Notes

Quantum Mechanics

Yi Huang[1]

University of Minnesota

E-mail: yihphysics@gmail.com

[1] *My Personal Website*

Contents

PART

I

Ando

1 Electronic properties of 2D systems

1.1 Derivation of Eq. (2.14)

The derivation follows from Stern[1]. We want to solve an equation

$$\boldsymbol{\nabla} \cdot (\kappa \boldsymbol{\nabla} \phi) - 2\bar{\kappa}\bar{q}_s \bar{\phi}(\mathbf{r})\delta(z) = -4\pi Z e \delta^2(\mathbf{r})\delta(z - z_0). \tag{1.1}$$

where $\mathbf{r}$ is the inplane radius vector, $\kappa = \kappa_{\text{ins}}$ if $z < 0$, $\kappa = \kappa_{\text{sc}}$ if $z > 0$ ($\kappa(z)$ is basically a theta function of z, so the derivative of $\kappa(z)$ will give $\delta(z)$). For cylindrically symmetric potential, $\phi = \phi(r, z)$, we use a Fourier-Bessel expansion for the potential

$$\phi(r, z) = \int_0^\infty k' \, \mathrm{d}k' \, A_{k'}(z) J_0(kr). \tag{1.2}$$

where J_0 is the Bessel function of the first kind. $A_k(z)$ is a function of k and z. To proceed further, we multiply Eq. (1.1) by $rJ_0(kr)$ and integrate over r. Notice that

$$\int_0^\infty r J_\nu(kr) J_\nu(k'r) \, \mathrm{d}r = \frac{1}{k}\delta(k - k'), \tag{1.3}$$

and the recursive relation of the Bessel function

$$\frac{\mathrm{d}J_\nu(z)}{\mathrm{d}z} = \frac{\nu}{z} J_\nu(z) - J_{\nu+1}(z), \tag{1.4}$$

and also the fact that $J_0(0) = 1$ and $J_0(\infty) = 0$, we have

$$\begin{aligned}
\int r \, \mathrm{d}r \, J_0(k'r) \boldsymbol{\nabla} \cdot (\kappa \boldsymbol{\nabla} \phi) &= \kappa \int A_{k'}(z) k' \, \mathrm{d}k' \int_0^\infty \mathrm{d}r \, \frac{\partial}{\partial r}\left(r \frac{\partial J_0(k'r)}{\partial r} \right) J_0(kr) \\
&+ (\kappa_{\text{sc}} - \kappa_{\text{ins}})\delta(z) \int \frac{\partial A_{k'}}{\partial z} k' \, \mathrm{d}k' \int_0^\infty \mathrm{d}r \, r J_0(kr) J_0(k'r) \\
&+ \kappa \int \frac{\partial^2 A_{k'}}{\partial z^2} k' \, \mathrm{d}k' \int_0^\infty \mathrm{d}r \, r J_0(kr) J_0(k'r) \\
&= -\kappa k^2 A_k(z) + (\kappa_{\text{sc}} - \kappa_{\text{ins}})\delta(z) \frac{\partial A_k(z)}{\partial z} + \kappa \frac{\partial^2 A_k(z)}{\partial z^2}.
\end{aligned}$$

Notice that $\delta^2(\mathbf{r}) = \delta(r)\delta(\theta)/r$, we have

$$\int_0^\infty \mathrm{d}r \, \delta^2(\mathbf{r}) r J_0(kr) = \delta(\theta). \tag{1.5}$$

And

$$\int_0^\infty \mathrm{d}r \, \bar{\phi}(r) r J_0(kr) = A_k(z = 0) = \bar{A}_k. \tag{1.6}$$

Eventually we have

$$-\kappa k^2 A_k(z) + (\kappa_{\text{sc}} - \kappa_{\text{ins}})\delta(z) \frac{\partial A_k(z)}{\partial z} + \kappa \frac{\partial^2 A_k(z)}{\partial z^2} - 2\bar{\kappa}\bar{q}_s \bar{A}_k \delta(z) = -4\pi Z e \delta(\theta)\delta(z - z_0) \tag{1.7}$$

References

[1] Frank Stern and W. E. Howard. Properties of semiconductor surface inversion layers in the electric quantum limit. *Phys. Rev.*, 163:816–835, Nov 1967.