Reproducing Malekakhlagh et al., PRA 102, 042605

Yutaro Iiyama

Code repo: https://github.com/yiiyama/symqudit

Perturbative description of transmon

$$H_q = 4E_C N^2 - E_J \cos(\phi)$$
$$= \frac{\omega_h}{4}\left[y^2 - \frac{2}{\epsilon}\cos(\sqrt{\epsilon x})\right]$$

$$\phi = \phi_{\text{zpf}} x = \left(\frac{2E_C}{E_J}\right)^{1/4} \underbrace{(b + b^\dagger)}_{x}$$

$$N = N_{\text{zpf}} y = \frac{1}{2}\left(\frac{E_J}{2E_C}\right)^{1/4} \underbrace{[-i(b - b^\dagger)]}_{y}$$

For transmons $\epsilon = \sqrt{2E_C/E_J} = \mathcal{O}(0.1)$ is a small parameter to expand on

$$H_q = \sum_{p=0}^{\infty} \epsilon^p H_q^{(p)} \qquad H_q^{(0)} = \omega_h b^\dagger b$$

$$\hat{\mathcal{H}}_{\text{q}}^{(p)} \equiv \omega_h \frac{(-1)^p}{2(2p+2)!}(\hat{b} + \hat{b}^\dagger)^{2p+2}$$
$$= \omega_h \sum_{m=0}^{p} \sum_{l=-(m+1)}^{l=m+1} \left[\frac{(-1)^p}{2^{p-m+1}(p-m)!} \times \frac{(\hat{b}^\dagger)^{m+1+l}}{(m+1+l)!}\frac{\hat{b}^{m+1-l}}{(m+1-l)!}\right],$$

Corresponding energy eigenvalues and eigenstates are

$$E_n = \sum_{p=0}^{\infty} \epsilon^p E_n^{(p)}, \tag{A6a}$$

$$|\psi_n\rangle = \sum_{p=0}^{\infty} \epsilon^p \big|\psi_n^{(p)}\big\rangle. \tag{A6b}$$

Spectrum

- In a perturbative expansion, zeroth-order expressions must coincide with those at $\epsilon \to 0$:
  $|\psi_n^{(0)}\rangle = |n\rangle$ (unnormalized), $E_n^{(0)} = E_{\text{zpf}} + n\omega_h$
- We can assume $\langle n \,|\, \psi_n^{(r)}\rangle = 0 \quad (r \geq 1)$ ($|n\rangle$ component can be renormalized into $|\psi_n^{(0)}\rangle$)
- Then at each order:
- $E_n^{(p)}$ determined from
  $|\psi_n^{(r)}\rangle \ (r < p)$
- $|\psi_n^{(p)}\rangle$ determined from
  $|\psi_n^{(r)}\rangle \ (r < p)$
  and
  $E_n^{(r)} \ (r \leq p)$
- Spectrum determined recursively

$$\sum_{m\neq n}\langle m|\left(H^{(0)}+\sum_{p=1}^{\infty}\epsilon^p H^{(p)}\right)\left(|n\rangle+\sum_{p=1}^{\infty}\epsilon^p|\psi_n^{(p)}\rangle\right)|m\rangle = \sum_{m\neq n}\langle m|\left(n\omega_h+\sum_{p=1}^{\infty}\epsilon^p E_n^{(p)}\right)\left(|n\rangle+\sum_{p=1}^{\infty}\epsilon^p|\psi_n^{(p)}\rangle\right)|m\rangle$$

$$\sum_{m\neq n}\left(m\omega_h\sum_{p=1}^{\infty}\epsilon^p\langle m|\psi_n^{(p)}\rangle+\sum_{p=1}^{\infty}\epsilon^p\langle m|H^{(p)}|n\rangle+\sum_{\substack{p=1\\q=1}}^{\infty}\epsilon^{p+q}\langle m|H^{(p)}|\psi_n^{(q)}\rangle\right)|m\rangle$$

$$= \sum_{m\neq n}\left(n\omega_h\sum_{p=1}^{\infty}\epsilon^p\langle m|\psi_n^{(p)}\rangle+\sum_{\substack{p=1\\q=1}}^{\infty}\epsilon^{p+q}E_n^{(p)}\langle m|\psi_n^{(q)}\rangle\right)|m\rangle$$

$$\therefore \sum_{m\neq n}\langle m|\psi_n^{(p)}\rangle|m\rangle=\sum_{m\neq n}\frac{1}{(n-m)\omega_h}\left[\langle m|H^{(p)}|n\rangle+\sum_{r=0}^{p-1}\langle m|\left(H^{(p-r)}-E_n^{(p-r)}\right)|\psi_n^{(r)}\rangle\right]|m\rangle$$

$\times\sum_{m\neq n}|m\rangle\langle m|$

$$E_n|\psi_n\rangle = H|\psi_n\rangle$$

$\times\langle n|$

$$E_n\langle n|\psi_n\rangle=\langle n|H|\psi_n\rangle$$

$$\left(\sum_{p=0}^{\infty}\epsilon^p E_n^{(p)}\right)\left(\sum_{p=0}^{\infty}\epsilon^p\overset{=1}{\langle n|\psi_n^{(p)}\rangle}\right)=\langle n|\left(\sum_{p=0}^{\infty}\epsilon^p H^{(p)}\right)\left(\sum_{p=0}^{\infty}\epsilon^p|\psi_n^{(p)}\rangle\right)$$

$$\left(\cancel{n\omega_h}+\sum_{p=1}^{\infty}\epsilon^p E_n^{(p)}\right)=\cancel{n\omega_h\left(\sum_{p}\epsilon^p|\psi_n^{(p)}\rangle\right)}+\sum_{\substack{p=1\\q=0}}^{\infty}\epsilon^{p+q}\langle n|H^{(p)}|\psi_n^{(q)}\rangle$$

$$E_n^{(p)}=\sum_{q=0}^{p-1}\langle n|H^{(p-q)}|\psi_n^{(q)}\rangle \qquad (p\geq 1)$$

Sympy implementation

- Class Transmon
  - ω_h, ϵ as symbolic parameters
  - eigenstate(level, pert_order)
    and
    eigenvalue(level, pert_order)
    performs the recursive spectrum calculation
  - State expanded in terms of
    sympy.physics.quantum.sho1d.SHOKet
  - Using sympy.physics.quantum.qapply to evaluate the inner products
  - Normalization computed at each order in perturbation (relevant for the leading state term)

Validation

```python
e0 = tr.eigenvalue(0, 3)
for level in range(1, 4):
    display(Math(fr'\frac{{E_{level} - E_0}}{{\omega_h}} = ' + latex(((tr.eigenvalue(level, 3) - e0) / tr._omegah).expand())))
```

$$\frac{E_1 - E_0}{\omega_h} = -\frac{21\epsilon^3}{512} - \frac{\epsilon^2}{16} - \frac{\epsilon}{4} + 1$$

$$\frac{E_2 - E_0}{\omega_h} = -\frac{123\epsilon^3}{512} - \frac{17\epsilon^2}{64} - \frac{3\epsilon}{4} + 2$$

$$\frac{E_3 - E_0}{\omega_h} = -\frac{213\epsilon^3}{256} - \frac{45\epsilon^2}{64} - \frac{3\epsilon}{2} + 3$$

$$\frac{E_1 - E_0}{\omega_h} = 1 - \frac{1}{4}\epsilon - \frac{1}{16}\epsilon^2 + O(\epsilon^3),$$

$$\frac{E_2 - E_0}{\omega_h} = 2 - \frac{3}{4}\epsilon - \frac{17}{64}\epsilon^2 + O(\epsilon^3),$$

$$\frac{E_3 - E_0}{\omega_h} = 3 - \frac{3}{2}\epsilon - \frac{45}{64}\epsilon^2 + O(\epsilon^3),$$

Validation

```python
def organize_by_ket(state):
    ket_coeffs = defaultdict(lambda: S.Zero)
    for term in state.expand().args:
        c, nc = term.args_cnc()
        ket_coeffs[nc[0].args] += Mul(*c)

    states = []
    for ket in sorted(ket_coeffs.keys()):
        coeff = ket_coeffs[ket]
        states.append(latex(coeff.expand() * SHOKet(*ket)))

    return '+'.join(states)

for level in range(4):
    display(Math(organize_by_ket(tr.eigenstate(level, 2))))
```

$$\left(1-\frac{13\epsilon^2}{3072}\right)|0\rangle+\left(\frac{13\sqrt{2}\epsilon^2}{768}+\frac{\sqrt{2}\epsilon}{16}\right)|2\rangle+\left(\frac{\sqrt{6}\epsilon^2}{96}+\frac{\sqrt{6}\epsilon}{96}\right)|4\rangle+\frac{23\sqrt{5}\epsilon^2|6\rangle}{3840}+\frac{\sqrt{70}\epsilon^2|8\rangle}{3072}$$

$$\left(1-\frac{35\epsilon^2}{1024}\right)|1\rangle+\left(\frac{37\sqrt{6}\epsilon^2}{768}+\frac{5\sqrt{6}\epsilon}{48}\right)|3\rangle+\left(\frac{41\sqrt{30}\epsilon^2}{1920}+\frac{\sqrt{30}\epsilon}{96}\right)|5\rangle+\frac{11\sqrt{35}\epsilon^2|7\rangle}{1280}+\frac{\sqrt{70}\epsilon^2|9\rangle}{1024}$$

$$\left(-\frac{5\sqrt{2}\epsilon^2}{192}-\frac{\sqrt{2}\epsilon}{16}\right)|0\rangle+\left(1-\frac{419\epsilon^2}{3072}\right)|2\rangle+\left(\frac{145\sqrt{3}\epsilon^2}{768}+\frac{7\sqrt{3}\epsilon}{24}\right)|4\rangle+\left(\frac{103\sqrt{10}\epsilon^2}{960}+\frac{\sqrt{10}\epsilon}{32}\right)|6\rangle+\frac{43\sqrt{35}\epsilon^2|8\rangle}{1920}+\frac{5\sqrt{14}\epsilon^2|10\rangle}{1024}$$

$$\left(-\frac{13\sqrt{6}\epsilon^2}{192}-\frac{5\sqrt{6}\epsilon}{48}\right)|1\rangle+\left(1-\frac{405\epsilon^2}{1024}\right)|3\rangle+\left(\frac{79\sqrt{5}\epsilon^2}{256}+\frac{3\sqrt{5}\epsilon}{8}\right)|5\rangle+\left(\frac{103\sqrt{210}\epsilon^2}{1920}+\frac{\sqrt{210}\epsilon}{96}\right)|7\rangle+\frac{53\sqrt{105}\epsilon^2|9\rangle}{1920}+\frac{5\sqrt{462}\epsilon^2|11\rangle}{3072}$$

$$\begin{aligned}|\psi_0\rangle &= \left(1-\frac{13}{3072}\epsilon^2\right)|0\rangle+\left(\frac{1}{8\sqrt{2}}\epsilon+\frac{13}{384\sqrt{2}}\epsilon^2\right)|2\rangle \\ &\quad+\left(\frac{\sqrt{6}}{96}\epsilon+\frac{\sqrt{6}}{96}\epsilon^2\right)|4\rangle+\frac{23}{768\sqrt{5}}\epsilon^2|6\rangle \\ &\quad+\frac{\sqrt{\frac{35}{2}}}{1536}\epsilon^2|8\rangle+O(\epsilon^3),\end{aligned} \tag{A10a}$$

$$\begin{aligned}|\psi_1\rangle &= \left(1-\frac{35}{1024}\epsilon^2\right)|1\rangle+\left(\frac{5}{8\sqrt{6}}\epsilon+\frac{37}{128\sqrt{6}}\epsilon^2\right)|3\rangle \\ &\quad+\left(\frac{1}{16}\sqrt{\frac{5}{6}}\epsilon+\frac{41}{64\sqrt{30}}\epsilon^2\right)|5\rangle+\frac{11}{256}\sqrt{\frac{7}{5}}\epsilon^2|7\rangle \\ &\quad+\frac{1}{512}\sqrt{\frac{35}{2}}\epsilon^2|9\rangle+O(\epsilon^3),\end{aligned} \tag{A10b}$$

$$\begin{aligned}|\psi_2\rangle &= \left(-\frac{1}{8\sqrt{2}}\epsilon-\frac{5}{96\sqrt{2}}\epsilon^2\right)|0\rangle+\left(1-\frac{419}{3072}\epsilon^2\right)|2\rangle \\ &\quad+\left(\frac{7}{8\sqrt{3}}\epsilon+\frac{145}{256\sqrt{3}}\epsilon^2\right)|4\rangle \\ &\quad+\left(\frac{1}{16}\sqrt{\frac{5}{2}}\epsilon+\frac{103}{96\sqrt{10}}\epsilon^2\right)|6\rangle \\ &\quad+\frac{43}{384}\sqrt{\frac{7}{5}}\epsilon^2|8\rangle+\frac{5}{512}\sqrt{\frac{7}{2}}\epsilon^2|10\rangle+O(\epsilon^3),\end{aligned} \tag{A10c}$$

$$\begin{aligned}|\psi_3\rangle &= \left(-\frac{5}{8\sqrt{6}}\epsilon-\frac{13}{32\sqrt{6}}\epsilon^2\right)|1\rangle+\left(1-\frac{405}{1024}\epsilon^2\right)|3\rangle \\ &\quad+\left(\frac{3\sqrt{5}}{8}\epsilon+\frac{79\sqrt{5}}{256}\epsilon^2\right)|5\rangle \\ &\quad+\left(\frac{1}{16}\sqrt{\frac{35}{6}}\epsilon+\frac{103}{64}\sqrt{\frac{7}{30}}\epsilon^2\right)|7\rangle \\ &\quad+\frac{53}{128}\sqrt{\frac{7}{15}}\epsilon^2|9\rangle+\frac{5}{512}\sqrt{\frac{77}{6}}\epsilon^2|11\rangle+O(\epsilon^3).\end{aligned} \tag{A10d}$$

Transition amplitudes

x, y are re-expressed in terms of renormalized raising and lowering operators

$$x = x^{-} + x^{+} \qquad\qquad y = -i(y^{-} - y^{+})$$

Matrix elements are straightforwardly obtained using the eigenstates

$$\mu_{mn} \equiv \langle \psi_m | \hat{x} | \psi_n \rangle, \tag{A12a}$$

$$\nu_{mn} \equiv \langle \psi_m | \hat{y} | \psi_n \rangle. \tag{A12b}$$

Validation

[33]:
```python
for row, col in [(0, 1), (1, 2), (2, 3), (0, 3)]:
    display(Math(fr'\mu_{{{row}{col}}} = ' + latex(tr.phase_matrix_element(row, col, 2))))
```

$$\mu_{01} = \frac{13\epsilon^2}{256} + \frac{\epsilon}{8} + 1$$

$$\mu_{12} = \frac{95\sqrt{2}\epsilon^2}{512} + \frac{\sqrt{2}\epsilon}{4} + \sqrt{2}$$

$$\mu_{23} = \frac{105\sqrt{3}\epsilon^2}{256} + \frac{3\sqrt{3}\epsilon}{8} + \sqrt{3}$$

$$\mu_{03} = -\frac{3\sqrt{6}\epsilon^2}{128} - \frac{\sqrt{6}\epsilon}{48}$$

$$\mu_{01} = 1 + \frac{1}{8}\epsilon + \frac{13}{256}\epsilon^2 + O(\epsilon^3), \qquad \text{(A14b)}$$

$$\mu_{12} = \left(1 + \frac{1}{4}\epsilon + \frac{95}{512}\epsilon^2\right)\sqrt{2} + O(\epsilon^3), \qquad \text{(A14c)}$$

$$\mu_{23} = \left(1 + \frac{3}{8}\epsilon + \frac{105}{256}\epsilon^2\right)\sqrt{3} + O(\epsilon^3), \qquad \text{(A14d)}$$

$$\mu_{03} = -\frac{\sqrt{6}}{48}\epsilon - \frac{3\sqrt{6}}{128}\epsilon^2 + O(\epsilon^3). \qquad \text{(A14e)}$$

[34]:
```python
for row, col in [(0, 1), (1, 2), (2, 3), (0, 3)]:
    display(Math(fr'\nu_{{{row}{col}}} = ' + latex((I * tr.charge_matrix_element(row, col, 2)).expand())))
```

$$\nu_{01} = -\frac{11\epsilon^2}{256} - \frac{\epsilon}{8} + 1$$

$$\nu_{12} = -\frac{73\sqrt{2}\epsilon^2}{512} - \frac{\sqrt{2}\epsilon}{4} + \sqrt{2}$$

$$\nu_{23} = -\frac{79\sqrt{3}\epsilon^2}{256} - \frac{3\sqrt{3}\epsilon}{8} + \sqrt{3}$$

$$\nu_{03} = -\frac{5\sqrt{6}\epsilon^2}{128} - \frac{\sqrt{6}\epsilon}{16}$$

$$\nu_{01} = 1 - \frac{1}{8}\epsilon - \frac{11}{256}\epsilon^2 + O(\epsilon^3), \qquad \text{(A15b)}$$

$$\nu_{12} = \left(1 - \frac{1}{4}\epsilon - \frac{73}{512}\epsilon^2\right)\sqrt{2} + O(\epsilon^3), \qquad \text{(A15c)}$$

$$\nu_{23} = \left(1 - \frac{3}{8}\epsilon - \frac{79}{256}\epsilon^2\right)\sqrt{3} + O(\epsilon^3), \qquad \text{(A15d)}$$

$$\nu_{03} = -\frac{\sqrt{6}}{16}\epsilon - \frac{5\sqrt{6}}{128}\epsilon^2 + O(\epsilon^3). \qquad \text{(A15e)}$$

Perturbative description of CR

CR Hamiltonian

$$\hat{\mathcal{H}}_0 = \sum_{j=c,t} \frac{\omega_{\mathrm{jh}}}{4}\left[\hat{y}_j^2 - \frac{2}{\epsilon_j}\cos(\sqrt{\epsilon_j}\hat{x}_j)\right], \tag{8a}$$

$$\hat{\mathcal{H}}_{\mathrm{int}}(t) = J\hat{y}_c\hat{y}_t + \Omega\hat{y}_c\sin(\omega_{\mathrm{d}}t + \phi_d), \tag{8b}$$

RWA upfront:

$$\begin{aligned}\hat{\mathcal{H}}_J &= -J(\hat{y}_c^- - \hat{y}_c^+)(\hat{y}_t^- - \hat{y}_t^+) \approx J(\hat{y}_c^-\hat{y}_t^+ + \hat{y}_c^+\hat{y}_t^-)\\ &\approx \sum_{\substack{m,n=0\\ m<n}}^{3}\sum_{\substack{l,r=0\\ l<r}}^{3} \nu_{c,mn}\nu_{t,lr}J(\hat{P}_{c,mn}\hat{P}_{t,rl} + \hat{P}_{c,nm}\hat{P}_{t,lr}),\end{aligned} \tag{10}$$

$$\begin{aligned}\hat{\mathcal{H}}_d &= \frac{\Omega}{2}(\hat{y}_c^- - \hat{y}_c^+)(e^{i\omega_d t} - e^{-i\omega_d t})\\ &\approx \frac{\Omega}{2}(\hat{y}_c^- e^{i\omega_d t} + \hat{y}_c^+ e^{-i\omega_d t})\\ &\approx \sum_{\substack{m,n=0\\ m<n}}^{3} \frac{1}{2}\nu_{mn}\Omega(\hat{P}_{c,mn}e^{i\omega_d t} + \hat{P}_{c,nm}e^{-i\omega_d t}).\end{aligned} \tag{11}$$

Introduce a fictitious expansion parameter λ

$$\hat{\mathcal{H}}_s(t) = \hat{\mathcal{H}}_0 + \lambda\hat{\mathcal{H}}_{\mathrm{int}}(t), \tag{12}$$

Perturbative description of CR

SWPT:

- Assume that the full Hamiltonian describes a static (dominant) + oscillatory evolution
- Apply a small oscillatory unitary transformation to extract the static terms
- Unitary-transformed states evolve according to the static effective Hamiltonian → true state oscillates around the static evolution

Work in the qudit frame:

$$\lambda\hat{\mathcal{H}}_{\mathrm{I}}(t) \equiv e^{i\hat{\mathcal{H}}_0 t}[\lambda\hat{\mathcal{H}}_{\mathrm{int}}(t)]e^{-i\hat{\mathcal{H}}_0 t}. \tag{13}$$

SW transformation

$$\hat{\mathcal{H}}_{\mathrm{I,eff}}(t) \equiv e^{i\hat{G}(t)}[\lambda\hat{\mathcal{H}}_{\mathrm{I}}(t) - i\partial_t]e^{-i\hat{G}(t)}, \tag{14}$$

Expand both G and H_{Ieff} in λ

$$\hat{G}(t) = \sum_{\lambda=1}^{\infty} \lambda^n \hat{G}_n(t), \tag{15a}$$

$$\hat{\mathcal{H}}_{\mathrm{I,eff}}(t) = \sum_{\lambda=1}^{\infty} \lambda^n \hat{\mathcal{H}}^{(n)}_{\mathrm{I,eff}}(t). \tag{15b}$$

Perturbative description of CR

BCH formula + some math for time derivative version

$$\hat{\mathcal{H}}^{(0)}_{\mathrm{I,eff}} = 0, \tag{C13b}$$

$$\hat{\mathcal{H}}^{(1)}_{\mathrm{I,eff}} = -\dot{\hat{G}}_1 + \hat{\mathcal{H}}_I, \tag{C13c}$$

$$\hat{\mathcal{H}}^{(2)}_{\mathrm{I,eff}} = -\dot{\hat{G}}_2 - \frac{i}{2}[\hat{G}_1, \dot{\hat{G}}_1] + i[\hat{G}_1, \hat{\mathcal{H}}_I], \tag{C13d}$$

$$\hat{\mathcal{H}}^{(3)}_{\mathrm{I,eff}} = -\dot{\hat{G}}_3 - \frac{i}{2}[\hat{G}_1, \dot{\hat{G}}_2] - \frac{i}{2}[\hat{G}_2, \dot{\hat{G}}_1] + \frac{1}{6}[\hat{G}_1, [\hat{G}_1, \dot{\hat{G}}_1]] + i[\hat{G}_2, \hat{\mathcal{H}}_I] - \frac{1}{2}[\hat{G}_1, [\hat{G}_1, \hat{\mathcal{H}}_I]], \tag{C13e}$$

$$\begin{aligned}\hat{\mathcal{H}}^{(4)}_{\mathrm{I,eff}} = {} & -\dot{\hat{G}}_4 - \frac{i}{2}[\hat{G}_1, \dot{\hat{G}}_3] - \frac{i}{2}[\hat{G}_2, \dot{\hat{G}}_2] - \frac{i}{2}[\hat{G}_3, \dot{\hat{G}}_1] + \frac{1}{6}[\hat{G}_1, [\hat{G}_1, \dot{\hat{G}}_2]] + \frac{1}{6}[\hat{G}_1, [\hat{G}_2, \dot{\hat{G}}_1]] + \frac{1}{6}[\hat{G}_2, [\hat{G}_1, \dot{\hat{G}}_1]] \\ & + \frac{i}{24}[\hat{G}_1, [\hat{G}_1, [\hat{G}_1, \dot{\hat{G}}_1]]] + i[\hat{G}_3, \hat{\mathcal{H}}_I] - \frac{1}{2}[\hat{G}_1, [\hat{G}_2, \hat{\mathcal{H}}_I]] - \frac{1}{2}[\hat{G}_2, [\hat{G}_1, \hat{\mathcal{H}}_I]] - \frac{i}{6}[\hat{G}_1, [\hat{G}_1, [\hat{G}_1, \hat{\mathcal{H}}_I]]].\end{aligned} \tag{C13f}$$

Dotted G_i terms are unconstrained a priori

→ Use to cancel the time-dependent terms order by order

→ Time dependent terms are cast up to the next order

H_I cut off at level 3:

$$J \sum_{\substack{0 \le k \le 3 \\ 0 \le l \le 3}} e^{-it\Delta^{k+1}_{l+1}} |kc\rangle\langle k+1c| \otimes |l+1t\rangle\langle lt| \nu^c{}_{k,k+1} \nu^t{}_{l,l+1} + \frac{\Omega \sum_{k=0}^{3} e^{-it\Delta^{k+1}_{1}} |kc\rangle\langle k+1c| \otimes \mathcal{I} \nu^c{}_{k,k+1}}{2} \quad +\text{h.c.}$$

All terms are derived from products and integrals of these terms

Perturbative description of CR

The only ways a static term appears from the products of these commutators are:

- Product between $Je^{\pm it\Delta_{l+1}^{k+1}}$ and $Je^{\mp it\Delta_{l+1}^{k+1}}$ terms
  - Acquires factor $J^2(\nu_{k,k+1}^c)^2(\nu_{l,l+1}^t)^2|kc\rangle\langle kc| \otimes |l+1t\rangle\langle l+1t|$ and potentially Δs in the denominator
  - Gives diagonal operators
- Product between $Je^{\pm it\Delta_{1}^{k+1}}$ and $\Omega e^{\mp it\Delta_{1}^{k+1}}$ terms
  - Acquires factor $J\Omega(\nu_{k,k+1}^c)^2\nu_{l,l+1}^t|kc\rangle\langle kc| \otimes |l+1t\rangle\langle l| + \mathrm{h.c.}$
  - Gives (diag)X

In both cases, <u>static terms are block diagonal</u>

Sympy implementation

- Using sympy expand() and qapply() was too slow
  → Since all terms are of form
  (coefficient) × TensorProduct(OuterProduct, OuterProduct or Id)
  converted the expressions into {ket: {bra: coeff}}
- Then recursive H_{Ieff} determination can be formally made into a loop:
  - $$O(\lambda):\begin{cases}\hat{\mathcal{H}}^{(1)}_{\mathrm{I,eff}}=0,\\ \dot{\hat{G}}_1=\hat{\mathcal{H}}_I.\end{cases}\qquad(16a)$$

Sympy implementation

- Other orders:
  - Replace all calculated commutators with cached expressions from previous orders
    → Should eliminate all nested commutators
  - Compute the commutators from the expression dicts & Cache the newly calculated commutators
  - Extract the static terms from the sum of the commutators
    → $H^{(k)}_{I\mathrm{eff}}$
  - Integrate the dynamic terms and pass them down to the next order
- Calculation highly parallelized and run on iutgpu01 (256 cores) but $\mathcal{O}(\lambda^4)$ still takes a few minutes

Validation

```
[9]: compos = tth.pauli_components(hieff[2], 2, 2)
     pauli_names = ['I', 'X', 'Y', 'Z']

     for idx in [(0, 1), (3, 0), (3, 1), (3, 3)]:
         display(Math(fr'\omega_{{{pauli_names[idx[0]]}{pauli_names[idx[1]]}}}^{{(2)}} = ' + latex(compos[idx])))
```

$$\omega_{IX}^{(2)} = -\frac{J\Omega\left({\nu^c}_{1,2}\right)^2 {\nu^t}_{0,1}}{2\Delta_1^2}$$

$$\omega_{ZI}^{(2)} = \frac{J^2\left({\nu^c}_{1,2}\right)^2\left({\nu^t}_{0,1}\right)^2}{2\Delta_1^2} - \frac{J^2\left({\nu^c}_{0,1}\right)^2\left({\nu^t}_{1,2}\right)^2}{2\Delta_2^1} - \frac{J^2\left({\nu^c}_{0,1}\right)^2\left({\nu^t}_{0,1}\right)^2}{\Delta_1^1} + \frac{\Omega^2\left({\nu^c}_{1,2}\right)^2}{4\Delta_1^2} - \frac{\Omega^2\left({\nu^c}_{0,1}\right)^2}{2\Delta_1^1}$$

$$\omega_{ZX}^{(2)} = \frac{J\Omega\left({\nu^c}_{1,2}\right)^2 {\nu^t}_{0,1}}{2\Delta_1^2} - \frac{J\Omega\left({\nu^c}_{0,1}\right)^2 {\nu^t}_{0,1}}{\Delta_1^1}$$

$$\omega_{ZZ}^{(2)} = -\frac{J^2\left({\nu^c}_{1,2}\right)^2\left({\nu^t}_{0,1}\right)^2}{2\Delta_1^2} + \frac{J^2\left({\nu^c}_{0,1}\right)^2\left({\nu^t}_{1,2}\right)^2}{2\Delta_2^1}$$

Operator	Coefficient (Kerr)	Estimate (MHz)	Coefficient (energy basis)	Estimate (MHz)
$\frac{1}{2}\hat{I}\hat{X}$	$-\frac{1}{\Delta_{ct}+\alpha_c}J\Omega$	1.462	$-\frac{\nu_{t,01}\nu_{c,12}^2}{2(\Delta_{ct}+\alpha_c)}J\Omega$	1.250
$\frac{1}{2}\hat{Z}\hat{I}$	$[\frac{1}{2(\Delta_{ct}+\alpha_c)} - \frac{1}{2\Delta_{ct}}]\Omega^2$	-15.865	$[\frac{\nu_{c,12}^2}{4(\Delta_{ct}+\alpha_c)} - \frac{\nu_{c,01}^2}{2\Delta_{ct}}]\Omega^2$	-14.371
$\frac{1}{2}\hat{Z}\hat{X}$	$(\frac{1}{\Delta_{ct}+\alpha_c} - \frac{1}{\Delta_{ct}})J\Omega$	-2.411	$\frac{1}{2}(\frac{\nu_{t,01}\nu_{c,12}^2}{\Delta_{ct}+\alpha_c} - \frac{2\nu_{t,01}\nu_{c,01}^2}{\Delta_{ct}})J\Omega$	-2.118
$\frac{1}{2}\hat{Z}\hat{Z}$	$(\frac{1}{\Delta_{ct}-\alpha_t} - \frac{1}{\Delta_{ct}+\alpha_c})J^2$	0.138	$\frac{1}{2}(\frac{\nu_{c,01}^2\nu_{t,12}^2}{\Delta_{ct}-\alpha_t} - \frac{\nu_{t,01}^2\nu_{c,12}^2}{\Delta_{ct}+\alpha_c})J^2$	0.114

J^2 terms in ZI come from state dressing → picked out in the paper

Validation

$$\omega^{(4)}_{ZX} = J\Omega^3\left(\frac{(\nu^c{}_{1,2})^2(\nu^c{}_{2,3})^2\nu^t{}_{0,1}}{96(\Delta^3_1)^2(\Delta^2_1+\Delta^3_1)} - \frac{(\nu^c{}_{1,2})^2(\nu^c{}_{2,3})^2\nu^t{}_{0,1}}{32\Delta^2_1\Delta^3_1(\Delta^2_1+\Delta^3_1)} + \frac{(\nu^c{}_{1,2})^2(\nu^c{}_{2,3})^2\nu^t{}_{0,1}}{48\Delta^2_1(\Delta^3_1)^2} + \frac{(\nu^c{}_{1,2})^2(\nu^c{}_{2,3})^2\nu^t{}_{0,1}}{48(\Delta^2_1)^2(\Delta^2_1+\Delta^3_1)} - \frac{(\nu^c{}_{1,2})^2(\nu^c{}_{2,3})^2\nu^t{}_{0,1}}{24(\Delta^2_1)^2\Delta^3_1} + \frac{(\nu^c{}_{1,2})^4\nu^t{}_{0,1}}{8(\Delta^2_1)^3} + \frac{(\nu^c{}_{0,1})^2(\nu^c{}_{1,2})^2\nu^t{}_{0,1}}{32\Delta^1_1\Delta^2_1(\Delta^1_1+\Delta^2_1)}\right.$$
$$\left. - \frac{(\nu^c{}_{0,1})^2(\nu^c{}_{1,2})^2\nu^t{}_{0,1}}{12\Delta^1_1(\Delta^2_1)^2} - \frac{(\nu^c{}_{0,1})^2(\nu^c{}_{1,2})^2\nu^t{}_{0,1}}{32(\Delta^1_1)^2(\Delta^1_1+\Delta^2_1)} + \frac{5(\nu^c{}_{0,1})^2(\nu^c{}_{1,2})^2\nu^t{}_{0,1}}{48(\Delta^1_1)^2\Delta^2_1} - \frac{(\nu^c{}_{0,1})^4\nu^t{}_{0,1}}{4(\Delta^1_1)^3}\right)$$

+ $J^3\Omega$ terms

$$\begin{aligned}\omega^{(4)}_{zx} = \Bigg[&\frac{\nu^4_{c,01}\nu_{t,01}}{2\Delta^3_{ct}} + \frac{-\nu^2_{c,01}\nu^2_{c,12}\nu_{t,01} + 3\nu^2_{c,12}\nu^2_{c,23}\nu_{t,01}}{4\Delta^2_{ct}(\Delta_{ct}+\alpha_c)} \\ &+ \frac{\nu^2_{c,01}\nu^2_{c,12}\nu_{t,01} - \nu^2_{c,12}\nu^2_{c,23}\nu_{t,01}}{4\Delta_{ct}(\Delta_{ct}+\alpha_c)^2} - \frac{\nu^4_{c,12}\nu_{t,01}}{4(\Delta_{ct}+\alpha_c)^3} \\ &- \frac{\nu^2_{c,01}\nu^2_{c,12}\nu_{t,01}}{4\Delta^2_{ct}(2\Delta_{ct}+\alpha_c)} + \frac{9\nu^2_{c,12}\nu^2_{c,23}\nu_{t,01}}{4\Delta^2_{ct}(2\Delta_{ct}+3\alpha_c)}\Bigg]J\Omega^3,\end{aligned} \tag{22}$$

Results don't match, but the expression in paper does not seem correct to me: terms like $\dfrac{(\nu^c_{12})^2(\nu^c_{23})^2\nu^t_{01}}{\Delta^2_{ct}(\Delta_{ct}+\alpha_c)}$ shouldn't exist. Δ_{ct} in the denominator must be accompanied by ν^c_{01}.