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SHORT COMMUNICATIONS

On an Algorithm for Constructing Uniformly Distributed Korobov Grids

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The problem to effectively construct uniformly distributed grids on a multidimensional unit cube is an actively developing and urgent direction with numerous applications and relations to other areas of mathematics (see, e.g., [1] and the bibliography therein).

In the present paper, by a *uniform distribution* of a sequence of grids (finite sets) $\{\xi_k^{(p)}\}_{k=1}^p$ in some s-dimensional unit cube $[0,1]^s$ that are indexed by a sufficiently dense increasing sequence of positive integers p we mean the existence of positive numbers $c(s)>0$ and $\beta(s)>0$ such that the inequality

$$\begin{aligned} D_s(\{\xi_k^{(p)}\}_{k=1}^p) &= \sup\biggl\{\biggl|\frac{1}{p}\sum_{k=1}^p \chi_I(\xi_k^{(p)}) - \prod_{j=1}^s (d_j-b_j)\biggr| : I=\prod_{j=1}^s [b_j,d_j]\subset[0,1]^s\biggr\} \\ &\le c(s)p^{-1}(\ln p)^{\beta(s)} \end{aligned} \qquad (1)$$

holds for any p, where χ_A is the characteristic function of the set A (for details, see, e.g., [2]).

In 1958, Korobov [3] proved that, for any positive integer p, there are positive integers

$$a_1=a_1(p), \quad a_2=a_2(p), \quad \dots, \quad a_s=a_s(p)$$

coprime to p and such that the grid

$$\xi_k(p,a_1,\dots,a_s)=\biggl(\biggl\{\frac{k}{p}a_1\biggr\},\dots,\biggl\{\frac{k}{p}a_s\biggr\}\biggr), \qquad k=1,\dots,p, \qquad (2)$$

is uniformly distributed on $[0,1]^s$ ($\{x\}$ stands for the fractional part of the number x).

This result is quite valuable from the computational point of view: it means that the grid (2) is completely determined by a given $(s+1)$-dimensional integral vector $(p, a_1(p), a_2(p), \dots, a_s(p))$ from which this grid is recovered by using $\asymp p$ elementary arithmetic operations, whereas an s-dimensional grid of volume p is composed by using sp real numbers.

We refer to the grids of the form (2) as *Korobov grids*.

Thus, the problem is to find a sufficiently dense sequence of positive integers p and of corresponding integers $a_1(p), a_2(p), \dots, a_s(p)$ coprime to p and such that the discrepancy D_s of the Korobov grid (2) satisfies inequality (1).

The main result of the present paper is the following theorem in which a definite answer to the problem posed above is given. Before formulating the theorem, we present another definition. Write

$$b_r(x)=\sum_{(m_1,\dots,m_s)\in\mathbb{Z}^s}(\overline{m}_1\cdots\overline{m}_s)^{-r}e^{2\pi i(m_1x_1+\dots+m_sx_s)},$$

where $r>1$ and $\overline{m}_j=\max(1,|m_j|)$, $j=1,\dots,s$.

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Theorem. *For given $r > 1$ and s, $s = 1, 2, \dots$, there are positive quantities c_1, c_2, c_3, β_1, and β_2 such that, for any positive integer p and for any integral vector $(a_1, \dots, a_s)$, the inequality*

$$c_3(s)\frac{(\ln p)^{(s-1)/2}}{p} \le D_s\left[\left(\left\{\frac{k}{p}a_1\right\}, \dots, \left\{\frac{k}{p}a_s\right\}\right)_{k=1}^{p}\right] \le c_1(s)\frac{(\ln p)^{\beta_1(s)}}{p} \tag{3}$$

holds if and only if

$$\left|\frac{1}{p}\sum_{k=1}^{p} b_r\left(\left\{\frac{k}{p}a_1\right\}, \dots, \left\{\frac{k}{p}a_s\right\}\right) - 1\right| \le c_2(r,s)\frac{(\ln p)^{\beta_2(r,s)}}{p^r}. \tag{4}$$

The lower bound in (3) follows from Roth's results in [1].

In this theorem, in addition to known criteria (see Theorem 22 (pp. 141–146) and Theorem 19 (pp. 126–130) in [3]), another criterion for the uniform distribution of Korobov grids is given.

A vast literature(see, e.g., [2]–[4] and the bibliography therein) is devoted to the problem of constructing optimal quadrature formulas, which is equivalent to inequality (3) (for details, see [3]). The results in these works are mainly of the type of existence theorems.

In [5]–[7], effective algorithms to construct uniformly distributed grids [7, Theorem 2] and quadrature formulas optimal in the power-law scale with equal weights and a grid of the form (2) [7, Theorem 4] are given; these algorithms are reduced to division of rational integers.

All these algorithms, even optimal ones (with respect to the number of elementary arithmetic operations $\asymp N \ln^\tau N$, $\tau \ge 0$), need a large amount of computational work for their realization with sufficiently many nodes N.

Therefore, other methods should be used to construct specific uniformly distributed grids. For instance, these are computational experiments using the theorem presented above.

Namely, in the case of integers $r > 1$, the function $b_r(x)$ in condition (4) is a 1-periodic algebraic Bernoulli polynomial; for example, for $r = 10$, we have

$$b_r(x) = \prod_{j=1}^{s}\left[1 + \frac{(2\pi)^{10}}{10!}\left(\frac{5}{66} - \frac{3}{2}x_j^2 + 5x_j^4 - 7x_j^6 + \frac{15}{2}x_j^8 - 5x_j^9 + x_j^{10}\right)\right].$$

Moreover, since the parameter $r > 1$ in (3) is involved in constants only, it follows that, to prove inequality (4) for any $r > 1$, it suffices to prove this inequality for some $r = r_0 > 1$ (up to constants).

Thus, the problem of constructing a uniformly distributed Korobov grid is reduced to the verification of inequality (4) for some integer $r > 1$ for the algebraic Bernoulli polynomial $b_r(x)$ (for one of the realizations, see [8]).

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