Notes

Quantum Mechanics

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PART

I

Caprice

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1.1 Derivation of I.2 (28) in Appendix 3 ïijĹNutshell)

To do an integral of form $I = \int_{-\infty}^{\infty} dq \exp[-f(x)/\hbar]$, we often have to resort to steepest-descent method. In limit of $\hbar \to 0$, the integral is dominated by the minimum of $f(a) = \min[f(q)]$, Expanding $f(q)$ around a

$$f(q) = f(a) + \tfrac{1}{2} f''(a)(q-a)^2 + \mathcal{O}[(q-a)^3] \tag{1.1}$$

This is a Gaussian integral

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \mathrm{d}q\, \mathrm{e}^{-f(x)/\hbar} \\ &= \mathrm{e}^{-f(a)/\hbar} \left[\frac{2\pi\hbar}{f''(a)}\right]^{\frac{1}{2}} \mathrm{e}^{-\mathcal{O}(\hbar^{\frac{1}{2}})} \end{aligned}$$

Suppose now $f(\mathbf{q})$ is a function of multiple variables $\mathbf{q} = (q_1, \dots, q_N)$, then the expandsion of $f(\mathbf{q})$ around the equilibrium position $\mathbf{a} = (a_1, \dots, a_N)$ can be written as

$$f(\mathbf{q}) = f(\mathbf{a}) + \tfrac{1}{2}(\mathbf{q}-\mathbf{a})^\top f''(\mathbf{a})(\mathbf{q}-\mathbf{a}) + \mathcal{O}[|\mathbf{q}-\mathbf{a}|^3] \tag{1.2}$$

where $f''(\mathbf{q})$ is understood as the Hessian of $f(\mathbf{q})$

$$[f''(\mathbf{q})]_{i,j} = \frac{\partial f}{\partial q_i \partial q_j} \tag{1.3}$$

Therefore we rewrite the integral

$$\begin{aligned} I &= \int \mathrm{d}\mathbf{q}\, \mathrm{e}^{-f(\mathbf{q}/\hbar)} \\ &= \mathrm{e}^{-f(\mathbf{a})/\hbar} \int \mathrm{d}\mathbf{q} \exp\left[\tfrac{1}{2\hbar}(\mathbf{q}-\mathbf{a})^\top f''(\mathbf{a})(\mathbf{q}-\mathbf{a})\right] \mathrm{e}^{-\mathcal{O}(\hbar^{\frac{1}{2}})} \end{aligned}$$

where

$$\begin{aligned} &\exp\left[\tfrac{1}{2\hbar}(\mathbf{q}-\mathbf{a})^\top f''(\mathbf{a})(\mathbf{q}-\mathbf{a})\right] \\ &= (\mathbf{q}-\mathbf{a})^\top \sum_{k=0}^{\infty} \frac{[f''(\mathbf{a})]^k}{k!(2\hbar)^k} (\mathbf{q}-\mathbf{a}) \\ &= (\mathbf{q}-\mathbf{a})^\top U^\top \sum_{k=0}^{\infty} \frac{[U f''(\mathbf{a}) U^\top]^k}{k!(2\hbar)^k} U(\mathbf{q}-\mathbf{a}) \\ &= \mathbf{y}^\top \exp\left[U f''(\mathbf{a}) U^\top / 2\hbar\right] \mathbf{y} \end{aligned}$$

where U is an orthogonal matrix such that $U^\top U = \mathbb{I}$, and $\mathbf{y} = U\mathbf{q}$. Choose U such that $Uf''(\mathbf{a})U^\top$ is a diagonal matrix, with the diagonal elements equal to its eigenvalues

$$\left[Uf''(\mathbf{a})U^\top\right]_{i,i} = \mu_i \tag{1.4}$$

Then

$$\mathbf{y}^\top \exp\left[Uf''(\mathbf{a})U^\top/2\hbar\right]\mathbf{y} = \exp\left(\tfrac{1}{2\hbar}\sum_i \mu_i y_i^2\right) \tag{1.5}$$

Under the change of variables, the differential elements $\mathrm{d}\mathbf{q} \to \mathrm{d}\mathbf{y} = |\frac{\partial \mathbf{y}}{\partial \mathbf{q}}|\mathrm{d}\mathbf{q}$, so the Jacobian is $\det(U) = 1$. Eventually we get the product of Gaussian integrals

$$\begin{aligned}
I &= \int \mathrm{d}\mathbf{q}\, \mathrm{e}^{-f(\mathbf{q}/\hbar)} \\
&= \int \mathrm{d}\mathbf{y} \exp\left(\tfrac{1}{2\hbar}\sum_i \mu_i y_i^2\right) \\
&= \prod_{i=1}^{N} \int_{-\infty}^{\infty} \mathrm{d}y_i \left[\exp\left(\tfrac{1}{2\hbar}\mu_i y_i^2\right)\right] \\
&= \left[\frac{(2\pi\hbar)^N}{\prod_i \mu_i}\right]^{\frac{1}{2}} = \left[\frac{(2\pi\hbar)^N}{\det[f''(\mathbf{a})]}\right]^{\frac{1}{2}}
\end{aligned}$$

1.2 Contraction identities

Show the following contraction identities are true:

$$\gamma_\lambda\gamma^\lambda = 4, \quad \gamma_\lambda\gamma^\alpha\gamma^\lambda = -2\gamma^\alpha$$
$$\gamma_\lambda\gamma^\alpha\gamma^\beta\gamma^\lambda = 4g^{\alpha\beta}, \quad \gamma_\lambda\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\lambda = -2\gamma^\gamma\gamma^\beta\gamma^\alpha$$
$$\gamma_\lambda\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\gamma^\lambda = 2(\gamma^\delta\gamma^\alpha\gamma^\beta\gamma^\gamma + \gamma^\gamma\gamma^\beta\gamma^\alpha\gamma^\delta)$$

Proof.

$$\begin{aligned}
\gamma_\lambda\gamma^\lambda &= \frac{1}{2}g_{\lambda\mu}\gamma^\mu\gamma^\lambda + \frac{1}{2}g_{\mu\lambda}\gamma^\lambda\gamma^\mu \\
&= \frac{1}{2}g_{\mu\lambda}[\gamma^\mu, \gamma^\lambda]_+ \\
&= g_{\mu\lambda}g^{\mu\lambda} = 4,
\end{aligned}$$

where we use $g_{\lambda\mu} = g_{\mu\lambda}$ is a symmetric tensor.

$$\begin{aligned}
\gamma_\lambda\gamma^\alpha\gamma^\lambda &= \gamma_\lambda(-\gamma^\lambda\gamma^\alpha + 2g^{\alpha\lambda}) \\
&= -4\gamma^\alpha + 2\gamma^\alpha = -2\gamma^\alpha.
\end{aligned}$$

$$\begin{aligned}
\gamma_\lambda\gamma^\alpha\gamma^\beta\gamma^\lambda &= \gamma_\lambda\gamma^\alpha(-\gamma^\lambda\gamma^\beta + 2g^{\beta\lambda}) \\
&= 2[\gamma^\alpha\gamma^\beta]_+ = 4g^{\alpha\beta}.
\end{aligned}$$

$$\begin{aligned}
\gamma_\lambda \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\lambda &= \gamma_\lambda \gamma^\alpha \gamma^\beta (-\gamma^\lambda \gamma^\gamma + 2g^{\gamma\lambda}) \\
&= -4\gamma^\gamma g^{\alpha\beta} + 2\gamma^\gamma \gamma^\alpha \gamma^\beta \\
&= -2(\gamma^\gamma \gamma^\alpha \gamma^\beta + \gamma^\gamma \gamma^\beta \gamma^\alpha) + 2\gamma^\gamma \gamma^\alpha \gamma^\beta \\
&= -2\gamma^\gamma \gamma^\beta \gamma^\alpha.
\end{aligned}$$

$$\begin{aligned}
\gamma_\lambda \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta \gamma^\lambda &= \gamma_\lambda \gamma^\alpha \gamma^\beta \gamma^\gamma (-\gamma^\lambda \gamma^\delta + 2g^{\delta\lambda}) \\
&= 2(\gamma^\delta \gamma^\alpha \gamma^\beta \gamma^\gamma + \gamma^\gamma \gamma^\beta \gamma^\alpha \gamma^\delta).
\end{aligned}$$

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