No: ______________________

Q3. Let $Y_1, Y_2, \dots, Y_n$ denote a random sample from the uniform distribution

$$f(y) = \begin{cases} 1, & \text{for } 0 \le y \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

$$F(y) = \begin{cases} 0, & \text{for } y < 0 \\ y, & \text{for } 0 \le y \le 1 \\ 1, & \text{for } y > 1. \end{cases}$$

Find the joint distribution of $Y_{(1)}$ and $Y_{(n)}$,

$$\cancel{F_{Y_{(1)}}(y_{(1)}) = 1 - [1 - y_{(1)}]^n}$$

$$\cancel{F_{Y_{(n)}}(y_{(n)}) = [y_{(n)}]^n}$$

$$P(Y_{(1)} > y_{(1)}, Y_{(n)} \le y_{(n)}) = P(y_{(1)} < Y_1 \le y_{(n)}, y_{(1)} < Y_2 \le y_{(n)}, \dots, y_{(1)} < Y_{(n)} \le y_{(n)})$$
$$= [F(y_{(n)}) - F(y_{(1)})]^n$$

$$F_{Y_{(1)}, Y_{(n)}}(y_{(1)}, y_{(n)}) = P(Y_{(1)} \le y_{(1)}, Y_{(n)} \le y_{(n)})$$
$$= P(Y_{(n)} \le y_{(n)}) - P(Y_{(1)} > y_{(1)}, Y_{(n)} \le y_{(n)})$$
$$= [F(y_{(n)})]^n - [F(y_{(n)}) - F(y_{(1)})]^n$$

$$f_{Y_{(1)}, Y_{(n)}}(y_{(1)}, y_{(n)}) = \frac{d}{dy_{(1)}} \cdot \frac{d}{dy_{(n)}} \{ [F(y_{(n)})]^n - [F(y_{(n)}) - F(y_{(1)})]^n \}$$
$$= \frac{d}{dy_{(1)}} \{ n[F(y_{(n)})]^{n-1} f(y_{(n)}) - n[F(y_{(n)}) - F(y_{(1)})]^{n-1} f(y_{(n)}) \}$$
$$= n(n-1) [F(y_{(n)}) - F(y_{(1)})]^{n-2} f(y_{(1)}) f(y_{(n)})$$
$$= n(n-1) [F(y_{(n)}) - F(y_{(1)})]^{n-2}$$
$$= n(n-1) [y_{(n)} - y_{(1)}]^{n-2}, \quad \text{for } 0 \le y_{(1)} \le 1, \ 0 \le y_{(n)} \le 1,$$

and $f_{Y_{(1)}, Y_{(n)}}(y_{(1)}, y_{(n)}) = 0$, elsewhere.

Let $U = Y_{(1)}$, $R = Y_{(n)} - Y_{(1)}$,

$\rightsquigarrow$ $Y_{(1)} = U$, $Y_{(n)} = R + U$

$$\frac{dy_{(1)}}{du} = 1, \quad \frac{dy_{(1)}}{dr} = 0, \quad \frac{dy_{(n)}}{du} = 1, \quad \frac{dy_{(n)}}{dr} = 1$$

$$J = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

$$g(u, r) = n(n-1)(r + u - u)^{n-2} \cdot |1|$$
$$= n(n-1) r^{n-2}, \quad \text{for } 0 \le u \le 1 - r, \ 0 \le r \le 1,$$

and $g(u, r) = 0$, elsewhere.

Probability density function for $R = Y_{(n)} - Y_{(1)}$,

$$h(r) = \int_0^{1-r} n(n-1) r^{n-2} \, du$$
$$= n(n-1) r^{n-2} \, u \Big|_0^{1-r}$$
$$= n(n-1)(1-r) r^{n-2}, \quad \text{for } 0 \le r \le 1,$$

and $h(r) = 0$ elsewhere.