Notes

Classical Electrodynamics

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PART

I

Caprice

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1.1 The angular integral in (4.16')

Show the following equation is true

$$\int \mathrm{d}\Omega\, \mathbf{n} \cos\gamma = \frac{4\pi \mathbf{n}'}{3}, \tag{1.1}$$

where $\mathbf{n} = \mathbf{i}\sin\theta\cos\phi + \mathbf{j}\sin\theta\sin\phi + \mathbf{k}\cos\theta$, and $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$.

Proof. First look at the x component of the integral:

$$\begin{aligned}
\int \mathrm{d}\Omega \sin\theta\cos\phi\cos\gamma &= \int \mathrm{d}\cos\theta\, \mathrm{d}\phi \sin\theta\cos\phi[cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')] \\
(\text{integrate over } \phi) &= \pi \int \mathrm{d}\cos\theta \sin^2\theta \sin\theta' \cos\phi' \\
(u = \cos\theta) &= \pi \sin\theta' \cos\phi' \int_{-1}^{1} \mathrm{d}u\, (1 - u^2) \\
&= \frac{4\pi}{3} \sin\theta' \cos\phi'.
\end{aligned}$$

Similarly we can complete the proof. □

1.2 One tricky point on partial derivative below (5.108)

Just below (5.108), it claims $\partial r/\partial z = \cos\theta$, this partial derivative treats x and y as constant. What is $\partial z/\partial r$? If we use expression $z = r\cos\theta$, then it is easy to go to result $\partial z/\partial r = \cos\theta$. But we have $\partial r/\partial z = \cos\theta$ already, how could $\partial z/\partial r = \partial r/\partial z$? To resolve this "paradox", we should notice that when we do the partial derivative, we always need to specify what variables we keep as constant. In the first calculation $\partial r/\partial z = \cos\theta$, we keep x, y as constant. However, in $\partial z/\partial r = \cos\theta$ we treat θ, ϕ as constant. That's the reason why we have such inconsistent results.

1.3 Derivation of (6.27) and (6.28)

$$
\begin{aligned}
\mathbf{J_l} &= \int \mathrm{d}^3x' \delta(\mathbf{x}-\mathbf{x}')\mathbf{J_l} \\
&= -\frac{1}{4\pi}\int \mathrm{d}^3x' \nabla'^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\mathbf{J_l} \\
&= -\frac{1}{4\pi}\int \mathrm{d}^3x' \boldsymbol{\nabla}'\cdot\left[\boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\mathbf{J_l}\right] + \frac{1}{4\pi}\int \mathrm{d}^3x' \boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)(\boldsymbol{\nabla}'\cdot\mathbf{J_l}) \\
&= -\frac{1}{4\pi}\int \mathrm{d}^3x' \boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)(\boldsymbol{\nabla}'\cdot\mathbf{J_l}) \\
&= -\frac{1}{4\pi}\boldsymbol{\nabla}\int \mathrm{d}^3x' \frac{\boldsymbol{\nabla}'\cdot\mathbf{J_l}}{|\mathbf{x}-\mathbf{x}'|} \\
&= -\frac{1}{4\pi}\boldsymbol{\nabla}\int \mathrm{d}^3x' \frac{\boldsymbol{\nabla}'\cdot(\mathbf{J_l}+\mathbf{J_t})}{|\mathbf{x}-\mathbf{x}'|} \\
&= -\frac{1}{4\pi}\boldsymbol{\nabla}\int \mathrm{d}^3x' \frac{\boldsymbol{\nabla}'\cdot\mathbf{J}}{|\mathbf{x}-\mathbf{x}'|}.
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{J_t} &= \int \mathrm{d}^3x' \delta(\mathbf{x}-\mathbf{x}')\mathbf{J_t} \\
&= -\frac{1}{4\pi}\int \mathrm{d}^3x' \nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\mathbf{J_t} \\
&= -\frac{1}{4\pi}\int \mathrm{d}^3x' \left[-\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{J_t}}{|\mathbf{x}-\mathbf{x}'|}\right)\right) + \nabla^2\left(\frac{\mathbf{J_t}}{|\mathbf{x}-\mathbf{x}'|}\right)\right] \\
&= \frac{1}{4\pi}\int \mathrm{d}^3x' \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\left(\frac{\mathbf{J_t}}{|\mathbf{x}-\mathbf{x}'|}\right) \\
&= \frac{1}{4\pi}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\int \mathrm{d}^3x' \frac{\mathbf{J}}{|\mathbf{x}-\mathbf{x}'|}
\end{aligned}
$$

1.4 Derivation of (9.37) in Jackson

Basically we need to show the symmetric term can be related to quadrupole moment

$$
\int \mathrm{d}^3x\,(x_iJ_j + x_jJ_i) = -\int \mathrm{d}^3x\, x_ix_j\,\boldsymbol{\nabla}\cdot\mathbf{J}. \tag{1.2}
$$

Proof. We use $\delta_{ij} = \partial_i x_j$ to transform the symmetric term

$$
x_iJ_j + x_jJ_i = \partial_k(x_ix_jJ_k) - x_ix_j\partial_kJ_k. \tag{1.3}
$$

The divergence term vanishes if the current distribution is localized. □

1.5 Derivation of (9.46) in Jackson

We have

$$
\begin{aligned}
|(\mathbf{n}\times\mathbf{Q})\times\mathbf{n}|^2 &= |\mathbf{Q}-(\mathbf{Q}\cdot\mathbf{n})\mathbf{n}|^2 \\
&= |\mathbf{Q}|^2 - 2|\mathbf{Q}\cdot\mathbf{n}|^2 + |\mathbf{Q}\cdot\mathbf{n}|^2 \\
&= |\mathbf{Q}|^2 - |\mathbf{Q}\cdot\mathbf{n}|^2.
\end{aligned}
$$

1.6 Angular momentum operator of wave mechanics (9.101) in Jackson

The infinitesimal distance in spherical coordinates is

$$\mathrm{d}^2 s = \mathrm{d}^2 r + r^2\,\mathrm{d}^2\theta + r^2 \sin^2\theta\,\mathrm{d}^2\phi\,, \tag{1.4}$$

so the gradient operator in spherical coordinates is

$$\boldsymbol{\nabla} = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r}\frac{\partial}{\partial\theta} + \frac{1}{r\sin\theta}\hat{\mathbf{e}}_\phi, \tag{1.5}$$

where $\hat{\mathbf{e}}_r = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, $\hat{\mathbf{e}}_\theta = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$, and $\hat{\mathbf{e}}_\phi = (-\sin\phi, \cos\phi, 0)$. Hence we can write the angular momentum operator as

$$\begin{aligned}
\mathbf{L} &= \frac{1}{i}\mathbf{r}\times\boldsymbol{\nabla},\\
\mathbf{r}\times\boldsymbol{\nabla} &= (\hat{\mathbf{e}}_r r)\times\left(\hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{1}{r\sin\theta}\hat{\mathbf{e}}_\phi\right)\\
&= \hat{\mathbf{e}}_\phi\frac{\partial}{\partial\theta} - \hat{\mathbf{e}}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}.
\end{aligned}$$

And the differential operator L^2 can be obtained as

$$\begin{aligned}
L^2 &= -\left(\hat{\mathbf{e}}_\phi\frac{\partial}{\partial\theta} - \hat{\mathbf{e}}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)\cdot\left(\hat{\mathbf{e}}_\phi\frac{\partial}{\partial\theta} - \hat{\mathbf{e}}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)\\
&= -\hat{\mathbf{e}}_\phi\cdot\frac{\partial}{\partial\theta}\left(\hat{\mathbf{e}}_\phi\frac{\partial}{\partial\theta} - \hat{\mathbf{e}}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right) + \hat{\mathbf{e}}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\cdot\left(\hat{\mathbf{e}}_\phi\frac{\partial}{\partial\theta} - \hat{\mathbf{e}}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)\\
&= -\frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2},
\end{aligned}$$

where we have used $\partial\hat{\mathbf{e}}_\theta/\partial\theta = -\hat{\mathbf{e}}_r$, $\partial\hat{\mathbf{e}}_\phi/\partial\theta = 0$, $\partial\hat{\mathbf{e}}_\theta/\partial\phi = \cos\theta\hat{\mathbf{e}}_\phi$, and $\partial\hat{\mathbf{e}}_\phi/\partial\phi = -(\cos\phi, \sin\phi, 0)$. Next let's look at $\mathbf{L}_{x,y,z}$

$$\begin{aligned}
\mathbf{r}\times\boldsymbol{\nabla} &= (-\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}})\frac{\partial}{\partial\theta}\\
&\quad - (\cos\theta\cos\phi\hat{\mathbf{i}} + \cos\theta\sin\phi\hat{\mathbf{j}} - \sin\theta\hat{\mathbf{k}})\frac{1}{\sin\theta}\frac{\partial}{\partial\phi},\\
L_x &= \frac{1}{i}\left(-\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right),\\
L_y &= \frac{1}{i}\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right),\\
L_z &= -i\frac{\partial}{\partial\phi}.
\end{aligned}$$

Therefore

$$\begin{aligned}
L_+ &= L_x + iL_y = \mathrm{e}^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right),\\
L_- &= L_x - iL_y = \mathrm{e}^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right).
\end{aligned}$$

1.7 Remark of delta function in page 120, Jackson

To express $\delta(\mathbf{x}-\mathbf{x}') = \delta(x_1 - x_1')\delta(x_2 - x_2')\delta(x_3 - x_3')$ in term of the coordinates (ξ_1, ξ_2, ξ_3), related to (x_1, x_2, x_3) via the Jacobian $J(x_i, \xi_i)$, we note that the meaningful quantity is

$\delta(\mathbf{x}-\mathbf{x}')\,\mathrm{d}^3x$. Hence

$$\delta(\mathbf{x}-\mathbf{x}') = \frac{1}{|J(x_i,\xi_i)|}\delta(\xi_1-\xi_1')\delta(\xi_2-\xi_2')\delta(\xi_3-\xi_3') \tag{1.6}$$

See problem 1.2.

1.8 Derivation of (9.120) and (9.121) in Jackson

Introduce the normalized form of spherical harmonics

$$\mathbf{X}_{lm}(\theta,\phi) = \frac{1}{\sqrt{l(l+1)}}\mathbf{L}Y_{lm}(\theta,\phi). \tag{1.7}$$

Using the completeness in a unit sphere

$$\mathbb{I} = \int \mathrm{d}\Omega\, |\hat{\mathbf{r}}\rangle\langle\hat{\mathbf{r}}|\,, \tag{1.8}$$

we can calculate the orthogonality of vector shperical harmonics

$$\begin{aligned}
\int \mathrm{d}\Omega\,(\mathbf{L}Y_{lm})^*\cdot(\mathbf{L}Y_{l'm'}) &= \langle lm|\mathbf{L}\cdot\mathbf{L}|l'm'\rangle \\
&= \int \mathrm{d}\Omega\,\langle lm|\hat{\mathbf{r}}\rangle\;\langle\hat{\mathbf{r}}|\mathbf{L}\cdot\mathbf{L}|\hat{\mathbf{r}}\rangle\;\langle\hat{\mathbf{r}}|l'm'\rangle \\
&= l(l+1)\int \mathrm{d}\Omega\,Y^*_{lm}(\theta,\phi)Y_{l'm'}(\theta,\phi) \\
&= l(l+1)\delta_{ll'}\delta_{mm'},
\end{aligned}$$

To prove (9.121), we need to show $\mathbf{L}\cdot(\mathbf{r}\times\mathbf{L}) = 0$.

$$\begin{aligned}
\mathbf{L}\cdot(\mathbf{r}\times\mathbf{L}) &= \epsilon_{ijk}L_i x_j L_k \\
&= \epsilon_{ijk}(x_j L_i + [L_i, x_j])L_k \\
&= i\epsilon_{ijk}\epsilon_{ijl}x_l L_k \\
&= 2i\delta_{kl}x_l L_k \\
&= 2i x_k L_k \\
&= \frac{2i}{\hbar}\epsilon_{ijk}x_k x_i p_j = 0,
\end{aligned}$$

where we used the commutator

$$[L_i, x_j] = \frac{\epsilon_{ikl}}{\hbar}x_k[p_l, x_j] = -i\epsilon_{ikl}x_k\delta_{lj} = i\epsilon_{ijk}x_k, \tag{1.9}$$

and the property of Levi-Civita symbol $\epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl}$.