \frac{5}{6}\theta \qquad \therefore \text{ biased}$$

Use gamma integrals.

10.5 $E\left[\frac{1}{n}\sum(x_i-\mu)^2\right]=\frac{1}{n}\left[\sum_{i=1}^{n}E\left[(x_i-\mu\right]^2\right]$

$$=\frac{1}{n}\sum_{i=1}^{n}\sigma^2=\frac{1}{n}\cdot n\sigma^2=\sigma^2$$

10.6 $E(\overline{x})=\mu \qquad \text{var}(\overline{x})=\frac{\sigma^2}{n}$

$$E(\overline{x}^2)=\frac{\sigma^2}{n}+\mu^2 \;\rightarrow\; \mu^2 \text{ as } n\rightarrow\infty$$

10.7 $E\left(\frac{x+1}{n+2}\right)=\frac{1}{n+2}E(x+1)=\frac{1}{n+2}(n\theta+1)=\frac{n}{n+2}\theta+\frac{1}{n+2}$

$E\rightarrow\theta$ when $n\rightarrow\infty$, so is asymptotically unbiased

10.8 $g_1(y_1)=n\;e^{-(y_1-\delta)}\left[\int_{y_1}^{\infty}e^{-(x-\delta)}\;dx\right]^{n-1}$

$$=n\;e^{-(y_1-\delta)}\cdot e^{-(n-1)(y_2-\delta)}$$

$$=n\;e^{-n(y_1-\delta)}$$

$$E(y_1)=n\int_{\delta}^{\infty}y_1\;e^{-n(y_1-\delta)}\;dy_1 \qquad \text{let } u=y_1-\delta$$

$$=n\int_{0}^{\infty}(u+\delta)e^{-nu}du=\frac{1}{n}+\delta$$

The unbiased estimate is $Y_1-\frac{1}{n}$ $\qquad E(Y_1)\rightarrow\delta$ as $n\rightarrow\infty$

10.9 $g_1(y_1)=n\cdot\frac{1}{\beta}\left[\int_{y_1}^{\beta}\frac{1}{\beta}\;dx\right]^{n-1}=\frac{n}{\beta^n}(\beta-y_1)^{n-1}$

$$E(Y_1)=\frac{n}{\beta^n}\int_{0}^{\beta}y_1(\beta-y_1)^{n-1}dy_1 \qquad u=\frac{y_1}{\beta} \qquad du=\frac{dy_1}{\beta}$$

$$=\frac{b}{\beta^n}\int_{0}^{1}\beta u(\beta-\beta u)^{n-1}\beta\;du=n\beta\int_{0}^{1}u(1-u)^{n-1}du=\frac{\beta}{n+1}$$

Unbiased estimate is $(n+1)Y_1$

10.10 $E\left[\sum_{i=1}^{n}\frac{x_i^2}{n}\right]=\frac{1}{n}\sum_{i=1}^{n}E\left(x_i^2\right)$

$$=\frac{1}{n}\sum_{i=1}^{n}(\sigma^2+\mu^2)=\frac{1}{n}\sum_{i=1}^{n}\sigma^2=\sigma^2$$

10.11 $E\left[n\cdot\frac{x}{n}\cdot\left(1-\frac{x}{n}\right)\right]=E(x)-\frac{1}{n}E(x^2)$

$$=n\theta-\frac{1}{n}\left[n\theta(1-\theta)+n^2\theta^2\right]$$

$$=(n-1)\theta(1-\theta)\neq n\theta(1-\theta)\qquad\text{biased}$$

10.12 (a) $n-1$ values before y_n in $\binom{y_n-1}{n-1}$ ways.

$$f(y_n)=\frac{\binom{y_n-1}{n-1}}{\binom{k}{n}}\qquad\text{for } y_n=n,\ldots,k$$

(b) $E(Y_n)=\sum_{y_n=n}^{k} y_n\cdot\frac{\binom{y_n-1}{n-1}}{\binom{k}{n}}=\frac{n}{\binom{k}{n}}\sum_{y_n=n}^{k}\binom{y_n}{n}=\frac{n}{\binom{k}{n}}\binom{k+1}{n+1}$

$=\frac{n(k+1)}{n+1}$ see Exercise 1.15 or Theorem 1.11, respectively

$$E\left[\frac{n+1}{n}\cdot Y_n-1\right]=\frac{n+1}{n}\cdot\frac{n(k+1)}{n+1}-1=k\qquad\text{QED}$$

10.13 $E(\hat{\theta}^2)=\text{var}(\hat{\theta})+E(\hat{\theta})^2=\text{var}(\hat{\theta})+\theta^2$

$E(\tilde{\theta}^2)>\theta^2$ since $\text{var}(\tilde{\theta})>0$

10.14 $f(x;\theta)=\theta^x(1-\theta)^{1-x}\qquad E(x)=\theta\qquad E(x^2)=\theta$

$\ln f(x;\theta)=x\ln\theta+(1-x)\ln(1-\theta)$

$\frac{\partial\ln f(x;\theta)}{\partial\theta}=\frac{x}{\theta}-\frac{1-x}{1-\theta}=\frac{x-\theta}{\theta(1-\theta)}$

$E\left[\left(\frac{\partial\ln f(x;\theta)^2}{\partial\theta}\right)\right]=\frac{1}{\theta^2(1-\theta)^2}E(x-\theta)^2=\frac{1}{\theta(1-\theta)}$

$\frac{1}{n\cdot E}=\frac{\theta(1-\theta)}{n}=\text{var}\left(\frac{x}{u}\right)$ when x is binomial random variable.

$\therefore\frac{x}{n}$ is minimum variance estimator

$E\left(\frac{x}{n}\right)=\frac{n\theta}{n}=\theta$

$\therefore$ unbiased

10.15 $f(x;\lambda)=\dfrac{\lambda^x e^{-\lambda}}{x!}$ $\quad \mu=\lambda \quad \sigma^2=\lambda \quad \text{var}(\bar{x})=\dfrac{\lambda}{n}$

$E(\bar{x})=\lambda \rightarrow$ unbiased

$\ln f = x\ln\lambda-\lambda-\ln x!$

$$\frac{\partial \ln f}{\partial\lambda}=\frac{x}{\lambda}-1 \qquad E\left[\left(\frac{\partial \ln f}{\partial\lambda}\right)^2\right]=\frac{E(x^2)}{\lambda^2}-\frac{2}{\lambda}E(x)+1$$

$$=\frac{\lambda+\lambda^2}{\lambda^2}-\frac{2}{\lambda}\lambda+1=\frac{1}{\lambda}$$

$$\frac{1}{nE}=\frac{\lambda}{n}=\text{var}(\bar{x})$$

$\therefore \bar{x}$ is minimum variance unbiased estimator

10.16 $\text{var}(\hat{\theta}_1)=3\,\text{var}(\hat{\theta}_2)$

$E(a_1\hat{\theta}_1+a_2\hat{\theta}_2)=a_1\theta+a_2\theta=\theta\rightarrow a_1+a_2=1$

$\text{var}=a_i^2\,\text{var}(\hat{\theta}_1)+a_2^2\,\text{var}(\hat{\theta}_2)$

$\text{var}=3a_i^2\,\text{var}(\hat{\theta}_2)+a_2^2\,\text{var}(\hat{\theta}_2)=(3a_1^2+a_2^2)\,\text{var}(\hat{\theta}_2)$

$=[3a_1^2+(1-a_1)^2]\,\text{var}(\hat{\theta}_2)$

$$\frac{\partial}{\partial a_1}=6a_2+2(1-a_1)(-1)$$

$$=8a_1-2=0 \qquad a_1=\frac{1}{4} \quad a_2=\frac{3}{4}$$

10.17 $f(x;\theta)=\dfrac{1}{\theta}e^{-x/\theta} \quad E(x)=\theta \quad E(x^2)=2\theta^2 \quad \sigma^2=\theta^2$

$E(\bar{x})=\theta\rightarrow$ unbiased $\quad \text{var}(\bar{x})=\dfrac{\theta^2}{n}$

$$\ln f=-\ln\theta-\frac{x}{\theta}$$

$$\frac{\partial \ln f}{\partial\theta}=-\frac{1}{\theta}+\frac{x}{\theta^2}=\frac{x-\theta}{\theta^2}$$

$$E\left[\left(\frac{\partial \ln f}{\partial\theta}\right)^2\right]=\frac{1}{\theta^4}E(x-\theta)^2-\frac{1}{\theta^2}$$

$\dfrac{1}{nE}=\dfrac{\theta^2}{n}=\text{var}(\bar{x}) \quad \therefore \quad \bar{x}$ is minimum variance unbiased estimator

10.18 $E(Y_n)=\dfrac{n}{n+1}\beta,\ E(Y_n^2)=\dfrac{n\beta^2}{n+2},\ \text{var}(Y_n)=\dfrac{n\beta^2}{(n+2)(n+1)^2}$

let $B=\dfrac{n+1}{n}\cdot Y_n$

$E(B)=\dfrac{n+1}{n}\cdot\dfrac{n}{n+1}\cdot\beta=\beta\rightarrow$ unbiased

$$\text{var}(B) = \frac{(n+1)^2}{n^2} \cdot \frac{n\beta^2}{(n+2)(n+1)^2} = \frac{\beta^2}{n(n+2)}$$

$$\frac{1}{nE\left(\frac{\partial \ln f(X)}{\partial \beta}\right)} = \frac{1}{n\frac{1}{\beta^2}} = \frac{\beta^2}{n} > \frac{\beta^2}{n(n+2)} = \text{var}(B)$$

so the Cramèr-Rao inequality is not satisfied.

10.19 (a) $\frac{\partial \ln f(x)}{\partial \theta} = \frac{1}{f(x)} \frac{\partial f(x)}{\partial \theta}$ $\qquad \frac{\partial f(x)}{\partial \theta} = \frac{\partial \ln f(x)}{\partial \theta} \cdot f(x)$

$$\therefore \int \frac{\partial \ln f(x)}{\partial \theta} \cdot f(x)dx = 0$$

(b) $\frac{\partial^2 \ln f(x)}{\partial \theta^2} \cdot f(x) + \frac{\partial \ln f(x)}{\partial \theta} \cdot \frac{\partial \ln f(x)}{\partial \theta} \cdot f(x)$

$$\int \frac{\partial^2 \ln f(x)}{\partial \theta^2} \cdot f(x)dx = -\int \left[\frac{\partial \ln f(x)}{\partial \theta}\right]^2 f(x)dx$$

$$E\left[\left(\frac{\partial \ln f(x)}{\partial \theta}\right)^2\right] = -E\left[\left(\frac{\partial \ln f(x)}{\partial \theta}\right)\right]^2$$

10.20 $\frac{\partial \ln f(x)}{\partial \mu} = \frac{1}{\sigma}\left(\frac{x-\mu}{\sigma}\right)$ from Example 10.5

$$\frac{\partial^2 \ln f(x)}{\partial \mu^2} = -\frac{1}{\sigma^2}$$

$$\frac{1}{nE\left[\left(\frac{\partial \ln f(x)}{\partial \mu}\right)^2\right]} = \frac{1}{n\left(-\frac{-1}{\sigma^2}\right)} = \frac{\sigma^2}{n}$$

10.21 (a) $E[w\bar{x}_1 + (1-w)\bar{x}_2] = w\mu + (1-w)\mu = \mu$

(b) $\text{var}[w\bar{x}_1 + (1-w)\bar{x}_2] = w^2 \frac{\sigma_1^2}{n} + (1-w)^2 \frac{\sigma_2^2}{n}$

$$\frac{d}{dw} = 2w\frac{\sigma_1^2}{n} + 2(1-w)(-1)\frac{\sigma_2^2}{n} = 0$$

$$w\left(\sigma_1^2 + \sigma_2^2\right) = \sigma_2^2 \qquad w = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

10.22 $\text{var}\,1 = w^2 \frac{\sigma_1^2}{n} + (1-w)^2 \frac{\sigma_2^2}{n}$

$$w = \frac{1}{2} \qquad \text{var} = \frac{\sigma_1^2}{4n} + \frac{\sigma_2^2}{4n} = \frac{1}{4n}(\sigma_1^2 + \sigma_2^2)$$

$$\text{var}\,2 = \left(\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)^2 \frac{\sigma_1^2}{n} + \left(\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2 \frac{\sigma_2^2}{n}$$

$$= \frac{\sigma_1^2\sigma_2^2}{n}\left[\frac{\sigma_2^2}{\left(\sigma_1^2+\sigma_2^2\right)} + \frac{\sigma_1^2}{\left(\sigma_1^2+\sigma_2^2\right)^2}\right] = \frac{\sigma_1^2\sigma_2^2}{n(\sigma_1^2+\sigma_2^2)}$$

$$\text{efficiency} = \frac{\dfrac{\sigma_1^2\cdot\sigma_2^2}{n(\sigma_1^2+\sigma_2^2)^2}}{\dfrac{1}{4n}(\sigma_1^2+\sigma_2^2)} = \frac{\sigma_1^2\sigma_2^2}{n(\sigma_1^2+\sigma_2^2)}\cdot\frac{4n}{\sigma_1^2+\sigma_2^2}$$

$$= \frac{4\sigma_1^2\sigma_2^2}{(\sigma_1^2+\sigma_2^2)^2}$$

10.23 $\text{var} = w^2\dfrac{\sigma^2}{n_1} + (1-w)^2\dfrac{\sigma^2}{n_2}$

$$\frac{d}{dw} = \frac{2w\sigma^2}{n_1} - \frac{2(1-w)\sigma^2}{n_2} = 0$$

$$\frac{w}{n_1} = \frac{1-w}{n_2} \qquad w = \frac{n_1}{n_1+n_2}$$

10.24 For $w = \dfrac{1}{2}$ $\qquad \text{var} = \dfrac{1}{4}\cdot\dfrac{\sigma^2}{n_1} + \dfrac{1}{4}\dfrac{\sigma^2}{n_2} = \dfrac{\sigma^2}{4}\left(\dfrac{1}{n_1}+\dfrac{1}{n_2}\right)$

For $w = \dfrac{n_1}{n_1+n_2}$ $\qquad \text{var} = \left(\dfrac{n_1}{n_1+n_2}\right)^2\dfrac{\sigma^2}{n_1} + \left(\dfrac{n_2}{n_1+n_2}\right)^2\dfrac{\sigma^2}{n_2}$

$$= \frac{\sigma^2}{(n_1+n_2)^2}(n_1+n_2) = \frac{\sigma^2}{n_1+n_2}$$

$$\text{Efficiency} = \frac{\dfrac{\sigma^2}{n_1+n_2}}{\dfrac{\sigma^2}{4}\left(\dfrac{1}{n_1}+\dfrac{1}{n_2}\right)} = \frac{4n_1n_2}{(n_1+n_2)^2}$$

10.25 $\text{var}\left(\dfrac{x_1+2x_2+x_3}{4}\right) = \dfrac{1}{16}\sigma^2 + \dfrac{1}{4}\sigma^2 + \dfrac{1}{16}\sigma^2 = \dfrac{3}{8}\sigma^2 \qquad \text{var}(\overline{x}) = \dfrac{\sigma^2}{3}$

$$\text{Efficiency} = \frac{\dfrac{\sigma^2}{3}}{\dfrac{3}{8}\sigma^2} = \frac{8}{9}$$

10.26 $\mu = \theta$ and $\sigma^2 = \theta^2$ $\quad \text{var}(\overline{x}) = \dfrac{\theta^2}{2}$

From Ex. 8.4 $\quad g_1(y_1) = \dfrac{2}{\theta} e^{-2y_1/\theta} \quad$ for $y_1 > 0$

$$\text{var}(Y_1) = \left(\frac{\theta}{2}\right) \qquad E(2Y_1) = \theta \quad \text{unbiased}$$

$$\text{var}(Y_1) = \left(\frac{\theta}{2}\right)^2 = \frac{\theta^2}{4} \qquad \text{var}(2Y_1) = 4 \cdot \frac{\theta^2}{4} = \theta^2$$

$$\text{Efficiency} = \frac{\theta^2/2}{\theta^2} = \frac{1}{2}$$

10.27 $g_n(y_n) = \dfrac{n}{\beta^n} y_n^{n-1}$

$$E(Y_n) = \frac{n}{\beta^n} \int y_n^n dy_n \qquad 0 < y_n < \beta$$

$$= \frac{n}{\beta^n} \cdot \frac{\beta^{n+1}}{n+1} = \frac{\beta n}{n+1}$$

$$E(Y_n)^2 = \frac{n}{\beta^n} \int_0^\beta y_n^{n+1} dy_n = \frac{n}{\beta^n} \cdot \frac{\beta^{n+2}}{n+2} = \frac{n\beta^2}{n+2}$$

$$\text{var}(Y_n) = \frac{n\beta^2}{n+2} - \frac{n^2\beta^2}{(n+1)^2} = \frac{\beta^2\left[n(n+1)^2 - n^2(n+2)\right]}{(n+2)(n+1)^2}$$

$$= \frac{n\beta^2}{(n+2)(n+1)^2}$$

$$Z = Y_n \cdot \frac{n+1}{n} \qquad E(Z) = \frac{n+1}{n} \cdot \frac{n\beta}{n+1} = \beta \quad \text{unbiased}$$

$$\text{var}(Z) = \left(\frac{n+1}{n}\right)^2 \cdot \frac{n\beta^2}{(n+2)(n+1)^2} = \frac{\beta^2}{n(n+2)} \quad \text{QED}$$

10.28 $Y = \overline{x} - 1 \qquad \text{var}(Y) = \text{var}(\overline{x}) = \dfrac{\theta^2}{n} = \dfrac{1}{n}$

$$Z = Y_1 - \frac{1}{n} \qquad g_1(y_1) = ne^{-n(y_1-\delta)}$$

$$E(Y_1) = \frac{1}{n} + \delta$$

$$E(Y_1^2) = n\int_\delta^\infty y_1^2 e^{-n(y_1-\delta)} dy_1 \qquad u = y_1 - \delta$$

$$= n\int_0^\infty (u+\delta)^2 e^{-nu} du = \frac{2}{n^2} + \frac{2\delta}{n} + \delta^2$$

$$\text{var}(Y_1) = \frac{2}{n^2} + \frac{2\delta}{n} + \delta^2 - \left(\frac{1}{n} + \delta\right)^2 = \frac{1}{n^2}$$

$$\text{efficiency} = \frac{\text{var}(Z)}{\text{var}(Y)} = \frac{\left(\frac{1}{n}\right)^2}{\frac{1}{n}} = \frac{1}{n}$$

10.29 Continue from Exercise 10.12

$$E\left[Y_n(Y_n+1)\right] = \frac{1}{\binom{k}{n}} \sum_{y_n=n}^{k} y_n(y_n+1)\binom{y_n-1}{n-1} = \frac{n(n+1)}{\binom{k}{n}} \sum_{y_n=n}^{k} \binom{y_n+1}{n+1}$$

$$= \frac{n(n+1)}{\binom{k}{n}} \cdot \binom{k+2}{n+2} \qquad \text{Exerxise 1.15 or } \sum_{i=n}^{k} \binom{i}{n} = \binom{k+1}{n+1}$$

$$= \frac{n(k+1)(k+2)}{n+2}$$

$$\text{var}(Y_n) = \frac{n(k+1)(k+2)}{n+2} - E(Y_n^2) - E(Y_n)^2$$

$$= \frac{n(k+1)(k+2)}{n+2} - \frac{n(k+1)}{n+1} - \frac{n^2(k+1)^2}{(n+1)^2}$$

$$\text{var}(\left(\frac{n+1}{n} \cdot Y_n - 1\right) = \frac{(n+1)^2}{n^2} \text{var}(Y_n)$$

$$= \frac{(k+1)\left[(k+2)(n+1)^2 - (n+1)(n+2) - (k+1)n(n+2)\right]}{n(n+2)}$$

$$= \frac{(k+1)(k-n)}{n(n+2)}$$

$$E(x) = \frac{k+1}{2},\ E(x^2) = \frac{(k+1)(2k+1)}{6},\ \sigma^2 = \frac{k^2-1}{12}$$

for population

$$E(2\overline{x}-1) = 2E(\overline{x}) - 1 = 2\frac{(k+1)}{2} - 1 = k \qquad \text{unbiased}$$

$$\text{var}(\overline{x}) = \frac{(k^2-1)}{12n} \cdot \frac{k-n}{k-1} = \frac{(k+1)(k-n)}{12n}$$

$$\text{var}(2\overline{x}-1) = \frac{(k+1)(k-n)}{3n},\ \text{efficiency } = \frac{\frac{(k+1)(k-n)}{n(n+2)}}{\frac{(k+1)(k-n)}{3n}} = \frac{3}{n+2}$$

(a) efficiency = $\frac{3}{4}$; **(b)** efficiency = $\frac{3}{5}$

10.30 (a) $E(x)=\int_0^1 x\,dx=\frac{1}{2},\ E(x^2)=\int_0^1 x^2\,dx=\frac{1}{3},\ \text{var}(x)=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}$

$$\text{var}(\overline{x})=\frac{1/12}{3}=\frac{1}{36}$$

(b) $g_1(y_1)=3(1-y_1)^2 \qquad 0<y_1<1$

$g_3(y_3)=3y_3^2 \qquad 0<y_3<1$

$f(y_1,y_3)=6(y_3-y_1) \qquad 0<y_1<y_3<1$

$$E(Y_1)=3\int_0^1 y_1(1-y_1)^2dy_1=\frac{1}{4}$$

$$E(Y_1^2)=3\int_0^1 y_1^2(1-y_1)^2dy_1=\frac{1}{10}$$

$$E(Y_3)=3\int_0^1 y_3^2\,dy_3=\frac{3}{4},\ E(Y_3^2)=3\int_0^1 y_3^4\,dy_3=\frac{3}{5}$$

$$E(Y_1Y_3)=b\int_0^1\int_0^{y_3} y_1y_3(y_3-y_1)\,dy_1\,dy_3=\frac{1}{5}$$

$$\text{var}(Y_1)=\frac{1}{10}-\frac{1}{16}=\frac{3}{80},\quad \text{var}(Y_3)=\frac{3}{5}-\frac{9}{16}=\frac{3}{80}$$

$$\text{cov}(Y_1,Y_3)=\frac{1}{5}-\frac{3}{16}=\frac{1}{80}$$

(c) $E\left(\frac{Y_1+Y_3}{2}\right)=\frac{1}{2}\left(\frac{1}{4}+\frac{3}{4}\right)=\frac{1}{2}\rightarrow$ unbiased

$$\text{var}\left(\frac{Y_1+Y_3}{2}\right)=\frac{1}{4}\cdot\frac{3}{80}+\frac{1}{4}\cdot\frac{3}{80}+\frac{1}{2}\cdot\frac{1}{80}=\frac{1}{40}$$

Since $\frac{1}{40}$ is less than $\frac{1}{36}$ midrange here is more efficient than the mean.

10.31 $E(\hat{\theta})=\theta+b(\theta)$

$$E\left[(\hat{\theta}-\theta)^2\right]=E(\hat{\theta}^2)-2\theta E(\hat{\theta})+\theta^2=E(\hat{\theta})^2-2\theta\left[\theta+b(\theta)\right]+\theta^2$$

$$=E(\hat{\theta}^2)-\theta^2-2\theta b(\theta)$$

$$\text{var}(\hat{\theta})=E(\hat{\theta}^2)-\left[\theta+b(\theta)\right]^2=E(\hat{\theta}^2)-\theta^2-2\theta b(\theta)-\left[b(\theta)\right]^2$$

$$=E\left[(\hat{\theta}-\theta^2)\right]-\left[b(\theta)\right]^2$$

$$\therefore E\left[(\hat{\theta}-\theta)^2\right]=\text{var}(\hat{\theta})-\left[b(\theta)\right]^2$$

10.32 $\operatorname{var}(\hat{\theta}_1) = \frac{\theta(1-\theta)}{n} = \frac{1}{4n}$

$E(\hat{\theta}_2) = \frac{n\theta+1}{n+2}$ for $\theta = \frac{1}{2}$ $E(\hat{\theta}_2) = \frac{1}{2} \rightarrow$ unbiased

$\text{variance}(\hat{\theta}_2) = \frac{n\theta(1-\theta)}{(n+2)^2} = \frac{n}{4(n+2)^2}$ = mean square error

$$E\left[(\hat{\theta}_2-\theta)^2\right] - \left(\frac{1}{3}-\frac{1}{2}\right)^2 = \frac{1}{36}$$

(a) $\frac{n}{4(n+2)^2} < \frac{1}{4n}$ $\qquad n^2 < (n+2)^2$

for all values of n

(b) $\frac{1}{36} < \frac{1}{4n}$, $4n < 36$, $n < 9$

10.33 $g_1(y_1) = n\left[\int_{y_1}^{\alpha+1} f(x)\,dx\right]^{n-1}$ $\qquad f(x) = 1 \quad \alpha < x < \alpha+1$

$\qquad 0 \quad$ elsewhere

$= n(\alpha+1-y_1)^{n-1}$ $\qquad$ for $\alpha < y_1 < \alpha+1$

0 $\quad$ elsewhere

$$P\left(|Y_1-\alpha| < c\right) = \int_{\alpha}^{\alpha+c} n(\alpha+1-y_1)^{n-1}dy_1$$

$= 1^n - (1-c)^n \rightarrow 1$ when $n \rightarrow \infty$ with c fixed. QED

10.34 $E(\alpha+1-Y_1) = n\int_{\alpha}^{\alpha+1}(\alpha+1-y_n)^n dy = \frac{n}{n+1}$

$E(\alpha+1-Y_1)^2 = n\int_{\alpha}^{\alpha+1}(\alpha+1-y)^{n+1}dy = \frac{n}{n+2}$

$E(Y_1) = \alpha+1-\frac{n}{n+2} = \alpha+\frac{1}{n+1}$

$E(\left(Y_1-\frac{1}{n+1}\right) = \alpha \rightarrow$ unbiased

$\operatorname{var}(\alpha+1-Y_1) = \frac{n}{(n+2)} - \left(\frac{n}{n+1}\right)^2 = \frac{n}{(n+2)(n+1)^2}$

$\operatorname{var}\left(Y_1-\frac{1}{n+1}\right) = \frac{n}{(n+2)(n+1)^2} \rightarrow 0$ as $n \rightarrow \infty$ $\qquad$ QED

10.35 $g_n(y_n) = \frac{n}{\beta^n} y_n^{n-1} \qquad 0 < y_n < \beta$

$$P\left(|Y_n - \beta| < c\right) = \frac{n}{\beta^n} \int_{\beta-c}^{\beta} y_n^{n-1} dy_n = \frac{1}{\beta^n}\left[\beta^n - \left(\beta - c\right)^n\right)\right]$$

$$= 1 - \left(\frac{\beta - c}{\beta}\right)^n \to 1$$

when $n \to \infty$ with c fixed.

10.36 $\bar{x}$ is consistent estimate of the mean of any population with a finite variance. Since θ is the mean and $\sigma^2 = \theta^2$ if follows that $\bar{x}$ is consistent estimate of θ.

10.37 For any single observation and for $c = \theta$, $P(|X - \theta| < \theta) = 1 - e^{-2\theta/\theta} = e^{-2}$ does not converge to 0, so X_n is not consistent for θ.

10.38 Shown is (a) of 10.21 that it is unbiased. From 10.22 variance is $\frac{\sigma_1^2 \sigma_2^2}{n\left(\sigma_1^2 + \sigma_2^2\right)} \to 0$

So it is consistent by Theorem 10.3.

10.39 $\text{Var}\left(\frac{X+1}{n+2}\right) = \frac{1}{(n+2)^2} Var(X) = \frac{n\theta(1-\theta)}{(n+2)^2} \qquad \to \ \theta \text{ as } n \to \infty$

asymptotically unbiased

$$\text{Var}\left(\frac{X+1}{n+2}\right) = \frac{1}{(n+2)^2}\text{var}(x) = \frac{n\theta(1-\theta)}{n^2}$$

$$= \frac{\theta(1-\theta)}{n} \to 0 \text{ as } n \to \infty \quad \text{QED}$$

10.40 $E(Y_n) = \frac{n}{n+1}\beta \to \beta$ as $n \to \infty$ $\therefore$ asymptotically unbiased

From Example 10.6 (see Exercise 10.27)

$$\text{var}(Y_n) = \frac{n}{n+1} \cdot \frac{\beta^2}{n(n+2)} = \frac{\beta^2}{(n+1)(n+2)} \to 0 \text{ as } n \to \infty$$

consistent by Theorem 10.3

10.41 (a) $P\left(|x - \mu| < c\right) \frac{n-1}{n} P\left(|x - \mu| < c\right) + \frac{1}{n} P\left(|n^2 - \mu| < c\right) \quad 1 + 0 = 1$

since $\bar{x}$ is known to be consistent and $\frac{n-1}{n} \to 1$

(b) Let estimate be x

$$E(x) = \mu \cdot \frac{n+1}{n} + n^2 \cdot \frac{1}{n} = \mu \cdot \frac{n+1}{n} + n \neq \mu$$

not unbiased and *not* asymptotically unbiased.

10.42 $$f(x_1,x_2,\ldots,x_n)=\frac{1}{\theta^n}e^{-\left[\frac{1}{\theta}\sum_{i=1}^{n}x_i\right]}=\underbrace{\frac{1}{\theta^n}e^{-(1/\theta)n\bar{x}}}_{g(\hat{\theta},\theta)}$$

Since the joint density depends only on θ and $\bar{x}$, $\bar{x}$ is a sufficient estimator of θ.

10.43 $$f(x_1,x_2)=\binom{n_1}{x_1}\binom{n_2}{x_2}\theta^{x_1+x_2}(1-\theta)^{(n_1+n_2)-(x_1+x_2)}$$

$$\hat{\theta}=\frac{x_1+x_2}{n_1+n_2}$$

$$=\underbrace{\binom{n_1}{x_1}\binom{n_2}{x_2}}_{h(x_1,x_2)}\underbrace{\theta^{(n_1+n_2)\hat{\theta}}(1-\theta)^{(n_1+n_2)(1-\hat{\theta})}}_{(\hat{\theta},\theta)}$$

$\therefore$ by theorem, estimator is sufficient.

10.44 Try $x_1=0$ and $x_2=1$

$$f(0,1)=\binom{2}{0}\binom{2}{1}\theta(1-\theta)^3=2\theta(1-\theta)^2$$

$$Y=\frac{x_1+2x_2}{n_1+2n_2}=\frac{2}{6}=\frac{1}{3}\quad\text{only possibilities}\quad\begin{array}{ll}x_1=0 & x_2=1\\ x_1=2 & x_2=0\end{array}$$

$\therefore$ by theorem, estimator is sufficient.

$$f(2,0)=\binom{2}{2}\binom{2}{0}\theta^2(1-\theta)^2=\theta^2(1-\theta)^2$$

$$f\left(0,1\middle|Y=\frac{1}{3}\right)=\frac{2\theta(1-\theta)^3}{2\theta(1-\theta)^2+\theta^2(1-\theta)^2}=\frac{2(1-\theta)}{2(1+\theta)+\theta}$$

$$=\frac{2-2\theta}{2-\theta}\qquad\text{not independent of }\theta$$

$\therefore$ Y not sufficient

10.45 $$f(x_1,\ldots x_n)=\frac{1}{\beta^n}\qquad g(y_n)=\frac{n}{\beta^n}y_n^{n-1}$$

$$f(x_1,\ldots x_n|Y_n)=\frac{\frac{1}{\beta^n}}{\frac{n}{\beta^n}y_n^{n-1}}=\frac{1}{n\,y_n^{n-1}}$$

independent of β
$\therefore$ sufficient

10.46 $$f(x_1,x_2)=\frac{\lambda^{x_1+x_2}\,e^{-2\lambda}}{x_1!\,x_2!}\qquad\bar{x}=\frac{x_1+x_2}{2}$$

$$\underbrace{\lambda^{2\bar{x}}\,e^{-\lambda}}_{g(\bar{x},\lambda)}\cdot\underset{(x_1,x_2)}{\frac{1}{x_1!\,x_2!}}$$

satisfies Theorem 10.3
$\therefore$ sufficient

10.47 Try $x_1 = 0,\ x_2 = 1,\ x_3 = 0,\ Y = 2$

The only possibility is $x_1 = 1,\ x_2 = 0,\ x_3 = 1$

$f(0,1,0) = \theta(1-\theta)^2$

$f(1,0,1) = \theta^2(1-\theta)$

$$f(0,1,0 \mid Y = 2) = \frac{\theta(1-\theta)^2}{\theta(1-\theta)^2 + \theta^2(1-\theta)} = 1-\theta$$

not independent of $\theta \rightarrow$ not sufficient

10.48 $f(x) = \theta(1-\theta)^{x-1}$

$f(x_1, \ldots x_n) = \theta^n (1-\theta)^{\sum x-n} = \theta^n (1-\theta)^{n\bar{x}-n}$

Depends only on θ and $\bar{x} \rightarrow$ sufficient

10.49 $$f(x_1 \ldots x_n) = \frac{1}{(2\pi)^{n/2}\sigma^n} e^{-(1/2)\left[\sum (x_i-\mu)^2\right]/\sigma^2} = \frac{1}{(2\pi)^{n/2}\sigma^n} e^{-(n/2\sigma^2)\hat{\sigma}^2}$$

Depends only on σ^2 and $\hat{\sigma}^2 \rightarrow$ sufficient.

10.50 $\hat{\mu} = m_1',\ \mu^2 + \sigma^2 = m_2'$

$\hat{\sigma}^2 = m_2' - (m_1')^2$

10.51 $m_1' = \mu = \theta \qquad \hat{\theta} = m_1'$

10.52 $\mu = \frac{p}{2},\ \hat{\beta} = 2m_1'$

10.53 $\mu = \lambda \qquad \hat{\lambda} = m_1'$

10.54 $\beta = 1 \qquad \mu = \frac{\alpha}{\alpha+1} \qquad \frac{\alpha}{\alpha+1} = m_1' \qquad \alpha = \alpha m_1' + m_1'$

$\alpha(1-m_1') = m_1',\ \hat{\alpha} = \frac{m_1'}{1-m_1'}$

10.55 $$\mu = \frac{2}{\theta^2}\int_0^\theta x(\theta - x)dx = \frac{\theta}{3},\ \hat{\theta} = 3m_1'$$

10.56 $$\mu = \frac{1}{\theta}\int_\delta^\infty xe^{-(1/\theta)(x-\delta)}dx = \frac{1}{\theta}\int_0^\infty (u+\delta)e^{-(1/\theta)u}du = \theta + \delta$$

$u = x - \delta$

$$\mu_2' = \frac{1}{\theta}\int_\delta^\infty x^2 e^{-(1/\theta)(x-\delta)}dx = \frac{1}{\theta}\int_0^\infty (u+\delta)^2 e^{-(1/\theta)u}du = 2\theta^2 + 2\delta\theta + \delta^2$$

$m_1' = \delta + \theta,\ m_2' = 2\theta^2 + 2\delta\theta + \delta^2 = \theta^2 + (\theta+\delta)^2 = \theta^2 + (m_1')^2$

$\hat{\theta} = \sqrt{m_2' - (m_1')^2}$ and $\hat{\delta} - m_1' = \sqrt{m_2' - (m_1')^2}$

10.57 $\frac{\alpha+\beta}{2} = m_1'$ $\qquad \frac{1}{12}(\beta-\alpha)^2 + \frac{1}{4}(\alpha+\beta)^2 = m_2'$

$$m_2' = \frac{1}{12}(\beta-\alpha)^2 + (m_1')^2 \quad (\beta-\alpha)^2 = 12[m_2' - (m_1')^2]$$

$$\beta - \alpha = 2\sqrt{3[m_2' - (m_1')^2]}$$

$$\beta + \alpha = 2m_1$$

add $\hat{\beta} = m_1' + \sqrt{3[m_2' - (m_1')^2]}$

subtract $\hat{\alpha} = m_1' - \sqrt{3[m_2' - (m_1')^2}$

10.58 $\mu = 38$ $\qquad m_1' = \frac{n_0 \cdot 0 + n_1 \cdot 1 + n_2 \cdot 2 + n_3 \cdot 3}{N} = 3\theta$

$$\hat{\theta} = \frac{n_1 + 2n_2 + 3n_2}{3N}$$

10.59 $L(\lambda) = \frac{\lambda^{\sum x} e^{-n\lambda}}{\prod x_i!}$ $\qquad \ln L(\lambda) = \left(\sum x\right) - (\ln \lambda) - n\lambda - \ln \prod x_i!$

$$\frac{d \ln L(\lambda)}{d\lambda} = \frac{\sum x}{\lambda} - n = 0$$

$$\hat{\lambda} = \frac{\sum x}{n} = \overline{x}$$

10.60 $b(x;\alpha) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\Gamma(1)} x^{\alpha-1} = \alpha x^{\alpha-1}$

$$L(\alpha) = \alpha^n (\prod x_i)^{\alpha-1} \qquad \ln L(\alpha) = n\ln(n) + (\alpha-1)\sum \ln x_i$$

$$\frac{d \ln L(\alpha)}{d\alpha} = \frac{n}{\alpha} + \sum \ln x_i$$

$$\alpha = \frac{-n}{\sum_{i=1}^{n} \ln x_i}$$

10.61 $f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$ $\qquad \alpha = 2$

$$= \frac{1}{\beta^2} x \, e^{-x/\beta}$$

$$L(\beta) = \frac{1}{\beta^{2n}}\left(\prod x_i\right) e^{-(1/\beta)\sum x} \qquad \ln L(\beta) = -2n\ln\beta + \ln\prod x_i - \frac{1}{\beta}\sum x$$

$$\frac{d \ln L(\beta)}{d\beta} = \frac{-2n}{\beta} + \frac{1}{\beta^2}\sum x = 0$$

$$\beta = \frac{\sum x}{2n} = \frac{\overline{x}}{2}$$

10.62 $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(1/2)[(x-\mu)/\sigma]^2} \qquad L(\sigma) = \frac{1}{(2\pi)^n \sigma^n} e^{-(1/2\sigma^2)\sum (x-\mu)^2}$

$$\ln L(\sigma) = -\frac{n}{2}\ln 2\pi - n\ln\sigma - \frac{1}{2\sigma^2}\sum (x-\mu)^2$$

$$\frac{d\ln L(\sigma)}{d\sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma_2^2}\sum (x-\mu)^2 = 0$$

$$\hat{\sigma}^2 = \frac{\sum (x-\mu)^2}{n} \text{ and } \hat{\sigma} = \sqrt{\frac{\sum (x-\mu)^2}{n}}$$

10.63 **(a)** $\mu = \frac{1}{8} = m_1' \qquad \hat{\theta} = \frac{1}{m_1'} = \frac{1}{\bar{x}}$

(b) $g(x) = \theta(1-\theta)^{x-1} \quad L(\theta) = \theta^n (1-\theta)^{\sum x - n}$

$$\ln L(\theta) = n\ln\theta + \left(\sum x - n\right)\ln(1-\theta)$$

$$\frac{d\ln L(\theta)}{d\theta} = \frac{n}{\theta} + \left(\sum x - n\right)\left(\frac{-1}{1-\theta}\right) = 0 \qquad \hat{\theta} = \frac{n}{\sum x} = \frac{1}{\bar{x}}$$

10.64 $f(x) = 2\alpha x e^{-\alpha x^2} \qquad L(\alpha) = 2^n \alpha^n \left(\prod x_i\right) e^{-\alpha\left(\sum x^2\right)}$

$$\ln L(\alpha) = n\ln 2 + n\ln\alpha + \ln\prod x_i - \alpha\left(\sum x^2\right)$$

$$\frac{\ln L(\alpha)}{d\alpha} = \frac{n}{\alpha} - \sum x^2 = 0 \qquad \hat{\alpha} = \frac{n}{\sum x^2}$$

10.65 $f(x) = \frac{\alpha}{x^{\alpha+1}} \qquad L(a) = \frac{\alpha^n}{\left(\prod x_i\right)^{\alpha+1}}$

$$\ln L(\alpha) = n\ln\alpha - (\alpha+1)\ln(\prod x_i)$$

$$\frac{dL(\alpha)}{d\alpha} = \frac{n}{\alpha} - \ln\prod x_i = \frac{n}{\alpha} - \sum \ln x_i = 0$$

$$\bar{\alpha} = \frac{n}{\sum \ln x_i}$$

10.66 $f(x) = \frac{1}{8} e^{-(x-\delta)/\theta}$

$$L(\theta,\delta) = \frac{1}{\theta^n} e^{-(1/\theta)\sum (x-\delta)}$$

Maximized with respect to δ let $\hat{\delta}$ be y_1 (smallest sample value)

$\hat{\delta} = y_1$

$$\ln L(\theta,\delta) = -n\ln\theta - \frac{1}{\theta}\sum (x-\delta)$$

$$\frac{d\ln L(\theta,6)}{\partial\theta} = -\frac{n}{\theta} + \frac{1}{\sigma^2}\cdot\sum (x-\delta) \qquad \hat{\theta} = \frac{\sum x}{n} - \hat{\delta} \qquad \hat{\theta} = \bar{x} - y_1$$

10.67 $f(x)=\dfrac{1}{\beta-\alpha}$ $\qquad L(\alpha,\beta)=\dfrac{1}{(\beta-\alpha)^n}$

To maximize $\hat{\alpha}=y_1$, and $\hat{\beta}=y_n$

10.68 $L=[(1-\theta)^3]^{n_0}[3\theta(1-\theta)^2]^{n_1}[3\theta^2(1-\theta)]^{n_2}[\theta^3]^{n_3}$

$$=3^{n_1+n_2}\theta^{n_1+2n_2+3n_3}(1-\theta)^{3n_0+2n_1+n_2}$$

$$\ln L=(n_1+n_2)\ln 3+(n_1+2n_2+3n_3)\ln\theta+(3n_0+2n_1+n_2)\ln(1-\theta)$$

$$\frac{dL}{d\theta}=\frac{n_1+2n_2+3n_3}{\theta}-\frac{3n_0+2n_1+n_2}{1-\theta}$$

$$(n_1+2n_2+3n_3)(1-\theta)=(3n_0+2n_1+n_2)\theta$$

$$\theta(3n_0+3n_1+3n_2+3n_3)=n_1+2n_2+3n_3$$

$$\hat{\theta}=\frac{n_1+2n_2+3n_3}{3N}$$

10.69 $f(x)=\dfrac{1}{\beta^{\alpha}\Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta}$

(a) $L(\beta)=\dfrac{1}{\beta^{n\alpha}[\Gamma(\alpha)]^n}\left(\prod x_i\right)^{\alpha-1}e^{-(1/\beta)\sum x_i}$

$$\ln L(\beta)=-n\alpha\ln\beta-n\ln\Gamma(\alpha)+(\alpha-1)\ln\prod x_i-\frac{1}{\beta}\sum x_i$$

$$\frac{d\ln L(\beta)}{d\beta}=\frac{-n\alpha}{\beta}+\frac{1}{\beta^2}\sum x_i \qquad \hat{\beta}=\frac{\sum x_i}{n\alpha}=\frac{\overline{x}}{\alpha}$$

(b) $\tau=\left(\dfrac{2\overline{x}}{\alpha}-1\right)^2$

10.70 $L(\alpha,\beta)=\left(\sqrt{2\pi}\right)^{-2n}e^{-(1/2)\sum[v-(\alpha+\beta)]^2-(1/2)\sum[w-(\alpha-\beta)]^2}$

$$\ln L(\alpha,\beta)=k-\frac{1}{2}\sum[v-(\alpha+\beta)]^2-\frac{1}{2}\sum[w-(\alpha-\beta)]^2$$

$$\frac{\partial\ln L}{\partial\alpha}=\sum(v-(\alpha+\beta)]+\sum(w-(\alpha-\beta)]=0$$

$$\sum v+\sum w-2n\alpha=0 \qquad \hat{\alpha}=\frac{\sum v+\sum w}{2n}=\frac{\overline{v}+\overline{w}}{2}$$

$$\frac{\partial\ln L}{\partial\beta}=\sum(v-(\alpha+\beta)]-\sum(w-(\alpha-\beta)]=0$$

$$\sum v+\sum w-2n\beta=0 \qquad \hat{\beta}=\frac{\sum v-\sum w}{2v}=\frac{\overline{v}-\overline{w}}{2}$$

10.71 $V\ n_1\,\mu_1\,\sigma$

$W\ n_2\,\mu_2\,\sigma$

$$L=\frac{1}{\left(\sqrt{2\pi}\right)^{n_1+n_2}\ \sigma^{n_1+n_2}}e^{-(1/2\sigma^2)\sum(v-\mu_1)^2-(1/2\sigma^2)\sum(w-\mu_2)^2}$$

$$\ln L=k-(n_1+n_2)\ln\sigma-\frac{1}{2\sigma^2}\sum(v-\mu_1)^2-\frac{1}{2\sigma^2}\sum(w-\mu_2)^2$$

$$\frac{\partial\ln L}{\partial\mu_1}=+\frac{1}{2\sigma^2}\cdot2\sum(v-\mu_1)=0\qquad\hat{\mu}_1=\bar{v}$$

$$\frac{\partial\ln L}{\partial\mu_2}=+\frac{1}{2\sigma^2}\cdot2\sum(w-\mu_2)=0\qquad\mu_2'=\bar{w}$$

$$\frac{\partial L}{\partial\sigma}=-\frac{n_1+n_2}{\sigma}+\frac{1}{\sigma^3}\left[\sum(v-\mu_1)^2+\sum(w-\mu_2)^2\right]$$

$$\hat{\sigma}^2=\frac{\sum(v-\bar{v})^2+\sum(w-\hat{w})^2}{n_1+n_2}$$

10.72 Any value $\hat{\theta}$ will do so long as

$$\hat{\theta}-\frac{1}{2}\le y_1\ \text{ and }\ y_n<\hat{\theta}+\frac{1}{2}$$

$$\hat{\theta}\le y_2+\frac{1}{2}\ \text{ and }\ \hat{\theta}\ge y_n-\frac{1}{2}$$

$$y_n-\frac{1}{2}\le\hat{\theta}\le y_1+\frac{1}{2}$$

10.73 **(a)** It is if $Y_n-\frac{1}{2}\le\frac{1}{2}(Y_1+Y_n)\le Y_1+\frac{1}{2}$

make use of $\boxed{Y_1\le Y_n\le Y_1+1}$

$$\frac{1}{2}(Y_1+Y_n)\le\frac{1}{2}(Y_1+Y_1+1)=Y_1+\frac{1}{2}$$

$$\frac{1}{2}(Y_1+Y_n)\ge\frac{1}{2}(Y_n+Y_n-1)=Y_n-\frac{1}{2}$$

both conditions are satisfied

(b) Suppose $Y_2=Y_1+1$ let $n=2$

$$\frac{1}{3}(Y_1+2Y_2)=\frac{1}{3}(3Y_1+2)=Y_1+\frac{2}{3}\nleq Y_2+\frac{1}{2}$$

not max likelihood estimate

10.74 $E(\theta|x) = \dfrac{x+\alpha}{\alpha+\beta+n}$ where $\alpha = \theta_0\left[\dfrac{\theta_0(1-\sigma_0^2)}{\sigma_0^2} - 1\right]$

$$\beta = (1-\theta_0)\left[\frac{\theta_0(1-\theta_0)}{\sigma_0^2} - 1\right] \qquad \alpha+\beta = \frac{\theta_0(1-\theta_0)}{\sigma_0^2} - 1$$

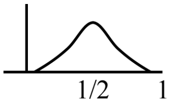
$$E(\theta|x) = \frac{x}{n}\cdot\frac{n}{\alpha+\beta+n} + \frac{\alpha}{\alpha+\beta+n}$$

$$= \frac{x}{n}\cdot\frac{n}{\dfrac{\theta_0(1-\theta_0)}{\sigma_o^2} - 1 + n} + \frac{\theta_0\left[\dfrac{\theta_0(1-\theta_0)}{\sigma_0^2} - 1\right]}{\dfrac{\theta_0(1-\theta_0)}{\sigma_0^2} - 1 + n}$$

$$= \frac{x}{n}\cdot w + \theta_0(1-w) \text{ where } w = \frac{n}{n + \dfrac{\theta_0(1-\theta_0)}{\sigma_0^2} - 1}$$

10.75 $\mu = \dfrac{\alpha}{\alpha+\beta} = \dfrac{40}{40+40} = \dfrac{1}{2}$ $\qquad \sigma^2 = \dfrac{40\cdot 40}{80^2\cdot 81} = \dfrac{1}{324}$

$$\sigma = \frac{1}{18}$$



Distribution is symmetrical about $x = \dfrac{1}{2}$

The function as well as its derivatives are 0 at $x = 0$ and 1, and with k = 3 in Chebyshev's Theorem

$\dfrac{8}{9}$ of area under curve falls between $\dfrac{1}{2} \pm \dfrac{1}{6} = \dfrac{1}{3}$ and $\dfrac{2}{3}$

10.76 $\mu_1 = \bar{x}\cdot\dfrac{n\sigma_0^2}{n\sigma_0^2+\sigma^2} + \mu_0\cdot\dfrac{\sigma^2}{n\sigma_0^2+\sigma^2} = \bar{x}w + \mu_0(1-w)$

$$w = \frac{n\sigma_0^2}{n\sigma_0^2+\sigma^2} = \frac{n}{n + \dfrac{\sigma^2}{\sigma_0^2}}$$

QED

10.77 $f(x|\lambda) = \dfrac{\lambda^x e^{-\lambda}}{x!}$

(a) $f(x,\lambda) = \dfrac{\lambda^x e^{-\lambda}}{x!}\cdot\dfrac{1}{\beta^\alpha\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\lambda/\beta}$

$g(x) = \dfrac{x^{\alpha-1}e^{-\beta}}{x!\beta^\alpha\Gamma(a)}\displaystyle\int_0^\infty \lambda^x e^{-\lambda}d\lambda$ gamma distribution with $\alpha = x+1$ and $\beta = 1$

$$= \frac{x^{\alpha-1}e^{-\beta}}{x!\beta^\alpha\Gamma(\alpha)}\cdot\Gamma(x+1) = \frac{x^{\alpha-1}e^{-\beta}x!}{x!\beta^\alpha\Gamma(\alpha)} = \frac{x^{\alpha-1}e^{-\beta}}{\beta^\alpha\Gamma(\alpha)}$$

$$f(x,\lambda) = \frac{\lambda^{x+\alpha-1}e^{-\lambda[1+(1/\beta)]}}{x!\beta^{\alpha}\Gamma(\alpha)}$$

$$g(x) = \frac{1}{x!\beta^{\alpha}\Gamma(\alpha)}\int_0^{\infty}\lambda^{x+\alpha-1}e^{-\lambda(\beta+1)/\beta}$$

gamma distribution with $x+\alpha$ and $\frac{\beta}{\beta+1}$

$$g(x) = \frac{\left(\frac{\beta}{\beta+1}\right)^{x+\alpha}\Gamma(x+\alpha)}{x!\beta^{\alpha}\Gamma(\alpha)}$$

$$\phi(\lambda|x) = \frac{\lambda^{x+\alpha-1}e^{-\lambda(\beta+1)/\beta}}{x!\beta^{\alpha}\Gamma(\alpha)}\cdot\frac{x!\beta^{\alpha}\Gamma(\alpha)}{\left(\frac{\beta}{\beta+1}\right)^{x+\alpha}\Gamma(x+\alpha)}$$

$$= \frac{1}{\left(\frac{\beta}{\beta+1}\right)^{x+\alpha}\Gamma(x+\alpha)}.\lambda^{x+\alpha-1}e^{-\lambda(\beta+1)/\beta}$$

gamma distribution with parameters

$x+\alpha$ and $\frac{\beta}{\beta+1}$

(b) $E(\Lambda|x) = \frac{(x+\alpha)\beta}{\beta+1}$ from Theorem 6.3

10.78 $\frac{25}{75}(27.6) + \frac{50}{75}(38.1) = 34.6$

10.79 $\frac{9}{13}(26.0) + \frac{4}{13}(32.5) = 28$

10.80 $\frac{4}{3}\cdot 210 - 1 = 279$

10.81 $\hat{\alpha} = \frac{n\bar{x}^2}{\sum(x-\bar{x})^2}$ $\qquad \hat{\beta} = \frac{\sum(x-\bar{x})^2}{n\bar{x}}$

or $\hat{\alpha} = \frac{(m_1')^2}{m_2' - (m_1')^2}$ $\qquad \hat{\beta} = \frac{m_2' - (m_1')^2}{m_1'}$

$\sum x = 86.4$ and $\sum x^2 = 756.52$

$m_1' = \frac{86.4}{12} = 7.2$ and $m_2' = \frac{756.52}{12} = 63.0433$

$$\hat{\alpha} = \frac{(7.2)^2}{63.0433 - (7.2)^2} = \frac{51.84}{63.0433 - 51.84} = 4.627$$

$$\hat{\beta} = \frac{63.0433 - (7.2)^2}{7.2} = 1.556$$

10.82 $\hat{\theta} = m_1'$ $\qquad \sum x = 201{,}000 \qquad \hat{\theta} = \frac{201{,}000}{5} = 40{,}200 \text{ miles}$

10.83 The likelihoods are $\dfrac{\binom{3}{1}\binom{N-3}{3}}{\binom{N}{4}}$

N	*Likelihood*	*N*	*Likelihood*
9	$\dfrac{\binom{3}{1}\binom{6}{3}}{\binom{9}{4}} = \dfrac{3\cdot 20}{126} = 0.4762$	12	$\dfrac{\binom{3}{1}\binom{9}{3}}{\binom{12}{4}} = \dfrac{3\cdot 84}{495} = 0.5091$
10	$\dfrac{\binom{3}{1}\binom{7}{3}}{\binom{10}{4}} = \dfrac{3\cdot 35}{210} = 0.5000$	13	$\dfrac{\binom{3}{1}\binom{10}{3}}{\binom{13}{4}} = \dfrac{3\cdot 120}{715} = 0.5035$
11	$\dfrac{\binom{3}{1}\binom{8}{3}}{\binom{11}{4}} = \dfrac{3\cdot 56}{330} = 0.5091$	14	$\dfrac{\binom{3}{1}\binom{11}{3}}{\binom{14}{4}} = \dfrac{3\cdot 165}{1001} = 0.4945$

Likelihood greatest for $N = 11$ or $N = 12$

10.84 $\hat{\theta} = 3m_1'$ $\qquad \sum x = 0.39 \qquad m_1' = \frac{0.39}{6} = 0.065 \qquad \hat{\theta} = 3\cdot\frac{0.39}{6} = 0.195$

10.85 $\sum x = 5524, \qquad \sum x^2 = 2{,}570{,}176 \qquad n = 12$

$m_1' = 460.3333 \qquad m_2' = 214{,}181.3333$

$\hat{\theta} = \sqrt{214{,}181.3333 - 211{,}906.7471} = 47.69$

$\hat{\delta} = 460.3333 - 47.69 = 412.64$

10.86 $\hat{\delta} = y_1 = 403 \qquad \hat{\theta} = 460.33 - 403 = 57.33$

10.87 $n = 8 \qquad \sum x = 63.1 \quad \sum x^2 = 541.55 \qquad m_1' = \frac{63.1}{8} = 7.8875$

$m_2' = \frac{541.55}{8} = 67.69375$

$\hat{\alpha} = 7.8875 - \sqrt{3(67.69375 - 62.2126)}$

$= 7.8875 - 4.0550 = 3.83$

$\hat{\beta} = 7.8875 + 4.0550 = 11.9427 = 11.95$

10.88 $\hat{\alpha} = 4.1$ and $\hat{\beta} = 11.5$

$\hat{\alpha} = y_1 \qquad \hat{\beta} = y_n$

10.89 $\hat{\alpha} = \dfrac{n}{\sum \ln x_i} = \dfrac{n(0.4343)}{\sum \log_{10} x}$

$\hat{\alpha} = \dfrac{15(0.4343)}{66.24567} = 0.098$

$\log_{10} x = 4.37840$

$n = 15$

$$\begin{array}{r} 4.37840 \\ 4.33244 \\ 4.42160 \\ 4.39445 \\ 4.52634 \\ 4.38917 \\ 4.46538 \\ 4.55871 \\ 4.35025 \\ 4.33244 \\ 4.45179 \\ 4.42813 \\ 4.49693 \\ 4.35603 \\ \underline{4.36361} \\ 66.24567 \end{array}$$

10.90 $n = 3 \qquad N = 20 \qquad n_0 = 11 \qquad n_1 = 7 \qquad n_2 = 2 \qquad n_3 = 0$

$\hat{\theta} = \dfrac{7 + 2 \cdot 2 + 3 \cdot 0}{3 \cdot 20} = \dfrac{11}{60}$

10.91 1, 3, 5, 1, 2, 1, 3, 7, 2, 4, 4, 8, 1, 3, 6, 5, 2, 1, 6, 2

$\sum x = 67 \qquad \hat{\theta} = \dfrac{20}{67} = 0.30$

10.92 $\sum v = 107.4 \qquad \sum v^2 = 116{,}108 \qquad n_1 = 10$

$\sum w = 674 \qquad \sum w^2 = 76{,}246 \qquad n_2 = 6$

$\hat{\mu}_1 = \dfrac{1074}{10} = 107.4 \qquad \hat{\mu}_2 = \dfrac{674}{6} = 112.3$

$\hat{\sigma}^2 = \dfrac{116{,}108 - 115{,}347.6 + 76{,}246 - 75{,}712.7}{16} = \dfrac{1{,}293.7}{16} = 80.86$

10.93 $n = 100 \qquad \theta_0 = 0.20 \qquad \sigma_0 = 0.04 \qquad x = 38$

$E(\theta|38) = \dfrac{38}{100} w + 0.20(1 - w)$

$w = \dfrac{100}{99 + \dfrac{(0.2)(0.8)}{(0.04)^2}} = \dfrac{100}{99 + 100} = 0.5025$

$E(\theta|38) = 0.38(0.5025) + 0.20(0.4975) = 0.29$

10.94 $\theta_0 = 0.74$ $\sigma_0 = 0.03$ $n = 30$ $x = 18$

(a) $\hat{\theta} = 0.74$

(b) $\hat{\theta}_n = \frac{x}{n} = \frac{18}{30} = 0.60$

(c) $$w = \frac{30}{29 + \frac{(0.74)(0.26)}{(0.03)^2}} = \frac{30}{29 + 213.8} = \frac{30}{242.8} = 0.1236$$

$$\hat{\theta} = (0.1236)(0.60) + (0.8764)(0.74) = 0.72$$

10.95 $\mu_1 = 715$ $\sigma_1 = 9.5$ $z = \frac{712 - 715}{9.5} = -0.32$

$$z = \frac{725 - 715}{9.5} = 1.05$$

$p = 0.1255 + 0.3531 = 0.4786$

10.96 $\mu_0 = 65.2$ $\sigma_0 = 1.5$ $z = \frac{63 - 65.2}{1.5} = -1.47$

$$z = \frac{68 - 65.2}{1.5} = 1.87$$

(a) $p = 0.4292 + 0.4693 = 0.8985$

(b) $w + \frac{40}{40 + \frac{7.4^2}{1.5^2}} = \frac{40}{64.34} = 0.62$ $\mu_1 = (0.62)72.9 + (0.38)65.2$
$= 69.97$

$$\frac{1}{\sigma_1^2} = \frac{40}{7.4^2} + \frac{1}{1.5^2} = 0.730 + 0.444 = 1.174 \qquad \sigma_1^2 = 0.92$$

$$z = \frac{63 - 70}{0.92} = -7.6$$

$$z = \frac{68 - 70}{0.92} = -2.18$$

$p = 0.5000 - 0.4854 = 0.0146$

10.97 (a) $\hat{\mu} = \alpha\beta = 50 \cdot 2 = 100$

(b) $\hat{\mu} = \bar{x} = 112$

(c) $\hat{\mu} = \mu_1 = \frac{2(50 + 112)}{3} = 108$

10.98 $n = \frac{z^2\sigma^2}{E^2} = \left(\frac{2.575 \cdot 4.2}{0.5}\right)^2 = 467.9$. Rounding up to the next integer, $n = 468$.

10.99 $z = \dfrac{E}{\sigma / \sqrt{n}} = \dfrac{6 \cdot 15}{1} = 9.0$; yes.

10.100 The sample is more likely to include longer sections than shorter ones; They take more time to pass the inspection station.

10.101 Heads of households may tend to have somewhat different political opinions than other members of the household who are likely to be younger and/or of a different sex.

Chapter 11

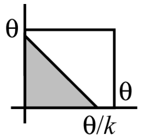
11.1 $P(0 < \theta < kx) = 1 - \alpha$

$$= p\left(x > \frac{\theta}{k}\right)$$

$$\int_{\theta/k}^{\infty} \frac{1}{\theta} e^{-x/\theta} dx = -e^{-x/\theta}\Big|_{\theta/k}^{\infty} = e^{-1/k} = 1 - \alpha$$

$$-\frac{1}{k} = \ln(1-\alpha) \text{ and } k = \frac{-1}{\ln(1-\alpha)}$$

11.2 **(a)**



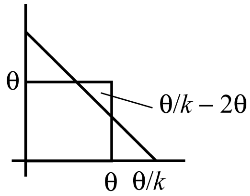
$$p[0 < \theta < k(x_1 + x_2)] = 1 - \alpha$$

$$p\left[(x_1 + x_2) > \frac{\theta}{k}\right] = 1 - \alpha$$

$$p\left[(x_1 + x_2) < \frac{\theta}{k}\right] = \alpha$$

$$\frac{1}{2} \cdot \frac{\theta^2}{k^2} \cdot \frac{1}{\theta^2} = \alpha \qquad \frac{1}{2k^2} = \alpha \qquad k^2 = \frac{1}{2\alpha} \qquad k = \frac{1}{\sqrt{2\alpha}}$$

(b)



$$p\left(x_1 + x_2 > \frac{\theta}{k}\right) = 1 - \alpha$$

$$\frac{1}{2}\left(\frac{\theta}{k} - 2\theta\right)^2 \frac{1}{\theta^2} = 1 - \alpha$$

$$\left(\frac{1}{k} - 2\right)^2 = 2(1-\alpha), \quad \frac{1}{k} - 2 = \pm\sqrt{2(1-\alpha)}$$

$$k = \frac{1}{2 \pm \sqrt{2(1-\alpha)}}$$

$$k = \frac{1}{2 - \sqrt{2(1-\alpha)}}$$

11.3 $p(R<\theta<cR)=1-\alpha$

$$p\left(\frac{\theta}{c}<R<\theta\right)=1-\alpha$$

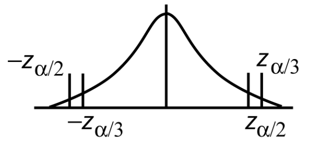
$$\frac{2}{\theta^2}\int_{\theta/c}^{\theta}(\theta-R)dR=1-\alpha \qquad \frac{2}{\theta^2}\left[\theta R-\frac{R^2}{2}\right]\Bigg|_{\theta/c}^{\theta}$$

$$\frac{2}{\theta^2}\left(\theta^2-\frac{\theta^2}{2}-\frac{\theta^2}{c}+\frac{\theta^2}{2c^2}\right)=1-\alpha$$

$$1-\frac{2}{c}+\frac{1}{2c^2}=1-\alpha,\ c^2-2c+1=(1-\alpha)c^2$$

$$ac^2-2c+1=0 \text{ and } c=\frac{2\pm\sqrt{4-4\alpha}}{2\alpha}=\frac{1\pm\sqrt{1-\alpha}}{\alpha}$$

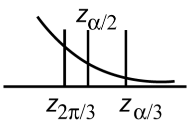
11.4



By inspection

$$z_{\alpha/3}-z_{\alpha/2}>z_{\alpha/2}-z_{2\alpha/3}$$
$$2z_{\alpha/2}<z_{\alpha/3}+z_{2\alpha/3}$$

length of first confidence interval is less than that of 2nd confidence interval



11.5 Length of confidence interval:

$$L=\overline{X}+z_{(1-k)\alpha}\cdot\frac{\sigma}{\sqrt{n}}-\left(\overline{X}-z_{k\alpha}\cdot\frac{\sigma}{\sqrt{n}}\right)$$
$$=(z_{(1-k)\alpha}+z_{k\alpha})\cdot\frac{\sigma}{\sqrt{n}}$$

If $k=1/2$, $\quad L_{1/2}=2z_{\alpha/2}\cdot\frac{\sigma}{\sqrt{n}}$

If $k<1/2$,

$$z_{k\alpha}=z_{a/2}+\delta_1>z_{\alpha/2} \qquad \delta_1>0;\ z_{(1-k)\alpha}<z_{(1-k)\alpha}+\delta_2=z_{\alpha/2} \text{ where } \delta_2>0$$

and $\quad L_k=[2z_{\alpha/2}+(\hat{\delta}_1-\hat{\delta}_2)]\cdot\frac{\sigma}{\sqrt{n}}$

Since the normal density function $f(x)$ is decreasing for $x>0$, $\delta_2<\delta_1$, thus

$$L_k>2z_{\alpha/2}\cdot\frac{\sigma}{\sqrt{n}}$$

By the symmetry of $f(x)$, for $k>1/2$, $L_k>2z_{\alpha/2}\cdot\frac{\sigma}{\sqrt{n}}$

11.6 $$p\left[|\bar{x}-\mu| < z_{\alpha/2}\cdot\frac{\sigma}{\sqrt{n}} = 1-\alpha\right]$$

$$z_{\alpha/2}\cdot\frac{\sigma}{\sqrt{n}} = E \text{ and } \sqrt{n} = z_{\alpha/2}\cdot\frac{\sigma}{E}$$

$$n = \left[z_{\alpha/2}\cdot\frac{\sigma}{E}\right]^2$$

11.7 Substitute $t_{\alpha/2,n-1}\cdot\frac{s}{\sqrt{n}}$ for $= z_{\alpha/2}\cdot\frac{\sigma}{\sqrt{n}}$

If $\bar{x}$, the mean of a random sample of size n from a formal population with the mean μ, is used as an estimate of μ, we can assert with $(1-\alpha)100\%$ confidence that the error is less than $t_{\alpha/2,n-1}\cdot\frac{s}{\sqrt{n}}$.

11.8 If $\bar{x}_1$ and $\bar{x}_2$ are the means of independent random samples of size n_1 and n_2 from normal populations with μ_1, μ_2, σ_1, and σ_2, and $\bar{x}_1-\bar{x}_2$ is to be used as an estimate if $\mu_1 - \mu_2$, the probability is $1-\alpha$ that error will be less than

$$z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}$$

11.9 $$E(S_p^2) = \frac{n_1-1}{n_1+n_2-2}\cdot\sigma^2 + \frac{n_2-1}{n_1+n_2-2}\cdot\sigma^2 = \frac{n_1+n_2-2}{n_1+n_2-2}\cdot\sigma^2 = \sigma^2$$

therefore unbiased

$$\frac{(n_1-1)s_1^2}{\sigma^2} \to \chi^2(n_1-1) \qquad \frac{(n_2-1)s_2^2}{\sigma^2} \to \chi^2(n_2-1)$$

$$\frac{(n_1-1)s_1^2}{\sigma^2} + \frac{(n_2-1)s_2^2}{\sigma^2} \to \chi^2(n_1+n_2-2) \qquad \text{var is } 2(n_1+n_2-2)$$

$$(n_1-1)s_1^2 + (n_2-1)s_2^2 \qquad \text{var is } 2\sigma^4(n_1+n_2-2)$$

$$\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2} \qquad \text{var is } \frac{2\sigma^4}{(n_1+n_2-2)}$$

11.10 $$T = \frac{Z}{\sqrt{\frac{Y}{n_1+n_2-2}}} = \frac{(\bar{x}_1-\bar{x}_2)-(\mu_2-\mu_1)}{\sigma\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}\cdot\frac{1}{\sqrt{\frac{(n_1+n_2-2)S_p^2}{n_1+n_2-2}}}$$

$$= \frac{(\bar{x}_1-\bar{x}_2)-(\mu_1-\mu_2)}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$$

11.11 $-z_{\alpha/2}\sqrt{n\theta(1-\theta)} = x-np$ and $z_{\alpha/2}\sqrt{n\theta(1-\theta)} = x-np$

$$z^2{}_{\alpha/2}n\theta(1-\theta) = (x-n\theta)^2 = x^2-2xn\theta+n^2\theta^2$$

$$n^2\theta^2+nz^2_{\alpha/2}-2xn\theta-nz^2_{\alpha/2}\theta+x^2=0$$

$$(n+z^2_{\alpha/2})\theta^2-(2x+z^2_{\alpha/2})\theta+\frac{x^2}{n}=0$$

by quadratic formula

$$\theta=\frac{2x+z^2_{a/2}\pm\sqrt{(2x+z^2_{\alpha/2})^2-4(n+z^2_{\alpha/2})\left(\frac{x^2}{n}\right)}}{2(\left(n+z^2_{\alpha/2}\right)}$$

11.13 $-z_{\alpha/2}<\frac{x-n\theta'}{\sqrt{n\theta'(1-\theta'')}}$; $\quad\frac{x-n\theta''}{\sqrt{n\theta''(1-\theta'')}}<z_{\alpha/2}$

Let θ^* = value of θ with $\theta'<\theta<\theta''$ closest to $\frac{1}{2}$. By Theorem 11.7,

$e<z_{\alpha/2}\sqrt{\frac{\theta^*(1-\theta^*)}{n}}$ and $n=\theta^*(1-\theta^*)\frac{z^2_{\alpha/2}}{e^2}$

11.15 By Theorem 11.8 with probability approximately $1-\alpha$

$$E<z_{\alpha/2}\sqrt{\frac{\hat{\theta}_1(1-\hat{\theta}_1)}{n_1}+\frac{\hat{\theta}_2(1-\hat{\theta}_2)}{n_2}}$$

11.16 If $n_1=n_2=n$, then $E<z_{\alpha/2}\sqrt{\frac{\hat{\theta}_1(1-\hat{\theta}_1)+\hat{\theta}_2(1-\hat{\theta}_2)}{n}}$

The right-hand side of this inequality is maximized when $\theta_1=\theta_2=\frac{1}{2}$.

Thus, $E<z_{\alpha/2}\sqrt{\frac{1}{2n}}$, $\quad E^2<\frac{z^2_{\alpha/2}}{2n}$, $\quad$ and $n=\frac{z^2_{\alpha/2}}{2E^2}$.

11.17 $\frac{1}{2n}\chi^2_{\alpha,2(x+1)}=\frac{1}{400}\chi^2_{0.01,8}=0.050$

11.18 $\frac{1}{f_{1-\alpha/2,n_1-1,n_2-1}}>\frac{\sigma_1^2s_2^2}{\sigma_2^2s_1^2}>\frac{1}{f_{\alpha/2,n_1-1,n_2-1}}$

$$\frac{s_1^2}{s_2^2}\cdot\frac{1}{f_{\alpha/2,n_1-1,n_2-1}}<\frac{\sigma_1^2}{\sigma_2^2}<\frac{s_1^2}{s_2^2}\cdot\frac{1}{f_{1-\alpha/2,n_1-1,n_2-1}}$$

$$\frac{s_1^2}{s_2^2}\cdot\frac{1}{f_{\alpha/2,n_1-1,n_2-1}}<\frac{\sigma_1^2}{\sigma_2^2}<\frac{s_1^2}{s_2^2}\cdot f_{\alpha/2,n_2-1,n_1-1}$$

11.19 $\sigma - z_{\alpha/2}\frac{\sigma}{\sqrt{2n}} < s < \sigma < z_{\alpha/2}\frac{\sigma}{\sqrt{2n}}$

$$\sigma\left(1-\frac{z_{\alpha/2}}{\sqrt{2n}}\right) < s < \sigma\left(1+\frac{z_{\alpha/2}}{\sqrt{2n}}\right)$$

$$\frac{1}{\sigma\left(1-\frac{z_{\alpha/2}}{\sqrt{2n}}\right)} > \frac{1}{s} > \frac{1}{\sigma\left(1+\frac{z_{\alpha/2}}{\sqrt{2n}}\right)}$$

$$\frac{s}{1+\frac{z_{\alpha/2}}{\sqrt{2n}}} < \sigma < \frac{s}{1-\frac{z_{\alpha/2}}{\sqrt{2n}}}$$

11.20 $n = 150$ $\sigma = 9.4$ $E = 1.96\frac{9.4}{\sqrt{150}} = \frac{1.96(9.4)}{12.247} = 1.50$

11.21 $61.8 \pm 2.575 \cdot \frac{9.4}{\sqrt{150}} = 61.8 \pm 1.98,\ 59.82 < \mu < 63.78$

11.22 $E = 2.575 \cdot \frac{10.5}{\sqrt{120}} = 2.575 \cdot \frac{10.5}{10.955} = 2.47$ mm

11.23 $141.8 \pm 2.33 \cdot \frac{10.5}{\sqrt{120}} = 141.8 \pm 2.33\frac{10.5}{10.955} = 141.8 \pm 2.23$

$$139.57 < \mu < 144.03$$

11.24 $\bar{x} \pm z_{0.005}\frac{s}{\sqrt{n}}$; $52.80 \pm 2.575\frac{45}{\sqrt{64}}$, or (51.35, 54.25).

11.25 $e < z_{0.025}\frac{s}{\sqrt{n}} = 1.96\frac{2.68}{\sqrt{40}} = 0.83$ min.

11.26 $e < z_{0.025}\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}} = 1.96\frac{9.4}{\sqrt{150}}\sqrt{\frac{900-150}{900-1}} = 1.37.$

11.27 $\bar{x} \pm z_{0.005}\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}$; $61.8 \pm 2.575\frac{9.4}{\sqrt{150}}\sqrt{\frac{900-150}{900-1}}$, or (60.01, 63.61).

11.28 $n = \left[z_{0.025}\frac{\sigma}{e}\right]^2 = \left[1.96\frac{12.2}{2.5}\right]^2 = 91.48$ or 92, rounded up to the nearest integer.

11.29 $n = \left[z_{\alpha/2}\frac{\sigma}{e}\right]^2 = 1.96\left[\frac{3.2}{1/3}\right]^2 = 354.04$ or 355, rounded up to the nearest integer.

11.30 $\bar{x} \pm t_{0.025,n-1} \frac{s}{\sqrt{n}}$; $\quad 5.68 \pm 2.262 \frac{0.29}{\sqrt{10}}$, or (5.47, 5.89)

11.31 $\bar{x} \pm t_{0.005,17} \frac{s}{\sqrt{n}}$; $\quad 63.84 \pm 2.898 \frac{2.75}{\sqrt{18}}$; or (61.96, 65.72).

11.32 $e < t_{0.025,11} \frac{s}{\sqrt{n}} = 2.201 \frac{0.625}{\sqrt{12}} = 0.40$

11.33 $(\bar{x}_1 - \bar{x}_2) \pm z_{0.05} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$; $\quad -5.2 \pm 1.645 \sqrt{\frac{4.8^2}{16} + \frac{3.5^2}{25}}$, or $(-7.49, -2.91)$.

11.34 $(\bar{x}_1 - \bar{x}_2) \pm z_{0.05} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$; $\quad -7.4 \pm 2.575 \sqrt{\frac{19.4^2 + 18.8^2}{61}}$, or $(-16.31, 1.51)$.

11.35 $s_p^2 = \frac{11(1.2)^2 + 14(1.5)^2}{25} = 1.8936 \qquad s_p = 1.376$

$$(13.8 - 12.9) \pm 2.060(1.376)\sqrt{\frac{1}{12} + \frac{1}{15}}$$
$$0.9 \pm 2.8346(0.387),\ 0.9 \pm 1.098$$
$$-0.198 < \mu_1 - \mu_2 < 1.998 \text{ feet}$$

11.36 $\bar{x}_1 = 8260,\ s_1 = 251.89,\ \bar{x}_2 = 7930,\ \mathrm{s}_2 = 206.52$

$$s_p^2 = \frac{4(251.89)^2 + 4(206.52)^2}{8} = 53{,}049.54 \qquad s_p = 230.32$$
$$8260 - 7930 \pm 3.355(230.32)\sqrt{\frac{1}{5} + \frac{1}{5}}$$
$$330 \pm 488.75$$
$$-158.75 < \mu_1 - \mu_2 < 818.75 \text{ million calorie per ton}$$

11.37 $E = 2.33\sqrt{\frac{(0.004)^2}{35} + \frac{(0.005)^2}{45}}$

$$= 2.33(0.001) = 0.0023 \text{ ohm}$$

11.38 $\hat{\theta} = \frac{204}{300} = 0.68$

$$0.68 \pm 1.96\sqrt{\frac{(0.68)(0.32)}{300}} \qquad 0.68 \pm 0.053$$
$$0.627 < \theta < 0.733$$

11.39 $e = 2.575\sqrt{\frac{(0.68)(0.32)}{300}} = 0.069$

11.40 **(a)** $\frac{190}{250} = 0.76 \qquad 0.76 \pm 2.575\sqrt{\frac{(0.76)(0.24)}{250}}$

$0.76 \pm 0.070 \qquad 0.690 < \theta < 0.830$

(b) $$\frac{190 + \frac{1}{2}(2.575)^2 \pm 2.575\sqrt{\frac{190(60)}{250} + \frac{1}{4}(2.575)^2}}{250 + (2.575)^2}$$

$$\frac{190 + 3.315 \pm 2.575\sqrt{45.6 + 1.658}}{250 + 6.631}$$

$$\frac{193.315 \pm 17.702}{256.631} \qquad 0.684 < \theta < 0.822$$

11.41 $e = 1.96\sqrt{\frac{(0.76)(0.24)}{250}} = 0.053$

11.42 $0.18 \pm 2.575\sqrt{\frac{(0.18)(0.82)}{100}} \qquad 0.18 \pm 0.099$

$0.081 < \theta < 0.279$

11.43 $\frac{54}{120} = 0.45 \qquad e = 1.645\sqrt{\frac{(0.45)(0.55)}{120}} = 0.075$

11.44 $0.05 = z\sqrt{\frac{(0.34)(0.66)}{300}} \qquad 0.05 = 0.02735z \qquad z = 1.83$

confidence is $2(0.4664) \cdot 100 = 93.3\%$

11.45 $n = \frac{(1.96)^2}{4(0.02)^2} = 2401$

11.46 $n = (0.03)(0.70)\left(\frac{1.96}{0.02}\right)^2 = (0.21)(9604) = 2017$

11.47 $n = \frac{(2.575)^2}{4(0.04)^2} = 1037$ rounded up

11.48 $n = (0.65)(0.35)\left(\frac{2.575}{0.04}\right)^2 = 943$

11.49 $\frac{84}{250} = 0.336$ $\frac{156}{250} = 0.624$

$$(0.336 - 0.624) \pm 1.96\sqrt{\frac{(0.336)(0.664)}{250} + \frac{(0.624)(0.376)}{250}}$$

-0.288 ± 0.084 $-0.372 < \theta_1 - \theta_2 < -0.204$

11.50 $\frac{48}{500} = 0.096, \quad \frac{68}{400} = 0.170$

$$0.096 - 0.170 \pm 2.575\sqrt{\frac{(0.096)(0.904)}{500} + \frac{(0.170)(0.830)}{400}}$$

-0.074 ± 0.059 $-0.133 < \theta_1 - \theta_2 < -0.015$

11.51 $e = 2.33\sqrt{\frac{(0.096)(0.904)}{500} + \frac{(0.170)(0.830)}{400}}$

$= 2.33(0.022939) = 0.053$

11.52 $n = \frac{(1.96)^2}{2(0.05)^2} = 769$

11.53 $\frac{9(0.29)^2}{19.023} < \sigma^2 < \frac{9(0.29)^2}{2.700}$

$0.04 < \sigma^2 < 0.28$

11.54 $\frac{11(0.625)^2}{19.675} < \sigma^2 < \frac{11(0.625)^2}{4.575}$

$0.2184 < \sigma^2 < 0.939$ $0.47 < \sigma < 0.97$

11.55 $\frac{4.5}{1 + \frac{2.575}{\sqrt{128}}} < \sigma < \frac{4.5}{1 - \frac{2.575}{\sqrt{128}}}$ $3.67 < \sigma < 5.83$

11.56 $\frac{2.68}{1 + \frac{2.33}{\sqrt{80}}} < \sigma < \frac{2.68}{1 - \frac{2.33}{\sqrt{80}}}$ $2.13 < \sigma < 3.62$

11.57 $\frac{19.4^2}{18.8^2} \cdot \frac{1}{f_{0.01,60,60}} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{19.4^2}{18.8^2} \cdot f_{0.01,60,60}$

$\frac{19.4^2}{18.8^2} \cdot \frac{1}{1.84} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{19.4^2}{18.8^2} \cdot 1.84$ $0.58 < \frac{\sigma_1^2}{\sigma_2^2} < 1.96$

11.58 $\frac{(1.2)^2}{(1.5)^2} \cdot \frac{1}{f_{0.01,11,14}} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{(1.2)^2}{(1.5)^2} \cdot f_{0.01,14,11}$

$\frac{0.64}{3.87} < \frac{\sigma_1^2}{\sigma_2^2} < (0.64)(4.30)$ $\qquad$ $0.165 < \frac{\sigma_1^2}{\sigma_2^2} < 2.752$

11.59 $\frac{(251.89)^2}{(206.52)^2} \cdot \frac{1}{f_{0.05,4,4}} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{(251.89)^2}{(206.52)^2} \cdot f_{0.05,4,4}$

$\frac{1.4876}{6.39} < \frac{\sigma_1^2}{\sigma_2^2} < 1.4876(6.39)$ $\qquad$ $0.233 < \frac{\sigma_1^2}{\sigma_2^2} < 9.506$

11.60 Using MINITAB we enter the data into C1 and we give the command

```
MTB> Tinterval 95.0 C1
```

Obtaining

```
N   MEAN    STDEV  SEMEAN  95.0 PERCENT C.I.
20  6.145   1.467   0.328    (5.458, 6.832)
```

11.61 Using MINITAB we enter the data into C1 and C2 and we give the command

```
MTB> St Dev C1   obtaining
ST DEV = 275.87
```

Then, with $\chi^2_{0.05,29} = 42.557$ and $\chi^2_{0.95,29} = 17.70$, we have

$$\frac{29(275.87)^2}{42.557} < \sigma^2 < \frac{29(275.87)^2}{17.78}$$

or $227.7 < \sigma < 352.3$ with 90% confidence.

Chapter 12

12.1 **(a)** simple; **(b)** composite (β *not* specified); **(c)** composite (parameter *not* specified); **(d)** composite (parameter *not* specified).

12.2 **(a)** simple; **(b)** composite (parameter *not* specified); **(c)** composite (σ *not* specified); **(d)** composite (θ *not* specified).

12.3
$$\alpha = -\frac{\binom{2}{2}\binom{5}{0}}{\binom{7}{2}} = \frac{1\cdot 1}{21} = \frac{1}{21}$$

$$\beta = \frac{\binom{4}{0}\binom{3}{2}}{\binom{7}{2}} + \frac{\binom{4}{1}\binom{3}{1}}{\binom{7}{2}} = \frac{1\cdot 3}{21} + \frac{4.3}{21} = \frac{15}{21} = \frac{5}{7}$$

12.4
$$\begin{aligned}
\alpha &= p(x \le 16; \theta = 0.90) = p(x \ge 4; \theta = 0.10)\\
&= 1 - (0.1216 + 0.2702 + 0.2852 + 0.1901)\\
&= 1 - 0.8671 = 0.1329\\
\beta &= p(x > 16; \theta = 0.60) = p(x < 4; \theta = 0.40)\\
&= 0.000 + 0.0005 + 0.0031 + 0.0123 = 0.0159
\end{aligned}$$

12.5
$$\alpha = p(x \ge k; \theta_0) = \frac{a}{1-r} = \frac{\theta_0(1-\theta_0)^{k-1}}{1-(1-\theta_0)} = (1-\theta_0)^{k-1}$$

$$\beta = p(x < k; \theta_1) = a\frac{1-r^n}{1-r} = \theta_1 \cdot \frac{1-(1-\theta_1)^{k-1}}{1-(1-\theta_1)} = 1-(1-\theta_1)^{k-1}$$

12.6
$$\begin{aligned}
\alpha &= p(x > 3; \theta = 2)\\
&= \int_3^{\infty} \frac{1}{2} e^{-x/2} dx = -e^{-x/2}\Big|_3^{\infty} e^{-1.5} = 0.223\\
\beta &= p(x \le 3; \theta = 5)\\
&= \int_0^{3} \frac{1}{5} e^{-x/5} dx = -e^{-x/5}\Big|_0^{3} = 1 - e^{-0.6} = 1 - 0.549 = 0.451
\end{aligned}$$

12.7
$$\bar{x} > \mu_0 + z_\alpha \cdot \frac{\sigma}{\sqrt{n}}$$

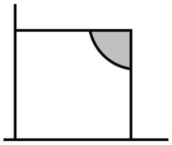
$$z_\alpha \cdot \frac{1}{\sqrt{2}} = 1 \qquad z_\alpha = \sqrt{2} = 1.414$$

$$a = 0.5000 - 0.4207 = 0.8$$

12.8 $p(x > \beta_0 + 1; \beta_0) = 0$

$$p(x \le \beta_0 + 1; \beta_0 + 2) = (\beta_0 + 1) \cdot \frac{1}{\beta_0 + 2} = \frac{\beta_0 + 1}{\beta_0 + 2}$$

12.9



$$1 - \beta = 4 \int_{3/4}^{1} x_2 \int_{3/4x_2}^{1} x_1 \; dx_1 \; dx_2$$

$$= 4 \int_{3/4}^{1} x_2 \left[\frac{1}{2} - \frac{9}{32x_2^2} \right] dx_2$$

$$1 - \beta = \int_{3/4}^{1} 2x_2 \; dx_2 - \frac{9}{8} \int_{3/4}^{1} \frac{dx_2}{x_2}$$

$$= 1 - \frac{9}{16} + \frac{9}{8} \ln 0.75$$

$$= \frac{7}{16} - \frac{9}{8}(0.28768) = 0.114$$

12.10 Proof same as in Example 12.4 except that the quantity $n(\mu_0 - \mu_1)$ is now *positive* and the inequalities are

$\bar{x} \le K$ inside c
$\bar{x} \ge K$ outside c

where $k = \mu_0 - z_\alpha \frac{1}{\sqrt{n}}$. So, critical region is

$$\bar{x} \le \mu_0 - z_\alpha \frac{1}{\sqrt{n}}$$

12.11 $L_0 = \frac{1}{\theta_0^n} e^{-(1/\theta)\sum x_i}$ $\qquad$ $L_1 = \frac{1}{\theta_1^n} e^{-(1/\theta_1)\sum x_i}$

$$\frac{L_0}{L_1} = \left(\frac{\theta_1}{\theta_0} \right)^n e^{-\sum x_i (1/\theta_0 - 1/\theta_1)} \le k$$

$$n \ln \frac{\theta_1}{\theta_0} - \sum x_i \left(\frac{1}{\theta_0} - \frac{1}{\theta_1} \right) \le \ln k$$

$$\sum x_i \ge \frac{n \ln \frac{\theta_1}{\theta_0} \ln k}{\frac{1}{\theta_0} - \frac{1}{\theta_1}} = K$$

Critical region is $\sum_{i=1}^{n} x_i \ge K$, where K can be determined by making use of fact that $\sum_{i=1}^{n} x_i$ has the gamma distribution with $\alpha = n$ and $\beta = \theta_0$.

12.12 $L_0 = \binom{n}{x}\theta_0^x(1-\theta_0)^{n-x} \qquad L_1 = \binom{n}{x}\theta_1^x(1-\theta_1)^{n-x}$

$$\frac{L_0}{L_1} = \left[\frac{\theta_0(1-\theta_1)}{\theta_1(1-\theta_0)}\right]^x \left(\frac{1-\theta_0}{1-\theta_1}\right)^n \le k$$

$$x \cdot \ln\frac{\theta_0(1-\theta_1)}{\theta_1(1-\theta_0)} + n \cdot \ln\frac{1-\theta_0}{1-\theta_1} \le \ln k$$

$$x \le \frac{\ln k - n\ln\frac{1-\theta_0}{1-\theta_1}}{\ln\frac{\theta_0(1-\theta_1)}{\theta_1(1-\theta_0)}} = K$$

Critical region is $x \le K$, where K can be determined from table of binomial probabilities.

12.13 $\frac{K-100(0.40)}{\sqrt{100(0.4)(0.6)}} = -1.645$, $K = 40\ -1.645(4.90) = 31.94$

Critical region $x \le 31$

$z = \frac{31.5-30}{\sqrt{100(0.3)(0.7)}} = \frac{1.5}{4.58} = 0.33 \qquad \theta = 0.5 - 0.1293 = 0.37$

12.14 $f(x) = \theta(1-\theta)^{x-1} \qquad x = 1,2,3,\ldots$

$L_0 = \theta_0(1-\theta_0)^{x-1} \qquad L_1 = \theta_1(1-\theta_1)^{x-1}$

$$\frac{L_0}{L_1} = \left[\frac{\theta_0(1-\theta_1)}{\theta_1(1-\theta_0)}\right]\left[\frac{1-\theta_0}{1-\theta_1}\right]^x \le k$$

$$\ln\left[\frac{\theta_0(1-\theta_1)}{\theta_1(1-\theta_0)}\right] + x \cdot \ln\frac{1-\theta_0}{1-\theta_1} \le \ln k$$

$$x \le \frac{\ln k - \ln\frac{\theta_0(1-\theta_1)}{\theta_1(1-\theta_0)}}{\frac{1-\theta_0}{1-\theta_1}} = K$$

Critical region is $x \le K$, where K can be determined using formula for sum of terms of geometric progression.

12.15 $L_0 = \frac{1}{(\sqrt{2\pi})^n\sigma_0^n}e^{-(1/2\sigma_0^2)\sum x^2} \qquad L_1 = \frac{1}{\sqrt{2\pi}^n\sigma_1^n}e^{-(1/2\sigma_1^2)\sum x^2}$

$$\frac{L_0}{L_1} = \left(\frac{\sigma_1}{\sigma_0}\right)^n e^{-\left(\sum x^2/2\right)\left(1/\sigma_0^2 - 1/\sigma_1^2\right)} \le k$$

$$n\ln\frac{\sigma_1}{\sigma_2} - \frac{\sum x^2}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right) \le \ln k$$

$$\sum x^2 \geq \frac{n \ln \frac{\sigma_1}{\sigma_0} - \ln k}{\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} \right)} = K$$

Critical region is $\sum x^2 \geq K$, where K is determined using the fact that $\sum x^2 = (n-1)s^2$ and $\frac{(n-1)s^2}{\sigma_0^2}$ is random variable having χ^2 distribution with $n-1$ degrees of freedom. Therefore, critical region is $\sum x^2 \geq \sigma_0^2 \cdot \chi^2_{\alpha,n-1}$.

12.16 The probabilities of making wrong decisions are

	$\theta = 0.9$	$\theta = 0.6$
d_1	0.0114	0.1255
d_2	0.0433	0.0509
d_3	0.0025	0.2499

(a) (0.0114)(0.8) + (0.1255)(0.2)=0.034

(b) (0.0433)(0.8) + (0.0509)(0.2) = 0.045

(c) (0.0025)(0.8) + (0.2499)(0.2) = 0.052

12.17 (a) $\frac{\binom{0}{2}\binom{7}{0}}{\binom{7}{2}} = 0$ $\qquad$ $\frac{\binom{1}{2}\binom{6}{0}}{\binom{7}{2}} = 0$ $\qquad$ $\frac{\binom{2}{2}\binom{5}{0}}{\binom{7}{2}} = \frac{1}{21}$

(b) $1 - \frac{\binom{4}{2}\binom{3}{0}}{\binom{7}{2}} = \frac{5}{7}$ $\qquad$ $1 - \frac{\binom{5}{2}\binom{2}{0}}{\binom{7}{2}} = \frac{11}{21}$ $\qquad$ $1 - \frac{\binom{6}{2}\binom{1}{0}}{\binom{7}{2}} = \frac{2}{7}$

$1 - \frac{\binom{7}{2}\binom{0}{0}}{\binom{7}{2}} = 0$

12.18
$\theta = 0.95$ $\quad \alpha = 0.0022 + 0.0003 = 0.0025$
$\theta = 0.90$ $\quad \alpha = 0.0319 + 0.0089 + 0.0020 + 0.0004 + 0.0001 = 0.0433$
$\theta = 0.85$ $\quad 1 - \beta = 1 - (0.0388 + 0.1368 + 0.2293 + 0.2428 + 0.1821) = 0.1702$
$\theta = 0.80$ $\quad 1 - \beta = 1 - (0.0115 + 0.0576 + 0.1369 + 0.2054 + 0.2182) = 0.3704$
$\theta = 0.75$ $\quad 1 - \beta = 1 - (0.0032 + 0.0211 + 0.0669 + 0.1339 + 0.1897) = 0.5852$
$\theta = 0.70$ $\quad 1 - \beta = 1 - (0.0008 + 0.0068 + 0.0278 + 0.0716 + 0.1304) = 0.7626$
$\theta = 0.65$ $\quad 1 - \beta = 1 - (0.0002 + 0.0020 + 0.0100 + 0.0323 + 0.0738) = 0.8817$
$\theta = 0.60$ $\quad 1 - \beta = 1 - (0.0005 + 0.0031 + 0.0123 + 0.0350) = 0.9491$
$\theta = 0.55$ $\quad 1 - \beta = 1 - (0.0001 + 0.0008 + 0.0040 + 0.0139) = 0.9812$
$\theta = 0.50$ $\quad 1 - \beta = 1 - (0.0002 + 0.0011 + 0.0046) = 0.9941$

12.19 $x_i - \mu_0 = (x_i - \overline{x}) + (\overline{x} - \mu_0)$

$$\begin{aligned}\sum (x_i - \mu_0)^2 &= \sum (x_i - \overline{x})^2 + 2\sum (x_i - \overline{x})(\overline{x} - \mu_0) + \sum (\overline{x} - \mu_0)^2 \\ &= \sum (x_i - \overline{x})^2 + 2\sum (\overline{x} - \mu_0)\sum (x_i - \overline{x}) + \sum (\overline{x} - \mu_0)^2 \\ &= \sum (x_i - \overline{x})^2 + \sum (\overline{x} - \mu_0)^2\end{aligned}$$

Therefore $\lambda = e^{-1/2\sigma^2}\left[\sum (x_i - \mu_0)^2 - \sum (x_i - x)^2\right]$

$$\begin{aligned}&= e^{-(1/2\sigma^2)}\sum (\overline{x} - \mu_0)^2 \\ &= e^{-(n/2\sigma^2)(\overline{x} - \mu_0)^2}\end{aligned}$$

12.20 (a) $L = \binom{n}{x}\theta^x (1-\theta)^{n-x} \qquad L_0 = \binom{n}{x}\left(\frac{1}{2}\right)^n$

$$\ln L = \ln\binom{n}{x} + x\ln\theta + (n-x)\ln(1-\theta)$$

$$\frac{d\ln L}{d\theta} = \frac{x}{\theta} - \frac{n-x}{1-\theta} = 0 \text{ yields } \theta = \frac{x}{n}$$

$$\max L = \binom{n}{x}\left(\frac{x}{n}\right)^x\left(\frac{n-x}{n}\right)^{n-x}$$

and $\lambda = \dfrac{\left(\frac{1}{2}\right)^n}{\left(\frac{x}{n}\right)^x\left(\frac{n-x}{n}\right)^{n-x}} = \dfrac{(n/2)^n}{x^x(n-x)^{n-x}} \le k$

(b)

$$-n\ln 2 + n\ln n - x\ln x - (n-x)\ln(n-x) \le \ln k$$

$$-x\ln x - (n-x)\ln(n-x) \le k'$$

$$x\ln x + (n-x)\ln(n-x) \ge K$$

(c)

$$f(x) = x\ln x + (n-x)\ln(n-x)$$

$$\frac{df(x)}{dx} = \ln x + 1 - \ln(n-x) - 1 = 0$$

$x = n - x$ and $x = \dfrac{n}{2}$ is minimum

Since $f(n-x) = f(x)$, symmetrical about $x = \dfrac{n}{2}$. Therefore critical region is $\left|x - \dfrac{n}{2}\right| \ge c$.

12.21 (a) $L = \frac{1}{\theta^n} e^{-(1/\theta)\sum x}$ $\qquad$ $\max L_0 = \frac{1}{\theta_0^n} e^{-(1/\theta_0)\sum x}$

$$\ln L = -n \ln \theta - \frac{1}{\theta} \sum x$$

$$\frac{d \ln L}{d\theta} = -\frac{n}{\theta} + \frac{\sum x}{\theta^2} = 0 \qquad \theta = \overline{x}$$

$$\lambda = \frac{\frac{1}{\theta_0^n} e^{-(1/\theta_0)\sum x}}{\frac{1}{\overline{x}^n} e^{-(1/\overline{x})\sum x}} = \left(\frac{\overline{x}}{\theta_0}\right)^n e^{-(n\overline{x}/\theta_0)+n}$$

(b) $\left(\frac{\overline{x}}{n}\right)^n e^{-(n\overline{x}/\theta_0)} \le \frac{k}{e^n} = k'$

$$\frac{\overline{x}}{n} e^{-\overline{x}/\theta_0} \le \sqrt[n]{k}$$

$$\overline{x} e^{-\overline{x}/\theta_0} \le n\sqrt[n]{k} = K$$

$$\overline{x} e^{-\overline{x}/\theta_0} \le K$$

12.22 Over Ω maximum likelihood estimates are $\hat{\mu} = \overline{x}$ and $\hat{\sigma}^2 = \frac{\sum (x-\overline{x})^2}{n}$

Over w maximum likelihood estimates are $\hat{\mu}_0 = \mu_0$ and $\hat{\sigma}_0^2 = \frac{\sum (x-\mu_0)^2}{n}$

$$\lambda = \frac{\frac{1}{(\sqrt{2\pi})^n \hat{\sigma}_0^n} e^{-(1/2\hat{\sigma}^2)}}{\frac{1}{(\sqrt{2\pi})^n \hat{\sigma}^2} e^{-(1/2\hat{\sigma}^2)}} = \left(\frac{\hat{\sigma}^2}{\hat{\sigma}_0^2}\right)^{n/2} = \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}^2}\right)^{-n/2}$$

$$\lambda^{-2/n} = \frac{\sum (x-\mu_0)^2}{\sum (x-\overline{x})^2} = \frac{\sum (x-\overline{x})^2 + n(\overline{x}-\mu_0)^2}{\sum (x-\overline{x})^2} = 1 + \frac{n(\overline{x}-\mu_0)}{\sum (x-\overline{x})^2}$$

$$= 1 + \frac{t^2}{n-1} \text{ where } t = \frac{\sqrt{n}(\overline{x}-\mu_0)^2}{s}$$

$$\lambda = 1 + \frac{t^2}{n-1}, \text{ where } t = \frac{\sqrt{n}(\overline{x}-\mu_0)}{s}$$

12.23 Use $\ln(1+\lambda) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \ldots$

$$\lambda^{-2} = \left(1 + \frac{t^2}{n-1}\right)^n$$

$$-2\ln\lambda = n\ln\left(1 + \frac{t^2}{n-1}\right) = n\left[\frac{t^2}{n-1} - \frac{1}{2}\left(\frac{t^2}{n-1}\right)^2 + \frac{1}{3}\left(\frac{t^2}{n-1}\right)^3 - \ldots\right]$$

$$\rightarrow \quad t^2$$

12.24 $\max L_0 = \dfrac{1}{(\sqrt{2\pi})^n\,\sigma_0^n}\,e^{-(1/2\sigma_0^2)\sum(x-\overline{x})^2}$

$$\max L = \frac{1}{(\sqrt{2\pi})^n\,\hat{\sigma}_0^n}\,e^{-(1/2\hat{\sigma}^2)\sum(x-\overline{x})^2}$$

$$\lambda = \left[\frac{\sum(x-\overline{x})^2}{n\sigma_0^2}\right]^{n/2} e^{-(1/2)\sum(x-\overline{x})^2(1/\sigma_0^2 - 1/\hat{\sigma}^2)}$$

$$\frac{1}{\sigma_0^2} - \frac{1}{\hat{\sigma}^2} = \frac{1}{\sigma_0^2} - \frac{n}{\sum(x-\overline{x})^2}$$

$$\lambda = \left[\frac{\sum(x-\overline{x})^2}{n\sigma_0^2}\right]^{n/2} e^{-(1/2)\left\{\left[\sum(x-\overline{x})^2/\sigma_0^2\right]-n\right\}}$$

12.25 (a) $L = \prod_{i=1}^{k} = \dfrac{1}{(\sqrt{2\pi})^{n_i}\,\sigma_i^{n_i}}\,e^{-\left[\frac{1}{2\sigma_i^2}\sum_{j=1}^{n_i}(x_{ij}-\mu_i)^2\right]}$

proceed as in Example 10.17

(b) $\max L_0 = \prod_{i=1}^{k} = \dfrac{1}{(\sqrt{2\pi})^{n_i}\,tc\hat{\sigma}_i^{n_i}}\,e^{-(1/2\hat{\sigma}_i^2)\sum_j(x_{ij}-\overline{x}_i)^2}$

$$\max L = \prod_{i=1}^{k} = \frac{1}{(\sqrt{2\pi})^{n_i}\,\hat{\hat{\sigma}}_i^{n_i}}\,e^{-(1/2\hat{\sigma}_i^2)\sum_j(x_{ij}-\overline{x}_i)^2}$$

$$\hat{\sigma}_i^2 = \sum_i \frac{(n_i-1)s_i^2}{\sum n_i} \qquad \hat{\hat{\sigma}}_i^2 = \frac{(n_i-1)s_i^2}{n_i}$$

$$\lambda = \frac{\prod_i \left[\frac{(n_i-1)s_i^2}{n_i}\right]^{n_i/2}}{\left[\sum_i \frac{(n_i-1)s_i^2}{n}\right]^{n/2}}$$

12.26 Dividing numerator and denominator by $\left(s_1^2\right)^{(n_1+n_2)/2}$ yields

$$\lambda = \frac{\left(\frac{n_1-1}{n_1}\right)^{n_1/2}\left(\frac{n_2-1}{n_2}\cdot\frac{s_2^2}{s_1^2}\right)^{n_2/2}}{\left(\frac{n_1-1}{n}+\frac{n_2-1}{n}\cdot\frac{s_2^2}{s_1^2}\right)^{n_2-2}}$$

QED

12.27 $L = 1+\theta^2\left(\frac{1}{2}-x\right)$

$$\pi(0) = \int_0^\alpha 1dx = \sigma$$

$$\beta = \int_\alpha^1 \left[1+\theta^2\left(\frac{1}{2}-x\right)\right]dx = 1-\alpha-\frac{1}{2}\theta^2\alpha(1-\alpha)$$

$$1-\beta = \alpha+\frac{1}{2}\theta^2\alpha(1-\alpha)$$

$$\pi(\theta) = \alpha+\frac{1}{2}\theta^2\alpha(1-\alpha)$$

Since $\frac{1}{2}\theta^2\alpha(1-\alpha) > 0$ for $0 < \alpha < 1$

$\pi(0)$ has minimum at $\theta = 0$

12.28 They would be committing a type I error if they erroneously reject the null hypothesis that 60% of their passengers object to smoking inside the plane.

They would be committing a type I error if they erroneously accept this null hypothesis.

12.29 The doctor would commit a type I error if he/she erroneously rejects the null hypothesis that the executive is able to take on additional responsibilities. The doctor would commit a type II error if he/she erroneously accepts this null hypothesis.

12.30 **(a)** The manufacturer should use the alternative hypothesis $\mu < 20$ and make the modification only if the null hypothesis can be rejected.

(b) The manufacturer should use the alternative hypothesis $\mu > 20$ and make the modification unless the null hypothesis can be rejected.

12.31 **(a)** $H_1 : \mu_2 > \mu_1$

(b) $H_1 : \mu_1 > \mu_2$

(c) $H_1 : \mu_1 \neq \mu_2$

12.32 With $\mu = 9.6,\ \bar{x} = 10.2,$ and $n = 80$

(a) Decision: reject H_0: since H_0 is true, decision is in error.

(b) Decision: reject H_0: since H_0 is false, decision is not in error.

(c) Decision: reject H_0: since H_0 is true, decision is in error.

(d) Decision: reject H_0: since H_0 is true, decision is not in error.

12.33 (a) $H_0 : \mu_1 = \mu_2$

(b) $H_1 : \mu_2 > \mu_1$

(c) $H_1 : \mu_2 < \mu_1$

12.34 (a) H_0: the antipollution device is effective. A type I error would be made if the device is effective and H_0 is rejected. A type II error would be made if the device is not effective and H_0 is not rejecte4d.

(b) H_0: The antipollution device is not effective.

12.35 (a) She will correctly reject the null hypothesis.
(b) She will erroneously reject the null hypothesis.

12.36 (a) He will erroneously accept the null hypothesis.
(b) He will correctly accept the null hypothesis.

12.37 (a) $-\sqrt{n} + 1.645 + -1.88$

$\sqrt{n} = 3.525 \qquad n = 12.43 \qquad n = 13$ rounded up to nearest integer

(b) $-\sqrt{n} + 1.645 = -2.33$

$\sqrt{n} = 3.975 \qquad n = 15.80 \qquad n = 16$ rounde4d up to nearest integer

12.38 (a) Yes; **(b)** Yes

12.39 (a)

$$1 - \int_8^{12} \frac{1}{10} e^{-x/10} dx = 1 + e^{-x/10}\Big|_8^{12} + 1 + e^{-1.2} - e^{-0.8}$$

$$= 1 + 0.3012 - 0.4493 = 0.852$$

(b)

$$\int_8^{12} \frac{1}{2} e^{-x/2} dx = -e^{-x/2}\Big|_8^{12} = e^{-4} - e^{-6} = 0.0183 - 0.0025 = 0.016$$

$$\int_8^{12} \frac{1}{4} e^{-x/4} dx = -e^{-x/4}\Big|_8^{12} = e^{-2} - e^{-3} = 0.1353 - 0.0448 = 0.086$$

$$\int_8^{12} \frac{1}{6} e^{-x/6} dx = -e^{-x/6}\Big|_8^{12} = e^{-1.33} - e^{-2} = 0.2645 - 0.1353 = 0.129$$

$$\int_8^{12} \frac{1}{8} e^{-x/8} dx = -e^{-x/8}\Big|_8^{12} = e^{-1} - e^{-1.5} = 0.3679 - 0.2231 = 0.145$$

$$\int_8^{12} \frac{1}{12} e^{-x/12} dx = -e^{-x/12}\Big|_8^{12} = e^{-0.67} - e^{-1} = 0.5117 - 0.3679 = 0.144$$

$$\int_8^{12} \frac{1}{16} e^{-x/16} dx = -e^{-x/16}\Big|_8^{12} = e^{-0.50} - e^{-0.75} = 0.6065 - 0.4724 = 0.134$$

$$\int_8^{12} \frac{1}{20} e^{-x/20} dx = -e^{-x/20}\Big|_8^{12} = e^{-0.40} - e^{-0.60} = 0.6703 - 0.5488 = 0.122$$

12.40 Reject if $\bar{x} > 43.5$ $\quad \sigma_{\bar{x}} = \sqrt{\frac{265}{64}} = 2$

(a)

$$z = \frac{43.5-37}{2} = 3.25, \quad P(\bar{X} > 43.5 | \mu = 37) = P(Z > 3.25) = 0.00058$$

$$z = \frac{43.5-38}{2} = 2.75, \quad P(\bar{X} > 43.5 | \mu = 38) = P(Z > 2.75) = 0.003$$

$$z = \frac{43.5-39}{2} = 2.25, \quad P(\bar{X} > 43.5 | \mu = 39) = P(Z > 2.25) = 0.0122$$

$$z = \frac{43.5-40}{2} = 1.75, \quad P(\bar{X} > 43.5 | \mu = 40) = P(Z > 1.75) = 0.04$$

(b)

$$z = \frac{43.5-41}{2} = 1.25, \quad P(\bar{X} \le 43.5 | \mu = 41) = P(Z \le 1.25) = 0.8944$$

$$z = \frac{43.5-42}{2} = 0.75, \quad P(\bar{X} \le 43.5 | \mu = 42) = P(Z \le 0.75) = 0.7734$$

$$z = \frac{43.5-43}{2} = 0.25, \quad P(\bar{X} \le 43.5 | \mu = 43) = P(Z \le 0.25) = 0.5987$$

$$z = \frac{43.5-44}{2} = -0.25, \quad P(\bar{X} \le 43.5 | \mu = 44) = P(Z \le -0.25) = 0.4103$$

$$z = \frac{43.5-45}{2} = -0.75, \quad P(\bar{X} \le 43.5 | \mu = 45) = P(Z \le -0.75) = 0.2266$$

$$z = \frac{43.5-46}{2} = -1.25, \quad P(\bar{X} \le 43.5 | \mu = 46) = P(Z \le -1.25) = 0.1056$$

$$z = \frac{43.5-47}{2} = -1.75, \quad P(\bar{X} \le 43.5 | \mu = 47) = P(Z \le -1.75) = 0.04$$

$$z = \frac{43.5-48}{2} = -2.25, \quad P(\bar{X} \le 43.5 | \mu = 48) = P(Z \le 2.25) = 0.0122$$

12.41 (a) Reject if $\sum x \le 5$ Use Table II

$\lambda = 11$	$p = 0.0375$	$\lambda = 12$	$p = 0.0203$
$\lambda = 13$	$p = 0.0107$	$\lambda = 14$	$p = 0.0055$
$\lambda = 15$	$p = 0.0027$		

(b) $\lambda = 10,\ 1 - 0.0671 = 0.9329,\ \lambda = 7.5,\ 1 - 0.2415 = 0.7585$

$\lambda = 5,\ 1 - 0.6160 = 0.3840,\ \lambda = 2.5,\ 1 - 0.9580 = 0.0420$

12.42 $\mu = 50, \quad \sigma = 5, \quad z = \dfrac{56.6 - 50}{5} = 1.3$

Probability of 57 or more heads is $0.500 - 0.4032 = 0.0968$
Since $0.0968 > 0.05$ null hypothesis cannot be rejected.

12.43 $$\lambda = \frac{\left(\frac{7 \cdot 16}{8}\right)^4 \left(\frac{9 \cdot 25}{10}\right)^5 \left(\frac{5 \cdot 12}{6}\right)^3 \left(\frac{7 \cdot 24}{8}\right)^4}{\left[(112 + 225 + 60 + 168) / 32\right]^{16}}$$
$$= \frac{14^4 \cdot 22.5^5 \cdot 10^3 \cdot 21^4}{17.656^{16}}$$

$$\ln \lambda = 4(2.63906) + 5(3.11352) + 3(2.30259) + 4(3.04452) - 16(2.8711)$$
$$= -0.712 \qquad\qquad -2 \ln \lambda = 1.424$$

Since this is less than $\chi^2_{0.05,3} = 7.815$, the null hypothesis cannot be rejected.

12.44 From Exercise 12.21
$$\lambda = \left(\frac{\overline{x}}{\theta_0}\right)^n e^{-(n\overline{x}/\theta_0)+n}$$
$$\ln \lambda = n \ln \frac{\overline{x}}{\theta_0} - \frac{n\overline{x}}{\theta_0} + n = 20 \ln \frac{529}{300} - \frac{529}{15} + 20$$
$$= 20(0.5670) - 15.27 = -3.93 \qquad -2 \ln \lambda = 2(3.93) = 7.86$$

Since 7.86 exceeds $\chi^2_{0.05,1} = 3.841$, the null hypothesis must be rejected.

Chapter 13

13.1 Test statistic $z = \dfrac{\overline{x} - \mu_0}{\sigma / \sqrt{n}}$

Then by Theorem 8.7 $\left(\dfrac{\overline{x} - \mu_0}{\sigma / \sqrt{n}} \right)^2$ is random variable having χ^2 distribution with $v = 1$. So rejection criterion becomes $\dfrac{n(\overline{x} - \mu_0)^2}{\sigma^2} \geq \chi^2_{\alpha,1}$

13.2 $K = \mu_0 + z_\alpha \dfrac{\sigma}{\sqrt{n}}$ and $K = \mu_1 - z_\beta \dfrac{\sigma}{\sqrt{n}}$

$$\mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} = \mu_1 - z_\beta \frac{\sigma}{\sqrt{n}}$$

$$\mu_1 - \mu_0 = (z_\alpha + z_\beta) \frac{\sigma}{\sqrt{n}} \rightarrow \sqrt{n} = \frac{\sigma(z_\alpha + z_\beta)}{\mu_1 - \mu_0}$$

and $n = \dfrac{\sigma^2 (z_\alpha + z_\beta)^2}{(\mu_1 - \mu_0)^2}$

13.3 $n = \dfrac{9^2(1.645 + 2.33)^2}{5^2} = \dfrac{81(3.975)^2}{25} = 51.19$ $\qquad n = 52$

13.4 $K = \delta + z_\alpha \sqrt{\dfrac{\sigma_1^2 + \sigma_2^2}{n}}$ $\qquad K = \delta' - z_\beta \sqrt{\dfrac{\sigma_1^2 + \sigma_2^1}{n}}$

$$\delta + z_\alpha \sqrt{\frac{\sigma_1^2 + \sigma_2^2}{n}} = \delta' - z_\beta \sqrt{\frac{\sigma_1^2 + \sigma_2^2}{n}}$$

$$\delta - \delta' = (z_\alpha + z_\beta) \sqrt{\frac{\sigma_1^2 + \sigma_2^2}{n}}$$

$\sqrt{n} = \dfrac{(z_\alpha + z_\beta)\sqrt{\sigma_1^2 + \sigma_2^2}}{\delta - \delta'}$ and $n = \dfrac{(\sigma_1^2 + \sigma_2^2)(z_\alpha + z_\beta)^2}{(\delta - \delta')^2}$

13.5 $n = \dfrac{(81 + 169)(2.33 + 2.33)^2}{6^2} = \dfrac{(250(21.7156)}{36} = 150.80 = 151$

13.6 $\frac{(n-1)s^2}{\sigma_0^2}$ has chi square distribution with $(n-1)$ degrees of freedom, so that according to corollary 2 to Theorem 6.3

$\mu = n-1$ and $\sigma = \sqrt{2(n-1)}$

Using normal approximation, critical region is

$$\frac{(n-1)s^2}{\sigma_0^2} \geq n-1+z_\alpha\sqrt{2(n-1)}$$

or $s^2 \geq \sigma_0^2\left[1+z_\alpha\sqrt{\frac{2}{n-1}}\right]$

For $H_1 : \sigma^2 < \sigma_0^2$ critical region is $s^2 \leq \sigma_0^2\left[1-z_\alpha\sqrt{\frac{2}{n-1}}\right]$

For $H_1 : \sigma^2 \neq \sigma_0^2$ critical region is $s^2 \leq \sigma_0^2\left[1-z_{\alpha/2}\sqrt{\frac{2}{n-1}}\right]$ or $s^2 \geq \sigma_0^2\left[1+z_{\alpha/2}\sqrt{\frac{2}{n-1}}\right]$

13.7 If x has χ^2 distribution with $n-1$ degrees of freedom, then according to Example 8.42 $\sqrt{2x}-\sqrt{2(n-1)} \to$ standard normal distribution.

Since $\frac{(n-1)s^2}{\sigma_0^2}$ has chi square distribution with $n-1$ degrees of freedom.

$\sqrt{\frac{2(n-1)s^2}{\sigma_0^2}}-\sqrt{2(n-1)}$ has approximately standard normal distribution

$\frac{s}{\sigma_0}\sqrt{2(n-1)}-\sqrt{2(n-1)}$ has approximately standard normal distribution

$\left(\frac{s}{\sigma_0}-1\right)\sqrt{2(n-1)}$ has approximately standard normal distribution

13.8 $e_{i1} = n_1\hat{\theta},\ e_{i2} = n_i(1-\hat{\theta}),\ f_{i1} = x_i,\ f_{i2} = n_i - x_i$

$$\chi^2 = \sum_{i=1}^{k}\sum_{j=1}^{2}\frac{(f_{ij}-e_{ij})^2}{e_{ij}} = \sum_{i=1}^{k}\frac{(x_i-n_i\hat{\theta})^2}{n_i\hat{\theta}}+\frac{[n_i-x_i-n_i(1-\hat{\theta})]^2}{n_i(1-\hat{\theta})}$$

$$=\sum_{i=1}^{k}\frac{(x_i-n_i\hat{\theta})^2+\hat{\theta}(x_i-n_i\hat{\theta})^2}{n_i\hat{\theta}(1-\hat{\theta})}$$

$$=\sum_{i=1}^{k}\frac{(x_i-n_i\hat{\theta})^2}{n_i\hat{\theta}(1-\hat{\theta})} \qquad \text{QED}$$

13.9 $H_1 : \lambda > \lambda_0$, Reject null hypothesis if $\sum_{i=1}^{n} x_i \ge k_\alpha$, where k_α is smallest integer for which

$$\sum_{y=k_\alpha}^{\infty} p(y;n\lambda_0) \le \alpha .$$

$H_1 : \lambda < \lambda_0$, Reject null hypothesis if $\sum_{i=1}^{n} x_i \le k'_\alpha$, where k'_α is smallest integer for which

$$\sum_{y=0}^{k'_\alpha} p(y;n\lambda_0) \le \alpha .$$

$H_1 : \lambda \ne \lambda_0$, Reject null hypothesis if $\sum x \le k'_{\alpha/2}$ or $\sum x \ge k_{\alpha/2}$

13.10 From Table II with $\lambda = 5(3.6) = 18$

$k_{0.025} = 25$ (Probability $X \ge 28 = 0.0173,\ x \ge 27 = 0.0282$)

$k'_{0.025} = 9$ (Probability $X \le 9 = 0.0153,\ x \le 10 = 0.0303$)

13.11 Substitute $e_{11} = \dfrac{n_1(x_1 + x_2)}{n_1 + n_2}, f_{11} = e_{21} = \dfrac{n_2(x_1 + x_2)}{n_1 + n_2}$

$f_{21} = x_2,\ e_{12} = \dfrac{n_1[(n_1 + n_2) - (x_1 + x_2)]}{n_1 + n_2},\ f_{12} = n_1 - x_1$

$e_{22} = \dfrac{n_2[(n_1 + n_2) - (x_1 + x_2)]}{n_1 + n_2},\ f_{22} = n_2 - x_2$ into

$\chi^2 = \sum_{i=1}^{k} \sum_{j=1}^{2} \dfrac{(f_{ij} - e_{ij})^2}{e_{ij}}$ and simplify algebraically

13.12 $E\left(\dfrac{x_1}{n_1} - \dfrac{x_2}{n_2}\right) = \theta_1 - \theta_2 = 0$

$$\text{var}\left(\frac{x_1}{n_1} - \frac{x_2}{n_2}\right) = \text{var}\left(\frac{x_1}{n_1}\right) + \text{var}\left(\frac{x_2}{n_2}\right)$$

$$= \frac{\theta_2(1-\theta_2)}{n_1} + \frac{\theta_2(1-\theta_2)}{n_2}$$

$$\theta_1 = \theta_2 = \theta \text{ estimated by } \hat{\theta} = \frac{x_1 + x_2}{n_1 + n_2}$$

$$\hat{\theta}(1-\hat{\theta})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$$

Thus, $z = \dfrac{\dfrac{x_1}{n_1} - \dfrac{x_2}{n_2} - 0}{\sqrt{\hat{\theta}(1-\hat{\theta})\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}} = \dfrac{\dfrac{x_1}{n_1} - \dfrac{x_2}{n_2}}{\sqrt{\hat{\theta}(1-\hat{\theta})\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$

has approximately standard normal distribution.

13.13 $\chi^2 = \frac{(x_1 - n_1\hat{\theta})^2}{n_1\hat{\theta}(1-\hat{\theta})} + \frac{(x_2 - n_2\hat{\theta})^2}{n_2\hat{\theta}(1-\hat{\theta})}$

$$= \frac{\left[x_1 - \frac{n_1(x_1+x_2)^2}{n_1+n_2}\right]^2}{n_1\hat{\theta}(1-\hat{\theta})} + \frac{\left[x_2 - \frac{n_2(x_1+x_2)^2}{n_1+n_2}\right]^2}{n_2\hat{\theta}(1-\hat{\theta})}$$

$$= \frac{\left[\frac{x_1 n_2}{n_1+n_2} - \frac{n_1 x_2}{n_1+n_2}\right]^2}{n_1\hat{\theta}(1-\hat{\theta})} + \frac{\left[\frac{x_2 n_1}{n_1+n_2} - \frac{n_2 x_1}{n_1+n_2}\right]^2}{n_2\hat{\theta}(1-\hat{\theta})}$$

$$= \frac{\frac{n_1^2 \cdot n_2}{n_1^2(n_1+n_2)^2}\left(\frac{x_1}{n_1} - \frac{x_2}{n_2}\right)^2 + \frac{n_2^2 \cdot n_1}{n_1^2(n_1+n_2)}\left(\frac{x_1}{n_1} - \frac{x_2}{n_2}\right)}{n_1 n_2\hat{\theta}(1-\hat{\theta})} = \frac{\left(\frac{x_1}{n_1} - \frac{x_2}{n_2}\right)^2}{\frac{(n_1+n_2)}{n_1 n_2}\hat{\theta}(1-\hat{\theta})}$$

$$= \frac{\left(\frac{x_1}{n_1} - \frac{x_2}{n_2}\right)^2}{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\hat{\theta}(1-\hat{\theta})} = Z^2 \qquad \text{QED}$$

13.14 $e_{ij} = \frac{\sum_i f_{ij} \sum_j f_{ij}}{n}$

$$\sum_i e_{ij} = \frac{\sum_i f_{ij} \cdot \sum_i \sum_j f_{ij}}{n} = \frac{\sum_i f_{ij} \cdot n}{n} = \sum_i f_{ij}$$

$$\sum_i e_{ij} = \frac{\sum_j f_{ij} \cdot \sum_i \sum_j f_{ij}}{n} = \frac{\sum_j f_{ij} \cdot n}{n} = \sum_j f_{ij}$$

13.15 Under $H_o : e_{1j} = \theta_{2j} = \ldots = \theta_{nj}$ for $j = 1, 2, \ldots.$

$$= \theta_j$$

$$\hat{\theta}_j = \frac{\sum_i f_{ij}}{n} \qquad e_{ij} = \frac{\sum_i f_{ij}}{n} \cdot \sum_j f_{ij} = \frac{\sum_i f_{ij} \cdot \sum_j f_{ij}}{n}$$

13.16 $\chi^2 = \sum_i \sum_j \frac{(f_{ij} - e_{ij})^2}{e_{ij}} = \sum_i \sum_j \frac{f_{ij}^2}{e_{ij}} - 2\sum_i \sum_j f_{ij} + \sum_i \sum_j e_{ij}$

$$= \sum_i \sum_j \frac{f_{ij}^2}{e_{ij}} - 2f + f \qquad \text{(see Ex 13.14)}$$

$$= \sum_i \sum_j \frac{f_{ij}^2}{e_{ij}} - f \qquad \text{QED}$$

13.17 $$\chi^2 = \frac{232^2}{212} + \frac{260^2}{265} + \frac{197^2}{212} + \frac{168^2}{188} + \frac{240^2}{235} + \frac{203^2}{188} - 1300$$
$$= 253.887 + 255.094 + 183.061 + 150.128 + 245.106 + 219.197 - 1300$$
$$= 6.473 \qquad \text{(differs due to rounding)}$$

13.18 (a)

$f/2$ $f/4$	0 $f/4$
0 $f/4$	$f/2$ $f/4$

$$\chi^2 = \frac{(f/4)^2}{f/4} + \frac{(f/4)^2}{f/4} + \frac{(f/4)^2}{f/4} + \frac{(f/4)^2}{f/4}$$
$$= f$$
$$C = \sqrt{\frac{f}{f+f}} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$$

(b)

$f/3$ $f/9$	0 $f/9$	0 $f/9$
0 $f/9$	$f/3$ $f/9$	0 $f/9$
0 $f/9$	0 $f/9$	$f/3$ $f/9$

$$\chi^2 = 3 \cdot \frac{\left(\frac{2f}{9}\right)^2}{f/9} + 6 \cdot \frac{\left(\frac{f}{9}\right)^2}{f/9}$$
$$= \frac{4}{3}f + \frac{2}{3}f = 2f$$
$$C = \sqrt{\frac{2f}{2f+f}} = \sqrt{\frac{2}{3}} = \frac{1}{3}\sqrt{6}$$

13.19 (a) not necessarily; **(b)** yes

13.20 (a) No, since $0.0316 > 0.01$
(b) Yes, since $0.0316 < 0.05$
(c) Yes, since $0.0316 < 0.10$

13.21 Normal curve area corresponding to $z = 2.84$ is 0.4977
p-value is $2(0.5000 - 0.4977) = 0.0046$

13.22 Normal curve area corresponding to 1.40 is 0.4192
p-value is $0.5000 - 0.4192 = 0.0808$

13.23 p-value is $\frac{1-0.3502}{2} = 0.3249$. As is exceeds 0.05, null hypothesis *cannot* be rejected.

13.24 $H_0: \mu = 10$; $H_1: \mu < 10$
$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{8.4 - 10}{3.2/\sqrt{16}} = -2.0$$

Since $z_{0.05} = 1.645$, we reject H_0 in favor of H_1.

13.25 1. $H_0: \mu = 84.3,\ H_1: \mu > 84.3,\ \alpha = 0.01$

2. Reject null hypothesis if $z \geq 2.33$

3. $z = \dfrac{87.5 - 84.3}{8.6/\sqrt{45}} = 2.73$

4. Since 2.73 exceeds 2.33, null hypothesis must be rejected.

13.26 2. $z = \dfrac{\overline{x} - \mu_0}{\sigma/\sqrt{n}}$

3. $z = \dfrac{87.5 - 84.3}{8.6/\sqrt{45}} = 2.73$, p-value = $0.5000 - 0.4968 = 0.0032$

4. Since $0.0032 < 0.01$, null hypothesis must be rejected.

13.27 1. $H_0: \mu = 30,\ H_1: \mu \neq 30,\ \alpha = 0.01$

2. Reject null hypothesis if $z \leq -2.575$ or $z \geq 2.575$

3. $z = \dfrac{30.8 - 30}{1.5/\sqrt{32}} = \dfrac{0.8\sqrt{32}}{1.5} = 3.02$

4. Since $3.02 > 2.575$, null hypothesis must be rejected.

13.28 2. $z = \dfrac{\overline{x} - \mu_0}{\sigma/\sqrt{n}}$

3. $z = \dfrac{30.8 - 30}{1.5/\sqrt{32}} = 3.02$, p-value = $2(0.5 - 0.4987) = 0.0026$

4. Since 0.0026 is less than 0.005, null hypothesis must be rejected.

13.29 1. $H_0: \mu = 35,\ H_1: \mu < 35,\ \alpha = 0.05$

2. Reject null hypothesis if $t \leq -t_{0.05,11} = -1.796$

3. $t = \dfrac{33.6 - 35}{2.3/\sqrt{12}} = \dfrac{-1.4}{2.3\sqrt{12}} = -2.11$

4. Since $-2.11 < -1.796$, the null hypothesis must be rejected.

13.30 $n = 5,\ \overline{x} = 14.4,\ s = 0.158$

1. $H_0: \mu = 14,\ H_1: \mu \neq 14,\ \alpha = 0.05$

2. Reject null hypothesis if $t \leq -2.776$ or $t \geq 2.776$

3. $t = \dfrac{14.4 - 14}{0.158/\sqrt{5}} = 5.66$

4. Since 5.66 exceeds 2.776, null hypothesis must be rejected.

13.31 $n = 5,\ \overline{x} = 14.7,\ s = 0.742$

3. $t = \dfrac{14.7 - 14}{0.742/\sqrt{5}} = 2.11$

4. Since $t = 2.11$ falls between -2.776 and 2.776, null hypothesis cannot be rejected.

$x - \mu_0$ has increased from 14.4 to 14.7 but s has increased from 0.158 to 0.742.

13.32 $t = 5.66$, d.f. $= 4$
p-value $= 1 - 0.9952 = 0.0048$
Since $0.0048 < 0.05$, null hypothesis must be rejected.

13.33 (a) $P(\text{reject } H_0 | H_0 \text{ is true}) = 0.05$ (by definition)

(b) $P(\text{reject } H_0$ on experiment 1 or experiment 2 (or both) $|H_0$ is true) =
$0.05 + 0.05 - .0025 = 0.0975$

(c) Reject H_0 on one or more of 30 experiments $|H_0$ is true) =
$1 - P($ do not reject H_0 on any experiment $|H_0$ is true)$=1-(0.95)^{30} = 0.79$.

13.34 (a) $P(\text{reject } H_0$ on exactly one factor $|H_0$ is true for all 48 factors) =
$$\binom{48}{1}(0.01)^1(0.99)^{47} = 0.30$$

(b) $P(\text{reject } H_0$ on more than one factor $|H_0$ is true for all 48 factors) $= 1 - 0.30 = 0.70$.

13.35 $$\frac{(\bar{x}_1 - \bar{x}_2) - 0.20}{\sqrt{\frac{(0.12)^2}{50} + \frac{(0.14)^2}{40}}} \le -1.96 \text{ or } \ge 1.96$$
$$\frac{(\bar{x}_1 - \bar{x}_2) - 0.20}{0.0279} \le -1.96 \text{ or } \ge 1.96$$
$\bar{x}_1 - \bar{x}_2 \le 0.20 - 0.0547 = 0.145$
or $\bar{x}_1 - \bar{x}_2 \ge 0.20 + 0.0547 = 0.255$

(a) $z = \frac{0.145 - 0.12}{0.0279} = 0.90$ and $z = \frac{0.255 - 0.12}{0.0279} = 4.84$
$\beta = 0.5 - 0.3159 = 0.1841 = 0.18$

(b) $z = \frac{0.145 - 0.16}{0.0279} = -0.54$ and $z = \frac{0.255 - 0.16}{0.0279} = 3.405$
$\beta = 0.2054 + 0.5 = 0.7054 = 0.71$

(c) $z = \frac{0.145 - 0.24}{0.0279} = -3.40$ and $z = \frac{0.255 - 0.24}{0.0279} = 0.54$
$\beta = 0.2054 + 0.5 = 0.7054 = 0.71$

(d) $z = \frac{0.145 - 0.28}{0.0279} = -4.84$ and $z = \frac{0.255 - 0.28}{0.0279} = -0.90$
$\beta = 0.5 - 0.3159 = 0.1841 = 0.18$

13.36 1. $H_0 : \mu_1 - \mu_2 = 0,\ H_1 : \mu_1 - \mu_2 \neq 0,\ \alpha = 0.05$

2. Reject null hypothesis if $z \leq -1.96$ or $z \geq 1.96$

3. $$z = \frac{9.1-8}{\sqrt{\frac{1.9^2}{40} + \frac{2.1^2}{50}}} = \frac{1.1}{0.4224} = 2.60$$

4. Since $2.60 > 1.96$, null hypothesis must be rejected.

13.37 $z = 2.60,\ p-\text{value} = 2(0.5 - 0.4953) = 0.0094$

Since $0.0094 < 0.05$, null hypothesis must be rejected.

13.38 1. $H_0 : \mu_1 - \mu_2 = -0.05,\ H_1 : \mu_1 - \mu_2 < -0.05,\ \alpha = 0.05$

2. Reject null hypothesis if $z \leq -1.645$

3. $$z = \frac{(53.8 - 54.5) + 0.05}{\sqrt{\frac{2.4^2}{400} + \frac{2.5^2}{500}}} = \frac{-0.20}{0.164} = -1.22$$

4. Since $-1.22 > -1.645$, null hypothesis cannot be rejected.

13.39 $z = -1.22,\ p-\text{value} = 0.5 - 0.3888 = 0.1112$

Since $0.1112 > 0.05$, null hypothesis cannot be rejected.

13.40 1. $H_0 : \mu_1 - \mu_2 = 0,\ H_1 : \mu_1 - \mu_2 \neq 0,\ \alpha = 0.01$

2. Reject null hypothesis if $t \leq -t_{0.005} = -3.169$ or $t > t_{0.005} = 3.169$

3. $$s_p^2 = \frac{5(3.3)^2 + 5(2.1)^2}{10} = 7.65 \text{ and } s_p = 2.766$$

$$t = \frac{77.4 - 72.2}{2.766\sqrt{\frac{1}{6} + \frac{1}{6}}} = \frac{5.2}{(2.766)(0.577)} = 3.26$$

4. Since $3.26 > 3.169$, null hypothesis must be rejected.

13.41 $t = 2.67,\ \text{d.f.} = 6,\ \alpha = 0.05$

$$p\text{-value} = \frac{1}{2}(1 - 0.9630) = 0.0185$$

13.42 $\bar{x}_1 = 144,\ s_1 = 19.06,\ \bar{x}_2 = 149,\ s_2 = 14.21$

1. $H_0 : \mu_1 = \mu_2,\ H_1 : \mu_1 \neq \mu_2,\ \alpha = 0.01$

2. Reject null hypothesis if $t \leq -3.169$ or $t \geq 3.169$

3. $$s_p^2 = \frac{5(19.06)^2 + 5(14.21)^2}{10} = 282.604 \text{ and } s_p = 16.802$$

$$t = \frac{144 - 149}{16.802\sqrt{\frac{1}{6} + \frac{1}{6}}} = \frac{-5}{(16.802)(0.577)} = -0.52$$

4. Since -0.52 falls between -3.169 and 3.169, null hypothesis cannot be rejected.

13.43 $t = -0.52,\ \text{d.f.} = 10$

p-value $= 1 - 0.3856 = 0.61$

Since $0.61 > 0.01$, null hypothesis cannot be rejected.

13.44 13, 7, -1, 5, 3, 2, -1, 0, 6, 1, 4, 3, 2, 6, 12, 4

$\overline{x} = 4.125,\ s = 4.064,\ n = 16$

1. $H_0 : \mu = 0,\ H_1 : \mu > 0,\ \alpha = 0.05$
2. Reject null hypothesis if $t \ge t_{0.05,15} = 1.753$
3. $t = \dfrac{4.125 - 0}{4.064 / \sqrt{16}} = 4.06$
4. Since $4.06 > 1.753$, null hypothesis must be rejected. Exercises are effective in reducing weight.

13.45 9, 13, 2, 5, -2, 6, 6, 5, 2, 6

$n = 10,\ \overline{x} = 5.2,\ s = 4.08$

1. $H_0 : \mu = 0,\ H_1 : \mu > 0,\ \alpha = 0.05$
2. Reject null hypothesis if $t > t_{0.05,9} = 1.833$
3. $t = \dfrac{5.2 - 0}{4.08 / \sqrt{10}} = 4.03$
4. Since $4.03 > 1.833$, null hypothesis must be rejected. Safety program is effective.

13.46 $t = 4.03,\ \text{d.f.} = 9$

$p\text{-value} = \dfrac{1}{2}(1 - 0.997) = 0.0015$

13.47
1. $H_0 : \sigma = 0.0100,\ H_1 : \sigma < 0.0100,\ \alpha = 0.05$
2. Reject null hypothesis if $\chi^2 \le \chi^2_{0.95,8} = 2.733$
3. $\chi^2 = \dfrac{8(0.0086)^2}{(0.0100)^2} = 5.92$
4. Since $5.92 > 2.733$, null hypothesis cannot be rejected.

13.48 $s = 238,\ n = 24$

1. $H_0 : \sigma = 250,\ H_1 : \sigma \ne 250,\ \alpha = 0.01$
2. Reject null hypothesis if $\chi^2 \le \chi^2_{0.995,23} = 9.260$ or $\chi^2 \ge \chi^2_{0.005,23} = 44.181$
3. $\chi^2 = \dfrac{23(238)^2}{(250)^2} = 20.84$
4. Since $9.260 < 20.84 < 44.181$, null hypothesis cannot be rejected.

13.49 $s = 2.53,\ n = 30,\ \alpha$=0.05

1. $H_0 : \sigma = 2.85,\ H_1 : \sigma < 2.85,\ \alpha = 0.05$
2. Reject null hypothesis if $\chi^2 \le \chi^2_{0.95,29} = 17.708$
3. $\chi^2 = \dfrac{29(2.53)^2}{(2.85)^2} = 22.85$
4. Since $22.85 > 17.708$, null hypothesis cannot be rejected.

13.50 1. $H_0: \sigma = \sigma_0,\ H_1: \sigma < \sigma_0,\ \alpha = 0.05$

2. Reject null hypothesis if $z \le -z_{0.05} = -1.645$

3. $z = \left(\frac{2.53}{2.85} - 1\right)\sqrt{2 \cdot 29} = -0.1123(7.616) = -0.85$

4. Since $-0.85 > -1.645$, null hypothesis cannot be rejected.

13.51 $n = 50,\ s = 0.49$

1. $H_0: \sigma = 0.41,\ H_1: \sigma > 0.41,\ \alpha = 0.05$

2. Reject null hypothesis if $z \ge z_{0.05} = 1.645$

3. $z = \left(\frac{0.49}{0.41} - 1\right)\sqrt{2 \cdot 49} = (0.1951)(9.8995) = 1.93$

4. Since $1.93 > 1.645$, null hypothesis must be rejected.

13.52 p-value = $0.5 - 0.4732 = 0.0268$

Since $0.0268 < 0.05$, null hypothesis must be rejected.

13.53 $n_1 = 4,\ s_1 = 31,\ n_2 = 4,\ s_2 = 26,\ \alpha = 0.05$

1. $H_0: \sigma_1 - \sigma_2 = 0,\ H_1: \sigma_1 - \sigma_2 > 0,\ \alpha = 0.05$

2. Reject null hypothesis if $\frac{s_1^2}{s_2^2} \ge F_{0.05,3,3} = 9.28$

3. $\frac{s_1^2}{s_2^2} = 1.42$

4. Since 1.42 does not exceed 9.28, null hypothesis cannot be rejected.

13.54 1. $H_0: \sigma_1 - \sigma_2 = 0,\ H_1: \sigma_1 - \sigma_2 \ne 0,\ \alpha = 0.10$

2. Reject null hypothesis if $\frac{s_1^2}{s_2^2} \ge F_{0.05,5,5} = 5.05$

3. $\frac{s_1^2}{s_2^2} = \frac{3.3^2}{2.1^2} = 2.47$

4. Since 2.47 < 5.05, null hypothesis cannot be rejected. Assumption was reasonable.

13.55 $s_1 = 19.06,\ s_2 = 14.21,\ n_1 = n_2 = 6$

1. $H_0: \sigma_1 - \sigma_2 = 0,\ H_1: \sigma_1 - \sigma_2 \ne 0,\ \alpha = 0.02$

2. Reject null hypothesis if $\max\left(\frac{s_1^2}{s_2^2}, \frac{s_2^2}{s_1^2}\right) \ge F_{0.01,5,5} = 11.0$

3. $\frac{s_1^2}{s_2^2} = 1.80$

4. Since 1.80 < 11.0, null hypothesis cannot be rejected.

13.56 $n = 20,\ \theta = 0.5$ against $\theta \neq 0.50,\ \alpha = 0.05$

$p(x \le 5) = 0.0207$ Critical region is $x \le 5$ or $x \ge 15$

$p(x \le 6) = 0.0507$ $a = 0.0207 + 0.0207 = 0.0414$

$p(x \ge 15) = 0.0207$

$p(x \ge 14) = 0.0507$

13.57 1. $H_0 : \theta = 0.40,\ H_1 : \theta > 0.40,\ \alpha = 0.05$

2. Observed number of successes in $n = 18$ trials
3. $x = 10$ $P(X \ge 10) = 0.1348$ p-value 0.1348
4. Since 0.1348 > 0.05, null hypothesis cannot be rejected.

13.58 $p(X \ge 12) = 0.0203$ Critical region is $x \ge 12$

$p(X \ge 11) = 0.0577$ $a = 0.0203$

13.59 1. $H_0 : \theta = 0.30,\ H_1 : \theta < 0.30,\ \alpha = 0.05$

2. Observed number of successes in $n = 19$ trials
3. $x = 1$ p-value is 0.0011 + 0.0093 = 0.0104
4. Since 0.0104 < 0.05, null hypothesis must be rejected.

13.60 $p(x \le 2) = 0.0462$ Critical region is $x \le 2$

$p(x \le 3) = 0.1331$ $\alpha = 0.0462$

13.61 1. $H_0 : \theta = 0.40,\ H_1 : \theta \neq 0.40,\ \alpha = 0.01$

2. Observed number of successions in $n = 14$ trials
3. $p(x \ge 12) = 0.0006$, p-value = 0.0012
4. Since 0.0012 < 0.01, null hypothesis must be rejected.

13.62 $P(x \le 0) = 0.0008,\ P(x \ge 11) = 0.0039$, Critical region is $x = 0$, or $x \ge 11$

$P(x \le 1) = 0.0081,\ P(x \ge 10) = 0.0175,\ \alpha = 0.008 + 0.0039 = 0.0047$

13.63 $H_0 : \theta = 0.35;\ H_1 : \theta < 0.35.$ Using the normal approximation

$$z = \frac{x - n\theta_0}{\sqrt{n\theta_0(1-\theta_0)}} = \frac{290 - 350}{\sqrt{(350)(0.65)}} = -3.98$$

Since $z_{0.05} = 1.645$, we reject H_0 at the 0.05 level of significance and conclude that $\theta < 0.35$; thus, the statement can be refuted.

13.64 1. $H_0 : \theta = 0.20,\ H_1 : \theta > 0.20,\ \alpha = 0.01$

2. Number of successes in $n = 12$ trials
3. $x = 6,\ p(X \ge 6) = 0.0194 = p$-value
4. Since 0.0194 > 0.01, null hypothesis cannot be rejected.

13.65 1. $H_0: \theta = 0.60,\ H_1: \theta \neq 0.60,\ \alpha = 0.05$

2. Number of failures in $n = 18$ trials
3. $x = 7,\ n - x = 18 - 7 = 11$

   $P(X \geq 11);\ \theta = 0.40) = 0.0577$

   p-value is 2(0.0577) = 0.1154
4. Since 0.1154 > 0.05, null hypothesis cannot be rejected.

13.66 1. $H_0: \theta = 0.30,\ H_1: \theta \neq 0.30,\ \alpha = 0.05$

2. Reject if $z \leq -1.96$ or $z \geq 1.96$
3. $$z = \frac{157 - 600(0.30)}{\sqrt{600(0.3)(0.7)}} = -2.05$$
4. Since $-2.05 < -1.96$, null hypothesis must be rejected.

13.67 1. $H_0: \theta = 0.90,\ H_1: \theta < 0.90,\ \alpha = 0.05$

2. Reject if $z < -1.645$
3. $$z = \frac{174 - 200(0.9)}{\sqrt{200(0.9)(0.1)}} = -\frac{6}{4.2426} = -1.41$$
4. Since $-1.41 > -1.645$, null hypothesis cannot be rejected.

13.68 1. $H_0: \theta_1 = \theta_2,\ H_1: \theta_1 \neq \theta_2,\ \alpha = 0.01$

2. Reject null hypothesis if $\chi^2 \geq \chi^2_{0.01,1} = 6.635$

74 83	92 83	166
176 167	158 167	334
250	250	500

$$e_{11} = \frac{166 \cdot 250}{500} = 83, \text{ others by subtraction}$$

$$\chi^2 = \frac{9^2}{83} + \frac{9^2}{83} + \frac{9^2}{167} + \frac{9^2}{167} = 2.92$$

4. Since $2.92 < 6.635$, null hypothesis cannot be rejected.

13.69 1. $H_0: \theta_1 = \theta_2,\ H_1: \theta_1 \neq \theta_2,\ \alpha = 0.01$

2. Reject null hypothesis if $z \leq -z_{0.005}$ or $z \geq z_{0.005}$

   $$\hat{\theta} = \frac{74 + 92}{500} = 0.332$$
3. $$z = \frac{\frac{74}{250} - \frac{92}{250}}{\sqrt{(0.332)(0.668)(0.008)}} = -\frac{0.072}{0.04212} = -1.71$$
4. Since -1.71 falls between -2.575 and 2.575, null hypothesis cannot be rejected.

13.70 1. $H_0: \theta_1 = \theta_2,\ H_1: \theta_1 \neq \theta_2,\ \alpha = 0.05$

2. Reject null hypothesis if $\chi^2 \geq \chi^2_{0.05,1} = 3.841$

3.

46 42.7	18 21.3	64
354 357.3	182 178.7	536
400	400	600

$$e_{11} = \frac{64 \cdot 400}{600} = 42.7 \text{, others by subtraction}$$

$$\chi^2 = \frac{3.3^2}{42.7} + \frac{3.3^2}{21.3} + \frac{3.3^2}{357.3} + \frac{3.3^2}{178.7}$$
$$= 0.255 + 0.511 + 0.030 + 0.061$$
$$= 0.86$$

4. Since 0.86 < 3.841, null hypothesis cannot be rejected.

13.71 1. $H_0: \theta_1 = \theta_2,\ H_1: \theta_1 \neq \theta_2,\ \alpha = 0.05$

2. Reject null hypothesis if $z \leq -1.96$ or $z \geq 1.96$

$$\hat{\theta} = \frac{74+92}{500} = 0.332$$

3. $$z = \frac{\frac{46}{400} - \frac{18}{200}}{\sqrt{(0.107)(0.893)(0.0075)}} = \frac{0.025}{0.0268} = 0.93$$

$$z^2 = (0.93)^2 = 0.8649 = 0.86 = \chi^2$$

13.72 $H_0: \theta_1 = \theta_2,\ H_1: \theta_1 > \theta_2,\ \alpha = 0.05$

2. Reject null hypothesis if $z \geq 1.645$

3. $\hat{\theta} = \frac{169}{500} = 0.338 \qquad z = \frac{\frac{82}{200} - \frac{87}{300}}{\sqrt{(0.338)(0.662)(0.00833)}} = 2.78$

4. Since 2.78 > 1.645, null hypothesis must be rejected.

13.73 $H_0: \theta_1 = \theta_2 = \theta_3 = \theta_4,\ H_1:$ not all equal, $\alpha = 0.05$

2. Reject null hypothesis if $\chi^2 \geq \chi^2_{0.05,3} = 7.815$

3.

26 24	23 24	15 24	32 24	96
174 176	177 176	185 176	168 176	704
200	200	200	200	800

$$e_{11} = \frac{96 \cdot 200}{800} = 24 \text{ etc.}$$

$$\chi^2 = \frac{4+1+81+64}{24} + \frac{4+1+81+64}{24} = 7.10$$

4. Since 7.10 < 7.818, null hypothesis cannot be rejected.

13.74 $H_0: \theta_1 = \theta_2 = \theta_3$, $H_1:$ not all equal, $\alpha = 0.05$

2. Reject null hypothesis if $\chi^2 \geq \chi^2_{0.05,2} = 5.991$

3. $e_{11} = \dfrac{360 \cdot 250}{600}$

155 150	118 120	87 90	360
95 100	82 80	63 60	240
250	200	150	600

$$\chi^2 = \frac{25}{100} + \frac{4}{120} + \frac{9}{90} + \frac{25}{100} + \frac{4}{80} + \frac{9}{60} = 0.75$$

4. Since 0.75 < 5.991, null hypothesis cannot be rejected.

13.75 In the following contingency table, the expected frequency is given below the observed frequency in each cell.

				TOTALS
	45 45.0	58 49.8	49 57.3	152
	21 21.0	15 23.2	35 26.7	71
TOTALS	66	73	84	223

The expected frequencies were calculated as $\dfrac{152 \times 66}{223} = 45.0$, etc.

Thus, $\chi^2 = \dfrac{(45-45.0)^2}{45.0} + \dfrac{(58-49.8)^2}{49.8} + \ldots + \dfrac{(35-26.7)^2}{26.7}$

$= 0.00 + 1.35 + 1.20 + 0.00 + 2.90 + 2.58 = 8.03$

Since $\chi^2_{0.01} = 9.210$, we cannot reject H_0, and we have no reason to conclude that the three processes have different probabilities of passing the strength standard.

13.76

48 44.4	40 38.6	12 17.0	100
55 60.9	53 52.9	29 23.2	137
57 54.7	46 47.5	20 20.8	123
160	139	61	360

1. $H_0:$ independent, $H_1:$ not independent, $\alpha = 0.05$
2. Reject null hypothesis, if $\chi^2 \geq \chi^2_{0.05,4} = 9.488$
3. $\chi^2 = 0.292 + 0.051 + 1.471 + 0.572$
   $+ 0.000 + 1.450 + 0.097$
   $+ 0.047 + 0.031$
   $= 4.01 = 4.0$
4. Since 4.0 < 9.488, null hypothesis cannot be rejected.

13.77

7 15	12 22.1	31 12.9	50
35 33.6	59 49.5	18 28.9	112
15 8.4	13 12.4	0 7.2	28
57	8.4	49	190

1. $H_0:$ independent, $H_1:$ not independent, $\alpha = 0.01$
2. Reject null hypothesis, if $\chi^2 \geq \chi^2_{0.01,4} = 13.277$
3. $\chi^2 = 4.27 + 4.62 + 25.40 + 0.06 + 1.82$
   $+ 4.11 + 5.19 + 0.029 + 7.2$
   $= 52.7$
4. Since 52.7 > 13.277, null hypothesis must be rejected.

13.78

12 13.5	23 21.4	89 89.1	124
8 8.9	12 14.2	62 58.9	82
21 18.6	30 29.4	119 122.0	170
41	65	270	376

1. $H_0:$ Venders ship equal quantities
   $H_1:$ Venders do not ship equal quantities; $\alpha = 0.01$
2. Reject null hypothesis, if $\chi^2 \geq \chi^2_{0.01,4} = 13.277$
3. $\chi^2 = 0.17 + 0.12 + 0.00 + 0.09 + 0.34$
   $+ 0.16 + 0.31 + 0.01 + 0.07$
   $= 1.27 = 1.3$
4. Since 1.3 < 13.277, null hypothesis cannot be rejected.

13.79

174 159.4	93 99.1	133 141.5	400
196 199.2	124 123.8	180 177.0	500
148 159.4	105 99.1	147 141.5	400
518	322	460	1300

1. $H_0:$ percentages same for three cities
   $H_1:$ percentages *not* same for three cities $\alpha = 0.05$
2. Reject null hypothesis, if $\chi^2 \geq \chi^2_{0.05,4} = 9.488$
3. $\chi^2 = 1.34 + 0.38 + 0.51 + 0.05 + 0.00$
   $+ 0.05 + 0.82 + 0.35 + 0.21$
   $= 3.71$
4. Since 3.71 < 9.488, null hypothesis cannot be rejected.

13.80

	f	prob	*e*
0	19	1/16	10
1	54	4/16	40
2	58	10/16	60
3	23	4/16	40
4	6	1/16	10

1. $H_0:$ coins are balanced
   $H_1:$ coins are *not* balanced
   $\alpha = 0.05$
2. Reject null hypothesis if $\chi^2 \geq \chi^2_{0.05,4} = 9.488$
3. $\chi^2 = \frac{81}{10} + \frac{196}{40} + \frac{4}{60} + \frac{289}{40} + \frac{16}{10} = 8.1 + 4.9 + 0.1 + 7.2 + 1.6 = 21.9$
4. Since 21.9 > 9.488, null hypothesis must be rejected.

13.81

	f		prob	e	
0	19		0.0907	27.2	
1	48		0.2177	65.3	
2	66		0.2613	78.4	
3	74		0.2090	62.7	
4	44		0.1254	37.6	
5	35		0.0602	18.1	
6	10	14	0.0241	7.2	10.7
7 or more	4		0.0117	3.5	
	300				

1. H_0: Poisson distribution with $\lambda = 2.4$
   H_1: *not* Poisson distribution with $\lambda = 2.4$
   $\alpha = 0.05$
2. Reject null hypothesis if
   $\chi^2 \ge \chi^2_{0.05,6} = 12.592$
3. $\chi^2 = 2.47 + 4.58 + 1.96 + 2.04 + 1.09 + 15.78 + 1.02 = 28.9$
4. Since 28.9 > 12.592, null hypothesis must be rejected.

13.82 $\bar{x} = \dfrac{0 \cdot 1 + 1 \cdot 16 + 2 \cdot 55 + 3 \cdot 228}{300} = \dfrac{810}{300} = 2.7$ $\qquad \hat{\theta} = \dfrac{2.7}{3} = 0.9$

	f		prob	e	
0	1	17	0.001	0.3	8.4
1	16		0.027	8.1	
2	55		0.243	72.9	
3	228		0.729	218.7	

1. H_0: binomial distribution
   H_1: *not* binomial distribution
   $\alpha = 0.05$
2. Reject null hypothesis if
   $\chi^2 \ge \chi^2_{0.05,1} = 3.841$
3. $\chi^2 = 8.80 + 4.40 + 0.40 = 13.6$
4. Since 13.6 > 3.841, null hypothesis must be rejected.

13.83 (a) $\bar{x} = 20$ and $s = 5.025 = 5$

using $\bar{x} = \dfrac{\sum xf}{n}$ and $s = \sqrt{\dfrac{n\left(\sum x^2 f\right) - \left(\sum xf\right)^2}{n(n-1)}}$

where x's are the class marks (midpoints)

(b)

	z			e
9.5	−2.1	0.4821	0.0179	1.8
14.5	−1.1	0.3643	0.1178	11.8
19.5	−0.1	0.0398	0.3245	32.4
24.5	0.9	0.3159	0.3557	35.5
29.5	1.9	0.4713	0.1554	15.5
34.5	2.9	0.4981	0.0268	2.7
			0.0019	0.2

Probabilities are 0.0179, 0.1178, 0.3245, 0.3557, 0.1554, 0.0268, 0.0019.

(c) Expected frequencies are 1.8, 11.8, 32.4, 35.6, 15.5, 2.7, 0.2

1. $H_0:$ normally distributed random variables
   $H_A:$ *not* normally distributed random variables, $\alpha = 0.05$

f	e
11	13.6
37	32.4
36	35.6
16	18.4

2. Reject null hypothesis if $\chi^2 \geq \chi^2_{0.05,1} = 3.841$
3. $\chi^2 = 0.50 + 0.65 + 0.00 + 0.31 = 1.46$
4. Since $1.46 < 3.841$, null hypothesis cannot be rejected.

13.84 $H_0: \mu = 300$; $H_1: \mu < 300$. Using MINITAB:

```
MTB> Ttest 300 C1;
SUBC> Alternative -1.
```

we get

```
N    MEAN     ST DEV    SEMEAN    T       P VALUE
38   284.553  104.220   16.907   -0.91    0.18
```

With a *P*-value of 0.18, the mean failure time is not significantly less than 300 hours at the 0.01 level of significance.

13.85 $H_0: \mu_1 = \mu_2$; $H_1: \mu_1 \neq \mu_2$ Using MINITAB:

```
MTB> TwosampleT for C1 vs C2
```

we get

```
      N    MEAN    ST DEV   SEMEAN
C1    20   57.76   3.66     0.82
C2    20   52.75   5.01     1.1

TTEST MUC1=MUC2: T = 3.61 P = 0.0009 DF = 38
```

With a *P*-value of 0.0009, we conclude that the difference between the mean drying times is significant at the 0.05 level of significance.

13.86 Using MINITAB, we enter the three columns in this table into C1, C2, and C3, respectively.

```
MTB> Chisquare C1 C2 C3
Expected counts are printed below observed counts.
         C1      C2      C3       Total
   1     36      22      18        76
         35.32   23.91   16.77
   2.    63      45      29        137
         63.68   43.09   30.23
Total    99      67      47        213
Chisq = 0.013 + 0.152 + 0.090 + 0.007 + 0.084 + 0.050 = 0.397
```

From Table V with df = 2, $\chi^2_{0.05,2} = 5.991$, and we cannot reject the null hypothesis that the three materials have the same probability of leaking at the 0.05 level of significance.

Chapter 14

14.1 $$h(y) = \int_0^\infty x e^{-x(1+y)} dy = \frac{1}{(1+y)^2}$$

$$\phi(x|y) = x e^{-x(1+y)} (1+y)^2$$

$$E(x|y) = (1+y)^2 \int_0^\infty x^2 e^{-x(1+y)} dx \qquad z = x(1+y)$$

$$= \int_0^\infty z^2 e^{-z} \frac{dz}{1+y} = \frac{\Gamma(3)}{1+y} = \frac{2}{1+y}$$

14.2 $$g(x) = \frac{2}{5}\int_0^1 (2x+3y)dy = \frac{2}{5}\left(2x + \frac{3}{2}\right)$$

$$w(y|x) = \frac{\frac{2}{5}(2x+3y)}{\frac{2}{5}\left(2x+\frac{3}{2}\right)} = \frac{2x+3y}{2x+\frac{3}{2}}$$

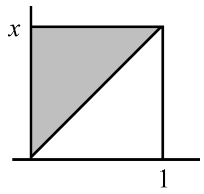
$$\mu_{Y|x} = \frac{1}{2x+\frac{3}{2}} \int_0^1 y(2x+3y)dy = \frac{x+1}{2x+\frac{3}{2}} = \frac{2(x+1)}{4x+3}$$

$$h(y) = \frac{2}{5}\int_0^1 (2x+3y)dx = \frac{2}{5}(1+3y)$$

$$\phi(x|y) = \frac{\frac{2}{5}(2x+3y)}{\frac{2}{5}(1+3y)} = \frac{2x+3y}{1+3y}$$

$$\mu_{x|Y} = \frac{1}{1+3y}\int_0^1 x(2x+3y)dx = \frac{\frac{2}{3}+\frac{3}{2}y}{1+3y} = \frac{4+9y}{6(1+3y)}$$

14.3



$$g(x) = \int_x^1 6x\, dy = 6x(1-x), \; w(y|x) = \frac{6x}{6x(1-x)} = \frac{1}{1-x}$$

$$E(Y|x) = \frac{1}{1-x}\int_x^1 y\, dy = \frac{1-x^2}{2(1-x)} = \frac{1+x}{2}$$

$$h(y) = \int_0^y 6x\, dx = 3y^2 \qquad \phi(x|y) = \frac{2x}{y^2}$$

$$E(x|y) = \frac{2}{y^2}\int_0^y x^2\, dx = \frac{2}{y^2}\cdot\frac{y^3}{3} = \frac{2y}{3}$$

14.4 $f(x,y)=\dfrac{2x}{(1+x+xy)^2}$

$$g(x)=\int_0^\infty \frac{2x}{(1+x+xy)^2}dy \qquad u=1+x+xy \qquad du=x\,dy$$

$$=\int_{1+x}^\infty \frac{2\,du}{u^2}=\frac{1}{u^2}\bigg|_{1x}^\infty=\frac{1}{(1+x)^2}$$

$$w(y|x)=\frac{2x(1+x)^2}{(1+x+xy)^3}$$

$$E(Y|x)=2x(1+x)^2\int_0^\infty \frac{y\,dy}{(1+x+xy)^2} \qquad u=1+x+xy$$

$$du=x\,dy$$

$$=2x(1+x)^2\int_{1+x}^\infty \frac{u-(1+x)}{x}\cdot\frac{du}{xu^3} \qquad v=\frac{u-(1+x)}{x}$$

$$=\frac{2(1+x)^2}{x}\left[-\frac{1}{u}+\frac{(1+x)}{2u^2}\right]_{1+x}^\infty=\frac{1+x}{x}$$

$$E(Y^2|x)=2x(1+x)^2\int_0^\infty \frac{y^2\,dy}{(1+x+xy)^3}\to\infty$$

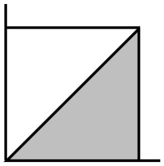
14.5 $\mu_{x|1}=0\cdot\dfrac{10}{21}+1\cdot\dfrac{10}{21}+2\cdot\dfrac{1}{21}=\dfrac{12}{21}=\dfrac{4}{7}$

$$\mu_{Y|0}=0\cdot\frac{5}{28}+1\cdot\frac{15}{28}+2\cdot\frac{15}{56}+3\cdot\frac{1}{56}=\frac{63}{56}=\frac{9}{8}$$

14.6 $m(x,y)=\dfrac{xy}{36}$, $g(x)=\dfrac{x}{6}$, so $w(y|x)=\dfrac{y}{6}$

$$E(Y|x)=\sum_{y=1}^{3}\frac{y^2}{6}=\frac{1}{6}(1+4+9)=\frac{14}{6}=\frac{7}{3}$$

14.7



$$f(x,y)=2 \qquad g(x)=2\int_0^x dx=2x$$

$$h(y)=2\int_y^1 dx=2(1-y)$$

(a) $w(y|x)=\dfrac{2}{2x}=\dfrac{1}{x}$, $\mu_{Y|x}=\dfrac{1}{x}\displaystyle\int_0^x y\,dy=\dfrac{1}{x}\cdot\dfrac{x^2}{2}=\dfrac{x}{2}$

$$\mu_{x|y}=\frac{1}{1-y}\int_y^1 x\,dx=\frac{1}{1-y}\cdot\frac{1}{2}(1-y^2)=\frac{1+y}{2}$$

(b) $$E(x^mY^n) = 2\int_0^1\int_0^x x^m y^n dy\ dx = 2\int_0^1 x^m \left[\frac{y^{n+1}}{n+1}\right]_0^x dx = \frac{2}{n+1}\int_0^1 x^{m+n+1}dx$$

$$= \frac{2}{(n+1)(m+n+2)}$$

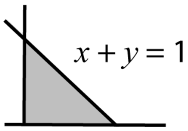
$$E(x) = \frac{2}{3},\ E(Y) = \frac{1}{3},\ E(x^2) = \frac{1}{2},\ E(Y^2) = \frac{1}{6},\ E(xY) = \frac{1}{4}$$

$$\sigma_1^2 = \frac{1}{18},\ \sigma_2^2 = \frac{1}{18},\ \sigma_{12} = \frac{1}{36},\ \rho = \frac{1/38}{\sqrt{\frac{1}{18}\cdot\frac{1}{18}}} = \frac{1}{2}$$

$$\mu_{Y|x} = \frac{1}{3} + \frac{1}{2}\left(x - \frac{2}{3}\right) = \frac{x}{2}$$

$$\mu_{x|y} = \frac{2}{3} + \frac{1}{2}\left(y - \frac{1}{3}\right) = \frac{1+y}{2}$$

14.8



$$g(x) = 24x\int_0^{1-x} y\ dy = 12x(1-x)^2$$

$$\phi(y|x) = \frac{24xy}{12x(1-x)^2} = \frac{2y}{(1-x)^2}$$

$$\mu_{Y|x} = \frac{2}{(1-x)^2}\int_0^{1-x} y^2\ dx = \frac{2}{(1-x)^2}\cdot\frac{(1-x)^3}{3} = \frac{2}{3}(1-x)$$

$$E(x^mY^n) = \int_0^1\int_0^{1-x} 24x^{m+1}y^{n+1}dy\ dx = \frac{24}{n+2}\int_0^1 x^{m+1}(1-x)^{n+2}dx$$

$$= \frac{24}{n+2}\cdot\frac{(m+1)!(n+2)!}{(m+n+4)!}$$ by definition of Beta function

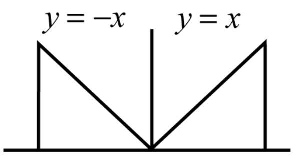
$$= \frac{24(m+1)!(n+1)!}{(m+n+4)!}$$

$$E(x) = \frac{2}{5},\ E(Y) = \frac{2}{5},\ E(x^2) = \frac{1}{5},\ E(Y^2) = \frac{1}{5},\ E(xY) = \frac{2}{15}$$

$$\sigma_1^2 = \frac{1}{25},\ \sigma_2^2 = \frac{1}{25},\ \sigma_{12} = -\frac{2}{75},\ \rho = -\frac{2}{3}$$

$$\mu_{Y|x} = \frac{2}{5} - \frac{2}{3}\left(x - \frac{2}{5}\right) = \frac{2}{3}(1-x)$$

14.9



$E(x)=0,\ E(xY)=0 \rightarrow$ uncorrelated

$$E(x^m y^n)=\int_0^1\int_0^x x^m y^n dy\ dx+\int_{-1}^0\int_0^{-x} x^m y^n dy\ dx$$

$$=\int_0^1 \frac{x^{m+n+1}}{n+1}dx+(-1)^{n+1}\int_{-1}^0 \frac{x^{m+n+1}}{n+1}dx=\frac{1-(-1)^{m+1}}{(n+1)(m+n+2)}$$

$$E(x)=0,\ E(Y)=\frac{1}{3},\ E(xY)=0$$

$$\therefore \sigma_{12}=0 \rightarrow \text{ uncorrelated}$$

$$h(y)=\int_{-y}^{y} dx=2y,\quad 0<y<1$$

$$g(x)=\begin{cases}\int_{-x}^{1} dy=1+x \text{ for } -1<x<0\\ \int_{x}^{1} dy=1-x \text{ for } 0<x<1\end{cases}$$

$$\phi(y|x)=\begin{cases}\frac{1}{1+x} \text{ for } -1<x\le 0 \text{ and } -x<y<1\\ \frac{1}{1-x} \text{ for } 0<x<1 \text{ and } x<y<1\end{cases}$$

14.10 $\text{var}(Y|x)=E(Y^2|x)-[E(Y|x)]^2$

multiply by $g(x)$ and integrate over x

$$\int \text{var}(Y|x)\ g(x)\ dx=\int\{g(x)\{E(Y^2|x)-[E(Y|x)]^2\}dx$$

$$\text{var}(Y|x)=E(Y^2)-\int g(x)[E(Y|x)]^2dx$$

$$=E(Y^2)-[E(Y)]^2-\left\{\int g(x)[E(Y|x)]^2dx-E(Y)^2\right\}$$

$$=\text{var}(Y)-\text{var}\,E(Y|x)$$

$$=\sigma_2^2-\text{var}\left[\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)\right]$$

$$=\sigma^2-\rho^2\frac{\sigma_2^2}{\sigma_1^2}\sigma_1^2=\sigma_2^2(1-\rho^2)$$

14.11 $\operatorname{var}\left(\frac{x}{\sigma_2}+\frac{Y}{\sigma_2}\right)=\frac{\sigma_1^2}{\sigma_1^2}+\frac{2\sigma_{12}}{\sigma_1\sigma_2}+\frac{\sigma_2^2}{\sigma_2}=2(1+\rho)$

$\operatorname{var}\left(\frac{x}{\sigma_1}-\frac{Y}{\sigma_2}\right)=\frac{\sigma_1^2}{\sigma_2^2}-\frac{2\sigma_{12}}{\sigma_1\sigma_2}+\frac{\sigma_2^2}{\sigma_2^2}=2(1-\rho)$

$1+\rho\geq 0 \qquad \rho\geq -1$ and $1-\rho\geq 0 \qquad \rho\leq 1$

$-1\leq\rho\leq 1$

14.12 $\int x_3 g(x_3|x_1,x_2)dx_3=\alpha+\beta_1(x_1-\mu_1)+\beta_2(x_2-\mu_2)$

multiply by $h(x_1,x_2)$ and integrate over x_1, x_2 and x_3

$\mu_2=\alpha+0+0=\alpha$

multiply by $(x_1-\mu_1)h(x_1,x_2)$ and integrate

$\sigma_{13}=\beta_1\sigma_1^2+\beta_2\sigma_{12}$

multiply by $(x_2-\mu_2)h(x_1,x_2)$ and integrate

$\sigma_{23}=\beta_1\sigma_{12}+\beta_2\sigma_2^2$

solve for β_1 and β_2

$\beta_1=\frac{\sigma_{23}\sigma_2^2-\sigma_{12}\sigma_{13}}{\sigma_1^2\sigma_2^2-\sigma_{12}^2}$ and $\beta_2=\frac{\sigma_{23}\sigma_1^2-\sigma_{12}\sigma_{13}}{\sigma_1^2\sigma_2^2-\sigma_{12}^2}$

14.13 $q=\sum_{i=1}^{n}[y_1=\hat{\beta}x_i]^2$

$$\frac{dq}{d\beta}=\sum_{i=1}^{n}(-2)x_i[y_i-\hat{\beta}x_i]=0 \qquad \hat{\beta}=\frac{\sum_{i=1}^{n}x_iy_i}{\sum_{i=1}^{n}x_i^2}$$

14.14 $\sum y=\hat{\alpha}n+\hat{\beta}\sum x$

$\sum xy-\hat{\alpha}\sum x+\hat{\beta}\sum x^2$

$$\hat{\alpha}=\frac{\begin{vmatrix}\sum y & \sum x\\ \sum xy & \sum x^2\end{vmatrix}}{\begin{vmatrix} n & \sum x\\ \sum x & \sum x^2\end{vmatrix}}=\frac{\left(\sum x^2\right)\left(\sum y\right)-\left(\sum x\right)\left(\sum xy\right)}{n\left(\sum x^2\right)-\left(\sum x\right)^2}$$

14.15 In previous exercise also $\hat{\beta} = \dfrac{\begin{vmatrix} n & \sum y \\ \sum x & \sum xy \end{vmatrix}}{n\left(\sum x^2\right) - \left(\sum x\right)^2} = \dfrac{n\left(\sum xy\right) - \left(\sum x\right)\left(\sum y\right)}{n\left(\sum x^2\right) - \left(\sum x\right)^2}$

letting $\sum x = 0$ yields $\hat{\alpha} = \dfrac{\left(\sum x^2\right)\left(\sum y\right)}{n\left(\sum x^2\right)} = \dfrac{\sum y}{n}$

$$\hat{\beta} = \frac{n\left(\sum xy\right)}{n\left(\sum x^2\right)} = \frac{\sum xy}{\sum x^2}$$

14.16 $q = \sum_{i=1}^{n} e_i^2 = 2\sum (y - \alpha - \beta x - \gamma x^2)$ differentiating partially with respect to α, β and γ and setting the resulting derivatives to zero to obtain the maximum likelihood estimates, we obtain

$$\frac{\partial q}{\partial \alpha} = 2\sum_{i=1}^{n}\left(y_i - \alpha - \beta x_i - \gamma x_i^2\right)(-1) = 0,$$

$$\frac{\partial q}{\partial \beta} = 2\sum_{i=1}^{n}\left(y_i - \alpha - \beta x_i - \gamma x_i^2\right)(-x_i) = 0, \text{ and}$$

$$\frac{\partial q}{\partial \gamma} = 2\sum_{i=1}^{n}\left(y_i - \alpha - \beta x_i - \gamma x_i^2\right)(-x_i^2) = 0.$$

Omitting the subscripts and limits of summation, we can write these equations in the usual normal-equation form:

$$\sum y = \alpha \cdot n + \beta \sum x + \gamma \sum x^2$$
$$\sum xy = \alpha \sum x + \beta \sum x^2 + \gamma \sum x^3$$
$$\sum x^2 y = \alpha \sum x^2 + \beta \sum x^3 + \gamma \sum x^4$$

14.17 $\sum [y - (\hat{\alpha} - \hat{\beta} x)]^2 = \sum (y_i - \overline{y} + \hat{\beta}\overline{x} - \hat{\beta} x_i]^2$

$$= \sum [(y_i - \overline{y}) - \hat{\beta}(x_i - \overline{x})]^2$$
$$= S_{yy} - 2\hat{\beta} S_{xy} + \hat{\beta}^2 S_{xx}$$
$$= S_{yy} - 2\hat{\beta} S_{xy} + \hat{\beta}\left(\frac{S_{xy}}{S_{xx}}\right) S_{xx}$$
$$= S_{yy} - \hat{\beta} S_{xy}$$

14.18 by Theorem 14.3 $E\left(\frac{n\hat{\sigma}^2}{\sigma^2}\right) = n-2$

(a) $E(\hat{\sigma}^2) = \frac{n-2}{n}\sigma^2 \neq \sigma^2$ QED

(b) $E\left(\frac{n\hat{\sigma}^2}{n-2}\right) = \frac{n}{n-2}\cdot\frac{n-2}{n}\cdot\sigma^2 = \sigma^2$

14.19 (a) $s_e = \hat{\sigma}\sqrt{\frac{n}{n-2}}$ $t = \frac{\hat{\beta}-\beta}{s_e / \sqrt{S_{xx}}}$

(b) $\hat{\beta} \pm t_{\alpha/2,n-2}\cdot\frac{s_e}{\sqrt{S_{xx}}}$

14.20 $\hat{\alpha} = \overline{y} - \hat{\beta}\overline{x}$ with $\hat{\beta} = \sum\left(\frac{x_i - \overline{x}}{S_{xx}}\right)y$ from text

(a) $\hat{\alpha} = \frac{\sum y_i}{n} - \sum \overline{x}\left(\frac{x_i - \overline{x}}{S_{xx}}\right)y_i$

$$\sum\left[\frac{1}{n} - \frac{(x_i - \overline{x})}{S_{xx}}\,y_i\;\overline{x}\right] = \sum_{i=1}^{n}\frac{S_{xx} + n\overline{x}^2 - n\overline{x}x_i}{nS_{xx}}y_i$$

(b) Use corollary to Theorem 4.14 and Exercise 7.58
Since $\hat{A}$ is linear combination of y's $\rightarrow \hat{\alpha}$ has normal distribution.

$$E(\hat{\alpha}) = \sum\left[\frac{SS_{xx} + n\overline{x}^2 - n\overline{x}x_i}{nS_{xx}}\right]E(Y_i)$$

$$= \sum\left[\frac{SS_{xx} + n\overline{x}^2 - n\overline{x}x_i}{nS_{xx}}\right](\alpha + \beta x_i)$$

$$= \frac{\alpha}{nS_{xx}}\sum[S_{xx} - n\overline{x}(x_i - \overline{x})] + \beta\sum\left[\frac{(S_{xx} + n\overline{x}^2)x_i}{nS_{xx}} - \frac{n\overline{x}x_i^2}{nS_{xx}}\right]$$

$$= \frac{\alpha}{nS_{xx}}\sum S_{xx} + \beta\sum\left[\frac{(S_{xx} + n\overline{x}^2)n\overline{x}}{nS_{xx}} - \frac{\overline{x}}{S_{xx}}\sum x_i^2\right]$$

$$= \alpha + \frac{\beta\overline{x}}{S_{xx}}\left[S_{xx} + n\overline{x}^2 - \sum x_i^2\right] = \alpha$$

$$\text{var}(\hat{\alpha}) = \sum\left[\frac{S_{xx} + n\overline{x}^2 - n\overline{x}x_i}{nS_{xx}}\right]^2\sigma^2$$

$$= \sum\left[\frac{S_{xx} + n\overline{x}(x_i - \overline{x})}{nS_{xx}}\right]^2\sigma^2 = \frac{1}{n} + \frac{n^2\overline{x}S_{xx}}{n^2S_{xx}^2}\cdot\sigma^2$$

$$= \frac{(S_{xx} + n\overline{x}^2)\sigma^2}{nS_{xx}}$$

14.21 $a_i = \dfrac{S_{xx} - n\overline{x}(x_i - \overline{x})}{nS_{xx}} \qquad b_i = \dfrac{x_i - \overline{x}}{S_{xx}}$

$$\operatorname{cov}(\hat{A},\hat{B}) = \sum a_i b_i \sigma^2 = \frac{\sigma^2}{nS_{xx}^2}\sum[S_{xx} - n\overline{x}(x_i - \overline{x})](x_i - \overline{x})$$

$$= \frac{\sigma^2}{nS_{xx}^2}[-n\overline{x}S_{xx}] = -\frac{\overline{x}}{S_{xx}}\sigma^2$$

14.22 $z = \dfrac{\hat{\alpha} - \alpha}{\sqrt{\dfrac{(S_{xx} + n\overline{x}^2)}{nS_{xx}}} \cdot \sigma} = \dfrac{(\hat{\alpha} - \alpha)\sqrt{nS_{xx}}}{\sigma\sqrt{S_{xx} + n\overline{x}^2}}$ has standard normal distribution and is independent of Z.

Also $\dfrac{n\hat{\sigma}^2}{\sigma^2}$ has χ^2 distribution with $n-2$ degrees of freedom.

$$t = \frac{(\hat{\alpha} - \alpha)\sqrt{nS_{xx}}}{\sigma\sqrt{S_{xx} + n\overline{x}^2}} + \sqrt{\frac{n\hat{\sigma}^2 / \sigma^2}{n-2}} = \frac{(\hat{\alpha} - \alpha)\sqrt{(n-2)S_{xx}}}{\hat{\sigma}^2\sqrt{S_{xx} + n\overline{x}^2}}$$

has t distribution with $n-2$ degrees of freedom

14.23 $\hat{Y}_0 = \hat{A} + \hat{B}x_0$ is sum of independent normal random variables and according to Ex. 7.58 has normal distribution

$$E(\hat{A}) + x_0 E(\hat{B}) = \alpha + x_0\beta = E(\hat{Y}_0 \,|\, x_0)$$

$$\operatorname{var}(\hat{Y}_0 \,|\, x_o) = \operatorname{var}(\hat{A}) + x_0^2 \operatorname{var}(\hat{B}) + 2x_0 \operatorname{cov}(\hat{A},\hat{B})$$

$$= \frac{(S_{xx} + n\overline{x}_2)\sigma^2}{nS_{xx}} + x_0^2 \cdot \frac{\sigma^2}{S_{xx}} + 2x_0\left(-\frac{\overline{x}}{S_{xx}}\sigma^2\right)$$

$$= \sigma^2\left[\frac{1}{n} + \frac{\overline{x}^2}{S_{xx}} + \frac{x_0^2}{S_{xx}} - \frac{2x_0\overline{x}}{S_{xx}}\right] = \sigma^2\left[\frac{1}{n} + \frac{(\overline{x} - x_0)^2}{S_{xx}}\right]$$

Using Theorem 14.3,

$$t = \frac{\hat{y}_0 - (\alpha + x_0\beta)}{\sigma\sqrt{\dfrac{1}{n} + \dfrac{(\overline{x} - x_0)^2}{S_{xx}}}} \div \sqrt{\frac{n\hat{\sigma}^2 / \sigma^2}{n-2}} = \frac{[\hat{y} + (\alpha + x_0\beta)]\sqrt{n-2}}{\hat{\sigma}\sqrt{1 + \dfrac{n(x - \overline{x}_0)^2}{S_{xx}}}}$$

has t distribution with $n-2$ degrees of freedom.

14.24 confidence limits are

$$\hat{y}_0 \pm t_{\alpha/2,n-2} \cdot \frac{\hat{\sigma}}{\sqrt{n-2}}\sqrt{1 + \frac{n(\overline{x} - x_0)^2}{S_{xx}}}$$

by substituting expression for t from Exercise 14.31 into $-t_{\alpha/2,n-2} < t < t_{\alpha/2,n-2}$ and solving by simple algebra.

14.25 $E[Y_0 - (\hat{A} + \hat{B}x_0)] = (\alpha + \beta x_0) - (\alpha + \beta x_0) = 0$

$$\begin{aligned}
\operatorname{var}[Y_0 - (A + Bx_0)] &= \sigma^2 + \operatorname{var}(\hat{A}) + x_0^2 \operatorname{var}(\hat{B}) - 2x_0 \operatorname{cov}(\hat{A}, \hat{B}) \\
&= \sigma^2 + \frac{(S_{xx} + n\overline{x}^2)\sigma^2}{nS_{xx}} + \frac{\sigma^2}{S_{xx}} x_0^2 - \frac{2x_0\overline{x}}{S_{xx}}\sigma^2 \\
&= \sigma^2\left[1 + \frac{1}{n} + \frac{\overline{x}^2 + x_0^2 - 2x_0\overline{x}}{S_{xx}}\right] = \sigma^2\left[1 + \frac{1}{n} + \frac{(\overline{x} - x_0)^2}{S_{xx}}\right]
\end{aligned}$$

$$t = \frac{[y_0 - (\hat{\alpha} + \hat{\beta}x_0)]}{\sigma\sqrt{1 + \frac{1}{n} + \frac{(\overline{x} - x_0)^2}{S_{xx}}}} + \sqrt{\frac{n\hat{\sigma}^2 / \sigma^2}{n-2}} = \frac{[\hat{y} - (\alpha + \beta x_0)]\sqrt{n-2}}{\hat{\sigma}\sqrt{1 + n + \frac{n(\overline{x} - x_0)^2}{S_{xx}}}}$$

14.26 Simple algebra leads to the following limits of prediction:

$$\hat{y}_0 \pm t_{\alpha/2,n-2} \cdot \frac{\hat{\sigma}}{\sqrt{n-2}} \sqrt{1 + n + \frac{n(\overline{x}_0 - x)^2}{S_{xx}}}$$

14.28 $t = \dfrac{\hat{\beta} - \beta}{\hat{\sigma}}\sqrt{\dfrac{(n-2)S_{xx}}{n}} = \left(1 - \dfrac{\beta}{\hat{\beta}}\right)\dfrac{\hat{\beta}}{\sigma}\sqrt{\dfrac{(n-2)S_{xx}}{n}}$

$$= \left(1 - \frac{\beta}{\hat{\beta}}\right)\frac{S_{xy}}{S_{xx}} \frac{\sigma^2}{\sqrt{1 - r^2}}\sqrt{\frac{(n-2)S_{xx}}{n}}$$

$$= \left(1 - \frac{\beta}{\hat{\beta}}\right)\frac{r}{\sqrt{1 - r^2}}\sqrt{n-2} \qquad \text{QED}$$

14.29 $1 - \dfrac{\beta}{\hat{\beta}} = \pm t_{a/2,n-2}\dfrac{\sqrt{1 - r^2}}{r\sqrt{n-2}}$

$$\frac{\beta}{\hat{\beta}} = 1 \pm t_{a/2,n-2}\frac{\sqrt{1 - r^2}}{r\sqrt{n-2}}$$

$$\beta = \hat{\beta}\left[1 \pm t_{\alpha/2,n-2}\frac{\sqrt{1 - r^2}}{r\sqrt{n-2}}\right] \qquad \text{QED}$$

14.30

$$\begin{array}{ll}
t = \dfrac{r\sqrt{n-2}}{\sqrt{1 - r^2}} & u = r^2 \\
& du = 2r \ \ dr \\
t^2 = \dfrac{r^2(n-2)}{1 - r^2} & r^2 = \dfrac{t^2}{n - 2 + t^2} \\
2t\dfrac{dt}{dr^2} = \dfrac{n-2}{(1 - r^2)^2} & \dfrac{dt}{dr^2} = \dfrac{(n-2)}{(1 - r^2)^2} \cdot \dfrac{\sqrt{1 - r^2}}{2r\sqrt{n-2}}
\end{array}$$

$$g(r^2) = \frac{\sqrt{n-2}}{2r(1-r^2)\sqrt{1-r^2}} \cdot k\left(1+\frac{t^2}{n-2}\right)^{-(n-1)/2}$$

$$= \frac{\sqrt{(n-2)}k}{2r(1-r^2)\sqrt{1-r^2}}\left[1+\frac{r^2}{1-r^2}\right]^{-(n-1)/2}$$

$$= \frac{K}{r(1-r^2)\sqrt{1-r^2}}(1-r^2)^{(n-1)/2}$$

$$= K(r^2)^{-1/2}(1-r^2)^{(n-4)/2} \qquad \text{beta distribution}$$

$$\alpha - 1 = -\frac{1}{2} \qquad \beta - 1 = \frac{n-4}{2}$$

$$\mu = \frac{\alpha}{\alpha+\beta} = \frac{\frac{1}{2}}{\frac{1}{2}+\frac{n-2}{n}} = \frac{1}{n-1}$$

14.31 $$-z_{\alpha/2} \le \frac{\sqrt{n-3}}{2}\ln\frac{(1+r)(1-\rho)}{(1-r)(1+\rho)} \le z_{\alpha/2}$$

$$-\frac{2z_{\alpha/2}}{\sqrt{n-3}} \le \ln\frac{(1+r)(1-\rho)}{(1-r)(1+\rho)} \le \frac{2z_{\alpha/2}}{\sqrt{n-3}}$$

$$e^{-(2z_{\alpha/2})/\sqrt{n-3}} \le \frac{(1+r)(1-\rho)}{(1-r)(1+\rho)} \le e^{(2z_{\alpha/2})/\sqrt{n-3}}$$

$$\frac{(1-r)}{(1+r)}e^{-(2z_{\alpha/2})/\sqrt{n-3}} \le \frac{1-\rho}{1+\rho} \le \frac{(1-r)}{(1+r)}e^{(2z_{\alpha/2})/\sqrt{n-3}}$$

$$1+\rho\cdot\frac{(1-r)}{1+r}e^{-(2z_{\alpha/2})/\sqrt{n-3}} \le \frac{1-\rho}{1+\rho} \le \frac{(1-r)}{(1+r)}e^{(2z_{\alpha/2})/\sqrt{n-3}}$$

$$\rho\left[1+\frac{1-r}{1+r}e^{-(2z_{\alpha/2})/\sqrt{n-3}}\right] \le 1-\frac{(1-r)}{(1+r)}e^{-(2z_{\alpha/2})/\sqrt{n-3}}$$

$$\rho \le \frac{1+r-(1-r)e^{-(2z_{\alpha/2})/\sqrt{n-3}}}{1+r+(1-r)e^{-(2z_{\alpha/2})/\sqrt{n-3}}} \quad \text{and}$$

$$\rho\left[1+\frac{1-r}{1+r}e^{(2z_{\alpha/2})/\sqrt{n-3}}\right] \ge 1-\frac{(1-r)}{(1+r)}e^{(2z_{\alpha/2})/\sqrt{n-3}}$$

$$p \ge \frac{1+r-(1-r)e^{(2z_{\alpha/2})/\sqrt{n-3}}}{1+r+(1-r)e^{(2z_{\alpha/2})/\sqrt{n-3}}}$$

$$\frac{1+r-(1-r)e^{(2z_{\alpha/2})/\sqrt{n-3}}}{1+r+(1-r)e^{(2z_{\alpha/2})/\sqrt{n-3}}} \le \rho \le \frac{1+r-(1-r)e^{-(2z_{\alpha/2})/\sqrt{n-3}}}{1+r+(1-r)e^{-(2z_{\alpha/2})/\sqrt{n-3}}}$$

14.32 Substitute $S_{xx} = \sum_{i=1}^{r} x_i^2 f_i - \frac{1}{n}\left[\sum_{i=1}^{r} x_i f_i\right]^2$

$$S_{yy} = \sum_{j=1}^{r} y_j^2 f_j - \frac{1}{n}\left[\sum_{j=1}^{r} y_j f_j\right]^2$$

and

$$S_{xy} = \sum_{i=1}^{r}\sum_{j=1}^{r} x_i y_j f_{ij} - \frac{1}{n}\left[\sum_{i=1}^{r} x_i f_i\right]\left[\sum_{j=1}^{r} y_j f_j\right]$$

into $r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$

14.33 $q = (Y - Xb)'(Y - Xb)$

$= \{Y' - (Xb)'\}\{Y - Xb\}$

$= Y'Y - Y'Xb - (Xb)'Y + (Xb)'Xb$

since $Y'Xb$ is $|X|$, a number, not a matrix, $Y'Xb = (Xb)'Y$

$q = Y'Y - 2Y'Xb + b'X'Xb$

vector of partial derivatives is

$-2(Y'X)' + 2X'Xb = -2X'Y = 2X'Xb$

put equal to zero yields

$-2X'Y + 2X'Xb = 0$

$b = (X'X)^{-1}X'Y$ QED

14.34 $L(b, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-(1/2\sigma^2)(Y-xb)'(Y-Xb)}$

To maximize L minimize $(Y - Xb)'(Y - Xb)$ as in Ex 14.33

(a) $\therefore$ maximum likelihood estimates = least square estimates

(b) as in simple regression

$$\ln L = -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}(Y - Xb)'(Y - Xb)$$

$$\frac{\partial \ln L}{\partial \sigma^2} = -\frac{n}{2}\cdot\frac{1}{\sigma^2} + \frac{1}{2\sigma^4}(Y - Xb)'(Y - Xb) = 0$$

together with $\frac{\partial \ln L}{\partial b} = 0$ we get

$\hat{\sigma}^2 = \frac{1}{n}(Y - XB)'(Y - XB)$ QED

14.35 $(Y - XB)'(Y - XB) = [(Y - X(X'X)^{-1}X'Y]'[Y - X(X'X)^{-1}X'Y]$

$= Y'[I - X(X'X)^{-1}X'][I - X(X'X)^{-1}X']Y$

$= Y'[I - X(X'X)^{-1}X']Y$

$= Y'Y - Y'X(X'X)^{-1}X'Y$

$= Y'Y - B'X'Y$ QED

14.36 $\hat{B} = (X'X)^{-1}X'Y$

(a) $$E(\hat{B}) = (X'X)^{-1}X'E(Y)$$
$$= (X'X)^{-1}X'XB = B$$
$$E(\hat{B}_i) = \hat{B}_i \text{ for } i = 0,1,2,\ldots k$$

(b) $$\operatorname{var}(\hat{B}) = (X'X)^{-1}X'\operatorname{var}(Y)[(X'X)^{-1}X']'$$
$$= (X'X)^{-1}X'\sigma^2[(X'X)^{-1}X']'$$
$$= \sigma^2(X'X)^{-1}$$
$$\operatorname{var}(\hat{B}_i) = c_{i1}\sigma^2 \text{ for } 0,1,2,\ldots k$$

(c) $$\operatorname{cov}(\hat{B}) = (X'X)^{-1}X\operatorname{cov}(Y)[(X'X)^{-1}X']'$$
$$= (X'X)^{-1}\sigma^2 I[(X'X)^{-1}X']'$$
$$= \sigma^2(x'x)^{-1}$$
$$\operatorname{cov}(\hat{B}_i, \hat{B}_j) = c_{ij}\sigma^2 \quad \text{for } i \neq j = 0,1,\ldots k$$

14.38 $$\hat{\beta}_i - t_{\alpha/2,n-k-1}\hat{\sigma}^2\sqrt{\frac{n|c_{ii}|}{n-k-1}} \le \beta_i \le \hat{\beta}_i + t_{\alpha/2,n-k-1}\hat{\sigma}\sqrt{\frac{n|c_{ii}|}{n-k-1}}$$

14.39 **(a)** $$B'X_0 = (\hat{\alpha}\hat{\beta})(X_0') = \hat{\alpha} + \hat{\beta}x_{01} = \hat{y}_0$$
$$(X'X)^{-1}\begin{pmatrix} \dfrac{S_{xx} + n\bar{x}^2}{nS_{xx}} & -\dfrac{\bar{x}}{S_{xx}} \\ -\dfrac{\bar{x}}{S_{xx}} & \dfrac{1}{S_{xx}} \end{pmatrix}$$
$$X_0'(X'X)^{-1} = \frac{S_{xx} + n\bar{x}^2 - nx_0\bar{x}}{nS_{xx}}, \frac{-\bar{x} + x_0}{S_{xx}}$$
$$X_0'(X'X)^{-1}X_0 = \frac{S_{xx} + n\bar{x}^2 - nx_0\bar{x} - nx_0\bar{x} + nx_0^2}{nS_{xx}}$$
$$n[X_0'(X'X)^{-1}X_0] = 1 + \frac{n(x_0 - \bar{x})^2}{S_{xx}}$$
$$t = \frac{(\hat{y}_0 - \mu_{Y|x_0})\sqrt{n-2}}{\hat{\sigma}\sqrt{1 + \dfrac{n(x_0 - \bar{x})^2}{S_{xx}}}}$$

(b) confidence limits are $B'X_0 \pm t_{\alpha/2,n-k-1}\hat{\sigma}\sqrt{\dfrac{n[X_0'(X'X)^{-1}X_0}{n-k-1}}$

14.40 (a) From 14.39

$$B'X_0 = \hat{\alpha} + \hat{\beta} X_0$$

$$X_0'(X'X)^{-1}X_0 = \frac{S_{xx} + n(x_0 - \bar{x})^2}{nS_{xx}}$$

$$n[1 + X_0'(X'X)^{-1}X_0] = \frac{nS_{xx} + S_{xx} + n(x_0 - \bar{x})^2}{S_{xx}} = n + 1 + \frac{n(x_0 - \bar{x})^2}{S_{xx}}$$

$$t = \frac{[(y_0 - (\hat{\alpha} + \hat{\beta} x_0))]\sqrt{n-2}}{\hat{\sigma}\left[1 + n\frac{n(x_0 - \bar{x})^2}{S_{xx}}\right]}$$

(b) confidence limits are $B'X_0 \pm t_{\alpha/2,n-k-1}\hat{\sigma}\sqrt{\frac{n[1 + X_0'(X'X)^{-1}X_0]}{n-k-1}}$

14.41 (a) $n = 5,\ \sum x = 7.69,\ \sum x^2 = 14.0225,\ \sum y = 447.9,\ \sum xy = 697.608$ Thus,

$S_{xx} = 14.0225 - (7.69)^2/5 = 2.1953 = 2.1953$ and

$S_{xy} = 697.608 - (7.69)(447.9)/5 = 8.7378.$

Finally, $\hat{\beta} = \frac{8.7378}{2.1953} = 3.98$ and $\hat{\alpha} = \frac{447.9}{5} - 3.98\frac{7.69}{5} = 83.46$.

(b) If x= 1.3, y is estimated as $\hat{y} = 83.46 + (3.98)(1.3) = 88.63$.

14.42 (a) $n = 7,\ \sum x = 70,\ \sum x^2 = 812,\ \sum y = 68,\ \sum y^2 = 952,\ \sum xy = 862$

$$S_{xx} = 812 - \frac{1}{7}(70)^2 = 812 - 700 = 112$$

$$S_{xy} = 862 - \frac{1}{7}(70)68 = 862 - 680 = 182$$

$$S_{yy} = 962 - \frac{1}{7}(68)^2 = 962 - 650.5714 = 301.4286$$

$$\hat{\beta} = \frac{182}{112} = 1.625 \qquad \hat{\alpha} = \frac{68}{7} - (1.625)10$$
$$= 9.7143 - 16.25 = -6.5357$$

(a) $\hat{y} = -6.5357 + 1.625x$

(b) $\hat{y} = -6.5357 + 1.625(7) = 4.8393$

14.43 $n = 12,\ \sum x = 854,\ \sum x^2 = 64.222,\ \sum y = 876,\ \sum y^2 = 65{,}850,\ \sum xy = 64{,}346$

$$S_{xx} = 64{,}222 - \frac{1}{12}(854)^2 = 64{,}222 - 60{,}776.333 = 3445.67$$

$$S_{xy} = 64{,}346 - \frac{1}{12}(854)(876) = 64{,}346 - 62{,}342 = 2004$$

$$\hat{\beta} = \frac{2004}{3445.67} = 0.5816 \qquad \hat{\alpha} = 73 - (0.5816)(71.1667) = 31.609$$

(a) $\hat{y} = 31.609 + 0.5816x$

(b) $\hat{y} = 31.609 + 0.5816(84) = 80.45$

14.44 $n = 12,\ \sum x = 507,\ \sum x^2 = 22{,}265,\ \sum y = 144,\ \sum y^2 = 1802,\ \sum xy = 6314$

$$S_{xx} = 22{,}265 - \frac{1}{12}(507)^2 = 844.25$$

$$S_{xy} = 6314 - \frac{1}{12}(507)(144) = 230$$

$$\hat{\beta} = \frac{230}{844.25} = 0.2724,\ \ \hat{\alpha} = \frac{144}{12} - (0.2724)\frac{507}{12} = 0.4911$$

(a) $\hat{y} = 0.4911 + 0.2724x$

(b) $\hat{y} = 0.4911 + (0.2724)(38) = 10.8423$

14.45 $n = 6,\ \sum x = 42,\ \sum x^2 = 364,\ \sum y = 7.8,\ \sum y^2 = 10.68,\ \sum xy = 48.6$

$$S_{xx} = 364 - \frac{1}{6}(42)^2 = 70,\ \ S_{xy} = 48.6 - \frac{1}{6}(42)(7.8) = -6$$

$$\hat{\beta} = \frac{-6}{70} = -0.0857 \text{ and } \hat{\alpha} = \frac{7.8}{6} - (-0.0857)\frac{42}{6} = 1.8999$$

(a) $\hat{y} = 1.8999 - 0.0857x$

(b) $\hat{y} = 1.8999 - 0.0857(5) = 1.4714$

14.46

x	y	$x'y$
−3	1	−3
−2	3	−6
−1	6	−6
0	8	
1	14	14
2	16	32
3	20	60
	68	91

$$\hat{\alpha} = \frac{68}{7} = 9.7143$$

$$\hat{\beta} = \frac{91}{28} = 3.25$$

(a) $\hat{y} = 9.7143 + 3.25x$ (coded)

(b) $\hat{y} = 9.7143 + 3.25(-1.5) = 9.7413 - 4.875$
$= 4.8393$

14.47

x	y	xy
−5	1.8	−9.0
−3	1.5	−4.5
−1	1.4	−1.4
1	1.1	1.1
3	1.1	3.3
5	0.9	4.5
	7.8	−6.0

$$\sum x^2 = 70,\ \hat{\alpha} = \frac{7.8}{6} = 1.3$$

$$\hat{\beta} = \frac{-6}{70} = -0.0857$$

(a) $\hat{y} = 1.3 - 0.0857x$ (coded)

(b) $\hat{y} = 1.3 - (0.0857)(-2) = 1.4714$

14.48

x	y	xy
-2	1.4	-2.8
-1	2.1	-2.1
0	2.6	
1	3.5	3.5
2	3.7	7.4
	13.3	6.0

$$\sum x^2 = 10,\ \hat{\alpha} = \frac{13.3}{5} = 2.66$$

$$\hat{\beta} = \frac{6}{10} = 0.6$$

$$\hat{y} = 2.66 + 0.6x \ \text{(coded)}$$

Sixth year $\hat{y} = 2.66 + 0.6(3) = 4.46$ million dollars

14.49

x	y	$y' = \log x$	xy'
1	2.0	0.3010	
2	2.4	0.3802	
4	5.1	0.7077	
5	7.3	0.8634	
6	9.4	0.9732	
8	18.3	1.2625	
26		4.4880	24.1484

$$\sum x^2 = 146$$

$$4.4880 = 6\log\hat{a} + 26\log\hat{\beta}$$

$$24.1484 = 25(\log\hat{\alpha}) + 146\log\hat{\beta}$$

$$\log\hat{\alpha} = \frac{\begin{vmatrix} 4.4880 & 26 \\ 24.1484 & 146 \end{vmatrix}}{\begin{vmatrix} 6 & 26 \\ 26 & 146 \end{vmatrix}} = \frac{27.3896}{200}$$

$$= 0.13695 \qquad \hat{\alpha} = 1.371$$

$$\log\hat{\beta} = \frac{\begin{vmatrix} 6 & 4.4880 \\ 26 & 24.1484 \end{vmatrix}}{200} = \frac{28.2024}{200} = 0.1410$$

$$\hat{\beta} = 1.383 \qquad \hat{y} = 1.371(1.383)^x$$

14.50 $x' = \log x$ $\qquad y' = \log y$

x	y	x'	y'
50	108	1.6990	2.0334
100	53	2.0000	1.7243
250	24	2.3679	1.3802
500	9	2.6990	0.9542
1,000	5	3.0000	0.6990

$$n = 5 \qquad \sum x' = 11.7659$$

$$\sum (x')^2 = 28.77815 \qquad \sum y' = 6.7911$$

$$\sum x'y' = 14.8439$$

$$S_{x'x'} = 28.77815 - \frac{1}{5}(11.7659)^2 = 28.77815 - 27.68728 = 1.0909$$

$$S_{x'y'} = 14.8439 - \frac{1}{5}(11.7659)(6.7911) = 14.8439 - 15.9807 = -1.1368$$

$$\hat{\beta} - \frac{-1.1368}{1.0909} = -1.0421 \qquad \log\hat{\alpha} = \frac{6.7911}{5} + (1.0421)\frac{11.7659}{5}$$

$$= 1.3582 + 2.4522 = 3.8104 \qquad \hat{\alpha} = 6{,}460$$

(a) $\hat{y} = 6{,}450x^{-1.0421}$

(b) $\log \hat{y} = 3.8104 - 1.0421(2.4771) = 3.8104 - 2.5814 = 1.2290$

$\hat{y} = 17.3 \quad (\$17.30)$

Since the calculations in Exercises 14.51 through 14.61 are fairly extensive, answers may differ substantially due to rounding.

14.51 $n = 7,\ \hat{\beta} = 1.625,\ S_{xx} = 112,\ S_{xy} = 182,\ S_{yy} = 301.4286$

1. $H_0 : \beta = 1.25,\ H_1 : \beta > 1.25,\ \alpha = 0.01$
2. Reject null hypothesis if $t \ge t_{0.01,5} = 3.365$
3. $\hat{\sigma} = \sqrt{\frac{1}{7}[301.4286 - (1.625)182]} = 0.9007$

   $t = \frac{(1.625 - 1.25)}{0.9007}\sqrt{\frac{5(112)}{7}} = (0.4163)(8.9443) = 3.7235$
4. Since $3.7235 > 3.365$, null hypothesis must be rejected.

14.52 $n = 12,\ \hat{\beta} = 0.2724,\ S_{xx} = 844.25,\ S_{xy} = 230$

$S_{yy} = 1802 - \frac{1}{12}(144)^2 = 1802 - 1728 = 74$ from Ex 14.18

$\hat{\sigma} = \sqrt{\frac{1}{12}[74 - (0.2724)230]} = 0.9725$

$t = \frac{0.2724 - 0.350}{0.9725}\sqrt{\frac{10(844.25)}{12}} = -\frac{0.0776}{0.9725}(26.5244) = -2.12$

1. $H_0 : \beta = 0.350,\ H_1 : \beta < 0.350,\ \alpha = 0.05$
2. Reject null hypothesis if $t \le -t_{0.05,10} = -1.812$
3. $t = -2.12$
4. Since $t = -2.12 < -1.812$, null hypothesis must be rejected.

14.53 $n = 8,\ \sum x = 1447.5,\ \sum x^2 = 264{,}290.5,\ \sum y = 1864.5,\ \sum y^2 = 439{,}901.6,$

$SS_{XX} = 264{,}290.5 - \frac{1}{8}(1447.5)^2 = 2383.469$

$SS_{XY} = 340{,}915.9 - \frac{1}{8}(1447.5)(1804.5) = 3557.911$

$S_{YY} = 439{,}901.6 - \frac{1}{8}(1864.5)^2 = 5356.599$

(a) $\hat{\beta} = \frac{3557.911}{2393.469} = 14{,}927$

$\hat{\alpha} = \frac{1864.5}{8} - (1.4927)\frac{1447.5}{8} = -37.023$

$\hat{y} = -37.023 + 1.4927x$

(b) 1. $H_0: \beta = 1.30,\ H_1: \beta > 1.30,\ \alpha = 0.05$

2. Reject null hypothesis if $t \geq t_{0.05,6} = 1.943$

3. $\hat{\sigma} = \sqrt{\frac{1}{8}[535.599 - (1.4927)(3557.911)]} = 2.3866$

$$t = \frac{1.4927 - 1.30}{2.3866}\sqrt{\frac{6}{8}(2383.469)} = 3.413$$

4. Since $t = 3.414 > 1.943$, null hypothesis must be rejected.

14.54 $n = 12,\ S_{xx} = 3445.67,\ S_{xy} = 2004$

$\hat{\beta} = 0.5816$ from Ex. 14.43

$S_{yy} = 65{,}850 - \frac{1}{12}(876)^2 = 1902$

$\hat{\sigma} = \sqrt{\frac{1}{12}[1902 - (0.5816)(2004)]} = 7.8341$

confidence limits are $0.5816 \pm (3.169)(7.8341)\sqrt{\frac{12}{10(3445.67)}}$

$0.5816 \pm (3.169)(7.8341)(0.01866)$

0.5816 ± 0.4632

$0.1184 < \beta < 1.0448$

14.55 $n = 6,\ \hat{\beta} = -0.0857,\ S_{xx} = 70,\ S_{xy} = -6$

$S_{yy} = 10.68 - \frac{1}{6}(7.8)^2 = 0.54$

$\hat{\sigma} = \sqrt{\frac{1}{6}(0.54 - (-0.0857)(-6)} = 0.06557$

confidence limits are $-0.0857 \pm (3.747)(0.06557)\sqrt{\frac{6}{4(70)}}$

-0.0857 ± 0.0360

$-0.1217 < \beta < -0.0497$

14.56 $n = 10,\ S_{xx} = 376,\ S_{xy} = 1305,\ \hat{\alpha} = 21.69,\ \hat{\beta} = 3.471$

$S_{yy} = 36{,}562 - \frac{1}{10}(564)^2 = 4752.4$

1. $H_0: \alpha = 21.50,\ H_1: \alpha \neq 21.50,\ \alpha = 0.01$

2. Reject null hypothesis if $t \leq -3.355$ or $t \geq 3.355$ $(t_{0.05,8})$

3. $\hat{\sigma} = \sqrt{\frac{1}{10}[4752.4 - (3.471)(1305)]} = 4.7196$

$$t = \frac{(21.69 - 21.50)\sqrt{8(376)}}{4.7196\sqrt{376 + 10(37.6)^2}} = 0.0183$$

4. Since $t = 0.0183$ falls between -3.355 and 3.355, null hypothesis cannot be rejected.

14.57 $n = 6,\ \sum x = 9,\ \sum x^2 = 16.94,\ \sum y = 20.9,\ \sum y^2 = 80.47,\ \sum xy = 36.45$

$$S_{xx} = 16.94 - \frac{1}{6}(9)^2 = 3.44$$

$$S_{xy} = 36.45 - \frac{1}{6}(9)(20.9) = 5.1$$

$$S_{yy} = 80.47 - \frac{1}{6}(20.9)^2 = 7.6683$$

(a) $\hat{\beta} = \frac{5.1}{3.44} = 1.4826$ and $\hat{\alpha} = \frac{20.9}{6} - (1.4826)(1.5) - 1.2594$

$\hat{y} = 1.2594 - 1.4826x$

(b) 1. $H_0: \alpha = 0.08,\ H_1: \alpha > 0.08,\ \alpha = 0.01$

2. Reject null hypothesis if $t \geq -t_{0.01,4} = 3.747$

3. $\hat{\sigma} = \sqrt{\frac{1}{6}[7.6683 - (1.4826)(5.1)]} = 0.1336$

$$t = \frac{(1.2594 - 0.8)\sqrt{4(3.44)}}{(0.1336)\sqrt{3.44 + 6(1.5)^2}} = 3.10$$

4. Since $t = 3.10$ is less than 3.747, null hypothesis cannot be rejected.

14.58 $n = 7,\ \hat{\alpha} = -6.5357,\ S_{xx} = 112,\ \bar{x} = \frac{70}{7} = 10,\ \hat{\sigma} = 0.9007,\ t_{0.025,5} = 2.571$

$$-6.5357 \pm \frac{(2.571)(0.9007)\sqrt{112 + 7(10)^2}}{\sqrt{5(112)}}$$

$$-6.5367 \pm \frac{(2.3157)(28.4956)}{23.6643}$$

$$-6.5357 \pm 2.7885$$

$$-9.3242 < \alpha < -3.7472$$

14.59 $n = 12,\ \hat{\alpha} = 31.609,\ \hat{\beta} = 0.5816,\ \mathrm{S}_{xx} = 3445.67,\ \hat{\sigma} = 7.8341,\ \bar{x} = \frac{854}{12} = 71.1667,$

$t_{0.005,10} = 3.169$

$$31.609 \pm \frac{(3.169)(7.8341)\sqrt{3445.67 + 12(71.1667)^2}}{\sqrt{10(3445.67)}}$$

$$31.609 \pm \frac{(24.8263)(253.4207)}{185.6252}$$

$$31.609 \pm 33.8936$$

$$-2.2846 < \alpha < 65.5026$$

14.60 **(a)** $$70.284 \pm (2.306)(4.720)\frac{\sqrt{1+\frac{10(14-10)^3}{376}}}{\sqrt{8}}$$

$$70.284 \pm (3.8482)\sqrt{1+0.4255}$$

$$70.284 \pm (3.8482)(1.1939)$$

$$70.284 \pm 4.5945$$

$$65.6895 < \mu_{Y|14} < 74.8785$$

(b) $$70.284 \pm (3.8482)\sqrt{11.4255}$$

$$70.284 \pm 13.0075$$

Limits of prediction are 57.2765 and 83.2915

14.61 $n = 7,\ S_{xx} = 112,\ \bar{x} = 10,\ x_0 = 9,\ t_{0.005,5} = 4.032,\ \hat{\sigma} = 0.9007$ **,**

$\hat{y}_0 = -6.5357 + 1.625(9) = 8.0893$

(a) $$8.0893 \pm \frac{(4.032)(0.9007)\sqrt{1+\frac{7(9-10)^2}{112}}}{\sqrt{5}}$$

$$8.0893 \pm (1.6421)\sqrt{1.0625}$$

$$8.0893 \pm 1.6741$$

$$6.452 < \mu_{Y|9} < 9.7634$$

(b) $$8.0893 \pm (1.6241)\sqrt{8.0625}$$

$$8.0893 \pm 4.6116$$

Limits of prediction are 3.4777 and 12.7009

14.62 $\hat{y}_0 = -6.537 + 1.625 \cdot 20 = 25.963$

(a) The confidence limits are $25.963 \pm \frac{4.032 \cdot 0.9007\sqrt{1+\frac{7(20-10)^2}{112}}}{\sqrt{5}}$ or 25.963 ± 4.373

(b) The limits of prediction are $25.963 \pm \frac{4.032 \cdot 0.9007\sqrt{1+7+\frac{7(20-10)^2}{112}}}{\sqrt{5}}$ or 25.963 ± 13.709 .

14.63 **(a)** Using MINITAB

```
MTB> Regress C2 on 1 C1
```

The regression equation is

```
C2 = 2.20 + 13.3 C1
```

(b) We calculate: $\sum x = 45.8 \quad \sum x^2 = 260.46 \quad \sum xy = 3{,}558.42$

$\sum y = 630.0 \quad \sum y^2 = 48{,}735.06$

Therefore, $S_{xx} = 260.46 - (45.8)^2/10 = 50.70$

$S_{yy} = 48,735.06 - (630.0)^2/10 = 9,045.06$

$S_{xy} = 3,558.42 - (45.8)(630.0)/10 = 673.02$

The 99% confidence limits for β are

$$\hat{\beta} = t_{\alpha/2,n-2}\hat{\sigma}\sqrt{\frac{n}{(n-2)S_{xx}}} : \text{ numerically, } 13.27 \pm (3.355)(3.38)\sqrt{\frac{10}{(8)(50.70)}}$$

where $t_{0.005,8} = 3.355$ Table IV) and

$$\hat{\sigma} = \sqrt{\frac{1}{10}[9,045.06 - (13.27)(673.02)]} = 3.38$$

Thus, 99% confidence limits for β are 13.27 ± 1.78, or (11.5, 15.1).

14.64 Using MINITAB

MTB> Regress C2 1 C1

The regression equation is

C2 = 1.09 + 0.0131 C1

(b) We calculate: $\sum x = 340$ $\sum x^2 = 15,500$ $\sum xy = 573.10$

$\sum y = 13.16$ $\sum y^2 = 21.9072$

Therefore, $S_{xx} = 15,500 - (340)^2/8 = 1,050$

$S_{yy} = 21.9072 - (13.16)^2/8 = 0.259$

$S_{xy} = 573.10 - (340)(13.16)/8 = 13.80$

To test $H_0 : \beta = 0.01;\ H_1 : \beta > 0.01$ we calculate

$$t = \frac{\hat{\beta} - \beta_0}{\hat{\sigma}}\sqrt{\frac{(n-2)S_{xx}}{n}} = \frac{0.013 - 0.010}{0.100}\sqrt{\frac{(6)(1,050)}{8}} = 0.84$$

where $\hat{\sigma} = \sqrt{\frac{1}{8}[0.259 - (0.013)(13.80)]} = 0.100$

Since $t_{0.05,6} = 3.707$, we cannot reject the null hypothesis at the 0.05 level of significance.

14.65 $n = 20,\ \sum x = 688,\ \sum x^2 = 24{,}282,\ \sum y = 703,\ \sum y^2 = 25{,}555,\ \sum xy = 24{,}582$

$$S_{xx} = 24{,}282 - \frac{1}{20}(688)^2 = 24{,}282 - 23{,}677.2 = 614.8$$

$$S_{yy} = 25{,}555 - \frac{1}{20}(703)^2 = 25{,}555 - 24{,}710.45 = 844.55$$

$$S_{xy} = 24{,}582 - \frac{1}{20}(688)(703) = 24{,}582 - 24{,}183.2 = 398.8$$

$$r = \frac{398.8}{\sqrt{(614)(844.55)}} = \frac{398.8}{720.5757} = 0.553$$

$$z = \frac{\sqrt{17}}{2}\ln\frac{1.553}{0.447} = (2.06)(\ln 3.474) = 2.06(1.24530) = 2.565$$

1. $H_0: \rho = 0;\ H_1: \rho \neq 0,\ \alpha = 0.05$
2. Reject null hypothesis is $z \leq -1.96$ or $z \geq 1.96$
3. $z = \frac{\sqrt{17}}{2}\ln\frac{1.553}{0.447} = 2.565$
4. Reject null hypothesis; value of *r* is significant.

14.66 $$\frac{1.553 - 0.447e^{2(1.96)/\sqrt{17}}}{1.553 + 0.447e^{0.951}} \leq \rho \leq \frac{1.553 - 0.447e^{-0.951}}{1.553 + 0.447e^{-0.951}}$$

$$\frac{1.553 - 0.447(2.59)}{1.553 + 0.447(2.59)} \leq \rho \leq \frac{1.553 - 0.447(0.386)}{1.553 + 0.447(0.386)}$$

$$\frac{0.395}{2.711} \leq \rho \leq \frac{1.380}{1.726} \qquad 0.15 \leq \rho \leq 0.80$$

14.67 $n = 33,\ \sum x = 2550,\ \sum x^2 = 238{,}960,\ \sum y = 861,\ \sum y^2 = 25{,}313,\ \sum xy = 74{,}476$

$$S_{xx} = 238{,}960 - 197.045.45 = 41{,}914.55$$

$$S_{yy} = 25{,}313 - 22{,}464.27 = 2{,}848.73$$

$$S_{xy} = 74{,}476 - 66{,}531.82 = 7{,}944.18$$

$$r = \frac{7944.18}{10927.18} = 0.727$$

1. $H_0: \rho = 0;\ H_1: \rho \neq 0,\ \alpha = 0.01$
2. Reject null hypothesis is $z \leq -2.575$ or $z \geq 2.575$
3. $z = \frac{\sqrt{30}}{2}\ln\frac{1.727}{0.273} = (2.739)\ln 6.326 = (2.739)(1.845) = 5.05$
4. Reject null hypothesis; value of *r* is significant.

14.68 $$\frac{1.727-(0.273)e^{0.94}}{1.727+(0.273)e^{0.94}} \le \rho \le \frac{1.727-(0.273)e^{-0.94}}{1.727+(0.273)e^{-0.94}}$$

$$\frac{1.727-0.699}{1.727+0.699} \le \rho \le \frac{1.727-0.107}{1.727-0.107}$$

$$\frac{1.028}{2.426} \le \rho \le \frac{1.620}{1.834} \qquad 0.42 \le \rho \le 0.88$$

14.69 $$\left(1-\frac{\beta}{3.471}\right)\frac{0.976\sqrt{8}}{\sqrt{1-0.976^2}} = \pm 2.306$$

$$\left(1-\frac{\beta}{3.471}\right)\frac{2.7605}{0.2178} = \pm 2.306$$

$$1-\frac{\beta}{3.471} = \pm 0.182 \qquad \frac{\beta}{3.471} = 1 \pm 0.182$$

$$2.84 \le \beta \le 4.10$$

14.70

x	y
12	27
26	36
0	9
24	25
39	53
1	16
20	32
–4	3
14	24
35	63

$n = 10,\ \sum x = 167,\ \sum x^2 = 4755,\ \sum y = 288,\ \sum y^2 = 11{,}374,$
$\sum xy = 7112$

$$S_{xx} = 4755 - \frac{1}{10}(167)^2 = 4755 - 2788.9 = 1966.1$$

$$S_{yy} = 11374 - \frac{1}{10}(288)^2 = 11374 - 8294.4 = 3079.6$$

$$S_{xy} = 7112 - \frac{1}{10}(167)(288) = 7112 - 4809.6 = 2302.4$$

$$r = \frac{2302.4}{\sqrt{(1966.1)(3079.6)}} = \frac{2302.4}{2460.65} = 0.936$$

14.71

	23	28	33	38	43	
23	1					1
28		3	1			4
33		2	5	2		9
38			1	4	1	6
43			1	3		4
48					1	1
	1	5	8	9	2	25

$n = 25$

$\sum xf = 855 \qquad \sum x^2 f = 29{,}855$

$SS_{xx} = 29{,}855 - 29.241 = 614$

$\sum yf = 880 \qquad \sum y^2 f = 31{,}830$

$S_{yy} = 31{,}830 - 30{,}976 = 854$

$\sum xyf = 30{,}655$

$$S_{xy} = 30{,}655 - \frac{1}{25}(855)(880)$$
$$= 30{,}655 - 30096 = 559$$

$$r = \frac{559}{\sqrt{(614)(854)}} = \frac{559}{724.1} = 0.772$$

1. $H_0 : \rho = 0;\ H_1 : \rho \neq 0,\ \ \alpha = 0.05$
2. Reject null hypothesis is $z \le -1.96$ or $z \ge 1.96$
3. $z = \dfrac{\sqrt{22}}{2} \ln \dfrac{1.772}{0.228} = 4.81 > 1.96$
4. Reject null hypothesis; the value of r is significant.

14.72

	−2	−1	0	1	2	
−2	1					1
−1		3	1			4
0		2	5	2		9
1			1	4	1	6
2			1	3		4
3					1	1
	1	5	8	9	2	

$n = 25,\ \sum x = 6,\ \sum x^2 = 26$

$\sum y = 11,\ \sum y^2 = 39$

$$S_{xx} = 26 - \frac{1}{25}(6)^2 = 26 - 1.44 = 24.56$$

$$S_{yy} = 39 - \frac{1}{25}(11)^2 = 39 - 4.84 = 34.16$$

$$\sum fxy = 4 + 3 + 4 + 6 + 2 + 6 = 25$$

$$S_{xy} = 25 - \frac{1}{25}(6)(11) = 25 - 2.64 = 22.36$$

$$r = \frac{22.36}{\sqrt{(24.56)(34.16)}} = \frac{22.36}{28.9650} = 0.772$$

14.73

			x		
		−1	0	1	
	−1	63	42	15	120
y	0	58	61	31	150
	1	14	47	39	90
		135	150	75	400

$\sum xf = -60,\ \sum x^2 f = 210$

$$S_{xx} = 210 - \frac{1}{360}(-60)^2 = 210 - 10 = 200$$

$\sum yf = -30,\ \sum y^2 f = 210$

$$S_{yy} = 210 - \frac{1}{360}(-30)^2 = 210 - 2.5 = 207.5$$

$$\sum xyf = 63 - 15 - 14 + 29 = 63$$

$$S_{xy} = 63 - \frac{1}{360}(-60)(-30) = 63 - 5 = 58$$

$$r = \frac{58}{\sqrt{200(207.5)}} = \frac{58}{203.7} = 0.285$$

$$z = \frac{\sqrt{357}}{2} \ln \frac{1.285}{0.715} = 9.447 \ln 1.80 = 9.45(0.58779) = 5.55$$

$z = 5.55 > 2.575$ is significant

14.74

			x		
		−1	0	1	
y	−1	67	64	25	156
	0	42	76	56	174
	1	119	23	37	70
		119	163	118	400

$$\sum xf = -1, \quad \sum x^2 f = 237$$

$$S_{xx} = 237 - \frac{1}{400}(-1)^2 = 237 - 0 = 237$$

$$\sum yf = -86, \quad \sum y^2 f = 226$$

$$S_{yy} = 226 - \frac{1}{400}(-86)^2 = 226 - 18.49 = 207.51$$

$$\sum xyf = 67 - 25 - 10 + 37 = 69$$

$$S_{yy} = 69 - \frac{1}{400}(-1)(-86) = 69 - 0.215 = 68.8$$

$$r = \frac{68.8}{\sqrt{237(207.51)}} = \frac{68.8}{221.8} = 0.31$$

$$z = \frac{\sqrt{397}}{2}\ln\frac{1.31}{0.69} = (9.96)\ln 1.90 = (9.96)(0.64185) = 6.39$$

$z = 6.39 > 1.96 \rightarrow$ significant

14.75 (a) Using the data of Exercise 14.63 and MINITAB:

MTB> Correlate C1 C2
Correlation of C1 and C2 = 0.994

(b) $$z = \frac{\sqrt{n-3}}{2}\cdot\ln\frac{1+r}{1-r} = \frac{\sqrt{10-3}}{2}\cdot\ln\frac{1.994}{0.006} = 7.68$$

Since $z > z_{0.02,5} = 1.96$, we reject the null hypothesis of no correlation.

14.76 (a) Using the data of Exercise 14.64 and MINITAB:

MTB> Correlate C1 C2
Correlation of C1 and C2 = 0.837

(b) $$z = \frac{\sqrt{n-3}}{2}\cdot\ln\frac{1+r}{1-r} = \frac{\sqrt{8-3}}{2}\cdot\ln\frac{1.837}{0.163} = 2.71$$

Since $z > z_{0.005} = 2.575$, we reject the null hypothesis of no correlation.

14.77 (a) $\hat{\beta}_0 = 14.56, \ \hat{\beta}_1 = 30.109, \ \hat{\beta}_2 = 12.16$

$\hat{y} = 14.56 + 30.109x_1 + 12.16x_2$

(b) $\hat{y} = \$101.41$

14.78 (a) $\hat{\beta}_0 = -0.627, \ \hat{\beta}_1 = 0.0972, \ \hat{\beta}_2 = 0.662$

(b) $\hat{y} = 29.05$

14.79 **(a)** $\hat{\beta}_0 = -124.57,\ \hat{\beta}_1 = 1.659,\ \hat{\beta}_2 = 1.439$
(b) $\hat{y} = 63.24$

14.80 $\hat{\beta}_0 = 197.68,\ \hat{\beta}_1 = 37.19,\ \hat{\beta}_2 = -0.120$
$\hat{y} = 197.68 + 37.19x_1 - 0.120x_2;\ \hat{y} = 70.89$

14.81 $\hat{\beta}_0 = 69.73,\ \hat{\beta}_1 = 2.975,\ \hat{\beta}_2 = -11.97$
$\hat{y} = 69.73 + 2.975z_1 - 11.97z_2$ where the $z_1's$ and $z_2's$ are the coded values;
$\hat{y} = 71.2$ (difference due to rounding) $z_1 = 0.5,\ z_2 = 0$

14.82 $\hat{\beta}_0 = -2.33,\ \hat{\beta}_1 = 0.90,\ \hat{\beta}_2 = 1.27,\ \hat{\beta}_3 = 0.90$
$\hat{y} = -2.33 + 0.90x_1 + 1.27x_2 + 0.90x_3$

14.83 $\hat{\beta}_0 = 10.5,\ \hat{\beta}_1 = -2.0,\ \hat{\beta}_2 = 0.2$
$y = 10.5 - 2.0x + 0.2x^2$
$y = 5.95$

14.84 $\hat{\beta}_0 = 384.39,\ \hat{\beta}_1 = -36.00,\ \hat{\beta}_2 = 0.896$
$\hat{y} = 384.39 - 36.00x + 0.896x^2$

14.85 $t = 2.94$; the null hypothesis $\beta_2 = 0$ cannot be rejected. It is worthwhile to fit a parabola.

14.86 $2723 < \hat{\beta}_2 < 10{,}957$

14.87 $t = 0.16$; null hypothesis cannot be rejected

14.88 $13.7 < \beta_1 < 46.5$

14.89 $t = -4.18$ reject the null hypothesis

14.90 $0.244 < \beta_2 < 1.08$

14.91 $288{,}650 < \mu_{Y|3,2} < 296{,}220$

14.92 $292{,}785 \pm 19{,}048,\ (273{,}737 - 311{,}833)$

14.93 $74.5 < \mu_{Y|2.4,1.2} < 128.3$ (in \$1000)

14.94 101.4 ± 57.4, 44.0 and 158.8 (in \$1000)

14.97 (a) Using MINITAB, we enter the values of y in C1 and $x_1,\ldots x_3$ in C2,...C4.

MTB> Regress C1 on C2 C3 C4

The regression equation is

C1 = $-2.33+0.900$ C2 + 1.27 C3 + 0.900 C4

14.98 (a) Using MINITAB, we enter the values of y in C1 and $x_1,\ldots x_3$ in C2,...C4.

The regression equation is

C1 = 2,906 + 5.46 C2 + 20.1 C3 − 120 C4

(b) $\hat{y}=2,906+5.46(90.0)+20.1(65)-120(20)=2,304$

14.99 (a) Using statistical software to fit the plane, we obtain $\hat{y}=170-1.39x_1+6.07x_2$.

(b) $R^2=0.367$; the regression equation explains only 36.7% of the variability of y.

(c) A computer-generated plot of the residuals against $\hat{y}$ shows an apparently random pattern.

(d) The correlation of x_1 and x_2 is -0.142, suggesting little or no multicollinearity, (This correlations is not significant at the 0.05 level of significance.

14.100(a) Using statistical software to fit the surface, we obtain

$$\hat{y}=2,097+6.34x_1+12.9x_2-61.5x_3.$$

(b) A computer generated normal-scores plot suggests little departure from normality.

(c) A computer-generated plot of the residuals against $\hat{y}$ shows an apparently random pattern.

(d) The correlations among the independent variables are $r_{x_1x_2}=0.133,\ r_{x_1x_3}=0.344,\ r_{x_2x_3}=0.192$. Since none of them is significant at the 0.05 level of significance, we conclude that there is little or no multicollinearity among the independent variables.

14.101(b) Using statistical software, we find $\hat{y}=86.9-0.904x_1+0.508x_2+2.06x_2^2$.

(c) The correlations among the independent variables are $r_{x_1x_2}=-0.142,\ r_{x_1x_2^2}=-0.218,\ r_{x_2x_2^2}=0.421$. Although the correlation between x_2 and x_2^2 is 0.421, a bit high, none of these correlations is significant at the 0.05 level.

(e) The standardized regression equation is

$$\hat{y}=47.5-24.84x_1'+15.0x_2'+70.2(x_2')^2$$

(f) A computer generated plot of the residuals seems to be random. It is noted that the residuals are much smaller than those of Exercise 14.99.

14.102(b) Using statistical software, we find

$$\hat{y}=11,024-98.2x_1-170x_2+2.70x_3+185x_1x_2.$$

(c) The correlation matrix is:

	x_1	x_2	x_3
x_2	0.133		
x_3	0.344	0.192	
x_1x_2	0.729	0.769	0.325

Standardization is strongly recommended as two of these correlations are high.

(e) The standardized regression equation is

$$\hat{y} = 2,218 - 261x_1' - 192x_2' + 4.2x_3' + 446x_1'x_2'.$$

The multiple correlation coefficient is 0.970, compared to 0.346 for Exercise.

(f) The new correlation matrix is:

	x_1'	x_2'	x_3'
x_2'	0.133		
x_3'	0.344	0.192	
$x_1'x_2'$	−0.515	−0.218	−0.452

Note the reduction in absolute value of the correlation coefficients involving $x_1'x_2'$.

Chapter 15

15.1 $$\frac{n\sum_{i=1}^{a}(\overline{x}_i - \overline{x}_{..})^2}{a-1} = \frac{n}{a-1}\sum_{i=1}^{a}[\overline{x}_{i.} - 2\overline{x}_{i.}\overline{x}_{..} + \overline{x}_{..}^2]$$

$$= \frac{n}{a-1}\sum_{i=1}^{a}\overline{x}_{i.} - \frac{an}{a-1}\overline{x}_{..}^2$$

$$E\left[\frac{n\sum_{i=1}^{a}(\overline{x}_{i.} - \overline{x}_{..})^2}{a-1}\right] = \frac{n}{a-1}\sum_{i=1}^{a}\left\{\frac{\sigma^2}{n} + (\mu + \sigma_i)^2\right\} - \frac{an}{a-1}\left(\frac{\sigma^2}{na} + \mu^2\right)$$

$$= \sigma^2 + \frac{n}{a-1}\sum_{i=1}^{a}\alpha_i^2$$

15.2 $$SST = \sum_{i=1}^{a}\sum_{j=1}^{n}(x_{ij} - \overline{x}_{..})^2$$

$$= \sum_{i=1}^{a}\sum_{j=1}^{n}x_{ij}^2 - 2\overline{x}_{..}\sum_{i=1}^{a}\sum_{j=1}^{n}x_{ij} + na\overline{x}_{..}^2$$

$$= \sum_{i=1}^{a}\sum_{j=1}^{n}x_{ij}^2 - 2\cdot\frac{T_{..}}{na}\cdot T_{..} + \frac{naT_{..}^2}{n^2a^2}$$

$$= \sum_{i=1}^{a}\sum_{j=1}^{n}x_{ij}^2 - \frac{1}{na}T_{..}^2$$

$$SS(Tr) = n\sum_{i=1}^{a}(\overline{x}_{i.} - \overline{x}_{..})^2$$

$$= n\sum_{i=1}^{a}\overline{x}_{i.}^2 - 2n\sum_{i=1}^{a}\overline{x}_{i.}\overline{x}_{..} + n\sum_{i=1}^{a}\overline{x}_{..}^2$$

$$= n\sum_{i=1}^{a}\frac{T_{i.}^2}{n^2} - 2n\overline{x}_{..}(a\overline{x}_{..}) + nax_{..}^2$$

$$= \frac{1}{n}\sum_{i=1}^{a}T_{i.}^2 - \frac{1}{na}T_{..}^2$$

15.3 $\sum_{i=1}^{a}\sum_{j=1}^{n_i}(x_{ij}-\bar{x}_{..})^2 = \sum_{i=1}^{a}\sum_{j=1}^{n_i}[(\bar{x}_{i.}-\bar{x}_{..})+(x_{ij}-\bar{x}_{i.}]^2$

$$= \sum_{i=1}^{a}\sum_{j=1}^{n_i}(\bar{x}_{i.}-\bar{x}_{..})^2+\sum_{i=1}^{a}\sum_{j=1}^{n_i}(x_{ij}-\bar{x}_{i.})^2$$

Since $\sum_{i=1}^{a}\sum_{j=1}^{n_i}(\bar{x}_j-\bar{x}_{..})(x_{ij}-\bar{x}_{..}) = \sum_{j=1}^{a_i}(\bar{x}_{i.}-\bar{x}_{..})\sum_{j=1}^{n_i}(x_{ij}-\bar{x}_{i.}) = 0$

$$\sum_{i=1}^{a}\sum_{j=1}^{n_i}(x_{ij}-\bar{x}_{..})^2\sum_{i=1}^{a}n(\bar{x}_{i.}-\bar{x}_{..})^2+\sum_{i=1}^{a}\sum_{j=1}^{n_i}(x_{ij}-\bar{x}_{i.})^2$$

SST is such that $\frac{SST}{\sigma^2}$ is value of random variable having χ^2 distribution with $\sum_{i=1}^{a}n_i-1=N-1$ degrees of freedom. For each *i*, $\frac{1}{\sigma^2}\sum_{j=1}^{n_i}(x_{ij}-\bar{x}_{i.})^2$ is value of random variable having χ^2 distribution with n_i-1 degrees of freedom, so that $\frac{1}{\sigma^2}SSE$ is value of random variable having $\sum_{i=1}^{a}(n_i-1)=N-a$ degrees of freedom. Also $\frac{SST}{\sigma^2}$ is value of random variable having χ^2 distribution with $a-1$ degrees of freedom.

15.4 $SST = \sum_{i=1}^{a}\sum_{j=1}^{n_i}(x_{ij}-\bar{x}_{..})^2$

$$= \sum_{i=1}^{a}\sum_{j=1}^{n_i}x_{ij}^2-2\bar{x}_{..}\sum_{i=1}^{a}\sum_{j=1}^{n_i}x_{ij}+\sum_{i=1}^{a}\sum_{j=1}^{n_i}.\bar{x}_{..}^2$$

$$= \sum_{i=1}^{a}\sum_{j=1}^{n_i}x_{ij}^2-\frac{2}{N}T_{..}^2+\frac{1}{N}T_{..}^2 = \sum_{i=1}^{a}\sum_{j=1}^{n_i}x_{ij}^2-\frac{1}{N}T_{..}^2$$

$$SS(Tr) = \sum_{i=1}^{n_i}n_i(\bar{x}_{i.}-\bar{x}_{..})^2 = \sum_{i=1}^{n_i}n_i\bar{x}_{i.}^2-2\bar{x}_{..}\sum_{i=1}^{n_i}n_i\bar{x}_i+\sum_{i=1}^{n_i}n_i\bar{x}_{..}^2$$

$$= \sum_{i=1}^{n_i}\frac{T_{i.}^2}{n_i}-2N\bar{x}_{..}^2+N\bar{x}_{..}^2 = \sum_{i=1}^{n_i}\frac{T_{i.}^2}{n_i}-\frac{1}{N}T_{..}^2$$

$SSE = SST - SS(Tr)$ from identities of Exercise 15.3

15.5 $SS(Tr) = n_1(\overline{x}_{1.} - \overline{x}_{..})^2 + n_2(\overline{x}_{2.} - \overline{x}_{..})^2 \qquad \overline{x}_{..} = \dfrac{n_1\overline{x}_{1.} + n_2\overline{x}_{2.}}{n_1 + n_2}$

$$= n_1\left(\overline{x}_{1.} - \frac{n_1\overline{x}_{1.} - n_2\overline{x}_{2.}}{n_1 + n_2}\right)^2 + n_2\left(\overline{x}_{2.} - \frac{n_1\overline{x}_{1.} - n_2\overline{x}_{2.}}{n_1 + n_2}\right)^2$$

$$= n_1\left(\frac{n_2\overline{x}_{1.} - n_2\overline{x}_{1.}}{n_1 + n_2}\right)^2 + n_2\left(\frac{n_1\overline{x}_{2.} - n_1\overline{x}_{1.}}{n_1 + n_2}\right)^2$$

$$= \frac{n_1 n_2^2}{(n_1 + n_2)^2}(\overline{x}_{1.} - \overline{x}_{2.})^2 + \frac{n_1^2 n_2}{(n_1 + n_2)^2}(\overline{x}_{1.} - \overline{x}_{2.})^2$$

$$= \frac{n_1 n_2}{(n_1 + n_2)}(\overline{x}_{1.} - \overline{x}_{2.})^2 = \frac{(\overline{x}_{1.} - \overline{x}_{2.})^2}{\dfrac{1}{n_1} + \dfrac{1}{n_2}}$$

$$SSE = \sum_{j=1}^{n_1}(x_{1j} - \overline{x}_{1.})^2 + \sum_{j=1}^{n_2}(x_{2j} - \overline{x}_{2.})^2 = (n_1 - 1)s_1^2 + (n_2 - 1)s_2^2$$

$$= (n_1 + n_2 - 2)s^2 p$$

$$F = \frac{(\overline{x}_1. - \overline{x}_2.)^2}{\dfrac{1}{n_1} + \dfrac{1}{n_2}} + \frac{(n_1 + n_2 - 2)s^2 p}{n_1 + n_2 - 2}$$

$$= \frac{(\overline{x}_1. - x_2.)^2}{s^2 p\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)} = t^2 \qquad \text{QED}$$

15.6 $u = \displaystyle\sum_{i=1}^{a}\sum_{j=1}^{n_i}[x_{ij} - (\mu + \alpha_i)]^2 + \lambda\sum\alpha_i$

$$\frac{\partial u}{\partial \mu} = 2\sum_{i=1}^{a}\sum_{j=1}^{n_i}[x_{ij} - (\mu + \alpha_i)](-1) = 0$$

$$= \sum_{i=1}^{a}\sum_{j=1}^{n_i}x_{ij} - a\left(\sum_{i=1}^{a}n_i\right)\mu = 0 \qquad \hat{\mu} = \overline{x}_{..}$$

$$\frac{\partial u}{\partial \alpha_i} = 2\sum_{j=1}^{n_i}[x_{ij} - (\mu + \alpha_i)](-1) + \lambda = 0$$

sum over i; $= -N\overline{x}_{..} + N\overline{x}_{..} + \lambda = 0 \qquad \lambda = 0$

$$\sum_{j=1}^{n_i}[x_{ij} - (\overline{x}_{..} + \alpha_i)] = 0$$

$$n_i x_{i.} - n_i\overline{x}_{..} - n_i\alpha_i = 0 \qquad \hat{\alpha} = \overline{x}_{i.} - \overline{x}_{..}$$

15.7 $\sum_{i=1}^{a}\sum_{j=1}^{n}(x_{ij}-\overline{x}_{..})^2$

$$= \sum_{i=1}^{a}\sum_{j=1}^{n}[(x_{i.}-\overline{x}_{..})+(\overline{x}_{.j}-\overline{x}_{..})+(x_{ij}-\overline{x}_{i.}-\overline{x}_{.j}+\overline{x}_{..})]^2$$

$$= n\sum_{i=1}^{a}(\overline{x}_{i.}-\overline{x}_{..})^2 + a\sum_{j=1}^{n}(x_{.j}-\overline{x}_{..})^2$$

$$+\sum_{i=1}^{a}\sum_{j=1}^{n}(x_{ij}-\overline{x}_{i.}-\overline{x}_{.j}+\overline{x}_{..})^2$$

$$+2\left[\sum_{i=1}^{a}(x_{i.}-\overline{x}_{..})\sum_{j=1}^{n}(x_{.j}-\overline{x}_{..})\right]$$

$$+2\sum_{i=1}^{a}\left[(x_{i.}-\overline{x}_{..})\sum_{j=1}^{n}(x_{ij}-\overline{x}_{i.}-\overline{x}_{.j}+\overline{x}_{..})\right]$$

$$+2\sum_{j=1}^{n}\left[(x_{.j}-\overline{x}_{..})\sum_{i=1}^{a}(x_{ij}-\overline{x}_{i.}-\overline{x}_{.j}+\overline{x}_{..})\right]$$

$$= \sum_{i=1}^{a}\sum_{j=1}^{n}(x_{ij}-\overline{x}_{..})^2 + n\sum_{i=1}^{a}(x_{ij}-\overline{x}_{..})^2 + a\sum_{j=1}^{n}[(x_{.j}-\overline{x}_{..})^2$$

$$+\sum_{i=1}^{a}\sum_{j=1}^{n}(x_{ij}-\overline{x}_{i.}-\overline{x}_{.j}+\overline{x}_{..})^2 \qquad \text{QED}$$

15.8 $\mu_{ij} = \mu + \alpha_i + \beta_j$

$$\frac{1}{na}\sum_{i=1}^{a}\sum_{j=1}^{n}(\mu+\alpha_i+\beta_j) = \frac{1}{na}\sum_{i=1}^{a}\sum_{j=1}^{n}\mu + \sum_{i=1}^{a}\sum_{j=1}^{n}\alpha_i + \sum_{i=1}^{a}\sum_{j=1}^{n}\beta_j$$

then since $\sum_{i=1}^{a}\alpha_i = 0$ and $\sum_{j=1}^{n}\beta_j = 0$

$$\frac{1}{na}\sum_{i=1}^{a}\sum_{j=1}^{n}\mu_{ij} = \frac{1}{na}\sum_{i=1}^{a}\sum_{j=1}^{n}\mu = \frac{1}{na}\cdot na\mu = \mu$$

15.9 $$\frac{a}{n-1}\sum_{j=1}^{n}(\overline{x}_{.j}-\overline{x}_{..})^2 = \frac{a}{n-1}\sum_{i=1}^{n}[\overline{x}_{.j}^2 - 2\overline{x}_{.j}x_{..} + \overline{x}_{..}^2]$$

$$= \frac{a}{n-1}\sum_{j=1}^{n}\overline{x}_{.j}^2 - \frac{an}{n-1}\overline{x}_{..}^2$$

$$E\left[\frac{a}{n-1}\sum_{j=1}^{n}(\overline{x}_{ij}-\overline{x}..)^2\right] = \frac{a}{n-1}\sum_{i=1}^{n}\left\{\frac{\sigma^2}{a}-(\mu+\beta_j)^2\right\} = \frac{an}{n-1}\left(\frac{\sigma^2}{na}+\mu^2\right)$$

$$= \sigma^2\frac{na}{(n-1)a} - \sigma^2\frac{1}{n-1} + \frac{a}{n-1}\sum_{j=1}^{n}\beta_j^2$$

$$= \sigma^2 + \frac{a}{n-1}\sum_{j=1}^{n}\beta_j^2 \qquad \text{(see also 15.1)}$$

15.10 $$SSB = a\sum_{j=1}^{n}(\overline{x}_{.j}-\overline{x}_{..})^2$$

$$= a\sum_{j=1}^{n}\overline{x}_{.j}^2 - 2a\sum_{j=1}^{n}\overline{x}_{.j}\overline{x}_{..} + a\sum_{j=1}^{n}\overline{x}_{..}^2$$

$$= a\sum_{j=1}^{n}\frac{T_{.j}^2}{k^2} - 2a\overline{x}_{..}(n\overline{x}_{..}) + na\overline{x}_{..}^2$$

$$= \frac{1}{a}\sum_{j=1}^{n}T_{.j}^2 - \frac{1}{na}(T_{..})^2 \qquad \text{QED}$$

15.11 $$\mu_{ijr} = \mu + \alpha_i + \beta_j + \rho_r + (\alpha\beta)_{ij}$$

$$\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{r}\mu_{ijk} = rba\mu + rb\sum_{i=1}^{a}\alpha_i + ar\sum_{j=1}^{b}\beta_j + ab\sum_{k=1}^{r}\rho_r + r\sum_{i=1}^{a}\sum_{j=1}^{b}(\alpha\beta)_{ij}$$

But

$$\sum_{i=1}^{a}\alpha_i = \sum_{j=1}^{b}\beta_j = \sum_{k=1}^{r}\rho_k = 0 \quad \text{also} \quad \sum_{j=1}^{b}(\alpha\beta)_{ij} = 0; \quad \therefore +r\sum_{i=1}^{a}\sum_{j=1}^{b}(\alpha\beta)_{ij} = 0$$

Finally

$$\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{r}\mu_{ijk} = rba\mu; \quad \mu = \frac{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{r}\mu_{ijk}}{rba}$$

15.12 Dropping the indexes of summation for simplicity, we have

$$\sum\sum\sum(x_{ijk}-\bar{x}_{...})^2 = \sum\sum\sum(\bar{x}_{i..}-\bar{x}_{...})^2 + \sum\sum\sum(\bar{x}_{.j.}-\bar{x}_{...})^{2.} + \sum\sum\sum(\bar{x}_{..k}-\bar{x}_{...})^2 +$$
$$\sum\sum\sum(\bar{x}_{ij.}-\bar{x}_{i..}-\bar{x}_{.j.}+\bar{x}_{...})^2 = \sum\sum\sum(\bar{x}_{ijk}-\bar{x}_{ij.}-\bar{x}_{..k}+\bar{x}_{...})^2 +$$

six cross-product terms.

To indicate the proof that all cross-product terms sum to zero, we take the following example:

$$2\sum_i\sum_j\sum_k(\bar{x}_{ij.}-\bar{x}_{i..}-\bar{x}_{.j.}+\bar{x}_{...})(x_{ijk}-\bar{x}_{ij.}-\bar{x}_{..k}+\bar{x}_{...})^2 =$$
$$2\sum_i\sum_j(\bar{x}_{ij.}-\bar{x}_{i..}-\bar{x}_{.j.}+\bar{x}_{...})\sum_k(\bar{x}_{ijk}-\bar{x}_{ij.}-\bar{x}_{..k}+\bar{x}_{...})$$

The summation on k equals zero, which completes the proof.

15.13 By Theorem 15.5,

$$SSA = rb\sum_{i=1}^{a}(\bar{x}_{i..}-\bar{x}_{...})^2 = rb\left[\sum_{i=1}^{a}\bar{x}_{i..}^2 - a\bar{x}_{...}^2\right] = rb\sum_{i=1}^{a}\bar{x}_{i..}^2 - rba\bar{x}_{..}^2$$

Now,

$$\bar{x}_{i..} = \frac{T_{i..}}{rb} \text{ and } \bar{x}_{...} = \frac{T_{...}}{rba}$$

Thus,

$$SSA = rb\sum_{i=1}^{a}\frac{T_{i..}^2}{(rb)^2} - rba\frac{T_{...}^2}{(rba)^2} = \frac{\sum_{i=1}^{a}T_{i..}^2}{rb} - C$$

The proofs for SSB and SSR are analogous. For *SSI*, we have

$$SSI = r\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{x}_{ij.}-\bar{x}_{i..}-\bar{x}_{.j.}+\bar{x}_{...})^2$$

Using the identity

$$\bar{x}_{ij.}-\bar{x}_{i..}-\bar{x}_{.j.}-\bar{x}_{...} = (\bar{x}_{ij.}-\bar{x}_{...})-(\bar{x}_{i..}-\bar{x}_{...})-(\bar{x}_{.j.}-\bar{x}_{...})$$

we can write

$$SSI = r\sum\sum\left[(\bar{x}_{ij.}-\bar{x}_{...})^2+(\bar{x}_{i..}-\bar{x}_{...})^2+(\bar{x}_{.j.}-\bar{x}_{...})^2\right]$$
$$-2r\sum\sum\left[(\bar{x}_{ij.}-\bar{x}_{...})(\bar{x}_{i..}-\bar{x}_{...})+(\bar{x}_{ij.}-\bar{x}_{...})^2(x_{.j.}-x_{...})+(\bar{x}_{i..}-\bar{x}_{...})(\bar{x}_{.j.}-\bar{x}_{...})\right]$$

$$= r\sum\sum(\bar{x}_{ij.}-\bar{x}_{...})^2 + SSA + SSB - 2SSA - 2SSB - 0$$

$$= \frac{\sum_{i=1}^{a}\sum_{j=1}^{b}T_{ij.}^2}{r} - C - SSA - SSB$$

15.14 First we write the identity

$$x_{ij(k)} - \bar{x}_{..} = (\bar{x}_{i..} - \bar{x}_{..}) + (\bar{x}_{.j} - \bar{x}_{..}) + (\bar{x}_{(k)} - \bar{x}_{..}) + (x_{ij(k)} - \bar{x}_{i.} - \bar{x}_{.j} - \bar{x}_{(k)} + 2\bar{x}_{..})$$

Then we square each side of the equation and sum each term on i and j from 1 to n. Recognizing that each of the cross-product terms sums to zero, we are left with

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(x_{ij(k)} - \bar{x}_{..})^2 = n\sum_{i=1}^{n}(\bar{x}_{i.} - \bar{x}_{..})^2 + n\sum_{j=1}^{n}(\bar{x}_{.j} - \bar{x}_{..})^2 + n\sum_{k=1}^{n}(\bar{x}_{(k)} - \bar{x}_{..})^2$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{n}(x_{ij(k)} - \bar{x}_{i.} - \bar{x}_{.j} - \bar{x}_{(k)} + 2\bar{x}_{..})^2 \qquad \text{QED}$$

15.15 The left-hand side of the identity in Exercise 15.14 is the total sum of squares, *SST*; the terms on the right-hand side are, respectively, the row sum of squares, *SSR*, the column sum of squares, *SSC*, the treatment sum of squares, *SS*(*Tr*) and the error sum of squares, *SSE*. Thus, we can write the following analysis-of-variance table for the Latin square of size *n*.

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F
Rows	$n-1$	SSR	$SSR/(n-1)$	MSR/MSE
Columns	$n-1$	SSC	$SSC/(n-1)$	MSC/MSE
Treatments	$n-1$	$SS(Tr)$	$SS(Tr)/(n-1)$	$MS(Tr)/MSE$
Error	$(n-1)(n-2)$	SSE	$SSE/(n-1)(n-2)$	
Total	n^2-1	SST		

where $SSR = \frac{1}{n}\left(\sum_{i=1}^{n}\bar{x}_{i.}\right)^2 - C;\quad SSC = \frac{1}{2}\left(\sum \bar{x}_{.j}\right)^2 - C;$

$SS(Tr) = \frac{1}{n}\left(\sum_{k=1}^{n}\bar{x}_{(k)}\right)^2 - C$ where $C = \frac{1}{n^2}\sum_{j=1}^{n}\sum_{i=1}^{n}x_{ij(k)}^2$

15.16 $a = 3,\ n = 8,\ T_{1.} = 456.8,\ T_{2..} = 473.4,\ T_{3..} = 547.6,\ T_{..} = 1477.8,$

and $\sum\sum x^2 = 91{,}939.96$

$SST = 91{,}939.96 - \frac{1}{24}(14.77.8)^2 = 944.425$ (d.f. = 23)

$SS(Tr) = \frac{1}{8}(732{,}639.56) - 90{,}995.535 = 584.41$ (d.f. = 2)

$SSE = 994.425 - 584.41 = 360.015$ (d.f. = 21)

$F = \frac{584.41/2}{360.015/21} = 17.0$

Since F = 17.0 exceeds $F_{0.01,2,21} = 5.78$, null hypothesis must be rejected. The difference in effectiveness are significant.

15.17 $a = 4,\ n = 5,\ T_{1.} = 70,\ T_{2.} = 75,\ T_{3.} = 79,\ T_{4.} = 69,\ T_{..} = 293,$

and $\sum\sum x^2 = 4407$

$$SST = 4407 - \frac{1}{20}(293)^2 = 4407 - 4292.45 = 114.45 \text{ (d.f.} = 19)$$

$$SS(Tr) = \frac{1}{5}(21{,}527) - 4292.45 = 12.95 \ \text{ (d.f.} = 3)$$

$$SSE = 114.55 - 12.95 = 101.6 \ \text{ (d.f.} = 16)$$

$$F = \frac{12.95/3}{101.6/16} = 0.68$$

Since $F = 0.68$ does not exceed $F_{0.05,3,16} = 3.24$, mull hypothesis cannot be rejected. Differences among the sample means are not significant.

15.18 $a = 3,\ n = 6,\ T_{1.} = 135,\ T_{2.} = 120,\ T_{3.} = 78,\ T_{..} = 333,\ \sum\sum x^2 = 6507$

$$SST = 6507 - \frac{1}{18}(333)^2 = 6507 - 6160.5 = 346.5 \text{ (d.f.} = 17)$$

$$SS(Tr) = \frac{1}{6}(38{,}709) - 6160.5 = 291.0 \ \text{ (d.f.} = 2)$$

$$SSE = 346.5 - 291.0 = 55.5 \ \text{ (d.f.} = 15)$$

$$F = \frac{291.0/2}{55.5/15} = 39.3$$

Since $F = 39.3$ exceeds $F_{0.05,2,15} = 3.68$, mull hypothesis must be rejected. Differences in dosage have an effect.

$$\hat{\mu} = \frac{133}{18} = 18.5$$

$$\hat{\alpha}_1 = \frac{135}{6} - 18.5 = 4.0,\ \ \hat{\alpha}_2 == \frac{120}{6} - 18.5 = 1.5,$$

$$\alpha_3 = \frac{78}{6} - 18.5 = -5.5$$

15.19 $a = 4,\ n_1 = 8,\ n_2 = 8,\ n_3 = 6,\ n_4 = 9,\ N = 31,\ T_{1.} = 574,\ T_{2.} = 547,$

$T_{3.} = 449,\ T_{4.} = 584,\ T_{..} = 2154$

$$\sum\sum x^2 = 41{,}386 + 37{,}491 + 33{,}683 + 38{,}064 = 150{,}624$$

$$SST = 150{,}624 - \frac{1}{31}(2154)^2 = 150{,}624 - 149{,}668.26 = 955.74$$

$$SS(Tr) = (41{,}184.5 + 37{,}401.125 + 33{,}600.17 + 37{,}895.11) - 149.668.26 = 412.645$$

$$SSE = 955.74 - 412.645 = 543.095$$

$$F = \frac{412{,}645/3}{543.095/27} = 6.84 \qquad F_{0.05,3,27} = 2.99$$

Differences cannot be attributed to chance.

15.20 $a = 3,\ n_1 = 4,\ n_2 = 2,\ n_3 = 3,\ N = 9,\ T_{1.} = 1908,\ T_{2.} = 990,\ T_{3.} = 1445,$
$T_{..} = 4343$

$$\sum\sum x^2 = 910{,}662 + 490{,}068 + 696{,}725 = 2{,}097{,}455$$

$$SST = 2{,}097{,}45 - \frac{1}{9}(4343)^2 = 2{,}097{,}415 - 2{,}095{,}738.8 = 1676.2 \text{ (d.f.} = 8)$$

$$SS(Tr) = 910{,}116 + 490{,}050 + 696{,}008.3 - 2{,}095{,}738.8 = 435.5 \text{ (d.f.} = 2)$$

$$SSE = 1676.2 - 435.5 = 1240.7 \text{ (d.f.} = 6)$$

$$F = \frac{435.5/2}{1240.7/6} = 1.05 \qquad F_{0.05,2,6} = 5.14$$

Null hypothesis cannot be rejected; differences can be attributed to chance.

15.21 $a = 3,\ n_1 = 400,\ n_2 = 500,\ n_3 = 400,\ N = 1300,\ T_{1.} = 81,\ T_{2.} = 72,$
$T_{3.} = 43,\ T_{..} = 196, \sum\sum x^2 = 840$

$$SST = 840 - \frac{1}{1300}(196)^2 = 840 - 29.95 = 810.45 \text{ (d.f.} = 1299)$$

$$SS(Tr) = (16.40 + 10.37 + 4.62) - 29.95 = 1.84 \text{ (d.f.} = 2)$$

$$SSE = 808.61 \text{ (d.f.} = 1297)$$

$$F = \frac{1.84/2}{808.61/1297} = 1.48 \qquad F_{0.05,2,1297} = 3.00$$

Null hypothesis cannot be rejected. Result same as in Ex. 13.74.

15.22

	−1	0	1	
A	12	23	89	124
B	8	12	62	82
C	21	30	119	170
	41	65	270	

$k = 3,\ n_1 = 124,\ n_2 = 82$
$n_3 = 170,\ N = 376$
$T_{1.} = 77,\ T_{2.} = 54,\ T_{3.} = 98$
$T_{..} = 229,\ \sum\sum x^2 = 311$

$$SST = 311 - \frac{1}{376}(229)^2 = 311 - 139.47 = 171.53 \quad \text{(d.f. = 375)}$$

$$SS(Tr) = (47.81 + 35.56 + 56.49) - 139.47 = 0.39 \quad \text{(d.f.)} = 2$$

$$SSE = 171.35 - 0.39 = 170.96 \text{ (d. f.} = 373)$$

$$F = \frac{0.39/2}{170.96/373} = 0.43 \qquad F_{0.01,2,373} = 4.61$$

Null hypothesis cannot be rejected. Result same as in Ex. 13.73.

15.23 $a = 3,\ n = 4,\ \mathrm{T}_{1.} = 197.4,\ T_{2.} = 185.9\ T_{3.} = 206.0,\ \mathrm{T}_{.1} = 137.6,$

$T_{.2} = 165.5,\ T_{.3} = 157.6,\ T_{.4} = 128.6,\ T_{...} = 589.3$

$\sum\sum x^2 = 9{,}888.3 + 8{,}732.45 + 10.697.8 = 29{,}318.55$

$SST = 29{,}318.55 - \frac{1}{12}(589.3)^2 = 29{,}318{,}55 - 28{,}939.54 = 379.01$ (d.f. = 11)

$SS(Tr) = \frac{1}{4}(115{,}961.57 - 28{,}939.54 = 50.85$ (d.f. = 2)

$SSB = \frac{1}{3}(87{,}699.73) - 28{,}939.54 = 293.70$ (d.f. = 3)

$SSE = 379.01 - 50.85 - 293.70 = 34.46$ (d.f. = 6)

$F_{Tr} = \frac{50.85/2}{34.46/6} = 4.43 \qquad F_B = \frac{293.70/3}{34.46/6} = 17.05$

$F_{0.01,2,6} = 10.9 \qquad F_{0.01,3,6} = 9.78$

Since $F = 4.43 < 10.9$, null hypothesis for launchers cannot be rejected. Since $F = 17.05 > 9.78$, mull hypothesis for fuels must be rejected. Difference among fuels is significant.

15.24 $a = 4,\ n = 3,\ \mathrm{T}_{1.} = 8.8,\ T_{2.} = 8.8,\ T_{3.} = 9.7,\ T_{4.}=10.3,\ \mathrm{T}_{.1} = 13.2,$

$T_{.2} = 11.4,\ T_{.3} = 13.0,\ T_{...} = 37.16$

$\sum\sum x^2 = 26.16 + 25.9 + 31.45 + 35.55 = 119.06$

$SST = 119.06 - \frac{1}{12}(37.6)^2 = 119.06 - 117.818 = 1.25$ (d.f. = 11)

$SS(Tr) = \frac{1}{3}(355.06) - 117.81 = 0.54$ (d.f. = 3)

$SSB = \frac{1}{4}(473.2) - 117.81 = 0.49$ (d.f. = 2)

$SSE = 1.25 - 0.54 - 0.49 = 0.22$ (d.f. = 6)

$F_{Tr} = \frac{0.54/3}{0.22/6} = 4.91 \qquad F_B = \frac{0.49/2}{0.22/6} = 6.68$

$F_{0.05,3,6,} = 4.76 \qquad F_{0.05,2,6} = 5.14$

Since 4.91 > 4.76, null hypothesis for laboratories must be rejected.
Since 6.68 > 5.14, null hypothesis for diet foods must be rejected.

15.25 $a = 5,\ n = 4,\ \mathrm{T}_{1.} = 83.1,\ T_{2.} = 103,\ T_{3.} = 94.5,\ \mathrm{T}_{4.} = 95.2,\ T_{5.} = 85,$

$T_{.1} = 115.8,\ T_{.2} = 112.1,\ T_{.3} = 114,\ T_{.4} = 118.9,\ T_{...} = 460.8$

$\sum\sum x^2 = 17.28.59 + 2655.48 + 2241.47 + 2277.22 + 1810.42 = 10{,}713.18$

$SST = 10{,}713.18 - \frac{1}{20}(460.8)^2 = 10{,}713.18 - 10{,}616.83 = 96.35$ (d.f. = 19)

$SS(Tr) = \frac{1}{4}(42{,}732.9) - 10{,}616.83 = 66.40$ (d.f. = 4)

$SSB = \frac{1}{5}(53{,}109.26) - 10{,}616.83 = 5.02$ (d.f. = 3)

$SSE = 96.35 - 66.40 - 5.02 = 24.93$ (d.f. = 12)

$F_{Tr} = \dfrac{66.40/4}{24.93/12} = 7.99$ $\qquad$ $F_B = \dfrac{5.02/3}{24.93/12} = 0.81$

$F_{0.05,4,12,} = 3.26$ $\qquad$ $F_{0.05,3,12} = 3.49$

$F_{Tr} = 7.99$ (for threads) is significant. $F_B = 0.81$ (for measuring instruments) is not significant.

15.26

	Teacher	Lawyer	Doctor
East	I	R	D
South	R	D	I
West	D	I	R

I = independent
R = Republican
D = Democrat

Completing the Latin Square, we find that Doctor who is a Western is a *Republican.*

15.27 Summing the observations in each replicate, we have $T_{..1} = 589.3$, $T_{..2} = 595.8$. Summing over the two replicates, we obtain the following two-way table:

	Fuels				
Launchers	1	2	3	4	Totals
X	92.0	113.5	104.8	86.0	396.3
Y	92.3	103.1	101.5	78.4	375.3
Z	91.5	114.8	111.5	95.7	413.5
Totals	275.8	331.4	317.8	260.1	1,185.1

$$C = \frac{(1{,}185.1)^2}{24} = 58{,}519.25$$

$$SS(\text{Total}) = (45.9)^2 + (57.6)^2 + \ldots + (47.6)^2 - C = 721.04$$

$$SS(\text{Launchers}) = [(396.3)^2 + (375.3)^2 + (413.4)^2]/8 - C = 91.50$$

$$SS(\text{Fuels}) = [(275.8)^2 + (331.4)^2) + (317.8)^2 + (260.1)^2]/6 - C = 570.83$$

$$SS(\text{Replicates}) = [(589.3)^2 + (595.8)^2]/12 - C = 1.76$$

$$SS(\text{Interaction}) = [(92.0)^2 + (113.5)^2 + \ldots + (95.7^2)]/2 - C - SS(\text{Launchers}) - SS(\text{Fuels}) = 50.94$$

$$SS(\text{Error}) = SST - SS(\text{Launchers}) - SS(\text{Fuels}) - SS(\text{Replicates}) - SS(\text{Interaction}) = 6.01$$

ANALYSIS OF VARIANCE

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F	Critical $F_{0.01}$
Launchers	2	91.40	45.75	83.2	7.21
Fuels	3	570.83	190.28	346.0	6.22
Replicates	1	1.76	1.76	3.2	9.65
Interaction	6	50.94	8.49	15.4	5.07
Error	11	6.01	0.55		
Total	23	721.04			

Thus, the Launchers, Fuels, and Interaction means are significantly different at the 0.01 level of significance.

15.28 Summing the observations in each replicate, we have $T_{..1} = 37.6$, $T_{..2} = 39.0$. Summing over the two replicates, we obtain the following two-way table:

	Foods			
Laboratories	A	B	C	Totals
1	6.9	5.1	5.7	17.7
2	6.0	5.6	6.3	17.9
3	6.9	6.4	7.2	20.5
4	6.8	6.6	7.1	20.5
Totals	26.6	23.7	26.3	76.6

$$C = \frac{(76.6)^2}{24} = 244.48$$

$SS(\text{Total}) = 247.28 - C = 2.80$

$SS(\text{Laboratories}) = 245.70 - C = 1.22$

$SS(\text{Foods}) = 245.12 - C = 0.64$

$SS(\text{Replicates}) = 244.56 - C = 0.08$

$SS(\text{Interaction}) = 246.89 - C - SS(\text{Laboratories}) - SS(\text{Foods}) = 0.55$

$SS(\text{Error}) = SST - SS(\text{Laboratories}) - SS(\text{Foods}) - SS(\text{Replicates}) - SS(\text{Interaction}) = 0.31$

ANALYSIS OF VARIANCE

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F	Critical $F_{0.05}$
Laboratories	3	1.22	0.41	13.7	3.59
Foods	2	0.64	0.32	10.7	3.98
Replicates	1	0.08	0.08	2.7	4.84
Interaction	6	0.55	0.09	3.0	3.09
Error	11	0.31	0.03		
Total	23	2.80			

Thus, the Laboratories and Foods means are significantly different at the 0.05 level of significance.

15.29 Summing the observations in each replicate, we have $T_{..1} = 122.8$, $T_{..2} = 122.7$. Summing over the two replicates, we obtain the following two-way table:

	Bonders				
Operators	A	B	C	D	Totals
1	22.4	21.5	22.4	20.1	86.4
2	22.4	22.7	21.0	22.1	88.2
3	21.4	20.1	20.5	8.9	70.9
Totals	66.2	64.3	63.9	51.1	245.5

$$C = \frac{(245.5)^2}{24} = 2{,}511.26$$

$SS(\text{Total}) = 2{,}609.51 - C = 98.25$

$SS(\text{Operators}) = 2{,}533.88 - C = 22.62$

$SS(\text{Bonders}) = 2{,}535.23 - C = 23.97$

$SS(\text{Replicates}) = 2{,}511.26 - C = 0.00$

$SS(\text{Interaction}) = 2{,}588.84 - C - SS(\text{Operators}) - SS(\text{Bonders}) = 30.99$

$SS(\text{Error}) = SST - SS(\text{Operators}) - SS(\text{Bonders}) - SS(\text{Replicates}) - SS(\text{Interaction}) = 20.67$

ANALYSIS OF VARIANCE

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F	Critical $F_{0.05}$
Operators	2	22.62	11.31	6.02	3.98
Bonders	3	23.97	7.99	4.25	3.59
Replicates	1	0.00	0.00	0.00	4.84
Interaction	6	30.99	5.17	2.75	3.09
Error	11	20.67	1.88		
Total	23	98.25			

Thus, the Operators and Bonders means are significantly different at the 0.05 level of significance.

15.30 Summing the observations in each replicate, we have $T_{..1} = 266.6$, $T_{..2} = 267.0$, $T_{..3} = 262.5$, $T_{..4} = 270.6$.

Summing over the two replicates, we obtain the following two-way table:

	DSS				
Time	0	50	100	150	Totals
1	138.1	140.3	141.9	144.1	564.4
2	112.8	123.4	131.5	134.6	502.3
Totals	250.9	263.7	273.4	278.7	1,066.7

$$C = \frac{(1{,}066.7)^2}{24} = 35{,}557.78$$

$SS(\text{Total}) = 35{,}765.15 - C = 207.37$

$SS(\text{DSS}) = 35{,}613.72 - C = 55.94$

$SS(\text{Time}) = 35{,}678.29 - C = 120.51$

$SS(\text{Replicates}) = 35{,}561.90 - C = 4.12$

$SS(\text{Interaction}) = 35{,}754.23 - C - SS(DSS) - SS(\text{Time}) = 20.00$

$SS(\text{Error}) = SST - SS(DSS) - SS(\text{Time}) - SS(\text{Replicates}) - SS(\text{Interaction}) = 6.80$

ANALYSIS OF VARIANCE

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F	Critical $F_{0.05}$
DSS Level	3	55.94	18.65	58.28	3.07
Time	1	120.51	120.51	376.59	4.32
Replicates	3	4.12	1.37	4.28	3.07
Interaction	3	20.00	6.67	20.84	3.07
Error	21	6.80	0.32		
Total	31	207.37			

Thus, the DSS, Time, Replicates, and Interaction means are all significantly different at the 0.05 level of significance.

15.31 The three detergent means are: A: 77.0 B: 68.0 C: 80.0

$$s_{\bar{x}} = \sqrt{\frac{MSE}{n}} = \sqrt{\frac{23}{5}} = 2.14$$

For Table IX, with $\alpha = 0.01$ and 12 d.f. and using $R_p = r_p \cdot s_{\bar{x}}$ we get

p	2	3
r_p	4.32	4.50
R_p	9.24	9.63

We obtain

Detergents:	B	A	C
Means:	68.0	77.0	80.0

and we conclude that detergents A and C do not give rise to significantly different means at the 0.01 level so significance.

15.32 The five block means are: 24.75, 27.50, 28.25, 27.75, and 30.75. Proceeding as in Exercise 15.31 with

$$s_{\bar{x}} = \sqrt{\frac{2.27}{4}} = 0.75$$

we obtain from Table IX, with $\alpha = 0.05$ and 12 d.f.

p	2	3	4	5
r_p	3.08	3.23	3.31	3.37
R_p	2.31	2.42	2.48	2.53

Thus,

Blocks:	Monday	Tuesday	Thursday	Wednesday	Friday
Means:	24.75	27.50	27.75	28.25	30.75

and we conclude that there is no significant difference among the means for Tuesday, Wednesday, and Thursday at the 0.05 level of significance.

15.33 The four compressor-design means are: 46.50, 22.63, 61.25, and 48.00. The four region means are: 52.88, 40.50, 52.88, and 32.13. With

$$s_{\bar{x}} = \sqrt{\frac{65}{8}} = 2.65$$

For both designs and regions, and from Table IX with $\alpha = 0.05$ and 15 d. f., we get

p	2	3	4
r_p	3.01	3.16	3.25
R_p	8.58	9.01	9.26

Thus,

$$\begin{array}{lccc} \text{Designs:} & \text{B} & \text{A} \qquad\quad \text{D} & \text{C} \\ \text{Means:} & 22.63 & \underline{46.50 \qquad 48.00} & 61.25 \end{array}$$

$$\begin{array}{lcc} \text{Regions:} & \text{Southwest} \qquad \text{Southeast} & \text{Northwest} \qquad \text{Northeast} \\ \text{Means:} & \underline{32.13 \qquad\qquad 40.50} & \underline{52.88 \qquad\qquad 52.88} \end{array}$$

We conclude, at the 0.05 level of significance, that designs A and D do not give rise to significantly different means and that the same is true for the Southwest and Southeast and for the Northwest and northeast regions.

15.34 The three diet-food means are: 3.33, 2.96, and 3.29. The four laboratory means are 2.95, 2.98, 3.42, and 3.42. With

$$\textit{Diet foods}: \ s_{\bar{x}} = \sqrt{\frac{0.03}{8}} = 0.06; \qquad \textit{Laboratories}: \ s_{\bar{x}} = \sqrt{\frac{0.03}{6}} = 0.07$$

and using Table IX with $\alpha = 0.05$ and 11 d.f., we get

	Diet Foods		Laboratories		
p	2	3	2	3	4
r_p	3.11	3.26	3.11	3.26	3.34
R_p	0.19	0.20	0.22	0.23	0.23

Thus

$$\begin{array}{lccccc} & \text{B} & \text{C} \qquad \text{A} & \text{1} \qquad \text{2} & \text{3} \qquad \text{4} \\ \text{Means:} & 2.96 & \underline{3.29 \quad 3.33} & \underline{2.95 \quad 2.98} & \underline{3.42 \quad 3.42} \end{array}$$

We conclude, at the 0.05 level of significance, that diet foods A and C, laboratories 1 and 2, and laboratories 3 and 4 do not give rise to significantly different means.

15.35 The three launcher means are: 49.54, 46.91, and 51.69. The four fuel means are: 45.97, 55.23, 52.97, and 43.35. With

$$\textit{Launchers}: \; s_{\bar{x}} = \sqrt{\frac{0.55}{8}} = 0.26; \qquad \textit{Fuels}: \; s_{\bar{x}} = \sqrt{\frac{0.55}{6}} = 0.30$$

and using Table IX with $\alpha = 0.01$ and 11 d.f., we get

	Launchers		Fuels		
p	2	3	2	3	4
r_p	4.39	4.58	4.39	4.58	4.70
R_p	1.14	1.19	1.32	1.37	1.41

Thus

$$\begin{array}{lccccccc} & \text{Y} & \text{X} & \text{Z} & 4 & 1 & 3 & 2 \\ \text{Means:} & 46.91 & 49.54 & 51.69 & 43.35 & 45.97 & \multicolumn{2}{c}{\underline{52.97 \qquad 55.23}} \end{array}$$

We conclude, at the 0.01 level of significance, that fuels 2 and 3 are not associated with significantly different means.

15.36 The DSS means are: 31.36, 32.96, 34.18, and 34.84. With

$$\textit{DSS Level}: \; s_{\bar{x}} = \sqrt{\frac{1.37}{8}} = 0.41; \qquad \textit{Time}: \; s_{\bar{x}} = \sqrt{\frac{1.37}{16}} = 0.29$$

and using Table IX with $\alpha = 0.05$ and 21 d.f., we get

	DSS Level			Time
p	2	3	4	2
r_p	2.95	3.10	3.19	2.95
R_p	1.21	1.27	1.31	0.86

Thus

$$\begin{array}{lcccccc} & 0 & 50 & 100 & 150 & 28 & 7 \\ \text{Means:} & 31.36 & 32.96 & \multicolumn{2}{c}{\underline{34.18 \qquad 34.84}} & 31.89 & 35.28 \end{array}$$

We conclude, at the 0.05 level of significance, that the means associated with DSS levels 100 and 150 are significantly different.

15.37 The Bonder means are: 11.03, 10.72, 10.65, and 8.52. The Operator means are: 10.80, 11.03, and 8.85. With

$$\textit{Bonders:}\ s_{\bar{x}} = \sqrt{\frac{1.88}{6}} = 0.56\,; \qquad \textit{Operators:}\ s_{\bar{x}} = \sqrt{\frac{1.88}{8}} = 0.48$$

And using Table IX with $\alpha = 0.05$ and 11 d.f., we get

	Bonders			Operators	
P	2	3	4	2	3
r_p	3.11	3.26	3.34	3.11	3.26
R_p	1.74	1.83	1.87	1.49	1.56

Thus,

	D	C	B	A	3	1	2
Means:	8.52	10.65	10.72	11.03	8.86	10.80	11.03

We conclude, at the 0.05 level of significance, that the mean bonding strengths for bonders A, B, and C are not significantly different, nor are those for operators 1 and 2.

15.38 $m = 3,\ T_{1.} = 230,\ T_{2.} = 260,\ T_{3.} = 246,\ T_{.1} = 240,\ T_{.2} = 248,$

$T_{.3} = 248,\ T_A = 244,\ T_B = 274,\ T_C = 218,\ T_{...} = 736$

$$\sum\sum x^2 = 17{,}782 + 22{,}662, + 20{,}438 = 60{,}882$$

$$SST = 60{,}882 - \frac{1}{9}(736)^2 = 60{,}882 \text{ - } 60{,}188.44 = 693.56 \ \ \text{(d.f.= 8)}$$

$$SSR = \frac{1}{3}(181{,}016) - 60{,}188.4 = 150.23 \text{ (d.f. = 2)}$$

$$SSC = \frac{1}{3}(180.608) - 60{,}188.44 = 14.23 \text{ (d.f. = 2)}$$

$$SS(Tr) = \frac{1}{3}(182{,}136) - 60{,}188.44 = 523.56 \text{ (d.f. = 2)}$$

$$SSE = 693.56 - 150.23 - 14.23 - 523.56 = 554 \text{ (d.f. = 2)}$$

$$F_R = \frac{150.23/2}{5.54/2} = 27.12,\ \ F_C = \frac{14.23/2}{5.54/2} = 2.57,\ \ F_{Tr} = \frac{5.23.56/2}{5.54/2} = 94.51$$

$$F_{0.05,2,2} = 19.0$$

(a) F_{Tr} (for instructor) = 94.5 is significant

(b) F_C = 2.57 (for ethnic background) is not significant

(c) F_R = 27.12 (for professional interest) is significant

15.39 (a) First we calculate the following totals: $T_{..} = 2{,}030,\ T_{1.} = 645,\ T_{2.} = 771,\ T_{3.} = 614$, $T_{.1} = 913,\ T_{.2} = 380,\ T_{.3} = T_{(1)} = 680,\ T_{(2)} = 646,\ T_{(3)} = 704$. The correction term is $C = T_{..}^2 / 9 = 457{,}878$. The total sum of squares is the sum of the squares of the nine observations, minus the correction term. The sums of squares for rows, columns, and treatments, respectively, is the sum of the squares of the corresponding totals, divided by 3, minus the correction term. For example, the sum of squares for rows is $(645^2 + 771^2 + 614^2)/3 - C = 4{,}609$. We then get the following analysis-of-variance table:

Source of Variability	Degrees of Freedom	Sum of Squares	Mean Square	f
Rows	2	4,609	2,305	10.4
Columns	2	49,168	24,584	111
Treatments	2	566	283	1.28
Error	2	441	221	
Total	8	54,784		

(b) No. With only 2 degrees of freedom for error, the f-tests have very little power.

15.40 (a) First we calculate the following totals: $T_{..} = 763.5,\ T_{1.} = 154.2,\ T_{2.} = 151.7,\ T_{3.} = 143.2$ $T_{4.} = 154.3,\ T_{5.}{=}150.1,\ T_{.1} = 161.4,\ T_{.2} = 164.8,\ T_{.3} = 152.1,\ T_{.4} = 124.1,\ T_{.5} = 161.1$, $T_{(1)} = 156.8,\ T_{(2)} = 150.9,\ T_{(3)} = 152.2,\ T_{(4)} = 154.1,\ \mathrm{T}_{(5)} = 149.5$, and $T_{..} = 763.5$. The correction term is $C = T_{..}^2 / 9 = 457{,}878$. The total sum of squares is the sum of the squares of the nine observations, minus the correction term. The sums of squares for rows, columns, and treatments, respectively, is the sum of the squares of the corresponding totals, divided by 5, minus the correction term. For example, the sum of squares for rows is $(154.2^2 + 151.7^2 + \ldots + 150.1^2)/5 - C = 244{,}76$. We then get the following analysis-of-variance table:

Source of Variability	Degrees of Freedom	Sum of Squares	Mean Square	f
Rows	4	2.56	0.64	<1
Columns	4	222.20	55.55	49.4
Treatments	4	6.50	1.63	1.45
Error	12	13.50	1.125	
Total	24	54,784		

15.41 (a)

Factor	Level 1	Level 2	Level 3	Level 4
A	1	2		
B	1	2	3	
C	1	2	3	4

(b) For r replicates, the total degrees of freedom is $24r - 1$. This leaves $24r - 1 - 23 - (r-1) = 23(r-1)$ degrees of freedom for error. For there to be at least 30 degrees of freedom for error, r must be at least 3 replicates.

(c) The only three-factor interaction is ABC, with 6 degrees of freedom. Without replication, and assuming ABC = 0, there would be only 6 degrees of freedom for error.

15.42 The analysis of variance shows the following significant effects (effects having P-values less than or equal to 0.05).

Effect	df	Mean Square	f	P
A	1	270.28	12.45	0.003
B	1	205.03	9.45	0.007
C	1	124.03	5.71	0.029
E	1	357.78	16.49	0.001
CE	1	157.53	7.26	0.016

15.43 There are 16 three-factor and higher-order interactions. If it is assumed that they do not exist, there will be 16 degrees for freedom for error.

15.44 MINITAB software provides a table of means for the main effects. Here are the means for the significant main effects.

Level	N	A	Level	N	B	Level	N	C	Level	N	E
1	16	44.063	1	16	38.625	1	16	43.125	1	16	37.812
2	16	38.250	2	16	43.688	2	16	39.188	2	16	44.500

Each main effect is the difference between its mean at level 2 and at level 1. Thus, the significant main effects are:

$$\text{A} = -5.813,\ \text{B} = 5.063,\ \text{C} = -3.937,\ \text{E} = 6.688$$

15.45. No. The effects C and E interact with each other.

15.47 Increasing temperature from 68° to 74°F decreases the gain by 5.813. Increasing the partial pressure from 10^{-15} to 10^{-4} increases the gain by 5.063. While there was only a negligible change in the gain when the relative humidity was increased in the laboratory from 1% to 30% (an increase of 0.5), the gain decreased by 20%, from 42.000 to 33.625, on the production line. (Confidence intervals should be constructed for these estimates.)

Chapter 16

16.1 **(a)** $$t = \frac{\overline{x}}{s/\sqrt{2}} = \frac{\sqrt{2}(x_1 + x_2)}{2\frac{\sqrt{(x_1 - x_2)^2 + (x_2 - x_1)^2}}{4}} = \frac{x_1 + x_2}{x_1 - x_2}$$

(b) Since the function $f(x) = \frac{x_1 + x_2}{x_1 - x_2}$ is decreasing for $x > x_2 > 0$ it follows that $\lim_{x \to \infty} f(x) = 1 < t' = f(10x_1) < t = f(x_1)$

16.2 When $T^+ = k$ then $T^- = \frac{n(n+1)}{2} - k$ and then

$$P(T^+ = k) = P\left(T^- = \frac{n(n+1)}{2} - k\right)$$
$$= P\left(T^+ = \frac{n(n+1)}{2} - k\right)$$

So that distribution is symmetrical about $\frac{n(n+1)}{4}$.

$$P\left(T^+ = \frac{n(n+1)}{4} + c\right) = P\left(T^- = \frac{n(n+1)}{4} - c\right)$$
$$= P\left(T^+ = \frac{n(n+1)}{4} - c\right)$$

16.3 $$T^+ - T^- = T^+ - \left[\frac{n(n+1)}{2} - T^+\right] = 2T^+ - \frac{n(n+1)}{2} = X$$

$E(X) = 2 \cdot \frac{n(n+1)}{4} - \frac{n(n+1)}{2} = 0$ by Theorem 16.1

$\text{var}(X) = 4 \cdot \frac{n(n+1)(2n+1)}{24}$ by Theorem 16.1

$$= \frac{n(n+1)(2n+1)}{6}$$

16.4 $n = 5,\ P(T = 0) = [P(x = 0)]^5 = (0.5)^5 = 0.031 > 0.02$, where x is a Bernoulli variable. Therefore, $T_{0.02}$ does not exist for $n = 5$.

16.5 **(a)** $$U_1 + U_2 = W_1 - \frac{n_1(n_1+1)}{2} + W_2 - \frac{n_2(n_2+1)}{2}$$
$$= \frac{(n_1 + n_2)(n_1 + n_2 + 1)}{2} - \frac{n_1(n_1+1)}{2} - \frac{n_2(n_2+1)}{2}$$
$$= n_1 n_2$$

(b) $$\min U_1 = \frac{n_1(n_1+1)}{2} - \frac{n_1(n_1+1)}{2} = 0$$

$$\max U_1 = \frac{n_1}{2}(n_1+1+n_2+n_1) - \frac{n_1(n_1+1)}{2} - \frac{n_2(n_2+1)}{2}$$
$$= n_1 n_2$$

Same proofs for U_2.

16.6 r_i

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$n_1+n_2+1-r_i$

From left to right we get $W_1 = \sum r_i$ and right to left we get $W_1 = n_1(n_1+n_2+1) - \sum r_i$.
Probabilities are the same.

$P(W_1) = P(n_1\{n_1+n_2+1\} - W_2)$ $\therefore$ symmetrical about $\frac{n_1(n_1+n_2+1)}{2}$

when $W_1 = \frac{n_1(n_1+n_2+1)}{2}$

$$U_1 = \frac{n_1(n_1+n_2+1)}{2} - \frac{n_1(n_1+1)}{2} = \frac{n_1 n_2}{2}$$

16.7 $$U_1 = W_1 - \frac{n_1(n_1+1)}{2} = \frac{(n_1+n_2)(n_1+n_2+1)}{2} - W_2 - \frac{n_1(n_1+1)}{2}$$
$$= \frac{n_1 n_2}{2} + \frac{n_2(n_1+n_2+1)}{2} - W_2$$
$$= n_1 n_2 + \frac{n_2(n_2+n_1)}{2} - W_2$$

Proof is same for U_2.

16.8 Ranking of *x*'s are $r_1 < r_2 < r_3 < \ldots < r_{n_1}$

r_1 r_2 r_3 r_{n_1}

y's ---------|--------------|----------------|----------------|----------------|----

$r_1 - 1$ $r_2 - r_1 - 1$ $r_3 - r_2 - 1$ ... $r_{n_1} - r_{n_1-1} - 1$

Number of *y*'s preceding r_1 is $r_1 - 1$
Number of *y*'s preceding r_2 is $(r_1-1)+(r_2-r_1-1) = r_2 - 2$
Number of *y*'s preceding r_3 is $(r_1-1)+(r_2-r_1-1)+(r_3-r_2-1) = r_3 - 3$
$\vdots$
Number of *y*'s preceding $r_{n_1} = r_{n_1} - n_1$

$$\sum\sum d_{ij}^2 = \sum_{i=1}^{n_1} r_i - (1+2+3+\ldots n_1) = W_2 - \frac{n_1(n_1+1)}{2}$$

16.9 $$H = \frac{12}{n(n+1)} - \sum_{i=1}^{k} n_1 \left[\frac{R_i}{n_i} - \frac{n+1}{2}\right]^2$$

$$= \frac{12}{n(n+1)}\left[\sum_{i=1}^{k}\frac{R_i^2}{n_i} - (n+1)\sum_{i=1}^{k} R_i + \left(\frac{n+1}{2}\right)^2 \sum_{i=1}^{k} n_i\right]$$

$$= \frac{12}{n(n+1)}\sum_{i=1}^{k}\frac{R_i^2}{n_i} - \frac{12}{n}\sum_{i=1}^{k} R_i + \frac{12}{n(n+1}\cdot\frac{(n+1)^2}{4}\cdot n$$

$$= \frac{12}{n(n+1)}\sum_{i=1}^{k}\frac{R_i^2}{n_i} - \frac{12}{n}\cdot\frac{n(n+1)}{2} + 3(n+1)$$

$$= \frac{12}{n(n+1)}\sum_{i=1}^{k}\frac{R_i^2}{n_i} - 6(n+1) + 3(n+1)$$

$$= \frac{12}{n(n+1)}\sum_{i=1}^{k}\frac{R_i^2}{n_i} - 3(n+1) \qquad \text{QED}$$

16.10 $T_i = R_i, \quad \sum n_i = n, \quad T_{..} = \frac{n(n+1)}{2}$

$$\sum\sum x^2 = \frac{n(n+1)(2n+1)}{6}$$

$$SST = \frac{n(n+1)(2n+1)}{6} - \frac{1}{n}\left[\frac{n(n+1)}{2}\right]^2 = \frac{n(n^2-1)}{12} \qquad (\text{d.f.} = n-1)$$

$$SST_r = \sum_{i=1}^{k}\frac{R_i^2}{n_i} - \frac{1}{n}\left[\frac{n(n+1)}{\text{`}2}\right]^2 = \sum_{i=1}^{k}\frac{R_i^2}{n_i} - \frac{n(n+1)^2}{4} \qquad (\text{d.f.} = k-1)$$

$$SSE = SST - SST_r = \frac{n(n+1)(2n+1)}{6} - \sum_{i=1}^{k}\frac{R_i^2}{n_i} \qquad (\text{d.f.} = n-k)$$

Since $\frac{n(n+1)}{12}H = \sum_{i=1}^{k}\frac{R_i}{n_i} - \frac{n(n+1)^2}{4}$

$$SST_r = \frac{n(n+1)}{12}H$$

$$SSE = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{12}H - \frac{n(n+1)^2}{4} = \frac{n(n^2-1)}{12} - \frac{n(n+1)}{12}H$$

$$F = \frac{\frac{n(n+1)}{12(k-1)}H}{\frac{n(n^2-1)}{12(n-k)} - \frac{n(n+1)}{12(n-k)}H} = \frac{\frac{n-k}{k-1}H}{(n-1)-H}$$

The test based on F is equivalent to test based on H.

16.11 k + 1 runs of first kind and k

runs of second kind in $\binom{n_1-1}{k}\binom{n_2-1}{k-1}$ ways

k runs of first kind and

k + 1 runs of second kind in $\binom{n_1-1}{k-1}\binom{n_2-1}{k}$ ways

In total $\binom{n_1-1}{k}\binom{n_2-1}{k-1}+\binom{n_1-1}{k-1}\binom{n_2-1}{k}$ ways

So $$f(2k+1)=\frac{\binom{n_1-1}{k}\binom{n_2-1}{k-1}+\binom{n_1-1}{k-1}\binom{n_2-1}{k}}{\binom{n_1+n_2}{n_1}}$$ QED

16.12 $n_1=7,\ n_2=3$

$$f(2)=\frac{2\binom{6}{0}\binom{2}{0}}{\binom{10}{7}}=\frac{2}{120}=\frac{1}{60};\ f(3)=\frac{\binom{6}{1}\binom{2}{0}+\binom{6}{0}\binom{2}{1}}{120}=\frac{8}{120}=\frac{4}{60}$$

$$f(4)=\frac{2\binom{6}{1}\binom{2}{1}}{120}=\frac{24}{120}=\frac{12}{60};\ f(5)=\frac{\binom{6}{2}\binom{2}{1}+\binom{6}{1}\binom{2}{2}}{120}=\frac{36}{120}=\frac{18}{60}$$

$$f(6)=\frac{2\binom{6}{2}\binom{2}{2}}{120}=\frac{30}{120}=\frac{15}{16};\ f(7)=\frac{\binom{6}{3}\binom{2}{2}+\binom{6}{2}\binom{2}{3}}{120}=\frac{20}{120}=\frac{10}{60}$$

16.13 $$f(8)=\frac{2\binom{5}{3}\binom{4}{3}}{\binom{11}{6}}=\frac{2\cdot 10\cdot 4}{462}=\frac{80}{462}$$

$$f(9)=\frac{\binom{5}{4}\binom{4}{3}+\binom{5}{3}\binom{4}{4}}{462}=\frac{5\cdot 4+10\cdot 1}{462}=\frac{30}{462}$$

$$f(10)=\frac{2\binom{5}{4}\binom{4}{4}}{462}=\frac{2\cdot 5\cdot 1}{462}=\frac{10}{462}$$

$$f(11)=\frac{\binom{5}{5}\binom{4}{4}+\binom{5}{4}\binom{4}{5}}{462}=\frac{1}{462}$$

$$f(8)+f(9)+f(10)+f(11)=\frac{121}{462}=\frac{11}{42}$$

16.14 $f(2)=\dfrac{\binom{7}{0}\binom{7}{0}}{\binom{16}{8}}=\dfrac{2}{12{,}870}=0.000155$

$$f(3)=\frac{\binom{7}{1}\binom{7}{0}+2\binom{7}{0}\binom{7}{1}}{12{,}870}=\frac{14}{12{,}870}=0.001088$$

$$f(4)=\frac{\binom{7}{1}\binom{7}{1}}{12{,}870}=\frac{98}{12{,}870}=0.007615$$

$f(2)+f(3)=0.001243$

$f(2)+f(3)+f(4)=0.008858$

$$f(16)=\frac{\binom{7}{7}\binom{7}{7}}{12{,}870}=\frac{2}{12{,}870}=0.000155$$

$$f(15)=\frac{\binom{7}{7}\binom{7}{6}+\binom{7}{6}\binom{7}{7}}{12{,}870}=\frac{14}{12{,}870}=0.001088$$

$$f(14)=\frac{\binom{7}{6}\binom{7}{6}}{12{,}870}=\frac{98}{12{,}870}=0.007615$$

Reject null hypothesis for $U = 2$, 3, 15, and 16

16.15 $W = 0$ makes $R_i=\dfrac{k(n+1)}{2}$ for each value of i; it reflects a complete lack of association.

There is complete agreement, for instance, when $R_i = ki$ and

$$\begin{aligned}
W &= \frac{12}{n(n^2-1)}\sum_{i=1}^{n}\left[i-\frac{n+1}{2}\right]^2 \\
&= \frac{12}{n(n^2-1)}\left[\sum_{i=1}^{n} i^2-(n+1)\sum_{i=1}^{n} i+\frac{n(n+1)^2}{4}\right] \\
&= \frac{12}{n(n^2-1)}\left[\frac{n(n+1)(2n+1)}{6}-\frac{(n+1)n(n+1)}{2}+\frac{n(n+1)^2}{4}\right] \\
&= \frac{1}{n-1}\{2(2n+1)-6(n+1)+3(n+1)\} \\
&= 1
\end{aligned}$$

16.16 $\mu = 20\cdot\dfrac{1}{2}=10$ $\qquad\sigma=\sqrt{20\dfrac{1}{2}\cdot\dfrac{1}{2}}=2.236$

$z=\dfrac{4-10}{2.236}=-2.68$ or $z=\dfrac{4.5-10}{2.236}=-2.46$

Since $z=-2.68$ (and $z=-2.46$) is less than -1.96, null hypothesis must be rejected.

16.17 Differences are

1.3	0.9	1.1	3.8	3.1	2.6	1.8	2.5	1.2	2.4
−	+	−	−	+	−	−	−	−	−
11	7.5	9	20	19	17	13	16	10	15

0.1	2.9	0.1	0.8	0.6	0.6	0.3	1.9	0.9	1.4
−	−	+	−	+	−	−	−	−	−
1.5	18	1.5	6	4.5	4.5	3	14	7.5	12

$T^+ = 7.5 + 19 + 1.5 + 4.5 = 32.5$ less than T^-
$T = 32.5$ $\qquad T_{0.05} = 52$ for $n = 20$
Since 32.5 < 52, null hypothesis must be rejected.

16.18 There are $x = 12$ plus signs among $n = 16$ $\qquad \alpha = 0.05$
$p = 0.5$ against $p > 0.50$, p-value $p(x \ge 12) = 0.0381$
Since p-value is less than 0.05, reject the null hypothesis.

16.19

1.15	0.85	4.75	−0.37	2.09	6.63	−2.35	0.27
8	6	14	4	11	16	12	3
−0.20	2.45	1.29	0.51	4.80	1	−1.52	0.11
2	13	9	5	15	1	10	1

$$T^- = 28,\ T^+ = \frac{16 \cdot 17}{2} - 28 = 108,\ T = 28 \qquad \alpha = 0.05$$

Reject if $T^- \le T_{0.10} = 36$

Since $T^{-1} = 28 < 36$; null hypothesis must be rejected.

16.20 $n = 10$, $\alpha = 0.05$

(a) based on T; reject if $T \le T_{0.05} = 8 \qquad T \le 8$

(b) based on T^-; reject if $T^- \le T_{0.10} = 11 \quad T^- \le 11$

(c) based on T^+; reject if $T^+ \le T_{0.10} = 11 \quad T^+ \le 11$

16.21 $n = 10$, $\alpha = 0.01$

(a) based on T; reject if $T \le T_{0.01} = 3$

(b) based on T^-; reject if $T^- \le T_{0.02} = 5$

(c) based on T^+; reject if $T^+ \le T_{0.02} = 5$

16.22 $\mu_0 = 35$ against $\mu \ne 35$, $\alpha = 0.05$, $n = 11$

3	8	1	−6	9	−7	5	15	4	12	−2
3	8	1	6	9	7	5	11	4	10	2

$T^- = 15,\ T^+ = 51,\ T = 15,\ T_{0.05} = 11$
Since $T = 15$ is not ≤ 11, null hypothesis cannot be rejected.

16.23

15	18	20	22	25	27	28	29	32	35	36	38
2	2	2	2	1	2	1	2	1	1	1	1
1	2	3	4	5	6	7	8	9	10	11	12

$W_2 = 1+2+3+4+6+8 = 24$

$u_1 = 24 - \frac{6\cdot 7}{2} = 3 \qquad \mu_1 > \mu_2 \qquad \alpha = 0.01 \qquad U_{0.02} = 3$

Since $U_2 = 3 = U_{0.02}$, null hypothesis must be rejected.

16.24 $\alpha = 0.05 \qquad \mu_1 < \mu_2 \qquad U_1 \le U_{0.01} = 10$

$W_1 = 8+1+3.5+5+2+7 = 26.5$

$U_1 = 26.5 - \frac{6\cdot 7}{2} = 5.5$

Since $U_1 = 5.5 < 10$, null hypothesis must be rejected.

16.25 $\alpha = 0.05 \qquad \mu_1 \ne \mu_2 \qquad n_1 = 10,\ n_2 = 12,\ U \le U_{0.05} = 49$

$W_1 = 18+2+9+10+5+16+27+11+9+20+14+23+6+25+23+3 = 208$

$U_1 = 208 - \frac{15\cdot 16}{2} = 88,\ U_2 = 15\cdot 12 - 88 = 92,\ U = 88$

Since $U = 88$ exceeds 49, null hypothesis cannot be rejected.

16.26 $\mu = \frac{15\cdot 12}{2} = 90,\ \sigma^2 = \frac{15\cdot 12\cdot 28}{12} = 420,\ \sigma = 20.5,\ z = \frac{88-90}{20.5} = -0.10$

Since $z = -0.10$ falls between -1.96 and 1.96, null hypothesis cannot be rejected.

16.27 A B A A A B A B B A B A A A B B B B B

$\sum\sum d = 1+4+5+5+6+9+9+9+9+9 = 66 = U_2$

$\sum\sum d = 0+1+1+1+2+4+5+5+5 = 24 = U_1$

16.28 B B B B A B A B A A A A

$U = 0+0+0+0+1+2 = 3$

16.29 $n_1 = 14,\ n_2 = 8,\ u = 5,\ \alpha = 0.05$

$u'_{0.025} = 6$ Since $u = 5 < 6$, null hypothesis of randomness must be rejected.

16.30 $n_1 = 12,\ n_2 = 10,\ u = 17,\ \alpha = 0.05$

Since $u = 17$ and $u_{0.025} = 17$, null hypothesis of randomness must be rejected.

16.31 $n_1 = 5,\ n_2 = 8,\ u = 4,\ \alpha = 0.05$

Since $u = 4$ falls between $u'_{0.025} = 3$ and $u_{0.025} = 11$, null hypothesis of randomness cannot be rejected.

16.32 $n_1 = 38,\ n_2 = 22,\ u = 28,\ \alpha = 0.05$

$$\mu = \frac{2 \cdot 38 \cdot 22}{60} + 1 = 28.87$$

$$\sigma^2 = \frac{2 \cdot 38 \cdot 22(2 \cdot 38 \cdot 22 - 60)}{60^2 \cdot 59} = \frac{1672 \cdot 1612}{212{,}400} = 12.69 \qquad \alpha = 3.56$$

$$z = \frac{28 - 28.87}{3.56} = 0.24 \text{ or } z = \frac{28.5 - 28.87}{3.56} = -0.10$$

Since $z = 0.24$ (or -0.10 with continuity correction) falls between -1.96 and 1.96, null hypothesis cannot be rejected.

16.33 $n_1 = 24,\ n_2 = 24,\ u = 30,\ \alpha = 0.01$

$$\mu = \frac{2 \cdot 24 \cdot 24}{48} + 1 = 25$$

$$\sigma^2 = \frac{2 \cdot 24 \cdot 24(2 \cdot 24 \cdot 24 - 48)}{48^2 \cdot 47} = \frac{1152 \cdot 1104}{108{,}288} = 11.74 \qquad \alpha = 3.43$$

$$z = \frac{30 - 25}{3.43} = 1.46 \text{ or } z = \frac{29.5 - 25}{3.43} = 1.31 \quad \text{(with continuity correction)}$$

Since $z = 1.46$ falls between -2.575 and 2.575, null hypothesis cannot be rejected.

16.35 Median is 30.5 and we get

b b a b b b a a a b a a a b a b a b b b a a a b

$n_1 = 12,\ n_2 = 12,\ u = 13,\ \alpha = 0.01$

Since $u = 13$ falls between 6 and 20, null hypothesis cannot be rejected.

16.36 Median is 99.7

b a a b a a a b b b a b a b a b a b a b a b a b

$n_1 = 12,\ n_2 = 12,\ u = 19,\ c = 0.05$

$$\mu = \frac{2 \cdot 12 \cdot 12}{24} + 1 = 13$$

$$\sigma^2 = \frac{2 \cdot 12 \cdot 12(2 \cdot 12 \cdot 12 - 24)}{24^2 \cdot 23} = \frac{288 \cdot 264}{13{,}248} = 5.739 \quad \sigma = 2.40$$

$$z = \frac{18.5 - 13}{2.40} = 2.29 \quad \text{(with continuity correction)}$$

Since 2.29 exceeds 1.645, null hypothesis must be rejected. There is a definite cyclical pattern.

16.37

11.3	12.2	13.0	13.2	14.1	14.7	14.9	15.2	15.3	15.4
A	B	A	A	A	B	A	B	B	A
16.2	16.6	16.9	17.0	18.3	18.9	19.4	19.8	21.2	
B	A	A	A	B	B	B	B	B	

$n_1 = 9,\ n_2 = 10,\ u = 10,\ \alpha = 0.05$

$$\mu = \frac{2 \cdot 9 \cdot 10}{19} + 1 = 10.47 \qquad \sigma^2 = \frac{180 \cdot 161}{19^2 \cdot 18} = \frac{28{,}980}{6498} = 4.46 \qquad \sigma = 2.11$$

$$z = \frac{10 - 10.47}{2.11} = -0.22$$

Since -0.22 is greater than -1.645, null hypothesis cannot be rejected.

16.38

R_x	R_y	d
13	12	1
14	11	3
1	2	−1
16.5	14.5	2
2.5	1	1.5
15	16	−1
16.5	17.5	−1
8	13	−5
6.5	8.5	−2
18	17.5	0.5
10.5	14.5	−4.0
2.5	8.5	−6.0
4	4.5	−0.5
5	3	2
10.5	6	4.5
10.5	8.5	2
6.5	4.5	2
10.5	8.5	2

$$\sum d^2 = 137$$

$$r_s = 1 - \frac{6(137)}{18.323} = 1 - 0.14 = 0.86$$

16.39 $z = \dfrac{0.86 - 0}{1/\sqrt{18-1}} = 0.86(4.423) = 3.55$

Since 3.55 exceeds 1.96, the value of r_s is significant.

16.40 $\sum d^2 = 138$ $\qquad r_s = 1 - \dfrac{6(138)}{15 \cdot 224} = 1 - 0.25 = 0.75$

16.41 $\sum d^2 = 130.5$ $\qquad r_s = 1 - \dfrac{6(130.5)}{12 \cdot 143} = 1 - 0.46 = 0.54$

$$z = \frac{0.54 - 0}{1/\sqrt{11}} = 0.54(3.3166) = 1.79$$

Since $z = 1.79$ falls between -1.96 and 1.96, null hypothesis cannot be rejected; $r_s = 0.54$ is not significant.

16.42 $R_1 = 15,\ R_2 = 12,\ R_3 = 7,\ R_4 = 15,\ R_5 = 29,\ R_6 = 10,\ R_7 = 11,\ R_8 = 25,$

$R_9 = 25,\ R_{10} = 15,\ \dfrac{k(n+1)}{2} = \dfrac{3 \cdot 11}{2} = 16.5$

$$W = \frac{12}{9 \cdot 10 \cdot 99}[(-1.5)^2 + (-4.5)^2 + (-9.5)^2 + (-1.5)^2 + (12.5)^2 + (-6.5)^2 + (-5.5)^2 + (9.5)^2 + (8.5)^2 + (-1.5)^2$$

$$= \frac{12}{90 \cdot 99}[508.5) = 0.685$$

16.43 A and B $\sum d^2 = 86$ $r_s = 1 - \frac{6(86)}{10 \cdot 99} = 1 - 0.521 = 0.479$

A and C $\sum d^2 = 40$ $r_s = 1 - \frac{6(40)}{990} = 1 - 0.242 = 0.758$

B and C $\sum d^2 = 108$ $r_s = 1 - \frac{6(108)}{990} = 1 - 0.655 = 0.345$

$\overline{r}_s = 0.527;\ \frac{kW - 1}{k - 1} = \frac{3(0.685) - 1}{2} = 0.5275$

16.44 Number of plus signs = 25 out of n = 36 $\alpha = 0.01$

$\mu = 35(0.5) = 18,\ \sigma = \sqrt{36(0.5)(0.5)} = 3,\ z = \frac{24.5 - 18}{3} = 2.16$ using continuity correction.

Since 2.16 is less than $z_{0.01}$ = 2.33, null hypothesis cannot be rejected.

16.45

0.1	1.1	0.3	1.1	1.0	0.7	0.6	0.4	0.8	1.0
−	+	+	+	+	+	+	+	+	−
3.5	33.5	12.5	33.5	30.5	25	23	17	26	30.5
0.3	0.2	0.1	0.1	0.1	0.6	0.6	0.2	0.4	0.5
+	+	−	−	+	+	−	−	+	+
12.5	8.5	3.5	3.5	3.5	23	23	8.5	17	20.5
1.6	1.4	1.0	0.1	0.3	0.4	0.2	0.1	0.3	0.5
+	+	+	+	+	−	−	−	+	+
36	35	30.5	3.5	12.5	17	8.5	3.5	12.5	20.5
0.2	0.4	1.0	0.4	0.9	0.9				
+	−	+	+	+	−				
8.5	17	30.5	17	27.5	27.5				

$$T^- = 3.5 + 30.5 + 3.5 + 3.5 + 23 + 8.5 + 17 + 8.5 + 3.5 + 17 + 27.5$$
$$= 146$$

$$\mu = \frac{36 \cdot 37}{4} = 333,\ \sigma^2 = \frac{36 \cdot 37 \cdot 73}{24} = 4051.5,\ \sigma = 63.65,\ z = \frac{146 - 333}{63.65} = -2.94$$

$\alpha = 0.01$

Since $-2.94 < -2.33$, null hypothesis must be rejected.

16.46 + + + − + + + − − + + − $x = 8$

For n = 12 and p = 0.5 $P(x \ge 8) = 0.1937$ $\alpha = 0.01$

Since 0.1937 > 0.01, null hypothesis cannot be rejected.

16.47

43	35	13	11	6	18	12	6	2	7	3	10
+	+	+	−	+	+	+	−	−	+	+	−
12	11	9	7	3.5	10	8	3.5	1	5	2	6

$T^- = 7 + 3.5 + 1 + 6 = 17.5$ $T_{0l02} = 10$

Since 17.5 > 10, null hypothesis cannot be rejected.

16.48 Number of plus signs $x = 7$ $\quad n = 24$ $\quad \alpha = 0.05$

$\mu = 24(0.5) = 12$ and $\sigma = \sqrt{24(0.5)(0.5)} = 2.45$

$$z = \frac{7-12}{2.45} = -2.04 \text{ or } z = \frac{7.5-12}{2.45} = -1.84 \text{ (with continuity correction)}$$

Since $-1.84 < -1.64$, null hypothesis must be rejected.

16.49

−5	−13	−6	−7	9	−8	−1	6	−7	7	−11		
9	24	12	15	20	18	1.5	12	15	15	21		
−1	−8	−3	4	−12	−3	6	−5	12	−8	−3	2	−5
1.5	18	5	7	22.5	5	12	9	22.5	18	5	3	9

$T^+ = 20 + 12 + 15 + 7 + 12 + 22.5 + 3 = 91.5$

$n = 24 \qquad T_{0.10} = 92$

Since $91.5 < 92$, null hypothesis must be rejected.

16.50

−5	9.4	11.1	−9.3	−1.5	15.6	29	4.3	12.9	−0.9
11	16	17	15	4	22	24	9	19	2
13	7.7	11.2	−0.1	3.8	−1.9	26.3		5.5	15.4
20	14	18	1	7	6	23		12	21
3.9	1.6	6.2	4.7	−1.4					
8	5	13	10	3					

$$T^- = 11 + 15 + 4 + 2 + 1 + 6 + 3 = 42, \quad T^+ = \frac{24 \cdot 25}{2} = 42 = 258$$

$T = 42$

(a) $T_{0.05} = 81$ Since $42 < 81$, null hypothesis must be rejected.

(b) $\mu = \dfrac{24 \cdot 25}{4} = 150 \qquad \sigma^2 = \dfrac{24 \cdot 25 \cdot 49}{24} = 1225 \qquad \sigma = 35$

$$z = \frac{258 - 150}{35} = 3.09$$

Since 3.09 exceeds 1.96, null hypothesis must be rejected.

16.51

5	−12	−3	8	11	−8	−16	13
7.5	16	4	11.5	15	11.5	19	17
3	5	−2	−10	−15	1	9	7
4	7.5	2	14	18	1	13	10
6	4	−3					
9	6	4					

(a) $T^+ = 7.5 + 11.5 + 15 + 17 + 4 + 7.5 + 1 + 13 + 10 + 9 + 6 = 101.5$

$T^- = \frac{19 \cdot 20}{2} - 101.5 = 98.5$, $\quad T = 98.5 \quad T_{0.05} = 45$

Since 98.5 is not $\leq$ 46, null hypothesis cannot be rejected.

(b) $\mu = \frac{19 \cdot 20}{4} = 95 \qquad \sigma^2 = \frac{19 \cdot 20 \cdot 39}{24} = 617.5 \qquad \sigma = 24.85$

$z = \frac{101.5 - 95}{24.85} = 0.26$

Since 0.26 falls between -1.96 and 1.96, null hypothesis cannot be rejected.

16.52 $\alpha = 0.05 \qquad \mu_1 \neq \mu_2 \qquad n_1 = n_2 = 20$

$\mu = \frac{20 \cdot 20}{2} = 200,\ \sigma^2 = \frac{20 \cdot 20 \cdot 41}{12} = 1366.7,\ \sigma = 36.97$

$W_1 = 499,\ U_1 = 499 - \frac{20 \cdot 21}{2} = 289,\ z = \frac{289 - 200}{36.97} = 2.41$

Since $z = 2.41$ exceeds 1.96, null hypothesis must be rejected.

16.53 $\alpha = 0.05 \qquad \mu_1 > \mu_2 \qquad n_1 = n_2 = 16 \qquad W_1 = 307$

$U_1 = 307 - \frac{16 \cdot 17}{2} = 171,\ \mu = \frac{16 \cdot 16}{2} = 128,\ \sigma^2 = \frac{16 \cdot 16 \cdot 33}{12} = 704$

and $\sigma = 26.53$

$z = \frac{171 - 128}{26.53} = 1.62$

Since $z = 1.62$ is less than 1.645, null hypothesis cannot be rejected.

16.54 $\alpha = 0.05 \qquad \chi^2_{0.05,3} = 7.815$

$R_1 = 4 + 7 + 10 + 14 + 18 = 53$

$R_2 = 5 + 12 + 15 + 16 + 20 = 68$

$R_3 = 1 + 3 + 6 + 9 + 11 = 30$

$R_4 = 2 + 8 + 13 + 17 + 19 = 59$

$$H = \frac{12}{20 \cdot 21}\left(\frac{53^2}{5} + \frac{68^2}{5} + \frac{30^2}{5} + \frac{59^2}{5}\right) - 3.21 = 4.51$$

Since $\chi^2 = 4.51$ is less than 7.815, null hypothesis cannot be rejected.

16.55 $n_1 = n_2 = n_3 = 10 \qquad \alpha = 0.05 \qquad \text{d.f.} = 2 \qquad \chi^2_{0.05,2} = 5.991$

$R_1 = 1.5 + 5 + 7.5 + 10.5 + 12 + 13 + 15.5 + 18 + 25 + 28 = 136$

$R_2 = 3 + 5 + 7.5 + 9 + 10.5 + 20 + 21 + 22.5 + 28 + 30 = 156.5$

$R_3 = 1.5 + 5 + 14 + 15.5 + 18 + 18 + 22.5 + 25 + 25 + 28 = 172.5$

$$H = \frac{12}{30 \cdot 31}\left[\frac{136^2}{10} + \frac{156.5^2}{10} + \frac{172.5^2}{10}\right] - 3.31 = 93.86 - 93 = 0.86$$

Since $H = 0.86$ is less than 5.991, null hypothesis cannot be rejected.

16.56 $n_1 = 8,\ n_2 = 10,\ n_3 = 8$ $\alpha = 0.01$ $\chi^2_{0.01,2} = 9.210$

$R_1 = 3+6+12+13+15+21+25+26 = 121$

$R_2 = 2+4+8+11+14+16+20+22+23+24 = 144$

$R_3 = 1+5+7+9+10+17+18+19 = 86$

$$H = \frac{12}{26\cdot 27}\left[\frac{121^2}{8}+\frac{144^2}{10}+\frac{86^2}{8}\right] - 3(27) = (0.017094)(4828.225)$$

$$= 82.53 - 81 = 1.53$$

Since H = 1.53 is less than 9.210, null hypothesis cannot be rejected.

16.57 Median = 21.5

b b b b b b a b b b b b b a a b b b b b b a a a b b b a a b a a a a a b a a a a a a a a a a a a a

$n_1 = 25,\ n_2 = 25,\ u = 12,\ \alpha = 0.025$

$$\mu = \frac{2\cdot 25\cdot 25}{50} + 1 = 26 \qquad \sigma^2 = \frac{2\cdot 25\cdot 25(2\cdot 25\cdot 25 - 50)}{50\cdot 50\cdot 49} = 12.24$$

$$\sigma = 3.50 \qquad z = \frac{12-26}{3.50} = -4 \text{ (−3.86 with continuity correction)}$$

Since z = −4 (or −3.86 with continuity correction) is less than −1.645, null hypothesis must be rejected; there is a trend.

16.58 Median is 5

$n_1 = 14,\ n_2 = 13,\ u = 5,\ \alpha = 0.01$

a a a a a a a a b b b b b b a a a a a b b b b b b a

Since u = 5 is less than 7, the null hypothesis must be rejected.

16.59 Median = 138 $\alpha = 0.05$

b b b b b a a b b a a a a b a b b b a a a a b b b a b b a a a a

$n_1 = 16,\ n_2 = 16,\ u = 12$

$$\mu = \frac{2\cdot 16\cdot 16}{32} + 1 = 17$$

$$\sigma^2 = \frac{2\cdot 16\cdot 16(2\cdot 16\cdot 16 - 32)}{32^2\cdot 31} = \frac{512\cdot 480}{31{,}744} = 7.742 \qquad \sigma = 2.78$$

$$z = \frac{12-27}{2.78} = -1.80$$

Since $z = -1.80$ is less than -1.645; the null hypothesis of randomness must be rejected; there seems to be a trend.