Notes

Tensors and Group theory

Yi Huang

University of Minnesota

E-mail: yihphysics@gmail.com

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Problems

PART

I

1 Chapter 2

This note is based on *An Introduction to Tensors and Group Theory for Physicsts* by Nadir Jeevanjee.

1.1 2.1

Prove that $L^2([-a,a])$ is closed under addition. You will need the triangle inequlity, as well as the following inequality, valid for all $\lambda \in \mathbb{R} : 0 \leq \int_{-a}^{a} (|f| + |g|)^2 \, \mathrm{d}x$.

Proof. Suppose $f, g \in L^2([-a,a])$, let $\lambda = -1$, and we have

$$2\int_{-a}^{a} |f||g| \, \mathrm{d}x \leq \int_{-a}^{a} |f|^2 \, \mathrm{d}x + \int_{-a}^{a} |g|^2 \, \mathrm{d}x < \infty, \tag{1.1}$$

so

$$\int_{-a}^{a} (|f| + |g|)^2 \, \mathrm{d}x = \int_{-a}^{a} (|f|^2 + |g|^2 + 2|f||g|) \, \mathrm{d}x < \infty. \tag{1.2}$$

Using the triangle inequality, we have

$$\int_{-a}^{a} |f+g|^2 \, \mathrm{d}x \leq \int_{-a}^{a} (|f| + |g|)^2 \, \mathrm{d}x < \infty. \tag{1.3}$$

So $L^2([-a,a])$ is closed under addition. □

1.2 2.2

In this problem we show that $\{r^l Y_m^l\}$ is a basis for $\mathcal{H}_l(\mathbb{R}^3)$, which implies that $\{Y_m^l\}$ is a basis for $\tilde{\mathcal{H}}_l(\mathbb{R}^3)$.

(a) Let $f \in \mathcal{H}_l(\mathbb{R}^3)$, and write f as $f = r^l Y(\theta, \phi)$. Then we know that

$$\Delta_{S^2} Y = -l(l+1)Y, \tag{1.4}$$

where

$$\Delta_{S^2} = \frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2}. \tag{1.5}$$

If you have never done so, use the expression for Δ_{S^2} and the expression for the angular momentum operators to show that

$$-\Delta_{S^2} = L_x^2 + L_y^2 + L_z^2 \equiv \mathbf{L}^2, \tag{1.6}$$

where

$$L_x = -i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right),$$
$$L_y = -i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right),$$
$$L_z = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

so that (1.4) says that Y is an eigenfuction of $\mathbf{L}^2$, as expected. You will need to convert between cartesian and spherical coordinates. The theory of angular momentum then tells us that $\mathcal{H}_l(\mathbb{R}^3)$ has dimension $2l+1$.

Proof. We need to express the cartesian coordinates $\{x, y, z\}$ and the corresponding derivatives $\{\partial_x, \partial_y, \partial_z\}$ in terms of spherical coordinates

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta,$$
$$r = \sqrt{x^2+y^2+z^2}, \quad \theta = \arccos\frac{z}{r}, \quad \phi = \arctan\frac{y}{x}.$$

Hence

$$\begin{aligned}
\mathrm{d}f &= \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z \\
&= \frac{\partial f}{\partial r}\,\mathrm{d}r + \frac{\partial f}{\partial \theta}\,\mathrm{d}\theta + \frac{\partial f}{\partial \phi}\,\mathrm{d}\phi \\
&= \left(\frac{\partial f}{\partial r}\right)_{\theta,\phi}\left(\frac{\partial r}{\partial x}\,\mathrm{d}x + \frac{\partial r}{\partial y}\,\mathrm{d}y + \frac{\partial r}{\partial z}\,\mathrm{d}z\right) \\
&+ \left(\frac{\partial f}{\partial \theta}\right)_{r,\phi}\left(\frac{\partial \theta}{\partial x}\,\mathrm{d}x + \frac{\partial \theta}{\partial y}\,\mathrm{d}y + \frac{\partial \theta}{\partial z}\,\mathrm{d}z\right) \\
&+ \left(\frac{\partial f}{\partial \phi}\right)_{r,\theta}\left(\frac{\partial \phi}{\partial x}\,\mathrm{d}x + \frac{\partial \phi}{\partial y}\,\mathrm{d}y + \frac{\partial \phi}{\partial z}\,\mathrm{d}z\right)
\end{aligned}$$

where we need to take care of what variables are kept constant when we take the partial derivatives, because we have two sets of variables (x, y, z) and (r, θ, ϕ). It is not difficult to show

$$\left(\frac{\partial r}{\partial x}\right)_{y,z} = \frac{x}{r}, \quad \left(\frac{\partial r}{\partial y}\right)_{x,z} = \frac{y}{r}, \quad \left(\frac{\partial r}{\partial z}\right)_{x,y} = \frac{z}{r},$$
$$\left(\frac{\partial \theta}{\partial x}\right)_{y,z} = \frac{\cos\theta\cos\phi}{r}, \quad \left(\frac{\partial \theta}{\partial y}\right)_{x,z} = \frac{\cos\theta\sin\phi}{r}, \quad \left(\frac{\partial \theta}{\partial z}\right)_{x,y} = -\frac{\sin\theta}{r},$$
$$\left(\frac{\partial \phi}{\partial x}\right)_{y,z} = -\frac{sin\phi}{r\sin\theta}, \quad \left(\frac{\partial \phi}{\partial y}\right)_{x,z} = \frac{\cos\phi}{r\sin\theta}.$$

Therefore we have

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \left(\frac{\partial r}{\partial x}\right)_{y,z}\left(\frac{\partial f}{\partial r}\right)_{\theta,\phi} + \left(\frac{\partial \theta}{\partial x}\right)_{y,z}\left(\frac{\partial f}{\partial \theta}\right)_{r,\phi} + \left(\frac{\partial \phi}{\partial x}\right)_{y,z}\left(\frac{\partial f}{\partial \phi}\right)_{r,\theta} \\
&= \left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)f, \\
\frac{\partial f}{\partial y} &= \left(\frac{\partial r}{\partial y}\right)_{x,z}\left(\frac{\partial f}{\partial r}\right)_{\theta,\phi} + \left(\frac{\partial \theta}{\partial y}\right)_{x,z}\left(\frac{\partial f}{\partial \theta}\right)_{r,\phi} + \left(\frac{\partial \phi}{\partial y}\right)_{x,z}\left(\frac{\partial f}{\partial \phi}\right)_{r,\theta} \\
&= \left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)f, \\
\frac{\partial f}{\partial z} &= \left(\frac{\partial r}{\partial z}\right)_{x,y}\left(\frac{\partial f}{\partial r}\right)_{\theta,\phi} + \left(\frac{\partial \theta}{\partial z}\right)_{x,y}\left(\frac{\partial f}{\partial \theta}\right)_{r,\phi} + \left(\frac{\partial \phi}{\partial z}\right)_{x,y}\left(\frac{\partial f}{\partial \phi}\right)_{r,\theta} \\
&= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)f,
\end{aligned}$$

Now we can calculate the angular momentum operators in spherical coordinates

$$\begin{aligned}
L_x &= -i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) = i\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right), \\
L_y &= -i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) = -i\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right) \\
L_z &= -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -i\frac{\partial}{\partial\phi},
\end{aligned}$$

and we define the ladder operators $L_\pm = L_x \pm iL_y$, so we have

$$L_\pm = \mathrm{e}^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right). \tag{1.7}$$

and

$$\begin{aligned}
\mathbf{L}^2 &\equiv L_x^2 + L_y^2 + L_z^2 \\
&= L_+L_- + i[L_x, L_y] + L_z^2 \\
&= L_+L_- + L_z^2 - L_z \\
&= -\frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.
\end{aligned}$$

where we use $L_z = i[L_x, L_y]$. □

(b) Exhibit a basis for $\mathcal{H}_l(\mathbb{R}^3)$ by considering the function $f_0^l \equiv (x+iy)^l$ and showing that

$$L_z(f_0^l) = lf_0^l, \quad L_+(f_0^l) \equiv (L_x + iL_y)(f_0^l). \tag{1.8}$$

The theory of angular momentum then tells us that $(L_-)^k f_0^l \equiv f_k^l$ satisfies $L_z f_k^l = (l-k)f_k^l$ and that $\{f_k^l\}_{0\le k\le 2l}$ is a basis for $\mathcal{H}_l(\mathbb{R}^3)$.

Proof. Rewrite f_0^l in spherical coordinates

$$f_0^l = r^l \sin^l \theta e^{il\phi}. \tag{1.9}$$

Hence

$$L_z f_0^l = -i \frac{\partial f_0^l}{\partial \phi} = l f_0^l. \tag{1.10}$$

Apply ladder operators to f_0^l, we have

$$\begin{aligned} L_+(f_0^l) &= e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\left(r^l \sin^l \theta e^{il\phi}\right) \\ &= r^l e^{i\phi} e^{il\phi}\left(l(\sin\theta)^{l-1}\cos\theta - l\cot\theta\sin^l\theta\right) = 0. \end{aligned}$$

□

1.3 Exercise 2.6

Suppose V is finite-dimensional and let $T \in \mathcal{L}(V)$. Show that T being one-to-one is equivalent to T being onto. Feel free to introduce a basis to assist you in the proof.

Proof. Choose a basis $\mathcal{B} = \{e_1, e_2, \dots, e_n\}$ of V, and $\dim V = n$.

1. Given T is one-to-one, we can prove that $\mathcal{B}' = \{T(e_i)\}$ is also a basis of V, i.e., $\mathcal{B}'$ is independent and spans V.

   We write the linear relation for $T(e_i)$

   $$\sum_{i=1}^{n} c_i T(e_i) = 0. \tag{1.11}$$

   Because T is linear, we can also write

   $$T(c_j e_j) = T\left(-\sum_{i\neq j} c_i e_i\right). \tag{1.12}$$

   Given T is injective, we have

   $$c_j e_j = -\sum_{i\neq j} c_i e_i, \quad \text{or} \quad \sum_{i=1}^{n} c_i e_i = 0. \tag{1.13}$$

   Then $c_i = 0$ for $i = 1, 2, \dots, n$, because $\mathcal{B}$ is linear independent. Hence $\mathcal{B}'$ is linear independent. Since the nember of elements of $\mathcal{B}'$ is n, $\mathcal{B}'$ is also a basis of V.

   For any vector $w \in V$, we can expand it in basis $\mathcal{B}'$, such that

   $$w = \sum_{i=1}^{n} a_i T(e_i) = T\left(\sum_{i=1}^{n} a_i e_i\right) = T(v), \tag{1.14}$$

   where $v = \sum_i a_i e_i \in V$. Hence $\exists v \in V$, such that $w = T(v)$, so T is onto.

2. Given T is onto, for any $w \in V$, there exists v such that $w = T(v)$. If we want to show T is also one-to-one, then we need to prove such v is unique. We use proof

by contradictions. Suppose $\exists v_1 \neq v_2$, such that $w = T(v_1) = T(v_2)$. Because T is linear, we have

$$T(v_1) - T(v_2) = T(v_1 - v_2) = 0. \tag{1.15}$$

Write $v_{1,2}$ in basis $\mathcal{B}$

$$v_1 = \sum_{i=1}^{n} c_{1i} e_i, \quad v_2 = \sum_{i=1}^{n} c_{2i} e_i,$$
$$v_1 - v_2 = \sum_{i=1}^{n} c_i e_i \neq 0.$$

where $c_i = c_{1i} - c_{2i}$, and there exists some $j \in \{1, 2, \ldots, n\}$ such that $c_j \neq 0$ because $v_1 - v_2 \neq 0$. Hence (1.15) can be written as

$$\sum_{i=1}^{n} c_i T(e_i) = 0, \quad \exists j \in \{1, 2, \ldots, n\} \text{such that} c_j \neq 0, \tag{1.16}$$

which means $\mathcal{B}' = \{T(e_i)\}$ is linearly dependent, $\dim(\mathcal{B}') < n$, so there exists $w' \notin V$, with w' can not be express in terms of a linear combination of basis $\mathcal{B}'$. In other words, we cannot find such $v' = \sum_i a_i e_i \in V$ such that $w' = \sum_i a_i T(e_i) = T(v')$. This contradicts our assumption that T is onto. Therefore $v_1 = v_2 = v$ is unique once we are given a $w \in V$. In other words

$$w = T(v_1) = T(v_2) \Longrightarrow v_1 = v_2, \tag{1.17}$$

if $w \in \text{Span}(\mathcal{B}')$, then $v \in \text{Span}(\mathcal{B})$, i.e. (1.17) is true for all $v_1 = v_2 \in V$. Eventually we prove that T is one-to-one.

□

1.4 Exercise 2.7

Suppose $T(v) = 0 \Longrightarrow v = 0$. Show that this is equivalent to T being one-to-one.

Proof. Write $v = v_1 - v_2$, because T is linear

$$T(v) = T(v_1 - v_2) = T(v_1) - T(v_2) = 0 \Longrightarrow v = v_1 - v_2 = 0. \tag{1.18}$$

Hence T is one-to-one. □

1.5 Exercise 2.10

By carefully working with the definitions, show that the e^i defined in (2.18) and satisfying (2.20) are linearly independent.

Proof. For any vector $v \in V$ such that

$$v = \sum_{i=1}^{n} v^i e_i, \tag{1.19}$$

a dual vector e^i is defined by

$$e^i(v) \equiv v^i. \tag{1.20}$$

If $v = e_j = \sum_i \delta^i_j e_i$, then $e^i(e_j) = \delta^i_j$. We are well-prepared to prove $\mathcal{B}^* = \{e^i\}$ is linearly independent.
First we write a linear relation for $\{e^i\}$

$$\sum_{i=1}^{n} c_i e^i = 0 \tag{1.21}$$

where $c_i \in C$ is a scalar. Remember that e^i is a C-valued function on V, so if we apply this linear relation (1.21) on any vector $v \in V$, we must still have zero. Let $v = e_j$, where $j = 1, 2, \ldots, n$, then

$$\sum_{i=1}^{n} c_i e^i(e_j) = \sum_{i=1}^{n} c_i \delta^i_j = c_j = 0. \tag{1.22}$$

In other words the linear relation (1.21) is true, unless all $c_i = 0$. Therefore $\mathcal{B}^*$ is linear independent. □

1.6 Excercise 2.11

Let $(\cdot|\cdot)$ be an inner product. If a set of non-zero vectors $e_1, \ldots, e_k$ is orthogonal, i.e. $(e_i|e_j) = 0$ when $i \neq j$, show that they are linearly independent. Note that an orthonormal set (i.e. $(e_i|e_j) = \pm\delta_{ij}$) is just an orthogonal set in which the vectors have unit length.

Proof. Similar to the previous exercise, we write a linear relation for $e_1, \ldots, e_k$

$$\sum_{j=1}^{k} c^j e_j = 0. \tag{1.23}$$

We have the following inner product between e_i ($i = 1, 2, \ldots, k$) and the left hand side of the linear relation

$$\left(e_i \middle| \sum_{j=1}^{k} c^j e_j \right) = c^i (e_i|e_i) = 0. \tag{1.24}$$

Notice that $(e_i|e_i) > 0$ since $(\cdot|\cdot)$ is an inner product, so we must have $c^i = 0$ where $i = 1, 2, \ldots, k$. Hence $\{e_1, \ldots, e_k\}$ is linear independent. □

1.7 Exercise 2.12

Let $A, B \in M_n(\mathbb{C})$. Define $(\cdot|\cdot)$ on $M_n(\mathbb{C})$ by

$$(A|B) = \frac{1}{2}\operatorname{Tr}\big(A^\dagger B\big). \tag{1.25}$$

Check that this is indeed an inner product. Also check that the basis $\{I, \sigma_x, \sigma_y, \sigma_z\}$ for $H_2(\mathbb{C})$ is orthonormal with respect to this inner product.

Proof. Condition 1, linearity in the second argument

$$\begin{aligned}(A|cB) &= \frac{1}{2}\operatorname{Tr}\left(A^\dagger cB\right)\\ &= c\frac{1}{2}\operatorname{Tr}\left(A^\dagger B\right)\\ &= c(A|B).\end{aligned}$$

Condition 2, Hermiticity

$$\begin{aligned}\overline{(B|A)} &= \frac{1}{2}\operatorname{Tr}\left[(B^\dagger A)^\dagger\right]\\ &= \frac{1}{2}\operatorname{Tr}\left(A^\dagger B\right)\\ &= (A|B).\end{aligned}$$

Condition 4, positive-definiteness

$$(A|A) = \frac{1}{2}\operatorname{Tr}\left(A^\dagger A\right) = \frac{1}{2}\sum_{i,j}|a_{ij}|^2 > 0 \quad \text{if} \quad A \neq 0. \tag{1.26}$$

Condition 4 implies condition 3 (non-degeneracy), so this is indeed a inner product. As for the basis of $H_2(\mathbb{C})$, recall the property of Pauli matrices

$$\sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k + \delta_{ij}, \tag{1.27}$$

which can be derived from

$$[\sigma_i, \sigma_j]_- = 2i\epsilon_{ijk}\sigma_k, \quad [\sigma_i, \sigma_j]_+ = 2\delta_{ij}. \tag{1.28}$$

Hence

$$(\sigma_i|\sigma_j) = \frac{1}{2}\operatorname{Tr}(\sigma_i\sigma_j) = \delta_{ij}. \tag{1.29}$$

And it is easy to show

$$(I|\sigma_j) = 0, \quad (I, I) = 1. \tag{1.30}$$

Therefore the basis $\{I, \sigma_x, \sigma_y, \sigma_z\}$ for $H_2(\mathbb{C})$ is orthonormal with respect to this inner product. □

1.8 Exercise 2.13

Let $v = (x, y, z, t)$ be an arbitrary non-zero vector in $\mathbb{R}^4$. Show that η is non-degenerate by finding another vector w such that $\eta(v, w) \neq 0$.

Proof. Since $v = (x, y, z, t)$ is non-zero, there exists at least one non-zero component in v. Without losing generality, say this non-zero component is space component x. Let $w = (x, 0, 0, 0)$, so we have

$$\eta(v, w) = x^2 > 0, \tag{1.31}$$

and it is easy to see that if the non-zero component is time, then we can set $w = (0, 0, 0, t)$ such that $\eta(v, w) = -t^2 < 0$. In either way we find another vector w such that $\eta(v, w) \neq 0$.

We can also prove its contrapositive proposition. If $\eta(v, w) = 0$ for any $w \in \mathbb{R}^4$, then $v = 0$.
Suppose $w = (1, 0, 0, 0)$, then $\eta(v, w) = x = 0$. Similarly we can prove all components of v are zero. $\square$

1.9 Exercise 2.16

Use the non-degeneracy of $(\cdot|\cdot)$ to show that L is one-to-one, i.e. that $L(v) = L(w) \implies v = w$. Combine this with the argument used in Exercise 2.7 to show that L is onto as well.

Proof. If $L(v) = L(w)$, then

$$L(u) = L(v - w) = L(v) - L(w) = 0, \tag{1.32}$$

where $u = v - w$. This is because L is conjugate linear. According to the definition of metric dual L, for any vector $x \in V$, we have

$$L(u)x = (u|x) = 0 \tag{1.33}$$

because $L(u) = 0$. Then by the non-degeneracy of $(\cdot|\cdot)$, (1.33) implies $u = 0$. Hence we prove $v = w$ if $L(v) = L(w)$, which means L is one-to-one.
We can employ similar argument shown in Excercise 2.6 and 2.7 to prove L is invertible, because V and V^* have the same dimensions and the metric dual of all basis vector $e_i \in V$ also forms a basis $\{L(e_i)\} \subset V^*$. $\square$

1.10 Inner product with a zero vector must be zero

Suppose we have an inner product $(\cdot|\cdot)$ defined in a vector space V, then the linearity implies the inner product with zero vector is zero. Explicitly if $c \in C$ is an arbitrary non-zero scalar in field C, we have

$$(v|0) = (v|c0) = c(v|0). \tag{1.34}$$

Suppose $(v|0) \neq 0$, then (1.34) means $c = 1$ which contradicts our assumption that c is an arbitrary scalar. Hence $(v|0) = 0$.

Actually this result is general for all linear or conjugate linear forms.