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Course : SCOR

No: UDPS 2013 Numerical Methods

Test :

Q1. (a) Solve the following linear system using LU factorization Method:

$$-4x + 2y - 4z = 0$$
$$6x + 3y + 2z = -5$$
$$2x + y - z = 1$$

$$\begin{bmatrix} -4 & 2 & -4 \\ 6 & 3 & 2 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 1 \end{bmatrix} \Rightarrow A\underline{x} = \underline{b}$$

$$A = \begin{bmatrix} -4 & 2 & -4 \\ 6 & 3 & 2 \\ 2 & 1 & -1 \end{bmatrix} \xrightarrow[R_3^1(0.5)]{R_2^1(1.5)} \begin{bmatrix} -4 & 2 & -4 \\ 0 & 6 & -4 \\ 0 & 2 & -3 \end{bmatrix} \xrightarrow{R_3^2(-1/3)} \begin{bmatrix} -4 & 2 & -4 \\ 0 & 6 & -4 \\ 0 & 0 & -5/3 \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} -4 & 2 & -4 \\ 0 & 6 & -4 \\ 0 & 0 & -5/3 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 1 & 0 \\ -1/2 & 1/3 & 1 \end{bmatrix}$$

For $A\underline{x} = \underline{b}$, $LU\underline{x} = \underline{b}$,

Let $\underline{y} = U\underline{x}$, $L\underline{y} = \underline{b}$,

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 1 & 0 \\ -1/2 & 1/3 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 1 \end{bmatrix}$$

$\therefore y_1 = 0$

$-3/2\,(0) + y_2 = -5 \Rightarrow y_2 = -5$

$-1/2\,(0) + 1/3(-5) + y_3 = 1 \Rightarrow y_3 = 8/3$

$$\underline{y} = \begin{bmatrix} 0 \\ -5 \\ 8/3 \end{bmatrix}$$

Then $\underline{y} = U\underline{x}$

$$\begin{bmatrix} 0 \\ -5 \\ 8/3 \end{bmatrix} = \begin{bmatrix} -4 & 2 & -4 \\ 0 & 6 & -4 \\ 0 & 0 & -5/3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\Rightarrow 8/3 = -5/3\, z \Rightarrow z = -1.6$

$-5 = 6y - 4(-1.6) \Rightarrow y = -1.9$

$0 = -4x + 2(-1.9) - 4(-1.6) \Rightarrow x = 0.65$

$\therefore x = 0.65,\ y = -1.9,\ z = -1.6$ *

No:

Q1. (b) Find the estimated solutions for the following linear system by using Gauss Elimination Method with scaled partial pivoting. All computations are to be carried out using four-digit rounding arithmetic.

$$-x + 10y - 2z = 7$$
$$-2y + 10z = 6$$
$$10x - y = 9$$

Comment on the accuracy of the estimated solutions obtained.

$$\begin{bmatrix} -1.000 & 10.00 & -2.000 \\ 0.000 & -2.000 & 10.00 \\ 10.00 & -1.000 & 0.000 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7.000 \\ 6.000 \\ 9.000 \end{bmatrix} \Rightarrow A\underset{\sim}{x} = \underset{\sim}{b}$$

$$\Rightarrow [A|b] = \left[\begin{array}{ccc|c} -1.000 & 10.00 & -2.000 & 7.000 \\ 0.000 & -2.000 & 10.00 & 6.000 \\ 10.00 & -1.000 & 0.000 & 9.000 \end{array}\right]$$

Using Gauss elimination with scaled partial pivoting,

$s_1 = 10.00$, $s_2 = 10.00$, $s_3 = 10.00$

and $\frac{|a_{11}|}{s_1} = \frac{1.000}{10.00} = 0.1000$, $\frac{|a_{21}|}{s_2} = \frac{0.000}{10.00} = 0.000$, $\frac{|a_{31}|}{s_3} = \frac{10.00}{10.00} = 1.000$

Interchange row 1 and row 3,

$$[A|b] = \left[\begin{array}{ccc|c} 10.00 & -1.000 & 0.000 & 9.000 \\ 0.000 & -2.000 & 10.00 & 6.000 \\ -1.000 & 10.00 & -2.000 & 7.000 \end{array}\right] \xrightarrow{R_3^1(0.1000)}$$

$$= \left[\begin{array}{ccc|c} 10.00 & -1.000 & 0.000 & 9.000 \\ 0.000 & -2.000 & 10.00 & 6.000 \\ 0.000 & 9.900 & -2.000 & 7.900 \end{array}\right]$$

Since $\frac{2.000}{10.00} = 0.2000$ and $\frac{9.900}{10.00} = 0.9900$, interchange row 2 and row 3.

$$[A|b] = \left[\begin{array}{ccc|c} 10.00 & -1.000 & 0.000 & 9.000 \\ 0.000 & 9.900 & -2.000 & 7.900 \\ 0.000 & -2.000 & 10.00 & 6.000 \end{array}\right] \xrightarrow{R_3^2(0.2020)}$$

$$= \left[\begin{array}{ccc|c} 10.00 & -1.000 & 0.000 & 9.000 \\ 0.000 & 9.900 & -2.000 & 7.900 \\ 0.000 & 0.000 & 9.596 & 7.596 \end{array}\right]$$

$\therefore$ $7.596 = 9.596z$ $\Rightarrow z = 0.7916$

$7.900 = 9.900y - 2.000(0.7916)$ $\Rightarrow y = 0.9579$

$9.000 = 10.00x - 1.000(0.9579)$ $\Rightarrow x = 0.9958$

$\therefore x = 0.9958,\ y = 0.9579,\ z = 0.7916$

To measure the accuracy of the estimated solutions, relative error is used:

Actual $x = \frac{473}{475}$, Actual $y = \frac{91}{95}$, Actual $z = \frac{376}{475}$

Relative Error of $x = 1.057 \times 10^{-5}$

Relative Error of $y = 5.495 \times 10^{-6}$

Relative Error of $z = 2.660 \times 10^{-5}$

$\therefore$ The accuracy of the estimated solutions are high since all the relative error are smaller than 10^{-4}.

No: ____________

Q1. (c) Estimate the solutions for the following linear system by implementing four iterations using SOR method with $w = 1.1$.

$$4x_1 + x_2 - x_3 = 5$$
$$-x_1 + 3x_2 + x_3 = -4$$
$$2x_1 + 2x_2 + 5x_3 = 1$$

All computational results are to be rounded up to four decimal places.

$$\begin{bmatrix} 4.0000 & 1.0000 & -1.0000 \\ -1.0000 & 3.0000 & 1.0000 \\ 2.0000 & 2.0000 & 5.0000 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5.0000 \\ -4.0000 \\ 1.0000 \end{bmatrix} \Rightarrow A\underline{x} = \underline{b}$$

$$x_1^{(k)} = 1.1000 \left[\tfrac{1}{4} \left(-x_2^{(k-1)} + x_3^{(k-1)} + 5.0000 \right) \right] - 0.1000\, x_1^{(k-1)}$$
$$x_2^{(k)} = 1.1000 \left[\tfrac{1}{3} \left(x_1^{(k)} - x_3^{(k-1)} - 4.0000 \right) \right] - 0.1000\, x_2^{(k-1)}$$
$$x_3^{(k)} = 1.1000 \left[\tfrac{1}{5} \left(-2x_1^{(k)} - 2x_2^{(k)} + 1.0000 \right) \right] - 0.1000\, x_3^{(k-1)} \quad , \; k = 1, 2, 3, \dots$$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0	0	0
1	1.3750	-0.9625	0.0385
2	1.5128	-0.8298	-0.0844
3	1.4287	-0.8289	-0.0355
4	1.4503	-0.8390	-0.0454

$\therefore \; x_1 \simeq 1.4503 \quad , \quad x_2 \simeq -0.8390 \quad , \quad x_3 \simeq -0.0454$ *

No: ____________________

Q2. (a) Comment on the accuracy of Lagrange Polynomial, Hermite Polynomial and Cubic Spline Polynomial used in the interpolated estimation for the curve $y=f(x)$ over an interval.

LAGRANGE POLYNOMIAL	HERMITE POLYNOMIAL	CUBIC SPLINE POLYNOMIAL
It passes through all the data points available over an interval.	It passes through all the data points available over an interval.	It passes through all the data points available over an interval.
All the data points are not required to be sorted out with respect to coordinate x.	All the data points are not required to be sorted out with respect to coordinate x.	All the data points are required to be sorted out with respect to coordinate x.
–	First derivative of Hermite Polynomial is the same as the first derivative of $f(x)$ at all the data points given.	It involves the piecewise-polynomial approximation where the cubic polynomial is used for each successive pair of nodes.
–	It has least estimation error compared to lagrange polynomial as $H'_{2n+1}(x_j) = f'(x_j)$, $j=0,1,2,\cdots,n$.	Its estimation error can be further reduced by allocating more data points on the part of the curve $y=f(x)$ when it is changing rapidly.

No: ____________

Q2 (b) The table below shows a sample of 5 nodes / points on the curve $y = f(x)$.

x	$f(x)$
0.2	0.059673
0.4	0.356087
0.6	1.195242
0.8	3.169941
1.0	7.389056

(i) Construct fourth Lagrange Polynomial from data points given above.

$$P_4(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} f(x_1) +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} f(x_2) + \cdots + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} f(x_4)$$

$$= \frac{(x-0.4)(x-0.6)(x-0.8)(x-1.0)}{(-0.2)(-0.4)(-0.6)(-0.8)}(0.059673) + \frac{(x-0.2)(x-0.6)(x-0.8)(x-1.0)}{(0.2)(-0.2)(-0.4)(-0.6)}(0.356087) +$$

$$\frac{(x-0.2)(x-0.4)(x-0.8)(x-1.0)}{(0.4)(0.2)(-0.2)(0.4)}(1.195242) + \cdots + \frac{(x-0.2)(x-0.4)(x-0.6)(x-0.8)}{(0.8)(0.6)(0.4)(0.2)}(7.389056)$$

$$= 1.5540(x-0.4)(x-0.6)(x-0.8)(x-1.0) - 37.0924(x-0.2)(x-0.6)(x-0.8)(x-1.0) +$$

$$186.7566(x-0.2)(x-0.4)(x-0.8)(x-1.0) - 330.2022(x-0.2)(x-0.4)(x-0.6)(x-1.0) +$$

$$192.4233(x-0.2)(x-0.4)(x-0.6)(x-0.8)$$

(ii) Estimate $f(0.5)$ using the interpolated polynomial in (b)(i).

$$f(0.5) \approx P_4(0.5)$$

$$= 1.5540(0.1)(-0.1)(-0.3)(-0.5) - 37.0924(0.3)(-0.1)(-0.3)(-0.5) +$$

$$186.7566(0.3)(0.1)(-0.3)(-0.5) - 330.2022(0.3)(0.1)(-0.1)(-0.5) +$$

$$192.4233(0.3)(0.1)(-0.1)(-0.3)$$

$$= 0.6829$$

No: ______________________

Q2. (c) The following table indicates a sample of experimental data between pressure x and reaction rate y and respective rates of change of y with respect to x.

x	y	y'
0.30	0.40496	1.75482
0.32	0.44068	1.81781
0.35	0.49667	1.91574

(i) Construct the Hermite polynomial using the divided - difference table from the sample data given in the table above.

(ii) Use the polynomial in (i) above to estimate y when x = 0.31.

All computations are performed up to five decimal places.

(i)

x	f(x)	1st Divided Difference	2nd Divided Difference	3rd Divided Difference	4th Divided Difference	5th Divided Difference
0.30	0.40496					
		1.75482				
0.30	0.40496		1.55900			
		1.78600		1.57500		
0.32	0.44068		1.59050		-20.76800	
		1.81781		0.53660		596.32000
0.32	0.44068		1.61733		9.04800	
		1.86633		0.98400		
0.35	0.49667		1.64700			
		1.91574				
0.35	0.49667					

$$H_5(x) = 0.40496 + 1.75482(x-0.30) + 1.55900(x-0.30)^2 + 1.57500(x-0.30)^2(x-0.32) - 20.76800(x-0.30)^2(x-0.32)^2 + 596.32000(x-0.30)^2(x-0.32)^2(x-0.35)$$

(ii) $f(0.31) \approx H_5(0.31)$

$$= 0.40496 + 1.75482(0.01) + 1.55900(0.0001) + 1.57500(0.0001)(-0.01) - 20.76800(0.0001)(0.0001) + 596.32000(0.0001)(0.0001)(-0.04)$$

$$= 0.42266$$

No:

Q3. (a) Use the first three iterations of Power Method with scaling to approximate the dominant eigenvalue of matrix $A = \begin{bmatrix} 3.556 & -1.778 & 0 \\ -1.778 & 3.556 & -1.778 \\ 0 & -1.778 & 3.556 \end{bmatrix}$. Use $x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Iteration 1: $y_1 = Ax_0 = \begin{bmatrix} 1.778 \\ 0 \\ 1.778 \end{bmatrix} \rightsquigarrow x_1 = \frac{1}{1.778} y_1 = \frac{1}{1.778} \begin{bmatrix} 1.778 \\ 0 \\ 1.778 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Iteration 2: $y_2 = Ax_1 = \begin{bmatrix} 3.556 \\ -3.556 \\ 3.556 \end{bmatrix} \rightsquigarrow x_2 = \frac{1}{3.556} y_2 = \frac{1}{3.556} \begin{bmatrix} 3.556 \\ -3.556 \\ 3.556 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Iteration 3: $y_3 = Ax_2 = \begin{bmatrix} 5.334 \\ -7.112 \\ 5.334 \end{bmatrix} \rightsquigarrow x_3 = \frac{1}{7.112} y_3 = \frac{1}{7.112} \begin{bmatrix} 5.334 \\ -7.112 \\ 5.334 \end{bmatrix} = \begin{bmatrix} 0.75 \\ -1 \\ 0.75 \end{bmatrix}$

$\therefore$ Dominant Eigenvalue $= \lambda \approx \dfrac{Ax_3 \cdot x_3}{x_3 \cdot x_3}$

$$= \frac{\begin{bmatrix} 4.445 & -6.223 & 4.445 \end{bmatrix} \begin{bmatrix} 0.75 \\ -1 \\ 0.75 \end{bmatrix}}{\begin{bmatrix} 0.75 & -1 & 0.75 \end{bmatrix} \begin{bmatrix} 0.75 \\ -1 \\ 0.75 \end{bmatrix}}$$

$= 12.8905 / 2.125$

$= 6.0661$

(b) Comment on the advantages and disadvantages of using Power Method with scaling over QR algorithm in estimating eigenvalues for matrix A above.

POWER METHOD WITH SCALING	QR ALGORITHM
Only approximate the dominant eigenvalue of matrix A.	Can approximate all the eigenvalues for matrix A.
Only applicable if matrix A has dominant eigenvalue.	Applicable when matrix A do not have dominant eigenvalue.
Applicable when matrix A is not in tridiagonal form.	Not applicable when matrix A is not in tridiagonal form.