Oleksandr Romanko, Ph.D.
Senior Research Analyst, Risk Analytics, Watson Financial Services, IBM Canada
Adjunct Professor, University of Toronto

MIE1624H – Introduction to Data Science and Analytics

Lecture 8 – Optimization

University of Toronto
November 13, 2018

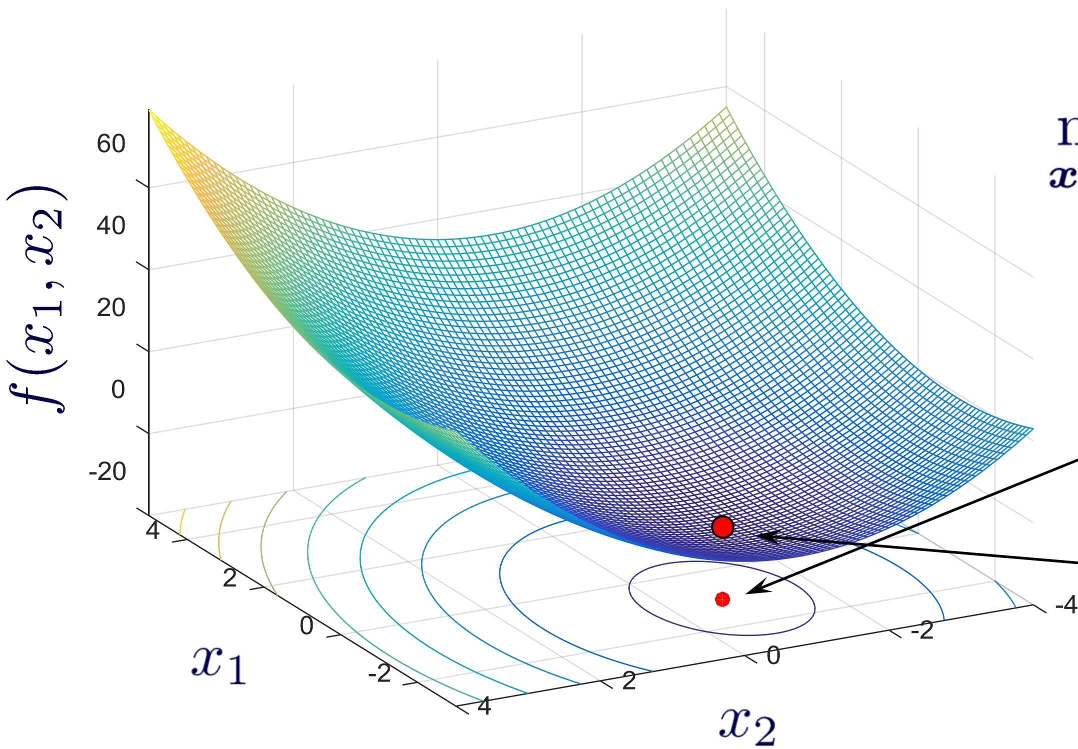
Overview of Optimization

Optimization

- **Optimization problem**

$$\begin{array}{ll} \text{minimize}_{\boldsymbol{x}} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \Omega \end{array}$$

- Examples:
  - Minimize **cost**
  - Maximize **profit**



$$\min_{\boldsymbol{x} \in \mathbb{R}^2} \quad x_1^2 + 2x_2^2 + 2x_1 + 3x_2$$

solve

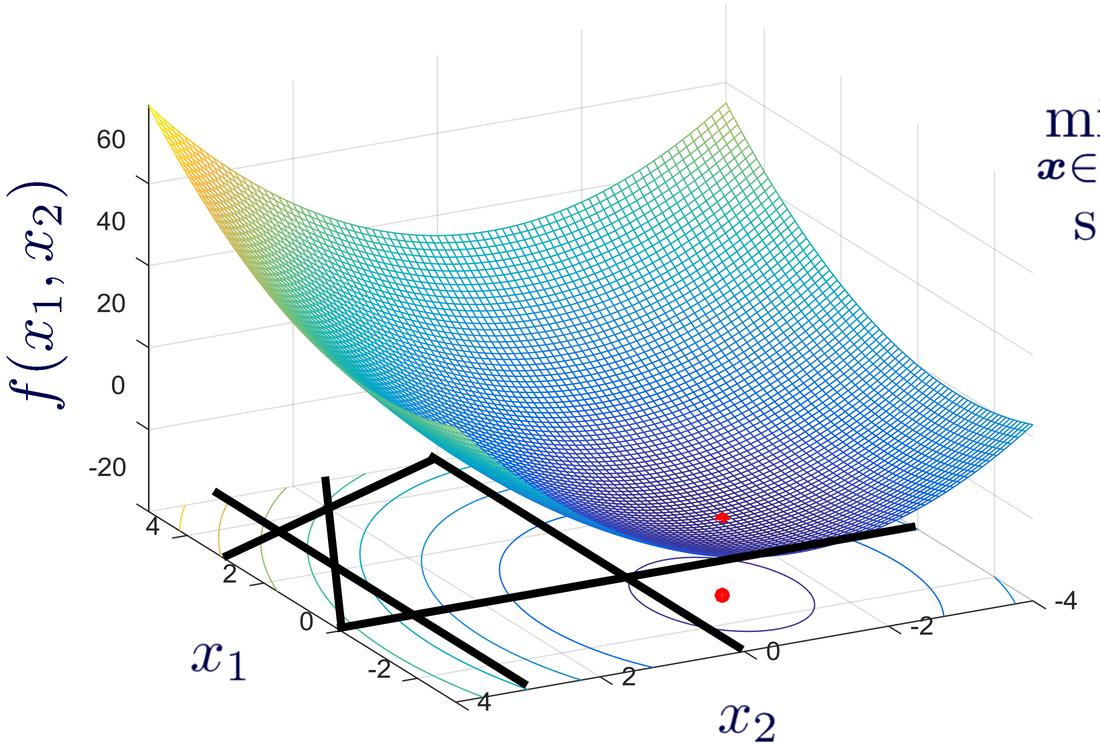
$$\boldsymbol{x}^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} -1.00 \\ -0.75 \end{pmatrix}$$

$$f(\boldsymbol{x}^*) = -2.125$$

Optimization

- **Optimization problem**

$$\begin{array}{ll} \text{minimize}_{\boldsymbol{x}} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \Omega \end{array}$$



$$\begin{array}{rl} \min\limits_{\boldsymbol{x}\in\mathbb{R}^2} & x_1^2 + 2x_2^2 + 2x_1 + 3x_2 \\ \text{s.t.} & x_1 + 2x_2 \leq 8 \\ & 2x_1 + \ \ x_2 \leq 10 \\ & x_2 \leq 3 \\ & x_1,\ x_2 \geq 0 \end{array}$$

Optimization

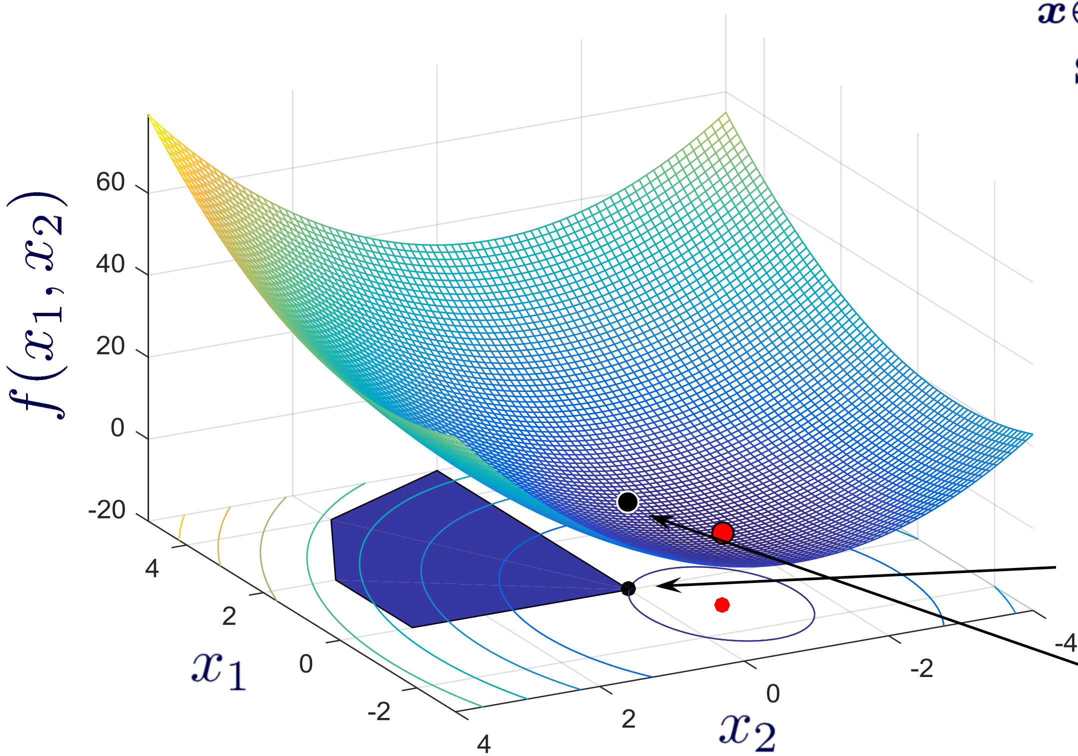
- **Optimization problem**

$$\begin{array}{ll} \text{minimize}_{\boldsymbol{x}} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \Omega \end{array}$$

$$\begin{array}{rl} \min\limits_{\boldsymbol{x}\in\mathbb{R}^2} & x_1^2 + 2x_2^2 + 2x_1 + 3x_2 \\ \text{s.t.} & x_1 + 2x_2 \leq 8 \\ & 2x_1 + x_2 \leq 10 \\ & x_2 \leq 3 \\ & x_1,\, x_2 \geq 0 \end{array}$$

solve

$$\boldsymbol{x}^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



$$f(\boldsymbol{x}^*) = 0$$

Optimization

- **Optimization problem**

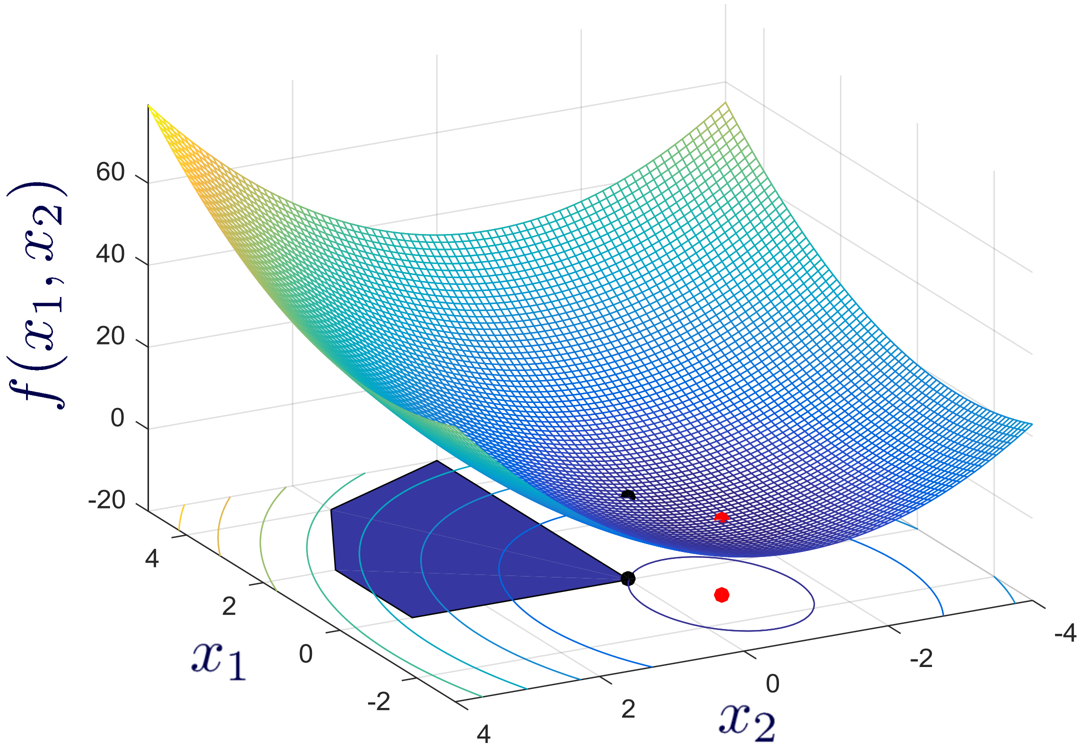
$$\begin{array}{ll} \text{minimize}_{\boldsymbol{x}} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \Omega \end{array}$$

- **Minimizing convex quadratic (QP) objective function** over a **polyhedron** (linear constraints)

$$\begin{array}{ll} \displaystyle\min_{\boldsymbol{x}\in\mathbb{R}^n} & \boldsymbol{c}^T\boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^T Q\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{l} \leq A\boldsymbol{x} \leq \boldsymbol{u} \\ & \boldsymbol{l}_b \leq \boldsymbol{x} \leq \boldsymbol{u}_b \end{array}$$

general form

$$\begin{array}{ll} \displaystyle\min_{\boldsymbol{x}\in\mathbb{R}^2} & x_1^2 + 2x_2^2 + 2x_1 + 3x_2 \\ \text{s.t.} & x_1 + 2x_2 \leq 8 \\ & 2x_1 + x_2 \leq 10 \\ & x_2 \leq 3 \\ & x_1,\, x_2 \geq 0 \end{array}$$



Optimization

- **Optimization problem**

$$\begin{aligned} \text{minimize}_{\boldsymbol{x}} \quad & f(\boldsymbol{x}) \\ \text{subject to} \quad & \boldsymbol{x} \in \Omega \end{aligned}$$

- Examples:
  - Minimize **cost**
  - Maximize **profit**

- **Multi-objective optimization**: simultaneously optimizing two or more conflicting objectives subject to certain constraints

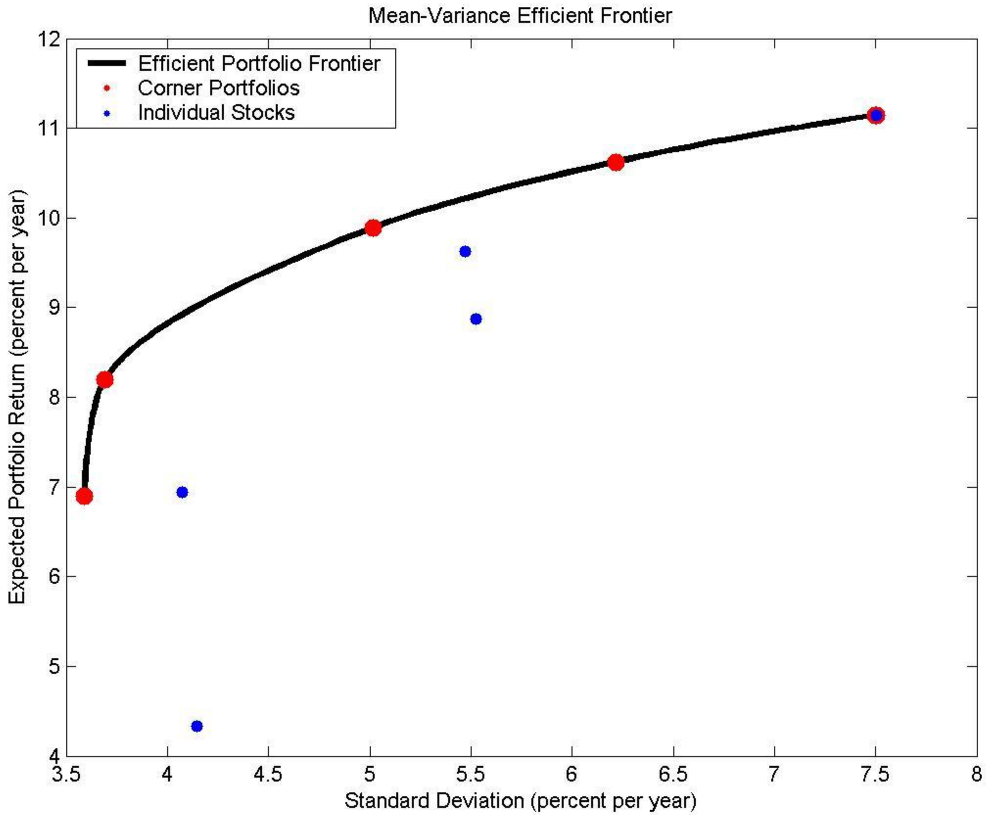
$$\begin{aligned} \min_{\boldsymbol{x}} \quad & F(\boldsymbol{x}) = [f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \ldots, f_k(\boldsymbol{x})]^T \\ \text{s.t} \quad & \boldsymbol{x} \in \Omega \end{aligned}$$

- Examples:
  - Minimize **cost** & Minimize **environmental impact**
  - Minimize **risk** & Maximize **return**

Multi-objective optimization

- Solving **multi-objective optimization** problems:

$$\begin{array}{rl} \text{minimize} & \textbf{risk} \\ \text{subject to} & \textbf{return} \geq \textbf{target} \\ & \textbf{other constraints} \end{array}$$



🕮 Linear Optimization Examples

Workforce planning

- Restaurant is open 7 days a week
- Based on past experience, the number of workers needed on a particular day:

Day	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Number	14	13	15	16	19	18	11

- Every worker works five consecutive days, and then takes two days off
- Minimize the number of workers that staff the restaurant
- Decision variables:
  - (**wrong**) x_i is the number of workers that work on day i
  - (**right**) x_i is the number of workers who begin their five-day shift on day i
- Objective function:

$$\min_{\boldsymbol{x} \in \mathbb{R}^7} \quad x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$$

- Bounds on variables:

$$x_i \geq 0 \qquad \forall\ i$$

Workforce planning

- Constraints:
  - Consider the constraint for Monday's staffing level of 14
  - Who works on Monday? Those who start their shift on Monday (x_1)
  - Those who start on Tuesday (x_2) do not work on Monday, nor do those who start on Wednesday (x_3)
  - Those who start on Thursday (x_4) do work on Monday, as do those who start on Friday (x_5), Saturday (x_6), and Sunday (x_7)
  - This gives the constraint:

$$x_1 + x_4 + x_5 + x_6 + x_7 \geq 14$$

- Similar argument give a total formulation

Workforce planning – expression formulation

- Constraints:
  - Consider the constraint for Monday's staffing level of 14
  - Who works on Monday? Those who start their shift on Monday (x_1)
  - Those who start on Tuesday (x_2) do not work on Monday, nor do those who start on Wednesday (x_3)
  - Those who start on Thursday (x_4) do work on Monday, as do those who start on Friday (x_5), Saturday (x_6), and Sunday (x_7)
- Similar argument give a total formulation:

$$
\begin{aligned}
\min_{\boldsymbol{x}\in\mathbb{R}^7} \quad & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \\
\text{s.t.} \quad & x_1 + x_4 + x_5 + x_6 + x_7 \geq 14 \\
& x_1 + x_2 + x_5 + x_6 + x_7 \geq 13 \\
& x_1 + x_2 + x_3 + x_6 + x_7 \geq 15 \\
& x_1 + x_2 + x_3 + x_4 + x_7 \geq 16 \\
& x_1 + x_2 + x_3 + x_4 + x_5 \geq 19 \\
& x_2 + x_3 + x_4 + x_5 + x_6 \geq 18 \\
& x_3 + x_4 + x_5 + x_6 + x_7 \geq 11 \\
& x_i \geq 0 \quad \forall\, i
\end{aligned}
$$

Workforce planning – matrix formulation

Mathematical formulation

$$\begin{aligned}
\min_{\boldsymbol{x}\in\mathbb{R}^7} \quad & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \\
\text{s.t.} \quad & x_1 + x_4 + x_5 + x_6 + x_7 \geq 14 \\
& x_1 + x_2 + x_5 + x_6 + x_7 \geq 13 \\
& x_1 + x_2 + x_3 + x_6 + x_7 \geq 15 \\
& x_1 + x_2 + x_3 + x_4 + x_7 \geq 16 \\
& x_1 + x_2 + x_3 + x_4 + x_5 \geq 19 \\
& x_2 + x_3 + x_4 + x_5 + x_6 \geq 18 \\
& x_3 + x_4 + x_5 + x_6 + x_7 \geq 11 \\
& x_i \geq 0 \quad \forall\, i
\end{aligned}$$

$$\Rightarrow \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} \quad \Rightarrow$$

Solver formulation

$$\begin{aligned}
\min_{\boldsymbol{x}\in\mathbb{R}^n} \quad & \boldsymbol{c}^T\boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{l} \leq A\boldsymbol{x} \leq \boldsymbol{u} \\
& \boldsymbol{l}_b \leq \boldsymbol{x} \leq \boldsymbol{u}_b
\end{aligned}$$

$$\boldsymbol{c} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^T \qquad \boldsymbol{l}_b = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T \qquad \boldsymbol{u}_b = \boldsymbol{u}$$

$$\underbrace{\begin{pmatrix} 14 \\ 13 \\ 15 \\ 16 \\ 19 \\ 18 \\ 11 \end{pmatrix}}_{\boldsymbol{l}} \leq \underbrace{\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}}_{A} \cdot \boldsymbol{x} \leq \underbrace{\begin{pmatrix} +\infty \\ +\infty \\ +\infty \\ +\infty \\ +\infty \\ +\infty \\ +\infty \end{pmatrix}}_{\boldsymbol{u}}$$

Road design problem

- Given point **A** and **B**, and a map, find a road that will be the cheapest to construct/maintain

Source: Warren L. Hare, A Multi-Haul Model for Vertical Alignment Optimization of Road Design

Road design problem

- Given point **A** and **B**, and a map, find a road that will be the cheapest to construct/maintain

Source: Warren L. Hare, A Multi-Haul Model for Vertical Alignment Optimization of Road Design

Road design problem

- Given point **A** and **B**, and a map, find a road that will be the cheapest to construct/maintain

Source: Warren L. Hare, A Multi-Haul Model for Vertical Alignment Optimization of Road Design

Road design problem - costs (easiest to hardest to compute)

- Cost depends on many factors
  - Land acquisition ≈ 0% to 25%
  - Bridges, tunnels, etc. ≈ 0% to 20%
  - Per km cost (final paving, maintenance) ≈ 5% to 15%
  - Earthwork ≈ 20% to 50%
  - Long term economic costs ≈ ?
  - Environmental issues ≈ ?

Source: Warren L. Hare, A Multi-Haul Model for Vertical Alignment Optimization of Road Design

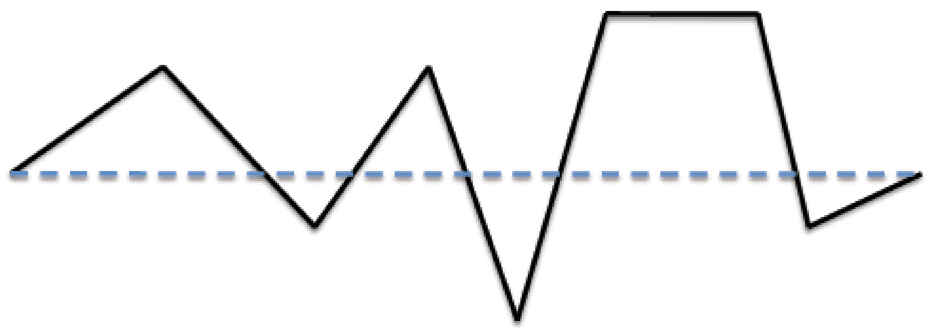
Road optimization

- Level 1: Horizontal Alignment (HA)
  - Look at a map and select a path for the road to follow
  - Evaluate HA based on safety constraints and Vertical Alignment
- Level 2: Vertical Alignment (VA)
  - Use HA to build a cross section of the terrain
  - Determine a VA for the future road
  - Evaluate VA based safety constraints and Earthwork
- Level 3: Earthwork (EW)
  - Determine how to move earth in order to make the terrain fit the VA
  - Cost based on minimization

Vertical Alignment and Earthwork can be modelled and solved simultaneously as a mixed-integer linear optimization problem

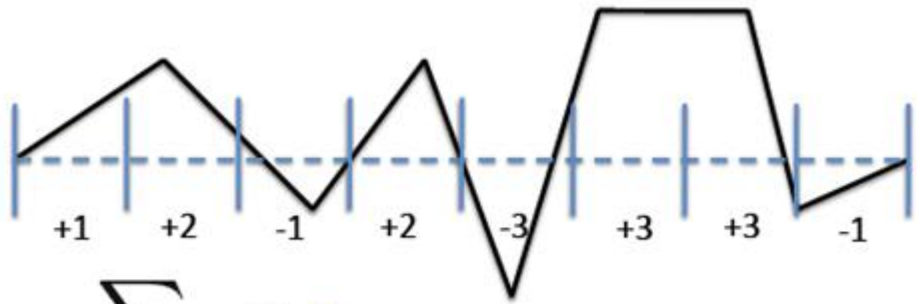
Source: Warren L. Hare, A Multi-Haul Model for Vertical Alignment Optimization of Road Design

Road optimization - earthwork



Source: Warren L. Hare, A Multi-Haul Model for Vertical Alignment Optimization of Road Design

Road optimization - earthwork



$$\min \sum_{(i,j)\in\mathcal{M}^2} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{j\in\mathcal{M}^i_{\rightarrow}} x_{ij} - \sum_{j\in\mathcal{M}^i_{\leftarrow}} x_{ji} = V_i \quad \text{for all } i \in \mathcal{S}$$

$$x_{ij} \geq 0 \quad \text{for all } i, j \in \mathcal{S} i \neq j$$

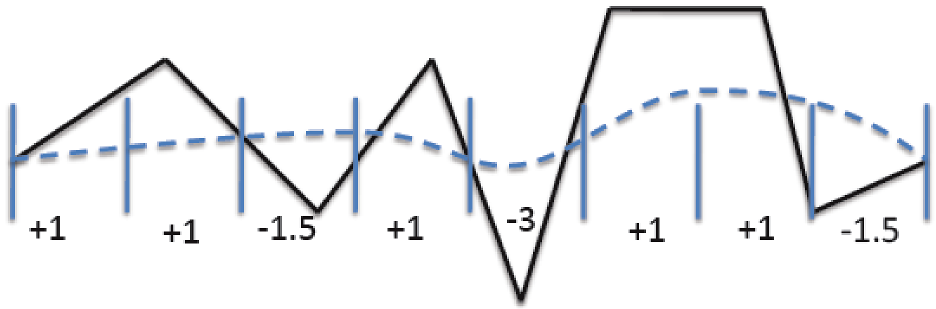
Feasibility:

- Only feasible if $\sum V_i = 0$
- Fixed by introducing “borrow pits” and “waste pits”
- Essentially extra sections that don't need to be balanced

Source: Warren L. Hare, A Multi-Haul Model for Vertical Alignment Optimization of Road Design

Road optimization - earthwork

A flat road is unlikely to minimize costs



$$\begin{aligned} \min \quad & \sum_{(i,j)\in\mathcal{M}^2} c_{ij} x_{ij} & \\ \text{s.t.} \quad & \textstyle\sum_{j\in\mathcal{M}^i_\rightarrow} x_{ij} - \sum_{j\in\mathcal{M}^i_\leftarrow} x_{ji} = V_i & \text{for all } i \in \mathcal{S} \\ & x_{ij} \geq 0 & \text{for all } i,j \in \mathcal{S}, i \neq j \\ & V_i = L(h_i, u_i) & \text{for all } i \in \mathcal{S} \end{aligned}$$

Notation: (variables in blue, constants in black)

x_{ij} = earth moved from i to j

c_{ij} = cost

V_i = target volume

L = a linear function

$\mathcal{S}$ = sections

$\mathcal{M}^i_\rightarrow, \mathcal{M}^i_\leftarrow, \mathcal{M}^2$ = move lists

u_i = height of ground at section i

h_i = height of road at section i

Source: Warren L. Hare, A Multi-Haul Model for Vertical Alignment Optimization of Road Design

Overview of Optimization Techniques

Optimization

- **Optimization problem**

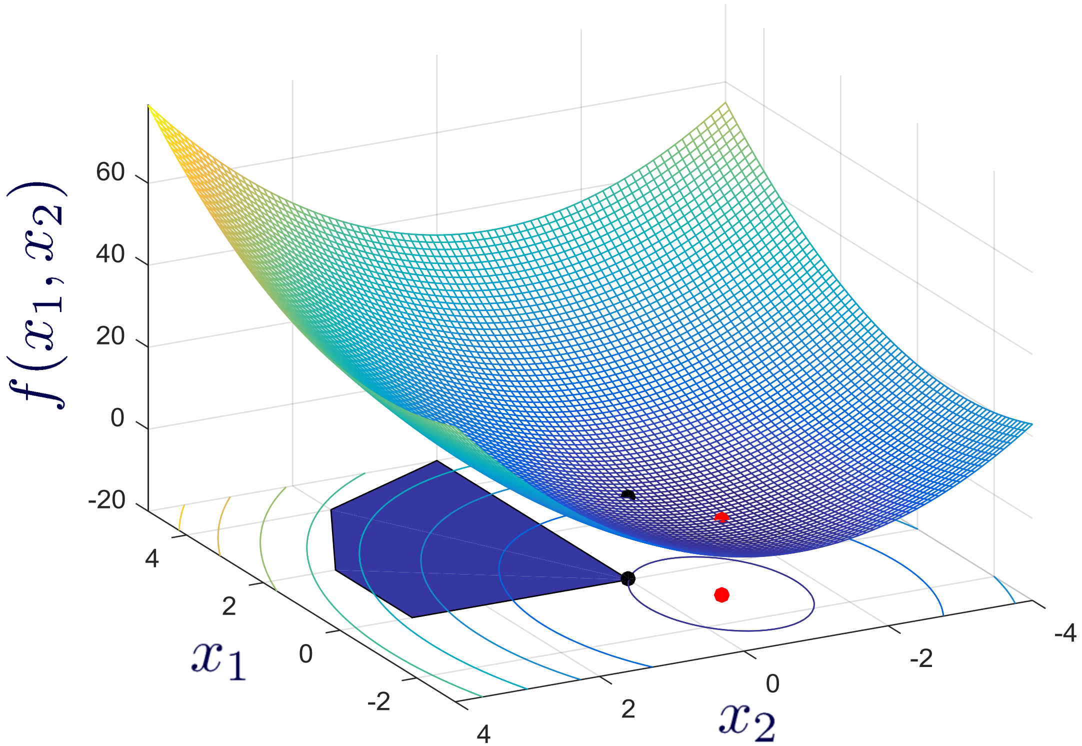
$$\begin{array}{ll} \text{minimize}_{\boldsymbol{x}} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \Omega \end{array}$$

- **Minimizing convex quadratic (QP) objective function** over a **polyhedron** (linear constraints)

$$\begin{array}{ll} \min\limits_{\boldsymbol{x}\in\mathbb{R}^n} & \boldsymbol{c}^T\boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^T Q \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{l} \le A\boldsymbol{x} \le \boldsymbol{u} \\ & \boldsymbol{l}_b \le \boldsymbol{x} \le \boldsymbol{u}_b \end{array}$$

general form

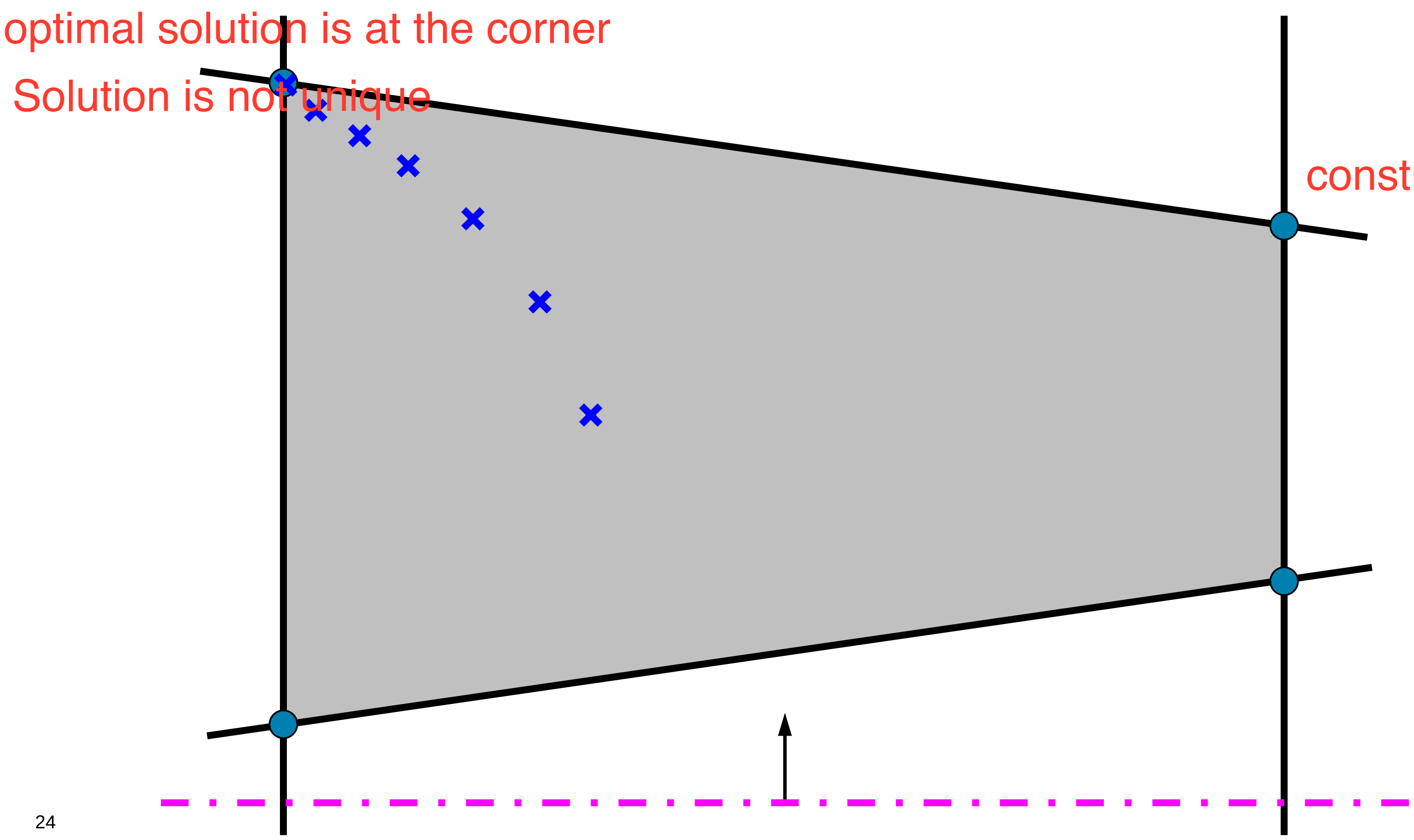
$$\begin{array}{ll} \min\limits_{\boldsymbol{x}\in\mathbb{R}^2} & x_1^2 + 2x_2^2 + 2x_1 + 3x_2 \\ \text{s.t.} & x_1 + 2x_2 \le 8 \\ & 2x_1 + x_2 \le 10 \\ & x_2 \le 3 \\ & x_1,\ x_2 \ge 0 \end{array}$$



Solving linear optimization problems

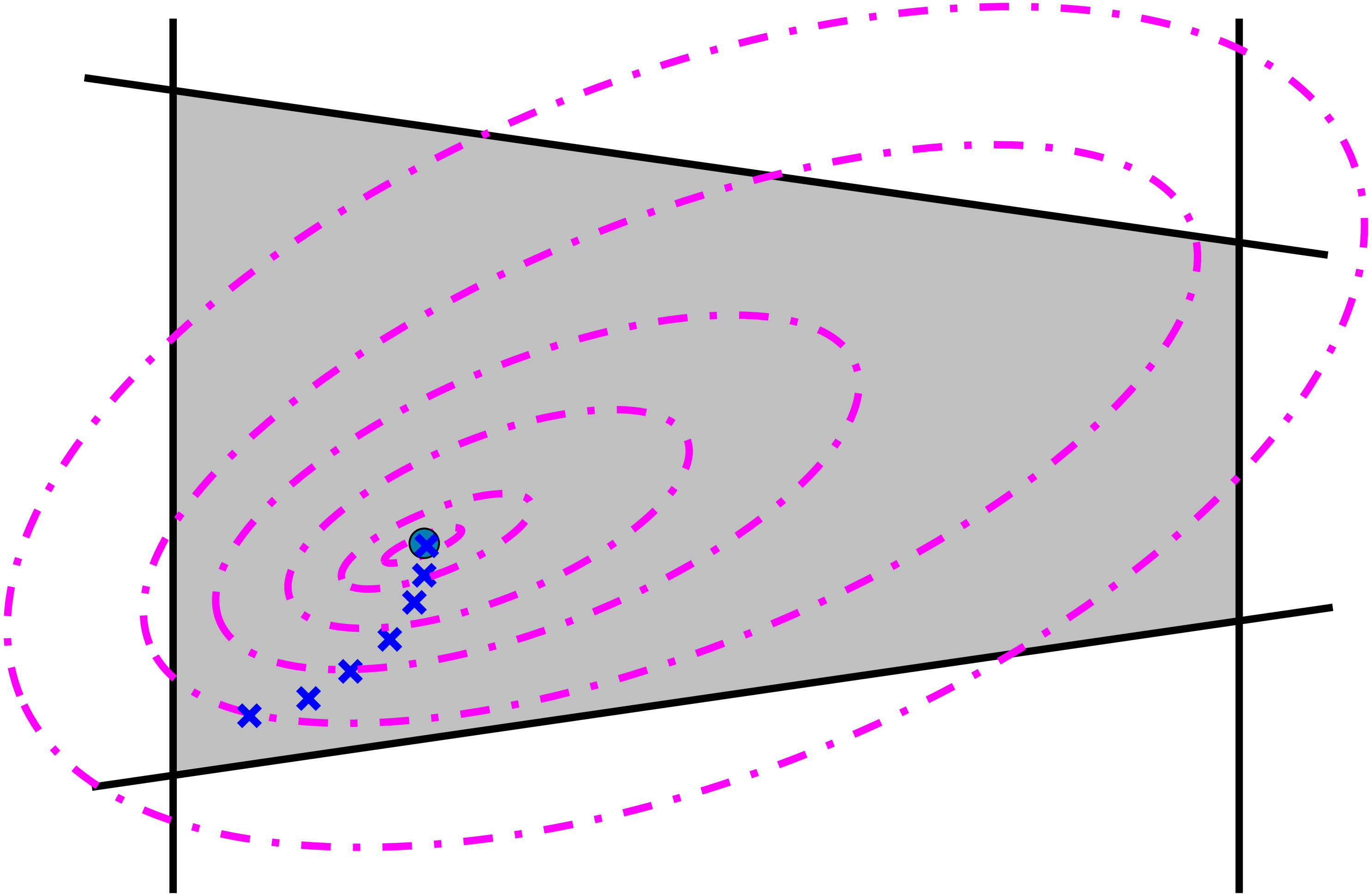
- Maximizing/minimizing linear (LP) function over a polyhedron
- Interior Point Methods vs. Simplex-type Methods

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad \boldsymbol{l} \le A\boldsymbol{x} \le \boldsymbol{u}$$



Solving non-linear optimization problems

- Convex non-linear objective function (NLP), linear or non-linear constraints
- Illustrated solution technique – Interior Point Methods
- Other solution techniques – gradient methods, Newton and Quasi-Newton



Solving linear optimization problems

Add new variables to make equation

- Simplex Method – graphical view

$$
\begin{aligned}
\max_{x \in \mathbb{R}^3} \quad & 3x_1 + 2x_2 + 2x_3 \\
\text{s.t.} \quad & x_1 + x_3 \le 8 \\
& x_1 + x_2 \le 7 \\
& x_1 + 2x_2 \le 12 \\
& x_1,\, x_2,\, x_3 \ge 0
\end{aligned}
$$

standard form

$$
\begin{aligned}
\max_{x \in \mathbb{R}^3} \quad & 3x_1 + 2x_2 + 2x_3 \\
\text{s.t.} \quad & x_1 + x_3 + x_4 = 8 \\
& x_1 + x_2 + x_5 = 7 \\
& x_1 + 2x_2 + x_6 = 12 \\
& x_1,\, x_2,\, x_3 \ge 0 \\
& x_4,\, x_5,\, x_6 \ge 0
\end{aligned}
$$

Source: Ed Klotz, Interior Point Algorithm Fundamentals

Solving linear optimization problems

- Simplex Method – graphical view

Current Basis

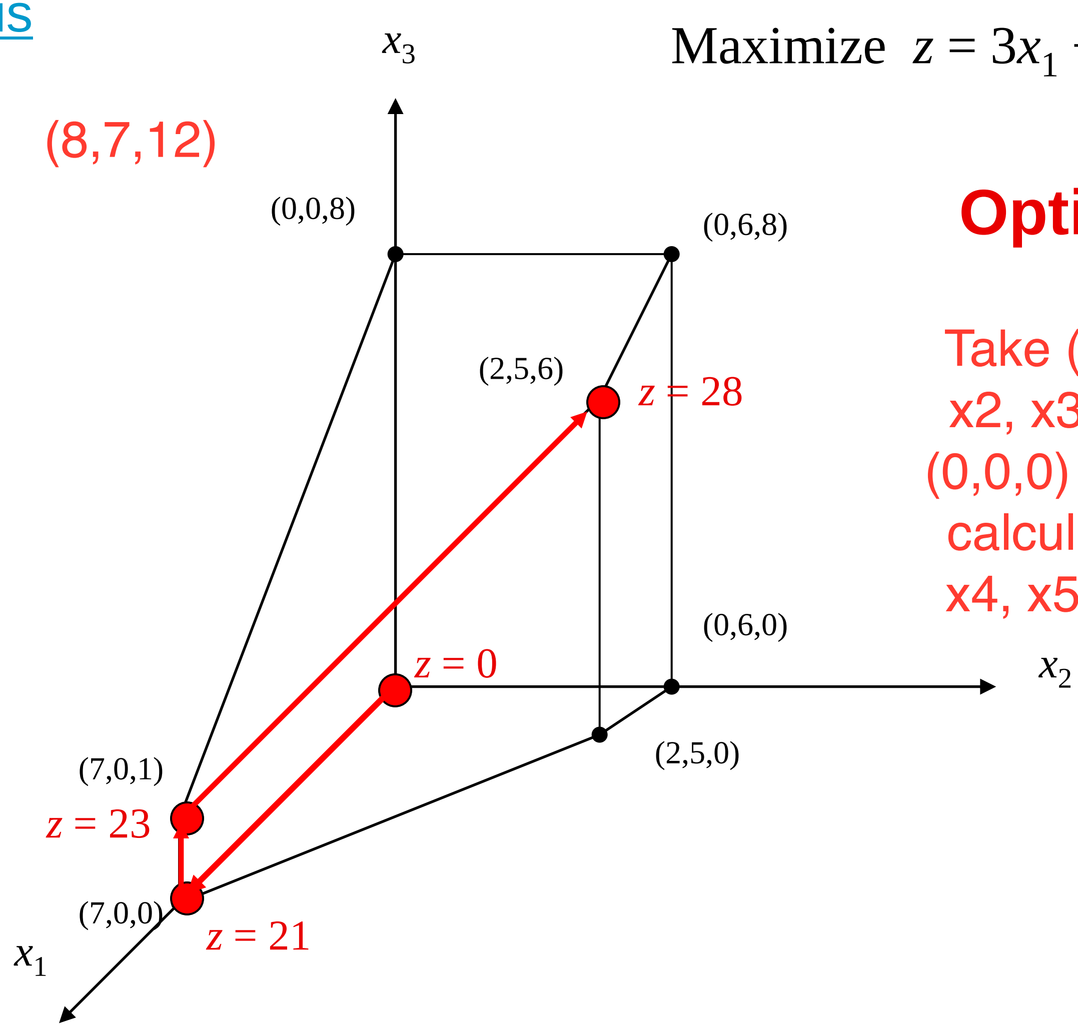
$\cancel{(x_4, x_5, x_6)}$ (8,7,12)

$\cancel{(x_4, x_1, x_6)}$

$\cancel{(x_3, x_1, x_6)}$

(x_3, x_1, x_2)

Maximize $z = 3x_1 + 2x_2 + 2x_3$



Optimal!

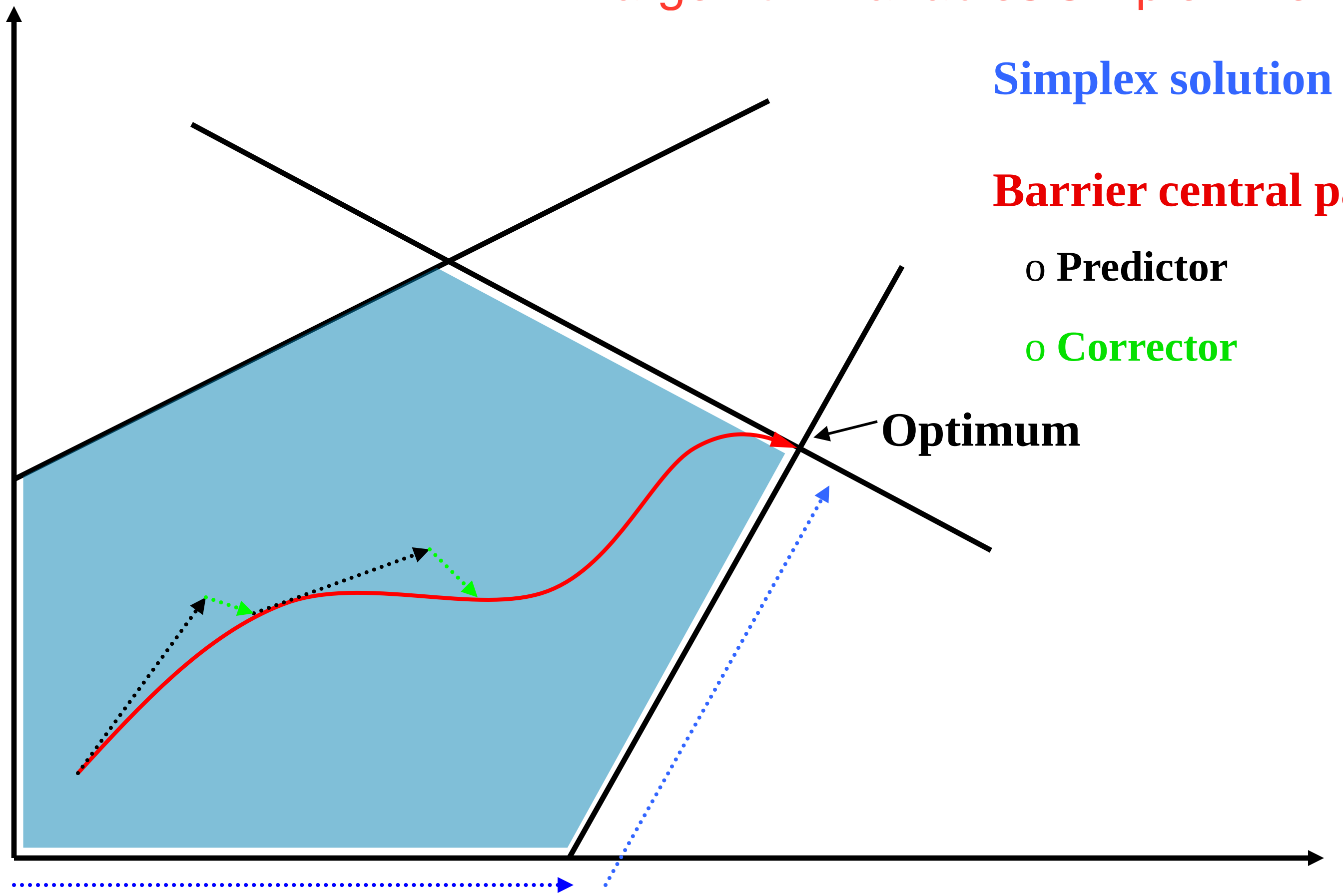
Take (x1, x2, x3) = (0,0,0) and calculate x4, x5,x6

Source: Ed Klotz, Interior Point Algorithm Fundamentals

Solving linear optimization problems

- Interior Point Method (barrier algorithm in CPLEX)

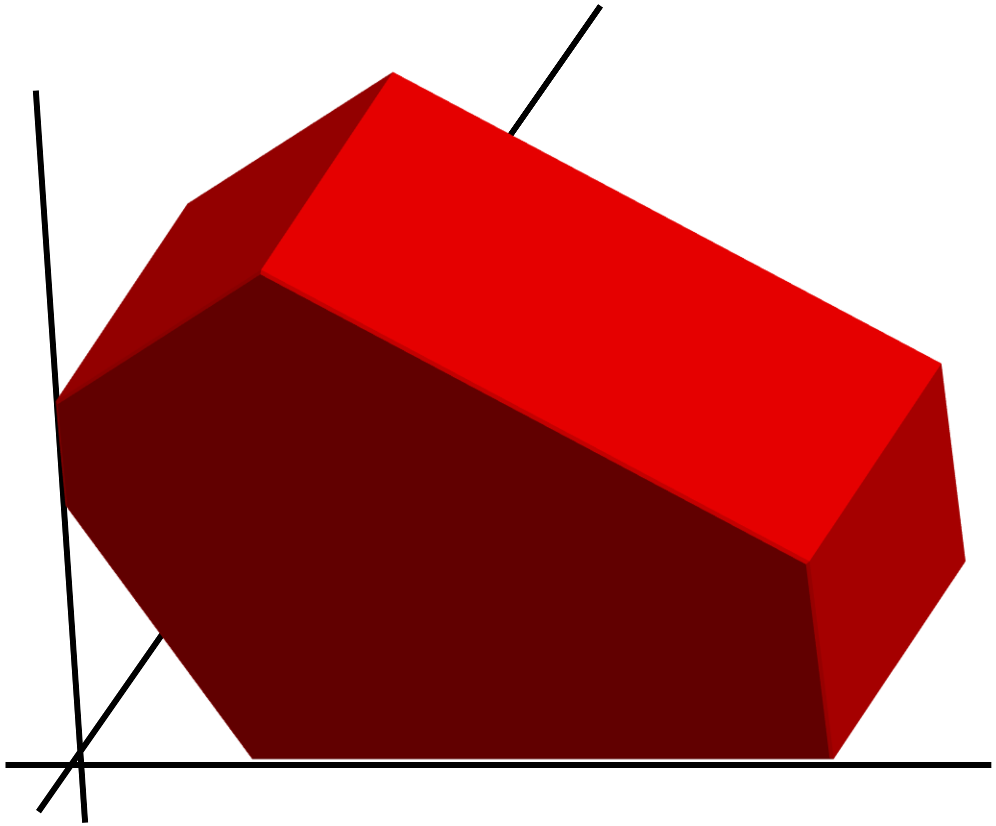
slower: small num variables interior point method
large num variables simplex method



Source: Ed Klotz, Interior Point Algorithm Fundamentals

Solving linear optimization problems

- Higher dimensions

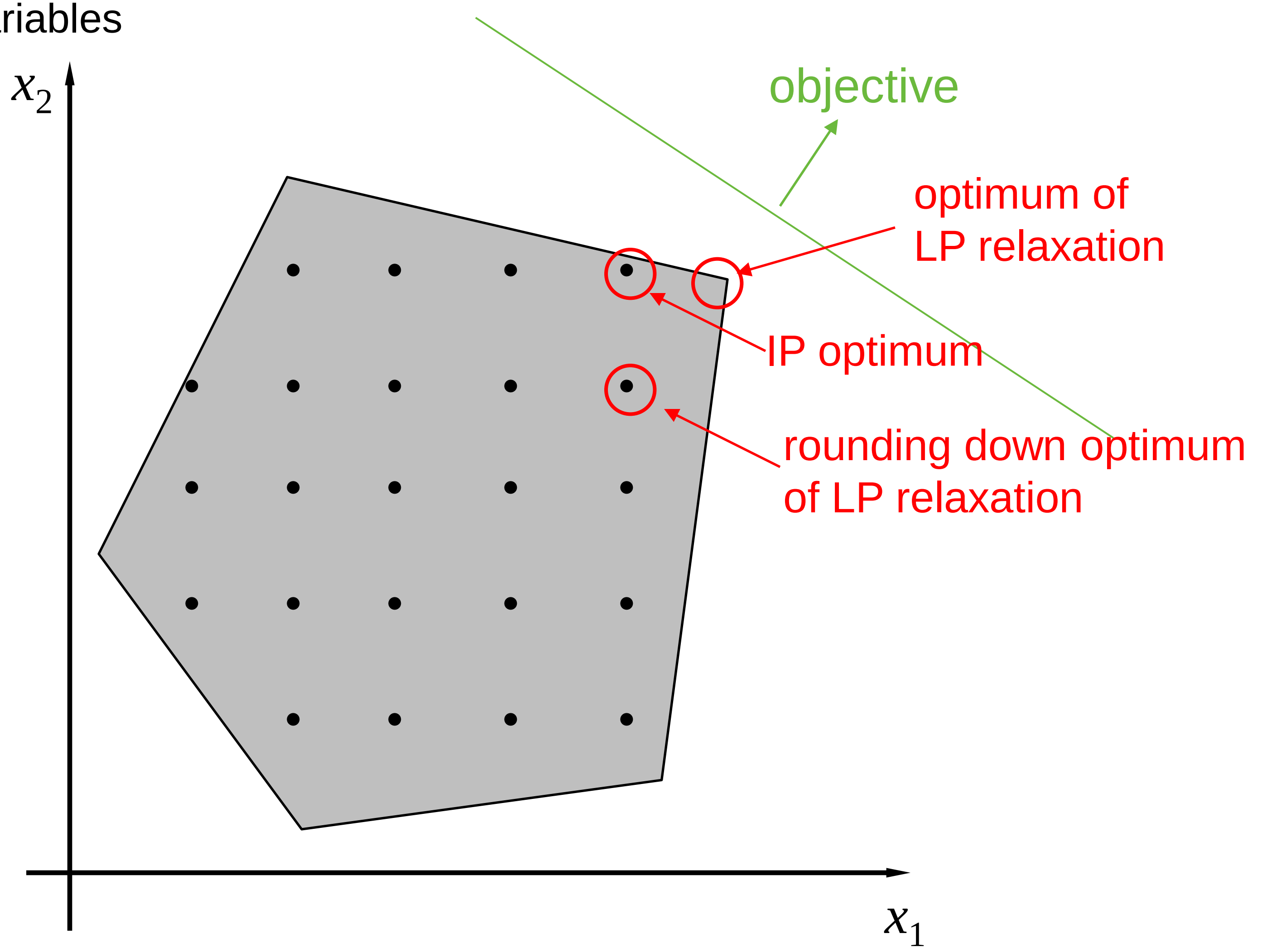


Solving mixed-integer optimization problems

- Mixed-integer optimization problems (MIP)
  - continuous variables
  - integer variables

more difficult because not a convex problem

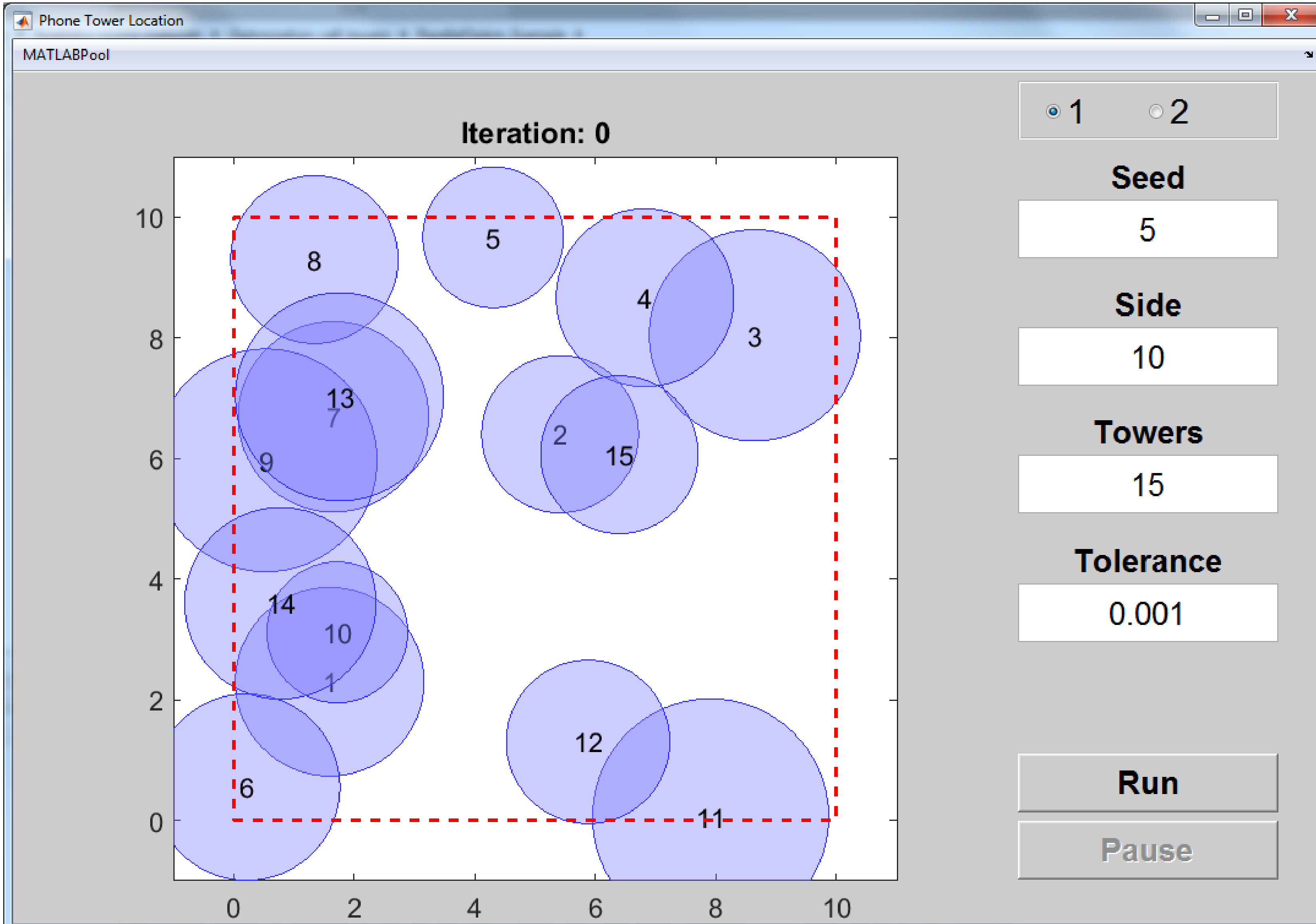
feasible solutions = •



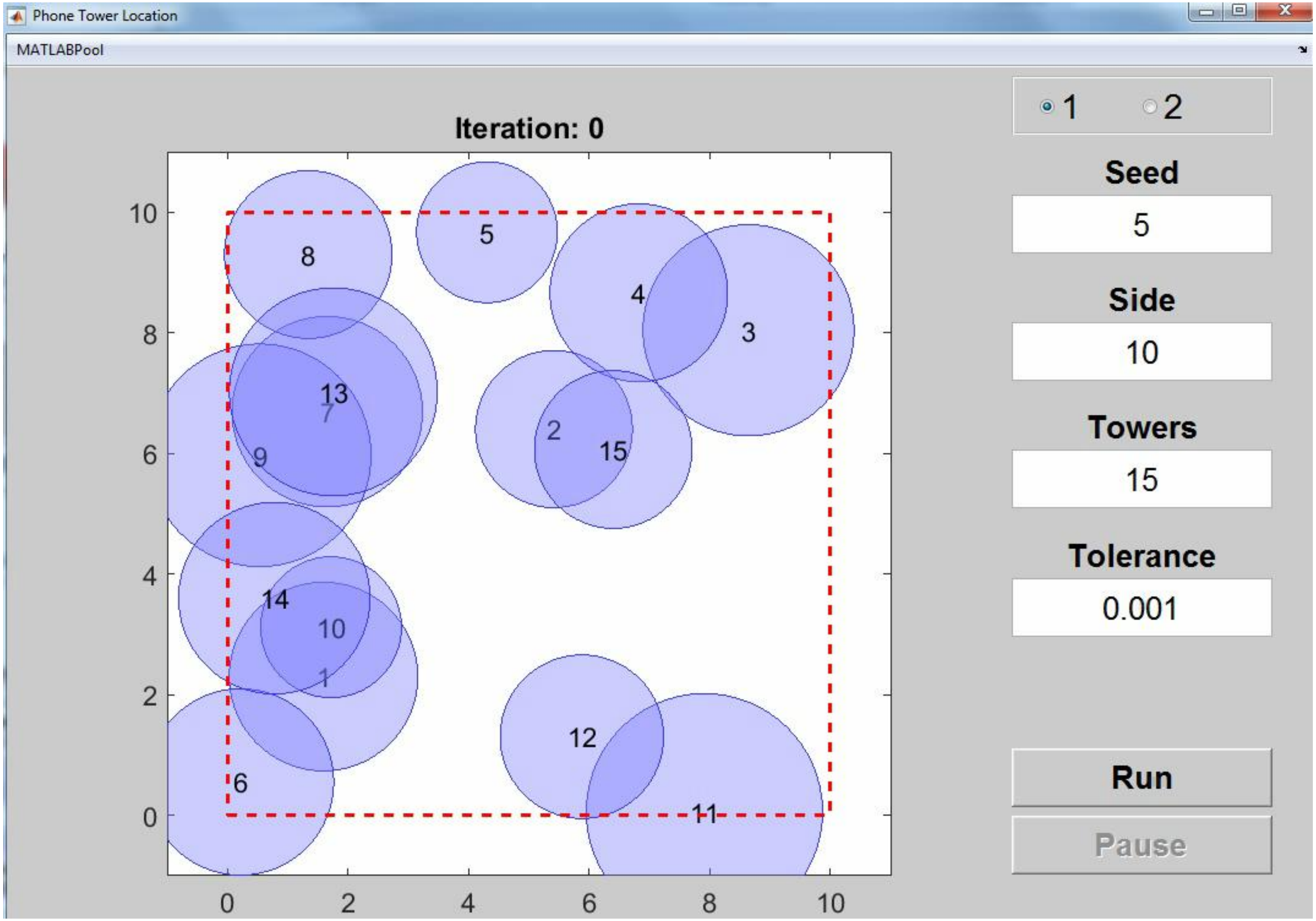
📖 Optimization Examples

Cell tower location optimization

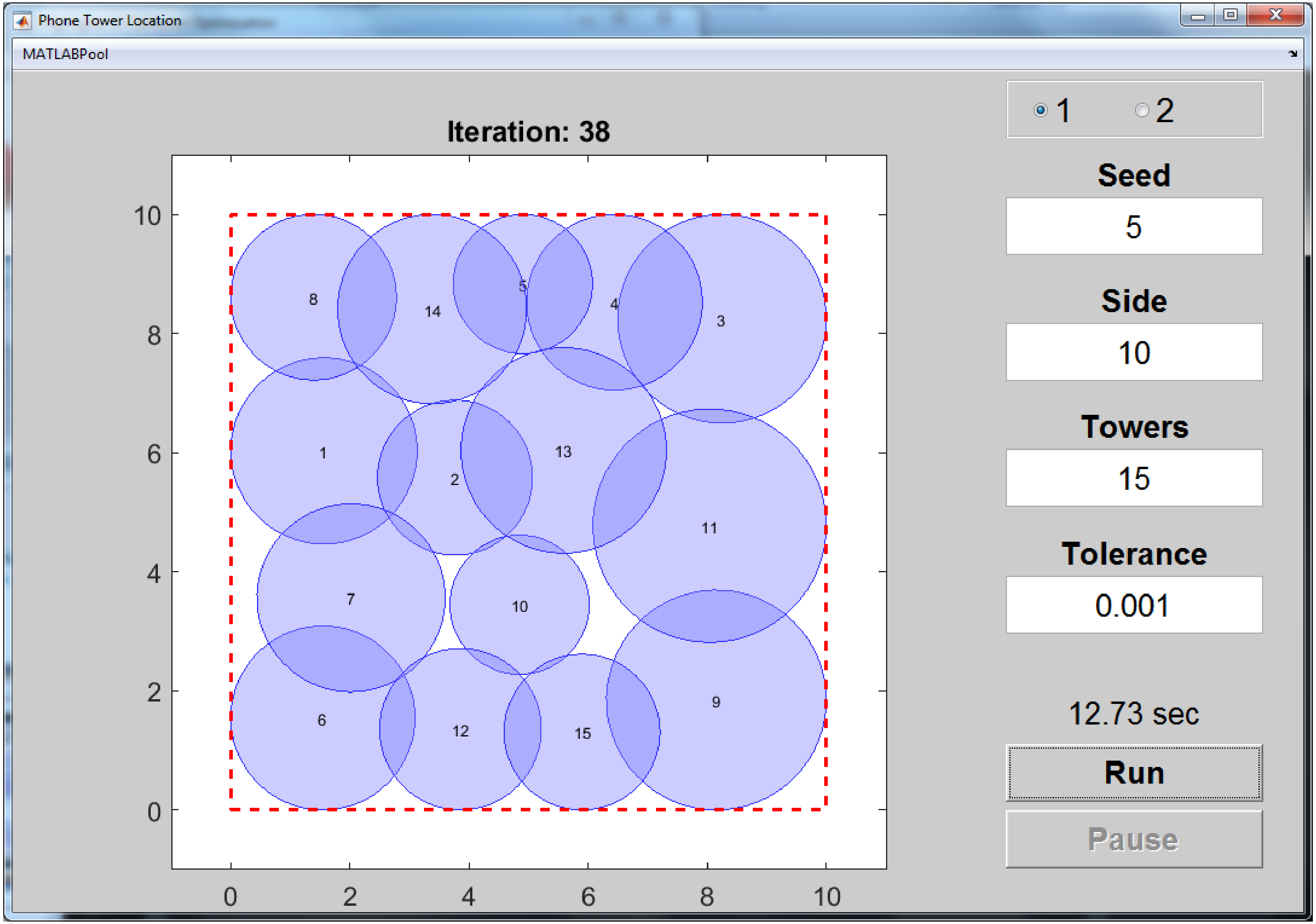
minimize the area of the intersection



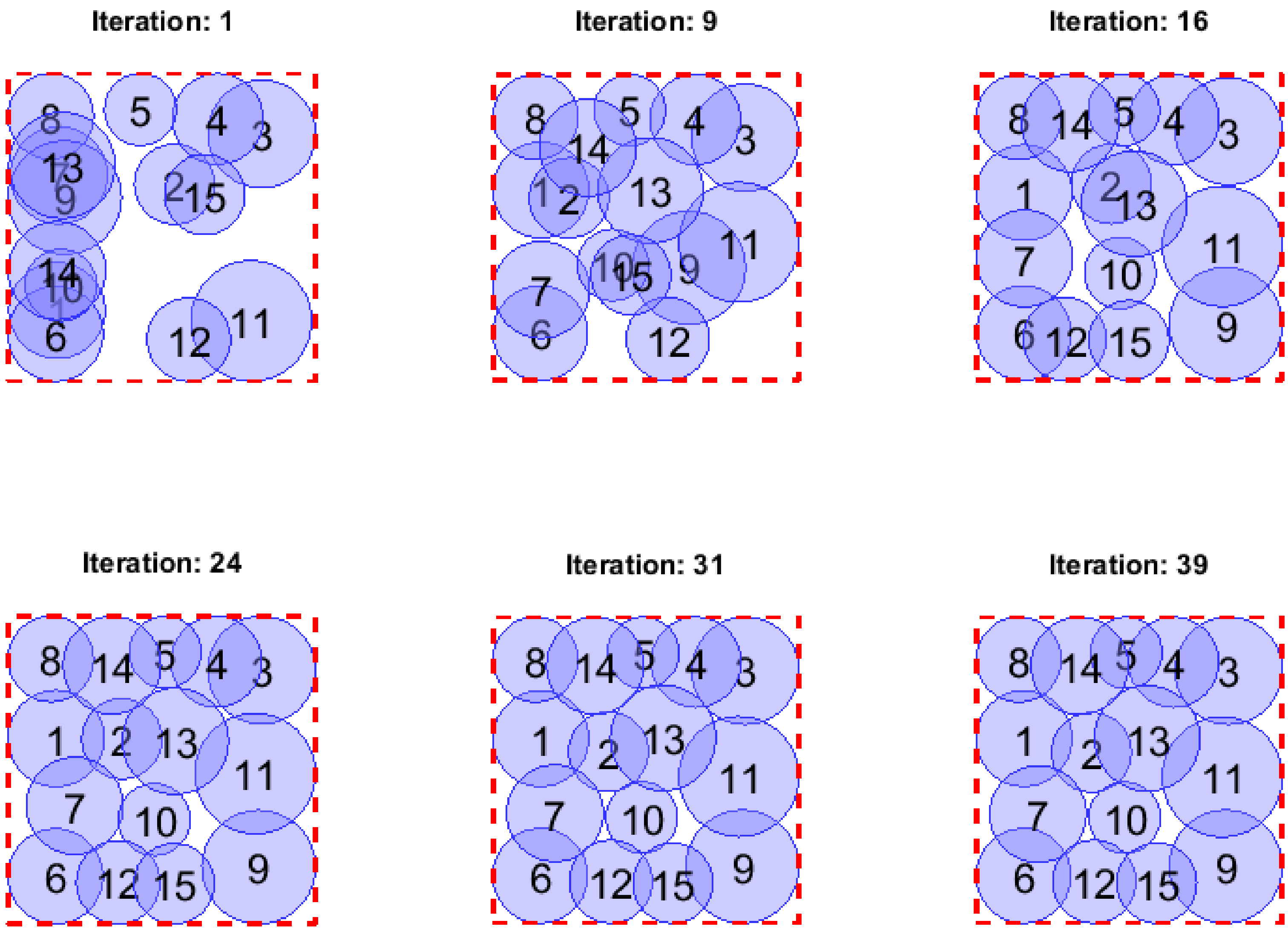
Cell tower location optimization



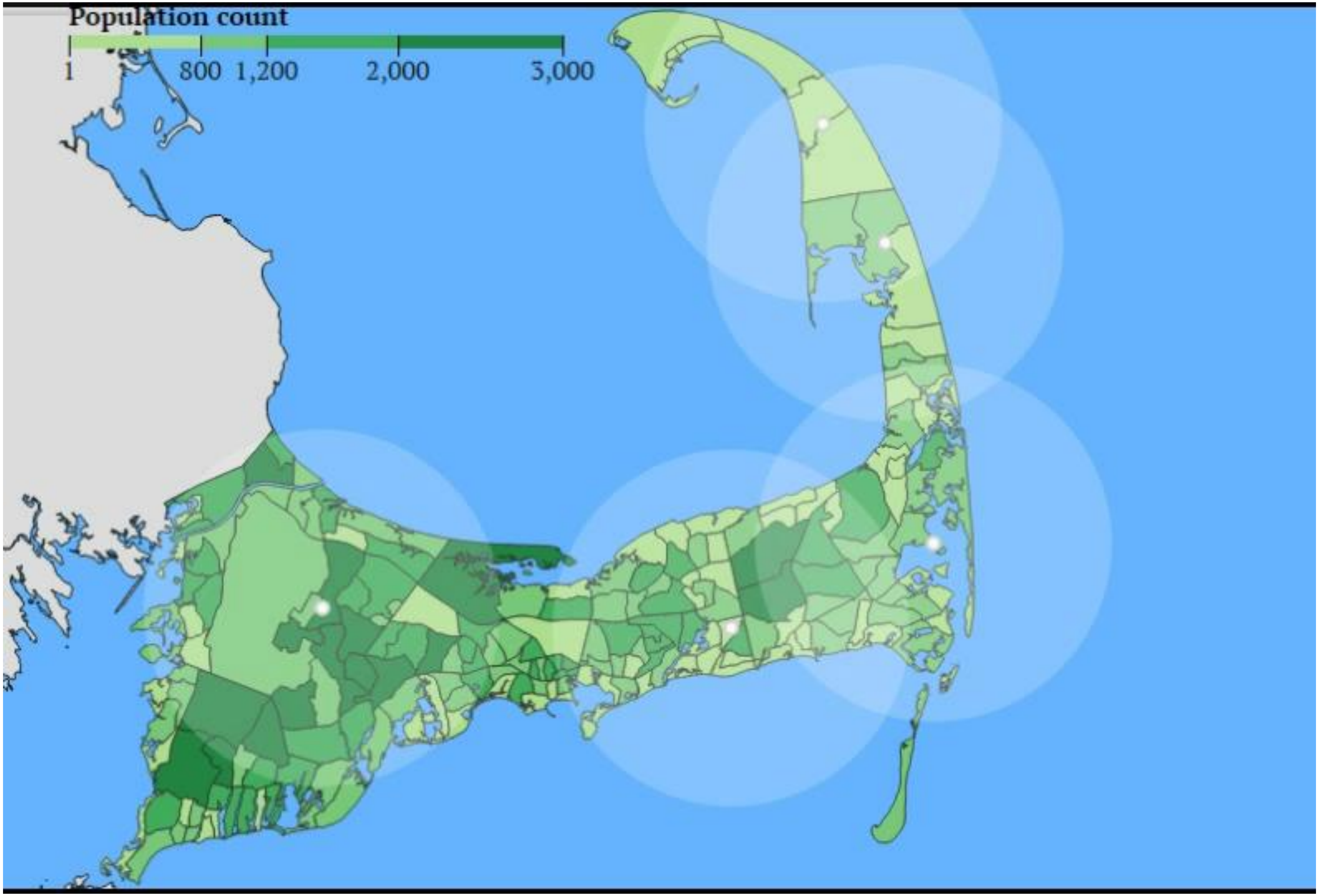
Cell tower location optimization



Cell tower location optimization



Cell tower location optimization take consideration of population density



Aircraft conflict avoidance

Aircraft i and j are **in conflict** if

- their horizontal distance is less than d:

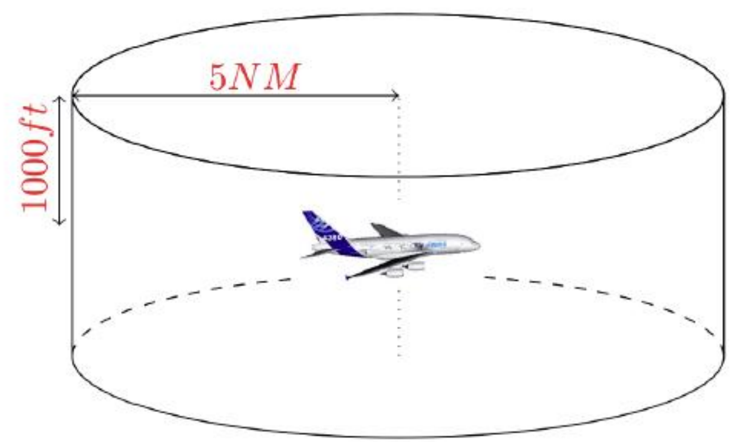
  $\|x_i(t) - x_j(t)\| \leq d \quad \forall t \qquad (d = 5\text{NM})$

- their altitude difference is less than h:

  $\|h_i(t) - h_j(t)\| \leq h \quad \forall t \qquad (h = 1000\text{ft})$

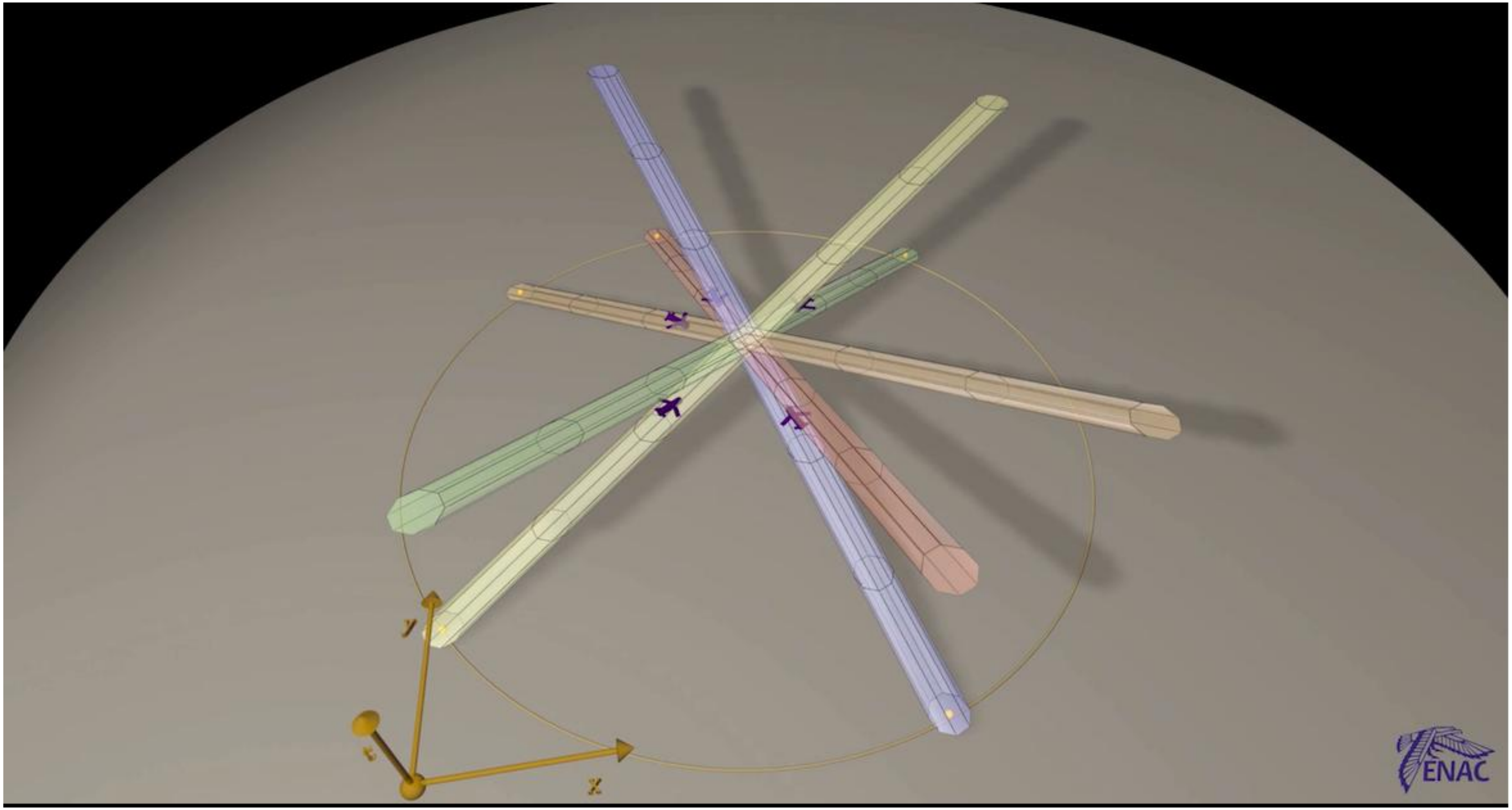
1 NM (nautical mile) = 1852 m

1 ft (feet) = 0.3048 m



Source: Marcel Mongeau, Solving aircraft conflicts by continuous optimization and mixed-integer nonlinear programming

Aircraft conflict avoidance change: speed, altitude, angle to avoid collapse



Source: Marcel Mongeau, Solving aircraft conflicts by continuous optimization and mixed-integer nonlinear programming