Beschrijving Algoritmes

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The concept of optimal coefficients was introduced in [1], and their significance for the approximate computation of multidimensional integrals of arbitrary multiplicity s was indicated. Various algorithms for computing s-dimensional optimal coefficients modulo p where p is he number of nodes of the quadrature formula were obtained in [1]-[3]. The realization of these algorithms required the execution of $O(p^2)$ or $O(p^{1+1/3})$ elementary arithmetic operations.
In this note we present more economical algorithms for $p = 2^n$ whose realization requires the execution of $O(p)$ or $O(p \ln p)$ operations.
Let n and s be positive integers, and $x_1, ..., x_s$ odd integers. Summations over odd integers m is indicated by $\sum_m^*$. For $v = 1, ..., n$ we define the function $h_v(x_1, x_2, ..., x_s)$ by

$$h_v(x_1, x_2, ..., x_s) = \frac{1}{2^v} \sum_{m=1}^{2^v}{}^* \left(2n - 2v + \frac{1}{||mx_1/2^v||}\right) \cdots \left(2n - 2v + \frac{1}{||mx_s/2^v||}\right)$$

where $||mx_j/2^v||$ is the distance from $mx_j/2^v$ to the nearest integer.
Take $a_{11} = ... = a_{s1} = 1$. Suppose that $v \geq 2$ and that the odd integers $a_{1v-1}, ..., a_{sv-1}$ are known for $2 \leq v \leq n$ we define $a_{1v}, ..., a_{sv}$ by the equalities

$$a_{1v} = a_{1v-1} + 2^{v-1}z_1', ..., a_{sv} = a_{sv-1} + 2^{v-1}z_s'$$

where $z_1', ...z_s'$ are the variables at which the function

$$h_v(a_{1v-1} + 2^{v-1}z_1', ..., a_{sv-1} + 2^{v-1}z_s')$$

attains a minimum as the variables $z_1, ..., z_s$ run through the values 0 and 1 independently.

THEOREM 1. *For an arbitrary positive integer n the integer $a_1, ..., a_s$ defined by the equalities $a_1 = a_{1n}, ..., a_s = a_{sn}$ are optimal coeffecients modulo $p = 2^n$.*

Proof. For $v = 1, ..., n$ we introduce the notation

$$h_v = h_v(a_{1v}, a_{2v}, ..., a_{sv}).$$
$$H_v = \sum_{k=1}^{v-1} \sum_{m=1}^{2^k}{}^* \frac{1}{||ma_{1k}/2^k|| \cdots ||ma_{sk}/2^k||} + (2^{n+1} - 2^v)h_v.$$

De notatie h_v duidt rekening houdend met de definities hierboven op de waarde van het minimum van $h_v(a_{1v-1} + 2^{v-1}z_1', ..., a_{sv-1} + 2^{v-1}z_s')$

Observing that if $v \geq 2$, then

$$\frac{1}{2}\sum_{z=0}^{1} \frac{1}{||m(a+2^{v-1}z)/2^v||} \leq 2 + \frac{1}{||ma/2^{v-1}||} \tag{1}$$

Dit is logisch omdat de maximale waarde van $1/||x||$ gelijk is aan 2. De maximale waarde van de som aan de linkerkant is dus gelijk aan 2, kleiner dan het rechterlid.
for odd a and m, we get

$$h_v \leq \frac{1}{2^s} \sum_{z_1,\dots z_s=0}^{1} h_v(a_{1v-1}+2^{v-1}z_1, \dots, a_{sv-1}+2^{v-1}z_s)$$

h_v is de waarde van het minimum, en is dus kleiner of gelijk aan de gemiddelde waarde van h_v. De som in het rechterlid kan dan opgesplitst worden in $\sum_{z_1,\dots,z_s=0}^{1} \frac{1}{2} h_v(\dots)$. Door alle paren samen te nemen die enkel verschillen in één z-waarde en (1) toe te passen

$$\leq \frac{1}{2^v} \sum_{m=1}^{2^v}{}^{*} \left(2n - 2v + 2 + \frac{1}{||ma_{1v-1}/2^{v-1}||}\right) \cdots \left(2n - 2v + 2 + \frac{1}{||ma_{sv-1}/2^{v-1}||}\right)$$

$2n-2v+2+\dots$ wordt $2n-2(v-1)+\dots$, en de termen voor $m = 2^{v-1}+1, 2^{v-1}+3, \dots$ zijn gelijk aan de termen voor $m = 1, 3, \dots$. De sommatie valt dus uiteen in twee gelijke van $m = 1..2^{v-1}$

$$= h_{v-1} \tag{2}$$

Since $a_{11} = \cdots = a_{s1} = 1$, it follows that

$$h_1 = \frac{1}{2} \sum_{m=1}^{2}{}^{*} \left(2n - 2 + \frac{1}{||m/2||}\right) \cdots \left(2n - 2 + \frac{1}{||m/2||}\right) = 2^{s-1} n^s,$$

and, consequently, 2 gives us that

$$h_n \leq h_{n-1} \leq \cdots \leq h_1 \leq 2^{s-1} n^s$$

We now estimate the quantities H_v. Obviously,

$$H_1 = (2^{n+1} - 2) h_1 = (2^n - 1) 2^s n^s < (2n)^s 2^n$$

Bij $v = 1$ valt de sommatie uit de definitie van H_v weg.
Since

$$h_v = \frac{1}{2^v} \sum_{m=1}^{2^v}{}^{*} \left(2n - 2v + \frac{1}{||ma_{1v}/2^v||}\right) \cdots \left(2n - 2v + \frac{1}{||ma_{sv}/2^v||}\right)$$

we get for $v \geq 2$ that

$$\begin{aligned} H_v &\leq \sum_{k=1}^{v-2} \sum_{m=1}^{2^k}{}^{*} \frac{1}{||ma_{1k}/2^k|| \cdots ||ma_{sk}/2^k||} + 2^{v-1} h_{v-1} + (2^{n+1} - 2^v) h_v \\ &\leq \sum_{k=1}^{v-2} \sum_{m=1}^{2^k}{}^{*} \frac{1}{||ma_{1k}/2^k|| \cdots ||ma_{sk}/2^k||} + (2^{n+1} - 2^{v-1}) h_{v-1} = H_{v-1} \end{aligned}$$

De eerste lijn volgt uit het weglaten van $k = v - 1$ uit de sommatie. De tweede lijn volgt dan uit het feit dat $h_v \leq h_{v-1}$. NOTA: mijns inziens zou de gelijkheid in de eerste lijn enkel opgaan in het geval $v = n + 1$

and, consequently

$$H_n \leq H_{n-1} \leq \cdots \leq H_1 < (2n)^s 2^n \tag{3}$$

According to the definition of a_j and a_{jk},

$$a_1 \equiv a_{1k}, \ldots, a_s \equiv a_{sk} \pmod{2^k}$$

for $k = 1, \ldots, n$. But then it is obvious that

$$\sum_{m=1}^{2^n-1} \frac{1}{||ma_1/2^n|| \cdots ||ma_s/2^n||} = \sum_{k=1}^{n} \sum_{m=1}^{2^k}{}^* \frac{1}{||ma_1/2^k|| \cdots ||ma_s/2^k||}$$

$$= \sum_{k=1}^{n-1} \sum_{m=1}^{2^k}{}^* \frac{1}{||ma_{1k}/2^k|| \cdots ||ma_{sk}/2^k||} + \sum_{m=1}^{2^n}{}^* \frac{1}{||ma_{1n}/2^n|| \cdots ||ma_{sn}/2^n||} = H_n$$

$\square$