

# ***Notes***

## ***Analysis***

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# Caprice

## 1 Fall 2018

This is my note for analysis. The textbook I use are Stein's classic four volumes on analysis.

| Section 1. Fall 2018

### 1.1 Cauchy criterion

**Definition 1.1.** If  $(a_n)$  is a sequence, then  $\lim_{n \rightarrow \infty} a_n = L$  means that for every  $\varepsilon > 0$  there exists a corresponding  $N \in \mathbb{N}^+$  such that if  $n \geq N$ , then  $|a_n - L| < \varepsilon$ . If this limit exists, then we say that the sequence  $(a_n)$  converges, and if this limit doesn't exist then we say the sequence  $(a_n)$  diverges.

**Proposition 1.1.** A converging sequence of  $\mathbb{R}$  has a unique limit.

**Proof.** Suppose both  $L_1$  and  $L_2$  are limits of  $(a_n)$  and  $L_1 \neq L_2$ , which means for any  $\varepsilon > 0$ , there exists  $N_1 \in \mathbb{N}^+$  such that

$$|a_n - L_1|, \quad \text{for every } n \geq N_1; \quad (1.1)$$

and for any  $\varepsilon > 0$ , there exists  $N_2 \in \mathbb{N}^+$  such that

$$|a_n - L_2|, \quad \text{for every } n \geq N_2. \quad (1.2)$$

Since  $L_1 \neq L_2$ , we can choose  $\varepsilon = |L_1 - L_2|/4$ ,  $N = \max\{N_1, N_2\}$ , then

$$\begin{aligned} |L_1 - L_2| &= |(a_n - L_2) + (L_1 - a_n)| \\ &\leq |a_n - L_1| + |a_n - L_2| \\ &\leq 2\varepsilon = |L_1 - L_2|/2, \quad \text{for every } n \geq N, \end{aligned}$$

which can be true only if  $L_1 = L_2$ , and this contradicts to the assumption  $L_1 \neq L_2$ .

Therefore we prove  $L_1 = L_2$  by contradiction.  $\square$

**Definition 1.2.** A sequence  $(a_n)$  is said to be a Cauchy sequence iff for any  $\varepsilon > 0$ , there exists  $N \geq \mathbb{N}^+$ , such that

$$|a_n - a_m| < \varepsilon, \quad (1.3)$$

for all  $n, m \geq N$ . In other words, a Cauchy sequence is one in which the terms eventually cluster together.

**Theorem 1.1.** (Cauchy Criterion.) A sequence is Cauchy iff it converges.

**Proof.** Sufficiency: We need to prove if a sequence converges, it is Cauchy. Given any  $\varepsilon > 0$ . Let

$$L := \lim_{n \rightarrow \infty} a_n. \quad (1.4)$$

There exists  $N$  such that for all  $n \geq N$ ,

$$|a_n - L| < \frac{\varepsilon}{2}. \quad (1.5)$$

Thus, for all  $m, n \geq N$ ,

$$|a_n - a_m| = |(a_n - L) + (L - a_m)| \leq |a_n - L| + |a_m - L| \leq \varepsilon. \quad (1.6)$$

Necessity: We need to show a Cauchy sequence must converge. The proof proceeds in several steps, which we isolate and prove subsequently.

Step 1: Since  $(a_n)$  is Cauchy, it must be bounded.

Step 2: Since  $(a_n)$  is bounded, it has a convergent subsequence  $(a_{n_k})$ . Let  $x := \lim_{k \rightarrow \infty} a_{n_k}$ .

Step 3: Show that  $\lim_{n \rightarrow \infty} a_n = x$ .

□

**Proposition 1.2.** (Step 1 in the proof of theorem 1.1.) If a sequence  $(a_n)$  is Cauchy, then it is bounded.

**Proof.** There exists  $N$  such that  $|a_n - a_m| < 1$  for all  $n, m \geq N$ . In particular, we have  $|a_n - a_m| < 1$  for all  $n \geq N$ , whence

$$a_n \in (a_N - 1, a_N + 1) \quad (1.7)$$

for all  $n \geq N$ . Thus  $\{a_n : n \geq N\}$  is bounded. Also,  $\{a_n : n < N\}$  is bounded since it is finite. We conclude that the entire range of the sequence  $(a_n)$  is bounded. □

Before we prove step 2 of theorem 1.1, we first prove three theorems as lemmas: monotone subsequence theorem, boundedness of convergent sequences, and monotone convergence theorem.

**Lemma 1.2.** (Monotone Subsequence Theorem.) Every sequence has a monotone subsequence.

**Proof.** Let  $(a_n)$  denote a sequence. We call a term  $a_k$  a peak iff  $a_k \geq a_m$  for all  $m > k$ . There are two cases:

1. There are infinitely many peaks.
2. There are finitely many peaks.

In the first case, the subsequence consisting of the peaks forms a monotonically decreasing sequence. In the second case, there exists some largest  $M$  such that  $a_M$  is a peak. Let  $m_1 = M + 1$ . Since  $a_{m_1}$  isn't a peak, there exists  $m_2 > m_1$  such that  $a_{m_1} < a_{m_2}$ . Since  $a_{m_2}$  isn't a peak, there exists  $m_3 > m_2$  such that  $a_{m_2} < a_{m_3}$ . We can continue this process indefinitely, thus creating a monotonically increasing subsequence  $(a_{m_k})$ . □

**Lemma 1.3.** (Boundedness of Convergent Sequences.) If  $(a_n)$  is a sequence of real numbers that is convergent to  $L \in \mathbb{R}$

$$\lim_{n \rightarrow \infty} a_n = L, \quad (1.8)$$

then  $(a_n)$  is also bounded.

**Proof.** Let  $\varepsilon_0 = 1$ . There exists  $N \in \mathbb{N}^+$  such that if  $n \geq N$  then  $|a_n - L| < \varepsilon = 1$ . Equivalently,  $L - 1 < a_n < L + 1$  for all  $n \geq N$ ,  $\{a_n : n \geq N\}$  is bounded. Also  $\{a_n : n \leq N\}$  is bounded since it is finite.

It is important to recognize in the theorem above that we could not have just taken the maximum value from the start because we cannot take the maximum of an infinite set, only a finite set.

It is also important to note that the converse of this theorem is not always true, that is just because a sequence  $(a_n)$  is bounded does not imply  $(a_n)$  to be convergent. For example, the sequence  $((-1)^n) = (-1, 1, -1, 1, \dots)$  is bounded, however, clearly  $((-1)^n)$  does not converge.  $\square$

**Lemma 1.4.** (Monotone Convergence Theorem.) If  $(a_n)$  is a monotone sequence of real numbers, then  $(a_n)$  is convergent iff  $(a_n)$  is bounded.

**Proof.** Let  $(a_n)$  be a monotone sequence.

Necessity: If  $(a_n)$  is convergent, then by the lemma 1.3 we just prove above,  $(a_n)$  is bounded.

Sufficiency: If  $(a_n)$  is bounded, there are two cases to consider.

Case 1. Suppose that  $(a_n)$  is an increasing sequence that is there exists  $M \in \mathbb{R}$  such that for any  $n \in \mathbb{N}^+$  we have  $a_n \leq M$ . By the completeness property of the real numbers,  $\{a_n\}$  has a supremum in  $\mathbb{R}$ , call it  $L = \sup\{a_n\}$ . Let  $\varepsilon > 0$  be given. Since  $L$  is the supremum of  $\{a_n\}$ , then  $L - \varepsilon$  is not an upper bound to  $\{a_n\}$  and so there exists  $N$  such that  $L - \varepsilon < a_N$ . Since  $(a_n)$  is an increasing sequence, then for all  $n \geq N$ , we have  $a_N \leq a_n$ , and so

$$L - \varepsilon < a_N \leq a_n \leq L < L + \varepsilon. \quad (1.9)$$

Omitting the unnecessary parts of the inequality, we see that for  $n \geq N$  we have  $L - \varepsilon < a_n < L + \varepsilon$ , and so  $|a_n - L| < \varepsilon$ . By definition of limit, since  $\varepsilon > 0$  is arbitrary, we have that  $\lim_{n \rightarrow \infty} a_n = L$ , that is  $(a_n)$  is convergent to  $L$ .

Case 2. Suppose that  $(a_n)$  is a decreasing sequence that is there exists  $n \in \mathbb{R}$  such that for any  $n \in \mathbb{N}^+$  we have  $a_n \geq M$ . By the completeness property of the real numbers,  $\{a_n\}$  has a infimum in  $\mathbb{R}$ , call it  $L = \inf\{a_n\}$ . Let  $\varepsilon > 0$  be given. Since  $L$  is the infimum of  $\{a_n\}$ , then  $L + \varepsilon$  is not a lower bound to  $\{a_n\}$  and so there exists  $N$  such that  $a_N < L + \varepsilon$ . Since  $(a_n)$  is an decreasing sequence, then for all  $n \geq N$ , we have  $a_n \leq a_N$ , and so

$$L - \varepsilon < a_n \leq a_N \leq L < L + \varepsilon. \quad (1.10)$$

Omitting the unnecessary parts of the inequality, we see that for  $n \geq N$  we have  $L - \varepsilon < a_n < L + \varepsilon$ , and so  $|a_n - L| < \varepsilon$ . By definition of limit, since  $\varepsilon > 0$  is arbitrary, we have that  $\lim_{n \rightarrow \infty} a_n = L$ , that is  $(a_n)$  is convergent to  $L$ .

In both cases, the bounded sequence  $(a_n)$  is convergent.  $\square$

**Corollary 1.1.** If  $(a_n)$  is a monotone sequence that's bounded, then

$$\lim_{n \rightarrow \infty} a_n = \sup\{a_n\} \quad \text{if } (a_n) \text{ is increasing;} \quad (1.11)$$

$$\lim_{n \rightarrow \infty} a_n = \inf\{a_n\} \quad \text{if } (a_n) \text{ is decreasing.} \quad (1.12)$$

The proof of corollary 1.1 is done in lemma 1.4.

**Proposition 1.3.** (Step 2 in the proof of theorem 1.1.) Every bounded sequence has a convergent subsequence.

**Proof.** Suppose  $(a_n)$  is a bounded sequence. By the monotone subsequence theorem (lemma 1.2), it has a monotone subsequence  $(a_{n_k})$ . But then this subsequence is both bounded and monotone, whence it is convergent by the monotone convergence theorem (lemma 1.4).  $\square$

Finally, we prove step 3 in the proof of theorem 1.1.

**Proposition 1.4.** (Step 3 in the proof of theorem 1.1.) If the subsequence of a Cauchy sequence converges to  $x$  then the sequence itself converges to  $x$ .

**Proof.** Let  $(a_n)$  be a Cauchy sequence, and let  $(a_{n_k})$  be a convergent subsequence. Set  $x = \lim_{k \rightarrow \infty} a_{n_k}$ . We wish to prove that  $a_n \rightarrow x$  as  $n \rightarrow \infty$ . Given  $\varepsilon > 0$ , there exists  $K$  such that

$$|a_{n_k} - x| < \varepsilon/2 \quad \text{for all } k \geq K. \quad (1.13)$$

Also, since  $(a_n)$  is Cauchy, there exists  $N$  such that

$$|a_n - a_m| < \varepsilon/2 \quad \text{for all } n, m \geq N. \quad (1.14)$$

Pick  $l > K$  large enough so that  $n_l > N$ . Then for all  $n > N$ , we have

$$\begin{aligned} |a_n - x| &= |(a_n - a_{n_l}) + (a_{n_l} - x)| \\ &\leq |a_n - a_{n_l}| + |a_{n_l} - x| \\ &< \varepsilon, \end{aligned}$$

which concludes the proof of theorem 1.1.  $\square$