# Notes

## Something

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# Caprice

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## 1 Fall 2018

This is my note for some non-trivial but not systematic problems which involves some interesting physics or maths.

Section 1. Fall 2018

### 1.1 Walkway equilibrium

Suppose the mass of the objects attached to each end of the rope are $m_1$ and $m_2$, The angles between each segment of the rope, bended by the central object which has mass $M$, with the horizontal plane are $\theta$ and $\phi$. The distance between two pulleys is $L$, and what we want to know is the vertical displacement $d$ of the central object. Thus we can obtaind the equations for $d$ when the system is at equilibrium.

$$L = d(\cot\theta + \cot\phi), \tag{1.1}$$
$$m_1 g\cos\theta = m_2\cos\phi, \tag{1.2}$$
$$m_1 g\sin\theta + m_2 g\sin\phi = Mg, \tag{1.3}$$

From (1.2), we have $\cos\phi = \frac{m_1}{m_2}\cos\theta$, thus (1.3) can be written as

$$m_1\sin\theta + m_2\sqrt{1 - \frac{m_1^2}{m_2^2}(1-\sin^2\theta)} = M, \tag{1.4}$$

such that we can solve for $\sin\theta$ and $\cos\theta$

$$\sin\theta = \frac{M^2 + m_1^2 - m_2^2}{2Mm_1}, \tag{1.5}$$
$$\cos\theta = \sqrt{1-\sin^2\theta} = \frac{1}{2Mm_1}\sqrt{[(m_1+m_2)^2 - M^2][M^2 - (m_1-m_2)^2]}, \tag{1.6}$$
$$\cot\theta = \frac{\sqrt{[(m_1+m_2)^2 - M^2][M^2 - (m_1-m_2)^2]}}{M^2 + m_1^2 - m_2^2}, \tag{1.7}$$

together with $\sin\phi$ and $\cos\phi$

$$\cos\phi = \frac{m_1}{m_2}\cos\theta = \frac{1}{2Mm_2}\sqrt{[(m_1+m_2)^2 - M^2][M^2 - (m_1-m_2)^2]}, \tag{1.8}$$
$$\sin\phi = \sqrt{1-\cos^2\phi} = \frac{M^2 - m_1^2 + m_2^2}{2Mm_2}, \tag{1.9}$$
$$\cot\phi = \frac{\sqrt{[(m_1+m_2)^2 - M^2][M^2 - (m_1-m_2)^2]}}{M^2 - m_1^2 + m_2^2}. \tag{1.10}$$

Therefore we can plug into (1.1) and obtain the expression of $d$ as follows

$$d = \frac{L[M^4 - (m_1^2 - m_2^2)^2]}{2M^2\sqrt{[(m_1+m_2)^2 - M^2][M^2 - (m_1-m_2)^2]}}. \tag{1.11}$$

The equilibrium condition in this case is

$$|m_1 - m_2| < M < (m_1 + m_2). \tag{1.12}$$

such that the argument under the square root is positive. Also we can easily check that if $m_1 = m_2 = m$ then this result reduces to our former result

$$d = \frac{LM}{2\sqrt{4m^2 - M^2}}. \tag{1.13}$$

## 1.2 A derivation of Gamma function from Fourier transfrom

It is well-known that $\Gamma(n+1) = n!$ for any natural number $n \in \mathbb{N}$. It is natural to ask what is $\Gamma(x)$ for any real number $x \geq 1$. Our purpose is to show that Gamma function can be express as an integral

$$\Gamma(x) = \int_0^\infty \mathrm{d}x\, t^{x-1}\mathrm{e}^t, \tag{1.14}$$

given that

$$\Gamma(x+1) = x\Gamma(x), \tag{1.15}$$

which is the most essential property and motivation to define the Gamma function.

Using Talor expansion we can easily show that

$$f(x+\Delta x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(\Delta x)^n = \exp\left(\Delta x \frac{\mathrm{d}}{\mathrm{d}x}\right) f(x) \tag{1.16}$$

where $\Delta x$ is a constant of translation. Therefore $\Gamma(x+1) = \mathrm{e}^{\frac{\mathrm{d}}{\mathrm{d}x}}\Gamma(x)$, and we can rewrite (1.17) as

$$\mathrm{e}^{\frac{\mathrm{d}}{\mathrm{d}x}}\Gamma(x) = x\Gamma(x). \tag{1.17}$$

Consider doing Fourier transform of (1.17), such that $\frac{\mathrm{d}}{\mathrm{d}x} \to i\omega$, $x \to i\frac{\mathrm{d}}{\mathrm{d}\omega}$, $\Gamma(x) \to \tilde{\Gamma}(\omega)$, and

$$\tilde{\Gamma}(\omega) = \mathcal{F}[\Gamma(x)] = \int_{-\infty}^{\infty} \mathrm{d}x\, \Gamma(x)\mathrm{e}^{-i\omega x}, \tag{1.18}$$

$$\Gamma(x) = \mathcal{F}^{-1}[\tilde{\Gamma}(\omega)] = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\tilde{\Gamma}(\omega)\mathrm{e}^{i\omega x}, \tag{1.19}$$

$$\mathrm{e}^{i\omega}\tilde{\Gamma}(\omega) = i\frac{\mathrm{d}}{\mathrm{d}\omega}\tilde{\Gamma}(\omega). \tag{1.20}$$

Solve the above differential equation of $\tilde{\Gamma}(\omega)$ we find

$$\tilde{\Gamma}(\omega) = C\exp\left(-\mathrm{e}^{i\omega}\right), \tag{1.21}$$

$$\Gamma(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} C\exp\left(-\mathrm{e}^{i\omega}\right)\mathrm{e}^{i\omega x}. \tag{1.22}$$

However, (1.22) does not converge, since (1.21) is a nonzero periodic function.

To resolve this difficulty of convergence, we expand the domain of Gamma function to the complex plane, such that $\Gamma(z+1) = z\Gamma(z)$, where $z \in \mathbb{C}$. Consider a pure imaginary number $z = ix$, where $x \in \mathbb{R}$, we can rewrite the recursion relation (1.15) as

$$\mathrm{e}^{-i\frac{\mathrm{d}}{\mathrm{d}x}}\Gamma(ix) = ix\Gamma(ix) \tag{1.23}$$

Again using Fourier transform we have

$$e^{\omega}\mathcal{F}[\Gamma(ix)] = -\frac{d}{d\omega}\mathcal{F}[\Gamma(ix)], \tag{1.24}$$

where $\mathcal{F}[\Gamma(ix)]$ is the Fourier transform of $\Gamma(ix)$

$$\mathcal{F}[\Gamma(ix)] = \int_{-\infty}^{\infty} dx\, \Gamma(ix) e^{-i\omega x}. \tag{1.25}$$

Solve (1.24) we have

$$\mathcal{F}[\Gamma(ix)] = C\exp(-e^{\omega}), \tag{1.26}$$

$$\Gamma(ix) = \frac{C}{2\pi}\int_{-\infty}^{\infty} d\omega \exp(-e^{\omega})e^{i\omega x}. \tag{1.27}$$

Thus

$$\begin{aligned}\Gamma(z) &= \frac{C}{2\pi}\int_{-\infty}^{\infty} d\omega \exp(-e^{\omega})e^{\omega z} \\ &= \frac{C}{2\pi}\int_{-\infty}^{\infty} de^{\omega} \exp(-e^{\omega})e^{\omega(z-1)} \\ &= \frac{C}{2\pi}\int_{-\infty}^{\infty} dt\, t^{z-1}e^{-t}.\end{aligned}$$

To determine the constant $C$ we use the fact that $\Gamma(1) = 0! = 1$, thus $C/2\pi = 1$, and we obtain the final integral expression of Gamma function

$$\Gamma(z) = \int_{-\infty}^{\infty} dt\, t^{z-1}e^{-t} \tag{1.28}$$

where $z \in \mathbb{C}$.

## 1.3 Euler's reflection formula

In mathematics, a reflection formula or reflection relation for a function f is a relationship between $f(a-x)$ and $f(x)$. A famous relationship is Euler's reflection formula

**Proposition 1.1.**

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad z \notin \mathbb{Z}. \tag{1.29}$$

**Proof.** To prove this reflection formula, we first notice a relationship between Gamma function and Beta function

$$B(q,p) = \frac{\Gamma(q)\Gamma(p)}{\Gamma(q+p)}, \tag{1.30}$$

where

$$B(q,p) = \int_0^1 dt\, t^{q-1}(1-t)^{p-1}, \quad q,p \neq 0,-1,-2,\ldots \tag{1.31}$$

This can be shown by performing a variable transformation of $\Gamma(q)\Gamma(p)$

$$
\begin{aligned}
\Gamma(q)\Gamma(p) &= \int_0^\infty \mathrm{d}u\, \mathrm{e}^{-u} u^{q-1} \int_0^\infty \mathrm{d}v\, \mathrm{e}^{-v} v^{p-1} \\
u = zt, v = z(1-t), \quad &= \int \mathrm{d}z\, \mathrm{d}t \left| \frac{\partial(u,v)}{\partial(z,t)} \right| \mathrm{e}^{-z} (zt)^{q-1} [z(1-t)]^{p-1} \\
&= \int_0^\infty \mathrm{d}z\, z^{q+p-1} \mathrm{e}^{-z} \int_0^1 \mathrm{d}t\, t^{q-1} (1-t)^{p-1} \\
&= \Gamma(q+p) B(q,p).
\end{aligned}
$$

Therefore

$$
\Gamma(z)\Gamma(1-z) = B(z, 1-z) = \int_0^1 \mathrm{d}t\, t^{z-1} (1-t)^{-z}. \tag{1.32}
$$

In order to prove Proposition 1.1, we only need to prove

$$
\int_0^1 \mathrm{d}t\, t^{z-1} (1-t)^{-z} = \frac{\pi}{\sin \pi z}. \tag{1.33}
$$

Perform a variable substitution $t \to \frac{x}{1+x}$, such that $\mathrm{d}t = \mathrm{d}x\,/(1+x)^2$ and

$$
\int_0^1 \mathrm{d}t\, t^{z-1} (1-t)^{-z} = \int_0^\infty \mathrm{d}x\, \frac{x^{z-1}}{1+x} \tag{1.34}
$$

Consider the following integral

$$
\int_0^\infty \mathrm{d}x\, \frac{x^{\alpha-1}}{x + \mathrm{e}^{i\phi}}, \quad 0 < \alpha < 1, \quad -\pi < \phi < \pi. \tag{1.35}
$$

□

## 1.4 $\chi^2$ distribution with $(n-1)$ degrees of freedom

Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables with $\mathrm{E}(Y_i) = \mu$ and $V(Y_i) = \sigma^2$ for $i = 1, 2, \ldots, n$. Let

$$
U_1 = \sum_{i=1}^n a_i Y_i \quad \text{and} \quad U_2 = \sum_{i=1}^n b_i Y_i, \tag{1.36}
$$

where $a_1, a_2, \ldots, a_n$, and $b_1, b_2, \ldots, b_n$ are constants. $U_1$ and $U_2$ are said to be orthogonal if $\mathrm{Cov}(U_1, U_2) = 0$.

1. Show that $U_1$ and $U_2$ are orthogonal if and only if $\sum_{i=1}^n a_i b_i = 0$.

   **Proof.** Sufficiency: If $\sum_{i=1}^n a_i b_i = 0$, then

$$
\begin{aligned}
\mathrm{Cov}(U_1, U_2) &= \mathrm{E}\left[\left(\sum_{i=1}^n a_i Y_i\right)\left(\sum_{j=1}^n b_j Y_j\right)\right] - \mathrm{E}\left(\sum_{i=1}^n a_i Y_i\right) \mathrm{E}\left(\sum_{j=1}^n b_j Y_j\right) \\
&= \sum_{i=1}^n a_i b_i [\mathrm{E}(Y_i^2) - \mathrm{E}(Y_i)^2] + \sum_{i \neq j} a_i b_j [\mathrm{E}(Y_i Y_j) - \mathrm{E}(Y_i)\,\mathrm{E}(Y_j)] \\
&= \sigma^2 \sum_{i=1}^n a_i b_i = 0.
\end{aligned}
$$

   Thus $U_1$ and $U_2$ are orthogonal. Necessity: If $U_1$ and $U_2$ are orthogonal, $\mathrm{Cov}(U_1, U_2) = \sigma^2 \sum_{i=1}^n a_i b_i = 0$, then we must have $\sum_{i=1}^n a_i b_i = 0$ if $\sigma \neq 0$. □

2. Show that if each component of independent random variables $Y_1, Y_2, \ldots, Y_n$ is normally distributed, then any linear combination $U = a_1Y_1 + a_2Y_2 + \cdots + a_nY_n$ is normally distributed.

   **Proof.** Proof by induction. We begin with two independent random variables $X_1 = a_1Y_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 = a_2Y_2 \sim N(\mu_2, \sigma_2^2)$, with $\mu_i = a_i\mu$ and $\sigma_i^2 = a_i^2\sigma^2$. Their sum $Z = a_1Y_1 + a_2Y_2$, which is a linear combination of $Y_1$ and $Y_2$. The characteristic function

   $$\varphi_Z(t) = \varphi_{X_1+X_2}(t) = \mathrm{E}(\mathrm{e}^{it(X_1+X_2)}) \tag{1.37}$$

   of the sum of two independent random variables is just the product of the characteristic functions of each random variable

   $$\varphi_{X_1+X_2}(t) = \varphi_{X_1}(t)\varphi_{X_2}(t) = \mathrm{E}(\mathrm{e}^{itX_1})\,\mathrm{E}(\mathrm{e}^{itX_2}). \tag{1.38}$$

   The characteristic function of the normal distribution with expected value $\mu$ and variance $\sigma^2$ is

   $$\varphi(t) = \exp\big[it\mu - \tfrac{1}{2}\sigma^2t^2\big]. \tag{1.39}$$

   Thus

   $$\begin{aligned}\varphi_{X_1+X_2}(t) &= \exp\big[it\mu_1 - \tfrac{1}{2}\sigma_1^2t^2\big]\exp\big[it\mu_2 - \tfrac{1}{2}\sigma_2^2t^2\big]\\ &= \exp\big[it(\mu_1+\mu_2) - \tfrac{1}{2}(\sigma_1^2+\sigma_2^2)t^2\big].\end{aligned}$$

   This is the characteristic function of the normal distribution with expected value $(\mu_1+\mu_2)$ and variance $(\sigma_1^2+\sigma_2^2)$. Finally, recall that no two distinct distributions can both have the same characteristic function, so the distribution of $Z$ must be just this normal distribution.
   Similarly we can prove that $W = Z + X_3$, which is a linear combination of $Y_1, Y_2$ and $Y_3$, is also normally distributed. By induction, we prove that every linear combination of $Y_1, Y_2, \ldots, Y_n$ is normally distributed. □

3. Suppose, in addition, that $Y_1, Y_2, \ldots, Y_n$ have a multivariate normal distribution. Then $U_1$ and $U_2$ have a bivariate normal distribution. Show that $U_1$ and $U_2$ are independent if they are orthogonal.

   **Proof.** If the random vector $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)^\top$ have the multivariate normal distribution, then every linear combination of its components $U = a_1Y_1 + a_2Y_2 + \cdots + a_nY_n$ is normally distributed. By definition, since the linear combination of $U_1$ and $U_2$ is still a linear combination of $\mathbf{Y}$, thus $c_1U_1 + c_2U_2$ is normally distributed, and we say $U_1$ and $U_2$ have a bivariate normal distribution.
   In general the bivariate density function of two random variables $Y_1$ and $Y_2$ has the following form

   $$f(y_1, y_2) = \frac{\mathrm{e}^{-Q/2}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}, \quad -\infty < y_1, y_2 < \infty, \tag{1.40}$$

   where

   $$Q = \frac{1}{1-\rho^2}\left[\frac{(y_1-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(y_1-\mu_1)(y_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(y_2-\mu_2)^2}{\sigma_2^2}\right]. \tag{1.41}$$

By doing a tedious integral exercise, we find $\mathrm{Cov}(Y_1, Y_2) = \rho\sigma_1\sigma_2$. If $\mathrm{Cov}(Y_1, Y_2) = 0$, i.e. if $\rho = 0$, then

$$f(y_1, y_2) = g(y_1)h(y_2), \tag{1.42}$$

where $g(y_1)$ is a nonnegative function of $y_1$ alone and $h(y_2)$ is a nonnegative function of $y_2$ alone. Therefore, if $U_1$ and $U_2$ are orthogonal, i.e. $\mathrm{Cov}(U_1, U_2) = 0$, then $U_1$ and $U_2$ are independent. Notice that in general $\mathrm{Cov}(U_1, U_2) = 0$ doesn't imply $U_1$ and $U_2$ are independent. It is only in the context of bivariate normal distribution that we have this conclusion. □

Suppose that $Y_1, Y_2, \dots, Y_n$ is a random sample from a normal distribution with mean $\mu$ and variance $\sigma^2$. The independence of $\sum_{i=1}^n (Y_i - \overline{Y})^2$ and $\overline{Y}$ can be shown as follows. Define an $n \times n$ matrix $A$ by

$$A = \begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{\sqrt{2\cdot 3}} & \frac{1}{\sqrt{2\cdot 3}} & \frac{-2}{\sqrt{2\cdot 3}} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{(n-1)n}} & \frac{1}{\sqrt{(n-1)n}} & & \cdots & & \frac{1}{\sqrt{(n-1)n}} & \frac{-(n-1)}{\sqrt{(n-1)n}} \end{bmatrix} \tag{1.43}$$

and notice that $A^\top A = I$, the identity matrix. Then,

$$\sum_{i=1}^n Y_i^2 = \mathbf{Y}^\top \mathbf{Y} = \mathbf{Y}^\top A^\top A \mathbf{Y}, \tag{1.44}$$

where $\mathbf{Y}$ is the vector of $Y_i$ values.

1. Show that

$$A\mathbf{Y} = \begin{bmatrix} \overline{Y}\sqrt{n} \\ U_1 \\ U_2 \\ \vdots \\ U_{n-1} \end{bmatrix}. \tag{1.45}$$

where $U_1, U_2, \dots, U_{n-1}$ are linear funcitions of $Y_1, Y_2, \dots, Y_n$. Thus,

$$\sum_{i=1}^n Y_i^2 = n\overline{Y}^2 + \sum_{i=1}^{n-1} U_i^2. \tag{1.46}$$

2. Show that the linear functions $\overline{Y}\sqrt{n}, U_1, U_2, \dots, U_{n-1}$ are pairwise orhogonal and hence independent under the normality assumption.

**Proof.** Notice that the coefficients of $\overline{Y}$ and $U_i$ are the elements of each row of matrix $A$, such that

$$A = \begin{bmatrix} \mathbf{a}_0^\top \\ \mathbf{a}_1^\top \\ \mathbf{a}_2^\top \\ \vdots \\ \mathbf{a}_{n-1}^\top \end{bmatrix}, \quad \overline{Y} = \mathbf{a}_0^\top \mathbf{Y}, \quad U_i = \mathbf{a}_i^\top \mathbf{Y}. \tag{1.47}$$

Since $\mathbf{a}_i^\top \mathbf{a}_j = \delta_{ij}$, $\overline{Y}\sqrt{n}, U_1, U_2, \ldots, U_{n-1}$ are pairwise orthogonal. In addition, $\mathbf{Y}$ has a multivariate normal distribution, because every linear combination of its components is normally distributed. Recall the previous exercise, if the multivariate normal distribution $\overline{Y}\sqrt{n}, U_1, U_2, \ldots, U_{n-1}$ are pairwise orthogonal, then they are independent random variables. □

3. Show that

$$\sum_{i=1}^{n}(Y_i - \overline{Y})^2 = \sum_{i=1}^{n-1} U_i^2 \tag{1.48}$$

and conclude that this quantity is independent of $\overline{Y}$.

**Proof.**

$$\begin{aligned}
\sum_{i=1}^{n}(Y_i - \overline{Y})^2 &= \sum_{i=1}^{n}(Y_i^2 - 2Y_i\overline{Y} + \overline{Y}^2) \\
&= \sum_{i=1}^{n} Y_i^2 - n\overline{Y}^2 \\
&= \mathbf{Y}^\top A^\top A\mathbf{Y} - n\overline{Y}^2 \\
&= \begin{bmatrix}\overline{Y}\sqrt{n}, \; U_1, \; U_2, \; \cdots, \; U_{n-1}\end{bmatrix} \begin{bmatrix}\overline{Y}\sqrt{n} \\ U_1 \\ U_2 \\ \vdots \\ U_{n-1}\end{bmatrix} - n\overline{Y}^2 \\
&= \sum_{i=1}^{n-1} U_i^2
\end{aligned}$$

Since $U_i$ is independent of $\overline{Y}$, $\sum_{i=1}^{n-1} U_i^2$ is also independet of $\overline{Y}$. □

4. Using the results of part (3), show that

$$\frac{\sum_{i=1}^{n}(Y_i - \overline{Y}^2)}{\sigma^2} = \frac{(n-1)S^2}{\sigma^2} \tag{1.49}$$

has a $\chi^2$ distribution with $(n-1)$ df.

**Proof.** First we show that $U_i \sim N(0, \sigma^2)$, and thus $Z_i = U_i/\sigma \sim N(0, 1)$, a standard normal distribution.

We have seen from the previous exercise that any linear combination of independent random variables with normal distribution is also normally distributed, so $U_i$ has normal distribution, with expected value

$$\mathrm{E}(U_i) = \mu\|\mathbf{a}_i\|_1 = 0, \tag{1.50}$$

and variance

$$\mathrm{Var}(U_i) = \sigma^2\|\mathbf{a}_i\|_2^2 = \sigma^2. \tag{1.51}$$

Next we prove that if each $Z_i$ is independent with standard normal distribution, then $\sum_{i=1}^{n} Z_i^2$ has a $\chi^2$ distribution with $n$ df.

The characteristic function of $Z_i^2$ is

$$\begin{aligned}\varphi_{Z_i^2}(t) &= \mathrm{E}\left(\mathrm{e}^{itZ_i^2}\right)\\ &= \int_{-\infty}^{\infty} \mathrm{d}z\, \frac{1}{\sqrt{2\pi}}\mathrm{e}^{itz^2}\mathrm{e}^{-z^2/2}\\ &= (1-2it)^{-1/2},\end{aligned}$$

and from the fact that $Z_i$ is independent with each other, the characteristic function of $V = \sum_{i=1}^{n} Z_i^2$ is the product of $n$ characteristic functions of $Z_i^2$

$$\varphi_V(t) = \prod_{i=1}^{n} \varphi_{Z_i^2}(t) = (1-2it)^{-n/2} \tag{1.52}$$

Because characteristic functions are unique, $V$ has a $\chi^2$ distribution with $n$ degrees of freedom.

Finally we prove that $\sum_{i=1}^{n}(Y_i - \overline{Y}^2)/\sigma^2$ has a $\chi^2$ distribution with $n$ df. We rewrite the (1.49) as follows

$$\frac{\sum_{i=1}^{n}(Y_i - \overline{Y}^2)}{\sigma^2} = \frac{\sum_{i=1}^{n-1} U_i^2}{\sigma^2} = \sum_{i=1}^{n-1} Z_i^2. \tag{1.53}$$

Thus it has a $\chi^2$ distribution with $(n-1)$ degrees of freedom. $\square$