Notes

Classical Electrodynamics

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Contents

Classical Mechanics

PART

I

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1.1 Problem 1 section 11 in Landau

The energy of this system is

$$E = \frac{1}{2} m l^2 \dot{\phi}^2 - mgl \cos phi = -mgl \cos \phi_0, \tag{1.1}$$

where ϕ_0 is the maximum angle of motion. Separate variables

$$\mathrm{d}t = \sqrt{\tfrac{l}{2g}} \int \frac{\mathrm{d}\phi}{\sqrt{\cos\phi - \cos\phi_0}}, \tag{1.2}$$

By symmetry, the period of motion is four times the time from angle $\phi = 0$ to $\phi = \phi_0$.

$$\begin{aligned}
T &= 4\sqrt{\tfrac{l}{2g}} \int_0^{\phi_0} \frac{\mathrm{d}\phi}{\sqrt{\cos\phi - \cos\phi_0}} \\
&= 2\sqrt{\tfrac{l}{g}} \int_0^{\phi_0} \frac{\mathrm{d}\phi}{\sqrt{\sin^2 \frac{1}{2}\phi_0 - \sin^2 \frac{1}{2}\phi}} \\
(\sin\xi = \frac{\sin\frac{1}{2}\phi}{\sin\frac{1}{2}\phi_0}) &= 2\sqrt{\tfrac{l}{g}} \int_0^{\phi_0} \frac{\mathrm{d}\phi}{\sin\frac{1}{2}\phi_0 \sqrt{1 - \sin^2\xi}} \\
&= 4\sqrt{\tfrac{l}{g}} \int_0^{\pi/2} \frac{\mathrm{d}\xi}{\sqrt{1 - \sin^2\frac{1}{2}\phi_0 \sin^2\xi}} \\
&= 4\sqrt{l/g} K\big(\sin\tfrac{1}{2}\phi_0\big).
\end{aligned}$$

where we use $\cos\phi = 1 - 2\sin^2\frac{1}{2}\phi$ in the third step, and change the variable in the 4th step s.t.

$$\mathrm{d}\phi = \frac{2\sin\frac{1}{2}\phi_0 \cos\xi}{\cos\frac{1}{2}\phi} \mathrm{d}\xi = \frac{2\sin\frac{1}{2}\phi_0\cos\xi}{\sqrt{1 - \sin^2\frac{1}{2}\phi_0\sin^2\xi}} \mathrm{d}\xi\,, \tag{1.3}$$

in the last step we use the definition of complete elliptic integral of the first kind

$$K(k) = \int_0^{\pi/2} \frac{\mathrm{d}x}{\sqrt{1 - k^2 \sin^2 x}}. \tag{1.4}$$

1.2 Problem 2 section 11 in Landau

The potential energy is

$$U = -U_0/\cosh^2 \alpha x, \quad U_0 > 0. \tag{1.5}$$

The shape of this potential can be inferred by its limit

$$\begin{aligned}
\lim_{x\to 0} U(x) &= -U_0, \\
\lim_{x\to\pm\infty} U(x) &= 0^-,
\end{aligned}$$

which is like an attractive potential well centered at $x = 0$ with minimum $-U_0$, and approches zero when $x \to \pm\infty$. The total energy E satisfy

$$-U_0 < E < 0, \tag{1.6}$$

which means the particle is bounded by potential $U(x)$. The positive turning point is

$$x_t = \cosh^{-1}\sqrt{U_0/|E|}. \tag{1.7}$$

Hence the period is

$$\begin{aligned}
T &= 4\sqrt{m/2}\int_0^{x_t}\frac{\mathrm{d}x}{\sqrt{E + U_0/\cosh^2\alpha x}} \\
&= 2\sqrt{2m}\int_0^{x_t}\frac{\cosh\alpha x\,\mathrm{d}x}{\sqrt{U_0 - |E|\cosh^2\alpha x}} \\
&= \frac{2\sqrt{2m}}{\alpha}\int\frac{\mathrm{d}\sinh\alpha x}{\sqrt{U_0 - |E|\big(1+\sinh^2\alpha x\big)}} \\
&= \frac{2}{\alpha}\sqrt{\frac{2m}{|E|}}\int_0^1\frac{\mathrm{d}(\eta\sinh\alpha x)}{\sqrt{1-\eta^2\sinh^2\alpha x}} \\
&= \frac{2}{\alpha}\sqrt{\frac{2m}{|E|}}\int_0^1\frac{\mathrm{d}u}{\sqrt{1-u^2}} \\
&= \frac{2}{\alpha}\sqrt{\frac{2m}{|E|}}\arcsin u\Big|_0^1 \\
&= \frac{\pi}{\alpha}\sqrt{\frac{2m}{|E|}}
\end{aligned}$$

where

$$\eta = \sqrt{\tfrac{|E|}{U_0-|E|}} \tag{1.8}$$