Notes

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1 Fall 2018

This is my note for some non-trivial but not systematic problems which involves some interesting physics or maths.

Section 1. Fall 2018

1.1 Walkway equilibrium

Suppose the mass of the objects attached to each end of the rope are m_1 and m_2, The angles between each segment of the rope, bended by the central object which has mass M, with the horizontal plane are θ and ϕ. The distance between two pulleys is L, and what we want to know is the vertical displacement d of the central object. Thus we can obtaind the equations for d when the system is at equilibrium.

$$L = d(\cot\theta + \cot\phi), \tag{1.1}$$

$$m_1 g\cos\theta = m_2\cos\phi, \tag{1.2}$$

$$m_1 g\sin\theta + m_2 g\sin\phi = Mg, \tag{1.3}$$

From (1.2), we have $\cos\phi = \frac{m_1}{m_2}\cos\theta$, thus (1.3) can be written as

$$m_1\sin\theta + m_2\sqrt{1 - \frac{m_1^2}{m_2^2}(1-\sin^2\theta)} = M, \tag{1.4}$$

such that we can solve for $\sin\theta$ and $\cos\theta$

$$\sin\theta = \frac{M^2 + m_1^2 - m_2^2}{2Mm_1}, \tag{1.5}$$

$$\cos\theta = \sqrt{1-\sin^2\theta} = \frac{1}{2Mm_1}\sqrt{[(m_1+m_2)^2 - M^2][M^2 - (m_1-m_2)^2]}, \tag{1.6}$$

$$\cot\theta = \frac{\sqrt{[(m_1+m_2)^2 - M^2][M^2 - (m_1-m_2)^2]}}{M^2 + m_1^2 - m_2^2}, \tag{1.7}$$

together with $\sin\phi$ and $\cos\phi$

$$\cos\phi = \frac{m_1}{m_2}\cos\theta = \frac{1}{2Mm_2}\sqrt{[(m_1+m_2)^2 - M^2][M^2 - (m_1-m_2)^2]}, \tag{1.8}$$

$$\sin\phi = \sqrt{1-\cos^2\phi} = \frac{M^2 - m_1^2 + m_2^2}{2Mm_2}, \tag{1.9}$$

$$\cot\phi = \frac{\sqrt{[(m_1+m_2)^2 - M^2][M^2 - (m_1-m_2)^2]}}{M^2 - m_1^2 + m_2^2}. \tag{1.10}$$

Therefore we can plug into (1.1) and obtain the expression of d as follows

$$d = \frac{L[M^4 - (m_1^2 - m_2^2)^2]}{2M^2\sqrt{[(m_1+m_2)^2 - M^2][M^2 - (m_1-m_2)^2]}}. \tag{1.11}$$

The equilibrium condition in this case is

$$|m_1 - m_2| < M < (m_1 + m_2). \tag{1.12}$$

such that the argument under the square root is positive. Also we can easily check that if $m_1 = m_2 = m$ then this result reduces to our former result

$$d = \frac{LM}{2\sqrt{4m^2 - M^2}}. \tag{1.13}$$

1.2 A derivation of Gamma function from Fourier transfrom

It is well-known that $\Gamma(n+1) = n!$ for any natural number $n \in \mathbb{N}$. It is natural to ask what is $\Gamma(x)$ for any real number $x \geq 1$. Our purpose is to show that Gamma function can be express as an integral

$$\Gamma(x) = \int_0^\infty \mathrm{d}x\, t^{x-1}\mathrm{e}^t, \tag{1.14}$$

given that

$$\Gamma(x+1) = x\Gamma(x), \tag{1.15}$$

which is the most essential property and motivation to define the Gamma function.

Using Talor expansion we can easily show that

$$f(x + \Delta x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(\Delta x)^n = \exp\left(\Delta x \frac{\mathrm{d}}{\mathrm{d}x}\right) f(x) \tag{1.16}$$

where Δx is a constant of translation. Therefore $\Gamma(x+1) = \mathrm{e}^{\frac{\mathrm{d}}{\mathrm{d}x}}\Gamma(x)$, and we can rewrite (1.17) as

$$\mathrm{e}^{\frac{\mathrm{d}}{\mathrm{d}x}}\Gamma(x) = x\Gamma(x). \tag{1.17}$$

Consider doing Fourier transform of (1.17), such that $\frac{\mathrm{d}}{\mathrm{d}x} \to i\omega$, $x \to i\frac{\mathrm{d}}{\mathrm{d}\omega}$, $\Gamma(x) \to \tilde{\Gamma}(\omega)$, and

$$\tilde{\Gamma}(\omega) = \mathcal{F}[\Gamma(x)] = \int_{-\infty}^{\infty} \mathrm{d}x\, \Gamma(x)\mathrm{e}^{-i\omega x}, \tag{1.18}$$

$$\Gamma(x) = \mathcal{F}^{-1}[\tilde{\Gamma}(\omega)] = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\tilde{\Gamma}(\omega)\mathrm{e}^{i\omega x}, \tag{1.19}$$

$$\mathrm{e}^{i\omega}\tilde{\Gamma}(\omega) = i\frac{\mathrm{d}}{\mathrm{d}\omega}\tilde{\Gamma}(\omega). \tag{1.20}$$

Solve the above differential equation of $\tilde{\Gamma}(\omega)$ we find

$$\tilde{\Gamma}(\omega) = C\exp\left(-\mathrm{e}^{i\omega}\right), \tag{1.21}$$

$$\Gamma(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} C\exp\left(-\mathrm{e}^{i\omega}\right)\mathrm{e}^{i\omega x}. \tag{1.22}$$

However, (1.22) does not converge, since (1.21) is a nonzero periodic function.

To resolve this difficulty of convergence, we expand the domain of Gamma function to the complex plane, such that $\Gamma(z+1) = z\Gamma(z)$, where $z \in \mathbb{C}$. Consider a pure imaginary number $z = ix$, where $x \in \mathbb{R}$, we can rewrite the recursion relation (1.15) as

$$\mathrm{e}^{-i\frac{\mathrm{d}}{\mathrm{d}x}}\Gamma(ix) = ix\Gamma(ix) \tag{1.23}$$

Again using Fourier transform we have

$$e^{\omega}\mathcal{F}[\Gamma(ix)] = -\frac{d}{d\omega}\mathcal{F}[\Gamma(ix)], \tag{1.24}$$

where $\mathcal{F}[\Gamma(ix)]$ is the Fourier transform of $\Gamma(ix)$

$$\mathcal{F}[\Gamma(ix)] = \int_{-\infty}^{\infty} dx\, \Gamma(ix)e^{-i\omega x}. \tag{1.25}$$

Solve (1.24) we have

$$\mathcal{F}[\Gamma(ix)] = C\exp(-e^{\omega}), \tag{1.26}$$

$$\Gamma(ix) = \frac{C}{2\pi}\int_{-\infty}^{\infty} d\omega \exp(-e^{\omega})e^{i\omega x}. \tag{1.27}$$

Thus

$$\begin{aligned}\Gamma(z) &= \frac{C}{2\pi}\int_{-\infty}^{\infty} d\omega \exp(-e^{\omega})e^{\omega z}\\ &= \frac{C}{2\pi}\int_{-\infty}^{\infty} de^{\omega} \exp(-e^{\omega})e^{\omega(z-1)}\\ &= \frac{C}{2\pi}\int_{-\infty}^{\infty} dt\, t^{z-1}e^{-t}.\end{aligned}$$

To determine the constant C we use the fact that $\Gamma(1) = 0! = 1$, thus $C/2\pi = 1$, and we obtain the final integral expression of Gamma function

$$\Gamma(z) = \int_{-\infty}^{\infty} dt\, t^{z-1}e^{-t} \tag{1.28}$$

where $z \in \mathbb{C}$.

1.3 Euler's reflection formula

In mathematics, a reflection formula or reflection relation for a function f is a relationship between $f(a-x)$ and $f(x)$. A famous relationship is Euler's reflection formula

Proposition 1.1.

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}, \quad z \notin \mathbb{Z}. \tag{1.29}$$

Proof. To prove this reflection formula, we first notice a relationship between Gamma function and Beta function

$$B(q,p) = \frac{\Gamma(q)\Gamma(p)}{\Gamma(q+p)}, \tag{1.30}$$

where

$$B(q,p) = \int_0^1 dt\, t^{q-1}(1-t)^{p-1}, \quad q,p \neq 0,-1,-2,\dots \tag{1.31}$$

This can be shown by performing a variable transformation of $\Gamma(q)\Gamma(p)$

$$
\begin{aligned}
\Gamma(q)\Gamma(p) &= \int_0^\infty \mathrm{d}u\, \mathrm{e}^{-u} u^{q-1} \int_0^\infty \mathrm{d}v\, \mathrm{e}^{-v} v^{p-1} \\
u = zt, v = z(1-t), \quad &= \int \mathrm{d}z\, \mathrm{d}t \left| \frac{\partial(u,v)}{\partial(z,t)} \right| \mathrm{e}^{-z} (zt)^{q-1} [z(1-t)]^{p-1} \\
&= \int_0^\infty \mathrm{d}z\, z^{q+p-1} \mathrm{e}^{-z} \int_0^1 \mathrm{d}t\, t^{q-1} (1-t)^{p-1} \\
&= \Gamma(q+p) B(q,p).
\end{aligned}
$$

Therefore

$$
\Gamma(z)\Gamma(1-z) = B(z, 1-z) = \int_0^1 \mathrm{d}t\, t^{z-1} (1-t)^{-z}. \tag{1.32}
$$

In order to prove Proposition 1.1, we only need to prove

$$
\int_0^1 \mathrm{d}t\, t^{z-1} (1-t)^{-z} = \frac{\pi}{\sin \pi z}. \tag{1.33}
$$

Perform a variable substitution $t \to \frac{x}{1+x}$, such that $\mathrm{d}t = \mathrm{d}x\,/(1+x)^2$ and

$$
\int_0^1 \mathrm{d}t\, t^{z-1} (1-t)^{-z} = \int_0^\infty \mathrm{d}x\, \frac{x^{z-1}}{1+x} \tag{1.34}
$$

Consider the following integral

$$
\int_0^\infty \mathrm{d}x\, \frac{x^{\alpha-1}}{x + \mathrm{e}^{i\phi}}, \quad 0 < \alpha < 1, \quad -\pi < \phi < \pi. \tag{1.35}
$$

$\square$