Notes

Classical Electrodynamics

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PART

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Classical Mechanics

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1.1 Problem 1 section 11 in Landau

The energy of this system is

$$E = \frac{1}{2} m l^2 \dot{\phi}^2 - mgl \cos phi = -mgl \cos\phi_0, \tag{1.1}$$

where ϕ_0 is the maximum angle of motion. Separate variables

$$\mathrm{d}t = \sqrt{\tfrac{l}{2g}} \int \frac{\mathrm{d}\phi}{\sqrt{\cos\phi - \cos\phi_0}}, \tag{1.2}$$

By symmetry, the period of motion is four times the time from angle $\phi = 0$ to $\phi = \phi_0$.

$$\begin{aligned}
T &= 4\sqrt{\tfrac{l}{2g}} \int_0^{\phi_0} \frac{\mathrm{d}\phi}{\sqrt{\cos\phi - \cos\phi_0}} \\
&= 2\sqrt{\tfrac{l}{g}} \int_0^{\phi_0} \frac{\mathrm{d}\phi}{\sqrt{\sin^2 \tfrac{1}{2}\phi_0 - \sin^2 \tfrac{1}{2}\phi}} \\
(\sin\xi = \frac{\sin\tfrac{1}{2}\phi}{\sin\tfrac{1}{2}\phi_0}) &= 2\sqrt{\tfrac{l}{g}} \int_0^{\phi_0} \frac{\mathrm{d}\phi}{\sin\tfrac{1}{2}\phi_0 \sqrt{1-\sin^2\xi}} \\
&= 4\sqrt{\tfrac{l}{g}} \int_0^{\pi/2} \frac{\mathrm{d}\xi}{\sqrt{1-\sin^2\tfrac{1}{2}\phi_0 \sin^2\xi}} \\
&= 4\sqrt{l/g} K\big(\sin\tfrac{1}{2}\phi_0\big).
\end{aligned}$$

where we use $\cos\phi = 1 - 2\sin^2\frac{1}{2}\phi$ in the third step, and change the variable in the 4th step s.t.

$$\mathrm{d}\phi = \frac{2\sin\tfrac{1}{2}\phi_0 \cos\xi}{\cos\tfrac{1}{2}\phi}\,\mathrm{d}\xi = \frac{2\sin\tfrac{1}{2}\phi_0\cos\xi}{\sqrt{1-\sin^2\tfrac{1}{2}\phi_0\sin^2\xi}}\,\mathrm{d}\xi\,, \tag{1.3}$$

in the last step we use the definition of complete elliptic integral of the first kind

$$K(k) = \int_0^{\pi/2} \frac{\mathrm{d}x}{\sqrt{1-k^2\sin^2 x}}. \tag{1.4}$$

1.2 Problem 2 section 11 in Landau

The potential energy is

$$U = -U_0/\cosh^2\alpha x, \quad U_0 > 0. \tag{1.5}$$

The shape of this potential can be inferred by its limit

$$\begin{aligned}
\lim_{x\to 0} U(x) &= -U_0, \\
\lim_{x\to\pm\infty} U(x) &= 0^-,
\end{aligned}$$

which is like an attractive potential well centered at $x = 0$ with minimum $-U_0$, and approches zero when $x \to \pm\infty$. The total energy E satisfy

$$-U_0 < E < 0, \tag{1.6}$$

which means the particle is bounded by potential $U(x)$. The positive turning point is

$$x_t = \cosh^{-1}\sqrt{U_0/|E|}. \tag{1.7}$$

Hence the period is

$$\begin{aligned}
T &= 4\sqrt{m/2}\int_0^{x_t}\frac{\mathrm{d}x}{\sqrt{E+U_0/\cosh^2\alpha x}} \\
&= 2\sqrt{2m}\int_0^{x_t}\frac{\cosh\alpha x\,\mathrm{d}x}{\sqrt{U_0-|E|\cosh^2\alpha x}} \\
&= \frac{2\sqrt{2m}}{\alpha}\int\frac{\mathrm{d}\sinh\alpha x}{\sqrt{U_0-|E|\big(1+\sinh^2\alpha x\big)}} \\
&= \frac{2}{\alpha}\sqrt{\frac{2m}{|E|}}\int_0^1\frac{\mathrm{d}(\eta\sinh\alpha x)}{\sqrt{1-\eta^2\sinh^2\alpha x}} \\
&= \frac{2}{\alpha}\sqrt{\frac{2m}{|E|}}\int_0^1\frac{\mathrm{d}u}{\sqrt{1-u^2}} \\
&= \frac{2}{\alpha}\sqrt{\frac{2m}{|E|}}\arcsin u\Big|_0^1 \\
&= \frac{\pi}{\alpha}\sqrt{\frac{2m}{|E|}}
\end{aligned}$$

where

$$\eta = \sqrt{\tfrac{|E|}{U_0-|E|}} \tag{1.8}$$

1.3 Kepler's problem, section 15 in Landau

Given an attractive potential

$$U(r) = -\frac{\alpha}{r}, \tag{1.9}$$

The motion of the particle is in a plane, which is defined by its initial velocity and the centrifugal force. We can write its Lagrangian

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{\alpha}{r}. \tag{1.10}$$

There's a cyclic coordinate ϕ, so the angular momentum is conserved

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\frac{\partial L}{\partial\dot{\phi}}}_{M} = \frac{\mathrm{d}}{\mathrm{d}t}\big(mr^2\dot{\phi}\big) = \frac{\partial L}{\partial\phi} = 0, \tag{1.11}$$

where M is the angular momentum of the particle. This point can also be seen from the fact that $U(r)$ is spherically symmetric, so the angular momentum must be conserved. The energy of this particle is

$$E = \frac{1}{2}m\dot{r}^2 + \frac{M}{2mr^2} - \frac{\alpha}{r} = \frac{1}{2}m\dot{r}^2 + U_{\text{eff}}(r), \tag{1.12}$$

where the effective potential is

$$U_{\text{eff}}(r) = \frac{M}{2mr^2} - \frac{\alpha}{r}. \tag{1.13}$$

From this we have the differential relation between $\mathrm{d}t$ and $\mathrm{d}r$

$$\mathrm{d}t = \sqrt{\frac{m}{2}} \frac{\mathrm{d}r}{\sqrt{E - U_{\text{eff}}(r)}}. \tag{1.14}$$

Also using the definition of angular momentum we have

$$mr^2 \frac{\mathrm{d}\phi}{\mathrm{d}t} = M \Longleftrightarrow \mathrm{d}t = \frac{mr^2}{M}\,\mathrm{d}\phi\,, \tag{1.15}$$

Hence we obtain the differential equation for orbital

$$\mathrm{d}\phi = \sqrt{\frac{M^2}{2m}} \frac{\mathrm{d}r\,/r^2}{\sqrt{E - U_{\text{eff}}(r)}}. \tag{1.16}$$

Change variable $u = 1/r$ we have

$$\mathrm{d}u = -\,\mathrm{d}r\,/r^2, \quad U_{\text{eff}}(r) = \frac{M^2}{2m}u^2 - \alpha u, \tag{1.17}$$

so we have the following integral

$$\phi = -\sqrt{\tfrac{M^2}{2m}} \int \frac{\mathrm{d}u}{\sqrt{E + \alpha u - \frac{M^2}{2m}u^2}}. \tag{1.18}$$

Change the variable again to complete the square in the denorminator

$$y = u - \frac{\alpha m}{M^2}, \quad z = y\sqrt{\frac{\frac{M^2}{2m}}{E + \frac{\alpha^2 m}{2M^2}}}, \tag{1.19}$$

we have

$$\begin{aligned}
\phi &= -\sqrt{\tfrac{M^2}{2m}} \int \frac{\mathrm{d}y}{\sqrt{E + \frac{\alpha^2 m}{2M^2} - \frac{M^2}{2m}y^2}} \\
&= -\int \frac{\mathrm{d}z}{\sqrt{1 - z^2}} \\
&= \arccos z + \text{const.}.
\end{aligned}$$

This is the result in Landau.

1.4 Two-body problem

The Lagrangian of a two-body system in an inertial frame K is as follows

$$L = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - U(\mathbf{x}_1 - \mathbf{x}_2), \tag{1.20}$$

where $\mathbf{x}_{1,2}$ are the coordinates of $m_{1,2}$ respectively, and the potential only depends on the relative position of the two bodies $\mathbf{x}_1 - \mathbf{x}_2$. This problem can be simplified by changing the variables from $\mathbf{x}_{1,2}$ to $\mathbf{R}$ and $\mathbf{r}$, where $\mathbf{R}$ is the center of mass of the two bodies, and $\mathbf{r}$ is the relative position

$$\mathbf{R} = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{m_1 + m_2}, \quad \mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2. \tag{1.21}$$

or inversely

$$\mathbf{x}_1 = \mathbf{R} + \frac{m_2}{m_1 + m_2}\mathbf{r} = \mathbf{R} + \frac{\mu}{m_1}\mathbf{r}, \tag{1.22}$$

$$\mathbf{x}_2 = \mathbf{R} - \frac{m_1}{m_1 + m_2}\mathbf{r} = \mathbf{R} - \frac{\mu}{m_2}\mathbf{r}, \tag{1.23}$$

where μ is the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{1.24}$$

Hence after a little calculation, we can rewrite our Lagrangian as follows

$$L = \frac{1}{2}(m_1 + m_2)V^2 + \frac{1}{2}\mu v^2 - U(\mathbf{r}), \tag{1.25}$$

where $\mathbf{V} = \dot{\mathbf{R}}$, and $\mathbf{v} = \dot{\mathbf{r}}$. If there's no external force, the total momentum $\mathbf{P} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = (m_1 + m_2)\mathbf{V}$ of this system is conserved, i.e. $\mathbf{V}$ is a constant vector. One remark is that this Lagrangian is still in the reference frame K, and all we did is changing the variables from $\mathbf{x}_{1,2}$ to $\mathbf{R}$ and $\mathbf{r}$. From this Lagrangian, we can interpret the original two-body problem as a free particle with mass $(m_1 + m_2)$ and the second particle with mass μ inside a potential $U(\mathbf{r})$.

If we go to the center-of-mass frame K' of the two bodies, then all we need to modify for the above Lagrangian is to set $\mathbf{R}$ as the origin, and the speed of reference frame K' relative to K is $\mathbf{V}$, which is a constant vector as mentioned before. Therefore K' is also an inertial frame. In this frame the coordinates of the two particles are colinear in $\mathbf{r}$

$$\mathbf{x}'_1 = \frac{m_2}{m_1 + m_2}\mathbf{r} = \frac{\mu}{m_1}\mathbf{r}, \tag{1.26}$$

$$\mathbf{x}'_2 = \frac{m_1}{m_1 + m_2}\mathbf{r} = -\frac{\mu}{m_2}\mathbf{r}, \tag{1.27}$$

For the velocity in K' we have

$$\mathbf{V} = \mathbf{V}' + \mathbf{V} \Longrightarrow \mathbf{V}' = 0, \tag{1.28}$$

$$\mathbf{v} = \mathbf{v}', \tag{1.29}$$

because $\mathbf{r}' = \mathbf{r}$ is invariant under this reference frame transformation. The Lagragian in K'

$$\begin{aligned} L' &= \frac{1}{2}(m_1 + m_2)V'^2 + \frac{1}{2}\mu v'^2 - U(\mathbf{r}') \\ &= \frac{1}{2}\mu v^2 - U(\mathbf{r}). \end{aligned}$$

Therefore we conclude that the center of the potential $U(\mathbf{r})$ coincides with center of mass of the two-body system. The motions of the two bodies in frame K' can be think of an effective one-body motion $\mathbf{r}(t)$, and their coordinates are given by (1.26) and (1.27).