01/19/2022. Wednesday.

complex Analysis = calculus w/ complex Numbers.

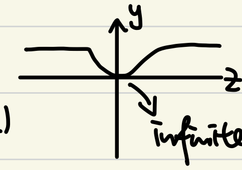
Example: $\sum_{n=0}^{\infty}|a_n||x|^n < \infty$. when $|x|<r$. $x \in (-r, r)$

if x is complex variable z, $|z|<r$. $z \in D(0,r)$ 

Example: $f(z) = \begin{cases} e^{-1/z^2} & z \neq 0 \\ 0 & z = 0. \end{cases}$

if z is real.

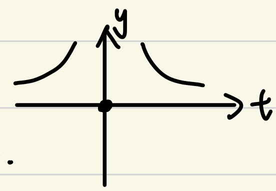
($f(z)$ viewed as a func $\mathbb{R} \to \mathbb{R}$)



infinitely differentiable at $z=0$.

Taylor series $f(0) + f'(0) + f''(0) + \cdots = 0 + 0 + \cdots = 0$. which is Not $f(z)$.

if $z = it$, $t \in \mathbb{R}$. $f(it) = \begin{cases} e^{1/t^2} & t \neq 0 \\ 0 & t = 0. \end{cases}$



$f(z)$ not conti. at $z=0$. $\Rightarrow$ $f(z)$ not complex-differentiable at $z=0$.

Example: let $z = x+iy$. $x, y \in \mathbb{R}$. $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(\mathbb{R}^2 \to \mathbb{R}^2)$.

$\frac{\partial z}{\partial x} = \frac{\partial x}{\partial x} + i\frac{\partial y}{\partial x} = 1 + i \cdot 0 = 1.$

$\frac{\partial z}{\partial y} = \frac{\partial x}{\partial y} + i\frac{\partial y}{\partial y} = 0 + i \cdot 1 = i.$

$\Rightarrow$ $\frac{\partial f(z)}{\partial x} = \frac{\partial f(z)}{\partial z} \cdot \frac{\partial z}{\partial x} = f'(z)$ $\quad$ $\frac{\partial f(z)}{\partial y} = \frac{\partial f(z)}{\partial z} \cdot \frac{\partial z}{\partial y} = i f'(z)$

$\frac{\partial^2 f(z)}{\partial x^2} = \frac{\partial f'(z)}{\partial x} = \frac{\partial f'(z)}{\partial z} \cdot \frac{\partial z}{\partial x} = f''(z)$ $\quad$ $\frac{\partial^2 f(z)}{\partial y^2} = i \frac{\partial f'(z)}{\partial z} \cdot \frac{\partial z}{\partial y} = i f''(z) \cdot i = -f''(z)$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(z) = 0$$

$f(z)$ satisfied the 2 dimensional Laplace equation.

Example: $\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx = \arctan(\infty) - \arctan(-\infty) = \pi.$

$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+1} dx$. $(a \in \mathbb{R})$. using complex technique, $\pi e^{-|a|}$.

01/21/2022 Friday.

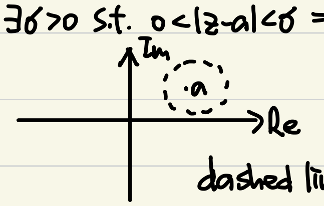
Derivative of complex-valued function.

$$f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

for $f(x)$ real-valued ($f: \mathbb{R}\to\mathbb{R}$). $\lim_{x\to a} f(x) = L$ means $\forall \varepsilon>0$, $\exists \delta>0$ s.t. $\forall x$ $0<|x-a|<\delta$, we have $|f(x)-L|<\varepsilon$.
(for any tolerance ε, we can guarantee $f(x)$ is within ε of L by forcing x to be close enough to a.)
if $\lim_{x\to a} f(x) = f(a)$, f is continuous (cts.) at a. $\Rightarrow |x-a|<\delta$ $|f(x)-f(a)|<\varepsilon$.

$a-\delta$ a $a+\delta$ $\to \mathbb{R}$ interval

for $f(z)$ complex-valued ($f: \mathbb{C}\to\mathbb{C}$). $\lim_{z\to a} f(z) = L$ means $\forall\varepsilon>0$, $\exists\delta>0$ s.t. $0<|z-a|<\delta \Rightarrow |f(z)-L|<\varepsilon$.



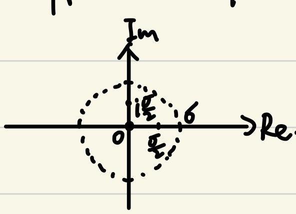
disk.

dashed line → x boundary.

Similarly, if $\lim_{z\to a} f(z) = f(a)$, we say f is cts. at $z=a$.

Example: $f(z)=z$. cont. at any point.
$\forall\varepsilon>0$, take $\delta=\varepsilon$, we have $|z-a|<\varepsilon$. so $|f(z)-f(a)|<\varepsilon$.

Example: $\lim_{z\to 0} \bar{z}/z$.
Suppose $\lim_{z\to 0} \bar{z}/z = L$ for some L. let $\varepsilon=1$, then $\exists\delta>0$ s.t. $0<|z-0|<\delta \Rightarrow |f(z)-L|<1$.



let $z=\frac{\delta}{2}$, $|f(z)-L|<1 \Rightarrow |1-L|<1$.
let $z=i\frac{\delta}{2}$, $|f(z)-L|<1 \Rightarrow |-1-L|<1$ contradicts. so such L exists.
No way to extend $\bar{z}/z$ to a cont. func at $z=0$.

if $\lim_{x\to a} f(x) = L_1$, $\lim_{x\to a} g(x) = L_2$, then $\lim_{x\to a}(f(x)+g(x)) = L_1+L_2$.
$\lim_{x\to a}(f(x)g(x)) = L_1L_2$.
$\lim_{x\to a}(f(x)/g(x)) = L_1/L_2$. (w/ $L_2\neq 0$).

if f, g are cts. at a, $L_1=f(a)$, $L_2=g(a)$. $f+g$, fg, f/g are cts.

if $f(x)$ is cts. at $x=a$, $g(x)$ is cts. at $x=f(a)$, then $g(f(x))$ is cts. at $x=a$.
proof: want to show $|g(f(x))-g(f(a))|<\varepsilon$.

by continuity of g at $f(a)$. $|g(w)-g(f(a))|<\varepsilon$ when $|w-f(a)|<\delta_1$.

want to take $w=f(x)$, so we need $|f(x)-f(a)|<\delta_1$. $\leftarrow \varepsilon$.

by continuity of f at $x=a$. $|f(x)-f(a)|<\delta_1$ holds for $|x-a|<\delta_2$.

$f(z)$ is differentiable at $z=a$ $\Longleftrightarrow$ the func $\frac{f(z)-f(a)}{z-a}$ extends to a cont. func at $z=a$. $f'(a)$.

Example: $f(z)=z$ is diff. w/ $f'(z)=1$.

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=\lim_{h\to 0}\frac{z+h-z}{h}=\lim_{h\to 0}1=1.$$

Example: $f(z)=\bar{z}$ is not diff. (but cts.)

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=\lim_{h\to 0}\frac{\overline{z+h}-\bar{z}}{h}=\lim_{h\to 0}\frac{\bar{z}+\bar{h}-\bar{z}}{h}=\lim_{h\to 0}\frac{\bar{h}}{h} \text{ doesn't exist.}$$

differentiable at $z=a \Rightarrow$ cts. at $z=a$.

pf: want $\lim_{z\to a} f(z)=f(a)$.

$$\lim_{z\to a} f(z)-f(a)=\lim_{z\to a}\frac{f(z)-f(a)}{z-a}(z-a). = \underset{\substack{\uparrow\\ \text{exists as } f \text{ is diff. at } a.}}{\lim_{z\to a}\frac{f(z)-f(a)}{z-a}}\cdot \underset{\text{exists.}}{\lim_{z\to a}(z-a)} = f'(a)\cdot 0=0$$

$\frac{d}{dz}\cdot cf(z)=c\cdot f'(z)$ $(c\in\mathbb{C})$.

$\frac{d}{dz}(f+g)=(\frac{d}{dz}f)+(\frac{d}{dz}g)$.

$\frac{d}{dz}(fg)=f'\cdot g+f\cdot g'$

$\frac{d}{dz}(f/g)=\frac{f'\cdot g-fg'}{g^2}$

$\frac{d}{dz}f(g(z))=g'(z)f'(g(z))$.

$\frac{d}{dz}z^n=nz^{n-1}$. for all integer n.

proof: for $n\geq 0$. Argue by induction. $n=0$ $\frac{d}{dz}1=0$.

by product rule, $\frac{d}{dz}z^n=1\cdot z^{n-1}+z\cdot\frac{d}{dz}z^{n-1}=1\cdot z^{n-1}+\underset{\text{by inductive assumption.}}{z((n-1)z^{n-2})}=nz^{n-1}$.

for $n<0$, by quotient rule, $\frac{d}{dz}z^n=\frac{d}{dz}(1/z^{-n})=\frac{-\frac{d}{dz}z^{-n}}{z^{-2n}}=\frac{-(-n)z^{(-n-1)}}{z^{-2n}}=\frac{nz^{-n-1}}{z^{-2n}}=nz^{n-1}$

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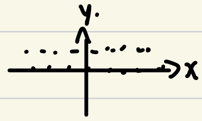
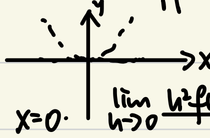
Example. $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$



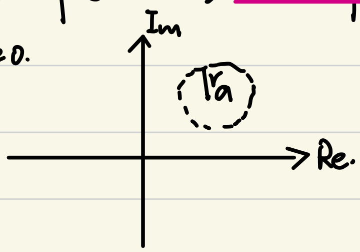
nowhere cts, so nowhere diff.

Consider $x^2 f(x)$



diff. at $x=0$. $\lim_{h \to 0} \frac{h^2 f(h) - 0^2 f(0)}{h} = \lim_{h \to 0} h \underbrace{f(h)}_{\text{bounded}} = 0$. Not diff. anywhere else.

Def: a function $f: \mathbb{C} \to \mathbb{C}$ is holomorphic at a point a, if it is diff. at z for all z s.t. $|z-a| < r$ for some $r \geq 0$.
(on an open disk centered at a).



$f(z)$ is diff. everywhere sufficiently close to a.

f is holomorphic $\to$ holomorphic at all points.

Def: The open disk (open balls) of radius r centered at $a \in \mathbb{C}$ is $D(a,r) = \{z \in \mathbb{C} \mid |z-a| < r\}$.

The closed disk (closed balls) is $\bar{D}(a,r) = \{z \in \mathbb{C} \mid |z-a| \leq r\}$.

Example: z^n is diff. everywhere for $n \geq 0$.

so $a_0 z^0 + a_1 z^1 + a_2 z^2 + \cdots + a_d z^d$ (every polynomials) is diff. everywhere. $\Rightarrow$ holomorphic

Example: $f(z) = |z|^2 = z\bar{z}$ is diff. at zero.

$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h\bar{h} - 0}{h} = \lim_{h \to 0} \bar{h} = 0$. Not diff. anywhere else.

diff at just one point $\to$ Not holomorphic.
diff. at infinite many pts.

To determine whether a fn is complex-differentiable.

let $z = x + iy$. x, y are real variables.

if $f: \mathbb{C} \to \mathbb{C}$, $\frac{\partial f}{\partial x}(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{f(x+iy+h) - f(x+iy)}{h}$ (h is real)

$\frac{\partial f}{\partial y}(z) = \lim_{h \to 0} \frac{f(x+i(y+h)) - f(z)}{h} = \lim_{h \to 0} \frac{f(x+iy+ih) - f(x+iy)}{h}$ (h is real).

Example: $f(z) = z^2$ so $f(x+iy) = (x+iy)^2 = x^2 - y^2 + 2ixy$.

$\frac{\partial f}{\partial x}(z) = \lim_{h \to 0} \frac{(x+h)^2 - y^2 + 2i(x+h)y - (x^2 - y^2 + 2ixy)}{h} = \lim_{h \to 0} \frac{h^2 + 2xh + 2ihy}{h} = \lim_{h \to 0} h + 2x + 2iy. = 2x + 2iy. = 2z = f'(z)$

$\frac{\partial f}{\partial y}(z) = \lim_{h\to 0} \frac{x^2-(y+h)^2+2ix(y+h)-(x^2-y^2+2ixy)}{h} = \lim_{h\to 0} \frac{-h^2-2hy+2ixh}{h} = \lim_{h\to 0} -h-2y+2ix = -2y+2ix$

$= 2i(x+iy) = 2iz = if'(z)$

$\frac{\partial f}{\partial x}(z) = -i\frac{\partial f}{\partial y}(z).$

Thm: (i) if $f: \mathbb{C} \to \mathbb{C}$ is complex differentiable, then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and obey $\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$.

(ii) if $f: \mathbb{C} \to \mathbb{C}$ is a function and $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ exist and are continuous on some open disk centered at z. If $\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$, then f is complex-differentiable at z.

Pf: (i) f is complex-differentiable, so $\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = f'(z)$

for every $\varepsilon > 0$, there is $\delta > 0$ s.t. $\left|\frac{f(z+h)-f(z)}{h} - f'(z)\right| < \varepsilon$ whenever $|h| < \delta$ $(h \in D(0,\delta))$.

Suppose h is real, $\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = f'(z)$.

h is real, so $f'(z) = \frac{\partial f}{\partial x}(z)$

Suppose h is imaginary, $h = ik$, $k \in \mathbb{R}$. $\lim_{k\to 0}\frac{f(z+ik)-f(z)}{ik} = f'(z)$

$\frac{1}{i}\frac{\partial f}{\partial y}(z) = f'(z) \Rightarrow \frac{\partial f}{\partial y}(z) = if'(z)$

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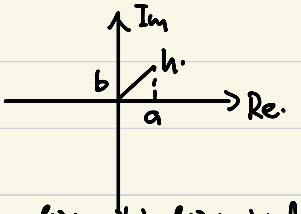
pf: (ii) want to show $\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=f'(z)$

instead, let's show $\lim_{h\to 0}\left|\frac{f(z+h)-f(z)}{h}-\frac{\partial f}{\partial x}(z)\right|=0$. (this shows $f'(x)$ exists and $f'(x)=\frac{\partial f}{\partial x}(z)$).

let $h=a+ib$ with $a,b\in\mathbb{R}$.



$\left|\frac{a}{h}\right|\le 1$. $\left|\frac{b}{h}\right|\le 1$.

$h=re^{i\theta}$. $a=r\cos(\theta)$. $b=r\sin(\theta)$.

$$\lim_{h\to 0}\frac{f(z+a+ib)-f(z+a)+f(z+a)-f(z)}{a+ib}-\frac{a+ib}{a+ib}\frac{\partial f}{\partial x}.$$

$$=\left(\lim_{h\to 0}\frac{f(z+a+ib)-f(z+a)}{ib}\cdot\frac{ib}{a+ib}-\frac{\partial f}{\partial x}\cdot\frac{ib}{a+ib}\right)+\left(\lim_{h\to 0}\frac{f(z+a)-f(z)}{a}\cdot\frac{a}{a+ib}-\frac{\partial f}{\partial x}\cdot\frac{a}{a+ib}\right)$$

the real part: $\lim_{h\to 0}\left(\frac{f(z+a)-f(z)}{a}-\frac{\partial f}{\partial x}\right)\frac{a}{a+ib}$

$h\to 0 \Rightarrow a\to 0$. $\frac{f(z+a)-f(z)}{a}$ converge to $\frac{\partial f}{\partial x}$. $\left|\frac{a}{h}\right|\le 1$.

for $a<\delta$, we have $\left|\frac{f(z+a)-f(z)}{a}-\frac{\partial f}{\partial x}(z)\right|<\varepsilon$.

so $\left|\frac{f(z+a)-f(z)}{a}-\frac{\partial f}{\partial x}\right|\left|\frac{a}{a+ib}\right|<\varepsilon$. for $|h|<\delta$. the limit $=0$.

the imaginary part. $\lim_{h\to 0}\left(\frac{f(z+a+ib)-f(z+a)}{ib}-\frac{\partial f}{\partial x}\right)\frac{ib}{a+ib}$.

as $\frac{\partial f}{\partial x}=-i\frac{\partial f}{\partial y}$. $\frac{f(z+a+ib)-f(z+a)}{ib}$ converge to $\frac{1}{i}\frac{\partial f}{\partial y}(z+a)$

as $h\to 0$, $a\to 0$ and $b\to 0$.

by the continuity of $\frac{\partial f}{\partial y}$. as $a\to 0$, $\frac{\partial f}{\partial y}(z+a)\to\frac{\partial f}{\partial y}(z)$. the limit $=0$.

the Cauchy–Riemann Equations are $\frac{\partial f}{\partial x}=-i\frac{\partial f}{\partial y}$.

write $f(x+iy)=u(x,y)+iv(x,y)$ where $u,v:\mathbb{R}^2\to\mathbb{R}$.

$u(x,y)=\operatorname{Re}(f(x+iy))$

$v(x,y)=\operatorname{Im}(f(x+iy))$.

so $u_x+iv_x=-i(u_y+iv_y)$. $\Rightarrow \begin{cases} u_x=v_y. \\ v_x=-u_y. \end{cases}$

Example: $f(z)=z^2$, $z=x+iy$. $\Rightarrow f(x+iy)=x^2-y^2+2ixy$.

$u(x,y)=x^2-y^2$. $u_x=2x$. $v_y=2x$.

$v(x,y)=2xy$. $v_x=2y$. $u_y=-2y$. so C-R equations hold.

Example: $f(z)=|z|^2$. $z=x+iy$. $\Rightarrow f(x+iy)=x^2+y^2$

$u_x=2x$. $v_y=0$ C-R equations hold iff $x=y=z=0$.

$v_x=0$ $-u_y=-2y$.

(this fn is only diff. at $z=0$.)

Suppose $f = u + iv$.

then $u_{xx} = \frac{\partial}{\partial x} u_x \overset{\text{C-R}}{=} \frac{\partial}{\partial x} v_y = \frac{\partial}{\partial x}\frac{\partial}{\partial y} v \overset{\text{conti. of 2nd derivative.}}{=} \frac{\partial}{\partial y}\frac{\partial}{\partial x} v = \frac{\partial}{\partial y} v_x = \frac{\partial}{\partial y}(-u_y) = -u_{yy}$.

so $u_{xx} + u_{yy} = 0$. $\frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u = 0$ Laplace's equation in 2D.

$v_{xx} + v_{yy} = 0$.

if $f'(z) = 0$. $f'(z) = \frac{\partial f}{\partial x}(z) = u_x + i v_x = 0$. take Re / Im parts: $u_x = v_x = 0$.

by C-R, $v_y = u_y = 0$.

as $u_x = 0$, $u(x,y) = g(y)$.
as $u_y = 0$, $u(x,y) = g(x)$. $\Rightarrow$ u is a constant.

WLOG, v is a constant. so f is a constant.

Def: Möbius Transformation (= linear fractional transformation) is a function of the form

$f(z) = \frac{az+b}{cz+d}$ $(a,b,c,d \in \mathbb{C}$ s.t. $ad - bc \neq 0)$.

if $ad = bc$, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$, so $\lambda(a,b) + \mu(c,d) = (0,0)$

$f(z) = \frac{-\frac{\mu}{\lambda}(cz+d)}{cz+d} = -\frac{\mu}{\lambda}$. which is just a constant.

suppose $f_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1}$ $f_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}$

$f_1(f_2(z)) = \dots = \frac{(a_1 a_2 + b_1 c_2) z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2) z + (c_1 b_2 + d_1 d_2)}$ $\rightarrow$ Also Möbius

$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}$ $\hookleftarrow$ coincide.

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$$\underset{M}{\underset{\parallel}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}} \mapsto f(z) = \underset{f_M}{\underset{\parallel}{\frac{az+b}{cz+d}}}.$$

$f_M(f_N(z)) = f_{M\cdot N}(z).$

$f_{\lambda M}(z) = f_M(z).$

$f_{M^{-1}}(z) = \frac{dz-b}{-cz+a}.$

if $c \neq 0$, we consider $z = \frac{-d}{c}$ to be ∞. if $c=0$, let $\frac{a}{c}$ be ∞.

And we can evaluate $f(z)$ at $z=\infty$: if $z \to \infty$, $\frac{1}{z} \to 0$, $f(z) = \frac{a+b/z}{c+d/z} \to \frac{a}{c}$.

So now we consider Möbius transformation as fns from $\mathbb{C} \cup \{\infty\}$ to $\mathbb{C} \cup \{\infty\}$. (bijective).

We call $\mathbb{C} \cup \{\infty\}$ to be extended complex plane = Riemann Sphere = 1D complex projective space ($\mathbb{CP}^1$, $\mathbb{P}_{\mathbb{C}}^1$).

Not $+\infty, -\infty$. $\hat{\mathbb{C}}$

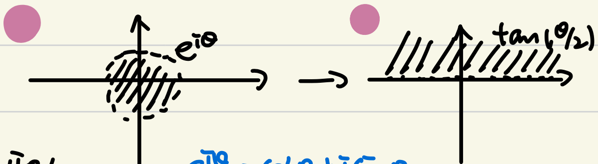
If we apply Möbius Transformation to a line or circle on the complex plane, the result is a line or circle.

Example: $f(z) = \frac{z-1}{iz+i} = \frac{z-1}{i(z+1)}$ apply to a unit circle ($z = e^{i\theta}$).

$$f(e^{i\theta}) = \frac{e^{i\theta}-1}{i(e^{i\theta}+1)} = \frac{\cos(\theta)-1+i\sin(\theta)}{i(\cos(\theta)+1+i\sin(\theta))} = \frac{-2\sin(\theta/2)^2 + i(2\sin(\theta/2)\cos(\theta/2))}{i(2\cos(\theta/2)^2 + i2\sin(\theta/2)\cos(\theta/2))} = \frac{2i(\cos(\theta/2)+i\sin(\theta/2))\sin(\theta/2)}{2i(\cos(\theta/2)+i\sin(\theta/2))\cos(\theta/2)}$$

$$= \frac{\sin(\theta/2)}{\cos(\theta/2)} = \tan(\theta/2)$$

↑ Real Number $\theta \in [0, 2\pi]$. or $(-\pi, \pi)$



$e^{i\theta} = \cos\theta + i\sin\theta$.

f sends the interior of unit disk to the interior of the upper half-plane.

If $g(z)$ is holomorphic on the upper half plane, then $g(f(z))$ is holomorphic on the unit disk.

We can use this to convert solutions of Laplace's equations between different domains.

prove: $\frac{az+b}{cz+d}$ can be written as
a translation. $z \mapsto z+a$.
then a dilation. $z \mapsto az$.
then a inversion. $z \mapsto \frac{1}{z}$.
} easily seen to perserve lines and circles.

In a 3D plane, consider the unit sphere and x-y plane. Let N be the North pole of the sphere $(0,0,1)$.

point on sphere $\to$ point on plane.

fn φ: sphere $\to$ plane: draw the line through N and the given point p on the sphere, then the intersection point of this line with the plane is $\varphi(p)$.

Example: if p = south pole $(0,0,-1)$, the line is the z-axis. $\varphi(p)=(0,0,0)\to(0,0)$.

Let $p=(x,y,z)$. $x^2+y^2+z^2=1$. the line is $(0,0,1)+\lambda(x,y,z-1)$

intersection: $1+\lambda(z-1)=0 \Rightarrow \lambda=\frac{1}{1-z}$. $\varphi(p)=\left(\frac{x}{1-z},\frac{y}{1-z},0\right)\to\left(\frac{x}{1-z},\frac{y}{1-z}\right)$.

when $\overset{p\to N}{z\to 1}$, $\left|\left(\frac{x}{1-z},\frac{y}{1-z}\right)\right|^2=|\varphi(p)|^2=\frac{x^2}{(1-z)^2}+\frac{y^2}{(1-z)^2}=\frac{x^2+y^2}{(1-z)^2}=\frac{1-z^2}{(1-z)^2}=\frac{1+z}{1-z}\to\infty$. so $\frac{1}{\varphi(p)}\to 0$.

include $p=N$

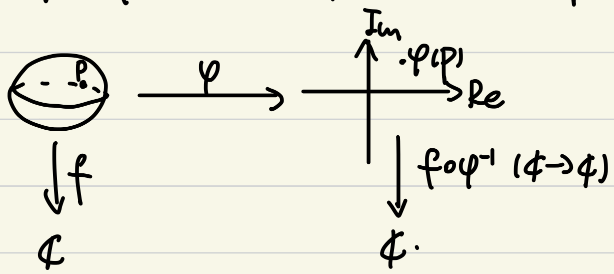
To extend φ ctsly to a fn. on the whole sphere, we take $\varphi(N)=\infty$. So φ: sphere $\to$ extended complex plane.

01/31/2022 Monday.

Stereographic bijection: φ is a bijection: sphere $\to \hat{\mathbb{C}}$.

Riemann Sphere is compact (closed and bounded).

φ define holomorphic on the sphere.



if $f\circ\varphi^{-1}$ is holomorphic at $\varphi(p)$, φ is holomorphic at p.

if $c=0$, $f(z)=\frac{a}{d}z+\frac{b}{d}$. dilation, translation.

otherwise, $f(z)=\frac{bc-ad}{c^2}\frac{1}{z+\frac{d}{c}}+\frac{a}{c}$. translation. inversion dilation. translation.

proof: inversion $(z\mapsto z^{-1})$ perserves inversion.

$(x-x_0)^2+(y-y_0)^2=r^2$. (circle centered at (x_0,y_0), radius r).

$\alpha(x^2+y^2)+\beta x+\gamma y+\delta=0$. if $\alpha=0$, it is the equation of a line.

To be nondegenerate, $\beta^2+\gamma^2>4\alpha\delta$

$z=x+iy$. $\quad z^{-1}=\frac{\bar{z}}{|z|^2}=\frac{x-iy}{x^2+y^2}=u+iv$. where $u=\frac{x}{x^2+y^2}$. $v=\frac{-y}{x^2+y^2}$

$\alpha+\beta u-\gamma v+\delta(u^2+v^2)=0$. as $\frac{1}{x^2+y^2}=|z^{-1}|^2=u^2+v^2$.

equation of another circle / line. $\beta^2+\gamma^2>4\alpha\delta$.

if $\delta=0$, the origin of plane lies on the circle / line.

$\alpha=\delta=0$, line through origin. $\mapsto$ $\alpha=\delta=0$, line through origin.

$\alpha\neq0$, $\delta=0$, circle through origin. $\mapsto$ $\delta\neq0$, $\alpha=0$, line not through origin.

$\alpha=0$, $\delta\neq0$. line not through origin. $\longmapsto$ $\alpha\neq0$, $\delta=0$. circle through origin.

$\alpha\neq0$, $\delta\neq0$. circle not through origin. $\longmapsto$ $\alpha\neq0$, $\delta\neq0$. circle not through origin.

circles and lines in the plane (in $\mathbb{C}$) are stereographic projections of circles on the sphere (Riemann sphere).

prove lines:

φ: projecting from N.

line L corresponds to the line segment on the sphere that connecting N and L, in particular the plane defined by L and N.

the intersection of sphere and plane is circle. $\to$ containing N.

Example: $f(z)=\frac{z-1}{iz+i}$. unit circle / "equator" $\longrightarrow$ real axis. / "prime meridian" (longitude = 0).

02/02/2022 Wednesday.

$e^{i\theta} = \cos\theta + i\sin\theta$.

$z = x+iy$. $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y)) = e^x\cos(y) + ie^x\sin(y)$ $\quad\forall z \in \mathbb{C},\ e^z \neq 0$.

we can define $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$ → holomorphic. $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$ → holomorphic as e^z is holomorphic. $\cos(iy) = \frac{e^{-y}+e^{y}}{2} = \cosh(y)$. $\sinh(y) = \frac{e^y - e^{-y}}{2}$

$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ (even if z is a $n\times n$ matrix).

$\sin(2z) = \frac{e^{2iz}-e^{-2iz}}{2i} = \frac{(e^{iz}+e^{iz})}{2}\frac{(e^{iz}-e^{iz})}{2i}2 = 2\sin(z)\cos(z)$.

$\sin^2(z) + \cos^2(z) = -\frac{e^{2iz}+e^{-2iz}-2}{4} + \frac{e^{2iz}+e^{-2iz}+2}{4} = 1$.

$\sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$

↑ Not bound by 1 $\quad \sin(iy) = i\sinh(y) = i\frac{e^y - e^{-y}}{2}$

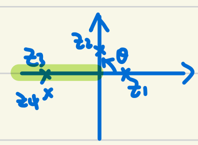
Logarithm.

we want $e^{\log(z)} = z$. but then $e^{\log(z)+2\pi i k} = z$. x make sense.

Def: Principal log is the fn $\log(re^{i\theta}) = \log(r) + i\theta$. $-\pi < \theta \leq \pi$. or we can choose $-\frac{\pi}{2} < \theta \leq \frac{3}{2}\pi$.

Example: $z_1 = 1$. $\log(z_1) = \log(1) = 0$. $\theta = 0$ $\quad z_2 = 3i$. $\log(z_2) = \log(3) + i\frac{\pi}{2}$. $\theta = \frac{\pi}{2}$

$z_3 = -6$ $\log(z_3) = \log(6) + i\pi$ $\theta = \pi$ $\quad z_4 = -6 - i\varepsilon$. $\log(z_4) \approx \log(6) - i\pi$. small



$\theta > \pi$ → On negative real axis, jumps by $2\pi i$. Not conti. So Not differentiable.

check if differentiable elsewhere:

if $z = x+iy = re^{i\theta}$. $\log(x+iy) = \underbrace{\frac{1}{2}\log(x^2+y^2)}_{u} + i\underbrace{\arctan\left(\frac{y}{x}\right)}_{v}$. → θ.

$u_x = \frac{x}{x^2+y^2}$ $\quad u_y = \frac{y}{x^2+y^2}$ $\quad v_x = -\frac{y}{x^2+y^2}$

$v_y = \frac{1}{x}\frac{1}{1+(y/x)^2} = \frac{x}{x^2+y^2}$ $\quad v_x = -\frac{y}{x^2}\frac{1}{1+(y/x)^2} = -\frac{y}{x^2+y^2}$ $\Big\}$ C-R holds

Holomorphic on $\mathbb{C}$ without negative real axis and 0.

$u_{xx} + u_{yy} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2} = 0$.

so Laplace equation holds.

$\arctan\left(\frac{y}{x}\right)$ isn't conti. As $\arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{y}{x} + 2\pi\right) = \arctan\left(\frac{y}{x} + 4\pi\right) = \cdots$

02/04/2022 Friday.

Complex Integration.

If $f:\mathbb{R}\to\mathbb{C}$ is a function defined on $\mathbb{R}$, taking values in $\mathbb{C}$ we define

$$\int_a^b f(x)\,dx = \underbrace{\int_a^b Re(f(x))\,dx}_{\text{Real-valued fn.}} + i\underbrace{\int_a^b Im(f(x))\,dx}_{}.$$

if we expect $\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$, $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$. $c\in\mathbb{C}$,

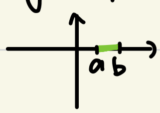
then the definition is forced.

for $f:\mathbb{R}\to\mathbb{R}$, only one path (real axis)

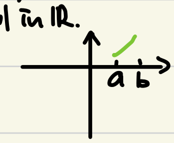
when differenciating $f:\mathbb{C}\to\mathbb{C}$, we need to specify the path from $\gamma(a)$ to $\gamma(b)$.

The path is the image of the fn. $\gamma:\underbrace{[a,b]}_{\text{interval in } \mathbb{R}}\to\mathbb{C}$.

Example: $\gamma(t)=t$.

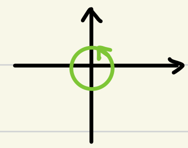


$\gamma(t)=t+it^2$



$\gamma(t)=e^{it}$

$0\le t\le 2\pi$



Def: the integral of the fn $f:\mathbb{C}\to\mathbb{C}$ along the path parametrised by $\gamma:[a,b]\to\mathbb{C}$

is $\int_a^b f(\gamma(t))\gamma'(t)\,dt. = \int_\gamma f(z)\,dz.$

Aside: $\int_{g(a)}^{g(b)} f(x)\,dx = \int_a^b f(g(x))\underbrace{g'(x)dx}_{dg}.$

Suppose we have a different parametisation, write this as $\gamma(\theta(t))$ where $\theta:[a,b]\to[a,b]$ is a reparametisation of the interval $[a,b]$.

[$\theta(a)=a$. $\theta(b)=b$. θ is cts. θ is increasing.] required by all reparametisation.

as t is increasing.

$$\int_a^b f(\gamma(\theta(t))\gamma'(\theta(t))\theta'(t)\,dt. = \int_{\theta(a)=a}^{\theta(b)=b} f(\gamma(u))\gamma'(u)\,du.\quad \begin{array}{l} u=\theta(t) \\ du=\theta'(t)dt. \end{array}$$

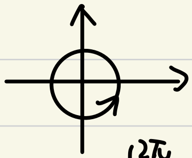
As $\gamma(\theta(t))=\gamma(t)$, the integral depends on the path in $\mathbb{C}$, not the parametisation.

Example: $\gamma(t)=t$. $\gamma'(t)=1$. $\int_\gamma f(z)\,dz = \int_a^b f(t)\,dt.$

Example: $\gamma(t)=t+it^2$ $\gamma'(t)=1+2it$. $\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_a^b (1+2it)\,dt$

$f(z)=1.$

$$= \int_a^b 1\,dt + i\int_a^b 2t\,dt = b-a+i(b^2-a^2) = \gamma(b)-\gamma(a)$$

Example: 

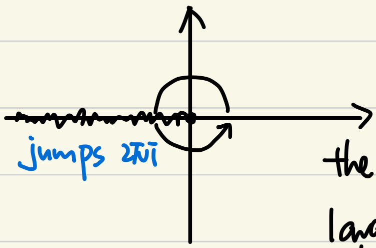
$\gamma(t) = e^{it} \cdot \quad 0 \le t \le 2\pi \cdot$ $\qquad \frac{d}{dt} e^{it} = \frac{d}{dt}(\cos t + i \sin t) = -\sin t + i \cos t = i e^{it} \cdot$

$f(z) = z^n \quad n \in \mathbb{Z} \cdot$

$$\int_\gamma f(z)\,dz = \int_0^{2\pi} f(\gamma(t))\,\gamma'(t)\,dt = \int_0^{2\pi} e^{int} \cdot (i e^{it})\,dt. = i\int_0^{2\pi} e^{it(n+1)}\,dt.$$

if $n \neq -1$, $= i\left[\frac{e^{it(n+1)}}{i(n+1)}\right]_{t=0}^{2\pi} = i\left(\frac{1}{i(n+1)} - \frac{1}{i(n+1)}\right) = 0.$

if $n = -1$, $= i\int_0^{2\pi} 1\,dt = 2\pi i.$ doesn't make sense as $\int x^n\,dx = \begin{cases} \frac{x^{n+1}}{n+1} + c & n \neq -1 \\ \log(x) + c & n = -1 \end{cases}$



the value we get back to $\theta = 0$ is $2\pi i$ larger than the initial value.

02/07/2022 Monday.

Basic Properties of Complex Integration.

$$\int_\gamma \lambda f(z)+\mu g(z)\,dz = \int_a^b (\lambda f(\gamma(t)) + \mu g(\gamma(t)))\gamma'(t)\,dt = \int_a^b \lambda f(\gamma(t))\gamma'(t)+\mu g(\gamma(t))\gamma'(t)\,dt$$

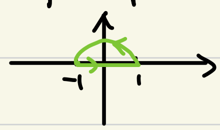
$\lambda,\mu\in\mathbb{C}$

$$= \lambda\int_a^b f(\gamma(t))\gamma'(t)\,dt + \mu\int_a^b g(\gamma(t))\gamma'(t)\,dt = \lambda\int_\gamma f(z)\,dz + \mu\int_\gamma g(z)\,dz.$$

for real integral, $\int_a^b g(x)dx = \int_a^c g(x)dx + \int_c^b g(x)dx$. $g(x)$ is often precisely defined.

for complex integral, only $\gamma(t)$ is precisely defined.

Example: $\gamma(t) = \begin{cases} 2t-1 & 0\le t\le 1 \\ e^{i(t-1)} & 1\le t\le 1+\pi \end{cases}$



$$\int_\gamma f(z)\,dz = \int_0^{1+\pi} f(\gamma(t))\gamma'(t)\,dt = \int_0^1 f(\gamma(t))\gamma'(t)\,dt + \int_1^{1+\pi} f(\gamma(t))\gamma'(t)\,dt$$

$$= \int_{\text{horizontal line}} f(z)\,dz + \int_{\text{arc part}} f(z)\,dz.$$

$$\int_a^a f(\gamma(t))\gamma'(t)\,dt = \int_a^c f(\gamma(t))\gamma'(t)\,dt + \int_c^a f(\gamma(t))\gamma'(t)\,dt = 0.$$

So $\int_a^c f(\gamma(t))\gamma'(t)\,dt = -\int_c^a f(\gamma(t))\gamma'(t)\,dt.$

proof: let $\delta:[a,b]\to\mathbb{C}$ $\quad \delta(t)=\gamma(a+b-t)$ reverse of γ.

$\delta(a)=\gamma(a+b-a)=\delta(b)$. $\quad \delta(b)=\gamma(a+b-b)=\gamma(a)$.

$\int_\delta f(z)\,dz = \int_a^b f(\delta(t))\delta'(t)\,dt = \int_a^b f(\gamma(a+b-t))\cdot(-\gamma'(a+b-t))\,dt.$

let $u=a+b-t$. $\int_{a=u(b)}^{b=u(a)} f(\gamma(u))(-\gamma'(u))\,dt = \int_b^a f(\gamma(u))\gamma'(u)\,du = -\int_\gamma f(z)\,dz$

$$\left|\int_a^b g(x)dx\right| \le \max_{x\in[a,b]}|g(x)|\cdot(b-a). \qquad \int_a^b|\gamma'(t)|dt$$

for complex integral: $\left|\int_\gamma f(z)dz\right| \le \max_{a\le t\le b}|f(\gamma(t))|\cdot \text{length}(\gamma).$

if $\gamma(t)$ is position at time t. $|\gamma'(t)|$ is the speed, $\int_a^b|\gamma'(t)|dt$ is total distance travelled.

$$\left|\int_a^b f(\gamma(t))\gamma'(t)dt\right| \le \int_a^b |f(\gamma(t))\gamma'(t)|dt.$$

proof: $\int_a^b g(x)dx = \left|\lim_{n\to\infty}\sum_{k=1}^n g\left(a+\frac{b-a}{n}k\right)\frac{b-a}{n}\right| \le \lim_{n\to\infty}\sum_{k=1}^n \left|g\left(a+\frac{b-a}{n}k\right)\right|\frac{b-a}{n}$ "=" $\int_a^b|g(x)|dx.$

$$\left|\int_a^b f(\gamma(t))\gamma'(t)dt\right| \le \int_a^b \underbrace{|f(\gamma(t))|}_{\text{bounded by } \max_{a\le t\le b}|f(\gamma(t))|=M}|\gamma'(t)|dt \le \int_a^b M|\gamma'(t)|dt = M\int_a^b|\gamma'(t)|dt = ML. \qquad L=\text{length}(\gamma).$$

$f(z)$

ML-lemma.

Fundamental theorem of Calculus. if $\frac{d}{dx}F(x) = f(x)$, $\int_a^b f(x)dx = F(b) - F(a)$.

Claim: if F is holomorphic on some subset $G \subseteq \mathbb{C}$. and $\frac{d}{dz}F(z) = f(z)$.

then $\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a))$.

Proof: $\int_\gamma f(z)dz = \int_a^b \underbrace{f(\gamma(t))\gamma'(t)}_{=\frac{d}{dt}F(\gamma(t))\ ?}dt$.

let $F(x+iy) = u(x,y) + iv(x,y)$

let $\gamma(t) = \alpha(t) + i\beta(t)$ so $F(\gamma(t)) = u(\alpha(t) + i\beta(t)) + iv(\alpha(t) + i\beta(t))$

$\frac{d}{dt}F(\gamma(t)) = u_x \cdot \alpha'(t) + u_y \cdot \beta'(t) + iv_x \cdot \alpha'(t) + iv_y \beta'(t)$.

by C-R equations. $u_x = v_y$. $u_y = -v_x$

$= u_x(\alpha'(t) + i\beta'(t)) + iv_x(\alpha'(t) + i\beta'(t))$. $= F'\gamma'(t)$. proved.

02/09/2022 Wednesday.

Recall if $F(z)$ is holomorphic on γ and $\gamma:[a,b]\to\mathbb{C}$ is a curve, then $\frac{d}{dt}F(\gamma(t)) = F'(\gamma(t))\gamma'(t)$

So if $f(z) = \frac{d}{dz}F(z)$, then $\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$

if $\gamma(a)=\gamma(b)$, $F(\gamma(a)) - F(\gamma(b)) = 0$.

Example: $\int_{\text{unit circle anticlockwise}} \frac{1}{z}\,dz = 2\pi i$. As $\gamma(a)=\gamma(b)$, $F(\gamma(b))-F(\gamma(a))$ will give 0.

Since this is not zero, we can't have $\frac{d}{dz}F(z)=\frac{1}{z}$ for a holomorphic fn defined on the unit circle.

in the region $\mathbb{C}\setminus\mathbb{R}_{\le 0}$, the principal logarithm $\log(z)$ is holomorphic.

$\log(re^{i\theta}) = \log(r)+i\theta \qquad \frac{d}{dz}(\log(x+iy)) = \frac{x}{x^2+y^2} + \frac{-iy}{x^2+y^2} = \frac{\bar z}{z\bar z} = \frac{1}{z}$.

In order for $F(z)$ s.t. $\frac{d}{dz}F(z)=f(z)$ to exist, we need $\int_\gamma f(z)\,dz=0$ for any closed curve. i.e. $\gamma(a)=\gamma(b)$.

Want $F(w)=\int_\gamma f(z)\,dz$ where γ is a curve from a fixed basepoint $q=\gamma(a)$ to $w=\gamma(b)$.

First, check $F(w)=\int_\gamma f(z)\,dz$ ($\gamma(a)=q$, $\gamma(b)=w$) doesn't depend on γ. w/ $\int_{\text{closed curve}}=0$.

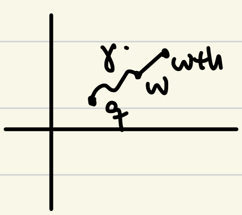
Suppose two such paths from q to w. call them σ_1, σ_2.

Want $\int_{\sigma_1} f(z)\,dz = \int_{\sigma_2} f(z)\,dz$.

Consider curve $\sigma_1-\sigma_2$. by assumption. $\int_{\sigma_1-\sigma_2} f(z)\,dz = 0 = \int_{\sigma_1} f(z)\,dz - \int_{\sigma_2} f(z)\,dz$. proved.

Next, verify $\frac{d}{dw}F(w)=f(w)$.

$\frac{d}{dw}F(w) = \lim_{h\to 0}\frac{F(w+h)-F(w)}{h}$



Since the choice of curve between q and $w+h$ doesn't matter, let's pick one goes to w and along the line segment (of length $|h|$).

As F is holomorphic at w, it is differentiable at disk $D(w,\varepsilon)$ for $\varepsilon>0$.

$F(w+h) - F(w) = \int_q^w - \int_q^{w+h} = \int_\ell f(z)\,dz$

line segment.

want $\lim_{h\to 0}\frac{1}{h}\int_\ell f(z)dz = f(w)=0$

observe $\int_\ell f(w)\,dz = f(w)\underbrace{\int_\ell dz}_{=[z]_w^{w+h}=h}$. So we want $\lim_{h\to 0}\frac{1}{h}\int_\gamma (f(\gamma)-f(w))dz=0$.

constant w.r.t. z.

By ML-lemma. $|\int_\gamma g(z)dz| \le \left(\max_{a\le t\le b}|g(\gamma(t))|\right)\cdot \text{length}(\gamma)$. $\gamma \to \ell$.

$|\frac{1}{h}\int_\ell f(z)-f(w)dw| \le \frac{|h|}{|h|}\max_{z\in\ell}|f(z)-f(w)|$

so enough to check $\lim_{h\to 0}\max_{z\in\ell}|f(z)-f(w)|$

so if $f(z)$ is cts. at w, $\forall\varepsilon>0, \exists\delta>0$ s.t. $|z-w|<\delta$, then $|f(z)-f(w)|<\varepsilon$.

so when $|h|<\delta$, any point z on ℓ (line segment starting from w, w/ length $|h|$) obeys $|z-w|<\delta$

and hence $|f(z)-f(w)|<\varepsilon$.

we have $|\frac{1}{h}\int_\ell f(z)dz - f(w)|<\varepsilon$. $\lim_{h\to 0}\frac{F(w+h)-F(w)}{h}=f(w)$.

So we need f to be cts. and $\int$closed curve $=0$.

Cauchy Theorem: If $\gamma:[a,b]\to\mathbb{C}$ is a closed curve. Assume $f(z)$ is holomorphic on γ and at every point in the interior region enclosed by the curve, then $\int_\gamma f(z)dz=0$.

Note: $\frac{1}{z}$ is not holomorphic at zero, and doesn't obey $\int$unit circle$=0$.

we assume $f'(z)$ is cts.

02/11/2022 Friday.

last time: for $f(z)$. we can find $F(z)$ s.t. $\frac{d}{dz}F(z) = f(z)$ if $\int_\gamma f(z)dz = 0$ for any closed curve.

today: show this property holds for $f(z)$ which is holomorphic on γ and inside γ (prove Cauchy Thm)

Recall: a vector field is a function, $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad \vec{F}(x,y) = (F_1(x,y), F_2(x,y))$.

$\int_\gamma \vec{F} \cdot d\vec{l} = \int_a^b \vec{F}(\gamma(t)) \cdot \gamma'(t)dt$ (dot product) $\quad$ suppose $\gamma(t) = (\alpha(t), \beta(t))$, then $\gamma'(t) = (\alpha'(t), \beta'(t))$.

Stokes' Thm (Green's Thm): $\int_\gamma \vec{F} d\vec{l} = \int_{\text{region enclosed by } \gamma} \vec{\nabla} \times \vec{F} dA \quad$ where $\vec{\nabla} \times \vec{F} = \frac{dF_2}{dx} - \frac{dF_1}{dy}$.

$\int_\gamma (\vec{F} \cdot \hat{n}) dl = \int_a^b F_1(\alpha(t), \beta(t)) \beta'(t) dt + F_2(\alpha(t), \beta(t))(-\alpha'(t)) dt$.

$\hat{n}$: normal vector to the curve.

Divergence Thm: $\int_\gamma (\vec{F} \cdot \hat{n}) dl = \int_{\text{area enclosed by } \gamma} \vec{\nabla} \cdot \vec{F} dA$. $\quad$ where $\vec{\nabla} \cdot \vec{F} = \frac{dF_1}{dx} + \frac{dF_2}{dy}$.

so want F_1, F_2 to have cts. partial derivatives. $\quad \int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt = 0$.

let $f(x+iy) = u(x,y) + iv(x,y) \quad \gamma(t) = \alpha(t) + i\beta(t)$.

$\int_a^b (u+iv)(\alpha'(t) + i\beta'(t))dt = \int_a^b u\alpha'(t) - v\beta'(t) dt + i\int_a^b v\alpha'(t) + u\beta'(t) dt$.

Note: $u\alpha' - v\beta' = (u, -v) \cdot (\alpha', \beta') \qquad v\alpha' + u\beta' = (u, -v) \cdot (\beta', -\alpha')$

Let's take $\vec{F}(x,y) = (u(x,y) - v(x,y))$

$$\int_\gamma f(z)dz = \int_\gamma \vec{F} \cdot d\vec{l} + i\int_\gamma (\vec{F} \cdot \hat{n}) d\vec{l}.$$

$$\vec{\nabla} \times \vec{F} = \frac{\partial(-v)}{\partial x} - \frac{\partial(u)}{\partial y} = -v_x - u_y = 0. \quad (u_y = -v_x \text{ by C-R}).$$

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial u}{\partial x} + \frac{\partial(-v)}{\partial y} = u_x - v_y = 0 \quad (u_x = v_y \text{ by C-R}).$$

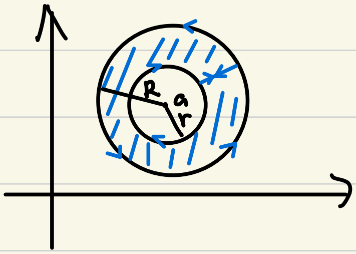
So $\int_\gamma f(z)dz = \int_{\text{area by } \gamma} 0 dA + i\int_{\text{area by } \gamma} 0 dA = 0 + i0 = 0$. $\qquad$ Proved.

Cauchy Integral formula. (first version).

Suppose $f(z)$ is holomorphic on the closed disk of radius R centered at $a \in \mathbb{C}$, then

$$\int_{\text{circle with radius } R \text{ centered at } a, \text{ anticlockwise}} \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

Proof: $\frac{f(z)}{z-a}$ may not be holomorphic at $z=a$. $\Rightarrow$ we need a curve not enclose a.



4 parts: 1) big circle, anticlockwise.

2) line segment in between

3) small circle, clockwise.

4) line segment, opposite direction.

cancel out.

by Cauchy Thm. $\int_{1)} + \int_{2)} + \int_{3)} + \int_{4)} = 0 \Rightarrow \int_{1)} + \int_{3)} = 0.$

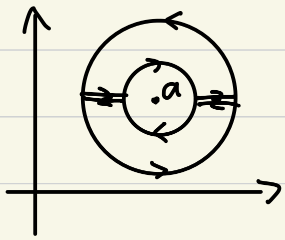
$\Downarrow$

$\int_{\text{big circle, clockwise}} = \int_{\text{small circle, clockwise}}.$

take $r \to 0$.

02/14/2022 Monday.

Another way of thinking.



By Cauchy's Thm: $\int_{\frown} = 0$ $\int_{\smile} = 0$.

Want to show $\displaystyle\int_{\text{circle of radius } r \text{, centered at } a} \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$.

$$\int_{\text{circle}} \frac{f(a)}{z-a}\,dt = f(a)\int_\gamma \frac{1}{z-a}\,dz = f(a)\int_0^{2\pi} \frac{1}{\gamma(t)-a}\gamma'(t)\,dt = f(a)\int_0^{2\pi} \frac{1}{re^{it}}\, ire^{it}\,dt = 2\pi i f(a).$$

$\gamma(t) = a + re^{it}$ $0 \le t \le 2\pi$

So we want $\displaystyle\int_\gamma \frac{f(z)}{z-a}\,dz = \int_\gamma \frac{f(a)}{z-a}\,dz$. i.e. $\displaystyle\int_\gamma \frac{f(z)-f(a)}{z-a}\,dz = 0$.

By ML-lemma, $\displaystyle\left|\int_\gamma \frac{f(z)-f(a)}{z-a}\,dz\right| \le \max_{z\in\gamma}\left|\frac{f(z)-f(a)}{z-a}\right| \cdot \text{length}(\gamma)$.

$|z-a| < r$. $\qquad 2\pi r$

$$\left|\int_\gamma \frac{f(z)-f(a)}{z-a}\,dz\right| \le \max_{z\in\gamma} \frac{|f(z)-f(a)|}{\cancel{r}} \cdot 2\pi \cancel{r}.$$

As $f(z)$ is diff., it is cts. $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|z-a| < \delta \Rightarrow |f(z)-f(a)| < \varepsilon$.

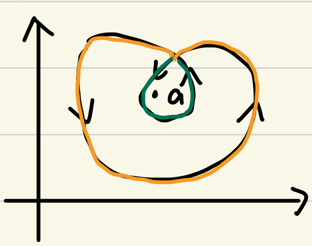
by making $r < \varepsilon$, we have $\displaystyle\left|\int_\gamma \frac{f(z)-f(a)}{z-a}\,dz\right| < 2\pi\varepsilon$. any positive number.

so $\displaystyle\int_\gamma \frac{f(z)-f(a)}{z-a}\,dz = 0$. doesn't depend on r.

Cauchy Integral Formula (Second Version).

Let γ to be a closed curve that encloses $a \in \mathbb{C}$ exactly once (anticlockwise).

Suppose $f(z)$ is holomorphic inside γ, then $\displaystyle\int_\gamma \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$.

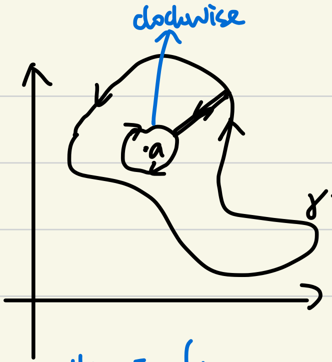


$\int_\gamma = \int + \int$ (both enclose point a).

so $\displaystyle\int_\gamma \frac{f(z)}{z-a}\,dz = \underbrace{2}_{\#\text{times } \gamma \text{ circles around } a.} \times 2\pi i f(a)$

(By Jordan Curve Theorem, if γ is injective, γ goes around any point in the interior region exactly once.)

Proof:



γ doesn't enclose a, so $\int_\gamma \frac{f(z)}{z-a}dz = 0$

($z=a$ is the only point where $\frac{f(z)}{z-a}$ is not holomorphic inside γ.)

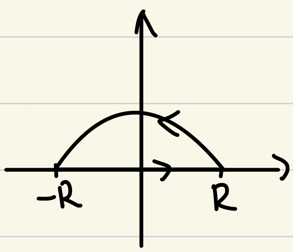
$\int_\gamma = -\int_{\text{small}\,\circ\,\text{clockwise}} = \int_{\text{small}\,\circ\,\text{anticlockwise}} = 2\pi i f(a)$.

first version.

this is $\int$ line segment.

Example: $\int_{-\infty}^{\infty} \frac{\cos(wx) + i\sin(wx)}{x^2+1}\,dx = \pi e^{-w}$ where $w \geq 0$, $w \in \mathbb{R}$. → cancel out by symmetry.

Consider $f(z) = \frac{e^{iwz}}{z^2+1} = \frac{e^{iwz}}{z+i} \Big/ z-i = \frac{g(z)}{z-i}$ where $g(z) = \frac{e^{iwz}}{z+i}$.



As $R \to \infty$, $\int_{\text{line segment}} f(z) \to \int_{-\infty}^{\infty} f(x)\,dx$.

Consider $\int_{\text{line segment}} = \int_\gamma \frac{g(z)}{z-i}\,dz - \int_{\text{semicircle}}$

By ML-lemma, for the semicircle, $\operatorname{Im}(z) \geq 0$, $\operatorname{Im}(wz) \geq 0$,

so $\operatorname{Re}(iwz) \leq 0$, $|e^{iwz}| = e^{\operatorname{Re}(iwz)} \leq e^0 = 1$.

$|z^2+1| \geq |z|^2 - 1 = R^2 - 1$.

$\left|\int_{\text{semicircle}}\right| \leq \frac{1}{R^2-1} \cdot 2\pi R$ which $\to 0$ as $R \to \infty$. $\Rightarrow \int_{\text{semicircle}} = 0$

By Cauchy Integral formula, $\int_\gamma \frac{g(z)}{z-i}\,dz = 2\pi i\, g(i) = 2\pi i \frac{e^{iwi}}{i+i} = \pi e^{-w}$.

$\pi e^{-w} - 0 = \int_{-\infty}^{\infty} \frac{\cos(wx)}{x^2+1}\,dx + i\int_{-\infty}^{\infty} \frac{\sin(wx)}{x^2+1}\,dx$.

by comparing the real & imaginary parts, $\int_{-\infty}^{\infty} \frac{\sin(wx)}{x^2+1}\,dx = 0$.

02/16/2022 Wednesday.

$\int_\gamma \frac{f(z)}{z-a}dz = \begin{cases} 2\pi i f(a) & \text{if } a \text{ in the region enclosed by } \gamma. \\ 0 & \text{otherwise.} \end{cases}$ $\Rightarrow$ $\frac{1}{z-a}$ holomorphic inside γ

$\Downarrow$

$\frac{f(z)}{z-a}$ holomorphic inside γ.

$\Downarrow$

Cauchy Thm applied.

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz = \frac{1}{2\pi i}\int_a^b \frac{f(\gamma(t))\gamma'(t)}{\gamma(t)-a}dt$$

only values of f on γ matters $\to$ determine values of f inside γ.

$\downarrow$

a difference between holomorphic and real-diff. fns.

Example: $f(e^{i\theta}) = 0$. (→ holomorphic) Compute $f(a)$ for $|a|<1$.

$$f(a) = \frac{1}{2\pi i}\int_{\text{unit circle}} \frac{f(z)}{z-a}dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\overset{=0}{f(e^{i\theta})}}{e^{i\theta}-1} ie^{i\theta}d\theta. = 0.$$

Example: $f(x+iy) = \frac{x^2+y^2-1}{x^2+y^2+1}$ on unit circle $(x^2+y^2=1)$

f vanishes on unit circle, but not elsewhere

Not holomorphic., Real-diff.

If holomorphic, for $|a|<1$.

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(e^{i\theta})}{e^{i\theta}-a} ie^{i\theta}d\theta = 0.$$

Show $f'(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^2}dz$:

Consider $\lim_{h\to 0} 2\pi i\frac{f(a+h)-f(a)}{h}$

$2\pi i f'(a) =$

$$= \lim_{h\to 0}\frac{\int_\gamma \frac{f(z)}{z-(a+h)}dz - \int_\gamma \frac{f(z)}{z-a}dz}{h}.$$

to be valid, we need $a+h$ inside γ.

Show $a+h$ inside γ:

for small h, true as the enclosed region is open.

$\gamma: [\alpha,\beta] \to \mathbb{C}$. Cts. so $|\gamma(t)-a|$ is cts.

set of valid values:

> nonnegative reals.

> Not zero. $|\gamma(t)-a| = 0 \Rightarrow a = \gamma(t)$ But a is inside γ, Not on γ.

Cts. image of a compact set is compact, and also closed by Heine-Borel Thm.

$[\alpha,\beta]$ is compact, so the set of values of $|\gamma(t)-a|$ is closed, so its complement

is open. 0 is in the complement, so does $(-\varepsilon,\varepsilon)$ for small $\varepsilon>0$.

if x in the set, so does $(x-\varepsilon, x+\varepsilon)$

therefore, $|\gamma(t)-a| \geq \varepsilon$. if $|h-a|<\varepsilon$, $a+h$ inside γ.

$$= \lim_{h\to 0} \frac{1}{h}\int_\gamma \frac{f(z)(z-a)-f(z)(z-a-h)}{(z-a)(z-a-h)}dz$$

$$= \lim_{h\to 0}\frac{1}{h}\int_\gamma \frac{hf(z)}{(z-a)(z-a-h)}dz. = \lim_{h\to 0}\int_\gamma \frac{f(z)}{(z-a)(z-a-h)}dz.$$

we want $\lim_{h\to 0}\int_\gamma \frac{f(z)}{(z-a)(z-a-h)}dz - \int_\gamma \frac{f(z)}{(z-a)^2}dz = 0.$

$\lim_{h\to 0} h\int_\gamma \frac{f(z)}{(z-a)^2(z-a-h)}dz = 0.$ So enough to show $\lim_{h\to 0}\int_\gamma \frac{f(z)}{(z-a)^2(z-a-h)}$ is bounded.

By ML-lemma, $\left|\int_\gamma\right| \leq \max_{z\in\gamma}\left|\frac{f(z)}{(z-a)^2(z-a-h)}\right| \cdot$ length.

for $|h| < \frac{\varepsilon}{2}$, $|z-a-h| \geq |z-a|-|h| \geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}$

$\max\left|\frac{f(z)}{(z-a)^2(z-a-h)}\right| \leq \max \frac{|f(z)|}{\varepsilon^2 \cdot \varepsilon/2}$ which is independent of h.

02/18/2022 Friday.

Liouville's Thm:

Suppose $f(z)$ is holomorphic on $\mathbb{C}$, and $f(z)$ is bounded i.e. $|f(z)| \le M$ for one fixed M. Then $f(z)$ is constant.

Note: $f(x+iy) = \frac{x^2+y^2-1}{x^2+y^2+1}$. real differentiable, bonded, but Not constant $\Rightarrow$ Not holomorphic.

Proof: $f'(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^2}dz$. $a \in \mathbb{C}$.

γ: circle of radius R centered at a.

by ML-lemma: $|f'(a)| \le \frac{1}{2\pi} \max_{z\in\gamma} \frac{|f(z)|}{|z-a|^2} \cdot \text{length} = \frac{1}{2\pi}\frac{|f(z)|}{R^2}\cdot 2\pi R \le \frac{1}{2\pi}\frac{M}{R^2}2\pi R = \frac{M}{R}$.

Since M doesn't depend on R, $|f'(a)| \le \frac{M}{R}$ for $R>0$.

As $R\to\infty$, $|f'(a)|=0 \Rightarrow f'(a)=0 \Rightarrow f(a) = \text{constant}$.

Suppose $f(z)$ is holomorphic inside γ, and a is inside γ.

Claim: $f''(a)$ exists and $f''(a) = \frac{1}{\pi i}\int_\gamma \frac{f(z)}{(z-a)^3}dz$.

Proof: $2\pi i \frac{f'(a+h)-f'(a)}{h} = \frac{1}{h}\left(\int_\gamma \frac{f(z)}{(z-a-h)^2}dz - \int_\gamma \frac{f(z)}{(z-a)^2}dz\right) = \frac{1}{h}\int_\gamma \frac{h\cdot(2z-2a-h)}{(z-a)^2(z-a-h)^2}f(z)dz$.

$2\pi i f''(a) =$ $\qquad 2\pi i f'(a) = \int_\gamma \frac{f(z)}{(z-a)^2}dz$.

check $\lim_{h\to0}\int_\gamma \frac{(2z-2a-h)}{(z-a)^2(z-a-h)^2}f(z)dz = 2\int_\gamma \frac{f(z)}{(z-a)^3}dz$.

$\lim_{h\to0}\int_\gamma \frac{(2z-2a-h)}{(z-a)^2(z-a-h)^2}f(z)dz - \int_\gamma \frac{2f(z)}{(z-a)^3}dz = 0$.

$\lim_{h\to0} 3h\int_\gamma \frac{f(z)}{(z-a)^2(z-a-h)^2}dz - 2h^2\int_\gamma \frac{f(z)}{(z-a)^3(z-a-h)^2}dz = 0$.

$\lim_{h\to0} h(3-2h)\int_\gamma \frac{f(z)}{(z-a)^3(z-a-h)^2}dz = 0$.

$\left|\int_\gamma \frac{1}{(z-a)^2(z-a-h)^2}f(z)dz\right| \le \max_{z\in\gamma}\left|\frac{f(z)}{(z-a)^2(z-a-h)^2}\right|\cdot\text{length}(\gamma) \le \max_{z\in\gamma}\frac{|f(z)|}{\varepsilon^2(\frac{\varepsilon}{2})^2}\cdot\text{length}(\gamma)$. Independent of h.

$\Downarrow$ bounded.

Suppose $f(z)$ is holomorphic at a, it is differ. on a disk centered at a of radius $\varepsilon>0$.

if $|z-a|<\varepsilon$, $f'(z)$ exists.

Let γ be the circle centered at a, radius $\frac{\varepsilon}{2}$, so $f(z)$ is diff. inside γ.

By Cauchy integral formula and its corollaries, $f''(a)$ exists. $\Rightarrow f'(z)$ is diff. at $z=a$.

$\Rightarrow f'(z)$ is holomorphic at a. $\Rightarrow f^{(n)}(z)$ is holomorphic at a.

(Not true for real diff. fns.)

02/23/2022 Wednesday.

$\deg(p)>0$, p depends on z. $p(z)\neq a_0$.

Fundamental theorem of algebra: if $p(z)$ is a nonconstant polynomial with complex coeffs.
$p(z)=a_d z^d + a_{d-1}z^{d-1}+\cdots+a_1 z^1 + a_0 z^0$ $a_i\in\mathbb{C}$ $a_d\neq 0$.
Any such polynomials can be written as a product of linear factors $(ax+b,\ a,b\in\mathbb{C})$.
Note: if $p(z)$ has real or even rational coeffs, it does not mean the linear factors do.

Weaker Version ⭐: if $p(z)$ is a nonconstant poly, $\exists w\in\mathbb{C}$ s.t. $p(w)=0$. $\xrightarrow{\text{inductively}}$ FtoA.

Lemma: Given $p(z)=a_d z^d+a_{d-1}z^{d-1}+\cdots+a_0$. $(a_d\neq 0)$. $\exists R\in\mathbb{R}$ s.t. $|z|\geq R$,
then $\frac{1}{2}|a_d z^d|\leq |p(z)|\leq \frac{3}{2}|a_d z^d|$.
equivalently, $\left|\frac{|p(z)|}{|a_d z^d|}-1\right|\leq\frac{1}{2}$.

Proof of Lemma: $\left|\frac{p(z)}{a_d z^d}\right|=\left|1+\frac{a_{d-1}}{a_d}z^{-1}+\frac{a_{d-2}}{a_d}z^{-2}+\cdots+\frac{a_0}{a_d}z^{-d}\right|$
if $|z|\to\infty$, $\left|\frac{p(z)}{a_d z^d}\right|\to 1$.
$\forall\varepsilon>0$, $\exists R$ s.t. $|z|\geq R$. s.t. $\left|\frac{p(z)}{a_d z^d}-1\right|\leq\varepsilon$. take $\varepsilon=\frac{1}{2}$.

Proof of ⭐: Consider $\frac{1}{p(z)}$. Suppose $p(z)\neq 0$ for any z, then $\frac{1}{p(z)}$ is a holomorphic fn.

Composition of holomorphic.

want to show $\frac{1}{p(z)}$ is constant.
By Liouville's thm, enough to show $\frac{1}{p(z)}$ is bounded.
if $|z|\geq R$, $\frac{1}{p(z)}\leq\frac{1}{\frac{1}{2}|a_d z^d|}=\frac{2}{|a_d|}|z|^{-d}\leq\frac{2}{|a_d|}R^{-d}$ bounded.
if $|z|\leq R$, $|z|\leq R$ is compact (closed and bounded)
$\frac{1}{p(z)}$ is holomorphic $\Rightarrow$ cts.
As $\left|\frac{1}{p(z)}\right|$ is composition of cts. fn. it is cts.
the cont. image of compact set is compact. $\Rightarrow$ bounded by M.
$\left|\frac{1}{p(z)}\right|\leq\max\left(\frac{2}{|a_d|}|z|^{-d}, M\right)$
$\frac{1}{p(z)}=\text{Constant}\Rightarrow p(z)=\text{Constant}.$

02/25/2022 Friday.

Recall Cauchy-Riemann Equations.

$f(x+iy) = u(x,y) + iv(x,y)$.

$u_x = v_y \quad u_y = -v_x$. if first derivative cont $\rightarrow$ f is holomorphic.

$\downarrow$

infinite differentiable.

$\downarrow$

$u_{xx} + u_{yy} = 0$ 2D Laplace Equations.

$\downarrow$

$u(x,y), v(x,y)$ are solutions of Laplace Equations. (n-dimensional)

Def: Solutions of the Laplace Equations are called Harmonic functions. $u: \mathbb{R}^2 \to \mathbb{R}$.

Task today: find conditions under which a harmonic fn is the real part of a holomorphic fn.

Assert: the second derivatives $u_{xx}, u_{xy}, u_{yy}, u_{yx}$ exist and are cts. $u_{xx} + u_{yy} = 0$.

$\Downarrow$

$u_{xy} = u_{yx}$.

Thm: if $u(x,y)$ is a harmonic fn on an open subset $G \subseteq \mathbb{R}^2$ with no holes,

($\forall$ closed γ in G, the region enclosed by γ is also contained in G.)

then $\exists$ harmonic fn $v: G \to \mathbb{R}$ s.t. $f(x+iy) = u(x,y) + i\underbrace{v(x,y)}$ is holomorphic on G.

"harmonic conjugate" of $u(x,y)$.

Proof: $f'(x+iy) = u_x(x,y) + iv_x(x,y)$

if f is holomorphic, $v_x = -u_y$, $f'(x+iy) = u_x(x,y) - iu_y(x,y)$

define $g(x+iy) = u_x(x,y) - iu_y(x,y)$

check g is holomorphic: u has cts. 2^{nd} derivative, so g has cts. 1^{st} derivative.

$(u_x)_x = u_{xx} \overset{\text{by assertion}}{=} -u_{yy} = (-u_y)_y$.

$(u_y)_x = u_{yx} \overset{\text{by assertion.}}{=} u_{xy} = -(-u_y)_x$

$u_x = v_y$

$u_y = -v_x$

C-R ✓ $\Rightarrow$ holomorphic.

suppose $f(z)$ is the antiderivative of $g(z)$ i.e. $f'(z) = g(z)$. $f(w) = \int_\gamma g(z)dz$

γ: any path from a fixed basepoint to w.

$f(z)$ exists if (1) $g(z)$ is cts.

(2) $\int_\gamma g(z)\,dz = 0$ for any closed curve γ in G.

(1): $g(z)$ holomorphic $\Rightarrow$ diff. $\Rightarrow$ cts.

(2): By Cauchy thm, $\int_\gamma g(z)\,dz = 0$

$f(x+iy) = a(x,y) + ib(x,y)$ where $a,b: \mathbb{R}^2 \to \mathbb{R}$.

as $f'(z) = g(z)$, $a_x + ib_x = u_x - iu_y$. $\Rightarrow \begin{cases} a_x = u_x \\ b_x = -u_y \\ \;\; \shortparallel \\ -a_y \end{cases}$ ($f(z)$ is holomorphic as $f'(z) = g(z)$)

$a_x = u_x \Rightarrow a(x,y) = u(x,y) + C(y)$.

$a_y = u_y \Rightarrow a(x,y) = u(x,y) + D(x)$. $\Rightarrow C(y) - D(x) = 0$

the constant doesn't depend on x or y.

$a(x,y) = u(x,y) + \text{const.}$

$f(x+iy) - \text{Const.} = a(x,y) - \text{Const.} + ib(x,y) = u(x,y) + ib(x,y)$ proved.

$f(z)$ is a holo fn whose real part is $u(x,y)$.

Ex. the thm doesn't apply to $G = \mathbb{C} \setminus \{0\}$

$u(x,y) = \log(r) = \log(\sqrt{x^2+y^2})$ which is a harmonic fn

is the real part of a holomorphic fn ($\log(z)$) on $G = \mathbb{C} \setminus \mathbb{R}_{\le 0}$.

02/28/2022 Monday.

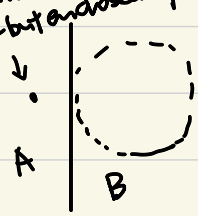
Corollary: Every harmonic function is infinitely differentiable.

pf: if $u: G \to \mathbb{R}$ (G is open subset of $\mathbb{C}$) and $z \in G$ is the pt where we want to check if u is infinitely diff.

Since G is open, it contains a disk centered at z. let's check this disk has no holes.

By Jordan Curve Theorem, a curve separates the plane into a bounded enclosed region (interior) and an unbounded region (exterior). → tangent line?

if the disk had a hole, $\exists\, \gamma$ inside the disk enclosing a point outside the disk.



Since γ is inside the disk, it cannot cross the tangent line. So the points must be in the same region (either A or B → bounded). However, A is unbounded. so the point is not enclosed by γ.

So we have our harmonic be the real part of a holo fn. Holo fn is infinitely complex-differentiable, so infinitely real-differentiable. Hence $\mathrm{Re}(f(z))$ is infinitely real-diff.

Laplace Equations. $\nabla \cdot \nabla u = 0$.

$\nabla \cdot$ ↑ net flow out; ∇u ↑ gradient. = change of u in every direction.

Prop: Suppose $u: G \to \mathbb{R}$ is a harmonic fn on an open set G containing the closed disk centered at w of radius r. Then $u(w) = \frac{1}{2\pi}\int_0^{2\pi} u(w+re^{i\theta})\,d\theta$.

↑ value of u at the center = average values of u on the boundary.

Proof: As the disk has no holes, $u(z) = \mathrm{Re}(f(z))$

By Cauchy Integral formula, $f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,dz$. γ is the circle of radius r centered at w.

$\gamma(\theta) = w + re^{i\theta}$. $\gamma'(\theta) = ire^{i\theta} = i(\gamma(\theta) - w)$

$f(w) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(\gamma(\theta))}{\gamma(\theta)-w}\cdot \gamma'(\theta)\,d\theta = \frac{1}{2\pi i}\int_0^{2\pi} i f(\gamma(\theta))\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(w+re^{i\theta})\,d\theta$

taking the real parts, $u(w) = \frac{1}{2\pi}\int_0^{2\pi} u(w+re^{i\theta})\,d\theta$.

if $u(w)$ is the maximum, values of u on the boundary are also maximums.

prop: Suppose $u: G \to \mathbb{R}$ is harmonic ($G \subseteq \mathbb{C}$ open), $w \in G$ s.t. $u(z) \le u(w)$ for all $z \in G$, then for any disk centered at w with radius r contained in G, $u(z) = u(w)$ for z in this disk.
inside + boundary.

proof: want to check $u(z) = u(w)$ for z inside the disk.
Suppose $\exists \theta_0$ s.t. $u(w+re^{i\theta_0}) < u(w)$. want to show this contradicts $u(w) = \frac{1}{2\pi}\int_0^{2\pi} u(w+re^{i\theta})d\theta$.
As a function of θ, $u(w+re^{i\theta})$ is cts. (composition of cts. fns, u diff. $\to$ cts.).
$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|\theta - \theta_0| < \delta$ implies $|u(w+re^{i\theta}) - u(w+re^{i\theta_0})| < \varepsilon$. let $\varepsilon = \frac{1}{2}(u(w) - u(w+re^{i\theta_0}))$
$u(w+re^{i\theta}) - u(w+re^{i\theta_0}) \le |u(w+re^{i\theta}) - u(w+re^{i\theta_0})| < \varepsilon = \frac{1}{2}(u(w) - u(w+re^{i\theta_0}))$
Add $\frac{1}{2}u(w) + \frac{1}{2}u(w+re^{i\theta_0}) - u(w+re^{i\theta})$ to both sides:
$\frac{1}{2}u(w) - \frac{1}{2}(u(w+re^{i\theta_0})) < u(w) - u(w+re^{i\theta}) \Rightarrow \varepsilon < u(w) - u(w+re^{i\theta}). \Rightarrow u(w+re^{i\theta}) < u(w) - \varepsilon.$
Now $\int_0^{2\pi} u(w+re^{i\theta})d\theta = \int_{\theta_0-\delta}^{\theta_0+\delta} u(w+re^{i\theta})d\theta + \int_{\text{rest in } [0,2\pi]} u(w+re^{i\theta})d\theta.$
By ML-Lemma: $\left|\int_{\theta_0-\delta}^{\theta_0+\delta} u(w+re^{i\theta})d\theta\right| \le |u(w+re^{i\theta})| 2\delta < (u(w)-\varepsilon)2\delta.$
$\left|\int_{\text{rest in } [0,2\pi]} u(w+re^{i\theta})d\theta\right| \le (u(w))(2\pi - 2\delta)$
So $\int_0^{2\pi} u(w+re^{i\theta})d\theta \le (u(w)-\varepsilon)2\delta + (u(w))(2\pi - 2\delta) = 2\pi u(w) - 2\delta\varepsilon.$
Thus $u(w) = \frac{1}{2\pi}\int_0^{2\pi} u(w+re^{i\theta})d\theta \le u(w) - \frac{\delta\varepsilon}{\pi} < u(w)$ contradicts $\Rightarrow$ such θ_0 doesn't exist

03/02/2022 Wednesday.

We know if $u(w) \geq u(z)$ for all $z \in G$, $u(z) = u(w)$ for $z \in$ disk.

However, $u(w') \geq u(z)$ for all $z \in G$, So we have $u(z) = u(w') = u(w)$ for $z \in$ new disk.

Def: A set G is path Connected if for any two points p, q in G, there is a curve from p to q contained in G. (i.e. $\gamma: [a,b] \to G$ cts. w/ $\gamma(a) = p$, $\gamma(b) = q$)

Example: Any disk is path connected as it is convex (contains the line segment joining any two points). the line segment provides the curve we need.

Non-Example: two disjoint open disks (G) centered at $2, -2$, with radius $=1$.

if γ exists, $\gamma: [a,b] \to G$ w/ $\gamma(a) = -2$ $\gamma(b) = 2$.

By Intermediate Value Theorem, for some $t \in [a,b]$, $Re(\gamma(t)) = 0$

But there is no $z \in G$ w/ $Re(z) = 0$ by inspection.

Consider $g: G \to \mathbb{R}$. define $g(z) = \begin{cases} 1 & \text{if } z \text{ in right disk.} \\ 0 & \text{if } z \text{ in left disk.} \end{cases}$ harmonic

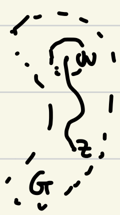
g attains maximum at the right disk, but g is not constant on G.

Therefore, If G is path connected, $u(z) = u(w)$ for $z \in G$.

03/04/2022 Friday.

A harmonic fn on a path-connected open set G that attains maximal value must be constant.

Prop: $u(z)=u(w)$ for all $z\in G$. $\Rightarrow$



Idea: Stepping along the path from w to z, show that u is constant along each step

if u is constant on a disk of radius ε centered at w, then we can step ε along the curve, u will still be constant.

→ strictly inside the disk.

step $\frac{\varepsilon}{2}$ each time, after n times, length$(\gamma)-n\frac{\varepsilon}{2}<\varepsilon$. z is inside the disk $u(z)=u(w)$

ε.

check we can use disks of same radius ε of any point on γ.

Want $\varepsilon>0$, and disk of radius ε centered at $\gamma(t)$ to be contained in G for all t.

i.e. $|\gamma(t)-z|<\varepsilon \Rightarrow z\in G$.

→ greatest lower bound.

$G\subseteq\mathbb{C}$ open, $\gamma:[a,b]\xrightarrow{\text{cts.}}G$. $d(t)=\inf_{z\in\mathbb{C}\setminus G}|\gamma(t)-z|$ if L is a lower bound of S, $L\le\inf S$.

As $\gamma(t)\in G$, G is open, $\exists$ disk centered at $\gamma(t)$ contained in G, radius $\varepsilon(t)$.

so any $z\in\mathbb{C}\setminus G$ is at least $\varepsilon\cancel{(t)}$ away from $\gamma(t)$. $|\gamma(t)-z|\ge\varepsilon\cancel{(t)}>0$.

$d(t)=\inf_{z\in\mathbb{C}\setminus G}|\gamma(t)-z|\ge\varepsilon\cancel{(t)}>0$.

↑ strictly.

Let's show $d(t)$ is cts. i.e. $\lim_{h\to0}d(h+t)=d(t)$

$|\gamma(t)-z|\ge d(t)$ $|\gamma(t+h)-z|\ge d(t+h)$

→ triangular inequality.

$d(t)\le|\gamma(t)-z|=|\gamma(t)-\gamma(t+h)+\gamma(t+h)-z|\le|\gamma(t)-\gamma(t+h)|+|\gamma(t+h)-z|$

so $d(t)-|\gamma(t)-\gamma(t+h)|\le|\gamma(t+h)-z|$ $\Rightarrow$ $\underbrace{d(t)-|\gamma(t)-\gamma(t+h)|}_{\text{lower bound}}\le\underbrace{d(t+h)}_{\text{inf.}}$

$d(t)-d(t+h)\le|\gamma(t)-\gamma(t+h)|$. ①

As $d(t+h)\le|\gamma(t+h)-z|=|\gamma(t+h)-\gamma(t)+\gamma(t)-z|\le|\gamma(t+h)-\gamma(t)|+|\gamma(t)-z|$.

WLOG, $d(t+h)-|\gamma(t+h)-\gamma(t)|\le d(t)$. $\Rightarrow$ $d(t+h)-d(t)\le|\gamma(t+h)-\gamma(t)|$. ②.

by ① and ②. $0\le|d(t+h)-d(t)|\le|\gamma(t+h)-\gamma(t)|$.

by squeeze thm, $h\to0$, by continuity of γ, $\lim_{h\to0}\gamma(h+t)=\gamma(t)$. $d(t+h)-d(t)\to0$

Therefore, we have $d:[a,b]\to\mathbb{R}$ with $d(t)>0$ and d cts, $[a,b]$ compact.

closed and bounded. → contains 0 as $d(t)>0\Rightarrow d(t)\ne0$ $(-\varepsilon,\varepsilon)$

So the set of values of $d(t)$ is compact. So its complement is open.

(By Weierstrass Thm, a cts fn on a compact set has a minimum which it attains.

the min value of $d(t)$ for $t\in[a,b]$ is some $\varepsilon>0$, $d(t)\ge\varepsilon>0$.)

Thm: <u>Maximum Modulus Principle.</u>

Suppose G is a path connected open set in $\mathbb{C}$, and f is a holomorphic function on G. Then if $|f|$ attains a maximum on G (i.e. $\exists w \in G$ s.t. $\forall z \in G$, $|f(z)| \le |f(w)|$.), and $f(z) \neq 0$ for all $z \in G$, f is constant on G.

03/07/2022 Monday.

power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$

Key example: Taylor Series $f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)(x-x_0)^2/2 + \cdots$

convergence of sequence of partial sums $\sum_{n=0}^{k} a_n(z-z_0)^n$ = convergence of the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$.

suppose $G \subseteq \mathbb{C}$, $f_n : G \to \mathbb{C}$ ($n \in \mathbb{Z}_{\geq 0}$) is a sequence of functions.

a sequence of complex number.

Def: $f_n(z)$ converges to $f(z)$ pointwise if $\forall z \in G$, the sequence $f_n(z)$ converges to $f(z)$.

i.e. $\forall z \in G$, $\forall \varepsilon > 0$, $\exists N \geq 0$ s.t. $n \geq N$ implies $|f_n(z) - f(z)| < \varepsilon$.

can depend on z.

Example: $f_n(z) = z^n$ $G = [0,1] \subseteq \mathbb{R}$.

when $z<1$, the limit is zero. Want to find N s.t. $|f_n(z) - 0| < \varepsilon$ for $n \geq N$.

since z is a nonnegative real number, $z^n < \varepsilon$. $\Rightarrow$ $n\underbrace{\log(z)}_{<0} < \log(\varepsilon)$. $\Rightarrow$ $n > \frac{\log(\varepsilon)}{\log(z)} \to N$.

$f_n(z)$ converges to $f(z) = \begin{cases} 1 & z=1 \\ 0 & 0 \leq z < 1 \end{cases}$ which is not cts.

Def: a sequence of functions $f_n(z)$ converges to $f(z)$ uniformly on G if $\forall \varepsilon > 0$,

$\exists N \geq 0$ s.t. $|f_n(z) - f(z)| < \varepsilon$ for all $n \geq N$ and $z \in G$.

can't depend on z.

Example: $f_n(z) = z^n$ $G = [0,r]$ $r<1$, fixed.

$|f_n(z) - f(z)| < \varepsilon$. $\Rightarrow z^n < \varepsilon$ when $n > \frac{\ln(\varepsilon)}{\ln(z)}$

need to find N s.t. $N \geq \frac{\ln(\varepsilon)}{\ln(z)}$ for all $z \in G = [0,r]$.

maximized when $z=r$. $\Rightarrow$ take $N = \frac{\ln(\varepsilon)}{\ln(r)}$

though $f_n(z) \to 0$ on $G = [0,1)$, it doesn't converge uniformly.

because as $z \to 1$, $\frac{\ln(\varepsilon)}{\ln(z)} \to \infty$, so no such N (which all z obey the inequality) exists.

Properties of uniform convergence:

1) if $f_n(z)$ are cts. and converge uniformly to $f(z)$, then $f(z)$ is cts.

proof: Fix $z_0 \in G$, need to show $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$.

Let N be such that $|f_n(z) - f(z)| < \varepsilon/3$ for $n \geq N$. by uniform convergence.

pick some $n \geq N$ and let δ be s.t. $|f_n(z) - f_n(z_0)| < \varepsilon/3$ when $|z - z_0| < \delta$. by conti. of $f_n(z)$.

$|f(z) - f(z_0)| = |f(z) - f_n(z) + f_n(z) - f_n(z_0) - f_n(z_0) - f(z_0)| \leq |f(z) - f_n(z)| + |f_n(z) - f_n(z_0)| + |f_n(z_0) - f(z_0)|$

$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$ $\Rightarrow$ $f(z)$ is cts.

2) if $f_n(z)$ converges to $f(z)$ uniformly on G, and γ is a curve in G, then

$\lim_{n \to \infty} \int_\gamma f_n(z)\,dz = \int_\gamma f(z)\,dz$. (so we can swap the order of limits and integration).

proof: for $\varepsilon > 0$, pick N s.t. $n \geq N \Rightarrow |f_n(z) - f(z)| < \frac{\varepsilon}{\text{length}(\gamma)}$

$\left|\int_\gamma f_n(z)\,dz - \int_\gamma f(z)\,dz\right| = \left|\int_\gamma f_n(z) - f(z)\,dz\right| \leq \max_{z \in \gamma} |f_n(z) - f(z)| \cdot \text{length}(\gamma)$

since $\forall z \in G$, $|f_n(z) - f(z)| < \varepsilon / \text{length}(\gamma)$ $\Rightarrow$ $\left|\int_\gamma f_n(z)\,dz - \int_\gamma f(z)\,dz\right| \leq \varepsilon$.

03/09/2022 Wednesday.

Thm: Weierstrass Test (M-test)

if $G \subseteq \mathbb{C}$, $f_k: G \to \mathbb{C}$ $(k \in \mathbb{Z}_{\geq 0})$ and $|f_k(z)| \leq M_k$ (nonnegative real) for all $z \in G$, then

if $\sum_{k=0}^{\infty} M_k$ converges, then $\sum_{k=0}^{\infty} f_k(z)$ converges uniformly on G.

(also implies $\sum_{k=0}^{\infty} |f_k(z)|$ converges i.e. $\sum_{k=0}^{\infty} f_k(z)$ converges absolutely.)

proof: $\sum_{k=0}^{\infty} |f_k(z)| \leq \sum_{k=0}^{\infty} M_k$ finite by assumption. so

As $\sum_{k=0}^{\infty} M_k$ converges, say M. $\forall \varepsilon > 0, \exists N > 0$ s.t. if $n \geq N$, $\left|\sum_{k=0}^{n} M_k - M\right| < \varepsilon$.

$\sum_{k=0}^{n} M_k - \sum_{k=0}^{\infty} M_k = -\sum_{k=n+1}^{\infty} M_k$ so $\left|\sum_{k=0}^{n} M_k - M\right| = \left|-\sum_{k=n+1}^{\infty} M_k\right| = \sum_{k=n+1}^{\infty} M_k < \varepsilon$.

As $\sum_{k=0}^{\infty} f_k(z)$ converges, $\forall \varepsilon > 0$. $\exists N > 0$ s.t. if $n \geq N$, $\left|\sum_{k=0}^{n} f_k(z) - \sum_{k=0}^{\infty} f_k(z)\right| < \varepsilon$.

Now $\left|\sum_{k=0}^{n} f_k(z) - \sum_{k=0}^{\infty} f_k(z)\right| = \left|-\sum_{k=n+1}^{\infty} f_k(z)\right| \leq \sum_{k=n+1}^{\infty} |f_k(z)| \leq \sum_{k=n+1}^{\infty} M_k < \varepsilon$. N doesn't depend on $z \Rightarrow$ uniform.

triangular inequality

By same argument, $\sum_{k=0}^{\infty} |f_k(z)|$ converges uniformly.

Example: $f_k(z) = z^k$. Consider $\sum_{k=0}^{\infty} f_k(z) = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$ if $|z| < 1$. (geometric series) Uniform Convergence?

Let $G = \{z \in \mathbb{C} \mid |z| < r\}$. $0 < r < 1$ Use M-test to check uniform convergence.

$|f_k(z)| = |z^k| = |z|^k < r^k$. So we take $M_k = r^k$. then $\sum_{k=0}^{\infty} M_k = \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$

is uniform convergence on open unit disk? NO. (r infinitely close to 1.)

if it is true, $\forall \varepsilon > 0, \exists N > 0$ s.t. $n \geq N$, $\left|\sum_{k=0}^{n} z^k - \frac{1}{1-z}\right| < \varepsilon$ for all $|z| < 1$.

i.e. $\left|\frac{1-z^{n+1}}{1-z} - \frac{1}{1-z}\right| = \frac{|z|^{n+1}}{|1-z|} < \varepsilon$. for all $|z| < 1$.

However, as $z \to 1$, this can be arbitrary large regardless of n.

Lemma: Suppose $\sum_{k=0}^{\infty} a_k (w - z_0)^k$ $(a_k \in \mathbb{C})$ converges, then $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ converges absolutely whenever $|z - z_0| < |w - z_0|$.

Moreover, $\forall \rho < |w - z_0|$, converges on $|z - z_0| \leq \rho$ is uniform.

fixed. closed.

proof: let $r = |w - z_0|$.

Since $\sum_{k=0}^{\infty} a_k (w - z_0)^k$ converges, its terms must converge to zero. $\lim_{k \to \infty} a_k (w - z_0)^k = 0$.

(if $\sum c_k$ converges, $\lim_{k \to \infty} c_k = \lim_{k \to \infty} \left(\sum_{n=0}^{k} c_n - \sum_{n=0}^{k-1} c_n\right) = 0$)

Pick $\varepsilon > 0$, $\exists N > 0$ s.t. if $n \geq N$, $|a_n(w-z_0)^n - 0| < \varepsilon$. $\Rightarrow |a_n||w-z_0|^n < \varepsilon$.

$\forall k$, $|a_n(w-z_0)^k| \leq \max\{|a_n(w-z_0)^0|, |a_n(w-z_0)^1|, \dots, |a_0(w-z_0)^N|, \varepsilon\}$.

So $\exists M$ s.t. $|a_k(w-z_0)^k| \leq M$ for all k.

$$\sum_{k=0}^{\infty} |a_k(z-z_0)^k| \leq \sum_{k=0}^{\infty} |a_k(w-z_0)^k| \left|\frac{z-z_0}{w-z_0}\right|^k \leq \sum_{k=0}^{\infty} \underbrace{M}_{\text{Constant}} \underbrace{\left|\frac{z-z_0}{w-z_0}\right|^k}_{\text{Converging geometric series as } |z-z_0| < |w-z_0|}$$

By M-test, Want some M_k s.t. $M_k \geq |a_k(z-z_0)^k| = |a_k(w-z_0)^k| \left|\frac{z-z_0}{w-z_0}\right|^k \Rightarrow M_k \geq M\left(\frac{t}{r}\right)^k$

Take $M_k = M\left(\frac{t}{r}\right)^k$, then $\sum_{k=0}^{\infty} M_k = M \frac{1}{1-t/r}$ converges.

03/11/2022 Friday.

Thm: For the power series $\sum a_k(z-z_0)^k$, there is $R \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ s.t. the series converges absolutely when $|z-z_0|<R$, diverges when $|z-z_0|>R$, converges uniform on $|z-z_0| \leq \rho$. $(\rho < R)$

proof: Consider $S = \{x \in \mathbb{R}_{\geq 0} \mid \sum_k a_k x^k \text{ converges}\}$.

As $0 \in S \Rightarrow S$ is nonempty. S has the supremum in $\mathbb{R}_{\geq 0} \cup \{\infty\}$. least upper bound.

By Lemma, if $x \in S$, $\sum_{k\geq 0} a_k x^k$ converges $\Rightarrow \sum_{k \geq 0} a_k (z-z_0)^k$ converges absolutely for $|z-z_0|<|x|=x$.

let $z = z_0 + y$. $0 \leq y < x \Rightarrow \sum a_k y^k$ converges absolutely. $\Rightarrow y \in S$.

y can be in $[0,x) \Rightarrow x \in S$, $[0,x] \subseteq S$. $\Rightarrow [0,x] \subseteq S$.

Case 1. $\sup S = \infty$. S contains a sequence x_i with limit $=+\infty$. so S contains $[0,x_i]$ for all i.

for any real t, eventually $x_i \geq t$, so $t \in [0,x_i] \subseteq S$, so $t \in S$ and $S = \mathbb{R}_{\geq 0}$. $(R=\infty)$

Case 2. S has a supremum $\in \mathbb{R}$. $R = \sup S$.

if $|z-z_0|>R$ and $\sum_{k \geq 0} a_k |z-z_0|^k$ converges then S contains all x with $0 \leq x < |z-z_0|$.

but R is an upper bound for S. e.g. $x = \frac{R+|z-z_0|}{2} > R$. contradicts

So if $|z-z_0|>R \Rightarrow$ divergence.

if $|z-z_0|<R$. S contains a limit x s.t. $|z-z_0| < x \leq R$. otherwise $|z-z_0|$ would be a upper bound $<R \Rightarrow$ the least upper bound $\Rightarrow |z-z_0|=R$ contradicts. S not necessarily contains R. $x \leq |z-z_0| < R$.

Now $\sum_{k \geq 0} a_k x^k$ converges and $|z-z_0|<x$.

by lemma, absolute convergence for $|z-z_0|<x$. i.e. $|z-z_0|<R$.

Def: this R is the radius of convergence of $\sum_{k\geq 0} a_k (z-z_0)^k$. (knows nothing when $|z-z_0|=R$).

Lemma: $\frac{1}{R} = \limsup \sqrt[k]{|a_k|}$ (if $R=0$, $\frac{1}{R}=\infty$; if $R=\infty$, $\frac{1}{R}=0$).

Proof: if $\limsup \sqrt[k]{|a_k|} = L$. Can have noise ($>\limsup$) at the beginning

$\forall \varepsilon>0, \exists N>0$ s.t. $k \geq N$ implies $\sqrt[k]{|a_k|} < L+\varepsilon$. so $|a_k| < (L+\varepsilon)^k$.

$\sum_{k\geq 0} |a_k (z-z_0)^k| \leq \sum_{k \geq 0}^{N} |a_k| |z-z_0|^k + \sum_{k > N} |a_k| |z-z_0|^k \leq [\text{some constant}] + \sum_{k > N} (L+\varepsilon)^k |z-z_0|^k$.

this converges when $(L+\varepsilon)|z-z_0|<1$ for some $\varepsilon>0$. $|z-z_0| < \frac{1}{L+\varepsilon}$.

As $0 < \frac{1}{L+\varepsilon} < \frac{1}{L}$, absolute convergence when $|z-z_0| < \frac{1}{L}$.

we conclude $R \geq \frac{1}{L}$. As if $\frac{1}{L} > R$, choose z s.t. $\underbrace{\frac{1}{L} > |z-z_0|}_{\text{absolute converge}}\overbrace{\, > R}^{\text{diverge}}$ contradiction.

$\forall \varepsilon>0$, there are infinitely many k s.t. $\sqrt[k]{|a_k|} > L-\varepsilon$.

Assume $L>0$, for ε small enough, $L-\varepsilon>0$. $|a_k| > (L-\varepsilon)^k$ $\quad |a_k||z-z_0|^k \ge (L-\varepsilon)^k|z-z_0|^k \ge 0$

if $\sum a_k(z-z_0)^k$ converges, $|a_k(z-z_0)^k| \to 0$

By squeeze Thm, $(L-\varepsilon)^k|z-z_0|^k \to 0$. So we have $|(L-\varepsilon)|z-z_0|| < 1$. $\Rightarrow |z-z_0| < \frac{1}{L-\varepsilon}$

divergence when $\frac{1}{L-\varepsilon} \le |z-z_0|$.

As $\frac{1}{L-\varepsilon} > \frac{1}{L}$, we can find ε s.t. $\frac{1}{L-\varepsilon} \le |z-z_0|$ when $\frac{1}{L} < |z-z_0|$. So $\frac{1}{L} < |z-z_0| \Rightarrow$ divergence.

Claim: $\frac{1}{L} \ge R$, if not, $\underbrace{R>|z-z_0|}_{\text{convergence}}\overbrace{> \frac{1}{L}}^{\text{divergence}}$, contradicts.

Therefore, $\frac{1}{L} = R$.

Note: if $L=0$, any $\varepsilon>0$ is an upper bound on all but finitely many terms of $\sqrt[k]{|a_k|}$

$\sum |a_k||z-z_0|^k \le (\text{finite sum}) + \sum_{\text{sufficient large } k} \varepsilon^k |z-z_0|^k$ So converges if $\varepsilon < \frac{1}{|z-z_0|}$ Converges for all

$R=\infty$.

03/14/2022 Monday.

Example: $a_k = \frac{1}{k!}$ $z=0$. $\sum_{k\geq 0} \frac{z^k}{k!} = e^z$ $\frac{1}{R} = \limsup \sqrt[k]{|1/k!|}$

Stirling's formula $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ $\Rightarrow$ $\sqrt[n]{n!} \approx \frac{n}{e}$ + lower terms. $\Rightarrow$ $\sqrt[n]{\frac{1}{n!}} \approx \frac{e}{n} \to 0$. $R=\infty$.

Or by Ratio Test, $\sum c_k$ converges if $\lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right| < 1$.

$c_k = z^k/k!$ $\left|\frac{c_{k+1}}{c_k}\right| = |z/k+1| = |z|/k+1$. As $k\to\infty$, $|z|/k+1 \to 0$. Converges for all z. $R=\infty$.

Example: $\sum a_k(z-z_0)^k = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots = \arctan(z)$ $z_0 = 0$ $a_k = \begin{cases} 0 & k \text{ even.} \\ \frac{(-1)^{\frac{k-1}{2}}}{k} & k \text{ odd.}\end{cases}$ want to show $R=1$.

$\limsup_{k \text{ odd}} \sqrt[k]{\left|\frac{(-1)^{\frac{k-1}{2}}}{k}\right|} \underset{k \text{ odd}}{=} \sqrt[k]{1/k}$.

take log, $\ln(\sqrt[k]{1/k}) = \frac{1}{k}(-\ln(k)) \to 0$. so $\sqrt[k]{1/k} \to 1$. (rely on ln/exp being cts. at 1/0).

Prop: if $\sum_{k\geq 0} a_k(z-z_0)^k$ has radius of convergence R, then the limit of f_n is cts. on the open disk centered at z_0 with radius R.

proof. $\sum_{k\geq 0} a_k (z-z_0)^k$ is poly, so cts. By property of uniform convergence 1), enough to show uniform convergence on open disk.

let z in the disk, $|z-z_0| < R$. $|z-z_0| \leq \frac{|z-z_0|+R}{2}$ ← open/close.

03/16/2022 Wednesday.

Prop: If γ is a curve in the open disk of radius R centered at z_0,

$\int_\gamma [\sum_{k\geq 0} a_k(z-z_0)^k] dz = \sum_{k\geq 0} a_k [\int_\gamma (z-z_0)^k dz]$; if γ is closed, this equals zero.

proof: (curve γ in an open set does not arbitrarily close to the boundary.)

$\inf_{z\in \mathbb{C}\setminus G} |\gamma(t)-z| \geq \varepsilon$. for some $\varepsilon>0$, G = open disk of radius R.

so z is contained in the closed disk of radius $R-\varepsilon/2$., so uniform, giving the stated formula.

if γ is closed, $\int_\gamma (z-z_0)^k dz = 0$ → holomorphic everywhere.

Morera's Thm: If $f: G \to \mathbb{C}$ is cts. and $\int_\gamma f(z) dz = 0$ for every closed curve γ in G, then f(z) is holomorphic.

Proof: $\exists F(z)$ s.t. $F'(z) = f(z)$. F(z) is complex-diff. ⇒ F(z) holo ⇒ F(z) infinitely diff. ⇒ f(z) diff. ⇒ f(z) holo.

Thm: if $\sum_{k\geq 0} a_k (z-z_0)^k$ has radius of convergence R, then it converges to a holomorphic fn in the open unit disk of radius R. by

Lemma: if $f(z) = \sum_{k\geq 0} a_k (z-z_0)^k$ with radius of convergence R, then for $|z-z_0|<R$, $f'(z) = \sum_{k\geq 1} k a_k (z-z_0)^{k-1}$.

03/18/2022 Friday.

last time: $\frac{d}{dz}\left(\sum_{k=0}^{\infty} a_k (z-z_0)^k\right) = \sum_{k=0}^{\infty} \frac{d}{dz} a_k (z-z_0)^k$. Can differentiate a power series termwise.

Example: $z_0=0$ $a_k = \frac{1}{k!}$ $f(z) = 1+z+\frac{z^2}{2}+\cdots$ want to show $f(z)=e^z$.

$f'(z) = \sum_{k=0}^{\infty} \frac{k}{k!}(z-z_0)^{k-1} = f(z)$ $f(0)=1$.

Consider $f(z)e^{-z}$. $f'(z)e^{-z} + f(z)(-e^{-z}) = 0$. $\Rightarrow f(z)e^{-z} = c$.

at $z=0$, $f(0)e^{-0} = c \Rightarrow c=1$. so $f(z)e^{-z}=1$.

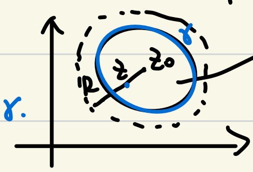
Thm: suppose $f(z)$ is holomorphic on $|z-z_0|<R$. then $f(z) = \sum_{k\geq 0} a_k (z-z_0)^k$ where $a_k = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{k+1}}dw$ and the series converges on $|z-z_0|<R$. (γ enclosing z_0 but contained in the disk).

proof: $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw$ now $\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{w-z_0}\cdot\frac{1}{1-\frac{z-z_0}{w-z_0}}$

want to expand as geo series to make sure $|w-z_0|>|z-z_0|$ on γ.

$|w-z_0| = \frac{|z-z_0|+R}{2}$

constant. radius of γ.



→ have uniform convergence on this disk (closed).

$\sum_{k\geq 0}\left(\frac{z-z_0}{w-z_0}\right)^k \to \frac{1}{1-\frac{z-z_0}{w-z_0}}$

$$f(z) = \frac{1}{2\pi i}\int_\gamma \underbrace{\frac{f(w)}{w-z_0}\sum_{k=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^k}_{\frac{f(w)}{w-z}} dw.$$

$\frac{f(w)}{w-z}$ is not defined on $w=z_0$, but defined and bounded on γ. (cts. γ is compact).

bounded fn $\times$ uniform converging series is still uniformly converging.

$$f(z) = \frac{1}{2\pi i}\sum_{k=0}^{\infty}\int_\gamma \frac{f(w)}{w-z_0}\cdot\frac{(z-z_0)^k}{(w-z_0)^k}dw. = \sum_{k=0}^{\infty}\underbrace{\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{k+1}}dw}_{a_k}(z-z_0)^k.$$

by interchange of Σ and $\int$.

$\int_\gamma \frac{f(w)}{(w-z_0)^{k+1}}dw$. is the same for any γ enclosing z_0.

* doesn't depend on z.

By Cauchy Thm, $\int_{\text{shaded region}}=0 \Rightarrow \int_\gamma - \int_{\gamma'} = 0$

Thm: The radius of convergence equals the largest number R' such that $f(z)$ can be extended to a holomorphic fn on $|z-z_0|<R'$.

proof. $R \geq R'$

if radius of convergence $= R$, the power series defines a holo fn (extending $f(z)$) on $|z-z_0|<R$, so $R' \geq R$.

Therefore, $R=R'$.

Example: $f(z) = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots = \arctan(z)$. $R=1$.

$\arctan(x)$ is infinitely differentiable, but still finite R.

$\arctan(x)$ doesn't extend holomorphically to $z = \pm i$.

if $\tan(z) = i$. $\sin(z) = i\cos(z) \Rightarrow \sin^2(z) + \cos^2(z) = 0$. Contradicts. So no such z.

we can't extend $\arctan(z)$ on $|z|<1$ holomorphically, so $R=1$.

$$\arctan(z) = \int_0^z \frac{1}{1+w^2} dw$$

blows up at $w = \pm i$.

Prop: $\frac{d^k f}{dz^k}(z_0) = \frac{k!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z_0)^{k+1}} dw = k! a_k$. $\Rightarrow f(z) = \sum_{k=0}^{\infty} \frac{\frac{d^k f}{dz^k}(z_0)}{k!} (z-z_0)^k$

γ encloses z_0.

$$a_k = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-z_0)^{k+1}} dw = \frac{d^k f}{dz^k}(z_0) / k!$$

So the power series is the Taylor series of f about z_0.

there is only one power series centered at z_0 and converging to $f(z)$.

03/28/2022 Monday.

holomorphic fn at $z_0 \Longleftrightarrow$ power series about z_0.

fns that can be written as a convergent power series on a disk centered at z_0 are called analytic (at z_0).

in complex analysis, holomorphic = analytic.

in other fields, analytic may make sense even if holomorphic does not.

do the same thing to holomorphic fns.

if $p(z)$ is a non-constant polynomial. if $p(z_0)=0$, then $p(z) = (z-z_0)\cdot\frac{p(z)}{z-z_0}$. → still polynomials. e.g. $\frac{z^2-1}{z+1} = z-1$

Thm: if $f(z)$ is non-constant holomorphic at z_0, $\exists m \in \mathbb{Z}_{\geq 0}$ s.t. $f(z) = g(z)\cdot(z-z_0)^m$ and

same. (1) $g(z)$ is holomorphic at z_0.

(2) $g(z_0) \neq 0$.

* Doesn't hold in real analysis. e.g. $\sqrt{|x|^3}$

proof: either all $a_k = 0 \Rightarrow$ constant fn.

or $\exists m$ s.t. $a_0, a_1, \cdots, a_{m-1} = 0$, $a_m \neq 0$. then $f(z) = (z-z_0)^m \underbrace{\sum_{k\geq m} a_k (z-z_0)^{k-m}}_{g(z)}$

$f(z) = a_m(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \cdots$

03/30/2022 Wednesday.

thm: If $f(z)$ is holomorphic at z_0 and $f(z_0)$, then either $f(z)$ is constant (on some neighborhood of z_0) or we have $f(z)=(z-z_0)^m g(z)$ with $m\in\mathbb{Z}_{\geq 0}$, $g(z)$ holomorphic at z_0, $g(z_0)\neq 0$.

same

m: the order of zero of $f(z)$ at z_0 (order of vanishing).

proof: $f(z)=\sum_{k\geq 0} a_k(z-z_0)^k$. let m be the least number s.t. $a_m\neq 0$, but $a_k=0$ for $k<m$. $f(z)=\sum_{k\geq m}(z-z_0)^k$.

$f(z)=(z-z_0)^m \underbrace{\sum_{k\geq 0} a_{k+m}(z-z_0)^k}_{g(z)}$. let's check $g(z)$ is holo. at z_0. and $g(z_0)\neq 0$.

Since convergent power series are holomorphic, it's enough to show that this has positive R.

① $R>0$: a_k comes from $f(z)=\sum_{k\geq 0} a_k(z-z_0)^k$ which is holo. at z_0. $\Rightarrow$ positive R. $\Rightarrow R>0$ ($f(z)$. $g(z)$)

② R finite: $f(z)$ holo. $\Rightarrow \limsup \sqrt[k]{|a_k|}<a$ for some $a>0$. $\Rightarrow |a_k|<a^k$. $\Rightarrow |a_{k+m}|<a^{k+m}$. $\Rightarrow \limsup \sqrt[k]{|a_{k+m}|} \leq a^{\frac{k+m}{k}} = a^{1+\frac{m}{k}} \leq a^{1+m}$.

$g(z_0)=a_m \neq 0$.

if no m s.t. $a_m\neq 0$, $f(z)=0 \to$ constant fn.

Corollary: if $f(z)$ is (nonconstant) holomorphic at z_0, $f(z_0)=0$, then there is an open disk centered at z_0 on which f takes the value 0 only at z_0. $\Rightarrow$ so zeros of holomorphic fns are isolated.

pf: $f(z)=(z-z_0)^m g(z)$ with $g(z_0)\neq 0$.

$\forall \varepsilon>0, \exists \delta>0$ s.t. if $|z-z_0|<\delta$, $|g(z)-g(z_0)|<\varepsilon$.

take $\varepsilon=|a_m|$, so $|g(z)-a_m|<|a_m|$ which forbids $g(z)=0$ for $|z-z_0|<\delta$.

Identity principle: if $f(z)$ is holomorphic on an open set G, and $z_1, z_2, z_3, \cdots$ are points in G with $f(z_i)=0$. if $\{z_i\}$ has an accumulation point in G, then $f(z)$ is the zero fn.

e.g. $\sin(\frac{1}{z})$. $w=0\notin G$.

Proof: let w be the accumulation point of z_i in G. so $\exists$ subsequence z_{i_j} converges to w.

f is holo. in $G \Rightarrow$ cts. at w. $f(w)=f(\lim_j z_{i_j})=\lim_j f(z_{i_j})=\lim_j 0=0$.

contradicts the corollary that $f(z)$ is nonzero on some disk centered at w

so constant fn $\Rightarrow$ zero fn.

Corollary: if $f(z), g(z)$ are holomorphic on G, and agree on a set $\{z_1, z_2, \cdots\}$ with an accumulation point, then $f(z)=g(z)$.

$\forall i, f(z_i)=g(z_i)$

(proof by applying $f(z)-g(z) \Rightarrow$ zero fn).

$f(z)=\sum_{k\geq 0} a_k (z-z_0)^k$. $a_k = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{k+1}}\,dw$.

$f(z)$ determined by power series $\Rightarrow a_k \Rightarrow$ Values of $f(z)$ along γ, a curve around z_0.

So f(z) is determined by its behavior on an open set containing z_0.

OR a set having z_0 as an accumulation point.

04/01/2022 Friday.

Def: A laurant series is a series of the form $\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$

$$\underbrace{\dots + a_{-2}(z-z_0)^{-2} + a_{-1}(z-z_0)^{-1}}_{\text{singular when } z=z_0} + \underbrace{a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots}_{\text{defines a holomorphic fn inside } R}$$

$\sum_{k=0}^{\infty} a_k(z-z_0)^k$ converges when $|z-z_0|<R_2$.

radius of convergence.

$\sum_{k=-\infty}^{-1} a_k(z-z_0)^k = \sum_{k=1}^{\infty} a_{-k}((z-z_0)^{-1})^k$ converges when $\left|\frac{1}{z-z_0}\right| < \frac{1}{R_1} \iff |z-z_0| > R_1$.

region where $R_1<|z-z_0|<R_2$ is called annulus.

if $R_1 \geq R_2$, the laurant series never converges.

e.g. $a_k=1$. $\sum_{k=-\infty}^{\infty} z^k = \sum_{k=-\infty}^{-1} z^k + \sum_{k=0}^{\infty} z^k$ $\Rightarrow$ never converge.

$\sum_{k=0}^{\infty} z^k$ converges if $|z|<1$.

$\sum_{k=-\infty}^{-1} z^k = \sum_{k=1}^{\infty} z^{-k}$ converges if $|z^{-1}|<1$

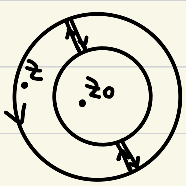
Thm: Suppose $f(z)$ is holomorphic in the annulus $R_1<|z-z_0|<R_2$, then $f(z)=\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$ with $a_k = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{k+1}}dw$.

γ: any curve inside the annulus enclosing z_0.

Since $\frac{f(w)}{(w-z_0)^{k+1}}$ is holomorphic in the annulus ($w \neq z_0$)

Proof: $f(z) = \frac{1}{2\pi i}\int_{\gamma'} \frac{f(w)}{w-z}dz$

γ': curve around z, inside holo region.



anticlockwise big circle enclosing z. $R<R_2$.

clockwise small circle doesn't enclose z. $R>R_1$.

$$f(z) = \frac{1}{2\pi i}\left[\int_{\text{big circle}} \frac{f(w)}{w-z}dw - \int_{\text{small circle}} \frac{f(w)}{w-z}dw\right]$$

$\int_{\text{small circle}} \frac{f(w)}{w-z}dw = 0$ if holomorphic $|z-z_0|<R_2$. $\Rightarrow$ power series

$$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{w-z_0}\,\frac{1}{1-\frac{z-z_0}{w-z_0}} = \frac{1}{w-z_0}\sum_{k\geq 0}\left(\frac{z-z_0}{w-z_0}\right)^k \quad \text{if } |z-z_0|<|w-z_0|.$$

$$= \frac{-1}{z-z_0}\,\frac{1}{1-\frac{w-z_0}{z-z_0}} = -\frac{1}{z-z_0}\sum_{k\geq 0}\left(\frac{w-z_0}{z-z_0}\right)^k \quad \text{if } |w-z_0|<|z-z_0|.$$

As uniform convergence,

interchange of $\int$ and Σ.

$$\frac{1}{2\pi i}\int_{\text{big circle}} \frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\int \frac{f(w)}{w-z_0}\sum_{k\geq 0}\left(\frac{z-z_0}{w-z_0}\right)^k = \sum_{k\geq 0} \underbrace{\frac{1}{2\pi i}\int_{\text{big circle}} \frac{f(w)}{(w-z_0)^{k+1}}dw}_{a_k}\,(z-z_0)^k.$$

$$\frac{-1}{2\pi i}\int_{\text{small circle}} \frac{f(w)}{w-z}dw = \sum_{k\geq 0}\frac{1}{2\pi i}\int_{\text{small circle}} \frac{f(w)}{(z-z_0)^{k+1}}dw\,(w-z_0)^k.$$

$$= \sum_{k=-\infty}^{-1}\frac{1}{2\pi i}\int_{\text{small circle}} \frac{f(w)}{(w-z_0)^{k+1}}dw\,(z-z_0)^k.$$

04/04/2022 Monday.

Laurent series: $\sum_{-\infty}^{\infty} a_k(z-z_0)^k = \sum_{k=-\infty}^{-1} a_k(z-z_0)^k + \sum_{k=0}^{\infty} a_k(z-z_0)^k$.

power series in $(z-z_0)^{-1}$ $\left|\frac{1}{z-z_0}\right| < R$.

power series

$R_1 < |z-z_0| < R_2$.

We can get uniform convergence in the region $A \le |z-z_0| \le B$, where $A > R_1$, $B < R_2$.

Thm: if $f(z) = \sum_{-\infty}^{\infty} a_k (z-z_0)^k$ for $A < |z-z_0| < B$, then a_k are uniquely determined by $f(z)$.

e.g. $\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}} dz = \sum_{k=-\infty}^{\infty} \int_\gamma a_k \frac{(z-z_0)^k}{(z-z_0)^{n+1}} dz = \sum_{k=-\infty}^{\infty} a_k \int_\gamma (z-z_0)^{k-(n+1)} dz = a_n 2\pi i = \begin{cases} 0 & n \ne k \\ 2\pi i & n = k \end{cases}$.

closed curve around z_0 and contained in the annulus.

if $n \ne k$, $k-(n+1) \ne -1$. $\int z^N dz = \frac{z^{N+1}}{N+1}$ given $N \ne -1$.

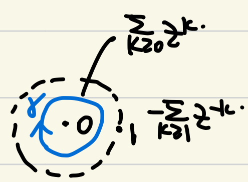
$\int_\gamma z^N dz = \frac{\gamma(b)^{N+1}}{N+1} - \frac{\gamma(a)^{N+1}}{N+1} = 0$ closed curve $\Rightarrow a = b$.

Conclusion: there is a unique laurent series in a given annulus.

Example: $\frac{1}{1-z} = f(z)$ singularity at $z=1$.

if $0 < |z| < 1$, $f(z) = \sum_{k=0}^{\infty} z^k$ power series

if $1 < |z| < \infty$, $f(z) = -\frac{1}{z} \cdot \frac{1}{1-z^{-1}} = -\frac{1}{z} \cdot \sum_{k=0}^{\infty} (z^{-1})^k = -\sum_{k=1}^{\infty} z^{-k}$



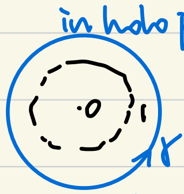
Note: $a_k = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{k+1}} dw$. $2\pi i a_{-1} = \int_\gamma f(w) dw$.

by Cauchy's thm. $\int_\gamma f(z) dz = 0$ $\Rightarrow$ for laurant series $\sum_{k\ge0} z^k$. $a_{-1}, a_{-2}, a_{-3}, \cdots = 0$.

in holo part.

$\Rightarrow 2\pi i a_{-1} = \int_\gamma f(z) dz = 0$.

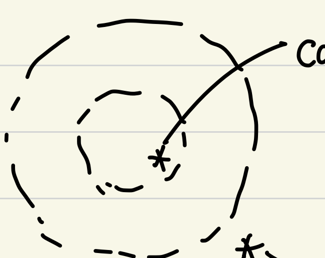
Now consider



$\int_\gamma f(z) dz = \int_\gamma \frac{-1}{z-1} dz = 2\pi i(-1) = 2\pi i a_{-1}$.

$\Rightarrow$ for laurant series $-\sum_{k\ge1} z^{-k}$ $a_{-1} = -1$.

by Cauchy integral formula.



can't have convergence inside this singularity.

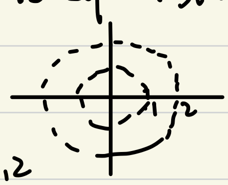
$\left|\frac{1}{z-z_0}\right| <$ distance to some singularity.

can't have convergence outside this singularity.

$|z-z_0| <$ distance to some singularity.

Example: $f(z) = \frac{1}{(z-1)(z-2)}$

singularities at $z = 1, 2$



three regions to consider

$0 < |z| < 1$, $1 < |z| < 2$, $2 < |z| < \infty$.

$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$

if $|z|>2$, $\frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\frac{2}{z}} = \frac{1}{z}\sum_{k\geq 0}\left(\frac{2}{z}\right)^k$ if $|z|<2$, $\frac{1}{z-2} = -\frac{1}{2}\frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{k\geq 0}\left(\frac{z}{2}\right)^k$

if $|z|>1$, $\frac{1}{z-1} = \sum_{k\geq 1} z^{-k}$ (by previous example) if $|z|<1$, $\frac{1}{z-1} = -\sum_{k\geq 0} z^k$

So we have if $|z|<1$, $f(z) = \sum_{k\geq 0} -2^{-(k+1)} z^k + \sum_{k\geq 0} z^k = \sum_{k\geq 0}\left(1-2^{-(k+1)}\right) z^k$

if $1<|z|<2$, $f(z) = \sum_{k\geq 0} -2^{-(k+1)} z^k - \sum_{k\geq 1} z^{-k}$

if $|z|>2$, $f(z) = \sum_{k\geq 0} 2^k z^{-(k+1)} - \sum_{k\geq 1} z^{-k} = \sum_{k\geq 0}\left(2^k-1\right) z^{-(k+1)}$

04/06/2022 Wednesday.

by identity principle, the zeros of a non-constant holomorphic fn are isolated.

for any z_0 with $f(z_0)=0$, $\exists\ \varepsilon>0$ s.t. $f(z)\neq 0$ for $0<|z-z_0|<\varepsilon$.

Def: a singularity z_0 of a function $f(z)$ is isolated if $\exists\ \varepsilon>0$ s.t. $f(z)$ is holomorphic on $0<|z-z_0|<\varepsilon$

Laurent Series.

Def: if z_0 is an isolated singularity of $f(z)$, then

(a) if $f(z)$ extends to a holomorphic fn on $0<|z-z_0|<\varepsilon$, we say z_0 is removable singularity.

$g(z)$ holo on $0<|z-z_0|<\varepsilon$ with $f(z)=g(z)$ for $0<|z-z_0|<\varepsilon$.

(b) if $\lim_{z\to z_0}|f(z)|=\infty$, $f(z)$ has a pole at z_0.

(c) otherwise z_0 is called an essential singularity.

Example of (a): $f(z)=\frac{e^z-1}{z}$ at $z=0$.

recall if $f(z_0)=0$. $f(z)=(z-z_0)^m g(z)$ for $g(z)$ holo at $z=z_0$, $g(z_0)\neq 0$.

let $f(z)=e^z-1$, $f(z)=zg(z)$ $\quad g(z)=\begin{cases}\frac{e^z-1}{z} & z\neq 0\\ 1 & z=0.\end{cases}$

Example of (b). $f(z)=\frac{1}{z^n}$ at $z=0$. $\lim_{z\to 0}\left|\frac{1}{z^n}\right|=\infty$.

Example of (c): $f(z)=e^{\frac{1}{z}}$ at $z=0$. $\lim_{z\to 0^+}e^{\frac{1}{z}}=\infty$. $\lim_{z\to 0^-}e^{\frac{1}{z}}=0$. $\Rightarrow$ no cont. extension.

Not a pole.

Prop: if z_0 is an isolated singularity of $f(z)$.

(a) z_0 is removable $\Longleftrightarrow \lim_{z\to z_0}(z-z_0)f(z)=0$.

(b) z_0 is a pole $\Longleftrightarrow$ it is not removable AND $\lim_{z\to z_0}(z-z_0)^{n+1}f(z)=0$ for some $n\in\mathbb{Z}_{\geq 0}$.

Proof: (a) $\lim_{z\to z_0}(z-z_0)f(z)=\lim_{z\to z_0}(z-z_0)g(z)=0\cdot g(z_0)=0$. $\square$

let $h(z)=\begin{cases}(z-z_0)^2f(z) & z\neq z_0\\ 0 & z=z_0\end{cases}$

$h'(z_0)=\lim_{z\to z_0}\frac{h(z)-h(z_0)}{z-z_0}=\lim_{z\to z_0}\frac{(z-z_0)^2f(z)-0}{z-z_0}=\lim_{z\to z_0}(z-z_0)f(z)=0$ so $h(z)$ is holo at z_0.

$h(z)=(z-z_0)^m k(z)$ $\quad k(z)$ holo at z_0, $k(z_0)\neq 0$.

claim $m\geq 2$: $h(z_0)=0$, so $m\geq 1$.

$h'(z_0)=0$, $0=m(z-z_0)^{m-1}k(z_0)+(z-z_0)^m k'(z_0)$ $\Rightarrow m-1\geq 1$, $m\geq 2$.

$f(z)=(z-z_0)^{m-2}k(z)$ is a holo extension of $f(z)$. $\square$.

(b) $\lim_{z\to z_0}|f(z)|=\infty$.

$\exists\varepsilon>0$ s.t. if $0<|z-z_0|<\varepsilon$, $|f(z)|>1$. $(f(z)\neq 0)$ $\Rightarrow$ $\frac{1}{f(z)}$ is holo for $0<|z-z_0|<\varepsilon$.

$\lim_{z\to z_0}\frac{1}{f(z)}=0$ $\Rightarrow$ $\lim_{z\to z_0}(z-z_0)\frac{1}{f(z)}=0$ $\Rightarrow$ z_0 is a removable singularity of $\frac{1}{f(z)}$.

let $\frac{1}{f(z)}=(z-z_0)^n \ell(z)$ $\ell(z)$ holo at z_0, $\ell(z_0)\neq 0$.

$f(z)=(z-z_0)^{-n}\frac{1}{\ell(z)}$ holo at z_0.

So $\lim_{z\to z_0}(z-z_0)^{n+1}f(z)=\lim_{z\to z_0}(z-z_0)\frac{1}{\ell(z)}=0$ $\square$.

Suppose $\lim_{z\to z_0}(z-z_0)^{n+1}f(z)=0$.

let $p(z)=(z-z_0)^n f(z)$, then $\lim_{z\to z_0}(z-z_0)p(z)=0$. z_0 is a removable singularity of $p(z)$.

so $p(z)$ extends to a fn holo at z_0. $\Rightarrow$ $\lim_{z\to z_0}(z-z_0)^n f(z)$ exists. $(=p(z_0))$.

nonzero by minimality of n.

then $\lim_{z\to z_0}|f(z)|=\lim_{z\to z_0}|p(z)(z-z_0)^{-n}|=\lim_{z\to z_0}|p(z)|\cdot\lim_{z\to z_0}|(z-z_0)^{-n}|=|p(z_0)|\cdot\infty=\infty$. $\square$.

Def: <u>n is the order of the pole of f(z) at z_0.</u>

Reminder: if $f(z)$ is holomorphic at z_0, we may write $f(z)=(z-z_0)^m g(z)$, where $g(z)$ is holomorphic at z_0 and $g(z_0)\neq 0$ in which case m is the order of the zero of $f(z)$ at z_0. Similarly if $f(z)$ has a pole at z_0, then we can write $f(z)=(z-z_0)^{-n}g(z)$ for a similar $g(z)$, and n is the order of the pole of $f(z)$ at z_0. So a zero is like a pole of negative order and vice versa.

04/08/2022 Friday.

Prop: Suppose z_0 is an isolated singularity of $f(z)$ and $f(z)=\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$ near z_0,

then (a) z_0 is removable $\Leftrightarrow a_k=0$ for $k<0$.

(b) z_0 is a pole of order $n \Leftrightarrow a_{-n}\neq 0$ and $a_k=0$ for $k<-n$.

(c) z_0 is essential $\Leftrightarrow a_{-n}\neq 0$ for infinitely many $n>0$.

proof: (a) power series $\Rightarrow$ defines a holo fn at z_0. □

holo fn at $z_0 \Rightarrow$ power series. □

(b) $\lim_{z\to z_0}(z-z_0)^{n+1}f(z)=0$, $(z-z_0)^n f(z)$ has a removable singularity at z_0, calling it $g(z)$.

$f(z)=(z-z_0)^{-n}g(z)$ for $g(z)$ holo at z_0, $g(z_0)\neq 0$.

$g(z)=\sum_{k\geq 0} a_k(z-z_0)^k$. with $a_0\neq 0$.

So $f(z)=(z-z_0)^{-n}\sum_{k\geq 0} a_k(z-z_0)^k = \sum_{k\geq 0} a_k(z-z_0)^{k-n}$. □

$f(z)=\sum_{k=-n}^{\infty} a_k(z-z_0)^k$ with $a_n\neq 0$.

$(z-z_0)^{n+1}f(z)=\underbrace{\sum_{k=-n}^{\infty} a_k(z-z_0)^{n+k+1}}_{\text{power series with constant term zero.}}$ Since $k\geq -n$, $n+k+1\geq 1$. $\Rightarrow$ define a holo fn s.t. $f(z_0)=0$. □

(c) (a) $\Leftrightarrow$ all negative terms $=0$.

(b) $\Leftrightarrow$ finite but nonzero number of negative index terms are zero.

$2\pi i a_{-1}=\int_\gamma f(w)\,dw$.

Def: the coeff a_{-1} in the laurent series of $f(z)$ at z_0 is called the residue of $f(z)$ at z_0.

$\operatorname{Res}_{z=z_0} f(z)$.

Thm: (Residue Thm) * Compared to the Cauchy Integral formula, singularities can be any kind. (formula: singularities with order=1 only.)

let γ be a closed curve and $f(z)$ a fn holo on γ and region enclosed by γ except for the isolated singularity, then there are infinitely many singularities inside the enclosed region and $\int_\gamma f(z)dz = 2\pi i \sum_k \operatorname{Res}_{z=z_k} f(z)$ (sum over singularities enclosed by γ).

Proof: let S be the set of singularities inside the enclosed region.

S is bounded. (contained in a bonded region, by Jordan curve thm).

S is closed. (complete of points s.t. $f(z)$ is holo → holo on the disk $\Rightarrow$ open).

Suppose infinite many pts in S, consider a sequence of distinct pts in S: $z_1, z_2, \cdots$

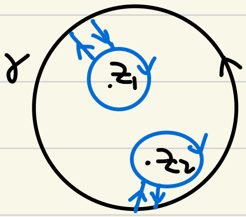
By Bolzano-Weierstrass, bounded sequence has a convergent subsequence.

$z_{i_1}, z_{i_2}, \cdots \to w$. S is closed, w is in S.

Claim: w is not an isolated singularity.

proof: $\forall \varepsilon > 0, \exists z_{i_j}$ within ε of w.

so $f(z)$ can't be holo on $0<|z-w|<\varepsilon$. $\Rightarrow$ only finite pts in S.



$\int_{\text{new curve}} f(z)\,dz = 0 = \int_\gamma + \sum \int_{\cancel{\text{line segments}}} + \sum \int_{\text{little circles anticlock}}$

so $\int_\gamma = \sum \int_{\text{little circles anticlockwise}}$.

$\int_\gamma f(z)\,dz = 2\pi i \sum_k \operatorname{Res}_{z=z_k} f(z)$.

all singularities are isolated.

04/11/2022 Monday.

Example: $\int_{-\infty}^{\infty}\frac{1}{(x^2+1)^n}dx$. → "ratio" of holo fns. → Residue form. But check singularity exists. Counter Example: $\frac{1}{\sin(z)}$

isolated.

Note if $f(z)=\frac{g(z)}{h(z)}$ with $g(z), h(z)$ holomorphic, $f(z)$ has singularities when $h(z)=0$.

$$\left.\begin{aligned} g(z)&=(z-z_0)^a\alpha(z)\\ h(z)&=(z-z_0)^b\beta(z)\end{aligned}\right\} f(z)=(z-z_0)^{a-b}\frac{\alpha(z)}{\beta(z)} \qquad \left.\begin{aligned}\alpha(z_0)&\neq0\\ \beta(z_0)&\neq0\end{aligned}\right\}\Rightarrow \frac{\alpha(z_0)}{\beta(z_0)}\neq0.$$

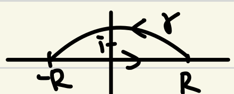
$f(z)$ has a zero of order $a-b$ if $a>b$.

$f(z)$ has a pole of order $b-a$ if $a<b$. so the singularities are poles.

Def: if all singularities of $f(z)$ are isolated poles (not essential), then $f(z)$ is called meromophic fn.

All meromorphic fns are of the form $\frac{g(z)}{h(z)}$ with g, h holo.

$\int_\gamma\frac{1}{(z^2+1)^n}dz$.



Singularities at $z=i, -i$

inside γ

$$\int_{-R}^{R}\frac{dx}{(x^2+1)^n}+\int_{arc}\frac{dz}{(z^2+1)^n}=2\pi i\operatorname*{Res}_{z=i}\frac{1}{(z^2+1)^n}$$

reverse △

$$\left|\int_{arc}\frac{dz}{(z^2+1)^n}\right|\le\pi R\cdot\max\left[\frac{1}{|z^2+1|^n}\right] \qquad \frac{\pi R}{\min|z^2+1|^n}\le\frac{\pi R}{\min(|z|^2-1)^n}=\frac{\pi R}{(R^2-1)^n}\to0 \text{ as } R\to\infty.$$

So $\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)^n}=2\pi i\operatorname*{Res}_{z=i}\frac{1}{(z^2+1)^n}$

$n=1$ $\frac{1}{z^2+1}=\frac{1}{z-i}\frac{1}{z+i}=\frac{1}{z-i}\frac{1}{(z-i)+2i}=\frac{1}{z-i}\frac{1}{2i}\frac{1}{1+\frac{z-i}{2i}}=\frac{1}{z-i}\frac{1}{2i}\sum_{k\ge0}\left(-\frac{z-i}{2i}\right)^k=\frac{1}{2i}\sum_{k\ge0}\left(\frac{-1}{2i}\right)^k(z-i)^{k-1}$.

Coef of $(z-i)^{-1}=\frac{1}{2i}\left(\frac{-1}{2i}\right)^0=\frac{1}{2i}$ So $\int_{-\infty}^{\infty}\frac{dx}{x^2+1}=2\pi i\frac{1}{2i}=\pi$.

$n=2$ $\frac{1}{(z^2+1)^2}=\frac{1}{(z-i)^2}\frac{1}{(z+i)^2}=\frac{1}{(z-i)^2}\left[\frac{1}{2i}\sum_{k\ge0}\left(-\frac{z-i}{2i}\right)^k\right]^2=\frac{1}{(z-i)^2}\left[\left(\frac{1}{2i}\right)^2+2\cdot\frac{1}{2i}\cdot\frac{1}{2i}\cdot\left(-\frac{z-i}{2i}\right)+\text{lower terms}\right]$.

$=\frac{1}{(2i)^2}\frac{1}{(z-i)^2}+\frac{2}{(2i)^2}\left(\frac{1}{-2i}\right)(z-i)^{-1}+\cdots$ Res $=\frac{2}{(2i)^2}\left(\frac{1}{-2i}\right)=\frac{1}{4i}$. So $\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)^2}=2\pi i\frac{1}{4i}=\frac{\pi}{2}$

$= \pi\binom{2n-2}{n-1}/4^{n-1}$.

Binomial thm: $(1+z)^n=\sum_{r=0}^{n}\binom{n}{r}z^r$.

not necessarily a positive integer.

$(1+z)^\alpha=\sum_{k\ge0}\binom{\alpha}{k}z^k$ $\quad \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k(k-1)\cdots1}$ if $\alpha=n$, $=\frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots1}\cdot\frac{(n-k)(n-k-1)\cdots1}{(n-k)(n-k-1)\cdots1}=\frac{n!}{k!(n-k)!}=\binom{n}{k}$

infinite sum

$(1+z)^\alpha=e^{\alpha\log(1+z)}$. holo for $|z|<1$.

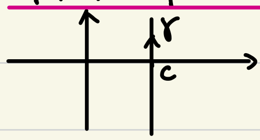
to compute Taylor series, differentiate $(1+z)^\alpha$ k times:

$\alpha(1+z)^{\alpha-1}, \alpha(\alpha-1)(1+z)^{\alpha-2}, \cdots \alpha(\alpha-1)\cdots(\alpha-k+1)(1+z)^{\alpha-k}\big|_{z=0} \Rightarrow \sum_{k\ge0}\frac{\frac{d^kf}{dz^k}}{k!}z^k=\sum_{k\ge0}\binom{\alpha}{k}z^k$ Converges for $|z|<1$.

04/15/2022 Friday.

Recall: Laplace transform. $\mathcal{L}(f)(s)=\int_0^\infty f(t)e^{-st}dt$ $\quad f:[0,\infty)\to\mathbb{C}$. $\quad s\in\mathbb{C}$.

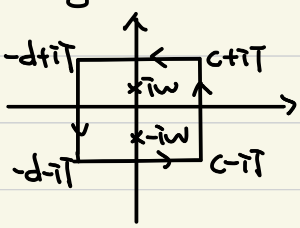
Laplace inversion formula: if $F(s)=\mathcal{L}(f)(s)$, then $f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)e^{st}ds$.



c is any real number s.t. all singularities of F(s) are on the left of the line.

Example: $f(t)=\cos(wt)$ $\quad F(t)=\mathcal{L}(f)(s)=\frac{s}{s^2+w^2}$

Singularities at $\pm iw$, so c can be any positive real.



top.

$$\left|\int_{c+iT}^{-d+iT}\frac{s}{s^2+w^2}e^{st}ds\right|\le|c+d|\max_{-d\le r\le c}\left|\frac{r+iT}{(r+iT)^2+w^2}e^{(r+iT)t}\right|.$$

$$|r+iT|\le|-d+iT|=\sqrt{d^2+T^2}$$

$$\frac{1}{|(r+iT)^2+w^2|}\le\frac{1}{|r^2-T^2+w^2+2riT|}\le\frac{1}{|r^2-T^2+w^2|}$$

Spoiler: $d=\sqrt{T}$. $|r^2|\le T$, as $T\to\infty$, $|r^2-T^2+w^2|=T^2-r^2-w^2$.

$$\frac{1}{|r^2-T^2+w^2|}\le\frac{1}{T^2-d^2-w^2}=\frac{1}{T^2-T-w^2}$$

$$|e^{(r+iT)t}|=e^{Re((r+iT)t)}=e^{rt}\le e^{ct}$$ independent of T or d.

So $$\left|\int_{c+iT}^{-d+iT}\right|\le(c+\sqrt{T})\frac{\sqrt{T^2+T}}{T^2-T-w^2}=\underbrace{\frac{T\sqrt{T}}{T^2}}_{\to 0\text{ as }T\to\infty}\underbrace{\frac{(\frac{c}{\sqrt{T}}+1)\sqrt{1+1/T}}{(1-1/T-w^2/T^2)}}_{\to 1\text{ as }T\to\infty}$$

Similarly, $\int_{-\sqrt{T}-iT}^{c-iT}=0$. bottom.

left

$$\left|\int_{-d+iT}^{-d-iT}\frac{s}{s^2+w^2}e^{st}ds\right|\le 2T\max_{-T\le r\le T}\left|\frac{(-d+ir)}{(-d+ir)^2+w^2}e^{(-d+ir)t}\right|$$

$$|-d+ir|=\sqrt{d^2+r^2}\le\sqrt{d^2+T^2}$$

$$\frac{1}{|(-d+ir)^2+w^2|}\le\frac{1}{|(-d+ir)^2|-w^2}\le\frac{1}{d^2-w^2}=\frac{1}{T-w^2}$$

$$|e^{(-d+ir)t}|=e^{Re((-d+ir)t)}=e^{-dt}=e^{-\sqrt{T}t}.$$

$$\left|\int_{-\sqrt{T}+iT}^{-\sqrt{T}-iT}\right|\le 2T\cdot\frac{\sqrt{T^2+T}}{T-w^2}e^{-\sqrt{T}t}=\frac{T^2}{T}e^{-\sqrt{T}t}\underbrace{\frac{2\sqrt{1+1/T}}{1-w^2/T}}_{\to\text{ Constant as }T\to\infty.}$$

$\lim_{T\to\infty}Te^{-\sqrt{T}t}=0$ as $e^{\sqrt{T}t}$ grows faster than any poly in T.

Suppose $x\ge0$, $e^x=\sum_{k\ge0}\frac{x^k}{k!}\ge\frac{x^4}{4!}$ 4th term of the sum.

$e^{t\sqrt{T}}\ge\frac{(t\sqrt{T})^4}{4!}=\frac{t^4T^2}{24}$ (any ≥ 4.), $\quad Te^{-\sqrt{T}t}=\frac{24}{t^4T}\to0$.

right

So we have $\int_{c-i\infty}^{c+i\infty}=2\pi i\left(\operatorname{Res}_{s=iw}\frac{s}{s^2+w^2}e^{st}+\operatorname{Res}_{s=-iw}\frac{s}{s^2+w^2}e^{st}\right)$

$\frac{s}{(s+iw)(s-iw)}e^{st}$ $\quad$ residue at $iw=\frac{iw}{(iw+iw)}e^{iwt}$. $\quad$ Residue at $-iw=\frac{-iw}{-iw-iw}e^{-iwt}$.

So $\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}=\frac{1}{2}e^{iwt}+\frac{1}{2}e^{-iwt}=\cos(wt)=f(t)$.

04/18/2022 Monday. (use Residue thm to solve discrete math problems.)

Q: How many ways to express k as a sum using some given numbers $a, b, c, \cdots$

$ax_1+bx_2+\cdots=k$. $x_1, x_2, \cdots$ nonnegative integers.

In case of only a, b.

the largest number that cannot be written is $ab-a-b$.

#expression for k using only a $=\begin{cases} 1 & \text{if } k \text{ is multiple of } a. \\ 0 & \text{otherwise.} \end{cases}$

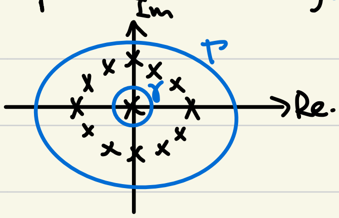
$1\cdot z^0+0\cdot z^1+0\cdot z^2+\cdots+1\cdot z^a+0\cdot z^{a+1}+\cdots$ (coeff of z^k is #expressions for k using a.)

$=z^0+z^a+z^{2a}+\cdots=\frac{1}{1-z^a}$. $|z^a|<1 \iff |z|<1$. Similarly $\frac{1}{1-z^b}$.

$\frac{1}{1-z^a}\frac{1}{1-z^b}=\sum_{k\ge0}a_kz^k$. ($a_k$ = #expressions for k using a and b. = #(x,y) s.t. $ax+by=k$ with $x,y\ge0$ integers).

since $(1+z^a+z^{2a}+\cdots)(1+z^b+z^{2b}+\cdots)=\sum_{x,y\ge0}z^{xa}z^{yb}=\sum_{x,y\ge0}z^{ax+by}$.

by residue thm, $\frac{1}{2\pi i}\int_\gamma \frac{1}{(1-z^a)(1-z^b)z^{k+1}}dz=a_k$. $\sum_{n\ge0}a_nz^{n-k-1}$. coeff of z^{-1} corres. to $n=k$ is a_k.



poles at $z=0$ or $z^a=1$, $z^b=1$

so $\frac{1}{2\pi i}\int_\Gamma \frac{1}{(1-z^a)(1-z^b)z^{k+1}}dz=\operatorname*{Res}_{z=0}f(z)+\sum_{\substack{w^a=1\\ \text{or } w^b=1}}\operatorname*{Res}_{z=w}f(z)$.

by ML-lemma. $\left|\int_\Gamma \frac{1}{(1-z^a)(1-z^b)z^{k+1}}dz\right|\le 2\pi R\cdot\max_{|z|=R}\left|\frac{1}{(1-z^a)(1-z^b)z^{k+1}}\right|$

$\le 2\pi R\cdot\frac{1}{(R^a-1)(R^b-1)R^{k+1}}$ which $\to 0$ as $R\to\infty$.

$a_k=\operatorname*{Res}_{z=0}f(z)=-\sum_{\substack{w^a=1\\ \text{or } w^b=1}}\operatorname*{Res}_{z=w}f(z)$.

assume $\gcd(a,b)=1$, otherwise the only sol to $w^a=1$ and $w^b=1$ is $w=1$.

so $f(z)$ has poles of order 1 at $e^{2\pi i r/a}$ $r=1,2,\cdots,a-1$. and $e^{2\pi i s/b}$ $s=1,2,\cdots,b-1$.

the pole at $z=1$ has order 2 $\Longrightarrow$ turns out to be $-\frac{a+b+2k}{ab}$.

$a_k = \sim\ >0$ for k large enough.

04/20/2022 Wednesday.

Def: gamma fn $\Gamma(s)=\int_0^\infty x^{s-1}e^{-x}dx$.

Converges for $Re(s)>0$.

Property 1: $\Gamma(s)$ is holomorphic for $Re(s)>0$.

Property 2: $\Gamma(s+1)=s\Gamma(s)$

Proof: $\Gamma(s)=\int_0^\infty x^{s-1}e^{-x}dx=\left[\frac{x^s}{s}e^{-x}\right]_0^\infty-\int_0^\infty\frac{x^s}{s}(-e^{-x})dx=\frac{1}{s}\Gamma(s+1)$

$\underset{0}{\shortparallel}$ as exponentials grow faster.

$\Gamma(1)=1$. $\Gamma(2)=1$. $\Gamma(3)=2$. $\Gamma(4)=6$. $\Gamma(n)=(n-1)!$

$\Gamma(\frac{1}{2})=\sqrt{\pi}$. $\Gamma(\frac{2n+1}{2})=\frac{(2n)!}{n!\,4^n}\sqrt{\pi}$.

$\Gamma(s)=\frac{1}{s}\Gamma(s+1)=\frac{1}{s}\frac{1}{s+1}\Gamma(s+2)=\cdots=\frac{1}{s}\frac{1}{s+1}\frac{1}{s+2}\cdots\frac{1}{s+n-1}\Gamma(s+n)$.

↳ holomorphic for $Re(s)>-n$ except $s=0,-1,-2,\cdots,-(n-1)$.

$\Gamma(s)$ has a pole of order 1 at $-k$ with residue $\lim_{s\to-k}(s+k)\Gamma(s)=\frac{(-1)^k}{k!}$

$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin(\pi s)}$

$\Gamma(s)\Gamma(s+\frac{1}{2})=2^{1-2s}\sqrt{\pi}\,\Gamma(2s)$

04/22/2022 Friday.

def: the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \frac{1}{1^s} + \frac{1}{2^s} + \cdots$ (any complex number $\neq 1$.)

Converges when $\mathrm{Re}(s) > 1$. $|n^{-s}| = |e^{\ln(n^{-s})}| = |e^{-s\ln(n)}| = e^{-\mathrm{Re}(s)\ln(n)} = n^{-\mathrm{Re}(s)}$

by integral test, $\zeta(s)$ converges when $s > 1$.

when $s=1$, $\zeta(1) =$ harmonic series $= \infty$.

for $\varepsilon > 0$, $\sum_{n\geq 1} n^{-s}$ converges uniformly on $\mathrm{Re}(s) \geq 1+\varepsilon$. By Morera's thm, $\zeta(s)$ is holomorphic for $\mathrm{Re}(s) > 1$.

$n^{-s} = s\int_n^{\infty} x^{-(s+1)}dx = s\left[\frac{1}{-s}x^{-s}\right]_{x=n}^{\infty} = (-\infty^{-s}) - (-n^{-s}) = n^{-s}$.

so $\zeta(s) = \sum_{n\geq 1} n^{-s} = \sum_{n\geq 1} s\int_n^{\infty} x^{-(s+1)}dx = s\int_1^{\infty} \lfloor x \rfloor x^{-(s+1)}dx = s\int_1^{\infty} \frac{x - \{x\}}{x^{s+1}}dx$

($\lfloor x \rfloor$: largest integer $\leq x$. $\{x\}$: fractional part of x, $0 \leq \{x\} < 1$.)

$= s\int_1^{\infty} \frac{x}{x^{s+1}}dx - s\int_1^{\infty} \frac{\{x\}}{x^{s+1}}dx$

$= \frac{s}{s-1} - s\int_1^{\infty} \frac{\{x\}}{x^{s+1}}dx$. first part needs $s \neq 1$. Second part needs $\mathrm{Re}(s) > 0$.

$\left|\int_1^{\infty} \frac{\{x\}}{x^{s+1}}dx\right| \leq \int_1^{\infty} \frac{|\{x\}|}{|x^{s+1}|}dx \leq \int_1^{\infty} \frac{1}{x^{\mathrm{Re}(s)+1}}dx$.

$\zeta(s)$ has a pole of order 1 with residue 1 at $s=1$.

$\zeta(s) = \frac{(2\pi)^s}{\pi}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$.

$\zeta(s)$ is famous for its connection to prime numbers.

$n = p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k} \longrightarrow n^{-s} = p_1^{-a_1 s}p_2^{-a_2 s}\cdots p_k^{-a_k s}$.

$\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + \cdots = \prod_{\text{prime } p}(1 + p^{-s} + p^{-2s} + \cdots) = \prod_{\text{prime } p}\frac{1}{1-p^{-s}}$

$\ln(\zeta(s)) = \sum_p -\ln(1-p^{-s})$

$\frac{\zeta'(s)}{\zeta(s)} = \sum_p -\frac{\ln(p)p^{-s}}{1-p^{-s}} = -\sum_p\left(\frac{\ln(p)}{p^s} + \frac{\ln(p)}{p^{2s}} + \cdots\right) = -\sum_{n\geq 1}\frac{-\Lambda(n)}{n^{-s}}$ where $\Lambda(n) = \begin{cases} \ln(p) & \text{if } n \text{ is a power of the prime } p. \\ 0 & \text{otherwise.} \end{cases}$

$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} n^{-s}\frac{x^s}{s}ds = \begin{cases} 1 & \text{if } x \geq n \\ 0 & \text{if } x < n. \end{cases}$ for $x > 0$.

so $\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\sum_{n\geq 1}\frac{\Lambda(n)}{n^s}\frac{x^s}{s}ds = \sum_{n\geq 1}\frac{\Lambda(n)}{2\pi i}\int_{c-i\infty}^{c+i\infty} n^{-s}\frac{x^s}{s}ds = \sum_{n\leq x}\Lambda(n)$.

let $\pi(x) = \#\text{primes} \leq x$.

$\sum_{n\leq x}\Lambda(n) = \sum_{p^k\leq x}\ln(p) = \sum_{p\leq x}\ln(p)\lfloor\log_p(x)\rfloor = \sum_{p\leq x}\ln(p)\left\lfloor\frac{\ln(x)}{\ln(p)}\right\rfloor \leq \sum_{p\leq x}\ln(p)\frac{\ln(x)}{\ln(p)} = \ln(x)\sum_{p\leq x}1 = \ln(x)\pi(x)$.

$\sum_{n\leq x}\Lambda(n) = x - \sum_{\rho}\frac{x^{\rho}}{\rho} - \ln(2\pi) - \frac{1}{2}\ln(1-x^{-2})$ (ρ = rho.) It turns out $\mathrm{Re}(\rho) < 1$ for all ρ, so x is the dominant term

(x: Res $s=1$; $\sum_\rho \frac{x^\rho}{\rho}$: sum across zeros ρ of $\zeta(s)$ with $0 < \mathrm{Re}(\rho) < 1$; $\ln(2\pi)$: from $s=0$; $\frac{1}{2}\ln(1-x^{-2})$: other zeros of $\zeta(s)$ $s=-2,-4,-6$.)

$\sum_{n\leq x}\Lambda(n) \approx x$, $\pi(x) \approx \frac{x}{\ln(x)}$ prime number theorem.

trivial zeros. negative even integers.

Riemann Hypothesis: $\mathrm{Re}(\rho) = \frac{1}{2}$.

Re(all nontrivial zeros) $= \frac{1}{2}$.

04/25/2022 Monday. Review.

$f'(z) = \lim_{\substack{h\to 0 \\ h\in\mathbb{C}}} \frac{f(z+h)-f(z)}{h}$ $\forall \varepsilon>0, \exists \delta>0$ s.t. $0<|h|<\delta \Rightarrow \left|\frac{f(z+h)-f(z)}{h} - L\right| < \varepsilon$. $L = f'(z)$.

$f(x+iy) = u(x,y) + iv(x,y)$. complex-diff if Cauchy-Riemann holds.

$u_x = v_y$. $u_y = -v_x$. $\Rightarrow$ obeys 2D Laplace equation.

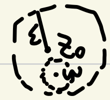
$u_{xx} + u_{yy} = 0$.

$f(z)$ is holomorphic at $z_0 \in \mathbb{C}$ if it is diff. for all z on a small disk centered at z_0.

$\exists \varepsilon>0$, s.t. $|z-z_0|<\varepsilon$ then f is diff. at z.

if f is holomorphic at z_0, it is also holomorphic on a small disk centered at z_0.

f diff. for $|z-z_0|<\varepsilon$. pick w with $|w-z_0|<\varepsilon$, $|z-w|<\varepsilon-|w-z_0|$.



$|w-z| = |(w-z_0)-(z-z_0)| \geq |z-z_0| - |w-z_0|$. reverse Δ. $\varepsilon - |w-z_0| > |z-z_0| - |w-z_0|$. $\Rightarrow \varepsilon > |z-z_0|$.

Product Rule / quotient rule. / chain Rule.

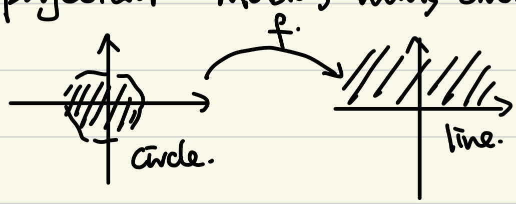
Möbius transformation. $f(z) = \frac{az+b}{cz+d}$ (w/ $ad-bc\neq 0$).

if we extend $\mathbb{C}$ to extended complex plane $\hat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$, $f:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$, $f(-\frac{d}{c}) = \infty$. $f(\infty) = \frac{a}{c}$.

f is a bijection. inverse (Möbius) = Möbius. composition (Möbius) = Möbius.

Stereographic projection. Möbius turns circles and lines into circles and lines.

e.g. $f(z) = \frac{z-1}{i(z+1)}$



the logarithm. $e^{x+iy} = e^x(\cos(y) + i\sin(y))$. $z = e^w$. (if $z\neq 0$, infinitely many w).

if z is real and positive, we can take $\log(z)$ to be a real, but otherwise there isn't a choice of w that is "better" than any other. $z = e^w$, $z = e^{w+2\pi i k}$ (k integer). no choice of w is "better".

let $-\pi < \operatorname{Im}(\log(z)) \leq \pi$. log become a fn on $\mathbb{C}\setminus\{0\}$, but Not conti. on negative real axis.

$\log(z_1) = i\theta \approx i\pi$. $\log(z_2) = -i\theta \approx -i\pi$.

04/29/2022. Friday.

$f(z)=\sum_{k\ge 0} a_k(z-z_0)^k$ $a_k=\frac{d^kf}{dz^k}(z_0)/k!$ order of zero of f at z_0.

if $a_0,\cdots,a_{m-1}$ are zeros, $a_m\neq 0$, $f(z)=(z-z_0)^m \underbrace{\sum_{k\ge m} a_k(z-z_0)^{k-m}}_{g(z) \text{ w/ } g(z_0)\neq 0}$

identity principle: $g(z)$ is nonzero on a small disk centered at z_0 by continuity.

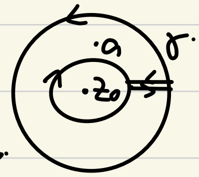
so $f(z)\neq 0$ on this disk except z_0 ⇒ the zeros of $f(z)$ are isolated.

if zeros of $f(z)$ have an accumulation point (limit point), contradicts having isolated zeros. ⇒ $a_m=0$ ⇒ $f(z)=0$

if f,g holomorphic, $S=\{z\in\mathbb{C} \mid f(z)=g(z)\}$. S is the zero set of $f(z)-g(z)$. if S is an accumulation point. $f-g$ is the zero fn, $f=g$.

Laurent Series.

$f(a)=\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz = \frac{1}{2\pi i}\left[\int_{\text{big circle}} - \int_{\text{small circle}}\right]$.



$\frac{1}{z-a}=\frac{1}{z}\frac{1}{1-\frac{a}{z}}=-\frac{1}{a}\frac{1}{1-\frac{z}{a}}$.

$f(a)=\sum_{k=0}^{\infty} c_k(a-z_0)^k+\sum_{k=-\infty}^{-1} c_k(a-z_0)^k$ ⇒ $R_1<|a-z_0|<R_2$ annulus.

e.g. $\frac{1}{1-z}$ expanded around $z=0$.

$=\sum_{k\ge 0} z^k$ when $0<|z|<1$.

$=-\sum_{k=-\infty}^{-1} z^k$ when $|z|>1$. $\frac{-1}{z}\sum_{k\ge 0}(z^{-1})^k$.

mostly we construct $0<|z-z_0|<\varepsilon$. (if z_0 is an isolated singularity).

- Removable. (extend to holo fn at the point). ⟺ $a_m=0$ for $m\ge 1$.
- Pole. (of order m) ($\lim_{z\to z_0}|f(z)|=\infty$). ⟺ $a_{-m}\neq 0$, $a_{-(m+1)}, a_{-(m+2)},\cdots=0$.
- Essential. ⟺ infinite many nonzero negative coeff. finite #nonzero negative coeff.

pole of order m: $(z-z_0)^m f(z)=(z-z_0)^m\sum_{k\ge -m} a_k(z-z_0)^k=\sum_{k\ge -m} a_k(z-z_0)^{k+m} := g(z)$ s.t. $g(z_0)\neq 0$.

Residue Thm.

$\int_\gamma f(z)\,dz=\int_\gamma \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k\,dz = \sum_{k=-\infty}^{\infty} a_k \underbrace{\int_\gamma (z-z_0)^k\,dz}_{\begin{cases}2\pi i & k=-1\\ 0 & k\neq -1\end{cases}} = 2\pi i\, a_{-1}$.

× Gamma. $\Gamma(s)$

× zeta $\zeta(s)$