MAT2041

Midterm Exam Sample Papers

Time: Nov/12/22 Sun 10:00am - 12:00pm University Stadium
DURATION OF EXAMINATION: 2 hours in-class

This examination paper includes 6 *pages and* 6 *problems. You are responsible for ensuring that your copy of the paper is complete. Bring any discrepancy to the attention of your invigilator.*

1. **(30 points)** *Solving a linear system of equations*
   For a real number c, consider the linear system:

$$x_1 + x_2 + cx_3 + x_4 = c \tag{1}$$

$$-x_2 + x_3 + 2x_4 = 0 \tag{2}$$

$$x_1 + 2x_2 + x_3 - x_4 = -c \tag{3}$$

   Do the following:

   (a) Write out the *coefficient matrix* of the system.

   (b) Write out the *augmented matrix* for this system and calculate its *row-reduced echelon form.*

   (c) Write out the complete set of solutions in *vector form.*

   (d) What is the *rank* of the coefficient matrix $\boldsymbol{A}$? Justify your answer.

   (e) Find a *basis* of the subspace of solutions when $c = 0$.

2. **(20 points)** *Linear space*
   Find a *basis* for each of the following spaces.

   - Space of $n \times n$ *skew symmetric matrices* (i.e. those matrix satisfying $\boldsymbol{A} = -\boldsymbol{A}^\top$)

   - The space of all *polynomials* of the form $ax^2 + bx + 2a + 3b$, where $a, b \in \mathbb{R}$.

   - $\text{Span}\{x - 1, x + 1, 2x^2 - 2\}$.

3. **(15 points)** *Matrix multiplications*
Prove the following statements:

(a) Define the set of $n \times n$ *diagonol matrices* to be K. Prove that for a diagonal matrix $\boldsymbol{D}$ with *distinct* elements (i.e. $\boldsymbol{D}_{ii} \neq \boldsymbol{D}_{jj}, \forall i \neq j$), the set $\{\boldsymbol{A} \in \mathbb{R}^{n \times n} | \boldsymbol{AD} = \boldsymbol{DA}\}$ is exactly K.

(b) If an $n \times n$ matrix $\boldsymbol{A}$ satisfies $\boldsymbol{AB} = \boldsymbol{BA}$ for any $n \times n$ matrix $\boldsymbol{B}$, then $\boldsymbol{A}$ must be of the form $c\boldsymbol{I}$, where c is a scalar.

4. **(10 points)** *Matrix Inverse*

(a) Compute the inverse of the matrix $\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$.

(b) Compute the inverse of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ if it exists. When does the inverse of the matrix exist?

5. **(15 points)** *Matrix rank*

(a) Suppose $\boldsymbol{u} \in \mathbb{R}^{n\times 1}$ satisfies $\|\boldsymbol{u}\| = 1$. What is the rank of the matrix $\boldsymbol{I} - \boldsymbol{u}\boldsymbol{u}^\top$?

(b) Suppose $\boldsymbol{u} \in \mathbb{R}^{n\times 1}$ satisfies $\|\boldsymbol{u}\| = 1$. Define $\boldsymbol{P} = \boldsymbol{I} - \boldsymbol{u}\boldsymbol{u}^\top$. What is the rank of $\boldsymbol{P}^2$? What about $\boldsymbol{P}^5$?

(c) Suppose $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{n\times 1}$. What is the rank of the matrix $\boldsymbol{I} - \boldsymbol{x}\boldsymbol{y}^\top$?

6. **(20 points)** *Orthogonal subspaces*

Let $\mathbf{v}$ be a nonzero vector in a linear space V. Let W be the set of all vectors in V that are orthogonal to $\mathbf{v}$. Prove that W is a subspace of V.