Notes

Analysis

Yi Huang[1]

University of Minnesota

E-mail: yihphysics@gmail.com

[1] *My Personal Website*

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This is my note for analysis. The textbook I use are Stein's classic four volumes on analysis.

1.1 Cauchy criterion

Definition 1.1. If (a_n) is a sequence, then $\lim_{n\to\infty} a_n = L$ means that for every $\varepsilon > 0$ there exists a corresponding $N \in \mathbb{N}^+$ such that if $n \geq N$, then $|a_n - L| < \varepsilon$. If this limit exists, then we say that the sequence (a_n) converges, and if this limit doesn't exists then we say the sequence (a_n) diverges.

Proposition 1.1. A converging sequence of $\mathbb{R}$ has a unique limit.

Proof. Suppose both L_1 and L_2 are limits of (a_n) and $L_1 \neq L_2$, which means for any $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}^+$ such that

$$|a_n - L_1|, \quad \text{for every} \quad n \geq N_1; \tag{1.1}$$

and for any $\varepsilon > 0$, there exists $N_2 \in \mathbb{N}^+$ such that

$$|a_n - L_2|, \quad \text{for every} \quad n \geq N_2. \tag{1.2}$$

Since $L_1 \neq L_2$, we can choose $\epsilon = |L_1 - L_2|/4$, $N = \max\{N_1, N_2\}$, then

$$\begin{aligned} |L_1 - L_2| &= |(a_n - L_2) + (L_1 - a_n)| \\ &\leq |a_n - L_1| + |a_n - L_2| \\ &\leq 2\varepsilon = |L_1 - L_2|/2, \quad \text{for every} \quad n \geq N, \end{aligned}$$

which can be true only if $L_1 = L_2$, and this contradicts to the assumption $L_1 \neq L_2$. Therefore we prove $L_1 = L_2$ by contradiction. $\square$

Definition 1.2. A sequence (a_n) is said to be a Cauchy sequence iff for any $\varepsilon > 0$, there exists $N \geq \mathbb{N}^+$, such that

$$|a_n - a_m| < \varepsilon, \tag{1.3}$$

for all $n, m \geq N$. In other words, a Cauchy sequence is one inwhich the terms eventually cluster together.

Theorem 1.1. (Cauchy Criterion.) A sequence is Cauchy iff it converges.

Proof. Sufficiency: We need to prove if a sequence converges, it is Cauchy. Given any $\varepsilon > 0$. Let

$$L := \lim_{n\to\infty} a_n. \tag{1.4}$$

There exists N such that for all $n \geq N$,

$$|a_n - L| < \frac{\varepsilon}{2}. \tag{1.5}$$

Thus, for all $m, n \geq N$,

$$|a_n - a_m| = |(a_n - L) + (L - a_m)| \leq |a_n - L| + |a_m - L| \leq \varepsilon. \tag{1.6}$$

Necessity: We need to show a Cauchy sequence must converge. The proof proceeds in several steps, which we isolate and prove subsequently.

Step 1: Since (a_n) is Cauchy, it must be bounded.

Step 2: Since (a_n) is bounded, it has a convergent subsequence (a_{n_k}). Let $x := \lim_{k\to\infty} a_{n_k}$.

Step 3: Show that $\lim_{n\to\infty} = x$.

□

Proposition 1.2. (Step 1 in the proof of theorem 1.1.) If a sequence (a_n) is Cauchy, then it is bounded.

Proof. There exits N such that $|a_n - a_m| < 1$ for all $n, m \geq N$. In particular, we have $|a_n - a_m| < 1$ for all $n \geq N$, whence

$$a_n \in (a_N - 1, a_N + 1) \tag{1.7}$$

for all $n \geq N$. Thus $\{a_n : n \geq N\}$ is bounded. Also, $\{a_n : n < N\}$ is bounded since it is finite. We conclude that the entire range of the sequence (a_n) is bounded. □

Before we prove step 2 of theorem 1.1, we first prove three theorems as lemmas: monotone subsequence theorem, boundedness of convergent sequences, and monotone convergence theorem.

Lemma 1.2. (Monotone Subsequence Theorem.) Every sequence has a momotone subsequence.

Proof. Let (a_n) denote a sequence. We call a term a_k a peak iff $a_k \geq a_m$ for all $m > k$. There are two cases:

1. There are infinitely many peaks.

2. There are finitely many peaks.

In the first case, the subsequence consisting of the peaks forms a monotonically decreasing sequence. In the second case, there exists some largest M such that a_M is a peak. Let $m_1 = M + 1$. Since a_{m_1} isn't a peak, there exists $m_2 > m_1$ such that $a_{m_1} < a_{m_2}$. Since a_{m_2} isn't a peak, there exists $m_3 > m_2$ such that $a_{m_2} < a_{m_3}$. We can continue this process indefinitely, thus creating a monotonically increasing subsequence (a_{m_k}). □

Lemma 1.3. (Boundedness of Convergent Sequences.) If (a_n) is a sequence of real numbers that is convergent to $L \in \mathbb{R}$

$$\lim_{n\to\infty} a_n = L, \tag{1.8}$$

then (a_n) is also bounded.

Proof. Let $\varepsilon_0 = 1$. There exists $N \in \mathbb{N}^+$ such that if $n \geq N$ then $|a_n - L| < \varepsilon = 1$. Equivalently, $L - 1 < a_n < L + 1$ for all $n \geq N$, $\{a_n : n \geq N\}$ is bounded. Also $\{a_n : n \leq N\}$ is bounded since it is finite.

It is important to recongnize in the theorem above that we could no t have just take the maximum value from the start because we cannot take the maximum of an infinite set, only a finite set.
It is also important to note that the converse of this theorem is not always true, that is just because a sequence (a_n) is bounded does not imply (a_n) to be convergent. For example, the sequence $((-1)^n) = (-1, 1, -1, 1, \dots)$ is bounded, however, clearly $((-1)^n)$ does not converge. $\square$

Lemma 1.4. (Monotone Convergence Theorem.) If (a_n) is a monotone sequence of real numbers, then (a_n) is convergent iff (a_n) is bounded.

Proof. Let (a_n) be a monotone sequence.
Necessity: If (a_n) is convergent, then by the lemma 1.3 we just prove above, (a_n) is bounded.
Sufficiency: If (a_n) is bounded, there are two cases to consider.
Case 1. Suppose that (a_n) is an increasing sequence that is there exists $M \in \mathbb{R}$ such that for any $n \in \mathbb{N}^+$ we have $a_n \leq M$. By the completeness property of the real numbers, $\{a_n\}$ has a supremum in $\mathbb{R}$, call it $L = \sup\{a_n\}$. Let $\varepsilon > 0$ be given. Since L is the supremum of $\{a_n\}$, then $L - \varepsilon$ is not an upper bound to $\{a_n\}$ and so there exists N such that $L - \varepsilon < a_N$. Since (a_n) is an increasing sequence, then for all $n \geq N$, we have $a_N \leq a_n$, and so

$$L - \varepsilon < a_N \leq a_n \leq L < L + \varepsilon. \tag{1.9}$$

Omitting the unnecessary parts of the inequality, we see that for $n \geq N$ we have $L - \varepsilon < a_n < L + \varepsilon$, and so $|a_n - L| < \varepsilon$. By definition of limit, since $\varepsilon > 0$ is arbitrary, we have that $\lim_{n\to\infty} a_n = L$, that is (a_n) is convergent to L.
Case 2. Suppose that (a_n) is a decreasing sequence that is there exists $n \in \mathbb{R}$ such that for any $n \in \mathbb{N}^+$ we have $a_n \geq M$. By the completeness property of the real numbers, $\{a_n\}$ has a infimum in $\mathbb{R}$, call it $L = \inf\{a_n\}$. Let $\varepsilon > 0$ be given. Since L is the infimum of $\{a_n\}$, then $L + \varepsilon$ is not a lower bound to $\{a_n\}$ and so there exists N such that $a_N < L + \varepsilon$. Since (a_n) is an decreasing sequence, then for all $n \geq N$, we have $a_n \leq a_N$, and so

$$L - \varepsilon < a_n \leq a_N \leq L < L + \varepsilon. \tag{1.10}$$

Omitting the unnecessary parts of the inequality, we see that for $n \geq N$ we have $L - \varepsilon < a_n < L + \varepsilon$, and so $|a_n - L| < \varepsilon$. By definition of limit, since $\varepsilon > 0$ is arbitrary, we have that $\lim_{n\to\infty} a_n = L$, that is (a_n) is convergent to L.
In both cases, the bounded sequence (a_n) is convergent. $\square$

Corollary 1.1. If (a_n) is a monotone sequence that's bounded, then

$$\lim_{n\to\infty} a_n = \sup\{a_n\} \quad \text{if } (a_n) \text{ is increasing;} \tag{1.11}$$

$$\lim_{n\to\infty} a_n = \inf\{a_n\} \quad \text{if } (a_n) \text{ is decreasing.} \tag{1.12}$$

The proof of corollary 1.1 is done in lemma 1.4.

Proposition 1.3. (Step 2 in the proof of theorem 1.1.) Every bounded sequence has a convergent subsequence.

Proof. Suppose (a_n) is a bounded sequence. By the monotone subsequence theorem (lemma 1.2), it has a monotone subsequence (a_{n_k}). But then this subsequence is both bounded and monotone, whence it is convergent by the monotone convergence theorem (lemma 1.4). $\square$

Finally, we prove step 3 in the proof of theorem 1.1.

Proposition 1.4. (Step 3 in the proof of theorem 1.1.) If the subsequence of a Cauchy sequence converges to x then the sequence itself converges to x.

Proof. Let (a_n) be a Cauchy sequence, and let (a_{n_k}) be a convergent subsequence. Set $x = \lim_{k \to infty} a_{n_k}$. We wish to prove that $a_n \to x$ as $n \to infty$.
Given $\varepsilon > 0$, there exists K such that

$$|a_{n_k} - x| < \varepsilon/2 \quad \text{for all } k \geq K \quad . \tag{1.13}$$

Also, since (a_n) is Cauchy, there exists N such that

$$|a_n - a_m| < \varepsilon/2 \quad \text{for all } n, m \geq N. \tag{1.14}$$

Pick $l > K$ large enough so that $n_l > N$. Then for all $n > N$, we have

$$\begin{aligned} |a_n - x| &= |(a_n - a_{n_l}) + (a_{n_l} - x)| \\ &\leq |(a_n - a_{n_l})| + |(a_{n_l} - x)| \\ &< \varepsilon, \end{aligned}$$

which concludes the proof of theorem 1.1. $\square$