Notes

Something

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This is my note for some non-trivial but not systematic problems which involves some interesting physics or maths.

1.1 Walkway equilibrium

Suppose the mass of the objects attached to each end of the rope are m_1 and m_2, The angles between each segment of the rope, bended by the central object which has mass M, with the horizontal plane are θ and ϕ. The distance between two pulleys is L, and what we want to know is the vertical displacement d of the central object. Thus we can obtaind the equations for d when the system is at equilibrium.

$$L = d(\cot\theta + \cot\phi), \tag{1.1}$$

$$m_1 g\cos\theta = m_2\cos\phi, \tag{1.2}$$

$$m_1 g\sin\theta + m_2 g\sin\phi = Mg, \tag{1.3}$$

From (1.2), we have $\cos\phi = \frac{m_1}{m_2}\cos\theta$, thus (1.3) can be written as

$$m_1\sin\theta + m_2\sqrt{1 - \frac{m_1^2}{m_2^2}(1-\sin^2\theta)} = M, \tag{1.4}$$

such that we can solve for $\sin\theta$ and $\cos\theta$

$$\sin\theta = \frac{M^2 + m_1^2 - m_2^2}{2Mm_1}, \tag{1.5}$$

$$\cos\theta = \sqrt{1-\sin^2\theta} = \frac{1}{2Mm_1}\sqrt{[(m_1+m_2)^2 - M^2][M^2-(m_1-m_2)^2]}, \tag{1.6}$$

$$\cot\theta = \frac{\sqrt{[(m_1+m_2)^2 - M^2][M^2-(m_1-m_2)^2]}}{M^2 + m_1^2 - m_2^2}, \tag{1.7}$$

together with $\sin\phi$ and $\cos\phi$

$$\cos\phi = \frac{m_1}{m_2}\cos\theta = \frac{1}{2Mm_2}\sqrt{[(m_1+m_2)^2 - M^2][M^2-(m_1-m_2)^2]}, \tag{1.8}$$

$$\sin\phi = \sqrt{1-\cos^2\phi} = \frac{M^2 - m_1^2 + m_2^2}{2Mm_2}, \tag{1.9}$$

$$\cot\phi = \frac{\sqrt{[(m_1+m_2)^2 - M^2][M^2-(m_1-m_2)^2]}}{M^2 - m_1^2 + m_2^2}. \tag{1.10}$$

Therefore we can plug into (1.1) and obtain the expression of d as follows

$$d = \frac{L[M^4 - (m_1^2 - m_2^2)^2]}{2M^2\sqrt{[(m_1+m_2)^2 - M^2][M^2-(m_1-m_2)^2]}}. \tag{1.11}$$

The equilibrium condition in this case is

$$|m_1 - m_2| < M < (m_1 + m_2). \tag{1.12}$$

such that the argument under the square root is positive. Also we can easily check that if $m_1 = m_2 = m$ then this result reduces to our former result

$$d = \frac{LM}{2\sqrt{4m^2 - M^2}}. \tag{1.13}$$

1.2 A derivation of Gamma function from Fourier transfrom

It is well-known that $\Gamma(n+1) = n!$ for any natural number $n \in \mathbb{N}$. It is natural to ask what is $\Gamma(x)$ for any real number $x \geq 1$. Our purpose is to show that Gamma function can be express as an integral

$$\Gamma(x) = \int_0^\infty \mathrm{d}x\, t^{x-1}\mathrm{e}^t, \tag{1.14}$$

given that

$$\Gamma(x+1) = x\Gamma(x), \tag{1.15}$$

which is the most essential property and motivation to define the Gamma function.

Using Talor expansion we can easily show that

$$f(x+\Delta x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(\Delta x)^n = \exp\left(\Delta x \frac{\mathrm{d}}{\mathrm{d}x}\right) f(x) \tag{1.16}$$

where Δx is a constant of translation. Therefore $\Gamma(x+1) = \mathrm{e}^{\frac{\mathrm{d}}{\mathrm{d}x}}\Gamma(x)$, and we can rewrite (1.17) as

$$\mathrm{e}^{\frac{\mathrm{d}}{\mathrm{d}x}}\Gamma(x) = x\Gamma(x). \tag{1.17}$$

Consider doing Fourier transform of (1.17), such that $\frac{\mathrm{d}}{\mathrm{d}x} \to i\omega$, $x \to i\frac{\mathrm{d}}{\mathrm{d}\omega}$, $\Gamma(x) \to \tilde{\Gamma}(\omega)$, and

$$\tilde{\Gamma}(\omega) = \mathcal{F}[\Gamma(x)] = \int_{-\infty}^{\infty} \mathrm{d}x\, \Gamma(x)\mathrm{e}^{-i\omega x}, \tag{1.18}$$

$$\Gamma(x) = \mathcal{F}^{-1}[\tilde{\Gamma}(\omega)] = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\tilde{\Gamma}(\omega)\mathrm{e}^{i\omega x}, \tag{1.19}$$

$$\mathrm{e}^{i\omega}\tilde{\Gamma}(\omega) = i\frac{\mathrm{d}}{\mathrm{d}\omega}\tilde{\Gamma}(\omega). \tag{1.20}$$

Solve the above differential equation of $\tilde{\Gamma}(\omega)$ we find

$$\tilde{\Gamma}(\omega) = C\exp\left(-\mathrm{e}^{i\omega}\right), \tag{1.21}$$

$$\Gamma(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} C\exp\left(-\mathrm{e}^{i\omega}\right)\mathrm{e}^{i\omega x}. \tag{1.22}$$

However, (1.22) does not converge, since (1.21) is a nonzero periodic function.

To resolve this difficulty of convergence, we expand the domain of Gamma function to the complex plane, such that $\Gamma(z+1) = z\Gamma(z)$, where $z \in \mathbb{C}$. Consider a pure imaginary number $z = ix$, where $x \in \mathbb{R}$, we can rewrite the recursion relation (1.15) as

$$\mathrm{e}^{-i\frac{\mathrm{d}}{\mathrm{d}x}}\Gamma(ix) = ix\Gamma(ix) \tag{1.23}$$

Again using Fourier transform we have

$$e^{\omega}\mathcal{F}[\Gamma(ix)] = -\frac{d}{d\omega}\mathcal{F}[\Gamma(ix)], \tag{1.24}$$

where $\mathcal{F}[\Gamma(ix)]$ is the Fourier transform of $\Gamma(ix)$

$$\mathcal{F}[\Gamma(ix)] = \int_{-\infty}^{\infty} dx\, \Gamma(ix)e^{-i\omega x}. \tag{1.25}$$

Solve (1.24) we have

$$\mathcal{F}[\Gamma(ix)] = C\exp(-e^{\omega}), \tag{1.26}$$

$$\Gamma(ix) = \frac{C}{2\pi}\int_{-\infty}^{\infty} d\omega \exp(-e^{\omega})e^{i\omega x}. \tag{1.27}$$

Thus

$$\begin{aligned}
\Gamma(z) &= \frac{C}{2\pi}\int_{-\infty}^{\infty} d\omega \exp(-e^{\omega})e^{\omega z} \\
&= \frac{C}{2\pi}\int_{-\infty}^{\infty} de^{\omega} \exp(-e^{\omega})e^{\omega(z-1)} \\
&= \frac{C}{2\pi}\int_{-\infty}^{\infty} dt\, t^{z-1}e^{-t}.
\end{aligned}$$

To determine the constant C we use the fact that $\Gamma(1) = 0! = 1$, thus $C/2\pi = 1$, and we obtain the final integral expression of Gamma function

$$\Gamma(z) = \int_{-\infty}^{\infty} dt\, t^{z-1}e^{-t} \tag{1.28}$$

where $z \in \mathbb{C}$.

1.3 Euler's reflection formula

In mathematics, a reflection formula or reflection relation for a function f is a relationship between $f(a-x)$ and $f(x)$. A famous relationship is Euler's reflection formula

Proposition 1.1.

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad z \notin \mathbb{Z}. \tag{1.29}$$

Proof. To prove this reflection formula, we first notice a relationship between Gamma function and Beta function

$$B(q,p) = \frac{\Gamma(q)\Gamma(p)}{\Gamma(q+p)}, \tag{1.30}$$

where

$$B(q,p) = \int_0^1 dt\, t^{q-1}(1-t)^{p-1}, \quad q,p \neq 0,-1,-2,\ldots \tag{1.31}$$

This can be shown by performing a variable transformation of $\Gamma(q)\Gamma(p)$

$$\begin{aligned}
\Gamma(q)\Gamma(p) &= \int_0^\infty \mathrm{d}u\, \mathrm{e}^{-u} u^{q-1} \int_0^\infty \mathrm{d}v\, \mathrm{e}^{-v} v^{p-1} \\
u = zt, v = z(1-t), \quad &= \int \mathrm{d}z\, \mathrm{d}t \left| \frac{\partial(u,v)}{\partial(z,t)} \right| \mathrm{e}^{-z} (zt)^{q-1} [z(1-t)]^{p-1} \\
&= \int_0^\infty \mathrm{d}z\, z^{q+p-1} \mathrm{e}^{-z} \int_0^1 \mathrm{d}t\, t^{q-1} (1-t)^{p-1} \\
&= \Gamma(q+p) B(q,p).
\end{aligned}$$

Therefore

$$\Gamma(z)\Gamma(1-z) = B(z, 1-z) = \int_0^1 \mathrm{d}t\, t^{z-1} (1-t)^{-z}. \tag{1.32}$$

In order to prove Proposition 1.1, we only need to prove

$$\int_0^1 \mathrm{d}t\, t^{z-1}(1-t)^{-z} = \frac{\pi}{\sin \pi z}. \tag{1.33}$$

Perform a variable substitution $t \to \frac{x}{1+x}$, such that $\mathrm{d}t = \mathrm{d}x\,/(1+x)^2$ and

$$\int_0^1 \mathrm{d}t\, t^{z-1}(1-t)^{-z} = \int_0^\infty \mathrm{d}x\, \frac{x^{z-1}}{1+x} \tag{1.34}$$

Consider the following integral

$$\int_0^\infty \mathrm{d}x\, \frac{x^{\alpha-1}}{x+\mathrm{e}^{i\phi}}, \quad 0 < \alpha < 1, \quad -\pi < \phi < \pi. \tag{1.35}$$

□

1.4 χ^2 distribution with $(n-1)$ degrees of freedom

Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables with $\mathrm{E}(Y_i) = \mu$ and $V(Y_i) = \sigma^2$ for $i = 1, 2, \ldots, n$. Let

$$U_1 = \sum_{i=1}^n a_i Y_i \quad \text{and} \quad U_2 = \sum_{i=1}^n b_i Y_i, \tag{1.36}$$

where $a_1, a_2, \ldots, a_n$, and $b_1, b_2, \ldots, b_n$ are constants. U_1 and U_2 are said to be orthogonal if $\mathrm{Cov}(U_1, U_2) = 0$.

1. Show that U_1 and U_2 are orthogonal if and only if $\sum_{i=1}^n a_i b_i = 0$.

   Proof. Sufficiency: If $\sum_{i=1}^n a_i b_i = 0$, then

$$\begin{aligned}
\mathrm{Cov}(U_1, U_2) &= \mathrm{E}\left[\left(\sum_{i=1}^n a_i Y_i\right)\left(\sum_{j=1}^n b_j Y_j\right)\right] - \mathrm{E}\left(\sum_{i=1}^n a_i Y_i\right) \mathrm{E}\left(\sum_{j=1}^n b_j Y_j\right) \\
&= \sum_{i=1}^n a_i b_i [\mathrm{E}(Y_i^2) - \mathrm{E}(Y_i)^2] + \sum_{i \neq j} a_i b_j [\mathrm{E}(Y_i Y_j) - \mathrm{E}(Y_i)\, \mathrm{E}(Y_j)] \\
&= \sigma^2 \sum_{i=1}^n a_i b_i = 0.
\end{aligned}$$

   Thus U_1 and U_2 are orthogonal. Necessity: If U_1 and U_2 are orthogonal, $\mathrm{Cov}(U_1, U_2) = \sigma^2 \sum_{i=1}^n a_i b_i = 0$, then we must have $\sum_{i=1}^n a_i b_i = 0$ if $\sigma \neq 0$. □

2. Show that if each component of independent random variables $Y_1, Y_2, \ldots, Y_n$ is normally distributed, then any linear combination $U = a_1Y_1 + a_2Y_2 + \cdots + a_nY_n$ is normally distributed.

   Proof. Proof by induction. We begin with two independent random variables $X_1 = a_1Y_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 = a_2Y_2 \sim N(\mu_2, \sigma_2^2)$, with $\mu_i = a_i\mu$ and $\sigma_i^2 = a_i^2\sigma^2$. Their sum $Z = a_1Y_1 + a_2Y_2$, which is a linear combination of Y_1 and Y_2. The characteristic function

$$\varphi_Z(t) = \varphi_{X_1+X_2}(t) = \mathrm{E}(\mathrm{e}^{it(X_1+X_2)}) \tag{1.37}$$

   of the sum of two independent random variables is just the product of the characteristic functions of each random variable

$$\varphi_{X_1+X_2}(t) = \varphi_{X_1}(t)\varphi_{X_2}(t) = \mathrm{E}(\mathrm{e}^{itX_1})\,\mathrm{E}(\mathrm{e}^{itX_2}). \tag{1.38}$$

   The characteristic function of the normal distribution with expected value μ and variance σ^2 is

$$\varphi(t) = \exp\bigl[it\mu - \tfrac{1}{2}\sigma^2t^2\bigr]. \tag{1.39}$$

   Thus

$$\begin{aligned}\varphi_{X_1+X_2}(t) &= \exp\bigl[it\mu_1 - \tfrac{1}{2}\sigma_1^2t^2\bigr]\exp\bigl[it\mu_2 - \tfrac{1}{2}\sigma_2^2t^2\bigr] \\ &= \exp\bigl[it(\mu_1+\mu_2) - \tfrac{1}{2}(\sigma_1^2+\sigma_2^2)t^2\bigr].\end{aligned}$$

   This is the characteristic function of the normal distribution with expected value $(\mu_1+\mu_2)$ and variance $(\sigma_1^2+\sigma_2^2)$. Finally, recall that no two distinct distributions can both have the same characteristic function, so the distribution of Z must be just this normal distribution.
   Similarly we can prove that $W = Z + X_3$, which is a linear combination of Y_1, Y_2 and Y_3, is also normally distributed. By induction, we prove that every linear combination of $Y_1, Y_2, \ldots, Y_n$ is normally distributed. □

3. Suppose, in addition, that $Y_1, Y_2, \ldots, Y_n$ have a multivariate normal distribution. Then U_1 and U_2 have a bivariate normal distribution. Show that U_1 and U_2 are independent if they are orthogonal.

   Proof. If the random vector $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)^\top$ have the multivariate normal distribution, then every linear combination of its components $U = a_1Y_1 + a_2Y_2 + \cdots + a_nY_n$ is normally distributed. By definition, since the linear combination of U_1 and U_2 is still a linear combination of $\mathbf{Y}$, thus $c_1U_1 + c_2U_2$ is normally distributed, and we say U_1 and U_2 have a bivariate normal distribution.
   In general the bivariate density function of two random variables Y_1 and Y_2 has the following form

$$f(y_1, y_2) = \frac{\mathrm{e}^{-Q/2}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}, \quad -\infty < y_1, y_2 < \infty, \tag{1.40}$$

   where

$$Q = \frac{1}{1-\rho^2}\left[\frac{(y_1-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(y_1-\mu_1)(y_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(y_2-\mu_2)^2}{\sigma_2^2}\right]. \tag{1.41}$$

By doing a tedious integral exercise, we find $\mathrm{Cov}(Y_1, Y_2) = \rho\sigma_1\sigma_2$. If $\mathrm{Cov}(Y_1, Y_2) = 0$, i.e. if $\rho = 0$, then

$$f(y_1, y_2) = g(y_1)h(y_2), \tag{1.42}$$

where $g(y_1)$ is a nonnegative function of y_1 alone and $h(y_2)$ is a nonnegative function of y_2 alone. Therefore, if U_1 and U_2 are orthogonal, i.e. $\mathrm{Cov}(U_1, U_2) = 0$, then U_1 and U_2 are independent. Notice that in general $\mathrm{Cov}(U_1, U_2) = 0$ doesn't imply U_1 and U_2 are independent. It is only in the context of bivariate normal distribution that we have this conclusion. □

Suppose that $Y_1, Y_2, \dots, Y_n$ is a random sample from a normal distribution with mean μ and variance σ^2. The independence of $\sum_{i=1}^n (Y_i - \overline{Y})^2$ and $\overline{Y}$ can be shown as follows. Define an $n \times n$ matrix A by

$$A = \begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{\sqrt{2\cdot 3}} & \frac{1}{\sqrt{2\cdot 3}} & \frac{-2}{\sqrt{2\cdot 3}} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{(n-1)n}} & \frac{1}{\sqrt{(n-1)n}} & & \cdots & & \frac{1}{\sqrt{(n-1)n}} & \frac{-(n-1)}{\sqrt{(n-1)n}} \end{bmatrix} \tag{1.43}$$

and notice that $A^\top A = I$, the identity matrix. Then,

$$\sum_{i=1}^n Y_i^2 = \mathbf{Y}^\top \mathbf{Y} = \mathbf{Y}^\top A^\top A \mathbf{Y}, \tag{1.44}$$

where $\mathbf{Y}$ is the vector of Y_i values.

1. Show that

$$A\mathbf{Y} = \begin{bmatrix} \overline{Y}\sqrt{n} \\ U_1 \\ U_2 \\ \vdots \\ U_{n-1} \end{bmatrix}. \tag{1.45}$$

where $U_1, U_2, \dots, U_{n-1}$ are linear funcitions of $Y_1, Y_2, \dots, Y_n$. Thus,

$$\sum_{i=1}^n Y_i^2 = n\overline{Y}^2 + \sum_{i=1}^{n-1} U_i^2. \tag{1.46}$$

2. Show that the linear functions $\overline{Y}\sqrt{n}, U_1, U_2, \dots, U_{n-1}$ are pairwise orhogonal and hence independent under the normality assumption.

Proof. Notice that the coefficients of $\overline{Y}$ and U_i are the elements of each row of matrix A, such that

$$A = \begin{bmatrix} \mathbf{a}_0^\top \\ \mathbf{a}_1^\top \\ \mathbf{a}_2^\top \\ \vdots \\ \mathbf{a}_{n-1}^\top \end{bmatrix}, \quad \overline{Y} = \mathbf{a}_0^\top \mathbf{Y}, \quad U_i = \mathbf{a}_i^\top \mathbf{Y}. \tag{1.47}$$

Since $\mathbf{a}_i^\top \mathbf{a}_j = \delta_{ij}$, $\overline{Y}\sqrt{n}, U_1, U_2, \ldots, U_{n-1}$ are pairwise orthogonal. In addition, $\mathbf{Y}$ has a multivariate normal distribution, because every linear combination of its components is normally distributed. Recall the previous exercise, if the multivariate normal distribution $\overline{Y}\sqrt{n}, U_1, U_2, \ldots, U_{n-1}$ are pairwise orthogonal, then they are independent random variables. □

3. Show that

$$\sum_{i=1}^{n}(Y_i - \overline{Y})^2 = \sum_{i=1}^{n-1} U_i^2 \tag{1.48}$$

and conclude that this quantity is independent of $\overline{Y}$.

Proof.

$$\begin{aligned}
\sum_{i=1}^{n}(Y_i - \overline{Y})^2 &= \sum_{i=1}^{n}(Y_i^2 - 2Y_i\overline{Y} + \overline{Y}^2) \\
&= \sum_{i=1}^{n} Y_i^2 - n\overline{Y}^2 \\
&= \mathbf{Y}^\top A^\top A \mathbf{Y} - n\overline{Y}^2 \\
&= \begin{bmatrix} \overline{Y}\sqrt{n}, & U_1, & U_2, & \cdots, & U_{n-1} \end{bmatrix} \begin{bmatrix} \overline{Y}\sqrt{n} \\ U_1 \\ U_2 \\ \vdots \\ U_{n-1} \end{bmatrix} - n\overline{Y}^2 \\
&= \sum_{i=1}^{n-1} U_i^2
\end{aligned}$$

Since U_i is independent of $\overline{Y}$, $\sum_{i=1}^{n-1} U_i^2$ is also independet of $\overline{Y}$. □

4. Using the results of part (3), show that

$$\frac{\sum_{i=1}^{n}(Y_i - \overline{Y})^2}{\sigma^2} = \frac{(n-1)S^2}{\sigma^2} \tag{1.49}$$

has a χ^2 distribution with $(n-1)$ df.

Proof. First we show that $U_i \sim N(0, \sigma^2)$, and thus $Z_i = U_i/\sigma \sim N(0,1)$, a standard normal distribution.

We have seen from the previous exercise that any linear combination of independent random variables with normal distribution is also normally distributed, so U_i has normal distribution, with expected value

$$\mathrm{E}(U_i) = \mu\|\mathbf{a}_i\|_1 = 0, \tag{1.50}$$

and variance

$$\mathrm{Var}(U_i) = \sigma^2\|\mathbf{a}_i\|_2^2 = \sigma^2. \tag{1.51}$$

Next we prove that if each Z_i is independent with standard normal distribution, then $\sum_{i=1}^{n} Z_i^2$ has a χ^2 distribution with n df.

The characteristic function of Z_i^2 is

$$\begin{aligned}\varphi_{Z_i^2}(t) &= \mathrm{E}\left(\mathrm{e}^{itZ_i^2}\right)\\ &= \int_{-\infty}^{\infty} \mathrm{d}z\, \frac{1}{\sqrt{2\pi}}\mathrm{e}^{itz^2}\mathrm{e}^{-z^2/2}\\ &= (1-2it)^{-1/2},\end{aligned}$$

and from the fact that Z_i is independent with each other, the characteristic function of $V = \sum_{i=1}^n Z_i^2$ is the product of n characteristic functions of Z_i^2

$$\varphi_V(t) = \prod_{i=1}^n \varphi_{Z_i^2}(t) = (1-2it)^{-n/2} \tag{1.52}$$

Because characteristic functions are unique, V has a χ^2 distribution with n degrees of freedom.
Finally we prove that $\sum_{i=1}^n (Y_i - \overline{Y})^2/\sigma^2$ has a χ^2 distribution with n df. We rewrite the (1.49) as follows

$$\frac{\sum_{i=1}^n (Y_i-\overline{Y})^2}{\sigma^2} = \frac{\sum_{i=1}^{n-1} U_i^2}{\sigma^2} = \sum_{i=1}^{n-1} Z_i^2. \tag{1.53}$$

Thus it has a χ^2 distribution with $(n-1)$ degrees of freedom. □

1.5 Exam 08

Exercise 1.1. Suppose that $X_1, X_2, \dots, X_n, X_{n+1}, \dots, X_{n+m}$ is a random sample from a normal distribution $N(0, \sigma^2)$.

1. Find a and b such that

$$V = a\left(\sum_{i=1}^n X_i\right)^2 + b\left(\sum_{i=n+1}^{n+m} X_i\right)^2 \tag{1.54}$$

has a χ^2 distribution.

Solution. Any linear combination of independent normally distributed random variables is also a random variable with normal distribution. Therefore

$$Z_1 = \frac{\sum_{i=1}^n X_i}{\sigma\sqrt{n}}, \quad Z_2 = \frac{\sum_{i=n+1}^{n+m} X_i}{\sigma\sqrt{m}}. \tag{1.55}$$

Both Z_1 and Z_2 have standard normal distributions $N(0,1)$. Thus

$$a\left(\sum_{i=1}^n X_i\right)^2 + b\left(\sum_{i=n+1}^{n+m} X_i\right)^2 = an\sigma^2 Z_1^2 + bm\sigma^2 Z_2^2. \tag{1.56}$$

If $a = 1/n\sigma^2$ and $b = 1/m\sigma^2$, then $V = Z_1^2 + Z_2^2$ has a χ^2 distribution with 2 df.

2. Find c such that

$$U = \frac{c\sum_{i=1}^n X_i}{\sqrt{\sum_{i=n+1}^{n+m} X_i^2}} \tag{1.57}$$

has a t distribution.

Solution. Recall the definition of t distribution

Definition 1.1. Let Z be a standard normal random variable and let W be a χ^2-distributed variable with ν df. Then, if Z and W are independent,

$$T = \frac{Z}{\sqrt{W/\nu}}$$

is said to have a t distribution with ν df.

From part (1) we know that

$$Z_1 = \frac{\sum_{i=1}^{n} X_i}{\sigma\sqrt{n}} \tag{1.58}$$

has a standard normal distribution. In additon

$$W = \sum_{i=n+1}^{n+m} (\frac{X_i}{\sigma})^2 \tag{1.59}$$

is a χ^2-distributed variable with m df. Thus

$$\begin{aligned} U &= \frac{c\sum_{i=1}^{n} X_i}{\sqrt{\sum_{i=n+1}^{n+m} X_i^2}} \\ &= c\sqrt{\frac{n}{m}}\frac{Z_1}{\sqrt{W/m}}. \end{aligned}$$

If $c = \sqrt{m/n}$, then U is a t-distributed variable with m df.

1.6 Shape of water in a rotating bucket

This is a static equilibrium problem in a rotating frame. Suppose the rotating bucket with radius R has an constant angular velocity ω. When the system arrives equilibrium, the differential volume element of the water will have the same angular velocity ω. We can write down its action in cylindrical coordinates

$$\begin{aligned} S &= \int_{t_0}^{t_1} \mathrm{d}t \int_V \mathrm{d}r^3\, \mathscr{L}(t, r, z) \\ &= \text{const.} \int_M \mathrm{d}m\, (\frac{1}{2}v^2 - gz) \\ &\propto \int_0^R \mathrm{d}r\, 2\pi r z(r) \left[\frac{1}{2}\omega^2 r^2 - \frac{1}{2}gz(r)\right] \\ &\propto \int_0^R \mathrm{d}r\, \left(\omega^2 r^3 z - grz^2\right), \end{aligned}$$

where $z(r)$ is the shape of the water, $\mathscr{L}$ is the Lagrangian density. There is an additional constraint of $z(r)$ for the conservation of volume

$$\int_0^R \mathrm{d}r\, 2\pi r z(r) = V = \text{const.} \tag{1.60}$$

So we have variation problem with additional condition. It is easy to solve using Lagrange multipliers' method. Necessary condition for S to have extrema is existing a multiplier λ

such that

$$F = \omega^2 r^3 z - grz^2 + \lambda rz, \tag{1.61}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{\partial F}{\partial \frac{\partial z}{\partial r}} - \frac{\partial F}{\partial z} = 0. \tag{1.62}$$

Since F does not depend on $\frac{\partial z}{\partial r}$ explicitly, we obtain

$$\omega^2 r^3 - 2grz + \lambda r = 0, \tag{1.63}$$

or equivalently

$$z = \frac{\omega^2 r^2}{2g} + \lambda, \tag{1.64}$$

where λ is determined by

$$\int_0^R \mathrm{d}r\, 2\pi r z(r) = \frac{\pi\omega^2 R^4}{4g} + \lambda\pi R^2 = V,$$

$$\lambda = \frac{V}{\pi R^2} - \frac{\omega^2 R^2}{4g}.$$

1.7 Derivative of inverse function

Suppose we are given a function f with its invers f^{-1}. Using a little geometry, we can compute the derivative $\mathrm{d}f^{-1}(x)/\mathrm{d}x$ in terms of f. See Fig(1).

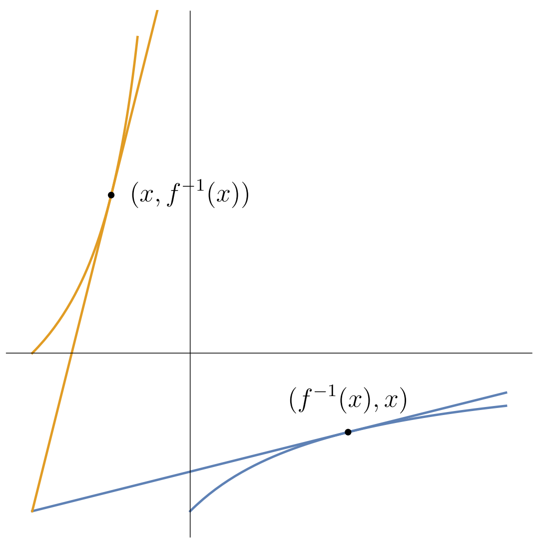

Figure 1. A schematic picture for function$f$ and its inverse f^{-1}. The slopes at points $(f^{-1}(x), x)$ and $(x, f^{-1}(x))$ are reciprocal to each other.

Hence we have

$$\frac{\mathrm{d}f^{-1}(x)}{\mathrm{d}x} = 1\Big/ \left.\frac{\mathrm{d}f(x')}{\mathrm{d}x'}\right|_{x'=f^{-1}(x)}. \tag{1.65}$$

Example 1.1.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\arcsin x &= 1\Big/\left.\frac{\mathrm{d}}{\mathrm{d}x'}\sin x'\right|_{x'=\arcsin x} \\
&= \frac{1}{\cos x'\big|_{x'=\arcsin x}} \\
&= \frac{1}{\sqrt{1-\sin^2 x'}\Big|_{x'=\arcsin x}} \\
&= \frac{1}{\sqrt{1-x^2}}.
\end{aligned}$$