

# ***Notes***

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# Caprice

## 1 Fall 2018

This is my note for some non-trivial but not systematic problems which involves some interesting physics or maths.

| Section 1. Fall 2018

### 1.1 Walkway equilibrium

Suppose the mass of the objects attached to each end of the rope are  $m_1$  and  $m_2$ , The angles between each segment of the rope, bended by the central object which has mass  $M$ , with the horizontal plane are  $\theta$  and  $\phi$ . The distance between two pulleys is  $L$ , and what we want to know is the vertical displacement  $d$  of the central object. Thus we can obtained the equations for  $d$  when the system is at equilibrium.

$$L = d(\cot \theta + \cot \phi), \quad (1.1)$$

$$m_1 g \cos \theta = m_2 g \cos \phi, \quad (1.2)$$

$$m_1 g \sin \theta + m_2 g \sin \phi = Mg, \quad (1.3)$$

From (1.2), we have  $\cos \phi = \frac{m_1}{m_2} \cos \theta$ , thus (1.3) can be written as

$$m_1 \sin \theta + m_2 \sqrt{1 - \frac{m_1^2}{m_2^2} (1 - \sin^2 \theta)} = M, \quad (1.4)$$

such that we can solve for  $\sin \theta$  and  $\cos \theta$

$$\sin \theta = \frac{M^2 + m_1^2 - m_2^2}{2Mm_1}, \quad (1.5)$$

$$\cos \theta = \sqrt{1 - \sin^2 \theta} = \frac{1}{2Mm_1} \sqrt{[(m_1 + m_2)^2 - M^2][M^2 - (m_1 - m_2)^2]}, \quad (1.6)$$

$$\cot \theta = \frac{\sqrt{[(m_1 + m_2)^2 - M^2][M^2 - (m_1 - m_2)^2]}}{M^2 + m_1^2 - m_2^2}, \quad (1.7)$$

together with  $\sin \phi$  and  $\cos \phi$

$$\cos \phi = \frac{m_1}{m_2} \cos \theta = \frac{1}{2Mm_2} \sqrt{[(m_1 + m_2)^2 - M^2][M^2 - (m_1 - m_2)^2]}, \quad (1.8)$$

$$\sin \phi = \sqrt{1 - \cos^2 \phi} = \frac{M^2 - m_1^2 + m_2^2}{2Mm_2}, \quad (1.9)$$

$$\cot \phi = \frac{\sqrt{[(m_1 + m_2)^2 - M^2][M^2 - (m_1 - m_2)^2]}}{M^2 - m_1^2 + m_2^2}. \quad (1.10)$$

Therefore we can plug into (1.1) and obtain the expression of  $d$  as follows

$$d = \frac{L[M^4 - (m_1^2 - m_2^2)^2]}{2M^2 \sqrt{[(m_1 + m_2)^2 - M^2][M^2 - (m_1 - m_2)^2]}}. \quad (1.11)$$

The equilibrium condition in this case is

$$|m_1 - m_2| < M < (m_1 + m_2). \quad (1.12)$$

such that the argument under the square root is positive. Also we can easily check that if  $m_1 = m_2 = m$  then this result reduces to our former result

$$d = \frac{LM}{2\sqrt{4m^2 - M^2}}. \quad (1.13)$$

## 1.2 A derivation of Gamma function from Fourier transform

It is well-known that  $\Gamma(n+1) = n!$  for any natural number  $n \in \mathbb{N}$ . It is natural to ask what is  $\Gamma(x)$  for any real number  $x \geq 1$ . Our purpose is to show that Gamma function can be express as an integral

$$\Gamma(x) = \int_0^\infty dx t^{x-1} e^{-t}, \quad (1.14)$$

given that

$$\Gamma(x+1) = x\Gamma(x), \quad (1.15)$$

which is the most essential property and motivation to define the Gamma function.

Using Talor expansion we can easily show that

$$f(x + \Delta x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} (\Delta x)^n = \exp\left(\Delta x \frac{d}{dx}\right) f(x) \quad (1.16)$$

where  $\Delta x$  is a constant of translation. Therefore  $\Gamma(x+1) = e^{\frac{d}{dx}} \Gamma(x)$ , and we can rewrite (1.17) as

$$e^{\frac{d}{dx}} \Gamma(x) = x\Gamma(x). \quad (1.17)$$

Consider doing Fourier transform of (1.17), such that  $\frac{d}{dx} \rightarrow i\omega$ ,  $x \rightarrow i\frac{d}{d\omega}$ ,  $\Gamma(x) \rightarrow \tilde{\Gamma}(\omega)$ , and

$$\tilde{\Gamma}(\omega) = \mathcal{F}[\Gamma(x)] = \int_{-\infty}^{\infty} dx \Gamma(x) e^{-i\omega x}, \quad (1.18)$$

$$\Gamma(x) = \mathcal{F}^{-1}[\tilde{\Gamma}(\omega)] = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\Gamma}(\omega) e^{i\omega x}, \quad (1.19)$$

$$e^{i\omega} \tilde{\Gamma}(\omega) = i \frac{d}{d\omega} \tilde{\Gamma}(\omega). \quad (1.20)$$

Solve the above differential equation of  $\tilde{\Gamma}(\omega)$  we find

$$\tilde{\Gamma}(\omega) = C \exp(-e^{i\omega}), \quad (1.21)$$

$$\Gamma(x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} C \exp(-e^{i\omega}) e^{i\omega x}. \quad (1.22)$$

However, (1.22) does not converge, since (1.21) is a nonzero periodic function.

To resolve this difficulty of convergence, we expand the domain of Gamma function to the complex plane, such that  $\Gamma(z+1) = z\Gamma(z)$ , where  $z \in \mathbb{C}$ . Consider a pure imaginary number  $z = ix$ , where  $x \in \mathbb{R}$ , we can rewrite the recursion relation (1.15) as

$$e^{-i\frac{d}{dx}} \Gamma(ix) = ix\Gamma(ix) \quad (1.23)$$

Again using Fourier transform we have

$$e^{\omega} \mathcal{F}[\Gamma(ix)] = -\frac{d}{d\omega} \mathcal{F}[\Gamma(ix)], \quad (1.24)$$

where  $\mathcal{F}[\Gamma(ix)]$  is the Fourier transform of  $\Gamma(ix)$

$$\mathcal{F}[\Gamma(ix)] = \int_{-\infty}^{\infty} dx \Gamma(ix) e^{-i\omega x}. \quad (1.25)$$

Solve (1.24) we have

$$\mathcal{F}[\Gamma(ix)] = C \exp(-e^{\omega}), \quad (1.26)$$

$$\Gamma(ix) = \frac{C}{2\pi} \int_{-\infty}^{\infty} d\omega \exp(-e^{\omega}) e^{i\omega x}. \quad (1.27)$$

Thus

$$\begin{aligned} \Gamma(z) &= \frac{C}{2\pi} \int_{-\infty}^{\infty} d\omega \exp(-e^{\omega}) e^{\omega z} \\ &= \frac{C}{2\pi} \int_{-\infty}^{\infty} dt e^{\omega} \exp(-e^{\omega}) e^{\omega(z-1)} \\ &= \frac{C}{2\pi} \int_{-\infty}^{\infty} dt t^{z-1} e^{-t}. \end{aligned}$$

To determine the constant  $C$  we use the fact that  $\Gamma(1) = 0! = 1$ , thus  $C/2\pi = 1$ , and we obtain the final integral expression of Gamma function

$$\Gamma(z) = \int_{-\infty}^{\infty} dt t^{z-1} e^{-t} \quad (1.28)$$

where  $z \in \mathbb{C}$ .

### 1.3 Euler's reflection formula

In mathematics, a reflection formula or reflection relation for a function  $f$  is a relationship between  $f(a-x)$  and  $f(x)$ . A famous relationship is Euler's reflection formula

**Proposition 1.1.**

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad z \notin \mathbb{Z}. \quad (1.29)$$

**Proof.** To prove this reflection formula, we first notice a relationship between Gamma function and Beta function

$$B(q, p) = \frac{\Gamma(q)\Gamma(p)}{\Gamma(q+p)}, \quad (1.30)$$

where

$$B(q, p) = \int_0^1 dt t^{q-1} (1-t)^{p-1}, \quad q, p \neq 0, -1, -2, \dots \quad (1.31)$$

This can be shown by performing a variable transformation of  $\Gamma(q)\Gamma(p)$

$$\begin{aligned}\Gamma(q)\Gamma(p) &= \int_0^\infty du e^{-u} u^{q-1} \int_0^\infty dv e^{-v} v^{p-1} \\ u = zt, v = z(1-t), &= \int dz dt \left| \frac{\partial(u, v)}{\partial(z, t)} \right| e^{-z} (zt)^{q-1} [z(1-t)]^{p-1} \\ &= \int_0^\infty dz z^{q+p-1} e^{-z} \int_0^1 dt t^{q-1} (1-t)^{p-1} \\ &= \Gamma(q+p) B(q, p).\end{aligned}$$

Therefore

$$\Gamma(z)\Gamma(1-z) = B(z, 1-z) = \int_0^1 dt t^{z-1} (1-t)^{-z}. \quad (1.32)$$

In order to prove Proposition 1.1, we only need to prove

$$\int_0^1 dt t^{z-1} (1-t)^{-z} = \frac{\pi}{\sin \pi z}. \quad (1.33)$$

Perform a variable substitution  $t \rightarrow \frac{x}{1+x}$ , such that  $dt = dx/(1+x)^2$  and

$$\int_0^1 dt t^{z-1} (1-t)^{-z} = \int_0^\infty dx \frac{x^{z-1}}{1+x} \quad (1.34)$$

Consider the following integral

$$\int_0^\infty dx \frac{x^{\alpha-1}}{x + e^{i\phi}}, \quad 0 < \alpha < 1, \quad -\pi < \phi < \pi. \quad (1.35)$$

□

## 1.4 $\chi^2$ distribution with $(n-1)$ degrees of freedom

Let  $Y_1, Y_2, \dots, Y_n$  be independent random variables with  $E(Y_i) = \mu$  and  $V(Y_i) = \sigma^2$  for  $i = 1, 2, \dots, n$ . Let

$$U_1 = \sum_{i=1}^n a_i Y_i \quad \text{and} \quad U_2 = \sum_{i=1}^n b_i Y_i, \quad (1.36)$$

where  $a_1, a_2, \dots, a_n$ , and  $b_1, b_2, \dots, b_n$  are constants.  $U_1$  and  $U_2$  are said to be orthogonal if  $\text{Cov}(U_1, U_2) = 0$ .

1. Show that  $U_1$  and  $U_2$  are orthogonal if and only if  $\sum_{i=1}^n a_i b_i = 0$ .

**Proof.** Sufficiency: If  $\sum_{i=1}^n a_i b_i = 0$ , then

$$\begin{aligned}\text{Cov}(U_1, U_2) &= E \left[ \left( \sum_{i=1}^n a_i Y_i \right) \left( \sum_{j=1}^n b_j Y_j \right) \right] - E \left( \sum_{i=1}^n a_i Y_i \right) E \left( \sum_{j=1}^n b_j Y_j \right) \\ &= \sum_{i=1}^n a_i b_i [E(Y_i^2) - E(Y_i)^2] + \sum_{i \neq j} a_i b_j [E(Y_i Y_j) - E(Y_i) E(Y_j)] \\ &= \sigma^2 \sum_{i=1}^n a_i b_i = 0.\end{aligned}$$

Thus  $U_1$  and  $U_2$  are orthogonal. Necessity: If  $U_1$  and  $U_2$  are orthogonal,  $\text{Cov}(U_1, U_2) = \sigma^2 \sum_{i=1}^n a_i b_i = 0$ , then we must have  $\sum_{i=1}^n a_i b_i = 0$  if  $\sigma \neq 0$ . □

2. Show that if each component of independent random variables  $Y_1, Y_2, \dots, Y_n$  is normally distributed, then any linear combination  $U = a_1 Y_1 + a_2 Y_2 + \dots + a_n Y_n$  is normally distributed.

**Proof.** Proof by induction. We begin with two independent random variables  $X_1 = a_1 Y_1 \sim N(\mu_1, \sigma_1^2)$ ,  $X_2 = a_2 Y_2 \sim N(\mu_2, \sigma_2^2)$ , with  $\mu_i = a_i \mu$  and  $\sigma_i^2 = a_i^2 \sigma^2$ . Their sum  $Z = a_1 Y_1 + a_2 Y_2$ , which is a linear combination of  $Y_1$  and  $Y_2$ . The characteristic function

$$\varphi_Z(t) = \varphi_{X_1+X_2}(t) = E(e^{it(X_1+X_2)}) \quad (1.37)$$

of the sum of two independent random variables is just the product of the characteristic functions of each random variable

$$\varphi_{X_1+X_2}(t) = \varphi_{X_1}(t)\varphi_{X_2}(t) = E(e^{itX_1})E(e^{itX_2}). \quad (1.38)$$

The characteristic function of the normal distribution with expected value  $\mu$  and variance  $\sigma^2$  is

$$\varphi(t) = \exp[it\mu - \frac{1}{2}\sigma^2 t^2]. \quad (1.39)$$

Thus

$$\begin{aligned} \varphi_{X_1+X_2}(t) &= \exp[it\mu_1 - \frac{1}{2}\sigma_1^2 t^2] \exp[it\mu_2 - \frac{1}{2}\sigma_2^2 t^2] \\ &= \exp[it(\mu_1 + \mu_2) - \frac{1}{2}(\sigma_1^2 + \sigma_2^2)t^2]. \end{aligned}$$

This is the characteristic function of the normal distribution with expected value  $(\mu_1 + \mu_2)$  and variance  $(\sigma_1^2 + \sigma_2^2)$ . Finally, recall that no two distinct distributions can both have the same characteristic function, so the distribution of  $Z$  must be just this normal distribution.

Similarly we can prove that  $W = Z + X_3$ , which is a linear combination of  $Y_1, Y_2$  and  $Y_3$ , is also normally distributed. By induction, we prove that every linear combination of  $Y_1, Y_2, \dots, Y_n$  is normally distributed.  $\square$

3. Suppose, in addition, that  $Y_1, Y_2, \dots, Y_n$  have a multivariate normal distribution. Then  $U_1$  and  $U_2$  have a bivariate normal distribution. Show that  $U_1$  and  $U_2$  are independent if they are orthogonal.

**Proof.** If the random vector  $\mathbf{Y} = (Y_1, Y_2, \dots, Y_n)^\top$  have the multivariate normal distribution, then every linear combination of its components  $U = a_1 Y_1 + a_2 Y_2 + \dots + a_n Y_n$  is normally distributed. By definition, since the linear combination of  $U_1$  and  $U_2$  is still a linear combination of  $\mathbf{Y}$ , thus  $c_1 U_1 + c_2 U_2$  is normally distributed, and we say  $U_1$  and  $U_2$  have a bivariate normal distribution.

In general the bivariate density function of two random variables  $Y_1$  and  $Y_2$  has the following form

$$f(y_1, y_2) = \frac{e^{-Q/2}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}, \quad -\infty < y_1, y_2 < \infty, \quad (1.40)$$

where

$$Q = \frac{1}{1-\rho^2} \left[ \frac{(y_1 - \mu_1)^2}{\sigma_1^2} - 2\rho \frac{(y_1 - \mu_1)(y_2 - \mu_2)}{\sigma_1\sigma_2} + \frac{(y_2 - \mu_2)^2}{\sigma_2^2} \right]. \quad (1.41)$$

By doing a tedious integral exercise, we find  $\text{Cov}(Y_1, Y_2) = \rho\sigma_1\sigma_2$ . If  $\text{Cov}(Y_1, Y_2) = 0$ , i.e. if  $\rho = 0$ , then

$$f(y_1, y_2) = g(y_1)h(y_2), \quad (1.42)$$

where  $g(y_1)$  is a nonnegative function of  $y_1$  alone and  $h(y_2)$  is a nonnegative function of  $y_2$  alone. Therefore, if  $U_1$  and  $U_2$  are orthogonal, i.e.  $\text{Cov}(U_1, U_2) = 0$ , then  $U_1$  and  $U_2$  are independent. Notice that in general  $\text{Cov}(U_1, U_2) = 0$  doesn't imply  $U_1$  and  $U_2$  are independent. It is only in the context of bivariate normal distribution that we have this conclusion.  $\square$

Suppose that  $Y_1, Y_2, \dots, Y_n$  is a random sample from a normal distribution with mean  $\mu$  and variance  $\sigma^2$ . The independence of  $\sum_{i=1}^n (Y_i - \bar{Y})^2$  and  $\bar{Y}$  can be shown as follows. Define an  $n \times n$  matrix  $A$  by

$$A = \begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{\sqrt{2 \cdot 3}} & \frac{1}{\sqrt{2 \cdot 3}} & \frac{-2}{\sqrt{2 \cdot 3}} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{(n-1)n}} & \frac{1}{\sqrt{(n-1)n}} & \cdots & \frac{1}{\sqrt{(n-1)n}} & \frac{-(n-1)}{\sqrt{(n-1)n}} \end{bmatrix} \quad (1.43)$$

and notice that  $A^\top A = I$ , the identity matrix. Then,

$$\sum_{i=1}^n Y_i^2 = \mathbf{Y}^\top \mathbf{Y} = \mathbf{Y}^\top A^\top A \mathbf{Y}, \quad (1.44)$$

where  $\mathbf{Y}$  is the vector of  $Y_i$  values.

1. Show that

$$A\mathbf{Y} = \begin{bmatrix} \bar{Y}\sqrt{n} \\ U_1 \\ U_2 \\ \vdots \\ U_{n-1} \end{bmatrix}. \quad (1.45)$$

where  $U_1, U_2, \dots, U_{n-1}$  are linear functions of  $Y_1, Y_2, \dots, Y_n$ . Thus,

$$\sum_{i=1}^n Y_i^2 = n\bar{Y}^2 + \sum_{i=1}^{n-1} U_i^2. \quad (1.46)$$

2. Show that the linear functions  $\bar{Y}\sqrt{n}, U_1, U_2, \dots, U_{n-1}$  are pairwise orthogonal and hence independent under the normality assumption.

**Proof.** Notice that the coefficients of  $\bar{Y}$  and  $U_i$  are the elements of each row of matrix  $A$ , such that

$$A = \begin{bmatrix} \mathbf{a}_0^\top \\ \mathbf{a}_1^\top \\ \mathbf{a}_2^\top \\ \vdots \\ \mathbf{a}_{n-1}^\top \end{bmatrix}, \quad \bar{Y} = \mathbf{a}_0^\top \mathbf{Y}, \quad U_i = \mathbf{a}_i^\top \mathbf{Y}. \quad (1.47)$$



Since  $\mathbf{a}_i^\top \mathbf{a}_j = \delta_{ij}$ ,  $\bar{Y}\sqrt{n}, U_1, U_2, \dots, U_{n-1}$  are pairwise orthogonal. In addition,  $\mathbf{Y}$  has a multivariate normal distribution, because every linear combination of its components is normally distributed. Recall the previous exercise, if the multivariate normal distribution  $\bar{Y}\sqrt{n}, U_1, U_2, \dots, U_{n-1}$  are pairwise orthogonal, then they are independent random variables.  $\square$

3. Show that

$$\sum_{i=1}^n (Y_i - \bar{Y})^2 = \sum_{i=1}^{n-1} U_i^2 \quad (1.48)$$

and conclude that this quantity is independent of  $\bar{Y}$ .

**Proof.**

$$\begin{aligned} \sum_{i=1}^n (Y_i - \bar{Y})^2 &= \sum_{i=1}^n (Y_i^2 - 2Y_i\bar{Y} + \bar{Y}^2) \\ &= \sum_{i=1}^n Y_i^2 - n\bar{Y}^2 \\ &= \mathbf{Y}^\top A^\top A \mathbf{Y} - n\bar{Y}^2 \\ &= \begin{bmatrix} \bar{Y}\sqrt{n}, U_1, U_2, \dots, U_{n-1} \end{bmatrix} \begin{bmatrix} \bar{Y}\sqrt{n} \\ U_1 \\ U_2 \\ \vdots \\ U_{n-1} \end{bmatrix} - n\bar{Y}^2 \\ &= \sum_{i=1}^{n-1} U_i^2 \end{aligned}$$

Since  $U_i$  is independent of  $\bar{Y}$ ,  $\sum_{i=1}^{n-1} U_i^2$  is also independent of  $\bar{Y}$ .  $\square$

4. Using the results of part (3), show that

$$\frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{\sigma^2} = \frac{(n-1)S^2}{\sigma^2} \quad (1.49)$$

has a  $\chi^2$  distribution with  $(n-1)$  df.

**Proof.** First we show that  $U_i \sim N(0, \sigma^2)$ , and thus  $Z_i = U_i/\sigma \sim N(0, 1)$ , a standard normal distribution.

We have seen from the previous exercise that any linear combination of independent random variables with normal distribution is also normally distributed, so  $U_i$  has normal distribution, with expected value

$$E(U_i) = \mu \|\mathbf{a}_i\|_1 = 0, \quad (1.50)$$

and variance

$$\text{Var}(U_i) = \sigma^2 \|\mathbf{a}_i\|_2^2 = \sigma^2. \quad (1.51)$$

Next we prove that if each  $Z_i$  is independent with standard normal distribution, then  $\sum_{i=1}^n Z_i^2$  has a  $\chi^2$  distribution with  $n$  df.

The characteristic function of  $Z_i^2$  is

$$\begin{aligned}\varphi_{Z_i^2}(t) &= \mathbb{E}\left(e^{itZ_i^2}\right) \\ &= \int_{-\infty}^{\infty} dz \frac{1}{\sqrt{2\pi}} e^{itz^2} e^{-z^2/2} \\ &= (1 - 2it)^{-1/2},\end{aligned}$$

and from the fact that  $Z_i$  is independent with each other, the characteristic function of  $V = \sum_{i=1}^n Z_i^2$  is the product of  $n$  characteristic functions of  $Z_i^2$

$$\varphi_V(t) = \prod_{i=1}^n \varphi_{Z_i^2}(t) = (1 - 2it)^{-n/2} \quad (1.52)$$

Because characteristic functions are unique,  $V$  has a  $\chi^2$  distribution with  $n$  degrees of freedom.

Finally we prove that  $\sum_{i=1}^n (Y_i - \bar{Y})^2 / \sigma^2$  has a  $\chi^2$  distribution with  $n$  df. We rewrite the (1.49) as follows

$$\frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{\sigma^2} = \frac{\sum_{i=1}^{n-1} U_i^2}{\sigma^2} = \sum_{i=1}^{n-1} Z_i^2. \quad (1.53)$$

Thus it has a  $\chi^2$  distribution with  $(n - 1)$  degrees of freedom.  $\square$

## 1.5 Exam 08

**Exercise 1.1.** Suppose that  $X_1, X_2, \dots, X_n, X_{n+1}, \dots, X_{n+m}$  is a random sample from a normal distribution  $N(0, \sigma^2)$ .

1. Find  $a$  and  $b$  such that

$$V = a \left( \sum_{i=1}^n X_i \right)^2 + b \left( \sum_{i=n+1}^{n+m} X_i \right)^2 \quad (1.54)$$

has a  $\chi^2$  distribution.

**Solution.** Any linear combination of independent normally distributed random variables is also a random variable with normal distribution.

Therefore

$$Z_1 = \frac{\sum_{i=1}^n X_i}{\sigma\sqrt{n}}, \quad Z_2 = \frac{\sum_{i=n+1}^{n+m} X_i}{\sigma\sqrt{m}}. \quad (1.55)$$

Both  $Z_1$  and  $Z_2$  have standard normal distributions  $N(0, 1)$ . Thus

$$a \left( \sum_{i=1}^n X_i \right)^2 + b \left( \sum_{i=n+1}^{n+m} X_i \right)^2 = a n \sigma^2 Z_1^2 + b m \sigma^2 Z_2^2. \quad (1.56)$$

If  $a = 1/n\sigma^2$  and  $b = 1/m\sigma^2$ , then  $V = Z_1^2 + Z_2^2$  has a  $\chi^2$  distribution with 2 df.

2. Find  $c$  such that

$$U = \frac{c \sum_{i=1}^n X_i}{\sqrt{\sum_{i=n+1}^{n+m} X_i^2}} \quad (1.57)$$

has a  $t$  distribution.

**Solution.** Recall the definition of  $t$  distribution

**Definition 1.1.** Let  $Z$  be a standard normal random variable and let  $W$  be a  $\chi^2$ -distributed variable with  $\nu$  df. Then, if  $Z$  and  $W$  are independent,

$$T = \frac{Z}{\sqrt{W/\nu}}$$

is said to have a  $t$  distribution with  $\nu$  df.

From part (1) we know that

$$Z_1 = \frac{\sum_{i=1}^n X_i}{\sigma\sqrt{n}} \quad (1.58)$$

has a standard normal distribution. In addition

$$W = \sum_{i=n+1}^{n+m} \left(\frac{X_i}{\sigma}\right)^2 \quad (1.59)$$

is a  $\chi^2$ -distributed variable with  $m$  df. Thus

$$\begin{aligned} U &= \frac{c \sum_{i=1}^n X_i}{\sqrt{\sum_{i=n+1}^{n+m} X_i^2}} \\ &= c \sqrt{\frac{n}{m}} \frac{Z_1}{\sqrt{W/m}}. \end{aligned}$$

If  $c = \sqrt{m/n}$ , then  $U$  is a  $t$ -distributed variable with  $m$  df.

## 1.6 Shape of water in a rotating bucket

This is a static equilibrium problem in a rotating frame. Suppose the rotating bucket with radius  $R$  has an constant angular velocity  $\omega$ . When the system arrives equilibrium, the differential volume element of the water will have the same angular velocity  $\omega$ . We can write down its action in cylindrical coordinates

$$\begin{aligned} S &= \int_{t_0}^{t_1} dt \int_V dr^3 \mathcal{L}(t, r, z) \\ &= \text{const.} \int_M dm \left( \frac{1}{2} v^2 - gz \right) \\ &\propto \int_0^R dr 2\pi r z(r) \left[ \frac{1}{2} \omega^2 r^2 - \frac{1}{2} g z(r) \right] \\ &\propto \int_0^R dr (\omega^2 r^3 z - g r z^2), \end{aligned}$$

where  $z(r)$  is the shape of the water,  $\mathcal{L}$  is the Lagrangian density. There is an additional constraint of  $z(r)$  for the conservation of volume

$$\int_0^R dr 2\pi r z(r) = V = \text{const.} \quad (1.60)$$

So we have variation problem with additional condition. It is easy to solve using Lagrange multipliers' method. Necessary condition for  $S$  to have extrema is existing a multiplier  $\lambda$

such that

$$F = \omega^2 r^3 z - grz^2 + \lambda rz, \quad (1.61)$$

$$\frac{d}{dr} \frac{\partial F}{\partial \frac{\partial z}{\partial r}} - \frac{\partial F}{\partial z} = 0. \quad (1.62)$$

Since  $F$  does not depend on  $\frac{\partial z}{\partial r}$  explicitly, we obtain

$$\omega^2 r^3 - 2grz + \lambda r = 0, \quad (1.63)$$

or equivalently

$$z = \frac{\omega^2 r^2}{2g} + \lambda, \quad (1.64)$$

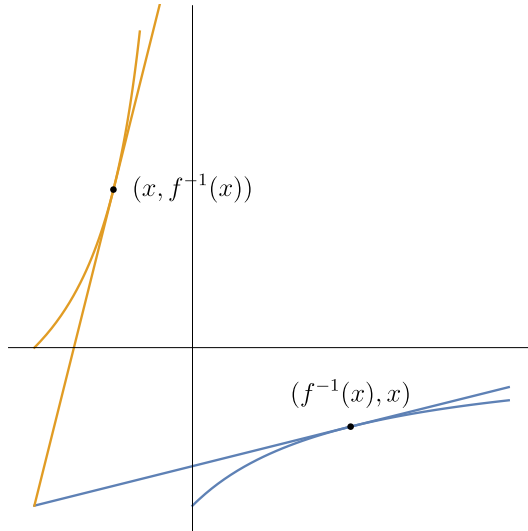
where  $\lambda$  is determined by

$$\int_0^R dr \, 2\pi r z(r) = \frac{\pi \omega^2 R^4}{4g} + \lambda \pi R^2 = V,$$

$$\lambda = \frac{V}{\pi R^2} - \frac{\omega^2 R^2}{4g}.$$

## 1.7 Derivative of inverse function

Suppose we are given a function  $f$  with its inverse  $f^{-1}$ . Using a little geometry, we can compute the derivative  $df^{-1}(x)/dx$  in terms of  $f$ . See Fig(1).



**Figure 1.** A schematic picture for function  $f$  and its inverse  $f^{-1}$ . The slopes at points  $(f^{-1}(x), x)$  and  $(x, f^{-1}(x))$  are reciprocal to each other.

Hence we have

$$\frac{df^{-1}(x)}{dx} = 1 / \left. \frac{df(x')}{dx'} \right|_{x'=f^{-1}(x)}. \quad (1.65)$$

**Example 1.1.**

$$\begin{aligned}
\frac{d}{dx} \arcsin x &= 1 \Big/ \left. \frac{d}{dx'} \sin x' \right|_{x'=\arcsin x} \\
&= \frac{1}{\left. \cos x' \right|_{x'=\arcsin x}} \\
&= \frac{1}{\left. \sqrt{1 - \sin^2 x'} \right|_{x'=\arcsin x}} \\
&= \frac{1}{\sqrt{1 - x^2}}.
\end{aligned}$$

**1.8 s-wave scattering and zero energy bound states**

We look at a rectangular potential which is attractive

$$V(r) = \begin{cases} -V_0, & r < r_0 \\ 0, & r > r_0 \end{cases}. \quad (1.66)$$

where  $V_0 > 0$ . For low energy, we focus on s-wave scattering. Inside the potential the wave function  $u(r) = rA_0(r) \rightarrow 0$ , as  $r \rightarrow 0$ , because  $A_0(r)$  must be finite at  $r = 0$ . Hence

$$u(r) = A \sin(k'r), \quad (1.67)$$

where  $k' = \sqrt{2m(E + V_0)/\hbar^2}$  is the wave number inside the potential. Outside the potential we have a phase shift  $\delta_0$ , so we have

$$u(r) = B \sin(kr + \delta_0), \quad (1.68)$$

where  $k = \sqrt{2mE/\hbar^2}$  is the wave number outside the potential. Next we match the boundary conditions

$$\begin{aligned}
A \sin(k'r_0) &= B \sin(kr_0 + \delta_0), \\
Ak' \cos(k'r_0) &= Bk \cos(kr_0 + \delta_0).
\end{aligned}$$

Divide the two equations above we have

$$\tan(kr_0 + \delta_0) = \frac{k}{k'} \tan k'r_0, \quad (1.69)$$

or

$$\delta_0 = -kr_0 + \tan^{-1} \left( \frac{k}{k'} \tan(k'r_0) \right). \quad (1.70)$$

**2 Summer 2019****2.1 Sommerfeld expansion**

For low-temperature ideal fermi gas, we always encounter integrals of the following forms

$$\int_0^\infty d\epsilon \frac{F(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1}. \quad (2.1)$$

where

$$f(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} + 1} \quad (2.2)$$

is the Fermi-Dirac distribution function,  $\mu(T)$  is the chemical potential which depends on temperature  $T$ , and  $\beta = 1/T$ . If  $T \ll \mu(0)$ , we may employ expansion around the fermi level  $\mu(0)$ , such that

$$\int_{-\infty}^{\infty} d\epsilon \frac{F(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1} = \int_{-\infty}^{\mu(0)} d\epsilon F(\epsilon) + \frac{1}{6}\pi^2 T^2 F'(\mu) + O(T/\mu)^4. \quad (2.3)$$

Or equivalently its variant form

$$\int d\epsilon F(\epsilon) \frac{\partial f}{\partial \epsilon} = -F(\mu) - \frac{1}{6}\pi^2 T^2 F''(\mu) + O(T/\mu)^4. \quad (2.4)$$

This is the well-known Sommerfeld expansion. The two variants (2.3) and (2.4) can be related by integration by parts. Now we want to show the coefficient of the second term is  $\frac{\pi^2}{6}$ .

**Proposition 2.1.** Show that the following definite integral is true.

$$\int_{-\infty}^{\infty} dz \frac{z^2}{\cosh^2 z} = \frac{1}{12}\pi^2. \quad (2.5)$$

**Proof.** We do integral by parts of the integral expression of Riemann zeta function  $\zeta(\nu)$  for  $\nu \neq 1$

$$\begin{aligned} \zeta(\nu) &= \frac{1}{\Gamma(\nu)} \int_0^{\infty} \frac{x^{\nu-1} dx}{e^x - 1} \\ &= \frac{1}{\Gamma(\nu)} \frac{1}{\nu} \left( \frac{x^{\nu}}{e^x - 1} \Big|_0^{\infty} + \int_0^{\infty} \frac{x^{\nu} e^x dx}{(e^x - 1)^2} \right) \\ &= \frac{1}{\Gamma(\nu + 1)} \int_0^{\infty} \frac{x^{\nu} e^x dx}{(e^x - 1)^2} \\ &= \frac{1}{\Gamma(\nu + 1)} \int_0^{\infty} \frac{x^{\nu} dx}{4 \sinh^2(x/2)}. \end{aligned}$$

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