Beschrijving Algoritmes

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The concept of optimal coefficients was introduced in [1], and their significance for the approximate computation of multidimensional integrals of arbitrary multiplicity s was indicated. Various algorithms for computing s-dimensional optimal coefficients modulo p where p is he number of nodes of the quadrature formula were obtained in [1]-[3]. The realization of these algorithms required the execution of $O(p^2)$ or $O(p^{1+1/3})$ elementary arithmetic operations.

In this note we present more economical algorithms for $p = 2^n$ whose realization requires the execution of $O(p)$ or $O(p \ln p$ operations.

Let n and s be positive integers, and $x_1, ..., x_s$ odd integers. Summations over odd integers m is indicated by $\sum_m^*$. For $v = 1, ..., n$ we define the function $h_v(x_1, x_2, ..., x_s)$ by

$$h_v(x_1, x_2, ..., x_s) = \frac{1}{2^v} \sum_{m=1}^{2^v}{}^* \left(2n - 2v + \frac{1}{||mx_1/2^v||}\right) \cdots \left(2n - 2v + \frac{1}{||mx_s/2^v||}\right) \tag{1}$$

where $||mx_j/2^v||$ is the distance from $mx_j/2^v$ to the nearest integer.

Take $a_{11} = ... = a_{s1} = 1$. Suppose that $v \geq 2$ and that the odd integers $a_{1v-1}, ..., a_{sv-1}$ are known for $2 \leq v \leq n$ we define $a_{1v}, ..., a_{sv}$ by the equalities

$$a_{1v} = a_{1v-1} + 2^{v-1} z_1', ..., a_{sv} = a_{sv-1} + 2^{v-1} z_s' \tag{2}$$

where $z_1', ...z_s'$ are the variables at which the function

$$h_v(a_{1v-1} + 2^{v-1} z_1', ..., a_{sv-1} + 2^{v-1} z_s' \tag{3}$$

attains a minimum as the variables $z_1, ..., z_s$ run through the values 0 and 1 independently.

THEOREM 1. *For an arbitrary positive integer n the integer $a_1, ..., a_s$ defined by the equalities $a_1 = a_{1n}, ..., a_s = a_{sn}$ are optimal coeffecients modulo $p = 2^n$.*

Proof. For $v = 1, ..., n$ we introduce the notation

$$h_v = \tag{4}$$

□