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Survey

Covering problems in facility location: A review

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ABSTRACT

In this study, we review the covering problems in facility location. Here, besides a number of reviews on covering problems, a comprehensive review of models, solutions and applications related to the covering problem is presented after Schilling, Jayaraman, and Barkhi (1993). This survey tries to review all aspects of the covering problems by stressing the works after Schilling, Jayaraman, and Barkhi (1993). We first present the covering problems and then investigate solutions and applications. A summary and future works conclude the paper.

1. Introduction

Facility location is a critical component of strategic planning for a broad spectrum of public and private firms (Owen & Daskin, 1998). For this, it is necessary to consider many criteria such as cost or distance from demand points. Many models have been made to help decision making in this area. The readers who are interested in learning about facility location models are referred to the works of Francis and White (1974), Handler and Mirchandani (1979), Love, Morris, and Wesolowsky (1988), Francis, McGinnis, and White (1992), Mirchandani and Francis (1990), Daskin (1995), Drezner (1995), Drezner and Hamacher (2002), Nickel and Puerto (2005), Church and Murray (2009) and Farahani and Hekmatfar (2009).

One of the most popular models among facility location models is covering problem. While covering models are not new they have always been very attractive for research. This is due to its applicability in real-world life, especially for service and emergency facilities. In some covering problems, a customer should be served by at least one facility within a given critical distance (not necessarily the nearest facility). In most of the covering problems, customers receive services by facilities depending on the distance between the customer and facilities. The customer can receive service from each facility which its distance from customer is equal or less than a predefined number. This critical predefined number is called coverage distance or coverage radius (Fallah, NaimiSadigh, & Aslanzadeh, 2009). Therefore, the concept of coverage is related to a satisfactory method rather than a best possible one. Many of the problems like determining the number and locations of public schools, police stations, libraries, hospitals, public buildings, post offices, parks, military vases, radar installations, branch banks, shopping centers and waste-disposal facilities can be formulated as covering problems (Francis & White, 1974). The scope of this survey is exclusively limited to the review of articles related to covering problem in facility location.

Schilling, Jayaraman, and Barkhi (1993) present a literature review on covering problems in facility location. Since they present a very comprehensive review considering publications up to 1991, we have tried to consider covering researches after this time. However, this paper also covers some older papers that are both very important and basic from classification point of view or have not been in the domain of Schilling et al. (1993).

Schilling et al. (1993) classify models which use the concept of covering in two categories: (1) *Set Covering Problem* (SCP) where coverage is required and (2) *Maximal Covering Location Problem* (MCLP) where coverage is optimized. For each category, they provide taxonomy according to topological structure, nature of demand, characteristic of facility to be sited and application in public or private sectors. Also, based on solution methods – either optimal or heuristic – a classification is proposed. Owen and Daskin (1998) present an overview of facility location literature considerings stochastic or dynamic problem characteristics. Conforti, Cornuéjols, Kapoor, and Vušković (2001) study results and also open problems on perfect, ideal and balanced metrics related to set packing and set covering problem. Also Berman,

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Drezner, and Krass (2010b) present an overview of covering model concentrate on three areas: (i) gradual covering model, (ii) cooperative covering model and (iii) variable radius model.

We have searched SCOPUS, one of the largest title, abstract and keyword databases to check the related works and to see the trend of those research works over time (since 1992 till 1 February 2011). Thus, we tested different keywords in this database. Finally, we made the following decision as a suitable combination:

(TITLE-ABS-KEY (covering AND location) AND TITLE-ABS-KEY (model OR problem OR facility)) AND PUBYEAR AFT 1991.

A 1531 cases were found. We have presented the most important parts of SCOPUS report in Table 1. Excluding 2011 (because we are in early February 2011 now) as the table shows, there is an increasing trend in the number of published documents on covering in different areas. The numbers in parentheses of the table show the number of found related issues. However, since some of the found articles in SCOPUS are not related to the scope of this paper we have not included them. Among the sources the highest priority has been given to journal papers, conference papers and books, respectively.

2. Basic models and classification

With respect to history and origination, for the first time Hakimi (1965) introduces covering problems. The model is aimed to

Table 1
SCOPUS's report based on search in "location" and "covering".

Source title	Author name	Year	Document type	Subject area
European Journal of Operational Research (30)	ReVelle, C. (14)	2010	Article	Engineering (409)
Computers and Operations Research (28)	Berman, O. (12)	(165)	(1072)	
Lecture Notes in Computer Science Including	Church, R. L. (11)			Earth and Planetary Sciences (326)
Subseries Lecture Notes in Artificial Intelligence and	Murray, A. T. (11)	2009	Conference	
Lecture Notes in Bioinformatics (27)	Marianov, V. (8)	(178)	Paper	Computer Science (253)
Astrophysical Journal (23)	Drezner, Z. (7)		(348)	
Proceedings of SPIE the International Society for	Krass, D. (7)	2008		Environmental Science (231)
Optical Engineering (18)	Lowe, T. J. (7)	(178)	Review	
Astronomy and Astrophysics (16)	Tamir, A. (6)		(40)	Mathematics (172)
Journal of Biogeography (16)	Galvao, R. D. (6)	2007		
Journal of the Operational Research Society (15)	Serra, D. (6)	(128)	Article in	Social Sciences (160)
Global Ecology and Biogeography (14)	Wesolowsky, G. O. (5)		Press (26)	
Annals of Operations Research (13)	Laporte, G. (5)	2006		Decision Sciences (159)
Forest Ecology and Management (11)	Kara, B. Y. (5)	(103)	Conference	
Journal of Geophysical Research D Atmospheres (9)	Francis, R. L. (5)		Review (6)	Agricultural and Biological Sciences (159)
Socio Economic Planning Sciences (9)	Plastria, F. (5)	2005		
Atmospheric Environment (9)	Lorena, L. A. N. (5)	(115)	Business	Medicine (128)
Location Science (8)	Kraemer, S. B. (4)		Article (3)	
Transportation Research Record (7)	Batta, R. (4)	2004		Physics and Astronomy (111)
Papers in Regional Science (7)	Hoefer, M. (4)	(93)	Short	
Computers and Industrial Engineering (7)	Crenshaw, D. M. (4)		Survey (3)	Biochemistry, Genetics and Molecular
Networks (7)	DeLand, M. T. (4)	2003		Biology (101)
Proceedings of the National Academy of Sciences of	Drezner, Z. (4)	(66)	Note (1)	
the United States of America (7)	Gencer, C. (4)			Business, Management and Accounting
Field Crops Research (7)	Gerrard, R. A. (4)	2002	Report (1)	(58)
Advances in Space Research (6)	Emir-Farinas, H. (4)	(62)		
International Geoscience and Remote Sensing	Lilley, D. G. (4)		Undefined	Energy (51)
Symposium IGARSS (6)	Saydam, C. (4)	2001	(16)	
Discrete Applied Mathematics (6)	Selim, H. (3)	(57)		Chemical Engineering (45)
Journal of Climate (5)	Cabello, S. (3)			
Remote Sensing of Environment (5)	Jogloy, S. (3)	2000		Materials Science (36)
International Journal of Climatology (5)	Van Dishoeck, E. F.	(58)		
Pure and Applied Geophysics (5)	(3)			Neuroscience (35)
Lecture Notes in Computer Science (5)	Vanhaverbeke, L. (3)	1999		
Annales Geophysicae (5)	Morabito, R. (3)	(51)		Economics, Econometrics and Finance
IIE Transactions Institute of Industrial Engineers (5)	Brandenberg, R. (3)			(23)
International Journal of Heat and Mass Transfer (4)	Schobel, A. (3)	1998		
Operations Research Letters (4)	Bottino, A. (3)	(55)		Chemistry (19)
Transportation Science (4)	Blake, G. A. (3)			
Icarus (4)	Schierbeck, J. (3)	1997		Immunology and Microbiology (18)
Journal of Vegetation Science (4)	Williams, J. C. (3)	(52)		
Theoretical and Applied Genetics (4)	Kotha, S. (3)			Multidisciplinary (16)
Annals of Glaciology (4)	Wu, N. W. (3)	1996		
Journal of Hydrology (4)	Gibson, A. (3)	(49)		Health Professions (15)
Water Science and Technology (4)	Yang, C. (3)			
Journal of Molecular Biology (4)	Mahlooji, H. (3)	1995		Psychology (13)
Monthly Weather Review (4)	Laurentini, A. (3)	(22)		
Monthly Notices of the Royal Astronomical Society	Roth, L. (3)			Pharmacology, Toxicology and
(4)	Dreyer, L. C. (3)	1994		Pharmaceutics (12)
Tectonophysics (4)		(31)		
Neuroimage (3)				Veterinary (6)
Health Physics (3)		1993		
Naval Research Logistics (3)		(25)		Nursing (3)
American Society of Mechanical Engineers Pressure				
Vessels and Piping Division Publication PVP (3)		1992		Dentistry (3)
Natural Hazards (3)		(28)		
Hydrology and Earth System Sciences Discussions (3)				Arts and Humanities (2)
				Undefined (9)

determine the minimum number of police needed to cover nodes (i) on a network of highways.

He formulates the problem as a vertex-covering problem in a graph. By considering graph G with the same weight assigned to its all branches (equal one), V as the set of vertices of graph G, W as a subset of V, d showing distance and S as a maximum acceptable service distance (or time), the subset of W covers G if:

$$d(v_i, w) \leqslant S \quad i = 1, \ldots, n \tag{1}$$

where

$$d(v_i, w) = \text{Min}[d(v_i, v_1), d(v_i, v_2), \ldots, d(v_i, v_q)] \tag{2}$$

The first mathematical model in covering problems was developed by Toregas, Swain, ReVelle, and Bergman (1971). They consider modeling the location of emergency service facilities as follows:

i: the index of demand nodes,
j: the index of facilities,
N_i: the set of potential locations within S so that $N_i = \{j | d_{ij} \leqslant S\}$,
x_j: a binary decision variable indicating whether the facility located at point j or not,
d_{ij}: the distance between demand node i and facility j, and
S: a maximum acceptable service distance. The model is as follows:

$$\text{Min } z = \sum_{j=1}^{n} x_j \tag{3}$$

S.T.

$$\sum_{j \in N_i} x_j \geqslant 1 \quad i = 1, \ldots, m \tag{4}$$

$$x_j \in \{0, 1\} \quad j = 1, \ldots, n \tag{5}$$

Objective function (3) minimizes the total number of located facilities. Constraint (4) shows the service requirement for demand node i and constraint (5) is the integrality constraint.

Among the books that have dealt with covering problems in detail including solution techniques, we believe that Francis and White (1974) contribute significantly. Francis and White (1974) present covering problems well; they divide it into two major categories that will be covered in this section, and Francis et al. (1992) talk about covering problems under two categories: *network location problem* and *cyclic network problems*. For the network location problem, they initially focus on a special kind of networks which is acyclic (so-called tree). Tree is an *acyclic network* with a unique shortest path between each two nodes that makes it easier than a general network to solve. Therefore, considering symmetry, positivity, triangle inequality and convexity properties, the problem can be solved efficiently. These properties and changing a general network to an acyclic one can be used as an approximation to create a "quick and dirty" solution for the problem.

Since set covering problems and maximal covering location problems are two traditional classifications for covering models, in this section we present the original mathematical formulations for these two categories and also direct extensions of these models.

2.1. The Set Covering Problem (SCP)

The set covering problem tries to minimize location cost satisfying a specified level of coverage. The mathematical formulation of this problem is as follows:

i: the index of demand nodes,
j: the index of facilities,
x_j: a binary decision variable indicating whether the facility located at point j or not,
S: the maximum acceptable service distance,
c_j: the fixed cost of locating facility at node j and
a_{ij}: a binary parameter is 1 if distance from candidate place j to the existing facility (customer) i is not greater than S. The model is as follows:

$$\text{Min} \sum_{j=1}^{n} c_j x_j \tag{6}$$

S.T.

$$\sum_{j=1}^{n} a_{ij} \cdot x_j \geqslant 1 \quad \forall\ i\ (i = 1, \ldots, m) \tag{7}$$

$$x_j \in \{0, 1\} \quad j = 1, \ldots, n \tag{8}$$

Objective function (6) minimizes the cost of locating facilities. The mentioned model is called Weighted Set Covering Problem (WSCP) and also non-unicost set covering problem. If the location costs for all the facilities are the same, the objective function can be simplified as (3).

Objective function (3) is equivalent to minimization of the number of located facilities. If we substitute objective function (3) with the objective of WSCP, the resulting model can be called Minimum Cardinality Set Covering Problem (MCSCP) and also unicost set covering problem. Vasko and Wilson (1986) show that, from computational time point of view, solving MCSCP is more difficult than WSCP.

In order to cover the nodes, constraint (7) enforces that for each demand node, at least one facility must be located within the set N_i of candidate facility sites. Constraint (8) is the integrality constraint.

The following subsections include several direct extensions of the SCP.

2.1.1. LSCP (location set covering problem) Implicit and Explicit

Murray, Tong, and Kim (2010) present two models called LSCP-Implicit and LSCP-Explicit. LSCP-Implicit model assumes that each demand area can be covered not only by one facility but also by two or more so that each facility covers a percentage of demand. The notation of the model is as follows:

i: the index of demand area,
j: the index of facilities,
k: the index of coverage levels,
x_j: a binary decision variable indicating whether the facility located at point j or not,
Y_{ik}: a binary decision variable equal one if area i is covered at level k and otherwise 0,
β_k: the minimum required percentage of coverage at level k,
α_k: the minimum number of required facilities for covering completely at kth level, and
Ω_{ik}: the set of potential facilities cover area i at least β_k.

LSCP-Implicit mathematical model is as follows:

$$\text{Min } z = \sum_{j} x_j \tag{9}$$

S.T.

$$\sum_{j \in \Omega_{ik}} x_j \geqslant \alpha_k Y_{ik} \quad \forall\ i, k \tag{10}$$

$$\sum_{k} Y_{ik} = 1 \quad \forall\ i \tag{11}$$

$$Y_{ik} \in \{0, 1\} \quad \forall\ i, k \tag{12}$$

$$x_j \in \{0, 1\} \quad \forall\ j \tag{13}$$

Objective function (9) minimizes the cost of locating facilities. Constraint (10) states that for completely covering demand area at level k, α_k facilities should be located. Constraint (11) ensures the existence of the coverage at level k. Constraints (12) and (13) are the integrality constraints.

Explicit coverage considers the coverage provided to a demand area by a specific set of facilities tracking facilities combination. Consider the following notation:

l: the index of facility configuration,
Ψ_{ik}: the set of k facility configurations completely cover area i,
Δ_{ikl}: the set of k facilities in lth configuration which covers area i completely, and
Z_{ikl}: a binary decision variable, it equals to 1 if area i is covered by configuration l at level k and otherwise 0.

LSCP-Explicit mathematical model is as follows:

$$\text{Min } z = \sum_j x_j \tag{14}$$

S.T.

$$\sum_k \sum_{l \in \Psi_{ik}} Z_{ikl} = 1 \forall\ i \tag{15}$$

$$x_j \geqslant Z_{ikl} \quad \forall\ i, k, l \in \Psi_{ik},\ \ j \in \Delta_{ikl} \tag{16}$$

$$Z_{ikl} \in \{0, 1\} \quad \forall\ i, k, l \in \psi_{ik} \tag{17}$$

$$x_j \in \{0, 1\} \quad \forall\ j \tag{18}$$

Constraint (15) states that in order to completely cover demand area, a k-facility configuration should be chosen. Constraint (16) indicates a configuration can be assumed only when the required facilities are located. Constraints (17) and (18) are the integrality constraints.

2.1.2. Capacitated SCP

In the above SCP formulation there is no distinction between nodes based on demand size. The located facilities are uncapacitated; i.e. there is no limitation for the capacity of a new located facility. Therefore, each node, whether it contains a single customer or a large number of customers (and consequently a large portion of the total demand), must be covered within the specified distance, regardless of cost (Owen & Daskin, 1998). Most of the papers in covering literature consider uncapacitated SCP. For instance, Kolen and Tamir (1990) concentrate only on the *uncapacitated* versions of the covering problems. They consider three types of costs with a given set of facility location problems with focus on covering, center and median problems: (i) set up cost, (ii) transportation cost and (iii) penalty cost. They believe that in covering problem common objective functions are minimizing "the sum of the set up costs and penalty costs". These researchers consider *budget constraint*, *client constraints* and *facility constraint* in their model. However, in most real-world applications of covering problems considering capacitated facilities are more realistic. For example, Current and Storbeck (1988) add facility capacity restrictions to the problem and present a *capacitated version of SCP* formulations.

2.1.3. Quadratics SCP

Bazaraa and Goode (1975) extend classical SCP to quadratic case. In other words, instead of the objective function (6) that can be expressed as $C \cdot X$, they are using $X^T \cdot C \cdot X$ where the constraints are of the inequality type. Hence, $C \cdot X$ is cost of locating new facilities and $X^T \cdot C \cdot X$ is cost of relationship between each pair of the new facilities. They do not talk about the application of their model but we may interpret that like in the traditional Quadratic Assignment Problem (QAP) in which there is a relation also between the new located facilities.

2.1.4. Multiple optimal SCP

Love et al. (1988) deal with covering models in a book chapter entitled "site-selecting location-allocation models". They introduce *set covering under the minimax criterion*. In this problem, knowing the optimal number of the new facilities needed for total coverage, say m, and provided that there are multiple optimal solutions for the problem, we are willing to find the location of m new facilities somehow it is more attractive for a decision maker. As such, a secondary criterion here minimizes maximum time (or distance) for all the demand points to their nearest facility.

2.1.5. Covering Tour Problem (CTP)

Gendreau, Laporte, and Semet (1997b) develop an integer linear programming formulation for *covering tour problem* on a graph. Consider the following notation:

i, j, k, l: the indices of vertices,
V: the set of vertices that can be covered,
W: the set of vertices that must be covered,
T: the set of vertices that must be visited,
S': any subset of V,
$S'_l : S'_l = \{v_k \in V | \delta_{ik} = 1\}$ where δ_{lk} is a binary coefficients, it equals to 1 when $v_l \in W$ can be covered by $v_k \in V$.
E: the set of edges ($E = \{(v_i, v_j) | v_i, v_j \in V \cup W, i < j\}$),
G: $G = (V \cup W, E)$,
x_{ij}: a binary decision variable indicates whether edge (v_i, v_j) belongs to the tour or not,
y_k: a binary decision variable, it equals to 1 if $v_k \in V$ belongs to the tour T, and
c_{ij}: the cost of belonging edge (v_i, v_j) to the tour.

The goal is determining a minimum length Hamiltonian cycle on a sub-set of V such that every vertex of W is within a pre-specified distance from the cycle. The mathematical model is as follows:

$$\text{Min } z = \sum_{i \prec j} c_{ij} x_{ij} \tag{19}$$

S.T.

$$\sum_{v_k \in S'_l} y_k \geqslant 1 \quad v_l \in W \tag{20}$$

$$\sum_{i \prec k} x_{ik} + \sum_{j \succ k} x_{kj} = 2y_k \quad v_k \in V \tag{21}$$

$$\sum_{\substack{v_i \in S', v_j \in V \setminus S' \\ \text{or } v_j \in S', v_i \in V \setminus S'}} x_{ij} \geqslant 2y_t (S' \subset V, 2 \leqslant |S'| \leqslant n-2, T \setminus S' \neq \phi, v_t \in S') \tag{22}$$

$$x_{ij} \in \{0, 1\} \quad 1 \leqslant i < j \leqslant n \tag{23}$$

$$y_k = 1 \quad v_k \in T \tag{24}$$

$$y_k \in \{0, 1\} \quad v_k \in V \setminus T \tag{25}$$

Objective function (19) minimizes tour total cost. Constraint (20) states that each vertex of W should be covered by the tour, constraints (21) and (22) are degree and connectivity constraints respectively and constraints (23)–(25) impose the integrality requirements.

2.1.6. Path covering problems

Boffey and Narula (1998) consider *path covering problems* with emphasizing on multi-path (especially double path) models. In most of the covering problems, the facilities are considered to be small in size in proportion to their location region and can be assumed as points. Therefore, we can term these problems as point cover. Sometimes, we face application of covering problems with one-dimensional facilities like paths and trees in which the shape of facilities cannot be neglected. Subway (e.g. metro, tube, underground) and highways are suitable examples for facilities of these kinds. They define *Maximum Population Shortest Path (MPSP)* as to find a path through the network such that the path length is minimized and the population coverage is maximized. Accordingly, they introduce 2-MPSP as the problem of finding two paths such that the combined path length is minimized and population coverage, at least once, is maximized. They also develop two Lagrangian solution approaches for solving the 2-MPSP problem. Consider the following notation for MPSP:

i, j, k, O, D: the indices of vertices,
V: the set of vertices ($V = \{1,\ldots,n\}$),
E: the set of edges,
G: G (V,E),
x_{ij}: a binary decision variable, it is 1 if the solution path contains arc ij, and 0 otherwise,
z_k: a binary decision variable, it is 1 if node k is covered by the path, and 0 otherwise,
d_{ij}: the weight (length) of arc $ij \in E$, and
h_k: the weight (population) of vertex k.

$$\text{Max } z_1 = \sum_{k\in V} h_k z_k \tag{26}$$

$$\text{Min } z_2 = \sum_{ij\in E} d_{ij} x_{ij}$$

S.T.

$$\sum_{i|ij\in E} x_{ij} - \sum_{i|ji\in E} x_{ij} = \begin{cases} -1 & \text{if } j = O \\ 0 & \text{if } j \neq O, D \\ 1 & \text{if } j = D \end{cases} \tag{27}$$

$$\sum_{j|ij\in E} x_{ij} - z_i \geqslant 0 \quad \forall\ i \neq D \tag{28}$$

$$x_{ij}, z_k \in \{0,1\} \quad \forall\ ij \in E,\ \ k \in V \tag{29}$$

Objective functions (26) minimize the length of path and also maximize the population coverage. Constraint set (27) identifies conditions which flow be conserved on the route taken by flowing of one unit from node O to node D. Constraint (28) considers the variable x_{ij} in terms of covering node j. Constraint (29) is the integrality constraint.

Here consider the following notation for 2-MPSP:

r: the index of path,
x_{ij}^r: a binary decision variable, it is 1 if path r contains arc ij, and 0 otherwise, and
z_k^r: a binary decision variable, it is 1 if node k is covered by path r, and 0 otherwise.

$$\text{Max } z_1 = \sum_{k\in V} h_k \left(z_k^1 + z_k^2\right) \tag{30}$$

$$\text{Min } z_2 = \sum_{ij\in E} d_{ij} \left(x_{ij}^1 + x_{ij}^2\right)$$

S.T.

$$\sum_i x_{ij}^r - \sum_k x_{jk}^r = \begin{cases} -1 & \text{if } j = O \\ 0 & \text{if } j \neq O; D, \quad r = 1,2 \\ 1 & \text{if } j = D \end{cases} \tag{31}$$

$$\sum_{j|ij\in E} x_{ij}^r = z_i^r \quad \forall\ i \neq j,\ \ r = 1,2 \tag{32}$$

$$z_k^1 + z_k^2 \leqslant 1 \quad \forall\ k \in V \tag{33}$$

$$x_{ij}^r, z_k^r \in \{0,1\} \quad \forall\ i,j,r \tag{34}$$

Objective functions (30) try to find two paths such that the length of path minimized and also cover (at least once) maximized. Constraints in set (31) determine flow conservation and also permission of loops. Constraint set (32) considers the variable x_{ij}^1 and x_{ij}^2 in terms of covering node. Constraint (33) states the interaction between the paths. Constraint (34) is the integrality constraint.

2.1.7. Probabilistic SCP

We can consider each of deterministic parameters of a SCP as probabilistic to create a probabilistic SCP. ReVelle and Hogan (1989b) present a probabilistic version of SCP and call it *probabilistic Location Set Covering Problem* (PLSCP). Their model considers dynamic aspect of location problems especially in emergency facilities. In emergency facilities sometimes vehicles are not available when they are called. They propose two mathematical formulations for this problem. The notation of the main model is as follows:

x_j: a binary decision variable indicating whether the facility located at point j or not,
α: the required amount of probability,
F_i: Cumulative the number of calls per hour at node i multiplied by the average duration of a call (h), and
b_i: the smallest integer satisfying $1 - (F_i/b_i)^{b_i} \geqslant \alpha$.

The model is as follows:

$$\text{Min } z = \sum_{j=1}^{n} x_j \tag{35}$$

S.T.

$$\sum_{j\in N_i} x_j \geqslant b_i \forall\ i \tag{36}$$

$$x_j \in \{0,1\} \quad \forall\ j \tag{37}$$

The first derivation of their model is "α-reliable P-center problem" aims to find the location of P facilities in order to minimizing maximum time between availability of α reliable service.

The second derivation of their model is "the maximum reliability location problem" which aims to find the location of P facilities in order to maximize the minimum reliability of services.

Beraldi and Ruszczynski (2002) develop probabilistic set covering problem. In this problem,

x_j: a binary decision variable indicating whether the facility located at point j or not,
A: 0–1 covering matrix; element ij is 1 if potentially the facility located at point j can cover demand point i,
ζ: a random binary vector and

$p \in (0,1)$: the prespecified reliability. The model is as follows:

$$\text{Min } z = \sum_{j=1}^{n} x_j \tag{38}$$

S.T.

$$P\{Ax \geqslant \zeta\} \geqslant p \tag{39}$$

$$x_j \in \{0,1\} \quad \forall\ j \tag{40}$$

Constraint (39) states that covering constraint must be satisfied with some prescribed probability.

Saxena, Goyal, and Lejeune (2010) present some mixed integer program formulations for the above probabilistic SCP. Consider the following notation:

u: a random vector $u = (u_1, \ldots, u_M)$ ($u \in \mathrm{R}^M$),
u^t: the sub-vector of u formed by components in M_t for $t = 1, \ldots, L$,
F: F:$\{0,1\}^M \to R$ is the cumulative distribution function of ξ and $u \in \{0,1\}^M$, $F(u) = p(\zeta \leqslant u)$,
F_t: the restriction of F to M_t for $t \in \{1, \ldots, L\}$ where $M_1, \ldots, M_L$ is a partition of M,
S_t: the set of binary vectors either p-efficient or dominate a p-efficient point of F_t,
I_t: the set of p-inefficient points of F_t,
u_i: $u_i = \min(a_i^T x, 1) \quad i \in M$ where a_i denotes the ith row of A,
η_t: $\eta_t = \ln F_t(u^t) \quad t \in \{1, \ldots, L\}$, and
v: $v \in S_t \cup I_t$.

The MIP model is as follows:

$$\text{Min } z = \sum_j x_j \tag{41}$$

S.T.

$$Ax \geqslant u \tag{42}$$

$$1 \leqslant \sum_{i \in M_t, v_i = 0} u_i \quad \forall\ v \in S_t,\ \forall\ t \in \{1, \ldots, L\} \tag{43}$$

$$\sum_{t=1}^{L} \eta_t \geqslant \ln p \tag{44}$$

$$\eta_t \leqslant (\ln F_t(v)) \left(1 - \sum_{i \in M_t, v_i = 0} u_i\right) \quad \forall\ v \in S_t,\ \forall\ t \in \{1, \ldots, L\} \tag{45}$$

$$u_i \in \{0,1\} \quad \forall\ i \in M \tag{46}$$

$$x_j \in \{0,1\} \quad \forall\ j \tag{47}$$

Objective function (41) and constraints (42) and (43) are basic relations in set covering problem. They prove (44) and (45) are an equivalent conversion of (39). They propose two other equivalent formulations for above model.

2.1.8. Stochastic SCP

Hwang (2002) aims to design a supply chain logistics system. The problem is solved in two stages. In the first stage a mathematical programming model is developed to determine minimum number of warehouses/distribution centers (W/D) among a number of discrete potential sites. This is a *stochastic set covering problem* so that the probability of each demand point being covered is not less than a critical level. This problem is solved using 0–1 programming method. Consider the following notation:

x_j: a binary decision variable indicating whether the facility located at point j or not,
c_{ij}: the logistic cost incurred between nodes i and j,
A_i: a required service level,
r_i: critical service level, and
a_{ij}: a binary parameter, it is 1 if $prob(c_{ij} \leqslant A_i) \geqslant r_i$ and otherwise 0.

The model is as follows:

$$\text{Min} \sum_{j=1}^{n} x_j \tag{48}$$

S.T.

$$\sum_{j=1}^{n} a_{ij}.x_j \geqslant 1 \quad \forall\ i \quad (i = 1, \ldots, m) \tag{49}$$

$$x_j \in \{0,1\} \quad j = 1, \ldots, n \tag{50}$$

Objective function (48) and constraints (49) and (50) are basic relations in set covering problem. But (49) is an extension of the traditional deterministic function into stochastic space considering the new probabilistic definition of a_{ij}. In the model, it is assumed that all warehouse and distribution centers are always available. Also Hwang proposes other models in which availability of all facilities is relaxed. The second stage is related to a vehicle routing problem that is solved using genetic algorithm.

Baron, Berman, Kim, and Krass (2009) develop a set covering problem to a general class of *Location Problems with Stochastic Demand and Congestion* (LPSDC). They deal with a location-allocation problem. As their model application is in emergency systems with mobile servers, it is very important that an arriving call be responded by a server. With respect to this constraint, the objective is minimizing the total number of servers. They consider the problem on an undirected network. Consider the following notation:

$x(j)$: the number of servers to be placed at site j,
$A_i(X)$: the availability for node i, and
α: $\alpha \in (0,1)$ is the minimum required availability for node i.

It is assumed that an allocation vector X is given and related queuing system represented by X is stable.

$$\text{Min } z = \sum_j x(j) \tag{51}$$

S.T.

$$A_i(X) \geqslant \alpha \quad \forall\ i \tag{52}$$

$$x(j) = 0, 1, \ldots \quad \forall\ j \tag{53}$$

The model tries to minimize the total number of located servers on a network while satisfying the needed availability α at all nodes.

2.1.9. Fuzzy SCP

In order to consider uncertainty in a classical SCP, Hwang, Chiang, and Liu (2004) propose *fuzzy set-covering model*. They try to find an optimal α-cover with respect to a given set N_i and the desired degree α, for each $j \in N_i$, the membership grade i is no less than the level degree α. This formulation can be reduced to a nonlinear integer programming problem that is solved using optimization software. The notation of the model is as follows:

I: the subset of integers ($I = \{1, \ldots, m\}$),
J: the subset of integers ($J = \{1, \ldots, n\}$),
c_j: the cost of locating facility at j,
α: the desired degree of fuzzy cover,
$\underset{\sim}{\mu_j}(i)$: the membership grade of $i \in I$,
$\widetilde{P}_j$: $\widetilde{P}_j = \{(i, \mu_j(i)) : i \in I\}$ is a fuzzy subset of i, and

x_j: $x_j = \begin{cases} 1 & \widetilde{P}_j \in \tilde{p}^* \\ 0 & \text{otherwise} \end{cases}$,

$$\text{Min } z = \sum_{j=1}^{n} c_j x_j \tag{54}$$

S.T.

$$1 - \prod_{j=1}^{n} (1 - \mu_j(i) x_j) \geqslant \alpha \quad \forall\ i \tag{55}$$

$$x_j \in \{0,1\} \quad \forall\ j \tag{56}$$

Chiang, Hwang, and Liu (2005) obtain a simplified model of Hwang et al. (2004) model. They reformulate the model as follows:

$$\text{Min } z = \sum_{j=1}^{n} c_j x_j \tag{57}$$

S.T.

$$\sum_{j=1}^{n} x_j [-\ln(1 - \mu_j(i))] \geqslant -\ln(1 - \alpha) \tag{58}$$

$$x_j \in \{0,1\} \quad \forall\ j \tag{59}$$

2.1.10. Multiple coverage SCP

Kolen and Tamir (1990) discuss *multiple* coverage problems. In this problem, each existing demand node must be served by a number of new facilities and this number depends on the type of new facilities. There is also an upper bound for the number of facilities that must be located at a given potential site.

2.1.11. Backup coverage SCPs

Erdemir et al. (2010) propose two models to locate aero medical and ground ambulance service which are based on SCP and MCLP. They define coverage as a combination of both response time and total service time and consider three type of it: (i) Ground emergency medical service coverage. (ii) Air emergency medical service coverage. (iii) Joint coverage of ground and air emergency medical service thorough transfer point. The node is covered by closet emergency medical service, if it is fulfilled the response time and service time limits. The proposed model covers both crash nodes and paths (nodes and links of network). Therefore the uncertainty in spatial distribution of demand nodes is addressed. Also by considering unavailability of ground emergency medical service, the backup coverage is modeled. Consider the following notation:

a: index of potential ground ambulance locations,
h: index of potential air ambulance locations,
r: index of potential transfer point locations,
j: index of crash nodes,
k: index of crash paths,
M_A: the set of potential ground ambulance locations,
M_H: the set of potential air ambulance locations,
M_R: the set of potential transfer point locations,
N: the set of crash nodes,
P: the set of crash paths,
x_a: a binary decision variable, it is 1 if a ground ambulance is located at a,
y_h: a binary decision variable, it is 1 if an air ambulance is located at h,
z_r: a binary decision variable, it is 1 if a transfer point is located at r,
u_j: a binary decision variable, it is 1 if node/path j is covered by at least one of located air ambulances, and if node/path j is covered by at least two ground ambulances and/or combinations,
v_{ja}: a binary decision variable, it is 1 if node/path j is covered by ground ambulance a,
l_{ahr}: a binary decision variable, it is 1 if ground ambulance, air ambulance and a transfer point are located at a, h and r respectively,
c_A: the cost of locating ground ambulance,
c_H: the cost of locating air ambulance,
c_R: the cost of locating transfer point,
$A_{aj}(A_{ak})$: a binary parameter, it is 1 if potential ground ambulance location a covers node j (path k),
$A_{hj}(A_{hk})$: a binary parameter, it is 1 if potential air ambulance location h covers node j (path k),
$A_{ahrj}(A_{ahrk})$: a binary parameter, it is 1 if potential ground a air h ambulance, and transfer point r covers node j (path k), and
ε: a very small number.

The Set Cover Backup Model (SCBM) is as follows:

$$\text{Min} \sum_{a\in M_A} c_A x_a + \sum_{h\in M_H} c_H y_h + \sum_{r\in M_R} c_R z_r - \sum_{j\in N\cup P} u_j \cdot \varepsilon \tag{60}$$

S.T.

$$\sum_{h\in M_H} A_{hj} y_h \geqslant u_j \quad \forall\, j \in N\cup P \tag{61}$$

$$A_{aj} x_a + \sum_{h\in M_H} \sum_{r\in M_R} A_{ahrj} l_{ahr} \geqslant v_{ja} \quad \forall\, j \in N\cup P,\ \forall\, a \in M_A \tag{62}$$

$$\sum_{a\in M_A} v_{ja} = 2\cdot(1-u_j) \quad \forall\, j\in N\cup P \tag{63}$$

$$x_a \geqslant l_{ahr} \quad \forall\, a\in M_A,\ h\in M_H,\ r\in M_R \tag{64}$$

$$y_h \geqslant l_{ahr} \quad \forall\, a\in M_A,\ h\in M_H,\ r\in M_R \tag{65}$$

$$z_r \geqslant l_{ahr} \quad \forall\, a\in M_A,\ h\in M_H,\ r\in M_R \tag{66}$$

$$x_a + y_h + z_r - l_{ahr} \leqslant 2 \quad \forall\, a\in M_A,\ h\in M_H,\ r\in M_R \tag{67}$$

$$x_a, y_h, z_r, l_{ahr}, u_j, v_{ja} \in \{0,1\} \quad \forall\, a\in M_A,\ h\in M_H,\ r\in M_R,\ j\in N\cup P \tag{68}$$

The objective function minimizes the total cost whereas prevents allotting two different ground ambulance to cover node/path j, if there is at least one air ambulance covering node j. Constraints (61)–(63) indicate that all crash nodes and paths are covered at least once by an air ambulance or twice by a ground ambulance or combination of ground and air ambulance. Constraints (64)–(67) ensure that all emergency medical service must be located to provide joint coverage of ground and air emergency medical service thorough transfer point.

2.1.12. Multi-Criteria SCP (MCSC)

Liu (1993) introduces multi-criteria SCP. In this problem, there are several predetermined attributes with separate covering matrix for each attribute. An existing facility will be covered if from either attribute point of view it is covered. Also, there is one objective function for each attribute in terms of cost. Liu, thus develops a greedy heuristic algorithm to solve this problem.

2.1.13. Covering games

Hoefer (2006) and Cardinal and Hoefer (2010) consider a non-cooperative games coming from combinatorial SCPs and investigate cost sharing between non-cooperative agents. There is a game for k players (each service facility that is potentially able to serve/ cover existing facilities, is a player) and each player wants to satisfy a subset of the constraints. Therefore, the strategy is chosen by each of them so that a subset of nodes on a network is covered whereas total contribution exceeds the cost. Hoefer (2006) presents a polynomial time algorithms to calculate Nash equilibrium for this problem. The notation of the model is as follows:

j: index of vertices on a network,
t: index of edges on a network,
V: the set of vertices on a network ($|V| = n$),
E: the set of edges on a network ($|E| = m$),
G: $G = (V,E)$,
x_j: the amount of bought units of resource j,
c_j: the cost of integral unit of resource j,
a_{tj}: The non-negative (rational) entry, and
b_t: The non-negative (rational) entry.

$$\text{Min} \sum_{j=1}^{n} c_j x_j \tag{69}$$

S.T.

$$\sum_{j=1}^{n} a_{tj} x_j \geqslant b_t \quad t = 1,\ldots,m \tag{70}$$

$$x_j \in \mathrm{N} \quad j = 1,\ldots,n \tag{71}$$

The objective function (54) minimizes the total cost of bought units. Constraint (70) ensures the necessary amount of bought units are provided.

2.1.14. Other variants of SCP

Daskin (1995) focuses on *variants of the set covering location model*in his book; whereby he includes secondary objectives that are important in facility location like: (i) one of the extensions tries to choose one of the optimal solutions that maximizes the number of demand nodes covered at least once (ex. twice); this can also be seen in Church and Murray (2009) in their book. (ii) In another extension, he considers that we are not facing a from-scratch location planning problem. In other words, there are several existing facilities and then we want to add new facilities for total coverage. Daskin (1995) explains a reformulation of the objective function by using maximum number of existing facilities trying to use maximum number of them.

Chen and Yuan (2010) study the coverage problem in cellular network. They formulate two models that minimize the required power while the service area is fully covered. Overlap constraint between adjacent cells is also considered which ensures handover between them. The notation of the first model is as follows:

i, h: indices of broadcast channels,
j, k: indices of test points,
$F'_i(j)$: index of jth test point position ($j \in J_i$) in the sorted sequence of cell i,
$F_i(j)$: the inverse of $F'_{i(j)}$,
I: the set of cells ($I = \{1,\ldots,n\}$),
J: the set of test points ($J = \{1,\ldots,m\}$),
J_i: the set of test points that can be potentially covered by cell i($m_i = |J_i|$),
I_j: the set of potentially covering cell of j($n_j = |I_j|$),
J_{ih}: the minimum amount to be covered by both cell i and h ($J_{ih} = J_i \cap J_h,\ m_{ih} = |J_{ih}|$),
D: the set of adjacent cells that overlap with each other,
F_i: a bijection such that sort P_{ij} in ascending order (F_i:$J_i \to \{1,\ldots,m_i\}$),
F'_i: a bijection such that sort P_{ij} in descending order (F_i:$J_i \to \{1,\ldots,m_i\}$),
x_{ij}: a binary decision variable, it is 1 if the power of cell i equals P_{ij},
z_j^{ih}: a binary decision variable, it is 1 if test point j is covered by both cell i and h,
d_{ih}: the minimum amount to be covered by both cell i and h,
P_{ij}: the minimum amount of power required to cover test point $j \in J$ by cell $i \in I$,
n: number of cells,
m: number of test points,
n_j: number of potentially covering cell o j, and
m_i: number of test points that can be potentially covered by cell i.

The first model is as follows:

$$\text{Min} \sum_{i \in I} \sum_{j \in J_i} P_{ij} x_{ij} \tag{72}$$

S.T.

$$\sum_{i \in I_j} \sum_{k=F'_i(j)}^{m_i} x_{i(k)} \geqslant 1 \quad j \in J \tag{73}$$

$$\sum_{j \in J_i} x_{ij} = 1 \quad i \in I \tag{74}$$

$$\sum_{k=F'_i(j)}^{m_i} x_{i(k)} + \sum_{k=F'_h(j)}^{m_h} x_{h(k)} = z_j^{ih} + 1 \ (i,h) \in D, \quad j \in J_{ih} : I_j = \{i,h\} \tag{75}$$

$$\sum_{k=F'_i(j)}^{m_i} x_{i(k)} + \sum_{k=F'_h(j)}^{m_h} x_{h(k)} \geqslant 2z_j^{ih} \ (i,h) \in D, \quad j \in J_{ih} : \{i,h\} \subset I_j \tag{76}$$

$$\sum_{j \in J_{ih}} z_j^{ih} \geqslant d_{ih} \ (i,h) \in D \tag{77}$$

$$x_{ij} \in \{0,1\} \quad i \in I,\ j \in J_i \tag{78}$$

$$z_j^{ih} \in \{0,1\}(i,h) \in D, \quad j \in J_{ih} \tag{79}$$

Constraints (73) and (74) ensure that all test points are covered by exactly one of broadcast channel. Constraints (75) and (76) indicate that $z_j^{ih} = 1$, if both cell i and h cover test point j. Constraint (77) ensures the required overlap.

The second model is more effective than the first one. Consider additional notation as follows:

l: index of coverage level ($l \in \{2,\ldots,m_i\}$),
$J_{ih}(i_{l-1}) : J_{ih}(i_{l-1}) = \{j \in J_{ih} | P_{ij} \leqslant P_{i(l-1)}\}$,
$L(i_{l-1},h)$: the minimum power of cell h that can meet the coverage and overlap constraints of test points in J_{ih}, provided that cell i uses level $l - 1$, and
$P_{i(l)}$: the lth level of cell i power in a sorted sequence.

Chen and Yuan (2010) prove that there is an equivalent for the above model which is as follows:

$$\text{Min} \sum_{i \in I} \sum_{j \in J_i} P_{ij} x_{ij} \tag{80}$$

S.T.

$$\sum_{k=1}^{l-1} x_{i(k)} \leqslant \sum_{k=L(i_{l-1},h)}^{m_h} x_{h(k)} \ (i,h) \in D, \quad l \in \{2,\ldots,n_i\} : F_i(l) \in J_{ih} \tag{81}$$

$$\sum_{i \in I_j} \sum_{k=F'_i(j)}^{m_i} x_{i(k)} \geqslant 1 \quad j \in J \tag{82}$$

$$z_j^{ih} \in \{0,1\} \ (i,h) \in D, j \in J_{ih} \tag{83}$$

2.2. The Maximal Covering Location Problem (MCLP)

In many practical applications, allocated resources (e.g. budget) are not sufficient to cover all of existing facilities (e.g. customers) with desired level of coverage. Therefore, Church and ReVelle (1974) develop maximal covering location model. This model maximizes the amount of demand covered within the acceptable service distance S by locating a given fixed number of new facilities.

In order to formulate this problem we use the following set of notations:

i: the index of demand nodes,
j: the index of facilities,
h_i: given number that shows the number of demands at node i (this is something like the number of population at node),
S: the time (or distance) standard within which a server is desired to be found,
P: the total required server to be located,
a_{ij}: A binary parameter is 1 if distance from candidate place j to the existing facility (customer) i is not greater than S,
x_j: binary variable indicating a facility is positioned at j or not, and
z_i: binary decision variable is 1 if node i is covered, otherwise 0.

Thus maximal covering location problem can be formulated as follows:

$$\text{Max} \sum_i h_i.z_i \tag{84}$$

S.T.

$$z_i \leqslant \sum_j a_{ij} \cdot x_j \quad \forall\ i \tag{85}$$

$$\sum_j x_j \leqslant P \tag{86}$$

$$z_i \in \{0,1\} \quad \forall\ i \tag{87}$$

$$x_j \in \{0,1\} \quad \forall\ j \tag{88}$$

Objective function (84) maximizes the covered demands. Constraint (85) explains relation between the coverage and location variables and states that demand node j is covered if at least one facility at one of the potential sites able to cover node j, is located. Constraint (86) limits the number of located facilities to P. Constraints (87) and (88) are integrality constraints.

White and Case (1974) consider the weights of all demand points as equal. Klastorin (1979) demonstrates how MCLP can be formulated as Generalized Assignment Problems (GAP). Therefore, they find maximum number of demand nodes that are covered. Daskin (1995) introduces the maximal coverage location model as one of the variants of set covering.

2.2.1. MCLP implicit and explicit

Murray et al. (2010) present MCLP-Implicit model as:

i: the index of demand area,
j: the index of facilities,
k: the index of coverage levels,
x_j: a binary decision variable indicating whether the facility located at point j or not,
Y_{ik}: a binary decision variable equal one if area i is covered at level k and otherwise 0,
z_i: a binary decision variable is 1 if node i is covered, otherwise 0,
β_k: the minimum required percentage of coverage at level k,
α_k: the minimum number of required facilities for covering completely at kth level,
Ω_{ik}: the set of potential facilities cover area i at least β_k,
h_i: given number that shows the number of demands at node i (this is something like the number of population at node), and
P: total required number of facilities to be located.

The model is as follows:

$$\text{Max} \sum_i h_i \cdot z_i \tag{89}$$

S.T.

$$\sum_{j \in \Omega_{ik}} x_j \geqslant \alpha_k Y_{ik} \quad \forall\, i,k \tag{90}$$

$$\sum_k Y_{ik} = z_i \quad \forall\, i \tag{91}$$

$$\sum_j x_j = P \tag{92}$$

$$z_i \in \{0,1\} \quad \forall\, i \tag{93}$$

$$x_j \in \{0,1\} \quad \forall\, j \tag{94}$$

$$Y_{ik} \in \{0,1\} \quad \forall\, i,k \tag{95}$$

Objective function (89) maximizes the total demand covered. Constraint (90) expresses the relation between facility location variable and coverage demand. Constraint (91) ensures provided coverage at level k. Constraint (92) states the total number of facilities. Constraints (93)–(95) are the integrality constraints.

Considering above and the following notation, the MCLP-Explicit model is as follows:

Ψ'_{ik}: the set of k facility configurations partially covers area i,
Δ'_{ikl}: a set of k facilities in lth configuration which covers area i partially,
c_{ikl}: a fraction of ith demand covered by k facilities in l configuration and
Z_{ikl}: a binary decision variable, it equals to 1 if area i is covered by configuration l at level k and otherwise 0.

The model is as follows:

$$\text{Max} \sum_i \sum_k \sum_{l \in \Psi'_{ik}} h_i c_{ikl} Z_{ikl} \tag{96}$$

S.T.

$$\sum_k \sum_{l \in \Psi'_{ik}} Z_{ikl} \leqslant 1 \tag{97}$$

$$x_j \geqslant Z_{ikl} \quad \forall\, i,k,l \in \Psi'_{ik},\ \ j \in \Delta'_{ikl} \tag{98}$$

$$\sum_j x_j = P \tag{99}$$

$$Z_{ikl} = \{0,1\} \quad \forall\, i,k,l \in \psi'_{ik} \tag{100}$$

$$x_j \in \{0,1\} \quad \forall\, j \tag{101}$$

Objective function (96) maximizes the total demand covered. Constraint (97) stipulates that at most one-level configuration combination can account for the coverage of demand area i. Constraint (98) limits coverage to that provide by sited facilities. Constraint (99) states the total number of facilities. Constraints (100) and (101) are the integrality constraints.

2.2.2. Planar maximal covering

Church (1984) introduces MCLP on the plane; in other words, the potential sites for locating the new facilities are not on the network or discrete (and finite). He develops the planar maximal covering location problem under (i) Euclidean distance measure (PMCE) and (ii) rectilinear distance measure (PMCR).

2.2.3. Capacitated MCLP

Like it was said for capacitated SCP, Current and Storbeck (1988) also apply the same facility capacity restrictions to the MCLP problem and present a *capacitated version of MCLP* formulation.

2.2.4. MCLP with a criticality index analysis metric

Oztekin, Pajouh, Delen, and Swim (2010) present RFID network design methodology for asset tracking in healthcare. In their model the reachable readers for completely coverage is less than the required number, so the goal is to find the best location for available reader so that maximize system efficiency.

Consider the following notation:

i: the index of demand square, $i = 1,\ldots,n$,
j: the index of reader nods, $j = 1,\ldots,m$,
k: the index of assets type, $k = 1,\ldots,l$,
z_i: a binary variable is 1 if demand square i is covered by at least one reader and 0 otherwise,
x_j: a binary decision variable is 1 if a reader is located at reader node j, otherwise 0,
N_i: the set of reader nodes (j) that can cover demand square i. $N_i = \{j/l_{ij} \leqslant s\}$,
P: the fixed number of readers,
s: read range of the reader,
l_{ij}: the distance between these reader nodes and demand square i,
w_1 and w_2: the weight of two part of the objective function,
s_k: the importance level of asset k in emergency cases evaluated by experts based on a five-point Likert scale,
f_{ki}: number of times asset k passes through demand square i in a day,
$(dt)_{ki}$: average of time that asset k spends in demand square i in a day, and
c_i: the criticality index of demand square$i$ calculated as:

$$c_i = \sum_{k=1}^{l} f_{ki} * (dt)_{ki} * s_k$$

The model is as follows:

$$\text{Max } w_1\left(\sum_i c_i.z_i\right) - w_2\left(\sum_{i=1}^{n}\left(\sum_{j\in N_i} x_j\right) - z_i\right) \tag{102}$$

S.T.

$$z_i \leqslant \sum_{j\in N_i} x_j \leqslant z_i \cdot m \quad \forall\ i \tag{103}$$

$$\sum_j x_j = P \tag{104}$$

$$z_i \in \{0,1\} \quad \forall\ i \tag{105}$$

$$x_j \in \{0,1\} \quad \forall\ j \tag{106}$$

Objective function (102) maximizes the total covered criticality indices of demand squares and also minimizes the reader collision. Constraint (103) assures coverage of demand square while having readers on reader nodes. Constraint (104) expresses total number of reader nodes. Constraints (105) and (106) are the integrality constraints.

2.2.5. MCLP with mandatory closeness constraints

Love et al. (1988) in their book chapter introduce *maximal covering with mandatory closeness constraints*. In this problem, knowing the optimal number of the new facilities needed for total coverage, say P, and provided that there are multiple optimal solutions for the problem, the decision maker ignores the average closeness of response to their nearest service center; but seeks location for the P facilities to maximize population covered within a desired time or distance. They do this using a binary variable to decide assignment demand point to a facility and adding a set of constraints.

2.2.6. Probabilistic MCLP

ReVelle and Hogan (1989a) present a probabilistic version of MCLP and call it *Maximum Available Location Problem* (MALP). They try to locate P facilities so that with the probability α maximize the covered population that can find an available server. They introduce MALP I that utilizes the local busy fraction and MALP II that uses a system wide busy probability.

2.2.7. MALP I

In this model, consider the following notation:

i: the index of demand area,
j: the index of facility sites,
S: maximum distance or time that a demand point can be from its nearest facility,
d_{ij}: the shortest time (or distance) from site j to demand i,
N_i: $N_i = \{j/d_{ij} \leqslant S\}$,
x_j: a binary decision variable is 1 if a facility is located at node j, otherwise 0,
z_{ik}: a binary variable indicates whether the demand area i has at least k servers within S or not,
P: total number of facilities,
h_i: the population at node i,
$\bar{d}$: the average duration of a call,
b: defined as $b = [\log(1-\alpha)/\log q]$ where α is the reliability *and* $q = \bar{d}\cdot\sum_{i\in I} h_i/24\sum_{j\in J} x_j$.

The model is as follows considering all variables are 0, 1:

$$\text{Max } z = \sum_i h_i z_{ib} \tag{107}$$

S.T.

$$\sum_{k=1}^{b} z_{ik} \leqslant \sum_{j\in N_i} x_j \quad \forall\ i \tag{108}$$

$$z_{ik} \leqslant z_{ik-1} \quad \forall\ i, k = 2,\ldots,b \tag{109}$$

$$\sum_j x_j = P \tag{110}$$

Objective function (107) maximizes the population covered with reliability α, while constraint (108) determines the required connection between variables and constraint (109) states credit higher levels of coverage in the suitable order. Constraint (110) expresses the total number of facilities.

2.2.8. MALP II

In MALP II, maximizing the population of demand areas having b_i servers within S or maximizing the population with service available with alpha reliability is considered. The presented model for MALP II is the same as MALP I by changing b into b_i defined as the smallest integer satisfying $1 - (h_i/b_i)^{b_i} \geqslant \alpha$ for each i.

Marianov and ReVelle (1994) consider dependency between server availabilities and extend PLSCP by using queuing theory. They call their model queuing probabilistic location set covering problem (Q-PLSCP) and their solution technique, Maximum Availability Sitting Heuristic (MASH). The presented model is the same as PLSCP discussed by ReVelle and Hogan (1989b) with the difference in definition of variable b_i as the smallest integer that satisfies $\left(\frac{1}{b_i!}\rho_i^{b_i}\right)\Big/\left(1+\rho_i+\frac{1}{2!}\rho_i^2+\ldots+\frac{1}{b_i!}\rho_i^{b_i}\right) \leqslant 1-\alpha$; ρ_i is the utilization ratio of queuing theory.

Drezner (1995) includes two book chapters regarding covering problems: (i) Marianov and ReVelle (1995) present models for sitting emergency services. They use in their book chapter the simple deterministic covering model, the maximum population coverage version, two kinds of models developed to address congestion (deterministic redundant coverage and probabilistic model); (ii) Ghosh, McLafferty, and Carig (1995) apply covering models for *service center location* in their book chapter. They present two case studies that can also be considered as the applications of location-allocation models for planning a network of service centers.

2.2.9. Maximum covering location-interdiction problem

O'Hanley and Church (2011) present a facility location-interdiction model maximizing initial coverage and minimizing coverage level following the worst-case interdiction pattern. Consider the following notation:

i: the index of demand nodes,
j: the index of all potential facility locations,
k: the index of p numbered facilities locations, $k = 1,\ldots,p$,
ω: the index of feasible interdiction template,
N_i: the set of sites covering demand node i,
Ω: the set of all possible interdiction templates,
G_ω: the set of facility indices k including interdiction template ω,
$\theta > 0$: weight given before interdiction to the demand-weighted coverage,
z': demand-weighted coverage level when the worst-case interdiction pattern occurs,
h_i: amount of demand at node i,
y_i: a binary variable shows if demand i covered before interdiction occurrence (=1) or not (=0),
x_{jk}: a binary variable shows the kth facility locate in jth location or not, and
$y'_{i\omega}$: a binary variable determines whether node i is covered when interdiction pattern ω occurs or not.

The MIP formulation of that model is as follows:

$$\text{Max } z = \theta \sum_{i \in I} h_i y_i + z' \tag{111}$$

S.T.

$$\sum_j x_{jk} = 1 \quad \forall\, k \in K \tag{112}$$

$$\sum_{k \in K} x_{jk} \leqslant 1 \quad \forall\, j \tag{113}$$

$$\sum_{k \in K} \sum_{j \in N_i} x_{jk} \geqslant y_i \quad \forall\, i \tag{114}$$

$$\sum_{k \in K \setminus G_\omega} \sum_{j \in N_i} x_{jk} \geqslant y'_{i\omega} \quad \forall\, \omega \in \Omega, i \tag{115}$$

$$\sum_i h_i y'_{i\omega} \geqslant z' \quad \forall\, \omega \in \Omega \tag{116}$$

$$x_{jk} \in \{0,1\} \quad \forall\, j,k; \quad y_i \leqslant 1 \quad \forall\, i; \quad y'_{i\omega} \leqslant 1 \quad \forall\, i,\omega \tag{117}$$

Objective function (111) maximizes covered demand before and after interdiction, constraint (112) expresses that each facility k should choose one location, constraint (113) ensures that it is not possible to have two facilities in one location, constraint (114) states coverage of node i with open facility before occurring interdiction, constraint (115) states coverage of node i with open non interdicted facility and constraint (116) specifies the minimum amount of demand-weighted coverage level. Constraint (117) express the integrality variable x_{jk}, while binding the pre-interdiction and postinterdiction coverage variables y_i and $y'_{i\omega}$ to be less than or equal to one.

The bi-level MIP formulation of above model is as follows:

r: the number of interdictions,
x_j: a binary decision variable is 1 if a facility is located at node j, otherwise 0,
y'_i: a binary variable shows if demand i is covered before interdiction occurrence (=1) or not (=0) and
u'_j: a binary variable shows if facility j is interdicted (=1) or not (=0).

The mathematical model is:

$$\text{Max } z = \theta \sum_{i \in I} h_i y_i + z' \tag{118}$$

S.T.

$$\sum_{j \in N_i} x_j \leqslant y_i \quad \forall\, i \tag{119}$$

$$\text{Min } z' = \sum_i h_i y'_i \tag{120}$$

$$\sum_j u'_j = r \tag{121}$$

$$y'_i + u'_j \geqslant x_j \quad \forall\, i, \ \forall\, j \in N_i \tag{122}$$

$$\sum_j x_j = P \tag{123}$$

$$u'_j \in \{0,1\} \quad \forall\, j \quad y'_i \geqslant 0 \quad \forall\, i \tag{124}$$

$$y_i \leqslant 1 \quad \forall\, i \quad x_j \in \{0,1\} \quad \forall\, j \tag{125}$$

Constraint (119) states the coverage of node i with open facility before occurring interdiction, while the objective function (120) minimizes the demand-weighted coverage after interdiction, constraint (121) limits the number of interdictions. Constraint (122) states that each facility covering customer i should be interdicted. Constraint (123) expresses the total number of facilities. Constraints (124) are the integrality constraint while binding the post-interdiction coverage variables to be greater than or equal to zero. Constraints (125) also are the integrality constraint and limit the pre-interdiction coverage variables to be less than or equal to one.

2.2.10. Median Tour Problem (MTP) and Maximal Covering Tour Problem (MCTP)

Current and Schilling (1994) introduce *Median Tour Problem* (MTP) and *Maximal Covering Tour Problem* (MCTP) which are bicriterion routing problems. These problems have applications in mobile service delivery systems (e.g. healthcare systems, overnight mail delivery) and distributed computer networks. In the problems, there is a tour that must visit only P of the n nodes on the network. One of the objectives minimizes of total tour length and the other maximizes access to the tour for the nodes not directly on it.

We explain MCTP which looks for a tour among n nodes; the tour must visit exactly P of the n nodes. Given a directed graph $G = (V,E)$:

$|S'|$: the cardinality of set S' (any subset of V),
Q: node set in S used for sub-tour elimination constrains
x_{ij}: the binary variable indicates whether the arc from node i to node j is on the tour or not,
z_i: the binary variable; it is 0 if the demand node i is covered by a stop and 1 otherwise,
h_i: demand at node i,
s: maximal covering distance
N_i: set of nodes within covering radius of node i, $N_i = \{j | d_{ij} \leqslant s\}$
c_{ij}: the cost of including the path connecting nodes i and j on the tour and
P: the number of stops on the tour.

Considering $z'_T = \sum_i \sum_j c_{ij} x_{ij}$, $z'_c = \sum_i h_i z_i \cdot i$, the mathematical model is as follows:

$$\text{Min } z = (z'_T, z'_c) \tag{126}$$

S.T.

$$\sum_i x_{ij} - \sum_k x_{jk} = 0 \quad \forall\, j \tag{127}$$

$$\sum_j \sum_i x_{ij} = P \tag{128}$$

$$\sum_{i \in Q} \sum_{j \in Q} x_{ij} \leqslant |S'| - 1 \quad S' \subset V, \quad 2 \leqslant |S'| < P \tag{129}$$

$$x_{ij} \in \{0,1\} \quad (i,j) \in E \tag{130}$$

$$\sum_l \sum_{j \in N_i} x_{lj} + z_i \geqslant 1 \quad \forall\, i \tag{131}$$

$$z_i \geqslant 0 \quad \forall\, i \tag{132}$$

Objective function (126) minimizes two objective functions including the total length of the tour and also the total demand weighted travel distance which the demand at the nodes encounters reaching their nearest stop on the tour. Constraint sets (127)–(130) show that the solution contains a single tour (constraint (129) is used for sub-tour elimination) with p stops and constraint sets (131) and (132) ensure that $z_i = 1$, for all nodes i which are not covered by the tour.

2.2.11. Partial coverage problem

One of the key assumptions of the MCLP is that coverage is binary; it means customer is either fully covered if there is a facility within distance S from the located facility; otherwise it is not covered at all. In some of the real-world applications, such binary assumptions can be released; this can lead to partial covering problem. In partial covering, the coverage level provided by the facility acts as a decreasing function of the distance from the facility to the customer's location. Therefore, we will have some fully covered customers (those are in S covering radius) and the others are partially covered customers (Berman & Krass, 2002). This model has applications in supermarkets like super-drugstore chains (Jones & Simmons, 1993).

2.2.12. Generalized MCLP (GMCLP)

Berman and Krass (2002) present *Generalized Maximal Cover Location Problem* (GMCLP). In this problem, partial coverage of customers is modeled where the level of coverage is a non-increasing step function of the distance to the nearest facility. Application of this model is in locating retail facilities.

Considering network $G = (N,E)$ and following notations:

N'': a set of potential locations,
S': the given set of facility locations,
h_i: a positive weight assigned to each node $i \in N$ of network,
$d(j,i)$: shortest distance between any two points $i, j \in G$,
k: the coverage radii such as $S_i^0 = 0 < S_i^1 < \cdots < S_i^k = \propto$ with the related coverage level as $r_i^1 = 1 > r_i^2 > \cdots > r_i^k \geqslant 0$,
$d(S',i) : d(S',i) = \min_{j \in S'} d(j,i)$
$N(S',l) : N(S',l) = \left\{ i \in N/S_i^{l-1} \leqslant d(S',i) \leqslant S_i^l \right\}$

For a given set of facility location S' and level $l \leqslant k$, the GMCLP is written as:

$$\underset{S' \subset N'', |S'| = m}{\text{Max}} z(S') = \sum_{l=1}^{k} \sum_{i \in N(S',l)} h_i r_i^l \tag{133}$$

They develop several integer programming formulations for this problem and also show that this problem is equivalent to the uncapacitated Facility Location Problem (UFLP).

Alexandris and Giannikos (2010) propose an integer programming model for partial coverage. By applying Geographic Information System (GIS), demand points are considered as a spatial objects rather than single points. Empirical results indicate that the proposed model present larger proportion of coverage than the traditional ones. The notation of the model is as follows:

i: index of demand areas,
j: index of candidate facility locations,
I: the set of demand areas,
J: the set of all candidate locations,
N_i: the set of locations that can cover demand area A_i,
W_i: the set of candidate locations j that partially cover demand area A_i at least b times but less than 100%,
x_j: a binary decision variable, it is 1 if a server is located in location j,
z_i: a binary decision variable, it is 1 if demand area A_i is covered by at least one server,
v_i: a binary decision variable, it is 1 if demand area A_i is partially covered at least θ times,
b: minimum acceptable coverage percent ($b \in [0,100]$),
θ: minimum number of partial coverage facilities needed for complete coverage,
a_{ij}: a binary decision parameter, it is 1 if server located at j fully covered demand area A_i,
D_i: maximum acceptable distance for demand area A_i,
h_i: the benefit of fully covering demand area A_i,
α: the proportion of fully coverage benefit, and
p: number of servers.

The model is as follows:

$$\text{Max} \sum_{i \in I} h_i z_i \tag{134}$$

S.T.

$$\sum_{j \in J} a_{ij} x_j \geqslant z_i - v_i \quad \forall\ i \in I \tag{135}$$

$$\sum_{j \in J} x_j = p \tag{136}$$

$$\sum_{j \in Wi} x_j \geqslant \theta \cdot v_i \quad \forall\ i \in I \tag{137}$$

$$x_j, z_i, v_i \in \{0,1\} \quad \forall\ i \in I,\ j \in J \tag{138}$$

Objective function (134) maximizes the coverage of demand areas regardless of whether the areas are fully or partially covered. Constraint (135) indicates that if demand area A_i is not partially covered, it is necessary to locate a fully covering facility. Constraint (136) determines the number of available servers. Constraint (137) ensures that if $v_i = 1$, then demand area A_i is partially covered by at least θ facilities.

In cases that there is difference between partial and fully coverage, the following model is considered:

$$\text{Max} \sum_{i \in I} h_i (z_i + \alpha v_i) \tag{139}$$

S.T.

$$\sum_{j \in J} a_{ij} x_j \geqslant z_i \quad \forall\ i \in I \tag{140}$$

$$\sum_{j \in J} x_j = p \tag{141}$$

$$\sum_{j \in Wi} x_j \geqslant \theta \cdot v_i \quad \forall\ i \in I \tag{142}$$

$$z_i + v_i \leqslant 1 \quad \forall\ i \in I \tag{143}$$

$$x_j, z_i, v_i \in \{0,1\} \quad \forall\ i,j \tag{144}$$

Constraint (140) indicates that if $z_i = 1$, then demand area A_i is covered by at least one server. Constraint (143) expresses that demand area may be fully or partially covered, but not both.

2.2.13. Gradual coverage

Berman, Krass, and Drezner (2003) extend GMCLP named *gradual coverage decay model*. They consider a generalization of MCLP with two coverage radii S^1 and S^2 ($S^1 < S^2$). If a demand point can be covered with its closest facility in less than distance S^1, it will be "fully covered", if distance is between S^1 and S^2, the demand point will be partially covered. Finally, if the distance is more than S^2, the demand point will never be covered. They consider their model on a network. Consider the following notation:

h_i: demand weight associated with node i,
S': the set of facilities located in network G,
P: the total number of facilities,
$d_i(j)$: the shortest distance between node i and j,
$d_i(S') : d_i(S') = \min_{j \in S'} d_i(j)$,
$f_i(d_i(S'))$: the coverage decay function, and
$c_i(d_i(S')) = \left\{ \begin{array}{cc} h_i & d_i(S') \leqslant S^1 \\ h_i f_i(d_i(S')) & S^1 < d_i(S') \leqslant S^2 \\ 0 & d_i(S') > S^2 \end{array} \right\}$: the covered demand weight of node i.

The mathematical model is as follows:

$$\underset{S' \subset G, |S'| = P}{\text{Max}} z = \sum_i c_i(d_i(S')) \tag{145}$$

Objective function (145) maximizes the total covered weighted demand by locating P facilities.

Drezner, Wesolowsky, and Drezner (2004) solve the *gradual coverage problem* on plane. They consider a minimum distance within which all the demand points are covered with negligible cost and; a maximum distance covered with constant cost. Between these minimum and maximum distances, there is a linear cost based on the distance (d). They show that this formulation can be converted into the Weber problem by imposing a special structure on the cost function. The presented model minimizes the Berman et al. (2003)'s objective function in which S_i^1, S_i^2 are the minimum and maximum distance associated with demand point i and $c_i(d) = \begin{Bmatrix} 0 & d \leqslant S_i^1 \\ h_i(d - S_i^1) & S_i^1 < d \leqslant S_i^2 \\ h_i(S_i^2 - S_i^1) & d > S_i^2 \end{Bmatrix}$. Drezner, Drezner, and Goldstein (2010) present stochastic gradual cover model considering discontinuity cover in the maximal cover model. In their model, the minimum and maximum distances for gradual cover model are random variables. The notation of the model is as follows:

i: index of demand points,
$\Phi_{S^1}(d)$: the density function of the probability that S^1 is at distance d,
$\Phi_{S^2}(d)$: the density function of the probability that S^2 is at distance d,
$d_i(X)$: the distance between demand point i and the facility,
h_i: the demand weight associated with node I, and
$c(d)$: the expected coverage at distance $d\left(c(d) = pr(S^1 \geqslant d) + \int_0^d \int_d^\infty \frac{u-d}{u-v} \Phi_{S^1}(v)\Phi_{S^2}(u)du\,dv\right)$

The model is as follows:

$$\operatorname*{Max}_X \left\{ z(X) = \sum_{i=1}^n h_i c[d_i(X)] \right\} \tag{146}$$

Karasakal and Karasakal (2004) also develop a formulation for a similar problem and call it *MCLP in presence of partial coverage*. However, we believe that their problem is a gradual coverage problem. Consider the following notation:

M_i: the set of facility locations that fully or partially covers the demand node i,
z_{ij}: a binary variable shows the demand at node i is fully or partially covered by a facility at j or not,
x_j: a binary variable equal 1 if facility located at j, otherwise 0,
P: the total number of facilities,
S and T: the maximum full and partial coverage distance respectively ($T > S$),
d_{ij}: the distance between the facility j and the demand node i,
$f(d_{ij})$: partial coverage function, and
c_{ij}: the level of coverage provided by the facility j to the demand node i.

$$c_{ij} = \begin{cases} 1 & \text{if } d_{ij} \leqslant S \\ f(d_{ij}) & \text{if } S < d_{ij} \leqslant T, \quad (0 < f(d_{ij}) < 1) \\ 0 & \text{otherwise} \end{cases}$$

The model is as follows:

$$\text{Max} \sum_i \sum_{j \in M_i} c_{ij} z_{ij} \tag{147}$$

S.T.

$$z_{ij} \leqslant x_j \quad \forall\, i, \ \forall\, j \in M_i \tag{148}$$

$$\sum_{j \in M_i} z_{ij} \leqslant 1 \quad \forall\, i \tag{149}$$

$$\sum_j x_j = P \tag{150}$$

$$z_{ij} \in \{0,1\} \quad \forall\, i, \ \forall\, j \in M_i \tag{151}$$

$$x_j \in \{0,1\} \quad \forall\, j \tag{152}$$

Objective function (147) maximizes the coverage level. Constraint (148) states that if there is no facility in$j$, all related z_{ij}'s will be forced to become 0. Constraint (149) expresses that demand node will be covered by at most one facility. Constraint (150) imposes the total number of facilities. Constraints (151) and (152) are integrality constraints.

Berman and Wang (2011) develop a gradual covering location model in which weights of demand nodes on a network are random variables following an unknown distribution. Given interval of all possible weights of nodes, the objective function minimizes maximum uncovered demand. Consider the following notation:

i, j: indices of nodes on a network,
V: the set of nodes on a network,
E: the set of links on a network,
G: $G = (V, E)$,
x, y: the location of facilities,
d_{ij}: the length of link $(i,j) \in E$,
$d_i(x)$: the shortest distance between node i and $x \in G$,
$C_i(x)$: a portion of demand weight at node i that is covered by the facility located at x,

$$C_i(x) = \begin{cases} 1 & d_i(x) \leqslant \widehat{S}_i \\ \frac{\widetilde{S}_i - d_i(x)}{\widetilde{S}_i - \widehat{S}_i} & \widehat{S}_i < d_i(x) < \widetilde{S}_i \\ 0 & d_i(x) \geqslant \widetilde{S}_i \end{cases}$$

h_i^-: given lower bound of demand weight at node i,
h_i^+: given upper bound of demand weight at node i,
h_i: the demand weight at node i $\left(h_i \in [h_i^-, h_i^+]\right)$,
R: the Cartesian product of interval $[h_i^-, h_i^+]$,
H: $H = \{h_i | i \in N\} \in R$ is a weight scenario and
$F(H,x) : F(H,x) = \sum_{i \in N} C_i(x) h_i$ is a total weight covered by the facility.

Berman and Wang (2011) model the problem as follows; it is minimization of a nonlinear objective function without any constraint:

$$\operatorname*{Min}_{x \in G}\{\operatorname*{Max}_{y \in G} \operatorname*{Max}_{H \in R}\{F(H,y) - F(H,x)\}\} \tag{153}$$

They show under some conditions, the problem is equivalent traditional location problems like to the minmax regret median problem.

2.2.14. Backup Coverage Location Problem (BCLP)

Hogan and ReVelle (1986) define *backup* coverage as a case in which an extra facility can cover a demand node so that the backup coverage of the node will equal the demand at that node. This concept is applicable for both SCP and MCLP.

Pirkul and Schilling (1989) present a model formulation where each new facility is capacitated and primary and backup services are provided to each demand node. When a demand arrives at the system it would not be covered when all the facilities that are capable of covering the demand, are engaged in serving other demands. They develop a solution procedure to solve this MCLP problem using Lagrangian relaxation.

Church and Murray (2009) in their book also introduce advance topics in covering models including backup coverage.

Curtin, Hayslett-McCall, and Qiu (2010) propose a method for integrating geographic information system with linear programming optimization in order to present alternative optimal solutions. Also they develop a backup coverage model in order to locate police patrol. In this model each demand node can be covered by any number of facilities. Considering the following notations:

i: the index of known incident locations,
j: the index of potential locations for police patrol command centers,
I: the set of known incident locations,
J: the set of potential locations for police patrol command centers,
N_i: $N_i = \{j \in J | d_{ij} \leqslant S\}$,
x_j: a binary decision variable, it is 1 if police patrol is located at location j,
z_i: a binary decision variable, it is 1 if an incident location at i is covered by at least one located police patrol area,
h_i: the weight or priority of crime incident at incident location i, and
p: the number of police patrol areas to be located.

The Police Patrol Area Covering (PPAC) model is as follows:

$$\text{Max} \sum_i h_i.z_i \tag{154}$$

S.T.

$$\sum_{j \in N_i} x_j \geqslant z_i \quad \forall\, i \in I \tag{155}$$

$$\sum_{j \in J} x_j = p \tag{156}$$

$$x_j \in \{0,1\} \quad \forall\, j \in J \tag{157}$$

$$z_i \in \{0,1,\ldots,p-1,p\} \quad \forall\, i \in I \tag{158}$$

As mentioned before, Erdemir et al. (2010) propose model which imposes limits on budget. The model maximizes a weighted combination of first coverage. Consider additional notation as follows:

a: index of potential ground ambulance locations,
h: index of potential air ambulance locations,
r: index of potential transfer point locations,
j: index of crash nodes,
k: index of crash paths,
M_A: the set of potential ground ambulance locations,
M_H: the set of potential air ambulance locations,
M_R: the set of potential transfer point locations,
N: the set of crash nodes,
P: the set of crash paths,
x_a: a binary decision variable, it is 1 if a ground ambulance is located at a,
y_h: a binary decision variable, it is 1 if an air ambulance is located at h,
z_r: a binary decision variable, it is 1 if a transfer point is located at r,
u_j: a binary decision variable, it is 1 if node/path j is covered by at least one of located air ambulances, and if node/path j is covered by at least two ground ambulances and/or combinations,
v_{ja}: a binary decision variable, it is 1 if node/path j is covered by ground ambulance a,
l_{ahr}: a binary decision variable, it is 1 if ground ambulance, air ambulance and a transfer point are located at a, h and r respectively,
f_j: a binary decision variable, it is 1 if node/path j is covered at least once,
b_j: a binary decision variable, it is 1 if backup coverage is assigned to node/path j,
g_j: a binary decision variable, it is 1 if backup coverage is not needed for node/ path j by locating at least one air ambulance that covers $j (g_j = u_j b_j \in \{0,1\})$,
c_A: the cost of locating ground ambulance,
c_H: the cost of locating air ambulance,
c_R: the cost of locating transfer point,
B: the maximum budget of locating emergency medical services;
d_j: the weight of node/ path j, and
θ: weight of first coverage $(0 < \theta < 1)$.
$A_{aj}(A_{ak})$: a binary parameter, it is 1 if potential ground ambulance location a covers node j (path k),
$A_{hj}(A_{hk})$: a binary parameter, it is 1 if potential air ambulance location h covers node j (path k),
$A_{ahrj}(A_{ahrk})$: a binary parameter, it is 1 if potential ground a air h ambulance, and transfer point r covers node j (path k), and
ε: a very small number.

The maximal cover for a given budget model is as follows:

$$\text{Max} \left(\theta \sum_{j \in N \cup P} d_j f_j + (1-\theta) \sum_{j \in N \cup P} d_j b_j \right) + \varepsilon \cdot \sum_{j \in N \cup P} u_j \tag{159}$$

S.T.

$$\sum_{a \in M_A} c_A x_a + \sum_{h \in M_H} c_H y_h + \sum_{r \in \{M_R - N_I\}} c_R z_r \leqslant B \tag{160}$$

$$\sum_{a \in M_A} A_{aj} x_a + \sum_{h \in M_H} A_{hj} y_h + \sum_{a \in M_A} \sum_{h \in M_H} \sum_{r \in M_R} A_{ahrj} l_{ahr} \geqslant f_j \quad \forall\, j \in N \cup P \tag{161}$$

$$\sum_{h \in M_H} A_{hj} y_h \geqslant g_j \quad \forall\, j \in N \cup P \tag{162}$$

$$A_{aj} x_a + \sum_{h \in M_H} \sum_{r \in M_R} A_{ahrj} l_{ahr} \geqslant v_{ja} \forall\, j \in N \cup P \tag{163}$$

$$\sum_{a \in M_A} v_{ja} = 2b_j - 2g_j \quad \forall\, j \in N \cup P \tag{164}$$

$$x_a \geqslant l_{ahr} \quad \forall\, a \in M_A,\ h \in M_H,\ r \in M_R \tag{165}$$

$$y_h \geqslant l_{ahr} \quad \forall\, a \in M_A,\ h \in M_H,\ r \in M_R \tag{166}$$

$$z_r \geqslant l_{ahr} \quad \forall\, a \in M_A,\ h \in M_H,\ r \in M_R \tag{167}$$

$$x_a + y_h + z_r - l_{ahr} \leqslant 2 \quad \forall\, a \in M_A,\ h \in M_H,\ r \in M_R \tag{168}$$

$$u_j \geqslant g_j \quad \forall\, j \in N \cup P \tag{169}$$

$$b_j \geqslant g_j \quad \forall\, j \in N \cup P \tag{170}$$

$$u_j + b_j - g_j \leqslant 1 \quad \forall\, j \in N \cup P \tag{171}$$

$$f_j, x_a, y_h, z_r, l_{ahr}, u_j, v_{ja}, b_j, g_j \in \{0,1\} \quad \forall\, a \in M_A, h \in M_H, r \in M_R,\ j \in N \cup P \tag{172}$$

Constraint (160) ensures that budget does not exceed B. Constraint (161) indicates that demand node is fulfilled if it is covered by at least one ground, air or joint ground and air emergency services. Constraints (162)–(164) satisfy backup coverage. Constraints (165)–(168) ensure that all emergency medical service must be located to provide joint coverage of ground and air emergency medical service thorough transfer point.

2.2.15. p-Maximal cover problem

Berman (1994) shows relationship between p-maximal cover problem and p-partial center problem on network. The p-maximal cover problem finds a set of locations for p new facilities to maximize total covered demand; this demand must be at most S units far from the closest facility. The p-partial center problem attempts to find a set of locations for P facilities to minimize maximum distance between the closest facility and the covered demand. Berman (1994) solves a bi-objective problem with these two objective functions. He presents an algorithm to obtain all Pareto locations on a tree network with one facility. He also shows that general networks' optimal solution will be a Pareto location.

2.2.16. Quadratic MCLP

Most of the available literature on covering problems focus on the demands that are produced on the nodes of a network. Erdemir et al. (2008) assume that demands can be originated from both nodes and paths. They present two kinds of formulations to consider this issue implicitly and explicitly. Consider the following notation:

i: the index of demand nodes,
j: the index of facility locations,
k: the index of demand paths,
x_j: a binary variable equal 1 if facility located at j, otherwise 0,
z_i: a binary variable equal 1 if node i is covered, otherwise 0,
l_k: a binary variable indicating path k is covered or not,
$y_{j_1j_2} = x_{j_1} \cdot x_{j_2}$: a binary variable defining whether the facilities are located at both j_1 and j_2 or not,
a_{ij}:is equal 1 if facility j covers point i, otherwise 0,
$a_{j_1j_2k}$: is equal 1 if facility j_1 and j_2 cover path k, otherwise 0,
P: the total number of facilities,
w_t: the assigned weight to time slot t,
h_{it}: the amount of demand at point i ($i \in N$) at time t, and
h_{kt}: the mean demand at path k ($k \in K$) at time t.

The nonlinear mathematical formulation is as follows:

$$\text{Max} \sum_i \sum_t w_t h_{it} z_i + \sum_{k \in K} \sum_t w_t h_{kt} l_k \tag{173}$$

S.T.

$$\sum_j a_{ij} x_j \geqslant z_i \quad \forall\ i \tag{174}$$

$$\sum a_{j_1j_2k} x_{j_1} x_{j_2} \geqslant l_k \quad \forall\ k \in K \tag{175}$$

$$\sum_j x_j \leqslant P \tag{176}$$

$$l_k \in \{0,1\} \quad \forall\ k \in K \tag{177}$$

$$x_j \in \{0,1\} \quad \forall\ j \tag{178}$$

$$z_i \in \{0,1\} \quad \forall\ i \tag{179}$$

Objective function (173) maximizes the total nodal and path demand coverage. Constraints (174) and (175) express that node i and path k can be covered when they are covered at least once. Constraint (176) states that at most P facilities can be located. Constraints (177)–(179) are integrality constraints.

They try to implement their solution for a case study to locate cellular base stations in Erie County, New York State, USA. Since the explicit model is quadratic, they call this model *quadratic maximal covering location problem*.

2.2.17. Multiple facility quantity-of-coverage

Jia, Ordonez, and Dessouky (2007) consider large-scale facility locations for medical supplies. In fact, they formulate their problem as a MCLP with multiple facility quantity-of-coverage and quality-of-coverage requirements. In addition to the formulation and solution techniques, to show an illustrative real-world example, they use the population density Pattern for Los Angeles County. In this large-scale example they consider the centroid of each census tract as a demand point, so that the number of demand points in this example is 2054 discrete points with different population densities. Consider the following notation:

i: the index of demand nodes,
j: the index of facility locations,
r: level of quality, $r = 1,\ldots,q$,
x_j: a binary variable equal 1 if facility located at j, otherwise 0,
z_i^r: a binary variable determining demand node i is covered at quality level r or not,
d_{ij}: the distance from eligible facility location j to demand node i,
P: the total number of facilities,
h_i: the population of demand node i,
Q_i^r: The minimum required assigned facilities to demand node i in order to satisfying quality level r at node i,
w^r: The assigned weight for the facilities having quality level r,
S_i^r: The required distance for demand node i serviced at quality level r, and

$$z_{ij}^r = \begin{cases} 1 & d_{ij} \leqslant S_i^r \\ 0 & \text{otherwise} \end{cases}$$

The model is as follows:

$$\text{Max} \sum_r \sum_i w^r h_i z_i^r \tag{180}$$

S.T.

$$\sum_j z_{ij}^r x_j \geqslant Q_i^r z_i^r \quad r = 1,\ldots,q, \quad \forall\ i \tag{181}$$

$$\sum_j x_j \leqslant P \tag{182}$$

$$z_i^r = \{0,1\} \quad r = 1,\ldots,q, \quad \forall\ i \tag{183}$$

$$x_j \in \{0,1\} \quad \forall\ j \tag{184}$$

Objective function (180) maximizes the total covered demand. Constraint (181) determines that demand node i is covered at quality level r only if there are more than Q_i^r facilities located within the related distance constraint servicing it. Constraint (182) states that at most P facilities can be located. Constraints (183) and (184) are integrality constraints.

2.2.18. Complementary edge covering problem

Naimi Sadigh, Mozafari, and husseinzadeh Kashan (2010) present a model for complementary edge covering problem. In this model edges are allowed to be partially covered. The notation of the model is as follows:

i, j, k: indices of vertices on a network,
V: the set of vertices on a network,
E: $E = \{(v_i,\ v_j) | v_i,\ v_j \in V,\ i \neq j\}$ is the set of edges on a network,
G: $G = (V, E)$,
x_k: a binary decision variable, it is 1 if a facility is located at vertex k,
z_{kij}: a binary decision variable, it is 1 if the facility located at vertex k covers edge (v_i, v_j) through v_i,
n: number of nodes on a network,
c_k: the cost of locating a facility at vertex k,
h_{kij}: the portion of edge $(v_i,\ v_j)$ which is covered by vertex k through v_i, and
d_{ij}: the length of edge (v_i, v_j).

The mathematical model is as follows:

$$\text{Min} \sum_{k=1}^{n} c_k x_k \tag{185}$$

S.T.

$$\sum_{k=1}^{n} h_{kij} z_{kij} + \sum_{k=1}^{n} h_{kji} z_{kji} \geqslant d_{ij} \quad (v_i,\ v_j) \in E, \quad i < j \tag{186}$$

$$\begin{cases} z_{kij} \leqslant x_k \\ z_{kji} \leqslant x_k \end{cases} \quad k = 1,2,\ldots,n \quad (v_i,\ v_j) \in E,\ \ i < j \tag{187}$$

$$\begin{cases} \sum_{k=1}^{n} z_{kij} \leqslant 1 \\ \sum_{k=1}^{n} z_{kji} \leqslant 1 \end{cases} \quad (v_i, v_j) \in E, \quad i < j \tag{188}$$

$$0 \leqslant x_k \leqslant 1 \quad k = 1, 2, \ldots, n \tag{189}$$

$$z_{kij} \in \{0, 1\} \quad k = 1, 2, \ldots, n \quad (v_i, v_j) \in E \tag{190}$$

$$z_{kij} = 0 \forall \ h_{kij} = 0 \tag{191}$$

The objective function minimizes the cost of locating facilities. Constraint (186) expresses that all of the edge are covered through the located facilities. Constraint (187) indicates that if a facility located at vertex k, edge is covered partially or completely. Constraint (188) is a supplement to constraint (186) and ensures that each edge (v_i, v_j) is covered by its end points v_i and v_j.

2.2.19. Other variants of the MCLP

Nozick (2001) develops fixed charge facility location problem with coverage restriction. It considers how to identify location for facilities that minimizes cost while maintaining an appropriate level of service. She presents two Lagrangian relaxation based heuristics for solving the problem. Consider the following notation:

i: the index of demand nodes,
j: the index of facilities,
x_j: a binary decision variable indicating whether the facility located at point j or not,
c_j:fixed cost of locating a facility at j,
h_i:demand at node i,
d_{ij}: distance between node i and facility j;
c': the cost of each demand for each distance,
V: the sum of permissible uncovered demands,
z_{ij}: is equal 0 if demand point i is covered by facility j, otherwise 1, and
Y_{ij}: a fraction of demand i served by jth facility.

The model is as follows:

$$\text{Min} \sum_j c_j x_j + c' \sum_{ij} h_i d_{ij} Y_{ij} \tag{192}$$

S.T.

$$\sum_j Y_{ij} = 1 \quad \forall \ i \tag{193}$$

$$Y_{ij} \leqslant x_j \quad \forall \ i, j \tag{194}$$

$$\sum_{ij} h_i z_{ij} Y_{ij} \leqslant V \tag{195}$$

$$Y_{ij} \geqslant 0 \quad \forall \ i, j \tag{196}$$

$$x_j \in \{0, 1\} \quad \forall \ j \tag{197}$$

Objective function (192) minimizes the total cost while constraints (193), (194), (196) and (197) are the same as fixed charge facility location model constraints and constraint (195) ensures that the solution covers at least minimum required amount of demands.

Xia et al. (2010) propose a model for banking facility location problem which maximizes the profit of facility network. The profit function consists of revenue and costs. The notation of the model is as follows:

i: index of demand nodes,
j: index of candidate facility locations,
k: index of facility type,
I: the set of demand nodes,
J: the set of candidate facility locations,
K: the set of facility type,
x_j^k: a binary decision variable, it is 1 if a k-type facility is located at j,
z_i: the percent of demands covered at node $i (z_i \in [0, 1])$,
N^k: the number of permitted the k-type facilities,
O^k: the operation cost for the k-type facilities,
R_j^k: the rental cost for the k-type facility at node j,
S^k: the maximum acceptable distance of k-type facility,
d_{ij}: the distance between node i and node j,
h_i: demand amount of node i, and
$f^k(d_{ij}) : f^k(d_{ij}) = \text{Max}\left\{0, 1 - \frac{d_{ij}}{S^k}\right\}$ is the coverage function.

$$\text{Max} \sum_{i \in I} h_i z_i - \sum_{j \in J} \sum_{k \in K} (O^k + R_j^k) x_j^k \tag{198}$$

S.T.

$$\sum_{k \in K} x_j^k \leqslant 1 \quad \forall \ j \in J \tag{199}$$

$$\sum_{j \in J} x_j^k \leqslant N^k \quad \forall \ k \in K \tag{200}$$

$$z_i \leqslant 1 \quad \forall \ i \in I \tag{201}$$

$$z_i \leqslant \sum_{j \in J} \sum_{k \in K} f^k(d_{ij}) x_j^k \quad \forall \ i \in I \tag{202}$$

Constraint (199) indicates that for each candidate location only one type of facility can be considered. Constraint (200) imposes limits on the number of each type of facility. Constraint (201) implies that each demand node can be covered fully. Constraint (202) ensures that maximum coverage of node i is equal to all coverage of facilities for node i.

Lee and Murray (2010) present a maximal wireless covering model with survivability constraint for Wi-Fi mesh network. In Wi-Fi mesh network, each Wi-Fi router connected to other Wi-Fi router without any wires if the distance between them is not more than given number. On the other hand, some of them must be connected to the infrastructure. The model proposes maximal coverage with reliable provision of services based on network connectivity. Consider the following notation:

i: index of demand nodes,
j, l, r, t: indices of potential sites,
c: index of DSL central office,
I: the set of demand nodes,
J: the set of potential sites for Wi-Fi routers,
M: the set of existing DSL central offices,
N_i: $N_i = \{j \in J | d_{ij} \leqslant S\}$,
φ: $\varphi = \{j \in J | d_{cj} \leqslant L, c \in M\}$,
Ω_j: $\Omega_j = \{l \in J | d_{jl} \leqslant W, j \neq l\}$,
x_j: a binary decision variable, it is 1 if a Wi-Fi routers is located at j,
z_i: a binary decision variable, it is 1 if a demand node i is covered,
f_{jl}^{rt}: a binary decision variable, it is 1 if a flow (j, l) exists along with path $\{r, t\}$,
λ^{rt}: a binary decision variable, it is 1 if a logical path $\{r, t\}$ exists,
h_i: the population at demand node i,
p: the required number of Wi-Fi routers,
q: the required number of Wi-Fi routers for wired connection to existing central offices,
k: the required number of disjoint paths,
d_{ij}: the shortest distance from demand node i to Wi-Fi router locate at j,
d_{jc}: the shortest distance from Wi-Fi router locate at j to DSL central offices located at c,
d_{jl}: the shortest distance between Wi-Fi router locate at j and l,
S: the coverage standard for Wi-Fi,
T: the Coverage standard for DSL central office, and
W: the maximum acceptable distance for Wi-Fi router point to point interconnection.

The model is as follows:

$$\text{Max} \sum_{i \in I} h_i z_i \tag{203}$$

S.T.

$$z_i \leqslant \sum_{j \in N_i} x_j \quad \forall\, i \in I \tag{204}$$

$$\sum_{j \in J} x_j = p \tag{205}$$

$$f_{jl}^{rt} \leqslant \begin{cases} x_j \\ x_l \end{cases} \quad \forall\, j, r \in J,\ \ t \in \varphi,\ \ l \in \Omega_j,\ \ r \neq t \tag{206}$$

$$2f_{jl}^{rt} - (x_r + x_t) \leqslant 0 \quad \forall\, j, r \in J,\ \ t \in \varphi, l \in \Omega_j, r \neq t \tag{207}$$

$$\lambda^{rt} \geqslant x_r + x_t - 1 \quad \forall\, r \in J,\ \ t \in \varphi,\ \ r \neq t \tag{208}$$

$$x_r \geqslant \lambda^{rt} \quad \forall\, r \in J,\ \ t \in \varphi,\ \ r \neq t \tag{209}$$

$$x_t \geqslant \lambda^{rt} \quad \forall\, r \in J,\ \ t \in \varphi,\ \ r \neq t \tag{210}$$

$$\sum_{l \in \Omega_j} f_{jl}^{rt} - \sum_{l \in \Omega_j} f_{lj}^{rt} = \begin{cases} k\lambda^{rt} & \text{if } j = r, \\ & \quad \forall\, j,\ \ r \in J,\ \ t \in \varphi,\ \ r \neq t \\ -k\lambda^{rt} & \text{if } j = t, \\ & \quad \forall\, j,\ \ r \in J,\ \ t \in \varphi,\ \ r \neq t \\ 0 & \text{otherwise} \\ & \quad \forall\, j,\ \ r \in J,\ \ t \in \varphi,\ \ j \neq r \neq t \end{cases} \tag{211}$$

$$\sum_{l \in \Omega_j} f_{jl}^{rt} \leqslant 1 \quad \forall\, j, r \in J,\ \ t \in \varphi,\ \ j \neq r \neq t \tag{212}$$

$$\sum_{t \in \varphi} x_t \geqslant q \tag{213}$$

$$z_i, x_j \in \{0, 1\} \quad \forall\, i \in I,\ \ j \in J \tag{214}$$

$$f_{jl}^{rt} \in \{0, 1\} \quad \forall\, j,\ \ r \in J,\ \ t \in \varphi,\ \ l \in \Omega_j,\ \ r \neq t \tag{215}$$

$$\lambda^{rt} \in \{0, 1\} \quad \forall\, r \in J,\ \ t \in \varphi,\ \ r \neq t \tag{216}$$

The objective function maximizes the coverage of population by establishing Wi-Fi routers. Constraint (204) indicates that demand node i is covered if at least one Wi-Fi routers covers that node. Constraint (206) expresses that flow between any Wi-Fi routers j and l exists if Wi-Fi routers located at j and l. Constraints (208)–(210) ensure that logical path to locate Wi-Fi routers exist. Constraint (211) expresses the survivability constraint. Constraint (212) ensures that if a node is not a source or destination, then at most one unit flow to it. Constraint (213) implies that the number of potential sites to locate wired connection to existing DSL is at least q.

3. Other extended models

In this section, we introduce other extensions of covering models that are not easily related to SCP or MCLP; but are somehow related to the nature of covering problems.

3.1. Anti-covering

Moon and Chaudhry (1984) introduce the *Anti-Covering Location Problem* (ACLP) that maximizes the set of selected location sites so that there are no two selected sites within a pre-specified distance. Consider the following notation:

i: the index of demand nodes,
j: the index of facilities,
d_{ij}: the distance between vertex i and vertex j,
S: a maximum acceptable service distance, and
N_i: the set of potential locations within S so that $N_i = \{j \in J | d_{ij} \leqslant S,\ i \neq j\}$,
x_j: a binary decision variable indicating whether the facility located at point j or not,
M: a large positive number. The model is as follows:

$$\text{Max } z = \sum_{j \in J} x_j \tag{217}$$

S.T.

$$M(1 - x_i) \geqslant \sum_{j \in N_i} x_j \quad i \in J \tag{218}$$

$$x_j \in \{0, 1\} \quad \forall\, j \tag{219}$$

Objective function (217) maximizes the set of selected location sites. Constraint (218) states that the value of x_j is not affected by $x_i = 0$ when $j \in N_i$, but when $x_i = 1$, all of the mentioned x_j must take a value 0. Constraint (219) is the integrality constraint.

3.2. Indirect covering tree problem

Hutson and ReVelle (1993) develop two types of covering problems on tree. They focus on a spanning tree of a network: (i) The Minimum Cost Covering Sub-tree (MCCS) is used in SCP to find the minimum cost collection of arcs in form of a sub-tree; where satisfaction of covering constraints for nodes of the network is observed. To solve this problem they use reduction techniques and (ii) The Maximal Indirect Covering Sub-tree (MICS) is used for MCLP; where the sub-tree maximizes the demand within a standard distance of the sub-tree's nodes.

Williams (2003) presents four new bi-objective binary programming models; one to minimize the total distance of the sub-tree, and other to maximize the sub-tree's coverage of demand at nodes. These two models are related to direct and indirect coverage on a single spanning tree parent. Indirect coverage needs that a node is within a threshold distance $S > 0$ of the sub-tree and direct coverage requires that a node connects to the sub-tree. Williams, thus, develops new formulations for the original problems posed by Hutson and ReVelle (1993). The other two models are related to the applications of direct and indirect coverage with multiple parents. Consider the following notation for multiple parents:

i, j, k: the indices of nodes,
t: the index of candidate parent spanning tree,
N: the set of nodes,
T: the set of candidate parent spanning tree,
T_{ij}: the set of spanning tree t that contain directed arc (i,j),
N': the union of the sets of non-leaf nodes, taken over all parent spanning tree t,
S_k: the set of node i that are within the distance standard S of node k,
A: the union of directed arc sets, taken over all candidate parent spanning tree t,
P_j: the set of directed arc (i,j) that are incident to and directed into node j in any candidate parent spanning tree t,
F_j: the set of arc (j,k) that are incident to and directed out of node j in any candidate parent spanning tree t,
x_{ij}: a binary decision variable, it is 1 if directed arc (i,j) and node i are selected for subtree,
y_j: a binary decision variable, it is 1 if node j is selected as terminus of the for subtree,
z_k: a binary decision variable, it is 1 if node k is covered either directly or indirectly by the subtree,
u_i: a binary decision variable, it is 1 if node i is covered either indirectly,
v_t: a binary decision variable, it is 1 if spanning tree t is selected as the parent of the subtree,

(i,j): the arc that directs node i to node j,
n: number of nodes on a network,
c_{ij}: the cost or length of arc (i,j),
h_i: the demand or population of node i, if node i is covered directly, and
h'_i: the demand or population of node i, if node i is covered indirectly ($h'_i \leqslant h_i$).

Maximum direct covering tree model for multiple parent trees is:

$$\text{Min} \sum_{(i,j)\in A} c_{ij}x_{ij} \tag{220}$$

$$\text{Max} \sum_{(i,j)\in A} h_i x_{ij} + \sum_{j\in N'} h_j y_j \tag{221}$$

S.T.

$$x_{ij} - \sum_{(j,k)\in F_j} x_{jk} - y_j \leqslant 0 \quad \forall\ (i,j) \in A \tag{222}$$

$$\sum_{j\in N'} y_j = 1 \tag{223}$$

$$y_j - \sum_{(i,j)\in P_j} x_{ij} \leqslant 0 \quad \forall\ j \in N' \tag{224}$$

$$x_{ij} - \sum_{t\in T_{ij}} v_t \leqslant 0 \quad \forall\ (i,j) \in A \tag{225}$$

$$\sum_{t\in T} v_t = 1 \tag{226}$$

$$x_{ij}, y_j, v_t \in \{0,1\} \quad \forall\ i,j,t \tag{227}$$

The first objective function minimizes the total cost of sub-tree over the entire arc set A and second objective function maximizes the total node population covered directly by the sub-tree.

Constraint (222) indicates that if arc (i,j) is selected for sub-tree, node j will be selected as the sub-tree's terminus. Constraint (223) expresses that exactly one eligible node will be selected as a sub-tree's terminus. Both of these constraints ensure the connectivity of nodes. Constraint (224) enforces that at least one arc will appear in sub-tree by requiring that if node j is selected as a sub-tree's terminus. Constraint (225) indicates that arc (i,j) can be selected only if one of candidate parent spanning trees containing that arc is selected. Constraint (226) requires that exactly one candidate spanning tree be selected as the sub-tree's parent. Also Maximum indirect covering tree model is:

$$\text{Min} \sum_{(i,j)\in A} c_{ij}x_{ij} \tag{228}$$

$$\text{Max} \sum_{i\in N} h'_i u_i + \sum_{(i,j)\in A} (h_i - h'_i)x_{ij} + \sum_{j\in N'} (h_j - h'_j)y_j \tag{229}$$

S.T.

$$x_{ij} - \sum_{(j,k)\in F_j} x_{jk} - y_j \leqslant 0 \quad \forall\ (i,j) \in A \tag{230}$$

$$\sum_{j\in N'} y_j = 1 \tag{231}$$

$$y_j - \sum_{(i,j)\in P_j} x_{ij} \leqslant 0 \quad \forall\ j \in N' \tag{232}$$

$$x_{ij} - \sum_{t\in T_{ij}} v_t \leqslant 0 \quad \forall\ (i,j) \in A \tag{233}$$

$$\sum_{t\in T} v_t = 1 \tag{234}$$

$$u_k - \sum_{i\in S_k} \sum_{(i,j)\in F_i} x_{ij} - \sum_{j\in S_k} y_j \leqslant 0 \quad \forall\ k \in N \tag{235}$$

$$x_{ij}, y_j, v_t \in \{0,1\} \quad \forall\ i,j,t \tag{236}$$

Objective function (229) maximizes combined direct and indirect coverage of population. Constraints (235) guarantee the indirect coverage conditions.

3.3. Hub covering problem

Campbell (1994) introduces hub covering problem. He presents an integer programming formulation for this problem. Hubs are facilities that serve as transshipment points in networks like transportation network and telecommunication network. In a hub network, nodes are divided into two groups: hubs and non-hubs (spokes) so that flow between each OD (origin–destination) pairs of non-hub nodes passes only through hubs. In hub covering problem, a pair of hub nodes (which are all connected together) cover an OD pair if the travel time (cost) from O to D is less than a pre-specified limit. Consider the following notation:

i, j, k, m: the indices of nodes on a network,
y_k: a binary decision variable, it is 1 if location k is a hub,
z_{ijkm}: a binary decision variable, it is 1 if hubs k and m covered O–D pair (i,j),
u_{ij}: a binary decision variable, it is 1 if O–D pair (i,j) uncovered,
x_{ijkm}: the fraction of flow from location i to location j that is occurred via hubs at location k and m in that order,
c_k: the fixed cost to establish a hub at location k,
p_{ij}: the penalty for leaving O–D pair (i,j) uncovered,
h_{ij}: the flow from location i to location j, and
p: the numbers of hubs to be located.

The hub set covering model tries to locate the hubs in order to cover all demands. The mathematical model is as follows:

$$\text{Min} \sum_{K} c_k y_k \tag{237}$$

S.T.

$$x_{ijkm} \leqslant y_k \quad \forall\ i,j,k,m \tag{238}$$

$$x_{ijkm} \leqslant y_m \quad \forall\ i,j,k,m \tag{239}$$

$$\sum_{k} \sum_{m} x_{ijkm} z_{ijkm} \geqslant 1 \quad \forall\ i,j \tag{240}$$

$$y_k \in \{0,1\} \quad \forall\ k \tag{241}$$

If the cost to cover all O–D pairs is greater than the available budget, some of O–D pairs must be uncovered. In this situation the model is as follows:

$$\text{Min} \sum_{k} c_k y_k + \sum_{i} \sum_{j} p_{ij} u_{ij} \tag{242}$$

S.T.

$$\sum_{k} \sum_{m} x_{ijkm} z_{ijkm} + u_{ij} \geqslant 1 \quad \forall\ i,j \tag{243}$$

$$x_{ijkm} \leqslant y_k \quad \forall\ i,j,k,m \tag{244}$$

$$x_{ijkm} \leqslant y_m \quad \forall\ i,j,k,m \tag{245}$$

$$y_k \in \{0,1\} \quad \forall\ k \tag{246}$$

This model minimizes the total cost while uncovered O–D pairs are also minimized.

The hub maximal covering location model is as follows:

$$\text{Max} \sum_{i} \sum_{j} \sum_{k} \sum_{m} h_{ij} x_{ijkm} z_{ijkm} \tag{247}$$

S.T.

$$\sum_{k} y_k = P \tag{248}$$

$$0 \leqslant x_{ijkm} \leqslant 1 \quad \forall\ i,j,k,m \tag{249}$$

$$\sum_k \sum_m x_{ijkm} = 1 \quad \forall\ i,j \tag{250}$$

$$x_{ijkm} \leqslant y_k \quad \forall\ i,j,k,m \tag{251}$$

$$x_{ijkm} \leqslant y_m \quad \forall\ i,j,k,m \tag{252}$$

$$y_k \in \{0,1\} \quad \forall\ k \tag{253}$$

This model maximizes the demand covered with a given number of hub facilities.

Campbell (1994) develops the basic formulation for the hub set covering problem to cover all demand nodes (HCV). In another variation, due to limited budget, he proposes another formulation; in this case like in MCLP there are some uncovered nodes (HCV-P). This formulation is very similar to Kolen and Tamir (1990). In another option, he designs a hub Maximal Covering Problem (HMCV) in which maximization of the demand covered with a given number of hub facilities is investigated. Finally, he introduces hub covering problems with spoke flow thresholds.

Kara and Tansel (2003) present a new integer programming formulation of the hub covering problem which is different from Campbell's formulation. They provide three linearizations for the Campbell's model and also one new model. Then they compare the formulations in terms of computational performance. The notation of the model is:

i, j, k, r: the indices of nodes on a network,
$a(i)$: the hub that serves node i ($a(i) \in H$),
N: the set of nodes on the network (cities),
H: the set of nodes that specify the location of hubs ($H \subset N$),
z_{ik}: a binary decision variable, it is 1 if node i is covered from a hub at node k,
d_{ij}: the travel time between node i to node j ($d_{ij} = d_{ji}$, $d_{ij} + d_{jk} \geqslant d_{ik}\ \forall\ i, j, k \in N$),
α: the discount factor for hub-to-hub transportation,
S: the predetermined bound for travel time between any pair of cities, and
n: number of nodes on a network.

The combinatorial formulation of the model is as follows:

$$\text{Min}|H|_{H\subset N} \tag{254}$$

S.T.

$$d_{ia(i)} + \alpha d_{a(i)a(j)} + d_{a(j)j} \leqslant S \quad \forall\ i,j \in N, a(i) \in H, \forall\ i \tag{255}$$

It is assumed that each node is served by a single hub. This model minimizes number of hubs whereas the travel time between any pairs of nodes is not more than the cover radius. Since this model is NP-complete, they propose an integer programming formulation as follows:

$$\text{Min} \sum_k z_{kk} \tag{256}$$

S.T.

$$(d_{ir} + \alpha d_{rk} + d_{jk}) z_{ir} z_{jk} \leqslant S \quad \forall\ i,j,k,r \tag{257}$$

$$\sum_k z_{ik} = 1 \quad \forall\ i \in N \tag{258}$$

$$z_{ik} \leqslant z_{kk} \quad \forall\ i,k \in N \tag{259}$$

$$z_{ik} \in \{0,1\} \quad \forall\ i,k \in N \tag{260}$$

Constraint (257) ensures that cover radius is met. Constraint (258) and (260) ensure that every node is assigned to exactly one hub. Constrain (259) ensures that assignment is made, since a hub locates at node k.

3.4. Hierarchical covering problem

Some of the facility location problems are hierarchical in nature. For example, the health care system consists of local clinics, hospitals and medical centers. Hierarchical facility location problems can be divided into two major groups: (i) *successively inclusive facility hierarchy (nested)* in which facility at level k provides services 1 through k and (ii) *successively exclusive facility hierarchy (non-nested)* in which a facility level k provides only services of level k.

Moore and ReVelle (1982) define a *nested hierarchical maximal covering problem* and develop an integer programming problem for it. The objective is the maximization of covered population given specific limits on either the number of each type of facility or on the total investment that can be made in all facility types. They use a relaxed LP supplemented by branch and bound to solve the model. Consider the following notation:

i: index of demand points,
j, k: Indices of potential facility location,
I: the set of demand points,
J: the set of potential facility locations,
MA_i: $MA_i = \{j \in J | d_{ij} \leqslant S^{IA}\}$,
MB_i: $MB_i = \{k \in K | d_{ik} \leqslant S^{IB}\}$,
NB_i: $NB_i = \{k \in K | d_{ik} \leqslant T^{IB}\}$,
z_i: a binary decision variable, it is 1 if demand node i is uncovered,
x_j: a binary decision variable, it is 1 if facility A is sited at node j,
x_k: a binary decision variable, it is 1 if facility B is sited at node k,
d_{ij}: the shortest distance from node i to node j,
S^{IA}: the maximum acceptable service level for type A facility offering type A services,
S^{IB}: the maximum acceptable service level for type B facility offering type A services,
T^{IB}: the maximum acceptable service level for type B facility offering type B services,
h_i: the population of node,
p: the number of A facilities to be sited, and
q: the number of B facilities to be sited.

The mathematical model is:

$$\text{Min} \sum_{i\in I} h_i z_i \tag{261}$$

S.T.

$$\sum_{j\in MA_i} x_j + \sum_{k\in MB_i} x_k + z_i \geqslant 1 \quad \forall\ i \in I \tag{262}$$

$$\sum_{k\in NB_i} x_k + z_i \geqslant 1 \quad \forall\ i \in I \tag{263}$$

$$\sum_{k\in J} x_k = q \tag{264}$$

$$\sum_{j\in J} x_j = P \tag{265}$$

$$x_j, x_k, z_i \in \{0,1\} \tag{266}$$

Constraint (262) ensures that demand of node i will be covered if there is type A facility within S^{IA} distance or type B facility within S^{IB} distance. Constraint (263) allows node i to be covered for type B services if there is a type B facility within T^{IB} distance. Constraints (264) and (265) limit the investment in facilities.

Lee and Lee (2010) propose hierarchical covering location model which is based on partial coverage. In this model if distance from demand node i to facility is less than a given threshold, the node i is fully covered. On the other hand if the distance is beyond prespecified range, it is partially covered. The notation of the model is as follows:

i: index of potential facility sites,
j: index of customer nodes,
k: index of service type,

l: index of facility level,
z_{jik}: a binary decision variable; it is 1 if a customer at node i is covered with service type k by a facility located at site j,
x_{jl}: a binary decision variable; it is 1 if a facility of level l is established at site j,
S_k: the maximum acceptable distance for full coverage of service type k,
T_k: the maximum acceptable distance for partial coverage of service type k,
c_{jik}: $c_{jik} = f(d_{ji}, S_k, T_k) = \begin{cases} 1 & d_{ji} \leqslant S_k \\ b & S_k < d_{ji} \leqslant T_k \\ 0 & d_{ji} > T_k \end{cases}$ is the coverage ratio provided by service type k of the facility j to the customer node i,
h_{ik}: the population of customers at node i who need service type k,
t_{jk}: the capacity of service type k at node j when the facility is established,
I: the number of potential sites,
J: the number of customer nodes,
K: the number of service type,
L: the number of facility level,
p_l: the number of established facilities with service level l, and
b: the portion of partial coverage ($0 \leqslant b < 1$).

The mathematical model is as follows:

$$\text{Max} \sum_{i\in I}\sum_{j\in J}\sum_{k\in K} c_{jik} h_{ik} z_{jik} \tag{267}$$

S.T.

$$z_{jik} \leqslant \sum_{l=k}^{L} x_{jl} \quad \forall\ i,j,k \tag{268}$$

$$\sum_{j\in J} z_{jik} \leqslant 1 \qquad \forall\ i,k \tag{269}$$

$$\sum_{j\in J} x_{jl} \leqslant p_l \qquad \forall\ l \tag{270}$$

$$\sum_{i\in I} h_{ik} z_{jik} \leqslant t_{jk} \quad \forall\ j,k \tag{271}$$

$$\sum_{l\in L} x_{jl} \leqslant 1 \qquad \forall\ j \tag{272}$$

$$z_{jik}, x_{jl} \in \{0,1\} \quad \forall\ i,j,k,l \tag{273}$$

The objective function maximizes all level of facility coverages. Constraint (268) ensures that demand node i is covered if a facility with at least the same level as customers need is located. Constraint (269) indicates that each demand node i will be covered by at most one facility. Constraint (270) imposes limitations on the number of facilities in each level. Constraint (271) states that the total available service of each facility does not exceed its capacity. Constraint (272) indicates that each facility can be located in one site.

3.5. Coherent covering location problem

Serra (1996) offers a formulation for the *hierarchical covering location problem*. Coherency is defined as the property in which all of the areas allocated to a facility at one hierarchical level should belong to one and the same district in the next level. The mathematical formulation is developed for two hierarchical facility levels by maximizing the coverage while ensuring coherency. Consider the following notation:

i: index of demand points,
j: index of potential facility locations of type A,
k: index of potential facility locations of type B,
I: the set of demand points,
J: the set of potential facility locations,
MA_i: $MA_i = \{j \in J | d_{ij} \leqslant S^{IA}\}$,
MB_i: $MB_i = \{k \in K | d_{ik} \leqslant S^{IB}\}$,
NB_i: $NB_i = \{k \in K | d_{ik} \leqslant T^{IB}\}$,
O_j: $O_j = \{k \in K | d_{jk} \leqslant S^{AB}\}$,
z_i^A: a binary decision variable; it is 1 if area is covered by type A facility,
z_i^B: a binary decision variable; it is 1 if area is covered by type B facility,
x_j: a binary decision variable, it is 1 if type A facility is located at j,
x_k: a binary decision variable, it is 1 if type B facility is located at k,
d_{ij}: the shortest distance from node i to node j,
S^{IA}: the maximum acceptable distance for type A facility offering type A services,
S^{IB}: the maximum acceptable distance for type B facility offering type A services,
T^{IB}: the maximum acceptable distance for type B facility offering type B services,
S^{AB}: the maximum acceptable distance from a type A to a type B facility,
h_i: the population at node i,
p: the number of type A facilities to be sited, and
q: the number of type B facilities to be sited.

The mathematical formulation of the problem is as follows:

$$\text{Max} \sum_{i\in I} h_i z_i^A \tag{274}$$

$$\text{Max} \sum_{i\in I} h_i z_i^B \tag{275}$$

S.T.

$$z_i^A \leqslant \sum_{j\in MA_i} x_j + \sum_{k\in MB_i} x_k \quad \forall\ i \in I \tag{276}$$

$$z_i^B \leqslant \sum_{k\in NB_i} x_k \quad \forall\ i \in I \tag{277}$$

$$x_j \leqslant \sum_{k\in O_j} x_k \quad \forall\ j \in J \tag{278}$$

$$\sum_{j\in J} x_j = p \tag{279}$$

$$\sum_{k\in K} x_k = q \tag{280}$$

$$z_i^A, z_i^B, x_j, x_k \in \{0,1\} \quad \forall\ i \in I,\ \ j \in J,\ \ k \in K \tag{281}$$

The objective functions maximize demand covered by type A and B facilities. The constraint (276) indicates that if distance from type A facilities is less than S^{IA} or distance from type B facilities is less than S^{IB}, the node is covered. The constraint (277) ensures node i to be covered for type B services if there is a type B facility within T^{IB}. The constraint (278) ensures coherency and expresses that a type A facility has to be within S^{AB} from any type B facility. Constraints (279) and (280) determine the number of type A and type B facilities.

3.6. Spatial covering problem

Murray (2005) develops a new binary set covering problem for the spatial objective. The model considers a minimum acceptable coverage percent and minimum number of partial coverage facilities needed for complete coverage. Consider the following notation:

i: index of demand areas,
j: index of potential facility sites,
Ω_i: the set of candidate facilities that at least β% partially cover node i,
x_j: a binary variable is equal to 1 if potential facility site j is selected for service establishment, otherwise 0,
τ_i: a binary variable equal 1 if node i covered partially at least α times and otherwise equal to 0,
a_{ij}: is 1 if area i is suitably served by a potential facility at site j; and 0 otherwise,
β: the minimum acceptable coverage percent,
α: the minimum number of partial coverage facilities required covering completely and
w_j: the weighted cost.

The model is as follows:

$$\text{Min} \sum_j w_j x_j \tag{282}$$

S.T.

$$\sum_j a_{ij} x_j \geqslant 1 - \tau_i \quad \forall\, i \tag{283}$$

$$\sum_{i \in \Omega_i} x_j \geqslant \alpha \tau_i \quad \forall\, i \tag{284}$$

$$\tau_j \in \{0,1\} \quad \forall\, j \tag{285}$$

$$x_j \in \{0,1\} \quad \forall\, j \tag{286}$$

The objective function (282) minimizes the total weighted cost. Constraint (283) ensures required coverage of each i provided by at least one located facility and constraint (284) ensures having minimum number of facilities to partially covered node i. Constraint (285) and (286) are the integral constraints.

Murray and Tong (2007) extend the model of Church (1984) to a *spatial covering problem*. Their model finds location of facilities in the plane. The model works for line or polygon features. They call the problem extended planar maximal covering location problem-Euclidean (EPMCE). They try to apply model to locate some sirens for emergency warning.

Murray, O'Kelly, and Church (2008) try to find a way to represent a study region in a mathematical environment. Instead of using usual historical approach which considers a region as an irregular pattern of points they try to represent a region as a regular pattern of points. They compare efficiency of considering different pattern on covering an area.

3.7. Set packing and the SCP

Conforti et al. (2001) in a review paper survey the matrices arising in connection with set covering problems. This is the left hand side matrix related to the integer programming model. They discuss this subject in terms of perfect, ideal and balanced matrices.

3.8. Maximum Expected Coverage Location Problem (MEXCLP)

Daskin (1982) introduces application of an "Expected Covering Model" to Emergency Medical Service (EMS) systems. Later, Daskin (1983) develops *Maximum Expected Coverage Location Problem (MEXCLP)*. Given the probability for each vehicle being busy, this model locates P facilities at potential locations on a network in order to maximize the covered expected population within a threshold service distance (S). Again, he considers the application of this model to EMS system.

Daskin (1995) in his book also presents MEXCLP in which a fixed number of facilities are located to maximize the expected covered demands by an available facility. In this problem he defines a parameter q that is system-wide average probability showing that a facility is busy. One of the applications for this problem can be in locating ambulances to serve people; this is one of the variants that are emphasized in Church and Murray (2009), too.

3.9. Merging local reliability and expected coverage: LR-MEXCLP

Sorensen and Church (2010) combine local-reliability estimates of MALP (Maximum Availability Location Problem) introduced by ReVelle and Hogan (1989a) with the objective of MEXCLP maximizing expected covered demands. Consider the following notation:

i: index of demand nodes,
j: index of facilities,
k: index of servers,
N_i: set of sites (or nodes) that can cover i,
x_j: a binary variable is equal 1 if a server locates at j, otherwise 0,
$Y_{i,k}$: a binary variable equal 1 if node i is covered by k servers and otherwise 0.
h_i: the demand of node i,
P: the number of servers to locate and
$q_{i,k}$: the reliability of service at node i given k servers.

The model is as follows:

$$\text{Max} \sum_{i=1}^{n} \sum_{k=1}^{p} h_i q_{i,k} Y_{i,k} \tag{287}$$

S.T.

$$\sum_{j \in N_i} x_j - \sum_{k=1}^{p} k Y_{i,k} \geqslant 0 \quad i = 1, \ldots, n \tag{288}$$

$$\sum_{k=1}^{p} Y_{i,k} \leqslant 1 \quad i = 1, \ldots, n \tag{289}$$

$$\sum_j x_j = P \tag{290}$$

$$Y_{i,k} = \{0,1\} \quad i = 1, \ldots, n; \quad k = 1, \ldots, p \tag{291}$$

$$x_j \in \{0,1\} \quad \forall\, j \tag{292}$$

The objective function (287) maximizes the total demand considering reliability of coverage. Constraint (288) determines the number of required servers for covering specific node. Constraint (289) assures existence of at least one $Y_{i,k}$ for any demand node. Constraint (290) states the total number of facilities. Constraints (291) and (292) are the integral constraints.

3.10. Integration of multiple and backup coverage models with MEXCLP

Daskin, Hogan, and ReVelle (1988) integrate multiple, backup coverage models with MEXCLP and develop a formulation which is modeled on backup coverage with probabilistic situation.

3.11. Double coverage

Gendreau, Laporte, and Semet (1997a) develop a coverage model for locating ambulances. They define two coverage radii S^1 and S^2 with $S^1 < S^2$. All demand points must be covered within S^2 and a proportion of demands, say α, must be covered within S^1. The notation and mathematical formulation of the model is:

i: index of demand points,
j: index of potential location sites,

V: the set of demand points ($V = \{v_1, \ldots, v_n\}$),
W: the set of potential location sites ($W = \{v_{n+1}, \ldots, v_{n+m}\}$),
E: the set of edges on a network ($E = \{(v_i, v_j) | v_i \in V$ *and* $v_j \in W\}$),
$G = (V \cap W, E)$,
x_j: an integer variable indicating the number of ambulances located at $v_{n+j} \in W$,
z_i^k: a binary decision variable; it is 1 if v_i is covered at least k times within radius S^1 ($k = 1,2$),
S^1: the coverage radius,
S^2: the coverage radius ($S^2 > S^1$), h_i: demand at vertex $v_i \in V$,
P_j: upper bound on number of facility to be located number of ambulances located at $v_{n+j} \in W$, α: the parameter which indicates that proportion of the total demands that must be covered within S^1 units,

$$a_{ij} = \begin{cases} 1 & d_{i,n+j} \leqslant S_1 \\ 0 & \text{otherwise} \end{cases} \quad \forall\ v_i \in V, \quad v_{n+j} \in W,$$

$$b_{ij} = \begin{cases} 1 & d_{i,n+j} \leqslant S^2 \\ 0 & \text{otherwise} \end{cases} \quad \forall\ v_i \in V, \quad v_{n+j} \in W,$$

n: number of demand points, and
m: number of potential location sites.

$$\text{Max} \sum_{i=1}^{n} h_i z_i^2 \tag{293}$$

S.T.

$$\sum_{j=1}^{m} b_{ij} x_j \geqslant 1 \quad \forall\ v_i \in V \tag{294}$$

$$\sum_{i=1}^{n} h_i z_i^1 \geqslant \alpha \sum_{i=1}^{n} h_i \tag{295}$$

$$\sum_{j=1}^{m} a_{ij} x_j \geqslant z_i^1 + z_i^2 \quad \forall\ v_i \in V \tag{296}$$

$$z_i^2 \leqslant z_i^1 \quad \forall\ v_i \in V \tag{297}$$

$$\sum_{j=1}^{m} x_j = p \tag{298}$$

$$x_j \leqslant P_j \quad \forall\ v_{n+j} \in W \tag{299}$$

$$z_i^1, z_i^2 \in \{0,1\} \quad \forall\ v_i \in V \tag{300}$$

$$x_j \text{ integer} \quad \forall\ v_{n+j} \in W \tag{301}$$

The model maximizes the total demand covered at least twice within S^1 units. Constraint (294) and (295) indicate the single and double coverage requirements. Constraint (296) calculates the number of ambulances covering v_i within S^1 units. Constraint (297) ensures that a vertex v_i cannot be covered at least twice, if it is not covered at least once. Constraints (298), (299) and (301) limit the number of ambulances at each site.

3.12. Partial set cover

This problem is defined on a hypergraph $H = (V,E)$ in which each node has a non-negative weight ($w: V \to R^+$), and each edge has a non-negative length ($l: E \to R^+$). For a given threshold ($S \in R^+$), the objective function finds a subset of nodes with minimum total cost; the constraint is to cover at least length of the edges (as long as the threshold) (Yehuda, 2001). Bläser (2003) also studies the k-partial (weighted) SCP and its special case, the k-partial vertex cover problem. The model finds a minimum subcolloction among all subcollections of finite universe that covered at least k elements.

3.13. Combination of MEXLCP and DSM

Chuang and Lin (2007) combine MEXLCP with DSM and formulate MEXCLP-DS. This formulation can model double standard coverage with probabilistic situation. They use this model for an ambulance location problem. This model aims at providing sufficient coverage of emergency medical services once an ambulance is dispatched in response to a call for a real case. The notation of the model is as follows:

i: index of demand points,
j: index of facility locations,
z_{ji}: a binary decision variable; it is 1 if node i is covered by at least j facilities,
x_i: the number of ambulances at node i,
h_i: the population of demand node i,
S^1: The coverage radius,
α: the proportion of the total demands that must be covered within S^1 units,
p: the probability of a facility being busy,
p_j: the probability of call occurred at node j,
q: total number of ambulances, and
m: the number of demand points.

The mathematical formulation of MEXCLP-DS model is:

$$\text{Max} \sum_{j=1}^{q} \sum_{i=1}^{m} h_j (1-p) p^{q-1} z_{ji} p_i \tag{302}$$

S.T.

$$\sum_{j=1}^{q} \sum_{i=1}^{m} h_j z_{ji} p_i \geqslant \alpha \sum_{j=1}^{q} \sum_{i=1}^{m} h_j p_i \tag{303}$$

$$\sum_{i=1}^{m} x_i \geqslant \sum_{j=1}^{q} \sum_{i=1}^{m} z_{ji} \tag{304}$$

$$\sum_{i=1}^{m} x_i \leqslant q \tag{305}$$

The objective function of the model maximizes expected number of covered demands. Constraint (303) ensures that proportion α portion of demand is covered. Constraint (304) indicates that the total number of ambulances at all nodes must be greater than the number of facilities covering node j within S^1 radius. Constraint (305) expresses that the total number of ambulances at all nodes must be less than the total number of ambulances.

3.14. Undesirable covering problem

Facilities can be divided into desirable and undesirable (like garbage dump sites, chemical plants, nuclear reactors, military installations, prisons and polluting plants). The second group itself can be categorized as noxious (hazardous to health) and obnoxious (nuisance to lifestyle) facilities.

Drezner and Wesolowsky (1994) consider an *obnoxious facility location* problem, which is used when some nodes with their weights are given and a boundary circle containing weighted points is defined. The problem finds the location of an interior covering circles of a given radius that encloses the smallest weight of points. They present an algorithm to solve this problem.

In the context of obnoxious covering problems, Berman, Drezner, and Wesolowsky (1996) try to find location of a new facility on a network in order to minimize total weight of nodes within a pre-specified distance S. They formulate the problem, extend an algorithm to solve it and do some sensitivity analysis on the value of S. The notation of the model is as follows:

i, j: indices of nodes on a network,
V: the set of nodes on a network,
E: the set of links on a network,
G: $G = (V,E)$,
$I(x)$: $I(x) = \{i|d_i(x) < S\}$,
x: the location of any point on network,
S: the minimum distance radius,
h_i: the weight of node i,
d_{ij}: the length of link $(i,j) \in E$, and
$d_i(x)$: the shortest distance between point $x \in G$ and node $i \in N$.

The model is as follows:

$$\operatorname*{Min}_{x\in G}\left\{\sum_{i\in I(x)} h_i\right\} \tag{306}$$

Plastria and Carrizosa (1999) formulate a bi-objective undesirable facility location problem in the plane. In this problem, an undesirable facility must be located within some feasible region; the region can have any shape in the plane or on a planar network. The notation of bi-objective model is as follows:

i: index of points on the plane,
I: a finite subset of R^2 which includes points of the plane (population centers),
J: the closed feasible region ($J \subset R^2$), $I(x,S)$: $I(x,S) = \{i \in I| \|i - x\| < S\}$,
S: the coverage radius,
x: the location of the facility ($x \in J$), and
h: $I \to R^+$ allocate each $i \in I$ positive weight $h(i)$.

The bi-objective model is as follows:

$$\text{Min cov}(x,S) = \sum_{i\in I(x,S)} h(i) \tag{307}$$

$$\operatorname*{Max}_{x\in J, S\in R^+} S \tag{308}$$

$I(x,S)$ considers open circular disks of radius S and center x. The objective functions of the problems are: (i) maximization of a radius of influence and (ii) minimization of the total covered population.

Berman and Huang (2008) investigate Minimum Covering Location Problem with Distance Constraints (MCLPDC). The objective function minimizes the total demand covered through locating a fixed number of facilities with distance constraints on a network. The major constraint of this problem is that no two facilities are allowed to be closer than a pre-specified distance. In order to model the problem, five mathematical programming formulations are presented. Also two Lagrangian relaxation heuristics, Tabu search and Greedy heuristic are proposed to solve it. Then from the viewpoint of running time comparison is made. The notation of the model is as follows:

i: index of demand points,
j, k: indices of facility locations,
V: The set of nodes on a network,
E: The set of edges on a network,
G: $G = (V,E)$,
M: the set of potential locations,
M_i: $M_i = \{j \in M|d(j,i) < S\}$,
x_j: a binary decision variable, it is 1 if a facility located at point $j \in M$,
X: $X = (x_1,\ldots,x_p)$ is the location set of facilities,
z_i: a binary decision variable, it is 1 if node $i \in N$ is covered by a facility with coverage radius S,
$d(j,i)$: the distance between node j and i,
$d(i,X)$: $d(i,X) = \min\{d(i,x_j)\}_{1\leqslant j\leqslant p}$,
S: the coverage radius,
R: the minimum distance between facilities,
h_i: the weight associated with node i,
n: number of nodes on network,
m: number of potential locations, and
p: the number of facilities to be located.

The first formulation of the problem (MCLPDC1) is:

$$\text{Min}\sum_{i=1}^{n} h_i z_i \tag{309}$$

S.T.

$$x_j + x_k \leqslant 1 \quad \forall\, j,k \in M \ni d(j,k) < R \tag{310}$$

$$z_i \geqslant x_j \quad \forall\, i \in V,\ j \in M_i \tag{311}$$

$$\sum_{j=1}^{m} x_j = p \tag{312}$$

$$z_i, x_j \in \{0,1\} \quad \forall\, i \in V,\ j \in M \tag{313}$$

Constraint (310) ensures that distance between two facilities must be less than R. Constraint (311) indicates that node i will be covered if distance with j is less than S. Constraint (312) determines number of facilities that must be located.

It is evident that pair wise constraints of (310) are a special case of clique constraints. So the second formulation of the problem (MCLPDC2) is proposed as follows:

V': $V' = M$,
E': $E' = \{(j,k)|d(j,k) < R, j \neq k\}$,
G': $G' = (V',E')$,
MC: the set of all maximal cliques on G', and
C: the maximal clique on G'.

$$\text{Min}\sum_{i=1}^{n} h_i z_i \tag{314}$$

S.T.

$$\sum_{k\in C} x_k \leqslant 1 \quad \forall\, C \in MC \tag{315}$$

$$z_i \geqslant x_j \quad \forall\, i \in V,\ j \in M_i \tag{316}$$

$$\sum_{j=1}^{m} x_j = p \tag{317}$$

$$z_i, x_j \in \{0,1\} \quad \forall\, i \in V, j \in M \tag{318}$$

The LP relaxation of MCLPDC2 presents at least as good and tighter bound than the LP relaxation of MCLPDC1.

The third formulation of the problem (MCLPDC3) is:

$$Q_j = \{k \in M|d(k,j) < R, k \neq j\}.$$

$$\text{Min}\sum_{i=1}^{n} h_i z_i \tag{319}$$

S.T.

$$p(1 - x_j) \geqslant \sum_{k\in Q_j} x_k \ \ j \in M \tag{320}$$

$$z_i \geqslant x_j \quad \forall\, i \in V,\ j \in M_i \tag{321}$$

$$\sum_{j=1}^{m} x_j = p \tag{322}$$

$$z_i, x_j \in \{0,1\} \quad \forall\, i \in V,\ j \in M \tag{323}$$

Constraint (320) ensures that distance between two facilities is at least R. It is clear that MCLPDC3 has a fewer constraints than MCLPDC1.

The fourth formulation of the problem (MCLPDC4) is:

n_j: minimum coefficient necessary to impose spatial restriction between members of the set Q_j.

$$\text{Min} \sum_{i=1}^{n} h_i z_i \tag{324}$$

S.T.

$$n_j(1 - x_j) \geqslant \sum_{k \in Q_j} x_k \quad j \in M \tag{325}$$

$$z_i \geqslant x_j \quad \forall\ i \in V,\ j \in M_i \tag{326}$$

$$\sum_{j=1}^{m} x_j = p \tag{327}$$

$$z_i, x_j \in \{0, 1\} \quad \forall\ i \in V,\ j \in M \tag{328}$$

The fifth formulation of the problem (MCLPDC5) is:

$Q'_j = \{k | k \in Q_j \cup \{j\}, k \notin C_j\}$,

L_1: the set of all $C_j(j \in M)$,
L_2: the set of all nonempty $Q'_j(j \in M)$,
n'_j: minimum coefficient necessary to impose spatial restriction between members of the set Q'_j, and
C_j: the maximal clique which includes node j of G'.

$$\text{Min} \sum_{i=1}^{n} h_i z_i \tag{329}$$

S.T.

$$\sum_{k \in C_j} x_k \leqslant 1 \quad \forall\ C_j \in L_1 \tag{330}$$

$$n'_j(1 - x_j) \geqslant \sum_{k \in Q'_j} x_k \quad \forall\ Q'_j \in L_2 \tag{331}$$

$$z_i \geqslant x_j \quad \forall\ i \in V,\ j \in M_i \tag{332}$$

$$\sum_{j=1}^{m} x_j = p \tag{333}$$

$$z_i, x_j \in \{0, 1\} \quad \forall\ i \in V,\ j \in M \tag{334}$$

3.15. Semi-obnoxious covering problem

Ohsawa and Tamura (2003) present a semi-obnoxious covering model in a continuous plane to locate one facility. They consider two objectives: (i) maximization of the distance to the nearest residents and (ii) minimization of the summation of distances to all the farthest users. They focus on two two-dimensional areas special cases including (1) the elliptic maximin and rectangular minisum criteria problem and (2) the rectangular maximin and minimax criteria problem. The polynomial-time algorithm is proposed to find the efficient set. The model of the problem is as follows:

$\Omega \subset R^n$ $(n \geqslant 3)$ (it is closed and bounded to $\partial\Omega$),

I^-: index set of inhabitant who repels the facility,
I^+: index set of distinct users who attracts the facility,
z: decision variable for location of facility,
x_i: the site of the ith inhabitant, and
y_i: the site of the ith user.

$$\underset{z \in \Omega}{\text{Max}} \left(\underset{i \in I^-}{\text{Min}} \|z - x_i\| \right) \tag{335}$$

$$\underset{z \in \Omega}{\text{Min}} \left(\underset{i \in I^+}{\text{Max}} \|z - y_i\| \right) \tag{336}$$

Ohsawa, Plastria, and Tamura (2006) in another research consider location of a semi-obnoxious facility within a convex polygon. They focus on the closest points and farthest points concurrently using Euclidean distances and call it push and pull partial covering criteria. For this problem they develop a polynomial algorithm, too. They also discuss the tradeoff for full and partial covering for this problem. The model is formulated as follows:

$\Omega \subset R^n (n \geqslant 3)$ (it is closed and bounded to $\partial\Omega$),

I: index set of inhabitants who repels the facility ($|I| \geqslant 2$),
z: the decision variable for location of facility,
$X = \{x_1, \ldots, x_{|I|}\}$: site of the ith inhabitant,
n^-: the number of inhabitants that are neglected farther from the facility, and
n^+: the number of inhabitants that are neglected.

$$\underset{z \in \Omega}{\text{Max}} \left(\text{Max}_{J^- \subseteq I, |J^-| = |I| - n^-} \left(\underset{j \in J^-}{\text{Min}} \|z - x_j\| \right) \right) \tag{337}$$

$$\underset{z \in \Omega}{\text{Min}} \left(\text{Min}_{J^+ \subseteq I, |J^+| = |I| - n^+} \left(\underset{j \in J^+}{\text{Max}} \|z - x_j\| \right) \right) \tag{338}$$

The first objective function maximizes the nearest distance from the facility to $|I| - n^-$ inhabitants and second objective function minimizes the farthest distance from the facility $|I| - n^+$.

3.16. Maximum Covering Route Extension Problem (MCREP)

Matisziw, Murray, and Kim (2006) design a bi-objective *Maximum Covering Route Extension Problem* (MCREP) to extend existing service network through prioritizing route and stop additions. The first objective function maximizes demand coverage and the second minimizes added value length. They test the model for the bus route system in the Columbus. Consider the following notation:

i,j: index of potential candidate stops,
k: index of demand areas,
r: index of clique sets (entire set = R),
T: the set of beginning terminal nodes t,
E: the set of destination nodes e,
M_k: set of stops j which cover demand k,
C_r: subset of stops for that one j covers same demand as other stops,
N_j: $\{i/$ arc$\{i, j\}$ is defined$\}$,
N_t: $\{t/$ arc$\{t, j\}$ is defined$\}$,
N_e: $\{e/$ arc$\{i, e\}$ is defined$\}$,
z_j: a binary variable equal 1 if potential stop j used as a stop and otherwise equal 0,
x_{ij}: a binary variable equal 1 if arc $\{i, j\}$ is on the solution path and otherwise equal 0,
y_k: a binary variable equal 1 if demand k is covered and otherwise equal 0,
a_k: the demand of area k,
d_{ij}: the network distance between stop i and j when directly connected,
p: the number of stops to locate.

The model is as follows:

$$\text{Max} \sum_{k} a_k \cdot y_k \tag{339}$$

$$\text{Min} \sum_{i} \sum_{j} d_{ij} x_{ij} \tag{340}$$

S.T.

$$\sum_{j \in N_t} x_{tj} = z_t \quad \forall\ t \in T \tag{341}$$

$$\sum_{i \in N_e} x_{ie} = z_e \quad \forall\ e \in E \tag{342}$$

$$\sum_{i \in N_j, i \neq e, j} x_{ij} - \sum_{l \in N_j, l \neq t, j} x_{jl} = 0 \quad \forall\ j, j \neq t, e \tag{343}$$

$$\sum_{j \neq t, e} z_j = P \tag{344}$$

$$z_j - \sum_{i \neq j} x_{ij} \leqslant 0 \quad \forall\ j, j \neq t \tag{345}$$

$$z_i - \sum_{i \neq j} x_{ij} \leqslant 0 \quad \forall\ i, i \neq e \tag{346}$$

$$y_k \leqslant \sum_{j \in M_k} z_j \quad \forall\ k \tag{347}$$

$$\sum_{j \in C_r} z_j \leqslant 1 \quad \forall\ r \in R \tag{348}$$

$$z_j = 0, 1; \quad y_k = 0, 1; \quad x_{ij} = 0, 1 \qquad \forall\ i, j, k \tag{349}$$

The objective functions (339) and (340) maximize total coverage demand and also minimize the length of route respectively. Constraint (341) ensures that in an existing route, a beginning node t can be considered as a potential starting point in the route extension. Constraint (342) states that an ending node e will be in established route if it is selected. Constraints (343) ensure the outgoing flows of all entering arcs related to each node. Constraints (344) state the number of required stops. Constraints (345) and (346) ensure that any chosen stops are located on connected paths. Constraints (347) states that demand k can be covered if there is one or more sited stops for serving it. Constraints (348) are pair wise clique constraints while the last constraints (349) are integral constraints.

3.17. Bi-objective: primary and secondary coverage

Kim and Murray (2008) develop a model to *maximize primary and secondary coverage* where the number of serving facilities on a continuous space is given. They represent a bi-objective model including binary variables. The objective functions maximize actual primary and backup coverage respectively, i.e. they are considering error and uncertainty reduction. They test the model for the bus route system in Columbus, Ohio. They, hence, deal with the response to the modifiable Areal Unit Problem (MAUP) in the context of multi-objective facility location problem (Farahani, SteadieSeifi, & Asgari, 2010). Consider the following notation:

i: index of demand
j: index of potential facility locations,
N_i: $\{j|\varepsilon_{ij}=1\}$,
ε_{ij}: a binary variable equal 1 if demand i is covered by a potential facility j, otherwise 0,
z_i: a binary variable showing the demand i is covered (=1) or not (=0),
x_j: a binary variable showing whether the facility is located at j (=1) or not (=0),
u_i: a binary variable showing whether the demand i is covered twice (=1) or not (=0),
a_i: the importance of demand i,
P: the number of facilities to be located.

The backup coverage location problem is as follows:

$$\text{Max} \sum_i a_i z_i \tag{350}$$

$$\text{Max} \sum_i a_i u_i \tag{351}$$

S.T.

$$\sum_{j \in N_i} x_j - z_i - u_i \geqslant 0 \quad \forall\ i \tag{352}$$

$$u_i - z_i \leqslant 0 \quad \forall\ i \tag{353}$$

$$\sum_j x_j = P \tag{354}$$

$$u_i = \{0, 1\} \quad \forall\ i \tag{355}$$

$$z_i \in \{0, 1\} \quad \forall\ i \tag{356}$$

$$x_j \in \{0, 1\} \quad j = 1, \ldots, n \tag{357}$$

The objective functions (350) and (351) maximize primary and backup coverages. Constraint (352) quantifies the number of facilities. Constraint (353) ensures that backup coverage can be existed only after receiving primary coverage. Constraint (354) states the number of facilities to be located. The last three constraints (355)–(357) are integral constraints.

3.18. Circular covering

Matisziw and Murray (2009) develop two algorithms to site a point facility with *circular covering radius* on a non-convex region with uniformly distributed demand in order to find the biggest radius. Naturally, their algorithm can also be used to solve the problem with convex region which, they call as *Continuous Maximal Covering Problem: 1-facility* (CMCP-1). The notation of the model is as follows:

k: index for continuous component of $\overline{\Omega}$,
$G \subset R^2$: the region that maximal coverage of demand is desired,
$\overline{\Omega} = V \cap G$: the portion (s) of region that facility covers,
Ω_k: kth continuous component of $\overline{\Omega}$ ($\overline{\Omega} = \cup_k \Omega_k$),
(x, y): the decision variable for location of facility,
$\delta(G)$: continuous demand distribution function within G, and
S: the service area of facility sited at (x, y).

$$\text{Max}_{(x,y)} \sum_k \int \int_{\Omega_k} \delta(G) d(G) \tag{358}$$

It is assumed that $\forall (x, y) \in G$ can be a potential facility sites. The CMCP-1 identifies the location of facility so that maximum amount of continuous demand $\delta(G)$ in G is covered within a specified service area.

3.19. Continuous demand on the plane

One of the derivations in covering problems can be continuous covering problems that can have many applications in real-world problems as well. While few updated references related to this area can be found (e.g. Drezner & Suzuki, 2009), most of the research works on covering problems have been limited to discrete and network space. Nevertheless, a discrete space can sometimes be estimated to be a continuous space. A fairly comprehensive reference regarding continuous covering problems is Plastria (2002). He lists in his book chapter variants of continuous location problem as follows: Full covering, Maximal covering, Empty covering, Minimal covering, Push–Pull covering models, Positioning models and multiple facility covering models.

Drezner and Suzuki (2009) solve a covering problem where demand is *continuous*. In the problem, radius and number of facilities are given. They find the location of facilities to maximize the covered area. They investigate on the *plane* and solve the problem for locating up to P = 50 facilities.

3.20. Covering reliability

Berman, Krass, and Wang (2006) present a model to obtain the minimum number of facilities whereas *lost demand* is reduced. They believe that there are two causes to loss demand:

(i) Insufficient coverage when a facility is located far from customers and (ii) Congestion when customers have to spend much time in queue. Therefore, they consider up-levels of lost demand for both causes. This problem is called *fixed facility Location Problem with Stochastic Demands and Congestion* (LPSDC). The notation of the model is as follows:

i: index of demand points,
j, l: indices of candidate facility locations,
k: index of coverage levels,
V: the set of nodes on a network,
E: the set of edges on a network,
G: $G = (V,E)$,
J: the set of candidate facility locations ($J \subset V$),
$\left\{b_i^k\right\}$: the set of K coverage levels for node i $\left(b_i^1 = 1 > b_i^2 > \cdots > b_i^k \geqslant 0\right)$,
$\left\{S_i^k\right\}$: the set of $K+1$ coverage radii for node $i\left(0 = S_i^0 < S_i^1 < S_i^2 < \cdots < S_i^k = \infty\right)$,
x_j: a binary decision variable, it is 1 if a facility is located at node $j \in J$,
z_{ij}: a binary decision variable, it is 1 if a facility is located at node j serves node i,
λ: the arrival rate of aggregate demands at nodes of the network,
μ: number of customers serves per unit of time by the server,
a: the capacity of the system,
h_i: the demand weight at node i,
d_{ij}: the shortest distance between any two nodes $i,j \in G$,
C_{ij}: the coverage level of node i from a facility located at $j \in G$,
$U1$: the upper bound for proportion of demands lost due to congestion,
$U2_i$: the upper bound on the fraction of demands that is lost due to insufficient coverage at node i,
$\lambda_{j,U1}$: the unique arrival rate which ensures the fraction of lost demand due to congestion equals to $U1$, and
M: $M = \text{Max}\ \{d_{ij},\ i \in V,\ j \in J\}$ is the maximum distance between any demand nodes, and
K: number of coverage level.

By considering following assumptions:

- Aggregate customers demand arrival is Poisson process.
- Depending on the distance between facility and node i, customers are covered at different levels which obtain as $C_{ij} = b_i^k\ if\ S_i^{k-1} \leqslant d_{ij} \leqslant S_i^k k = 1,\ldots,K$.
- Each Facility serves the customers as a $M/M/1/a$ queuing system.

The model is as follows:

$$\text{Min} \sum_{j \in J} x_j \tag{359}$$

S.T.

$$\sum_{j \in J} z_{ij} = 1 \quad \forall\ i \in V \tag{360}$$

$$x_j \geqslant z_{ij} \quad \forall\ i \in V,\ j \in J \tag{361}$$

$$\sum_{j \in J} z_{ij} d_{ij} + (M - d_{il}) z_l \leqslant M \quad \forall\ i \in V,\ l \in J \tag{362}$$

$$\sum_{j \in J} z_{ij}(1 - C_{ij}) \leqslant U2_i \quad \forall\ i \in V \tag{363}$$

$$\sum_{i \in N} z_{ij} C_{ij} h_i \lambda \leqslant \lambda_{j,U1} \quad \forall\ j \in J \tag{364}$$

$$x_j, z_{ij} \in \{0,1\} \quad \forall\ i \in N,\ j \in J \tag{365}$$

Constraint (360) expresses that each node is served by only one facility. Constraint (361) guarantees that a node is served by only open facility. Constraint (362) ensures that customers are served from the closest facility. Constraints (363) and (364) limit demand lost due to insufficient coverage.

Berman, Huang, Kim, and Menezes (2007) consider problem of locating set of facilities on a network to maximize captured demand. Since demands are stochastic and also congestion exists at facilities, maximization of expected captured demand is investigated. In this problem, customers travel to their closest facility to get service. In case the facility is full, they will go to the next-closest facility. They propose an iterative approximation procedure to calibrate the demand rates called *Demand Assignment and Calibration* (DAC) and develop two heuristics for the problem. The notation of the model is as follows:

i: index of nodes,
j, k: indices of facilities,
V: set of nodes on a network,
E: set of edge on a network,
G: $G = (V,E)$,
S: set of facilities ($S = \{1,\ldots,p\}$),
L: multi-set of locations ($L = \{L(1),\ldots,L(p)\} = \{L_1,\ldots,L_p\}$),
$L(j)$: location of facility $j \in S$,
$S^{(i)}$: set of sequence of facilities to be visited by customer from node i $\left(S^{(i)} = \left\{S_1^{(i)}, S_2^{(i)}, \ldots, S_p^{(i)}\right\}\right)$,
$L^{(i)}$: set of locations correspond to $S^{(i)}\left(L^{(i)} = \left\{L_1^{(i)}, L_2^{(i)}, \ldots, L_p^{(i)}\right\}\right)$,
$B_i\left(S_j^{(i)}\right)$: long run fraction of time that facility $S_j^{(i)}$ presents service to demand originating from node i, while facilities $S_0^{(i)}, S_1^{(i)}, \ldots, S_{j-1}^{(i)}$ are not available,
λ: the arrival rate of demand process,
μ: the number of customers serves per unit of time,
h_i: the fraction of total population associated with node i,
n: number of nodes on a network,
p: number of facilities to be located,
d_{ij}: the shortest distance between any two point $i, j \in G$,
$f(d_{ij}) : f(d_{ij}) = \max\{1 - d_{ij}/(\alpha d_{\max}), 0\}$ is the fraction of demands from any node willing to visit facility j from customer's current location i,
c: the capacity of the system,
α: the distance sensitivity of customers, and
$\alpha d_{\max}$: the threshold distance.

Consider the following assumptions:

- demand arrival processes at node i is distributed according to Poisson process with mean λh_i (it is assumed that $\lambda = 1$),
- each facility serves the customers as a $M/M/1/c$ queuing system, and
- customers do not visit the facilities beyond the threshold distance.

The objective function of the model is as follows:

$$\text{Max} \sum_{i \in N} h_i \sum_{j=1}^{p} B_i\left(S_j^{(i)}\right) \prod_{k=1}^{j} f\left(d\left(L_{k-1}^{(i)}, L_k^{(i)}\right)\right) \tag{366}$$

This function maximizes the total expected demand captured by P facilities.

3.21. Defensive covering

Church, Scaparra, and Middleton (2004) propose that problems in the facilities and their services can be *lost* due to natural or man-made *disasters* (international strike). They call this intentional strike against a system "interdiction". They also deal with supply or service system including components that are vulnerable to interdiction. They propose two new spatial optimization models called (i) the Maximal Covering Problem and the *r-Interdiction Covering* (RIC) *problem* (ii) the *p*-Median Problem and the *r*-Interdiction/ Median Problem. The first model is related to this review paper. The *r*-Interdiction Covering problem is defined as follows: "Of the *p* different service locations, find the subset of *r* facilities, which when removed, maximizes the resulting drop in coverage".

Consider the following notation:

i: The index of demand points,
j: index of existing facility locations,
J: the set of existing facilities locations,
N_i: $N_i = \{j|\text{site } j \text{ covers demand } i\}$,
x_j: a binary decision variable; it is 1 if a facility located at $j \in J$ is eliminated,
z_i: a binary decision variable; it is 1 if demand point i is no longer covered,
h_i: measure of demand at node i, and
r: the number of facilities that must be eliminated. The RIC model is formulated as follows:

$$\text{Max} \sum_i h_i z_i \tag{367}$$

S.T.

$$z_i \leqslant x_j \quad \forall\, i, \ \ \forall\, j \in N_i \cap J \tag{368}$$

$$\sum_{j \in J} x_j = r \tag{369}$$

$$z_i, x_j \in \{0,1\} \quad \forall\, i,j \tag{370}$$

The objective function maximizes the amount of demands that are not covered after interdiction. Constraint (368) indicates that $z_i = 0$ unless each facility site in J that covers i is eliminated. Constraint (369) imposes limits on the number of facilities that must be eliminated.

They use CPLEX and the Swain data set (Swain, 1971) including 55 demand points to solve both problems.

Berman, Drezner, Drezner, and Wesolowsky (2009) present *defensive maximal covering problem*. They consider locating *p* facilities on the nodes of a network by constructing a leader–follower problem. The problem assumes that the leader wants to locate facilities to maximize coverage of demand existing on the nodes whereas the follower likes disconnect the most damaging link when the leader's *p* locations are determined. Consider the following notation:

i, j: indices of nodes on a network,
k: index of links on a network,
V: The set of nodes on a network ($|V| = n$),
E: The set of links on a network ($|E| = m$),
G: $G = (V,E)$,
z_{ij}: a binary decision variable; it is 1 if i covers node$j$,
z_{ijk}: a binary decision variable; it is 1 if i covers node j when link k is missing,
x_i: a binary decision variable; it is 1 if a facility is located at node i,
y_j: a binary decision variable; it is 1 if node j is covered when all links are usable and is not covered if the selected link is missing,
y_{jk}: a binary decision variable; it is 1 if node j is covered when link k is missing,
I_k: a binary decision variable; it is 1 if k is removed,
M: the maximum cover,
p: number of facilities,
n: number of nodes, and
m: number of links.

The follower's problem is formulated as:

$$\text{Max} \sum_{j \in V} h_j y_j \tag{371}$$

S.T.

$$1 - \frac{1}{p} \sum_{k \in E} \left\{ I_k \sum_{i \in V} z_{ijk} x_i \right\} \geqslant y_j \quad \forall\, j \in V \tag{372}$$

$$\sum_{i \in V} z_{ij} x_i \geqslant y_j \quad \forall\, j \in V \tag{373}$$

$$\sum_k I_k = 1 \tag{374}$$

$$y_j, I_k \in \{0,1\} \quad \forall\, j \in V, \ \ k \in E \tag{375}$$

The leader's problem is as follows:

$$\text{Max } \{M\} \tag{376}$$

S.T.

$$\sum_{j \in V} h_j y_{jk} \geqslant M \quad \forall\, k \in E \tag{377}$$

$$\sum_{i \in V} z_{ijk} x_i \geqslant y_{jk} \quad \forall\, j \in V, \ \ k \in E \tag{378}$$

$$\sum_i x_i = P \tag{379}$$

$$x_i, y_{jk} \in \{0,1\} \quad \forall\, j \in V, \ \ k \in E \tag{380}$$

3.22. Cooperative covering problem

Berman, Drezner, and Krass (2010a) focus on cooperative location problems including covering problems. In this problem, each facility sends a signal, the strength of which depends on the distance. On the other hand, demand points receive the aggregated emitted signals. A demand point is covered if its aggregate signal is more than a given threshold. Their paper focuses on cooperation of facilities and shows that the results in this situation, where each demand point receives signal from all combination of facilities, are better than coverage in traditional case where coverage is only provided by the closest facility. They mention different applications for this problem and solve the problem on the plane using real data in North Orange County, California. They call these problems *Cooperative Location Set Cover Problem* (CLSCP) and *Cooperative Maximum Covering Location Problem* (CMCLP). The notation of the model is as follows:

i: index of demand points,
j, k: indices of candidate facility locations,
N: The set of demand points,
X: the location space,
$N(x,S,p)$: $N(x,S,p) = \{i \in N | \phi_i(x) \geqslant S\}$ is the set of all covered points,
x_j: the unknown location of facility $j \in \{1,\dots,p\} (x_j \in X)$,
x: the facility location vector $x = (x_1,\dots,x_p)$,
p: the number of facilities to be located (known or unknown),
h_i: the weight associated with demand point $i \in N$,
H: $H = \sum_{i \in N} h_i$ is the total available demand,
$d_i(x_j)$: the distance between demand point $i \in N$ and facility j,
$\phi(d)$: the strength of the signal at distance d from the facility,
$\phi_i(x)$: $\phi_i(x) = \sum_{k=1}^{p} \phi(d_i(x_k))$ is an overall signal at point $i \in N$,
S: The threshold of coverage,
$C(x,S,p)$: $C(x,S,p) = \sum_{i \in N(x,S,p)} h_i$ is the total coverage, and
α: the proportion of total demand covered.

The CLSCP model minimizes the number of facilities p and their location x required to cover a specified proportion of total demands. The model is as follows:

$$\underset{x}{\text{Min}}\{P\} \tag{381}$$

S.T.

$$C(x,S,p) \geqslant \alpha H \tag{382}$$

The CMCLP, for a given number p of facilities and threshold S, finds the location x to cover maximum $C(x,S,p)$. The model is as follows:

$$\underset{x}{\text{Max}}\{C(x,S,p)\} \tag{383}$$

3.23. Variable covering radii

Berman, Drezner, Krass, and Wesolowsky (2009) consider a covering problem where *covering radius* is a *variable*. They interest to find optimum radii beside number and location of facilities. The objective function minimizes the cost of locating facilities with respect to covering all demand points. They present mathematical planer and discrete models and solutions for the discrete problem and heuristic approaches to solve large scale problems in the plane. The planer model is as follows:

i: index of demand points,
j: index of facility locations,
p: the unknown number of facilities,
X_j:$X_j = (x_j, y_j)$ is the unknown location of facilities j ($j = 1,\ldots,p$),
X: The vector $\{X_j\}$,
Z_{ij}: a binary decision variable; it is 1 if demand point i is assigned to facility j,
Z: the matrix $\{Z_{ij}\}$,
S_j: $S_j = Max_{1\leqslant i\leqslant n}\{Z_{ij}d_i(X_j)\}$ is the unknown coverage radius for facility j,
P: $P = \{p, X, (S_1,\ldots,S_p)\}$ is the facility set which indicates the number, locations and coverage radii of the facilities,
n: the number of demand points,
(a_i, b_i): the location of demand point i,
$d_i(X_j)$: the Euclidean distance between demand point i and facility j,
F: the non-negative fixed cost of locating one facility, and
$\phi(S)$: the variable cost of locating a facility of radius S.

The model is as follows:

$$\text{Min } F(X,Y,P) = PF + \sum_{j=1}^{p} \varphi(Max_{1\leqslant i\leqslant n}\{Z_{ij}d_i(X_j)\}) \tag{384}$$

S.T.

$$\sum_{j=1}^{p} Z_{ij} = 1 \tag{385}$$

$$Z_{ij} \in \{0,1\} \quad \forall\, i \in \{1,\ldots,n\},\ j \in \{1,\ldots,p\} \tag{386}$$

For discrete variable radius covering model, it is assumed that there is a finite set of potential facility locations. Consider the following notation:

i, j: indices of nodes on a network,
N: the set of nodes on a network,
x_j: a binary decision variable, it is 1 if a facility is located at node $j \in N$,
z_{ij}: a binary decision variable, it is 1 if node i is assigned to node j,
$\phi_j(S)$: the cost of locating a facility with coverage radius S at node j,
F_j: the fixed cost of locating a facility at node j, and
d_{ij}: the shortest distance between $i, j \in N$.

The mathematical model is as follows:

$$\text{Min} \sum_{j=1}^{n} F_j x_j + \sum_{j=1}^{n} \phi_j(\text{Max}_{i=1,\ldots,n} d_{ij} z_{ij}) \tag{387}$$

S.T.

$$\sum_{j=1}^{n} z_{ij} = 1 \quad \forall\, i \in N \tag{388}$$

$$z_{ij} \leqslant x_j \quad \forall\, i,j \in N \tag{389}$$

$$x_j, z_{ij} \in \{0,1\} \quad \forall\, i,j \in N \tag{390}$$

Constraint (388) indicates that each node is assigned to a facility. Constraint (389) ensures that node i is not assigned to node j unless there is a facility on node j.

3.24. Disk cover

Berg, Cabello, and Har-Peled (2006) study two continuous facility-location problems in their book chapter: (i) placing m unit disks, to maximize the total weight of the points inside the union of the disks and (ii) placing a single disk with center in a given constant-complexity region X to minimize the total weight of the points inside the disk. They develop approximation algorithms for these problems.

Table 2
Characteristics to classify covering problems.

Characteristics			
Model notes	Sub-characteristics	Description or classification	Symbol
Coverage	–	Total	T
		Partial	P
System parameters	Topological structure	General network	N (G)
		Tree network	N (T)
		Plane	P
	Travel time or distance	Deterministic	D
		Probabilistic	P
		Stochastic	S
		Fuzzy	F
Characteristics of facilities	Potential location	Finite	F
		Infinite	I
	Facilities capacity	Capacitated	C
		Uncapacitated	U
	Differentiation between facility	Homogenous	H
		Heterogeneous	N
	Desirability	Desirable	D
		Undesirable	U
		Semi-obnoxious	S
Characteristics of models	Number of periods	Single period	S
		Multi periods	M
	Objective function	Deterministic	D
		Probabilistic	P
		Stochastic	S
		Fuzzy	F
	Constraints	Deterministic	D
		Probabilistic/ Stochastic	P
		Fuzzy	F

3.25. Demand aggregation

In most covering problems, we need to know demand points, but usually aggregation of demand points is used in modeling. Naturally there will be an error in the aggregation. Emir-Farinas and Francis (2005) identify the magnitude of the error for planar covering location models. Francis, Lowe, Rayco, and Tamir (2009) discuss about aggregation approaches and present different aggregation error measures to identify their effectiveness.

Table 2 presents some characteristics to classify the above mentioned covering problems, which is mostly based on Schilling et al. (1993). Thereafter, the above papers are summarized in Table 3 based on these characteristics. Table 3 also includes researches that focus on defining a new covering problem. In other words, these works contribute towards changing assumptions, objective functions and constraints of classical covering models. In addition, most of them present mathematical models as a part of their contribution rather than stressing on solution techniques.

4. Solutions

4.1. Focus on models and solutions

- Bazaraa and Goode (1975) develop a heuristic for their quadratic SCP. This problem is a non-linear binary model; their iterative algorithm uses cutting planes to exclude both integer and non-linear solutions in each iteration.
- Handler and Mirchandani (1979) in their book "Location on networks" have not dedicated a chapter to covering problems. But in chapter 3 "Minimax locations (Centers) on a network" when they are concentrating on solution techniques for the classic "center problem" they also describe a technique that solves their problem in large scale through solving a series of small covering problems sequentially. It can be said that they are talking about a transformation between center problem and covering problem; with changing covering radius this technique can solve the center problem with minor computational efforts. We note that they assume there is a routine to solve covering problem in advance.
- Vasko and Wilson (1986) develop hybrid algorithms for WSCP and MCSCP problems and compare their efficiency against two common greedy heuristics algorithms. While the hybrid algorithms performance is better, they show that in the worst case WSCP is as difficult as MCSCP but usually MCSCP is more difficult than WSCP.
- Love et al. (1988) introduce some solution procedures like reduction rules, cutting plane procedure and Kolesar-Walker heuristic.
- Kolen and Tamir (1990) present mathematical models for the above mentioned multiple coverage problems and widely use balanced matrix (for the 0–1 covering matrix) to develop solution technique.
- For the acyclic network problems, Francis et al. (1992) present some rules including row infeasibility, row feasibility, row dominance and column dominance to reduce the size of the set covering problem.
- Current and Schilling (1994) develop a heuristic to generate a set of Pareto solutions to solve their bi-objective maximal covering tour problem. Since, the problem is a multi-objective decision making problem, they find a set of efficient solutions not just one ideal solution as there can be no ideal solution for the problem.
- Daskin (1995) describes set covering problem in detail and shows how this problem can be solved using branch-and-bound method. He also uses some heuristic methods like greedy adding algorithm and Lagrangian relaxation in solving the maximum covering location model.
- Gendreau et al. (1997a) develop a tabu search heuristic for their double coverage ambulance location problem using randomly generated data and real data that provides near-optimal solutions in the modest computing times.
- Gendreau et al. (1997b) develop an exact branch-and-cut algorithm and a heuristic for the covering tour problem.
- Plastria and Carrizosa (1999) design a low complexity algorithm for their bi-objective undesirable facility location problem. Complexity of this algorithm is $O(n^3)$ on the plane for a fixed feasible region and $O(n^2)$ for a planar network. Their algorithm can find all non dominated solutions.
- Yehuda (2001) presents an $O(|V|^2 + H)$ time algorithm for the partial set cover problem where V is the number of vertices and H is the size of the given hypergraph.
- Beraldi and Ruszczynski (2002) propose a special branch and bound algorithm and also several greedy heuristics to solve their probabilistic set covering problem.
- Bläser (2003) designs an exact algorithm for study of the k-partial vertex cover problem and shows this problem can be solved in polynomial time for a certain k.
- Karasakal and Karasakal (2004) develop a Lagrangian relaxation solution procedure for their gradual coverage problem; however they call their problem as MCLP in the presence of partial coverage.
- Berman and Krass (2005) propose an Integer Programming (IP) for Uncapacitated Facility Location Problem (UFLP). MCLP can be considered in this category of location problems. Their new formulation leads to improvement in computational time.
- Berman et al. (2006) develop an exact linear integer formulation, probabilistic search and tabu search heuristics for their *fixed facility* LPSDC problem.
- Huang, Liu, and Chandramouli (2006) develop *Linear Feature Covering Problem* (LFCP) with distance constraints. This problem consists of m polylines and n potential points located at a Cartesian plane. They consider polylines as lines composed of linear line segments. Also a critical distance S, a desired balance distance D, and an integer number P are given. They want to find P out of n new facilities so that the total length of those polylines lying within distance R of at least one point is maximized; this is the first objective function. The second objective function is distance balancing objective. They use a GIS to transform their bi-objective continuous problem to a discrete problem. They solve the discrete optimization problem by an ant algorithm. Since their problem is named Linear Feature Covering Problem (LFCP) they call the algorithm the LFCP-Ant algorithm.
- Chuang and Lin (2007) develop a genetic algorithm for their MEXLCP-DS model.
- Using "Big Triangle Small Triangle" (BTST) method, Drezner (2007) suggests a general upper bound for location problems; he applies the proposed procedure on nine different location problems including gradual covering problem and demonstrates the efficiency of his procedure.
- Jia et al. (2007) develop a genetic algorithm, a locate-allocate heuristic and a Lagrangian relaxation solution to solve their large scale maximal covering problem with multiple facility quantity-of-coverage and quality-of-coverage requirements.
- Berman and Huang (2008) develop three heuristics and metaheuristics for their Minimum Covering Location Problem with Distance Constraints (MCLPDC) problem: (i) a greedy algorithm, (ii) a tabu search and (iii) a Lagrangian relaxation technique.
- As mentioned before, Erdemir et al. (2008) present two kinds of formulations to consider demand produced in both nodes and paths. They show this implicitly and explicitly. For the explicit model, they develop *quadratic MCLP* formulation and solve it

using a greedy heuristic supported by simulated annealing. For the implicit model they develop a geometric approach and a heuristic to solve it.

- Rajagopalan, Saydam, and Xiao (2008) extend Marianov and ReVelle (1994) Q-PLSCP to consider dynamic situations. They call this model *Dynamic Available Coverage Location* (DACL). They use a reactive tabu search to solve their problem and use a real-world data from Mecklenburg County, North Carolina.
- Berman, Drezner, Drezner, et al. (2009) present an efficient solving approach for the follower's problem and an efficient algorithm to solve leader's problem by an ascent algorithm, simulated annealing and tabu search.
- Berman et al. (2010a) formulate CMSCP and CMCLP mathematically and solve them in the plane for the Euclidean distance measure. They develop optimal and heuristic algorithms for these problems. They also test the problems using a real-case data with discrete demand and many random test problems in the plane.
- Berman, Drezner, Krass, et al. (2009) extend maximal covering problem on a network that some of the weights can be negative. This problem can be considered as a combination of attractive and obnoxious facilities. In fact their problem is a gradual coverage that is solved using integer programming, heuristic and meta-heuristic (simulated annealing). Their research result distinguishes the simulated annealing is more efficient solution technique than tabu search and ascent algorithm to solve the problem.
- Berman et al. (2009) discuss MCLP on a network with negative weights. They formulate the problem using integer programming and compare the solutions resulted from ILOG CPLEX against some proposed heuristic algorithms (namely an ascent algorithm, and simulated annealing) over 40 test problems. They show that the simulated annealing approach provides better results.
- Chen and Yuan (2010) develop a tabu search algorithm for minimum Power Covering with Overlap (MPCO) problem. Computation results indicate that the proposed algorithm is capable of solving large problem.
- Curtin et al. (2010) propose solution process for PPAC model which is based on constraint method of multi objective programming.
- Drezner et al. (2010) solve the stochastic gradual cover location problem by the global optimization technique "Big Triangle Small Triangle" (BTST) approach, based on branch and bound algorithm and show the effectiveness of it by some computational results.
- Erdemir et al. (2010) propose a greedy heuristic algorithm on order to SCBM and MCGBM. Comparison with CPLEX indicates that the algorithm presents high quality solution much faster than CPLEX.
- Lee and Lee (2010) propose solution procedure for hierarchical facility location problem partial coverage. It consists of two phases. In the first phase, by employing hierarchical allocation method, initial solution is obtained and in second phase, based on tabu search, the solution is improved with location and allocation heuristic method.
- Naimi Sadigh et al. (2010) develop tabu search algorithm to solve complementary edge covering problem. Computational results indicate that proposed algorithm solve the problem up to 40 vertices and 456 edges optimally.
- O'Hanley and Church (2011) present a MIP and also bi-level-MIP formulation for maximum covering location-interdiction problem and use a decomposition algorithm to solve its bi-level-MIP formulation by showing that its speed is faster than branch and bound algorithm.
- Oztekin et al. (2010) present a methodology for RFID network design in healthcare facilities and use genetic algorithm to solve it. They test their methodology and present GA in Stillwater Medical Center.
- Sivan, Harini, and Rangan (2010) study the Conditional Covering Problem (CCP) on the paths and also interval graphs. The CCP considers a graph with the set of its vertexes represent as the demand nodes and also potential facility location. The model minimizes the location cost by covering all demand points in which each vertex cannot cover itself but only can cover the other vertexes lie in defined radius cover. They present a $O(n^3)$ algorithm for CCP on path, $O(\mathrm{n}^2)$ algorithm to solve CCP on paths when the cost is unity and also extension of it for CCP on interval graphs.
- Sorensen and Church (2010) use ILOG CPLEX mixed-integer program solver to solve MEXCLP, MALP and also LR-MEXCLP, result in convergence to optimality in all cases.
- Berman and Wang (2011) develop an algorithm for gradual covering location problem with complexity $O(n^4l^2)$. Also, by employing linear coverage decay function, an improved solution approach with complexity $O(n^2l^2 \log n)$ is proposed.
- Xia et al. (2010) develop hybrid nested partitions algorithm for banking facility location problem.

Table 4 presents the literatures that have concentrated on solution techniques for solving a variation of covering problems. Some of them do not have any contribution on defining a new problem. However, the papers that have contribution not only on problem definition and mathematical formulation but also in solution techniques are also included in Table 4. Therefore, Table 4 includes features and attributes more than that of Table 3. The additional parts consider solution specifications. However, we have tried to avoid duplication in references while preparing Tables 3 and 4. The contents of Table 4 are the same as in Table 3 except for few additions that are self-explanatory. However, the terms "Unit cost" and "Non-unit cost" in Table 4 are to be read thus: The papers including objective function that either clearly talking about cost (e.g. location fixed cost) or do not say anything about the dimension of weights (but the weights can be interpreted as cost) are categorized as "Unit cost". The other papers that either clearly mentioning the meaning of the weights (e.g. population) or are using equal costs/ weights so that WSCP is transformed to MCSCP are lying under "Non-unit cost".

4.2. Focus on solution

Since covering problems are very hard to solve (e.g. the simple SCP is a NP complete problem; Garey & Johnson, 1979) many algorithms have been developed to solve it.

4.2.1. The SCP

- Roth (1969) develops a probabilistic approach to obtain optimal solutions for large covering problem (up to $m \cdot n \leqslant 0.5 \times 10^6$) – refer to Section 2 for definition of m and n. However, this paper is not in a location context.
- Based on Bellmore and Ratlief (1971) an optimal solution to a weighted SCP is a basic feasible solution to its corresponding LP. They prove a property of the weighted SCP and propose a heuristic algorithm for it that can solve the problem up to 30×90.
- Guha (1973) present a list of reduction and additional rules to solve a set covering problem in which constraints (4) are the equality constraints.
- Church and ReVelle (1974) offer relaxed linear programming followed by branch and bound as needed to solve MCLP.

Table 3
List of literature with focus on new covering models.

Paper description	Model notes	The used symbol	Coverage	System parameters		Characteristics of facilities				Characteristics of models		
				Topological structure	Travel time/ distance/ cost	Potential locations	Facilities capacity	Differentiation between facility	Desirability	Number of periods	Objective function	Constraints
Alexandris and Giannikos (2010)	Partial coverage with spatial object	–	P	N (G)	D	F	U	H	D	S	D	D
Baron et al. (2009)	Stochastic demand and congestion	LPSDC	T	N (G)	D	F	U	H	D	S	D	P
Berman (1994)	Relation between p-maximal cover and p-partial center problems in a bi-objective model	–	P	N (T, G)	D	F	U	H	D	S	D	D
Berman and Krass (2002)	Generalized maximal covering location problem	GMCLP	T	N (G)	D	F	U	H	D	S	D	D
Berman and Wang (2008)	Probabilistic 1-maximal covering problem on a network	–	P	N (G)	D	F	C	H	D	S	P	D
Berman et al. (1996)	Minimum covering for obnoxious covering	–	T	N (G)	D	F	U	H	U	S	S	D
Berman et al. (2003)	Gradual covering decay location problem	–	P	N (G)	D	F	U	H	D	S	D	D
Berman, Huang, et al. (2007)	Maximize captured Poisson demand	DAC	P	N (G)	D	F	C	H	D	S	D	P
Berman, Drezner, Krass, et al. (2009)	Variable coverage radii	–	T	N (G), P	D	F, I	U	H	D	S	D	D
Boffey and Narula (1998)	Bi-objective multiple path or maximal population shortest path	2-MPSP	P	N (G)	D	F	U	H	D	S	D	D
Bläser (2003)	A polynomial time exact algorithm for the k-partial vertex cover problem	–	T, P	N (G)	D	F	U	H	D	S	D	D
Campbell (1994)	Integer programming formulation for hub covering	HCV, HCV-P, HMCV,	T	N (G)	D	F	U	H	D	S	D	D
Chiang et al. (2005)	An alternative formulation for Hwang et al. (2004)	–	T	N (G)	D	F	U	H	D	S	D	F
Church (1984)	The planar MCLP under Eclidean and rectilinear distances	PMCE, PMCR	P	D, P	D	F	U	H	D	S	D	D
Church et al. (2004)	r-interdiction covering in identifying critical infrastructure	RIC	P	D	D	F	U	H	D	S	D	D
Conforti et al. (2001)	Review on SCP matrix	–	T	N (G)	D	F	U	H	D	S	D	D
Current and Storbeck (1988)	Capacitated MCLP and SCP	–	T, P	N (G)	D	F	C	H	D	S	D	D
Daskin et al. (1988)	Backup coverage	MEXLCP	T	N (G)	D	F	U	H	D	S	P	D
Drezner and Suzuki (2009)	Continuous demand in a convex polygon	–	P	P	D	I	U	H	D	S	D	D
Drezner et al. (2004)	Gradual covering on the plane	–	P	P	D	I	U	H	D	S	D	D
Hogan and ReVelle (1986)	Introducing backup coverage	BACOP 1 & BACOP 2	T	N (G)	D	F	U	H	D	S	D	D
Hutson and ReVelle (1993)	Indirect set covering and maximal covering	MCCS, MICS	T, P	N (T)	D	F	U	H	D	S	D	D
Hwang (2002)	Stochastic set covering problem	–	T	N (G)	D	F	U	H	D	S	D	P
Hwang et al. (2004)	Fuzzy SCP	–	T	N (G)	D	F	U	H	D	S	D	F
Kara and Tansel (2003)	Formulations for hub covering integer programming	–	T	N (G)	D	F	U	H	D	S	D	D
Kim and Murray (2008)	Maximizing primary coverage and secondary backup coverage objectives in actual area	MAUP, BCLP	P	P, D	D	F	U	H	D	S	D	D
Lee and Murray (2010)	Maximize coverage with survivability constraint	MCSC	P	N (G)	D	F	U	H	D	S	D	D
Liu (1993)	Multi-criteria SCP	MCSC	T	N (G)	D	F	U	H	D	S	D	D
Marianov and ReVelle (1994)	Independence server availability in PLSCP	Q-PLSCP	T	N (G)	D	F	U	H	D	S	P	D

Matisziw and Murray (2009)	A facility on continuous non-convex area	CMCP-1	P	P	D	I	U	1H	D	S	D	D
Matisziw et al. (2006)	Route extension in transit service	MCREP	P	N (G)	D	F	U	H	D	S	D	D
Moore and ReVelle (1982)	Hierarchical service covering	–	P	D	D	F	U	N	D	S	D	D
Murray (2005)	Set covering with spatial objectives	SCP-SO	T	D	D	F	U	H	D	S	D	D
Murray and Tong (2007)	Continuous space covering	EPMCE	P	P	D	F	U	H	D	S	D	D
Murray et al. (2008)	Representation a study region in a digital environment	LSCP	T	P	D	I	U	H	D	S	D	D
Murray et al. (2010)	Enhancing classic coverage location models		T,P	N (G)	D	F	U	H	D	S	D	D
Nozick (2001)	Consider fixed charge and coverage restriction	–	P	N (G)	D	F	U	H	D	S	D	D
Ohsawa and Tamura (2003)	Bi-criteria semi-obnoxious covering to locate a facility on a continuous plane	–	P	P	D	I	U	1H	S	S	D	D
Ohsawa et al. (2006)	Bi-criteria semi-obnoxious covering to locate a facility on a continuous convex polygon and Euclidian distances	–	P	P	D	I	U	1H	S	S	D	D
ReVelle and Hogan (1989a)	Maximum available location problem	MALP I, MALP II	P	N (G)	D	F	U	H	D	S	P	D
ReVelle and Hogan (1989b)	Probabilistic location set covering problem	PLSCP	T	N (G)	D	F	U	H	D	S	D	P
Serra (1996)	Coherent covering location problem	–	T	N (G)	D	F	U	H	D	S	D	D
Toregas et al. (1971)	A formulation for locating emergency service facilities	–	T	N (G)	D	F	U	H	D	S	D	D
Williams (2003)	Four formulations for direct and indirect covering threes considering multiple parents	–	T, P	N (T)	D	F	U	H	D	S	D	D
Yehuda (2001)	An algorithm for partial set cover problem	–	T, P	N (G)	D	F	U	H	D	S	D	D

They propos greedy adding method and greedy adding with substitution.

- Francis and White (1974) explain mathematical models, exact solutions (in particular a special branch-and-bound method and also cutting plane approach) and a heuristic method for two main group of covering problems (set covering and maximal covering).
- Johnson (1974) analyzes simple polynomial time heuristic algorithms for several certain problems including SCP with respect to their worst case behaviors. Johnson shows that using simple algorithms for SCP, the worst case ratio grows with the log of the problem size.
- Etcheberry (1977) presents a special branch and bound algorithm for SCP problem based upon Lagrangian relaxation with sub-gradient optimization that can solve problems up to 70×400.
- Chvatal (1979) develops a simple greedy heuristic to solve SCP and compares the value of objective function for the heuristic to the optimum. Chvatal shows that in the worst case the ratio between these two grows logarithmically in terms of the largest column sum of the covering matrix.
- Beasley (1987) presents a three-stage to solve SCP consisting of a dual ascent procedure, a sub-gradient (which is started using Lagrange multipliers equal to the dual variables from the dual ascent procedure) and solving the dual of the LP relaxation of the SCP. This algorithm can solve the problem up to 400×4000.
- Beasley (1990) presents a heuristic algorithm for SCP based upon Lagrangian relaxation and sub-gradient optimization. This algorithm can solve the problems up to $1000 \times 10{,}000$.
- Fisher and Kedia (1990) develop a branch-and-bound algorithm for a mixed set covering/ partitioning model. They use a continuous heuristic and apply it to the dual of LP relaxed model to find a lower bound in the branch-and-bound algorithm. Using this algorithm they solve the problems up to 200×5000.
- Beasley and JØrnsten (1992) enhance Beasley's (1987) algorithm to solve SCP. They compare these two algorithms using the same test problems (up to 400×4000). This improvement is based upon Lagrangian heuristic, exclusion of feasible solution constraints, using Gomory f-cuts and improving the branching strategy.
- Christofides and Paixão (1993) try to solve large-scale SCP. They consider large problems with sparse matrix up to 20% non-zero elements. First, they do some reduction on columns and rows using Lagrangian Relaxation, linear programming and heuristics method. Then they develop the algorithm which is a tree-search procedure; this procedure works based on a combination of decomposition and state space Relaxation (SSR). They use this algorithm to solve the problems up to 400×4000.
- Lorena and Lopes (1994) develop a heuristic to solve SCP. This heuristic combines three routines: reduction tests, step size control and continuous surrogate relaxation. They solve large scale problems up to $1000 \times 12{,}000$ with 5% density before reduction.
- Afif, Hifi, Paschos, and Zissimopoulos (1995) develop a heuristic to solve SCP. In fact, they develop a transformation to convert SCP to flow problem that can be solved using traditional Ford and Fulkerson algorithm. This algorithm is a polynomial time algorithm.
- Jacobs and Brusco (1995) develop a local-search heuristic large non-unicost SCP. This algorithm works based on Simulated Annealing (SA). They test efficiency of their algorithm using large SCP test problem in the literature.

Table 4
Literature with focus on new covering models and solutions.

Paper description	Model notes	The used symbol	Coverage	System parameters		Characteristics of facilities				Characteristics of models			Objective function		Solution technique	The largest (test) problem solved (m rows: # demand points × n columns: # potential location)
				Topological structure	Travel time/ distance	Potential locations	Facilities capacity	Differentiation between facility	Desirability	Number of periods	Objective function	Constraints	Non-cost	Cost		
Abravaya and Segal (2010)	Solution for an unknown number of obnoxious facilities	–	P	P	D	I	U	H	U	S	D	D	√		H	
Battino and Laurentini (2011)	Algorithm for locating visual sensors	–	T	P	D	I	C	H	D	S	D	D		√	H	–
Bazaraa and Goode (1975)	Quadratic SCP	–	T	N (G)	D	F	U	H	D	S	D	D		√	H	60×35
Beraldi and Ruszczynski (2002)	A probabilistic SCP using a special branch and bound and greedy heuristics	–	T	N (G)	D	F	U	H	D	S	D	P	√	√	E, H	–
Berman and Huang (2008)	Solutions for minimum weighted undesirable covering	MCLPDC	P	N (G)	D	F	U	H	U	S	D	D	√		H, M, E	15×500
Berman and Krass (2005)	Partitioning potential location for each customer	UFLP	T	N (G)	D	F	U	H	D	S	D	D		√	E	–
Berman and Wang (2011)	Gradual covering location with probabilistic demand weight	–	P	N (G)	D	F	U	H	D	S	P	D	√		H	–
Berman et al. (2006)	A Heuristic and a tabu search for location problems with stochastic demands and congestion	fixed facility LPSDC	P	N (G)	D	F	C	H	D	S	D	D		√	H, M, E	100×100
Berman, Drezner, Drezner, et al. (2009)	Bi-level defensive covering and possibility of link removal	–	P	N (G)	D	F	U	H	D	S	D	D	√		H, M	900×200
Berman et al. (2010a)	Cooperative covering problem	CLSCP, CMCLP	P, T	N (G), P	D	F, I	U	H	D	S	D	D	√		E, H	1000×100
Berman et al. (2009)	Covering with negative weights using TS, SA and ascent	–	P	N (G)	D	F	U	H	S	S	D	D	√		H, M	900×90
Cardinal and Hoefer (2010)	Non-cooperative SCP	–	T	N (G)	D	F	U	H	D	S	D	D		√	H	
Chen and Yuan (2010)	Minimum required power with overlap constraint	MPCO	T	P	D	I	U	H	D	S	D	D		√	H	–
Chuang and Lin (2007)	GA for MEXCLP in an ambulance location problem	MEXCLP-DS	T	N (G)	D	F	U	H	D	S	P	D	√		M	–
Current and Schilling (1994)	A bi-objective maximal covering tour problem using a heuristic approach	MCTP	P	N (G)	D	F	U	H	D	S	D	D	√		H	681 nodes
Curtin et al. (2010)	Back up coverage model	PPAC	P	N (G)	D	F	U	H	D	S	D	D	√		H	–
Drezner (2007)	General approach for constructing bounds in gradual covering	BTST	P	P	D	I	U	H	D	S	D	D	√		E	The number of demand points up to 10,000
Drezner et al. (2010)	A stochastic gradual cover location problem	BTST	p	P	S	I	U	H	D	S	S	D	√		E	The number of demand points up to 10,000

Drezner and Wesolowsky (1994)	Propose an algorithm	MCLP	T	P	D	I	U	H	U	S	D	D			H	
Erdemir et al. (2008)	Quadratic maximal covering location problem	MCLP	T	N (G)	D	F	U	H	D	S	D	D	√		M, H	150 × 120
Erdemir et al. (2010)	Set cover backup coverage model; Maximal cover for a given budget model	SCBM MCGBM	T P	N (G)	D	I	U	H	D	S	D	D	√	√	H	–
Gendreau et al. (1997a)	Double coverage ambulance location problem using tabu search	DSM	T	N (G)	D	F	U	H	D	S	D	D	Total demand		M	400 × 70
Gendreau et al. (1997b)	Covering tour problem using a heuristic and exact branch and cut	CTP	T	N (G)	D	F	U	H	D	S	D	D			E, H	600 × 100
Hakimi (1965)	Graph theoretic approach using a Boolean function	–	T	N (G)	D	F	U	H	D	S	D	D	√		E	–
Huang et al. (2006)	Fuzzy bi-objective linear feature covering problem with poly lines distance constraints using ant algorithm	LFCP	P	P	D	F	U	H	D	S	F	F	√		M	–
Jia et al. (2007)	Heuristics for large scale multiple facility quantity-of-coverage and quality-of-coverage requirements	MCLP	P	N (G)	D	F	U	N	D	S	D	D	√		H, M	2054 × 200
Karasakal and Karasakal (2004)	Partial coverage in MCLP using Lagrangian relaxation	MCLP-P	P	N (G)	D	F	U	H	D	S	D	D		√	H	1000 × 40
Kolen and Tamir (1990)	Introduction to covering	SCP, multiple covering	–	N (G)	D	F	U	H	D	S	D	D		√	–	–
Lee and Lee (2010)	Partial coverage for hierarchical location problem	–	P	N (G)	D	F	C	H	D	S	D	D	√		M	–
Naimi Sadigh et al. (2010)	Complementary edge covering problem with partial coverage	–	P	N (G)	D	F	C	H	D	S	D	D		√	M	Up to 40 vertices and 456 edges.
O'Hanley and Church (2011)	robust coverage networks to hedge against worst-case facility losses	MCLIP	P	N (G)	D	F	U	N	D	S	D	D	√		E	
Oztekin et al. (2010)	Asset tracking in healthcare using MCLP	–	P	N (G)	D	F	U	H	D	S	D	D	√		M	–
Plastria and Carrizosa (1999)	Bi-objective undesirable covering problem on continuous plane	MCLP	T	P	D	I	U	H	U	S	D	D	√		E	$O(n^2)$ and $O(n^3)$
Rajagopalan et al. (2008)	Dynamic available coverage location using a reactive TS	DACL	T	N (G)	D	F	U	H	D	M	P	D	√		M	–
Vasko and Wilson (1986)	Hybrid heuristic	MCSCP, WSCP	T	N (G)	D	F	U	H	D	S	D	D	√	√	H	200 × 500
Saxena et al. (2010)	MIP formulation for probabilistic SCP	–	T	N (G)	D	F	U	H	D	S	D	P	√		E	
Sivan et al. (2010)	Conditional Covering Problem on path	CCP	T	N (G)	D	F	U	H	D	S	D	D		√	E	$O(n^2)$ and $O(n^3)$
Sorensen and Church (2010)	Integrating expected coverage and local reliability for emergency medical services location problems	LR-MEXCLP	P	N (G)	D	F	U	H	D	S	D	D	√		H	–
Xia et al. (2010)	Hybrid nested partition algorithm	–	P	N (G)	D	F	U	H	D	S	D	D	√	√	H	The number of demand points is equal to 45,000

- Balas and Carrera (1996) present a branch-and-bound algorithm to solve SCP. This algorithm combine the standard subgradient method with primal and dual heuristics, thereby enabling it to solve problems up to 1000 × 10,000 with 5% density.
- Beasley and Chu (1996) develop a Genetic Algorithm (GA) metaheuristic for non-unicost SCP. They can solve problems up to 1000 × 10,000. Al-Sultan, Hussain, and Nizami (1996) also propose a genetic algorithm that can solve SCP problems up to 1000 × 12,000.
- Grossman and Wool (1997) try to compare efficiency of different algorithms to solve uni-cost SCP. They compare nine approximation algorithms including several greedy variants, fractional relaxations, randomized algorithms and a neural network algorithm. In this study, they consider test problems up to 500 × 5000 and two sets up to 28,160 × 5000. The results show that round and greedy algorithms perform better.
- Haddadi (1997) propose a Lagrangian heuristic for large but low density SCPs. This method is based on Lagrangian duality, greedy heuristic, sub gradient optimization and redundant covers that can solve problems as large as 1000 × 10,000.
- Ceria, Nobili, and Sassano (1998) develop a Lagrangian-based heuristic for solving large-scale SCP that can solve problems up to 4872 × 968,672. Their problem originates from a real-world case in crew scheduling of the Italian Railways Company. They make use of variable fixing in designing their algorithm.
- Brusco, Jacobs, and Thompson (1999) develop a Simulated Annealing (SA) meta-heuristic for a non-unicost SCP. This algorithm solves problems with up to 800 × 8000 with 10% density in covering matrix.
- Caprara, Fischetti, and Toth (1999) present a Lagrangian-based heuristic large scale SCP. In fact, they solve Ceria et al. (1998) problem for the Italian Railways Company in the same size, but their algorithm is more efficient. In the same way, based on the initial definition of a core problem (which here is updated dynamically), and on a primal–dual sub-gradient combined with column fixing they design the Lagrangian based heuristic.
- Caprara, Toth, and Fischetti (2000) do a survey with focus on the most recent and effective heuristics and exact approaches developed to solve SCP. They use the test problems on Beasley's OR Library which are up to 1000 × 10,000 with 5% density in covering matrix. They show that the state-of-the-art general-purpose ILP solvers like CPLEX (they had 4.0.8) and MINTO (they had 2.3) are competitive with the best exact algorithms in the literature. They also show that the performance of these ILP solvers can be improved using an external pre-processing procedure.
- Hifi, Paschos, and Zissimopoulos (2000) develop an approximate algorithm to solve WSCP. They consider ideas from neural network model, the Boltzmann Machine (BM), and combinatorial optimization. They show that the developed BM-model is more powerful than both GH (greedy SC algorithm) and LH (Lagrangian heuristic) when dealing with instance smaller than 100 nodes. The percentage of the optimal solutions provided by BM is largely higher than the others. Also, BM converges very fast and with linear function to the size of the instances.
- Ohlsson, Peterson, and Söderberg (2001) develop an Artificial Neural Network (ANN) to solve non-unicost SCP. This algorithm is a mean field feedback ANN that uses multi-linear penalty function to encode the inequality constraint. They solve some available test problems with sizes from 200 × 1000 up to 5000 × 1,000,000 with acceptable error in reasonable time.
- Solar, Parada, and Urrutia (2002) present a Parallel Genetic Algorithm (PGA) to solve the SCP. This algorithm tries to take the advantage of the inherent the parallelism GA to improve performance. They compare this algorithm with SA and TS using available test problems in OR Library with sizes from 200 × 1000 up to 500 × 5000.
- Gomes, Meneses, Pardalos, and Viana (2006) compare performances of different approximate algorithms including round, dual-LP, primal–dual, and greedy to solve the Vertex Cover Problem (VCP) and the SCP. For the VCP they used three algorithms and the performance ranking is greedy, dual-LP and round, respectively. For the SCP they implement all of the four algorithms. All of the algorithms are fast, and the greedy algorithm performs well in all instances. On average, the round and primal–dual perform worse than the two others.
- Lan and DePuy (2006) actually discuss several randomization methods and memory mechanisms for meta-heuristic for randomized property search (Meta-RaPS). They look for several effective algorithms to maintain a good balance between randomness and memory. They test their achievement on the SCP test instances. Based on their results, for the memory mechanisms, the method of partial construction performs better than priority rule with element fitness. Incorporating memory into Meta-RaPS can improve the solution quality for the construction phase.
- Yagiura, Kishida, and Ibaraki (2006) develop a local search algorithm, which is based on the 3-flip neighborhood (3-FNLS), for the SCP. This algorithm which includes implementation of the neighborhood search, reducing the neighborhood size, strategic oscillation mechanism based on the adaptive control of penalty weights, the size reduction and approach by using Lagrangian relaxation can solve problems from 200 × 1000 to 4872 × 968,672 and density from 2% (in smaller sizes) downwards to 0.2% (in larger sizes).
- Bautista and Pereira (2007) develop a GRASP algorithm to solve the unicost SCP. The algorithm uses a local improvement procedure to solve problems up to 28,160 × 11,264.
- Crema, Loreto, and Raydan (2007) develop a method to solve integer programming problems and Lagrangean dual problems; they combine the sub-gradient method with the spectral choice of step length and a momentum term. To show the efficiency of their algorithm they use small and medium sizes SCPs.
- Galinier and Hertz (2007) propose three exact algorithms for solving LSCP which is the large SCP (kind of SCP that potential locations are very large sets that are possibly infinite). Two of the algorithms (removal and insertion) determine minimal covers and the third one (hitting set) produces minimum covers. They also developed heuristic versions of these algorithms and analyzed them.
- Lan and DePuy (2007) present a heuristic algorithm to solve both unicost and non-unicost SCP. They apply basic Meta-RaPS (it has already been discussed) and a heuristic that introduces randomness into a greedy heuristic to construct a feasible solution. Test problems up to 100 × 10,000 (non-unicost) and 28,160 × 11,264 (unicost) has been tested on this algorithm.
- Saxena et al. (2010) present an algorithm to solve probabilistic set covering problem and implement it for the MIP form of it using linear programming module (OsiClp) of COIN-OR and also CPLEX (version 9.0) testing 10,000 probabilistic examples.
- Straitiff and Cromley (2010) present a methodology for solving location set covering problem considering distributed demand over a limited area and nondefined potential location for facilities. They use GIS and $K = 3$ central place lattices.
- Battino and Laurentini (2011) propose an algorithm to locate visual sensors in order to cover interior of polynomial environment. The algorithm is based on Edge Covering (EC) algorithm. By employing EC algorithm, the set S which covers edges is obtained and then if it does not cover interior of polynomial environment, S is modified

4.2.2. The MCLP

- Downs and Camm (1996) develop an exact algorithm to solve MCLP. In fact, they design a special branch-and-bound that uses dual-based method and a greedy heuristic. They test their algorithm on large networks with up to 25,000 nodes existing in the literature and also on three real cases.
- Galvão and ReVelle (1996) develop a Lagrangian heuristics for MCLP. They define upper bounds using vertex addition/ substitution method and lower bounds using a sub-gradient algorithm and can be used to solve problem on networks with up to 150 nodes.
- Resende (1998) develops a Greedy Randomized Adaptive Search Procedure (GRASP) for MCLP. Resende also presents an upper bound for the problem that is extracted using a LP. This algorithm solves problems with up to 9996 demand points with 0.75653% density in covering matrix.
- Galvão, Espejo, and Boffey (2000) try to compare heuristic algorithms designed for the MCLP which are based on Lagrangian and surrogate relaxations. They use test problems in the literature that are networks with 55–900 vertices. They show that there is no significant difference between these methods.
- Gandhi, Khuller, and Srinivasan (2001) develop an *f*-approximation polynomial time algorithm for partial covering that combines primal dual algorithm with a threshold method. Later, Gandhi, Khuller, and Srinivasan (2004) improve this algorithm by applying "level-sets" distribution.
- Abravaya and Segal (2010) propose an optimization algorithm and an efficient $\frac{1}{2}$-approximation algorithm for locating an unknown number of obnoxious facilities on the plane as many as possible. By applying optimization algorithm, based on shifting strategy, a PTAS is designated. In this problem minimum distance between each demand point and facilities is at least S and minimum distance between each pair of facilities is at least T.

4.2.3. Other extensions

- Aytug and Saydam (2002) develop genetic algorithm for solving large-scale MEXCLP against Daskin's heuristic. Their algorithm can solve problems with up to 1600 nodes on a network.
- Berman, Verter, and Kara (2007) show that in the area of HazMat transportation, the problem of quick arrival of response teams at the accident site can be represented as a maximal arc-covering model.
- ReVelle, Scholssberg, and Williams (2008) apply the meta-heuristic concentration to solve large MCLPs. They solve problems with 900 nodes and 20 existing facilities.

5. Applications and real-world cases

5.1. Applications

This section is dedicated to the literature that addresses application of covering problems in facility location without focusing on a special real problem in certain place of the world. In fact, covering problems were originally introduced by Toregas et al. (1971) to model and locate emergency service facilities (and especially fire stations) as a SCP with equal costs.

Schilling et al. (1993) in their review paper divide the applications into two main categories: (i) public sector and (ii) private sector, as decision makers in these two categories are looking for different objectives (Farahani, Asgari, & Davarzani, 2009). Public sectors like governmental organizations look for maximizing servicing to the people using limited resources like in MCLP; while private sectors look for maximizing (minimizing) their profit (cost) like in SCP.

- There are many instances of covering-type considerations incorporated in the literature like network interdiction, forest management, and ecological conservation. However, Schilling et al. (1993) mention following applications for covering problems:
  - Bus stop location.
  - Terrain visibility.
  - Fire company relocation.
  - Fire service sitting.
  - Fire equipment allocation.
- Pastor (1994) uses MCLP to locate a network of bank branches.
- Daskin (1995) explains two applications of set covering problem:
  - Airline crew scheduling.
  - Tool selection in flexible manufacturing systems.
- Boffey and Narula (1998) list examples of applications such as:
  - Oil spill equipment.
  - Archaeological settlement pattern.
  - Rural health centers.
  - Direct mail marketing.
  - Health care units.
  - Essential air services.
  - Ambulances.
  - Early detection of glaucoma.
  - Health care centers.
  - Disease eradication.
  - Emergency Medical Services (EMS) vehicles.

- Mandell (1998) develops a MEXCLP model for locating EMS systems.
- Badri, Mortagy, and Colonel (1998) develop a multiple conflictive objective model to locate fire stations.
- Freling, Huisman, and Wagelmans (2003) use SCP in vehicle and crew scheduling.
- Drezner et al. (2004, 2010) present some general applications of the gradual covering problem and it's stochastic kind including
  - The delivery problem.
  - Competitive location.
  - Dense competition.
  - The radio, TV, or cellular transmitter problem.
  - Medical facility location problem.
- Coslovich, Pesenti, and Ukovich (2006) use SCP as part of a solution technique for fleet management problem in container trucking industry.
- Hollis, Forbes, and Douglas (2006) use SCP as part of a solution technique for multi-depot combined vehicle and crew scheduling problem.
- Chuang and Lin (2007) use a combination of MEXLCP with DSM for an ambulance location problem to prove sufficient coverage of EMSs.
- Jans and Degraeve (2008) present use of SCP for lottery problem.
- Mirchandani and Li (2010) use SCP and MCLP models to locate a Surveillance Infrastructure in and Near Ports or on Other Planar Surfaces to Monitor Flows and develop Lagrangian heuristic and a two-stage procedure with a conquer-and-divide scaling to solve them.
- Sorensen and Church (2010) use LR-MEXCLP for emergency medical services location problems.
- Wang, Ghosh, and Daskin (2010) review on sensor localization problem categorized into proximity-based localization, range-based localization and angle-based localization.

Table 5
A summary of case studies in covering problems.

Paper description	Subject	Place
Current and O'Kelly (1992)	Emergency warning sirens	A mid western city
Osleeb and McLafferty (1992)	Facilities to fight Dracunculiasis (Guinea worm disease)	The Zou province, Benin
Repede and Bernardo (1994)	Emergency medical vehicles	Louisville, Kentucky
Ceria et al. (1998) and Caprara et al. (1999)	Crew scheduling	The Italian Railways Company
Murray (2005)	Emergency warning siren.	Dublin, Ohio
Farahani and Asgari (2007)	Locating military warehouses as distribution centers	Persia
Berman, Kalcsics, Krass, et al. (2009)	Specialized teams like police and firefighter	Gasoline transportation in Quebec and Ontario
Alexandris and Giannikos (2010)	Locating bank branches	Greece, Athens
Curtin et al. (2010)	Designing police patrol area	Dalas, Tx
Erdemir et al. (2010)	Locating aero medical and ground ambulances	USA, State of New Mexico
Lee and Murray (2010)	Locating Wi-Fi equipment	Dublin, Ohio
Oztekin et al. (2010)	An RFID network design methodology for asset tracking in healthcare	Stillwater Medical Center
Straitiff and Cromley (2010)	Location Set covering problem	Dublin, Ohio
Bell et al. (2011)	Locating air craft alert sites	USA

5.2. Case studies

In this section we summarize the papers that clearly state that they have applied covering models using real-world data (Table 5).

6. Conclusions and directions for further research

We review literature of covering problems with more focus on the research works published after Schilling et al. (1993). We classify the works not only with respect to the two main categories (set covering and maximal covering) but also based on several detailed characteristics that show other extensions in covering problems. The growing attention and interest into covering problems is due to its application in real-world problems. However, the developed models are far from the real-world problems and there is still much work to do (Berman et al., 2010b).

With respect to aforementioned literature and the table, we consider two attributes to prepare future works directions: (i) vacancy of the research works which means investigating the columns of the tables in order to find the notations that have used less than the others and (ii) trends which is related to orientation and trajectory of researches over time. The areas that can be considered for further research are as follows:

- Considering the following under-explored areas with respect to the tables can be important:
  - Covering problems on plane.
  - Different facilities with different covering radius.
  - Capacitated facilities.
  - Fuzzy parameters.
  - Probabilistic parameters and using stochastic programming solution techniques.
  - Multi-objective covering problems (several bi-objective models are already well covered) and considering non-cost objective functions.
- One of the features in real-world covering problems is that the covering radius can be varying over a time horizon. For example, if covering radius of ambulances is 10 min in a city the covering matrix in peak hours (say 7:00–9:00 AM and 17:00–19:00) will be something different from the other times. This can lead us to define time-dependant covering problems. In this problem, if the located facilities are mobile (like ambulances) their location can be changed. Otherwise, the facilities must be located one time; but the assignments to the demand points can be dynamic. One of the popular objective functions in the latter case can be the balance between workloads of the located facilities. Such problem can be called ''Dynamic Covering Allocation Problem''. The model looks like the models in Sections 2.1 and 2.2. The main differences are using time indices and consequently defining a_{ij}^t as a parameter that is 1 if in period t candidate locating point j can cover the demand point i, otherwise is 0. In addition we have a variable y_{ij}^t for assignment that is 1 if in period t the demand point i is assigned to located facility j, otherwise is 0. Naturally, if a demand point is inside the covering radius of potential facility and if the facility is located in this point, this demand point ''can'' be allocated to the facility.
- We see several models dealing with the emergency facilities as serving facilities. But this emergency service facility can be busy serving another region when a new demand arrives. This can bring another important idea in mind Snyder, Scaparra, Daskin, and Church (2006). Every located facility faces disruptions like natural disasters (like earth quakes and hurricanes) or man-made disasters (like terrorist attacks and strike). In other words, the emergency service facility is not busy and it can be failed intentionally or by chance. Therefore, we have to be prepared in advance to change locations, assignments, to ask help for relief, use redundancy and excess inventory etc. (Chopra & Sodhi, 2004).
- Three future developments possible from Church and Murray (2009) that some of them are in accordance with the above suggestions:
  - Backup coverage or multiple-coverage is still very important.
  - For emergency facilities like fire stations and hospitals, service availability is applicable as an important issue.
  - Spatial representation with focusing on line, polygon (or other) shape facilities, not point facilities.
- Berman et al. (2010b) talk about new developments in location models. They bring forth an idea on the basis that some of the underlying assumptions in common covering models are not realistic enough. They summarize these assumptions into three groups:
  - All or nothing coverage means that each demand point is completely covered or not covered at all. To overcome this problem they introduce ''the gradual cover models'', which is somewhat similar to the concept of fuzzy theory against crisp.
  - Individual coverage means demand points receive their service from only one of the facilities and usually from the nearest one. Against this assumption they suggest ''the cooperative cover model'' which seeks to react based upon the amount of received demands. If the requested demand from a facility is more than a threshold, then another facility (like the second nearest) can help to cover the demand.

– Fixed coverage radius that is the opposite of "variable radius model". Variable radius model assumes that due to the fact that constructing the facilities are very costly and most of the time there is budget constraint for locating facilities, the new model tries to identify where and what kind of facilities should be installed. Naturally, different facilities have different costs and different covering radius.

- Berman, Drezner, Drezner, et al. (2009) consider a defensive maximal covering model on a network; it is assumed that one of the links of the network will become unusable due to an attack (terrorist or a natural disaster). We can consider an attack to our new located facilities. i.e. while we prefer to locate new facilities somehow to cover more populations an attacker in the worst case decides to attack to the crowded centers.
- In location-reliability problems most of the time we consider a failure probability as given. The question is how we can calculate this probability in real-world problems. These events are low-probability high-consequence events like incidents involving hazardous materials. In other words, these probabilities are usually unknown, since these events are rare and we cannot find sufficient data to estimate these probabilities. On the other hand, if we consider a long term historical data that are not reliable because of the drastic changes that occur in world, the probability would not be the same as in the past. One solution to this problem is considering a very risk-averse planner who does not know this probability but who wants to consider the worst case scenario (Szeto, Farahani, & Sumalee, 2010). Another extension of this problem is considering more than one attack simultaneously. Moreover, we can consider a mixed strategy; i.e. since these problems are solvable using game theoretic approach and those are usually non-cooperative, maybe there is not any equilibrium point for the problem; but we will have different probability for attack to different facilities.

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