01/19/2022 Wednesday.

Mechanism Design.

Example: Olympic Tennis.

① Round Robin Phase.

16 teams → 4 groups play w/ each other → group top 2.

② Single elimination "knockout" phase.

4×2=8 teams quarterfinals → semifinals → final → winner

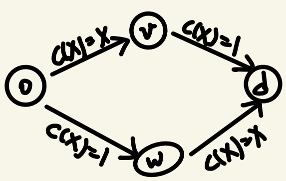
Goal of participants → get best medals.
Goal of committee → play hard on every matches.

∇: be a_2 instead of a_1 to guarantee the silver. (avoid d_1)

Solutions: × Round Robin Phase.

× public phase 2 schedule until Phase 2.

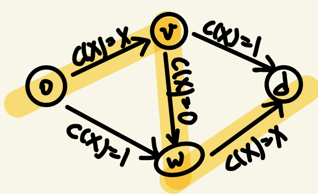
Example: Selfish Routing & Braess' Paradox. Cost minimization Game.



if $\frac{1}{4}$ of them take top path, traffic $= \frac{1}{4}+1 = \frac{5}{4}$. bottom path: $1+\frac{3}{4}=\frac{7}{4}$.

⇓ eventually.

equilibrium: $\frac{1}{2}$ top path; $\frac{1}{2}$ bottom path. optimum travel time (avg) $= \frac{3}{2}$.



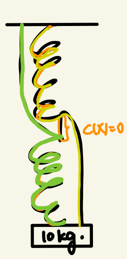
Eventually every one choose green path. traffic = 2 hrs.
↳ "Braess' Paradox".

Price of Anarchy (PoA) $= \dfrac{\text{equilibrium travel time}}{\text{optimal travel time}} = \dfrac{2}{3/2} = \dfrac{4}{3}$. (Always ≥ 1)

↖ w/ altruistic dictator.

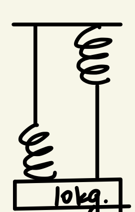
well-designed → close to 1.
as long as (close to) linear cost fns, PoA won't be large.

Braess' Paradox on elastic string. Hooke's law: displacement = k · force.
let k=1.



(three paths.)

if cut the strings,



displacement halved.

01/21/2022 Friday.

Pure and Mixed Nash Equilibrium.

Example: Rock Paper Scissor's Game.

Bimatrix Representation.

	Rock	Paper	Scissors
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissors	-1,1	1,-1	0,0

Utility Maximization Game.

Zero-Sum Game.

Def: Pure Nash Equilibrium (PNE):

A choice of action for each player such that no player can increase her utility by a unilateral deviation.

w/ pure strategy (fix player 2's action), RPS doesn't have PNE.

Mixed Nash Equilibrium (MNE) - mixed strategy: think a player's actions as a RV.

RPS: MNE w/ expected utility = 0. $((\frac{1}{3},\frac{1}{3},\frac{1}{3}),(\frac{1}{3},\frac{1}{3},\frac{1}{3}))$.

"σ_2" = mixed strategy of player 2.

$P(R)=P(P)=P(S)=\frac{1}{3}$.

cost of player 2. → $\frac{1}{3}$ chance he plays R

→ $\frac{1}{3}$ chance his opponent plays R.

Expected utility of player 2 (σ_1,σ_2) $= C_2(\frac{1}{3}R,\frac{1}{3}R)+C_2(\frac{1}{3}R,\frac{1}{3}P)+C_2(\frac{1}{3}R,\frac{1}{3}S)+C_2(\frac{1}{3}P,\frac{1}{3}R)+C_2(\frac{1}{3}P,\frac{1}{3}P)$
$+C_2(\frac{1}{3}R,\frac{1}{3}S)+C_2(\frac{1}{3}S,\frac{1}{3}R)+C_2(\frac{1}{3}S,\frac{1}{3}P)+C_2(\frac{1}{3}S,\frac{1}{3}S)$
$=\frac{1}{9}\times(0-1+1+1+0-1-1+1+0)=0.$

Expected utility of player 2 w/ $\sigma_1, \sigma_2'=R$ = $C_2(1R,\frac{1}{3}R)+C_2(1R,\frac{1}{3}P)+C_2(1R,\frac{1}{3}S)=\frac{1}{3}\times(0-1+1)=0.$

so w/ σ_1, utility of player 2 doesn't increase with any strategy, so Nash Equilibrium.

Single item auction.

setup: - seller w/ 1 item.
- n bidders
- bidder i has valuation v_i. (max willing to pay for an item).
- valuations is private. (to seller & other bidders).

Utility model - "quasilinear utility"

Utility (lose) = 0.

Utility (gain at price p) = $v_i - p$.

Sealed Bid auction. (most style item auctions are sealed bid auctions).

each bidder submits bid b_i.

decide (on) a winner. (usually the highest bidder)

decide (on) a selling price. (pays the second highest price.).
→ "2nd price / Vickrey Auction"

Prop: in a 2nd price auction, every bidder i has a dominant strategy: set $b_i = V_i$. (bidding truthfully).

doesn't care about < item. V_i.

Proof: Fix b_i, V_i, and $b_1, b_2, \cdots, b_{i-1}, b_{i+1}, b_{i+2}, \cdots, b_n$.

Want to show Utility(player i) is maximized when $b_i = V_i$.

let $B = \max_{j \neq i} b_j$

Utility(player i) < 0 / $V_i - B$.

! different V_i

$b_i > B \Rightarrow i$ wins. utility $= V_i - B$.

$b_i < B \Rightarrow i$ loses. utility $= 0$.

$b_i = B \Rightarrow i$ wins, utility $= V_i - B$

i loses, utility $= 0$.

as long as $b_i = V_i$, utility $= \max(0, V_i - B)$

if $V_i > B$, $\max(0, V_i - B) = V_i - B$.

win, $V_i - B$.

if $V_i < B$, $\max(0, V_i - B) = 0$.

lose, 0.

if $V_i = B$, $\max(0, V_i - B) = 0$.

win, 0

lose, 0.

01/24/2022 Monday.

fourth price Auction. / third price Auction.

bidding truthfully is Not a weakly dominant strategy. = dominant strategy. (utility ≥ others).

CounterExample: $b_i > v_i$ to win. (weakly dominant strategy Not existed.)

overbidding may cause negative utility.

Prop2: in a 2^{nd} price auction, every truthful bidder is guaranteed nonnegative utility.

first price Auction.

want $b_i < v_i$.

suppose n people bidding for an item. Suppose each person's valuation is iid uniformly distributed on $[0,1]$.

If $\underset{\checkmark}{n=3}$ / $n=2$, should bid more conservatively (higher / closer to v_i) ?

On average, utility is maximized if $b_i = (\frac{n-1}{n}) v_i$

Ideal Auction.

Def: An auction is dominant-strategy incentive compatible (DSIC) if truthful bidding is always a dominant strategy for every bidder and if truthful bidders always obtain nonnegative utility.

Only Assumption: players will follow the dominant strategy.

Example: 2^{nd} price Auction;

an auction that gives away its item for free to a random bidder.
(bidding strategy is irrelevant.)

Def: Social Welfare at a single-item auction is $\sum_{i=1}^{n} v_i x_i$ where $x_i = \begin{cases} 1 & \text{if } i \text{ wins.} \\ 0 & \text{else.} \end{cases}$

allocation vector $x = (x_1, x_2, \cdots, x_n)$ (all zeros OR one one.)

feasibility constraint: $\sum_{i=1}^{n} x_i = 1$.

Def: An auction is Welfare maximizing if, when bid truthfully, the auction outcome has the max possible social welfare.

Example: 2^{nd} price auction.

Adv. of DSIC, welfare maximizing auction:

item goes to bidder who wants it the most.

the seller knows how people will behave (dominant strategy.)

levels the playing field between simple and sophisticated bidders.

Def. An auction that can be implemented in time polynomial in the size of the input is called computationally efficient.

#bidders.

Ex. all auctions so far.

Def: ideal auction: DSIC + welfare maximizing + computationally efficient.

Ex. 2^{nd} price auction. (single item)

Sponsored Search Auctions. (multiple items auction).

- k slots
- click-through rate (CTR) at slot j is α_j.

  assume $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots \geq \alpha_k$.

  assume CTR is indep. of the actual ads.
- n bidders (advertisers) bid on some key words. top k bidders win.
- one slot can only goes to one bidder. one bidder can only get one slot.
- bidder i gets benefit of v_i per click. $\alpha_j v_i$.

To make it an ideal auction:

01/26/2022 Wednesday.

Design Approach.

STEP 1: Assume truthful bids and find an allocation rule to maximize social welfare in a computational efficient way.

STEP 2: Find a payment rule to the seller that makes it DSIC

Sponsored Search Algorithm:

Social welfare $= \sum_{i=1}^{n} v_i x_i$ where $x_i = \begin{cases} \alpha_j. & \text{if get slot } j. \\ 0 & \text{else.} \end{cases}$

$\sum_{i=1}^{n} x_n \le 1.$

STEP 1: use a greedy algorithm to assign best slots to highest bidders.

↘ Poly.

STEP 2:

Single Parameter Environment.

unit value of stuff.

Def single parameter environment (SPE): n bidders; bidder i has private valuation v_i; there is a feasible set $X \subseteq \mathbb{R}^n$. $\forall x \in X$, $i \in \{1, 2, \cdots, n\}$. x_i = amount of stuff bidder i gets

Example: k-unit auction with k identical items. + utility is linear in both x_i and v_i.

Allocation and Payment Rule.

Ex. Sealed bid Auction.

(1). collect bid profile $b = (b_1, \cdots, b_n)$.

(2). Allocation Rule. choose $x(b) \in X \subseteq \mathbb{R}^n$. $x(b) = (x_1(b), x_2(b), \cdots, x_n(b))$.

(3) Payment Rule: choose $p(b) \in \mathbb{R}^n$. $p(b) = (p_1(b), p_2(b), \cdots, p_n(b))$.

$u_i(b) = v_i x_i(b) - p_i(b)$ require $p_i(b) \in [0, b_i x_i(b)]$.

Statement of Myerson's Lemma.

Def. Implementable Allocation Rule.

An allocation rule, x, for a single parameter environment is implementable if $\exists p$. s.t. (x, p) is DSIC.

Example: in single item auction, is the x that awards the highest bidder implementable?

Yes. e.g. 2^{nd} price auction.

Example: in single item auction, is the x that awards the second highest bidder implementable?

No. Myerson's Lemma: implementable iff monotone.

Not monotone

↘ You bid more you get more.

b_i x_i

01/28/2022 Friday.

Example: two bidders. utility of getting y_i share of good $= v_i \log(1+y_i) - p_i$.

Change into single-parameter environment: let $x_i = \log(1+y_i)$.

$X = \{(x_1, x_2) : x_1 \geq 0,\ x_2 \geq 0,\ e^{x_1} + e^{x_2} \leq 3.\}$.

$y_i = e^{x_i} - 1$.

In DSIC SPE, the bid profile directly reveals players valuation.

We call $(x(b), p(b))$ Direct Revelation Mechanism. / indirect mechanism.

Def: Monotone Allocation Rule.

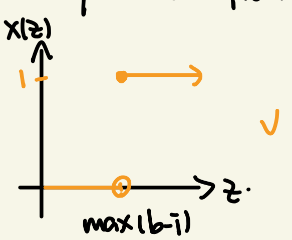
An allocation rule for a single parameter environment is monotone if $\forall$ bidder i, b_{-i} (other bids), $x_i(z, b_{-i})$ is nondecreasing in z.

$z \to b_i$

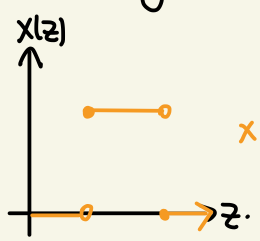
Express $x(z)$ for $x_i(z, b_{-i})$.

$p(z)$ for $p_i(z, b_{-i})$.

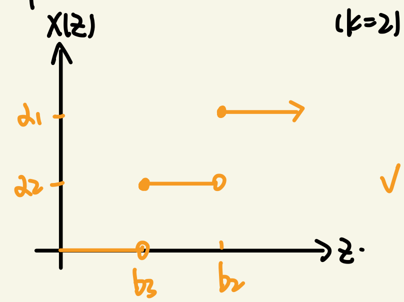
Example: 2^{nd} price auction.



auction that awards the second highest bidder.



Sponsored search Auction. (k=2)



Def: Myerson's Lemma.

Fix a SPE. (a). An allocation rule x is implementable iff it is monotone.

(b). If x is monotone, then $\exists$ unique p s.t. (x, p) is DSIC. (and $p_i(b) = 0$ when $b_i = 0$).

(c). The payment rule p in (b) is given by an explicit formula in the proof.

Proof: (implementable $\Rightarrow$ monotone).

$\exists p$. s.t. (x,p) is DSIC. $\Rightarrow$ truthful bidding gives the maximum utility.

let $0 \leq z < y$. (two fixed numbers).

if $v_i = z$. $u_i(z, b_{-i}) \geq u_i(y, b_{-i})$ overbidding.

$x(z) \cdot z - p(z) \geq x(y) \cdot z - p(y)$. $\Rightarrow z(x(y) - x(z)) \leq p(y) - p(z)$.

if $v_i = y$. $u_i(y, b_{-i}) \geq u_i(z, b_{-i})$ under bidding.

$x(y) \cdot y - p(y) \geq x(z) \cdot y - p(z) \Rightarrow p(y) - p(z) \leq y(x(y) - x(z))$.

So $z(x(y) - x(z)) \leq p(y) - p(z) \leq y(x(y) - x(z))$. $\Rightarrow$ Monotone.

$z(x(y) - x(z)) \leq y(x(y) - x(z))$.

$0 \leq z < y \Rightarrow x(y) - x(z) \geq 0$.

01/31/2022 Monday.

For a single item auction, w/ allocation rule that awards the highest bidder, the payment rule that makes it DSIC is <u>uniquely</u> charging the second price.

disprove other payment rules:
Counterexample – $U(b_i \neq v_i) > U(b_i = v_i)$.

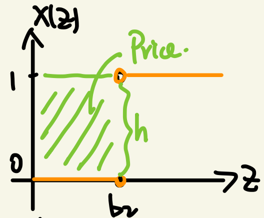
Proof (cont.): $z(x(y) - x(z)) \le p(y) - p(z) \le y(x(y) - x(z))$

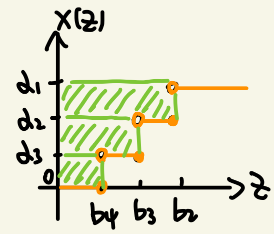
as $y \to x$, we have $p(y) - p(z) = z(x(y) - x(z))$.

if $\lim_{y \to z} x(y) - x(z) = 0$, then $p(y) - p(z) = 0$.

otherwise if $\lim_{y \to z} x(y) - x(z) = h$ for $h > 0$, then $p(y) - p(z) = z \cdot h$.

A typical monotone fn looks like:





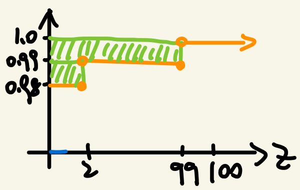
w/ $p(0) = 0$, $p(b_1) = h \cdot b_2 \cdot (1-0) = b_2$.

$p(b_3) = (d_3 - 0) \cdot b_4$. $p(b_2) = (d_2 - d_3) \cdot b_3$. $p(b_1) = (d_1 - d_2) \cdot b_2$.

Generally, order $b_1 \ge b_2 \ge b_3 \ge \dots \ge b_n$. $p(b_i) = \sum_{j=1}^{i} z_j h_j$. where h_j = jump at z_j (break point).
before b_i.
sum!
$p(b_i, b_{-i}) = 0$ if $b_i = 0$.

Therefore gives the unique payment rule.

Example: Sponsored Search auction with 3 bidders. 3 slots. \$100, \$99. \$2. 1.0. 0.99. 0.98.



if b_4 exists, start at 0.

$p(b_3) = 0$

$p(b_2) = (0.99 - 0.98) \times 2 = 0.02$.

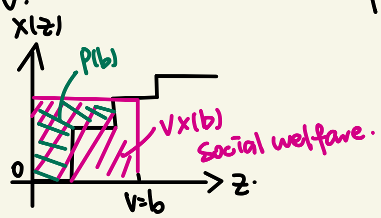
$p(b_1) = 0.02 + (1.0 - 0.99) \times 99 = 1.01$.

02/02/2022 Wednesday.

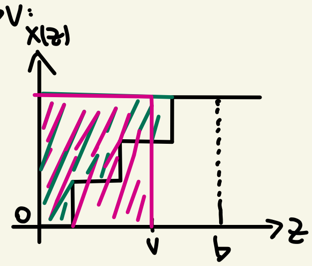
Critical bid is the lowest bid she could make and continue to win (fixing b_{-i}).
e.g. Second price in single item auction.

proof (cont.): (monotone $\Rightarrow$ implementable).

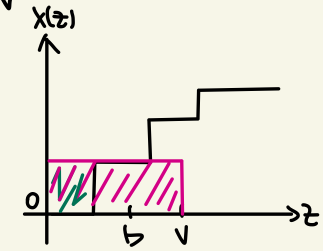
if $b = v$: (b_i, v_i)



if $b > v$:



if $b < v$:



Utility $= vx(b) - p(b)$

So utility is maximised when $b = v$. + Nonnegative $\Rightarrow$ DSIC. $\Rightarrow$ implementable.

Applications of Myerson's Lemma:

Example: Generalised Second Price Auction. (GSP).

k slots w/ CTRs $\alpha_1 \geq \alpha_2 \cdots \geq \alpha_k$. $n \geq k$ bidders.

$b_1 \geq b_2 \geq \cdots b_n$. assign bidder i to slot i at price b_{i+1}.

Not DSIC. as the payment rule here is not the unique payment rule given by the Lemma.

Exercise 4.1.

Thm: SPE w/ n bidders, w/ bids $b_1, b_2, \cdots, b_n$, and a set X of feasible allocations.
Then if $x \in X$ is welfare maximizing, x is monotone. $\Rightarrow$ implementable.

Proof: Fix i and b_{-i}. truthful bidding.

Let $x(b) = \underset{x=(x_1,\dots,x_n)\in X}{\arg\max} \sum_{i=1}^{n} b_i x_i(b_i, b_{-i})$ be the welfare maximizing allocation.

Let $0 < z' < z$.

Let $x' = x(z', b_{-i})$ and $x = x(z, b_{-i})$ be the welfare maximizing allocations for bid profiles (z', b_{-i}) and (z, b_{-i}).

x maximizes social welfare w/ (z, b_{-i}): $z x_i + \sum_{j \neq i} b_j x_j \geq z x_i' + \sum_{j \neq i} b_j x_j'$ ①

x' maximizes social welfare w/ (z', b_{-i}): $z' x_i' + \sum_{j \neq i} b_j x_j' \geq z' x_i + \sum_{j \neq i} b_j x_j$ ②

(① + ② to show monotone.)

Sec 4.1 The knapsack Problem. (limited space, largest value possible.)

think of it as an allocation rule problem:

let $v_1, \cdots, v_n$ be n values. $w_1, \cdots, w_n$ be n weights. W capacity.

let X be the feasible set $(x_1, \cdots, x_n)$ s.t. $\sum_{i=1}^{n} x_i w_i \leq W$. and $x_i \in \{0, 1\}$.

NP-Hard.

otherwise, $x_i \in [0, 1]$. ploynomial.

$x = \underset{x \in X}{\operatorname{argmax}} \sum_{i=1}^{n} v_i x_i$.

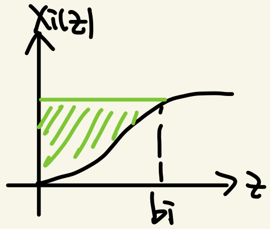
- exhaustively checking 2^n subsets — exponential time.
- dynamic programming — $O(nW)$

02/04/2022 Friday.

integral version of Myerson's Lemma.

$p_i(b_i, b_{-i}) = \int_0^{b_i} z x_i'(z) dz$

$= b_i x_i(b_i) - \int_0^{b_i} x_i(z) dz$



The greedy knapsack Heuristic: fast, myopic, almost optimal decisions

Assume $w_i \le W$, WLOG. $v_1/w_1 \ge v_2/w_2 \ge \cdots \ge v_n/w_n$.
→ otherwise ith item never fit in knapsack.

Let i^* be the maximum i s.t. $\sum_{i=1}^{i^*} w_i \le W$.

Max Value $(x = (\underbrace{1,1,1,\cdots,1}_{i^* \text{ items}}, 0,0,\cdots,0),\ x = (0,0,\cdots,0,\underset{\downarrow \text{ most valuable item.}}{1},0,\cdots,0))$

polynomial time $- O(\underline{n\log n})$ sort + extra linear processing time.

Thm: The greedy knapsack Heuristic is a computationally efficient approximation to the knapsack Problem that gives a total value at least $\frac{1}{2}$ OPT. ← the optimal value.

Proof: if $\sum_{i=1}^{i^*} v_i \ge \frac{1}{2}OPT$, return $x = (1,\cdots,1,0,\cdots,0)$ proved.

if $\sum_{i=1}^{i^*} v_i < \frac{1}{2}OPT$, if we allow a fraction of $(i^*+1)^{th}$ item, we must exceed OPT.

$$OPT < \sum_{i=1}^{i^*} v_i + v_{i^*+1} < \frac{1}{2}OPT + v_{i^*+1}$$

$$v_{i^*+1} > \frac{1}{2}OPT. \quad \text{so } \max\{v_i\} > \frac{1}{2}OPT. \text{ proved.}$$

The Knapsack Auction. (SPE)

Each bidder i has a publically known weight w_i, and a private valuation v_i.

The feasible set X is the 0-1 vectors $(x_1, x_2, \cdots, x_n)$ s.t. $\sum_{i=1}^{n} w_i x_i \le W$.

Example: shared resources w/ limited capacity.

welfare maximizing → NP-hard. DSIC − ✓

Not computational efficient → Not Ideal.

Thm: The greedy knapsack Heuristic has a monotone allocation.
→ bid truthfully, max value = max social welfare.

Pf: Fix bidder i and b_{-i}. Let $0 < z < y$. $\frac{b_1}{w_1} \ge \frac{b_2}{w_2} \ge \cdots \ge \frac{b_i}{w_i} \ge \cdots \ge \frac{b_n}{w_n}$.

Want to show $x_i(y, b_{-i}) \ge x_i(z, b_{-i})$

if $x_i(z, b_{-i}) = 0$, then clearly $x_i(y, b_{-i}) \ge x_i(z, b_{-i})$.

if $x_i(z, b_{-i}) = 1$, then must show $x_i(y, b_{-i}) = 1$.

if top i^* bidders win, if bidder i with $b_i = z$ wins, bidder i with $b_i = y$ must win, as $\frac{y}{w_i}$ is

(HW2) more dense than $\frac{z}{w_i}$ given $y > z$.

if the highest bidder win, then $x_i(y, b_{-i}) = 1$ as $y > z$ assume bid truthfully.

Therefore, the greedy knapsack Heuristic is

- DSIC ✓ w/ critical bid payment.
- welfare maximizing. ✗
- computational efficient ✓.

02/07/2022 Monday.

pricing the knapsack auction with greedy heuristic:

* critical bid payment.

Example:

i	v	w	v/w
1	90	5	18
2	6	1	6

$W=5$.

$i^*=1$ value $=90$, highest bid $=90$ } bidder 1 wins.

$\frac{x}{5}=\frac{6}{1} \Rightarrow x=30$ but bidder 1 still has the highest value. bidder 1 pays 6.

Example:

i	v	w	v/w
1	4	2	2
2	4	3	4/3
3	5	4	5/4
4	6	6	1

$W=6$.

$i^*=2$. value $=8$. highest bid $=6$ } $(1,1,0,0)$.

bidder 1 pays $\frac{x}{2}=\frac{5}{4} \Rightarrow x=\frac{5}{2}$.

bidder 2 pays $\frac{x}{3}=\frac{5}{4} \Rightarrow x=\frac{15}{4}$.

In SPE, the seller doesn't need to know v_i to maximize social welfare.

Example: single bidder, single item.

social welfare $= v \cdot \underbrace{x(b)}_{0 \text{ or } 1}$. to maximize it, seller sets price at 0.

Example: $n>1$ bidders, single item.

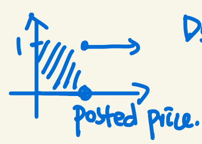
to maximize welfare $\rightarrow$ DSIC, so must 2^{nd} price. (no need to know v_i)

However, the seller needs to know v_i to make x and p that maximize revenue.

(Focusing on revenue maximizing DSIC Auctions.)

Example: single bidder, single item.

posted price $= v_i$ allocation: wins if $v_i \geq$ posted price.



DSIC. price $=1 \times$ posted price

Bayesian Analysis: (measure revenue without knowing v_i).

Assumptions: (1) $v_i \sim \underbrace{F_i}_{\text{cdf.}}$ with cts. density f_i $F(x)=P(v \leq x)$.

(2) $v_1, \cdots, v_n$ are independent.

(3) $\forall i$. f_i covers $[0, v_{max}]$

(4) The seller knows F_i. $\forall i$.

revenue = expected revenue over $F_1 \times F_2 \times \cdots \times F_n$

02/09/2022 Wednesday.

Externality Pricing.

Individual objective (maximize utility) is in alignment with the collective objective (maximize social welfare).

Another way to think about critical bid pricing for social welfare maximizing DSIC Auctions.

Example: 3 bidders, 2 identical items.

Soobin $b_1=10$

Adam $b_2=8$

Mikhail $b_3=5$.

(max welfare with Soobin − max welfare without Soobin).

Soobin pays = Soobin's bid − the social welfare Soobin brings

$$\| \quad = b_1x_1 - \left(\sum_{i=1}^{3} x_i b_i - \max_{(y_2,y_3)\in Y} \sum_{i=2}^{3} b_i y_i\right) = 10-(18-13)=5.$$

Soobin's externality = welfare loss the winner imposes on others.

= other's welfare without Soobin − other's welfare with Soobin

$$= \max_{(y_2,y_3)\in Y} \sum_{i=2}^{3} b_i y_i - \sum_{i=2}^{3} b_i x_i = 13-8=5.$$

Revenue-Maximizing.

Ex. one bidder, one item. revenue-maximizing $v\sim F$. posted price r.

expected revenue $= r\cdot P(r\le v) + 0\cdot P(r>v) = r\cdot(1-F(r))$.

Suppose F is standard uniform distribution on $[0,1]$, so $F(x)=x$.

$r=\frac{1}{2}$ maximizes revenue.

Ex. two bidders, one item. $V_1\ge V_2$ $V_1,V_2 \overset{iid}{\sim} \text{Uniform}(0,1)$, $V_2\sim \text{Beta}(1,2)$ w/ $f(v)=2(1-v)$.

expected revenue w/ second-price auction $= E(V_2) = \int_0^1 x f(x)\,dx = 2\int_0^1 x(1-x)\,dx = \frac{1}{3}$.

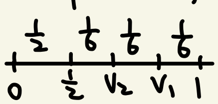
if put a reserve price ($\nearrow$ min bid) $r=\frac{1}{2}$:

if $V_1>V_2>r$. V_1 wins. revenue $=V_2$. Prob $=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$.

if $V_1>r>V_2$. V_1 wins, revenue $=r$. prob $=\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}=\frac{1}{2}$.

if $r>V_1>V_2$. nobody wins. prob $=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$.

$$E(\text{revenue}) = E(V_2 \mid \tfrac{1}{2}<V_2<V_1)\cdot\tfrac{1}{4} + \tfrac{1}{2}\cdot\tfrac{1}{2} + 0\cdot\tfrac{1}{4} = \tfrac{4}{6}\cdot\tfrac{1}{4}+\tfrac{1}{4} = \underset{\triangle}{\tfrac{5}{12}} > \tfrac{1}{3}.$$



Vickey Auction w/o reserve price: $E(\text{revenue})=\frac{1}{3}$

Is $r=\frac{1}{2}$ an optimal reserve price?
$\quad\quad\quad\quad \|$
max revenue

02/11/2022 Friday.

Ex. One item, two bidders, $V_i \sim Exp(1)$, Second price Auction.

Expected Revenue $= E(\min(V_1, V_2))$.

$1 - F_{\min(V_1,V_2)}(t) = P(\min(V_1, V_2) > t) = P(V_1 > t, V_2 > t) = P(V_1 > t)^2 = (e^{-t})^2 = e^{-2t}$.

so $\min(V_1, V_2)$ has survival fn e^{-2t}, $\min(V_1, V_2) \sim Exp(2)$.

As $E(Exp(\lambda)) = \frac{1}{\lambda}$, $E(Exp(2)) = \frac{1}{2}$.

<u>Explicitly describe how to design a DSIC, revenue-maximizing SPE:</u>

$\mathbb{V} = (V_1, V_2, \cdots, V_n)$ $\mathbb{F} = (F_1, \cdots, F_n)$. truthful bidding.

Want to optimize $E_{\mathbb{V} \sim \mathbb{F}}(\sum_{i=1}^n P_i(\mathbb{V}))$

STEP 1: By Myerson's Lemma, $P_i(\mathbb{V}) = \int_0^{V_i} z x_i'(z, \mathbb{V}_{-i})\, dz$.

$$E_{F_i}(P_i(\mathbb{V}) | \mathbb{V}_{-i}) = \int_0^{V_{max}} P_i(\mathbb{V}) f_{V_i|V_{-i}}(\mathbb{V})\, dV_i.$$

(by independence of $F_1, \cdots, F_n$ and $\mathbb{V}_{-i}$ fixed.)

$$= \int_0^{V_{max}} \int_0^{V_i} z x_i'(z, \mathbb{V}_{-i})\, dz\, f_i(V_i)\, dV_i$$

$$= \int_{V_i=0}^{V_{max}} \int_{z=0}^{V_i} z x_i'(z, \mathbb{V}_{-i}) f_i(V_i)\, dz\, dV_i.$$

(introduce allocation rule x into the formula).

STEP 2: x is monotone, $x_i' \geq 0$.

$0 \leq z \leq V_i \leq V_{max}$. fixing V_i, z runs between 0 to V_i.

$$\int_{V_i=0}^{V_{max}} \int_{z=0}^{V_i} \underbrace{z x_i'(z, \mathbb{V}_{-i}) f_i(V_i)}_{a(z, V_{-i})}\, dz\, dV_i$$

By Fubini's thm, we can switch the order of integration.

fixing z, let V_i runs between. $\int_0^{V_{max}} \int_z^{V_{max}} a(z, V_{-i})\, dV_i\, dz = \int_0^{V_{max}} \int_z^{V_{max}} z x_i'(z, \mathbb{V}_{-i}) f_i(V_i)\, dV_i\, dz$.

$$= \int_{z=0}^{V_{max}} z x_i'(z, \mathbb{V}_{-i}) \int_{V_i=z}^{V_{max}} f_i(V_i)\, dV_i\, dz$$

$$= \int_{z=0}^{V_{max}} z x_i'(z, \mathbb{V}_{-i}) (1 - F(z))\, dz.$$

STEP 3: $\int_{z=0}^{V_{max}} x_i'(z, \mathbb{V}_{-i}) z(1 - F(z))\, dz = x_i(z, \mathbb{V}_{-i}) z (1 - F(z)) \Big|_{z=0}^{z=V_{max}} - \int_{z=0}^{V_{max}} x_i(z, \mathbb{V}_{-i}) [1 - F(z) - z f(z)]\, dz$.

$\int g' h = gh - \int g h'$

$$= \int_{z=0}^{V_{max}} x_i(z, \mathbb{V}_{-i}) \underbrace{\left[z - \frac{1 - F(z)}{f(z)}\right]}_{\varphi_i(z)} f(z)\, dz.$$

Def: Virtual Valuation $\varphi_i(v_i) = v_i - \frac{1-F(v_i)}{f_i(v_i)}$ = max revenue − revenue lost by not knowing v_i in advance.

Example: let $v_i \sim U(0,1)$

$$\varphi_i(v_i) = v_i - \frac{1-v_i}{1} = 2v_i - 1$$

if v_i is a constant RV (known to seller), $f(v_i)$ is a point mass ($f(z) = \begin{cases} \infty & \text{if } z = v_i \\ 0 & \text{else} \end{cases}$).

$$\varphi_i(v_i) = v_i - \frac{1-F(v_i)}{f_i(v_i)} = v_i.$$

STEP 4: $E_{F_i}(p_i(V) \mid V_{-i}) = \int_{z=0}^{v_{max}} x_i(z, V_{-i}) \varphi_i(z) f_i(z) dz. = E_{F_i}(x_i(V)\varphi_i(v_i) \mid V_{-i})$

STEP 5: $E_{\mathbb{F}}(\sum_{i=1}^{n} p_i(V)) = \sum_{i=1}^{n} E_{\mathbb{F}}(p_i(V))$

$$E_{\mathbb{F}}(p_i(V)) = E_{\mathbb{F}_{-i}}(E_{\mathbb{F}_i}(p_i(v_i) \mid V_{-i})). = E_{\mathbb{F}_{-i}}(E_{\mathbb{F}_i}(x_i(V)\varphi_i(v_i) \mid V_{-i}) = E_{\mathbb{F}}(x_i(V)\varphi_i(v_i))$$

So $E_{\mathbb{F}}(\sum_{i=1}^{n} p_i(V)) = E_{\mathbb{F}}(\underbrace{\sum_{i=1}^{n} x_i(V)\varphi_i(v_i)}_{\text{Virtual social welfare.}})$.

*doesn't depend on p_i

Conclusion: Maximizing total revenue => maximizing virtual social welfare.

02/14/2022 Monday.

Def. A distribution F is "regular" if its virtual valuation is strictly increasing.

Example: $V \sim Exp(\lambda)$ $\lambda > 0$. $f(v) = \lambda e^{-\lambda v}$. $F(v) = 1 - e^{-\lambda v}$.

$$\varphi(v) = v - \frac{1-F(v)}{f(v)} = v - \frac{1-1+e^{-\lambda v}}{\lambda e^{-\lambda v}} = v - \frac{1}{\lambda} \text{ strictly increasing.}$$

$\varphi^{-1}(0) = \frac{1}{\lambda}$

$v - \frac{1}{\lambda} = 0$

Maximize expected virtual welfare.

Assumption: 1) single item auction.

2) $V_1, V_2, \cdots, V_n \overset{i.i.d.}{\sim} F \Rightarrow$ all φ_i are the same.

Find allocation rule x that maximize $\mathbb{E}_F[\sum_{i=1}^n x_i(v)\varphi(v_i)]$.

$\Downarrow$

Pointwise: $\sum_{i=1}^n \underbrace{x_i(v)}_{\{0,1\}} \varphi(v_i)$ Award item to bidder w/ highest nonnegative virtual valuation if any.

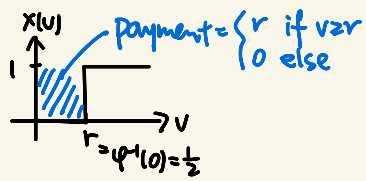
⭐ optimal single item auction

Not necessarily the highest bidder.

Example: one-bidder, one-item.

$$x(v) = \begin{cases} 1 & \text{if } \varphi(v) \geq 0 \iff v \geq \varphi^{-1}(0) \text{ assuming } \varphi \text{ is strictly increasing.} \\ 0 & \text{else} \end{cases}$$

if $F \sim U[0,1]$, $\varphi^{-1}(0) = \frac{1}{2}$



expected optimal revenue $= \frac{1}{2} \cdot P(v \geq \frac{1}{2}) = \frac{1}{4}$.
(optimal = revenue-maximizing)

Exercise 5.5: SPE, regular distribution $F_1, F_2, \cdots, F_n$, the virtual welfare maximizing $x(v)$ is monotone.

Thm: SPE, regular distribution $F_1, F_2, \cdots, F_n$, the allocation rule that maximizes virtual welfare is implementable and maximizes expected revenue. this characterizes optimal DSIC Mechanisms.

Example: n bidders, single item. Let $V_1, \cdots, V_n \overset{iid}{\sim} U[0,1]$

Assume $V_1 \geq V_2 \geq \cdots \geq V_n$. Find the optimal DSIC Auctions.

regular distribution $\Rightarrow$ want to find virtual welfare maximizer.

bidder 1 wins if $\varphi(v_1) \geq \max(\varphi(v_j), 0)$ $\forall j \neq i = \max(\varphi(v_2), 0)$.

$v_1 \geq \max(v_2, \varphi^{-1}(0) = \frac{1}{2})$.

Therefore, the optimal DSIC auction is the Vickrey Auction w/ $r = \frac{1}{2}$.

Example: k-unit item. regular i.i.d. distribution.

$r = \varphi^{-1}(0)$ depends on F. $\max(v_{k+1}, \varphi^{-1}(0))$.

02/16/2022 Wednesday.

Example: two bidders. $V_1, V_2 \overset{iid}{\sim} Exp(1)$

$$\varphi(v) = v - \frac{1-(1-e^{-v})}{e^{-v}} = v-1 \Rightarrow \varphi^{-1}(0) = 1.$$

Revenue maximizing auction. 2^{nd} price auction w/ $r=1$.

$F(v) = 1-e^{-v} \Rightarrow F(1) = P(v \le 1) = 1-e^{-1}$

$V_1, V_2 < r$. $p = (1-e^{-1})^2$. $E(\text{revenue}) = 0$.

$V_1/V_2 < r$, $V_1/V_2 > r$ $\quad p = 2(1-(1-e^{-1}))(1-e^{-1}) = 2(e^{-1}-e^{-2})$ $\quad E(\text{revenue}) = 1$. $\max(V_2, \varphi^{-1}(0) = 1)$.

$V_1, V_2 > r$. $p = (1-(1-e^{-1}))^2 = e^{-2}$ $\quad E(\text{revenue}) = E(\min(V_1,V_2) \mid \min(V_1,V_2) \ge 1) = 1 + \frac{1}{2} = \frac{3}{2}$.

$\min(V_1,V_2) \sim Exp(2)$. memoryless property of Exp.

Therefore, $E(\text{revenue}) = 2(e^{-1}-e^{-2}) \cdot 1 + e^{-2} \cdot \frac{3}{2}$

Def: An auction is Prior independent if it doesn't reference $F_1, \cdots, F_n$.

Ex. Vickrey auction w/o reserve price is prior independent.

$V_1, V_2, \cdots, V_n \overset{iid}{\sim} F$, $V_1 \ge V_2 \ge \cdots \ge V_n$. pays V_2. → No F.

Ex. Optimal Second price, single item auctions are Not prior independent.

$\cdots$ if $V_1 \ge \varphi^{-1}(0)$, pays $\max(V_2, \varphi^{-1}(0))$. → F otherwise, no winner.

Example: Single item. $F_1, \cdots, F_n$ are independent but not identically distributed. $\varphi_1 = \varphi_2 = \cdots = \varphi_n$.

Assume $V_1 \ge V_2 \ge \cdots \ge V_n$.

ith bidder wins if $\varphi_i(V_i) \ge \max_{j \ne i} (\varphi_j(V_j), 0)$

$$V_i \ge \max_{j \ne i} (\varphi_i^{-1}\varphi_j(V_j), \varphi_i^{-1}(0)).$$

Bulow-Klemperer Theorem: Single item auction, valuations iid, regular.

Expected revenue of Vickrey Auction (winner always exist) (n+1 bidders) welfare-maximizing $\ge$ Expected revenue of Optimal Auction (can be no winner). A^* (n bidders). revenue-maximizing.

Bringing extra people is more important than setting the optimal reserve price.

Proof: Consider Auction A. (create a third auction to bridge the gap.)

n+1 bidders. Simulates A^* Auction. If no winner, $n+1^{st}$ bidder gets it for free.

Vickrey Auction: n+1 bidders; always sells ; DSIC; Not nessarily maximize revenue.

Auction A: n+1 bidders; always sells; DSIC; Maximize revenue for n bidders.

Auction A^*: n bidders; Not always sells; DSIC; Maximize revenue.

Lemma: Given n+1 bidders w/ iid, regular valuations, the Vickrey auction maximizes expected revenue among all DSIC Auctions that always sells the item.

e.g. give to a random bidder.

Proof of Lemma: DSIC optimal single item auction is uniquely Vickrey Auction w/ $r = \varphi^{-1}(0)$.

always sell the item $\Rightarrow$ remove r.

Proof cont.: E(revenue A) = E(revenue A^*) $\leq$ E(Vickrey Auction).

02/18/2022 Friday

the optimal expected revenue of single item auctions of size n-1 ≥ that of size n.

$$E_{v\sim F^{n-1}}\left[\sum_{i=1}^{n-1} p_i^*(v)\right] \geq \frac{n-1}{n} E_{v\sim F^n}\left[\sum_{i=1}^{n} p_i^*(v)\right]. \quad A^*$$

Proof: Consider Auction A w/ n-1.

A simulates A^* for n bidders. $\begin{cases} \text{sell the item to } 1,2,\cdots,n-1 \text{ OR no one} \Rightarrow \text{copy outcome of } A^*. \\ \text{sell the item to } n. \Rightarrow \text{doesn't sell.} \end{cases}$

As valuations are i.i.d, $p=\frac{1}{n}$

So Expected Revenue A $= \frac{n-1}{n}$ Expected Revenue A^*.

As A^* is DSIC, A is DSIC. So Expected Revenue (optimal auction n-1) ≥ Expected Revenue (A).

General mechanism design environments

now we allow each bidder to have multiple private parameters (Not SPE) only v_i - valuation/unit

Setup: - n bidders

- a finite set Ω of outcomes (feasible allocations). a vector indicating who wins what; an individual can win multiple items.
- $v_i : \Omega \to \mathbb{R}_{\geq 0}$
- Social welfare of $w \in \Omega$ is $\sum_{i=1}^{n} v_i(w)$

Example: $\Omega = \{(1,0), (0,1), (0,0)\}$. $v_1(1,0)=5$. $v_1(0,1)=4$. $v_1(0,0)=0$. (allow preference for who wins).

Single Parameter Environment is a special case of GMDE.

Ω = feasible set X.

for $w \in \Omega$, $w=(x_1, x_2, \cdots, x_n)$.

$v_i(w) = v_i x_i$.

Combinatorial Auctions.

- n bidders
- a set of items M, $|M|=m$.
- $\Omega = \{(S_1, S_2, \cdots, S_n) \mid S_i \subseteq M, S_i \cap S_j = \phi\}$. $\sum_{i=1}^{n} S_i = M$. "bundle"
- $V_i : S_i \to \mathbb{R}$. So 2^m private valuations for bidder i. (No preference) each item either in the bundle or not.

Def: A VCG Auction is any welfare maximizing DSIC auction of a general mechanism design environment. * No myerson's Lemma.

Ex. Vickrey Auction. Sponsored Search Auction. Knapsack Auction.

To design a DSIC, VCG mechanism:

> find a welfare maximizing allocation rule (assume truthful bidding).

Def: In a general mechanism design environment, a welfare maximizing allocation rule is: for a fixed $\mathbb{b}$: $X(\mathbb{b}) = \underset{w \in \Omega}{\operatorname{argmax}} \sum_{i=1}^{n} b_i(w)$ where $b_i: \Omega \to \mathbb{R}_{\geq 0}$.

Example: Shoe auction $M = \{L, R\}$.

$\mathbb{b} = (b_1, b_2)$ where $b_1(w) = \begin{cases} 1 & \text{if } w = (M, \phi) \\ 0 & \text{otherwise.} \end{cases}$ $b_2(w) = \begin{cases} 1 & \text{if } w = (\phi, M) \text{ or } (\phi, L) \text{ or } (R, L) \\ 0 & \text{otherwise.} \end{cases}$

$X(\mathbb{b}) = \{(M, \phi), (\phi, M), (\phi, L), (R, L)\}$. all equal.

02/23/2022 Wednesday.

the VCG Mechanism Vickrey, Clarke, Groves

let $w^* \in X(\mathbb{b})$ be the outcome. "$w^*(\mathbb{b})$"

Example: Shoes Auction.

w^* is randomly chosen from $X(\mathbb{b}) = \{(M,\phi), (\phi, M), (\phi, L), (R, L)\}$. say (M, ϕ).

Revenue = $p(\text{Danny}) + P(\text{Gum}) = 1-(1-1) + 0-(1-1) = 1$

if there is a 3rd bidder, Amanda, who only wants a Right shoe.

$w^* = (\phi, L, R)$. Revenue $= P_1 + P_2 + P_3 = V_1(\phi) - (\overset{w/}{2} - \overset{w/o}{2}) + V_2(L) - (2-1) + V_3(R) - (2-1) = 1$

VCG Mechanisms can have less revenue with more competition. (Bad Revenue Property)

VCG auctions usually aren't computationally efficient. (welfare maximizing is NP-hard)

Ex. Knapsack Auction.

Ex. Combinatorial Shoe auction with n shoes has 2^n private valuations.

Maximizing social welfare involves finding max of an exponential big set → cannot be computed in polynomial time. (preference elicitation.)

VCG Theorem: In every general mechanism design environment, there is a DSIC welfare-maximizing mechanism.

Proof: fix $\mathbb{b}$. want to show that the welfare-maximizing allocation rule $\underset{w^*(b_1, b_{-1})}{w^*} = \underset{w \in \Omega}{\operatorname{argmax}} \sum_{i=1}^{n} b_i(w)$ with externality payment $P_i(\mathbb{b}) = \max_{w \in \Omega} \sum_{j \neq i} b_j(w) - \sum_{j \neq i} b_j(w^*)$ is DSIC.

let $i=1$, $U(b_1) = V_1(w^*) - P_1(\mathbb{b})$

$= V_1(w^*) - \left(\max_{w \in \Omega} \sum_{j=2}^{n} b_j(w) - \sum_{j=2}^{n} b_j(w^*)\right)$

$= \underbrace{V_1(w^*) + \sum_{j=2}^{n} b_j(w^*)}_{\text{need to maximize}} - \underbrace{\max_{w \in \Omega} \sum_{j=2}^{n} b_j(w)}_{\text{independent of bidder 1.}}$

As $w^*(V_1, b_{-1})$ maximizes $V_1(w^*(V_1, b_{-1})) + \sum_{j=2}^{n} b_j(w^*(b_j, b_{-j}))$.

i.e. $V_1(w^*(V_1, b_{-1})) + \sum_{j=2}^{n} b_j(w^*(b_j, b_{-j})) \geq V_1(w^*(b_1, b_{-1})) + \sum_{j=2}^{n} b_j(w^*(b_j, b_{-j}))$.

we have bidding truthfully is a dominant strategy.

$P_1(\mathbb{b}) = \underset{\substack{\uparrow \\ \text{include } w^*}}{\max_{w \in \Omega}} \sum_{j=2}^{n} b_j(w) - \sum_{j=2}^{n} b_j(w^*)$. so $P_1(\mathbb{b}) \geq 0$.

$\sum_{j=1}^{n} \bar{b_j}(w^*) = \max_{w\in\Omega} \sum_{j=1}^{n} \bar{b_j}(w) \geq \max_{w\in\Omega} \sum_{j=2}^{n} \bar{b_j}(w)$ as $b_1 \geq 0$

$\sum_{j=1}^{n} \bar{b_j}(w^*) - \max_{w\in\Omega} \sum_{j=2}^{n} \bar{b_j}(w) \geq 0 \Rightarrow -P_1(b) + b_1(w^*) \geq 0 \Rightarrow \underline{P_1(b) \leq b_1(w^*)}$

So $U_1(b) = b_1(w^*) - P_1(b) \geq 0$ the utility is nonnegative

02/25/2022 Friday.

Example: $m = \{L, R\}$.

bid truthfully: $V_1(m)=5$ $V_2(L)=2$ $V_3(R)=2$ $\Rightarrow$ welfare-maximizing $X=\{m,0,0\}$.

Collusion: $V_1(m)=5$. $b_2(L)=4$. $b_3(R)=4$. $\Rightarrow X=\{0,L,R\}$

$p_2=p_3=4-(8-5)=1$. $U_2=U_3=2-1=1>0$.

truthful bidding is still a weakly dominant strategy for bidder 2.3.

VCG gives a way for players to collude (Bad Incentive Property)

Mechanism Design Without Money. (e. voting, organ donation, school choice).

Example: House Allocation Problem.

N agents. Each initially owns one house. Each Agent's perferences are represented by a total ordering over the N houses rather than by numerical valuations.

An agent does not need to prefer her own house over the others.

How to reallocate the houses to make the agents better off?

Top Trading Cycle (TTC) Algorithm.

initialize N. While $N \neq \emptyset$ do

form the directed Graph G with vertex set N and edge set $\{(i,l)$: i's favorite house within N is owned by $l\}$. Compute the directed circles $C_1, \cdots, C_h$ of G.

available houses

self-loops; circles are disjoint

for each edge (i,l) of each cycle: reallocate l's house to agent i.

remove agents of $C_1, \cdots, C_h$ from N.

Each iteration of TCC produce at least one cycle.

proof: 1^{st} point $\to$ itself done.

$\searrow$ 2^{nd} point $\to$ 1^{st} point / itself done.

$\searrow$ 3^{rd} point .... the last point must point to a previous point done.

Lemma: Let N_k denote the set of agents removed in the k^{th} iteration. Every agent of N_k receives her favorite house outside those owned by $N_1 \cup \cdots \cup N_{k-1}$, and the original owner of this house is in N_k.

swap houses within N_k.

Thm: The TCC algorithm induces a DSIC Mechanism.

Proof: Fix agent i and reports (ordering) by the other. Assume i reports truthfully and $i \in N_j$.

Want to show no misreport can get i a house originally owned by $N_1 \cup \cdots \cup N_{j-1}$.

Suppose $\ell \in N_1$ points to i, then $i \in N_1$ regardless of what i reports.

However, $i \notin N_1$, so no one in N_1 points to i. $\Rightarrow$ No one in $N_1 \cup \cdots \cup N_{j-1}$ points to i.

Def: Consider the assignment of one distinct house to each agent. A subset of agents form a blocking coalition if they can internally reallocate their original homes to make some member better off while making no member worse off.

Def: A core allocation (= Pareto optimal allocation) is an assignment with no blocking coalitions.

(= "maximizing social welfare")

Thm: The allocation computed by the TCC Algorithm is the unique core allocation.

Proof: First, uniqueness. (if there is a core allocation, it is the TCC.)

Everyone in N_1 gets their first choice. $\Rightarrow$ N_1 forms a blocking coalition for all allocations differ from N_1.

Next, TCC is a core allocation.

Everyone in N_2 is maximally happy outside N_1.

if we combining N_1 and N_2, someone in N_1 must point to N_2 and be unhappy.

Def: An allocation is Pareto Optimal if there is no way to make someone better off without making someone else worse off.

Ex. TCC is Pareto optimal in the house allocation problem.

02/28/2022 Monday.

Example:

		Player 2		
		L	M	H
player 1	L	6,6	2,8	0,4
	M	8,2	4,4	1,3
	H	4,0	3,1	2,2

Pareto optimal allocations = (L,L), (M,L), (L,M).

Def: A strategy profile is a Nash Equilibrium if for every player, no unilateral deviation increases his payoff.

Example: Prisoner's Dillema Game. (payoff maximization game).

		player 2	
		Mum	Fink
player 1	Mum	-2,-2	-5,-1
	Fink	-1,-5	-4,-4

Pareto optimal allocation = (M,M), (M,F), (F,M)

Nash Equilibrium: (F,F)

Stable Matchings

(e. matching patients with hospitals, students to elementary schools).

Let M be a matching of V and W.
whole set

Def: Vertices $v \in V$ and $w \in W$ form a blocking pair for M if they are not matched in M, v prefers w to his match in M, and w prefers v to her match in M.

Def: A matching is stable if it has no blocking pairs.

Deferred Acceptance Algorithm. / Grale Shapley (GS) Algorithm.

While there is unmatched $v \in V$ do
    v attempts to match with her favorite w who has not reject him yet.
    if w is unmatched then u and w are tentatively matched.
    else if w is tentatively matched to v' then w rejects whomever of v, v' it likes less and is tentatively matched to the other one.
All tentative matches are made final.

Example:

boys' lists	V boys		W girls	girls' lists
D, E, F	A	—	D	A, B ×, C ×
~~D~~, E, F	B	—	E	B, C ×, A
~~D~~, ~~E~~, F	C	—	F	C, A, B

Iterations: C→B→A→C→B→C

① each boy systematically goes through his list from top to bottom.

② Since girls only reject a boy in favor of a better one, girls match with better and better boys.

③ At all times boys match with one girl and girls match with one boy.

fast computation

Thm: The G-S Algorithm finds a stable matching after at most n^2 iterations, where n is the number of vertices on each side.

proof: First, show the upper bound is n^2.

Each boy works down his list of n preferences never repeating, resulting in at most n iterations per boy.

Next, show the G-S Algorithm finds a matching.

If not, a boy is rejected by all girls. A boy can only be rejected in favor of a better boy, and once a girl is matched she is never alone. Hence all girls are matched, contradict.

Last, show the matching is stable.

Consider an arbitrary v and w not matched to each other. How can this occur?

(1) v is matched to w' who v prefers to w.

(2) w rejects v for a boy she prefers.

Hence no blocking pairs.

Corollary: (Existence of a stable Matching) For every collection of preference lists for boys and girls, there exists at least one stable matching.

different stable matching if girls do the process.

Corollary: The stable matching is unique if boys are doing the process. The stable matching in the G-S Algorithm doesn't depend on how the unmatched boy is chosen in each iteration.

03/07/2022 Monday.

(most precisely about the last corollary).

For a boy v, let $h(v)$ denote the highest ranked girl (in v's preference list) to which v is matched in any stable matching from a run of GS.

Thm: (Boy-optimality). The stable matching computed by the GS Algorithm matches every $v \in V$ to $h(v)$.

proof. Lemma A: If I am ever rejected by a girl (in some rounds of a GS run), then I can't be in a stable matching with that girl in any run of GS.

Assume Lemma A, prove the Thm:

Suppose h(Adam) = Mindy. If there exist a stable matching matching Adam with a girl he less like, Adam is rejected by Mindy. By lemma A, contradicted.

Prove Lemma A by strong induction:

Want to prove if Adam is rejected by Mindy for Troy in round i of a GS Run, Adam can't be in a SM with Mindy in any run of GS.

$i=1$ (impossible).

	round 1	round 2	
$i=2$.	A — M	R — M A	M is R's top choice.

If A—M, Mindy is matched with someone less like than Troy. Troy is matched with someone less like than Mindy. ⇒ Blocking pair.

Generally, if Adam is rejected by Mindy in $i>2$. Either M is T's top choice OR Troy was rejected by all girls he prefers to M in round $<i$.

by induction Troy can't match those girls ⇒ blocking pair.

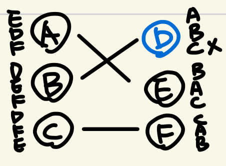
Corollary: The GS Algorithm outputs the worst possible Stable Matching for girls.

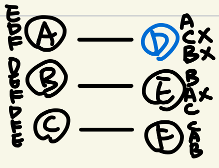
Corollary: Image the preferences for boys and girls are private.

(a) The GS is DSIC for boys.

(b) The GS is Not always DSIC for girls.

Example:



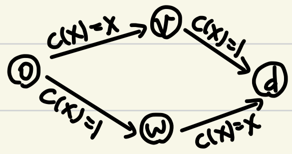


Nonatomic Selfish routing.

Example. Eligible size of cars. Want to get from origin o to destination d in shortest time.

Cost fn $c(x)$ is publically known, so no private info.

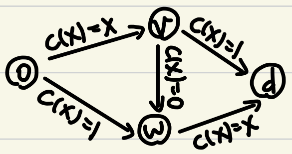
optimal flow VS. equilibrium flow.



Let $G=(E,V)$ be this selfish routing network having $r=1$ unit of total traffic.

traffic rate.

by symmetry, common traffic time $=\frac{3}{2}$ hrs.



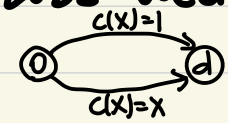
o-v-w-d is the dominant strategy when $x<1$. Eventually $x=1$.

New equilibrium = 2hrs.

Braess' Paradox: by increasing your feasible set of possible paths, you have a worse equilibrium.

the Price of Anarchy (PoA) $= \frac{\text{equilibrium travel time}}{\text{optimal travel time}} = \frac{2}{3/2} = \frac{4}{3}.$

03/09/2022 Wednesday.

Example: 

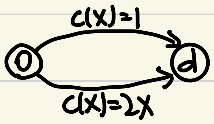
$(\frac{1}{4}, \frac{3}{4})$ $\frac{1}{4}$ unit traffic top edge, $\frac{3}{4}$ unit traffic bottom edge.

equilibrium flow: bottom path dominant $f=(0,1)$ $c(f)=0\cdot1+1\cdot1=1$

optimal flow: $c(f^*)=x\cdot1+(1-x)(1-x)=x^2-x+1$ $c'(f^*)=2x-1=0 \Rightarrow x=\frac{1}{2}$.

$f^*=(\frac{1}{2},\frac{1}{2})$. $c(f^*)=\frac{1}{2}\cdot1+\frac{1}{2}\cdot\frac{1}{2}=\frac{3}{4}$

$PoA=\frac{c(f)}{c(f^*)}=\frac{4}{3}$.

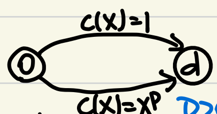


equilibrium flow: $C_{top}(f)=C_{bot}(f) \Rightarrow f=(\frac{1}{2},\frac{1}{2})$ $c(f)=1$

optimal flow: $c(f^*)=2x^2-3x+2$ $c'(f^*)=4x-3 \Rightarrow x=\frac{3}{4}$.

$f^*=(\frac{3}{4},\frac{1}{4})$ $c(f^*)=\frac{7}{8}$.

$PoA=\frac{8}{7}<\frac{4}{3}$.

Pigou's example: 

two edge selfish routing network. $r=1$ (i.e. $x\le1$)

equilibrium flow: bottom path dominant. $f=(0,1)$ $c(f)=1$.

optimal flow: $f^*=(\varepsilon,1-\varepsilon)$ $c(f^*)=\varepsilon\cdot1+(1-\varepsilon)(1-\varepsilon)^p=\varepsilon+(1-\varepsilon)^{p+1}$ As $p\to\infty$, $\to\varepsilon$.

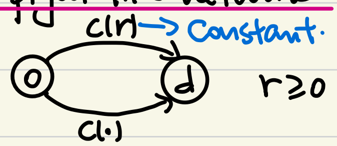
$PoA=\frac{1}{\varepsilon}\to\infty$ As $\varepsilon\to0, p\to\infty$. highly nonlinear cost fns are the only obstacle to a small PoA.

PoA is close to 1 for affine (ax+b) cost fns.

Tight PoA bounds for selfish routing.

assumptions: Cost fns are assumed to be nonnegative, continuous and nondecreasing. C.

informal statement: Among all networks with cost fns in a set C, the largest PoA is achieved in a pigou-like network.

Pigou-like network: 

$r\ge0$ * same cost fns at top and bottom.

two free parameters: C - set of permissible cost fns. r.

bottom path is strictly (except when x=r) dominant. $f=(0,1)$ $c(f)=r\cdot c(r)$

let $f=(r-x,x)$ $c(f)=(r-x)\cdot c(r)+x\cdot c(x)$ $c(f^*)=\inf_{0\le x\le r}\{(r-x)c(r)+xc(x)\}$

if $0\le x\le r$, $(r-x)c(r)+xc(x)=rc(r)+x[c(x)-c(r)]\le rc(r)$

if $x\ge r$, $(r-x)c(r)+xc(x)\ge rc(r)$ doesn't take inf.

$c(f^*)=\inf_{0\le x}\{(r-x)c(r)+xc(x)\}$. $\Rightarrow PoA=\sup_{x\ge0}\left\{\frac{rc(r)}{(r-x)c(r)+xc(x)}\right\}$.

Def: The Pigou bound $\alpha(C)$ is the largest PoA in a Pigou-like network in which the lower edge's cost fn belongs to C.

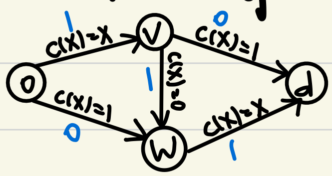
$$\alpha(C) = \sup_{c \in C} \sup_{r \geq 0} \sup_{x \geq 0} \left\{ \frac{r c(r)}{x c(x) + (r-x) c(r)} \right\}.$$

assume C contains all constant fns.

Thm: For every C of cost fns and every selfish routing networks with cost fns in C, the worst PoA is $\alpha(C)$.

constant

03/11/2022 Friday.

Example: 

$P_1: o \to v \to w \to d$ $\quad P_2: o \to v \to d$ $\quad P_3: o \to w \to d.$

let $f = (1, 0, 0)$

$C_{P_1}(f) = 1 + 0 + 1 = 2$ $\quad C_{P_2}(f) = \underset{c(1)=1}{1} + \underset{c(0)=1}{1} = 2$ $\quad C_{P_3}(f) = \underset{c(0)=1}{1} + \underset{c(1)=1.}{1} = 2$

$C(f) = \underline{2 \cdot 1 + 2 \cdot 0 + 2 \cdot 0 = 2.}$

f equilibrium? Yes, since no one in path 1 wants to make a unilateral deviation.

Since only $f_{P_1} > 0$, we must confirm that P_1 is the shortest path.

check $\underline{C_{P_1}(f) \le C_{P_2}(f)}$. $\quad \underline{C_{P_1}(f) \le C_{P_3}(f)}$

let $f = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$

$C_{P_1}(f) = \frac{3}{4} + 0 + \frac{3}{4} = \frac{3}{2}$ $\quad C_{P_2}(f) = \frac{3}{4} + 1 = \frac{7}{4}$ $\quad C_{P_3}(f) = 1 + \frac{3}{4} = \frac{7}{4}.$ $\quad C(f) = \frac{3}{2} \cdot \frac{1}{2} + \frac{7}{4} \cdot \frac{1}{4} + \frac{7}{4} \cdot \frac{1}{4} = \frac{13}{8}$

Not equilibrium flow since P_1 is the only shortest path and people on P_2, P_3 would want to unilaterally switch to P_1.

Exercise 11.1. Prove that if C is the set of cost fns of the form $\underline{c(x) = ax + b \text{ with } a, b \ge 0}$, then $\underline{\text{the Pigou bound } \alpha(C) \text{ is } \frac{4}{3}}$.

WLOG assume $r = 1$ since $\dfrac{\frac{1}{r^2}(r(ar+b))}{\frac{1}{r^2}(x(ax+b) + (r-x)(ar+b))} = \dfrac{1(a \cdot 1 + \frac{b}{r})}{\frac{x}{r}(a\frac{x}{r} + \frac{b}{r}) + (1 - \frac{x}{r})(a \cdot 1 + \frac{b}{r})}$

$b' = \frac{b}{r}, x' = \frac{x}{r}$ $\quad c'(x') = ax' + b'$

$= \dfrac{a + b'}{x'(ax' + b') + (1 - x')(a + b')} = \dfrac{1 \cdot c'(1)}{x' c'(x') + (1 - x') c'(1)}$

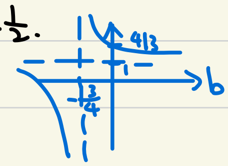
WLOG, we can assume $a = 1$.

Hence $\alpha(C) = \sup_{b \ge 0} \sup_{x \ge 0} \dfrac{1 + b}{x(x+b) + (1-x)(1+b)}$

Find x that minimizes $x(x+b) + (1-x)(1+b) = x^2 + bx + 1 + b - x - bx = x^2 - x + (1+b)$

$\frac{d}{dx}(x^2 - x + (1+b)) = 2x - 1 = 0 \quad x = \frac{1}{2}.$

$\alpha(C) = \sup_{b \ge 0} \dfrac{1+b}{\frac{3}{4} + b} = \frac{4}{3}$



Let $G = (E, V)$ be a selfish routing network, with r units of traffic traveling from o to d.

let $P = \{\text{path from } o \text{ to } d \text{ of } G\}$.

Def: A flow is a nonnegative vector, over paths in G, $\{f_p\}_{p \in P}$, with $\sum_{p \in P} f_p = r$. $\quad (f_{P_1}, f_{P_2}, \cdots)$ how traffic is split over.

Def: For an edge $e \in E$ and flow f, $\underline{f_e = \sum_{p \in P: e \in p} f_p}$ (the amount of traffic on edge e).

e.g. $f_{(o,v)}$ $\quad$ *All paths.

the cost of a flow
$\shortparallel$

Def: let $C(f)$ be the total travel time incurred by traffic in a flow f. $C(f) = \sum_{e \in E} f_e c_e(f_e)$

Ex. $C(f) = \frac{3}{4} \times \frac{3}{4} + \frac{1}{4} \times 1 \times 2 + \frac{1}{2} \times 0 + \frac{3}{4} \times \frac{3}{4} = \frac{13}{8}$.

Def: $C_p(f) = \sum_{e \in E : e \in p} C_e(f_e)$ cost of flow on a path per unit of traffic. $C(f) = \sum_{p \in P} f_p \cdot C_p(f)$

Ex. $C(f) = \frac{1}{2} \cdot \left(\frac{3}{4} + 0 + \frac{3}{4}\right) + \frac{1}{4} \cdot \left(\frac{3}{4} + 1\right) + \frac{1}{4} \cdot \left(1 + \frac{3}{4}\right) = \frac{13}{8}$.

03/14/2022 Monday.

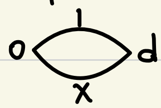
$C(f) = \sum_{p\in P} f_p C_p(f) = \sum_{p\in P} f_p \sum_{e\in E: e\in p} C_e(f_e) = \sum_{p\in P}\sum_{e\in E: e\in p} f_p C_e(f_e) = \sum_{e\in E} f_e C_e(f_e)$

Similarly, $\sum_{p\in P} f_p^* C_p(f) = \sum_{e\in E} f_e^* \cdot C_e(f_e)$.

Def: A flow, f, is an Equilibrium flow, if no unilateral switch by an individual results in a faster commute.

a positive component of flow f.
if $f_{\tilde{p}} > 0$, then $C_{\tilde{p}}(f) \le C_p(f)$ for any $p \in P$.

Equilibrium flow V.S. Optimal flow.

Example: pigou Network. 

$f = (0,1)$ $f^* = (\frac{1}{2}, \frac{1}{2})$

$C(f) = 0\cdot 1 + 1\cdot 1 = 1$ $C(f^*) = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\frac{1}{2} = \frac{3}{4}$.

optimal flow obtains the shortest average travel time possible. (maximize social welfare at the sacrafice of some individuals.)

Equilibrium flow makes everyone happy and everyone has the same travel time.

Prove ♡: Fix a selfish routing network, $G(V,E)$, with cost fns $c \in C$ and r unit of traffic.

let f be the equilibrium flow, f^* be the minimum cost flow (optimal flow)

Want to show $\frac{C(f)}{C(f^*)} \le \alpha(C)$ $\Rightarrow$ $f_e^* C_e(f_e^*) \ge \frac{1}{\alpha(C)} f_e C_e(f_e) + (f_e^* - f_e) C_e(f_e)$

STEP 1: Show $\sum_{e\in E} (f_e^* - f_e) C_e(f_e) \ge 0$

As f is a equilibrium flow, $f_{\tilde{p}} > 0 \Rightarrow C_{\tilde{p}}(f) \le C_p(f)$ for any $p \in P$.

if $f_p = 0$.
i.e. all path p in f have a common cost $C_{\tilde{p}}(f) = L$. $C_p(f) \ge L$. ($C_p(f) = L$ if $f_p > 0$.)

thus $C(f) = \sum_{p\in P} f_p \cdot C_p(f) = r\cdot L$.

f_p not necessarily >0.
$C(f) = \sum_{p\in P} f_p^* \cdot C_p(f) \ge r\cdot L$ since $C_p(f) \ge L$ when $f_p^* > 0$.

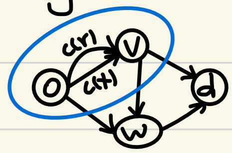
so we have $C(f) = \sum_{p\in P} (f_p^* - f_p) C_p(f) = \sum_{e\in E} (f_e^* - f_e) C_e(f_e) \ge 0$

As f routes all traffic on the shortest path, no other flow can be better with the given fixed cost.

03/16/2022 Wednesday.

All equilibrium flows (can be more than one) have the same cost, so POA is well-defined.

STEP 2: for every edge between two vertices in the network make a pigou like network.



run f_e on the top edge, f_e^* on the bottom edge.

$c(r) \to C_e(f_e)$ $\quad c(x) \to C_e(f_e^*)$

$$\alpha(C) = \sup_{c \in C} \sup_{r \geq 0} \sup_{x \geq 0} \left\{ \frac{r \cdot c(r)}{x \cdot c(x) + (r-x)c(r)} \right\}.$$ Instantiating C_e for c, f_e for r, f_e^* for x.

$$\alpha(C) \geq \frac{f_e C_e(f_e)}{f_e^* C_e(f_e^*) + (f_e - f_e^*) C_e(f_e)}$$

$$f_e^* C_e(f_e^*) + (f_e - f_e^*) C_e(f_e) \geq \frac{1}{\alpha(C)} f_e C_e(f_e)$$

$$f_e^* C_e(f_e^*) \geq \frac{1}{\alpha(C)} f_e C_e(f_e) + (f_e^* - f_e) C_e(f_e)$$

Summing over all edges, $C(f^*) \geq \frac{1}{\alpha(C)} C(f) + \underbrace{\sum_{e \in E} (f_e^* - f_e) C_e(f_e)}_{\geq 0}$

so $C(f^*) \geq \frac{1}{\alpha(C)} C(f)$, $POA = \frac{C(f)}{C(f^*)} \leq \alpha(C)$

markov = exponential

Modeling an M/M/1 Queue.

single server.

Example: An airport runway for arrivals only.

Arriving aircraft join a single queue for the runway.

Service time $\sim Exp(\mu = 27$ arrivals / hour)

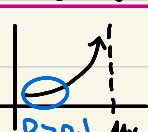
Poisson arrivals with a rate $\lambda = 20$ arrivals / hour.

$\Rightarrow$ Wait time $\sim Exp(27 - 20 = 7)$ average wait time $= \frac{1}{7}$ hrs.

(communication networks) the more you over provision the closer to optimal the network will function without management. (POA $\to$ 1)

POA bonds for over provision networks.

$$C_e(x) = \begin{cases} \frac{1}{\mu_e - x} & \text{if } x < \mu_e \\ \infty & \text{if } x \geq \mu_e. \end{cases}$$



nonnegative, continuous, nondecreasing cost fns.

(~ low degree poly x^d).

For network with $\beta > 0.1$, the cost fn isn't very steep

Application: A querying system consists of customers arriving at random times to some facility where they receive service and they depart

arrive at rate x, helping at rate $\mu_e \Rightarrow$ service time $\sim Exp(\mu_e - x)$ Avg $= \frac{1}{\mu_e - x}$

Def: For a parameter $\beta \in (0,1)$, call a selfish routing network with cost fns of the form $C_e(x) = \begin{cases} \frac{1}{\mu_e - x} & \text{if } x < \mu_e. \\ \infty & \text{if } x \geq \mu_e. \end{cases}$ β-overprovisioned if $f_e \leq (1-\beta)\mu_e$ for every edge e,

f_e is at most $(1-\beta) \times 100\%$ capacity.

where f is some equilibrium flow.

The worst case PoA in β-overprovisioned networks is $\frac{1}{2}(1+\sqrt{\frac{1}{\beta}})$.

overprovision.

prove in a pigou-like network with traffic rate r and cost fn $\frac{1}{\mu - x}$ ($\mu > r$) on the lower edge, the PoA $= \frac{1}{2}(1+\sqrt{\frac{1}{\beta}})$, where $\beta = 1 - \frac{r}{u}$:

worst PoA

need to verify concave down.

let $c(x) = \frac{1}{\mu - x}$, fix r and $\mu > r$, find x s.t. $\left(\frac{rc(r)}{xc(x)+(r-x)c(r)}\right)' = 0 \rightsquigarrow x = \mu - \sqrt{\mu^2 - r\mu}$

$$\alpha(c) = \frac{r/\mu - r}{\frac{\mu - \sqrt{\mu^2 - r\mu}}{\sqrt{\mu^2 - r\mu}} + \frac{r - \mu + \sqrt{\mu^2 - r\mu}}{\mu - r}} = \frac{r}{u-r} \cdot \frac{u^2 - ru}{2u(\sqrt{u^2 - ur} - \mu^2 + r\mu)} = \frac{r}{2(\sqrt{u^2 - ur} - \mu^2 + r\mu)} = 2\left(1 + \sqrt{\frac{1}{1 - \frac{r}{u}}}\right)$$

$\beta = 0.1$. PoA ≈ 2.1 which is sufficient for near optimal selfish routing.

Adding capacity VS smart management of the network...

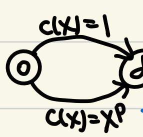
In m/m/1 routing network, double capacity with equilibrium routing $\geq$ optimal routing in original network.

Thm 12.1 (resource augmentation bound).

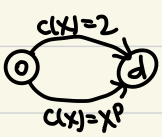
For every selfish routing network and $r > 0$, the cost of an equilibrium flow with traffic rate r is at most the cost of an optimal flow with traffic rate $2r$.

C nonnegative, continuous, nondecreasing.

Example:



p large



$f = (0,1)$ $\quad f^* = (1,1)$ $\quad C(f^*) = 1 \cdot 2 + 1 \cdot 1^p = 3$.

$C(f) = 0 \cdot 1 + 1 \cdot 1^p = 1$. $\quad f^* = (1+\varepsilon, 1-\varepsilon)$ $\quad C(f^*) = (1+\varepsilon) \cdot 2 + (1-\varepsilon)(1-\varepsilon)^p \approx 2$.

(equilibrium in a faster network is better than optimal in original network.)

03/18/2022 Friday.

Proof: f-equilibrium flow with r f^*- optimal flow with $2r$.

Want $\sum_e f_e C_e(f_e) \le \sum_e f_e^* C_e(f_e^*)$

Let L = Common cost of all shortest path. $\sum_P f_P C_P(f_e) = rL$. $\sum_P f_P^* C_P(f_e) \ge 2rL$.

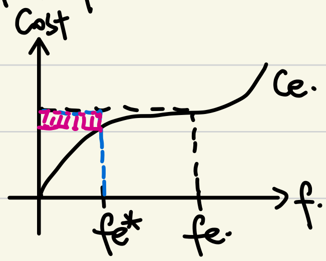
Want to prove $\sum_e f_e^* C_e(f_e^*) \ge \underbrace{\sum_e f_e^* C_e(f_e)}_{\ge 2rL} - \underbrace{\sum_e f_e C_e(f_e)}_{=rL} \ge rL = \sum_e f_e C_e(f_e)$

Prove by edge: $f_e^*[C_e(f_e) - C_e(f_e^*)] \le f_e C_e(f_e)$

Suppose $f_e^* \ge f_e$. $C_e(f_e^*) \ge C_e(f_e)$ and $f_e^* \ge 0$, hence $f_e^*[C_e(f_e) - C_e(f_e^*)] \le 0$.

$f_e C_e(f_e) \ge 0$ ✓.

Suppose $f_e^* < f_e$.



$f_e^*[C_e(f_e) - C_e(f_e^*)] \le f_e C_e(f_e)$

By Exercise 12.3. for M/M/1 queue, the total cost of equilibrium flow of network with double capacity $\le \cdot\cdot f^* \cdot\cdot$ 1x capacity.

Atomic Selfish Routing (no longer have negligible size)

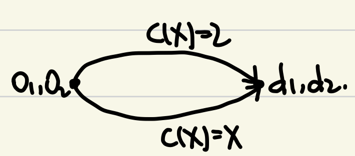
- $G(V,E)$ is a directed graph with nonnegative, nondecreasing, continuous edge cost fns.
- k players, each with own origin o_i, destination d_i.
- each player is one unit of traffic.
- $\mathcal{P}_i$ denotes the set of o_i-d_i paths of G.
- A flow can be represented as a vector $(\overset{\in \mathcal{P}_1}{P_1}, \overset{\in \mathcal{P}_2}{P_2})$

→ no agent can decrease her cost by unilateral deviation.

Def: A flow $(P_1, \cdots, P_k)$ is an equilibrium if, for every agent i and path $\hat{P}_i \in \mathcal{P}_i$,

$$\sum_{e\in P_i} C_e(f_e) \le \sum_{\underset{\text{*both}}{e\in \hat{P}_i \cap P_i}} C_e(f_e) + \sum_{e\in \hat{P}_i \setminus P_i} C_e(f_e+1)$$

Example:



PoA $= \frac{4}{3}$.

$\{$top, bottom$\}$ equilibrium? Yes. $c(1)=3$

if o_1 switches to bottom. $c(x)=2$ unchanged.

if o_2 switches to top. $c(x)=2$ increase.

$c(f) = 4.$

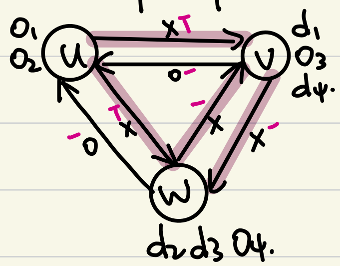
{bottom, bottom} is also an equilibrium.

$\Rightarrow$ we can have equilibrium flow with different costs $c(f)$. Not in nonatomic.

Def: $\text{PoA} = \dfrac{\text{cost of worst equilibrium flow}}{\text{cost of optimum flow}}$.

With affine cost fns, PoA can be as large as $5/2$.

Example:



each agent has two options (one hop path, two hop path).

if everyone chooses one hop path, $c(f) = 1+1+1+1 = 4$.

if everyone chooses two hop path, $(2+1)+(2+1)+(2+0)+(2+0) = 10$.

$\forall$ agent, one hop path is the optimal.

Verify one-hop is equilibrium:

agent	one hop	two hop (deviation.)
1	1	4
2	1	4
3	1	2
4	1	2

Verify two-hop is equilibrium:

agent	two hop	one hop (deviation.)
1	3	3
2	3	3
3	2	2
4	2	2

these are the only two equilibrium.

$\text{PoA} = \frac{10}{4} = \frac{5}{2}$ Worst PoA - affine atomic with any number of agents.

03/28/2022 Monday.

Thm 12.3. (PoA bound for atomic selfish routing)

For all atomic selfish routing networks with affine cost function, PoA $\le 5/2$.

Proof: let f be an equilibrium flow, f^* be OPT flow, $C_e(x) = a_e x + b_e$.

STEP 1: Want to get an upper bound on equilibrium cost for each agent.

let P_i be the path of agent i in f. P_i^* be the path of agent i in f^*.

suppose i deviates from P_i to P_i^*.

$$\sum_{e\in P_i} C_e(f_e) \le \sum_{e\in P_i\cap P_i^*} C_e(f_e) + \sum_{e\in P_i^*\setminus P_i} C_e(f_e+1)$$

STEP 2: Sum over i to get upper bound on $C(f)$.

$$C(f) = \sum_{e\in E} f_e C_e(f_e) = \sum_{i=1}^{k}\sum_{e\in P_i} C_e(f_e) \le \sum_{i=1}^{k}\Big(\sum_{e\in P_i\cap P_i^*} C_e(f_e) + \sum_{e\in P_i^*\setminus P_i} C_e(f_e+1)\Big)$$

$$\le \sum_{i=1}^{k}\sum_{e\in P_i^*} C_e(f_e+1) = \sum_{e\in E} f_e^* C_e(f_e+1)$$

$$= \sum_{e\in E} f_e^*(a_e(f_e+1)+b_e),\quad a_e, b_e \ge 0.$$

STEP 3: Lemma: $\forall y, z \in \{0,1,2,\cdots\}$, $y(z+1) \le \frac{5}{3}y^2 + \frac{1}{3}z^2$

$y = f_e^*$, $z = f_e$.

$$C(f) = \sum_{e\in E} f_e^*(a_e(f_e+1)+b_e) = \sum_{e\in E}(a_e f_e^*(f_e+1) + b_e f_e^*) \le \sum_{e\in E}\Big(a_e\big(\tfrac{5}{3}(f_e^*)^2 + \tfrac{1}{3}f_e^2\big) + b_e f_e^*\Big)$$

$$= \sum_{e\in E}\Big(\tfrac{5}{3}a_e(f_e^*)^2 + b_e f_e^*\Big) + \sum_{e\in E} a_e \tfrac{1}{3} f_e^2.$$

$$C(f^*) = \sum_{e\in E} f_e^* C_e(f_e^*) = \sum_{e\in E} f_e^*(a_e f_e^* + b_e) = \sum_{e\in E}(a_e(f_e^*)^2 + b_e f_e^*)$$

$$C(f) = \sum_{e\in E}(a_e(f_e)^2 + b_e f_e).$$

$$\text{so } C(f) \le \sum_{e\in E}\Big(\tfrac{5}{3}a_e(f_e^*)^2 + \tfrac{5}{3}b_e f_e^*\Big) + \sum_{e\in E} a_e \tfrac{1}{3} f_e^2 + \sum_{e\in E} b_e \tfrac{1}{3} f_e.$$

$$= \tfrac{5}{3}C(f^*) + \tfrac{1}{3}C(f).$$

STEP 4: $\frac{2}{3}C(f) \le \frac{5}{3}C(f^*) \Rightarrow \text{PoA} = \frac{C(f)}{C(f^*)} \le \frac{5}{2}$

03/30/2022 Wednesday.

Def: A strategy profile s of a cost minimization game is a PNE (→ Pure Nash Equilibrium.) if for every agent $i \in \{1,2,\cdots,k\}$ and every unilateral deviation $s_i' \in S_i$ $C_i(s) \le C_i(s_i', s_{-i})$

s_i is the best response to s_{-i}. s_i minimizes $C_i(s_i', s_{-i})$ for $s_i' \in S_i$.

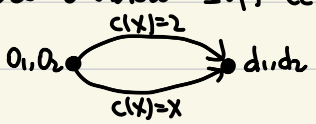
ex. games w/ PNE: atomic selfish routing.

games w/o PNE: rock paper scissors.

Def: A potential function for an atomic selfish routing game satisfies

$\Phi(\hat{f}) - \Phi(f) = C_i(\hat{f}) - C_i(f)$ for agent i where $\hat{f}$ is a unilateral deviation of agent i from f.

Example: Consider $\Phi(f) = \sum_{e\in E}\sum_{i=1}^{f_e} c_e(i)$



$f_1 = (top, top)$ $\quad \Phi(f_1) = 2+2=4.$

$f_2 = (top, bot)$ $\quad \Phi(f_2) = 2+1=3.$ $\qquad f_2, f_3, f_4$ are minimums of Φ $\Downarrow$ PNE.

$f_3 = (bot, top)$ $\quad \Phi(f_3) = 1+2=3.$

$f_4 = (bot, bot)$ $\quad \Phi(f_4) = 1+2=3.$

let $f = (top, bot)$ $\quad \hat{f} = (bot, bot)$ (→ deviate.) $\quad \Phi(\hat{f}) - \Phi(f) = 3-3 = C_1(\hat{f}) - C_1(f) = 2-2 = 0 \ge 0$

Def: A potential game is one for which there exists a potential function Φ s.t.

$\Phi(s_i', s_{-i}) - \Phi(s_i, s_{-i}) = C_i(s_i', s_{-i}) - C_i(s_i, s_{-i})$

in the context of atomic, $\Phi(\hat{f}) - \Phi(f) = C_i(\hat{f}) - C_i(f)$

finite potential games always have a minimum $\Phi \Rightarrow$ PNE. (minimum cost $\Rightarrow$ equilibrium).

Example: show $\Phi = \sum_{e\in E}\sum_{i=1}^{f_e} c_e(i)$ is a potential function.

want to show $\Phi(\hat{f}) - \Phi(f) = \sum_{e\in \hat{P}_i} c_e(\hat{f}_e) - \sum_{e\in P_i} c_e(f_e)$

switch from P_i to $\hat{P}_i$: $\hat{f}_e = f_e$ if $e \in \hat{P}_i \cap P_i$.

($\hat{f}_e$ - f_e after switch) $\quad \hat{f}_e = f_e + 1$ if $e \in \hat{P}_i \setminus P_i$.

$\hat{f}_e = f_e - 1$ if $e \in P_i \setminus \hat{P}_i$.

$$\Phi(\hat{f}) - \Phi(f) = \sum_{e\in E}\left(\sum_{i=1}^{\hat{f}_e} c_e(i) - \sum_{i=1}^{f_e} c_e(i)\right)$$

$$= \sum_{e\in \hat{P}_i \setminus P_i} c_e(f_e+1) + \sum_{e\in \hat{P}_i \cap P_i} c_e(f_e) - \sum_{e\in P_i\setminus \hat{P}_i} c_e(f_e) - \sum_{e\in \hat{P}_i \cap P_i} c_e(f_e)$$

$$= \sum_{e\in \hat{P}_i \setminus P_i} c_e(f_e+1) - \sum_{e\in P_i \setminus \hat{P}_i} c_e(f_e) = \sum_{e\in \hat{P}_i} c_e(\hat{f}_e) - \sum_{e\in P_i} c_e(f_e)$$

So atomic is a potential game.

finite number of different flow $\Rightarrow$ minimum $\Rightarrow$ equilibrium.

Thm 13.6. (Existence of PNE in atomic routing games.)

Every atomic selfish routing networks has at least one equilibrium flow.

Thm: PNE of nonatomic routing exists and is unique. $\rightarrow$ any two PNE have the same cost.

$$\Phi(f) = \sum_{e \in E} \int_{x=0}^{f_e} c_e(x)\,dx.$$

set of flow

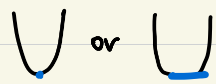
P is a compact subspace of $\mathbb{R}^n$ where $n = |P|$ $\Rightarrow$ $\Phi(f)$ is a conti. fn in a compact set.

By Extreme Value Thm, $\Phi(f)$ has a local minimum $\Rightarrow$ PNE exists.

$$\Phi(f) = \sum_{e \in E} \int_{x=0}^{f_e} c_e(x)\,dx$$ integral of nondecreasing conti. fn is convex.

$\downarrow$

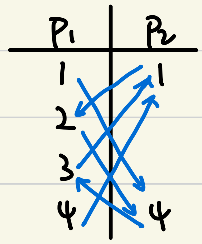
Sum of convex fns $\rightarrow$ convex. $\rightarrow$ all min is global min.



unique f many f with same c(f).

04/01/2022 Friday.

Example:



(If player 2 takes path P_1 or P_2, the unique response by player 1 that minimizes its cost is P_4.) $C_1(P_4,P_1) \le C_1(P_3,P_1), C_1(P_2,P_1), C_1(P_1,P_1)$

PNE exists? No, no mutual best response. $\downarrow\uparrow$

Since PNE not always exist, we want to enlarge the set of equilibria so that an equilibrium is guaranteed.

Generalize selfish routing networks to cost minimization / payoff maximization games.

C_i - payoff fn; all inequality reversed.

- A finite number k of agents.
- A finite set S_i of pure strategies for each agent i.
- A nonnegative cost function $C_i(s_i, s_{-i})$ for each agent i.

Def: A mixed strategy σ_i, is a randomization of pure strategies $s_i \in S_i$.

e.g. $f=(p_1,p_2,\cdots,p_n)$ of a nonatomic routing with $r=1$ $\to$ infinite person game where everyone plays the same fixed strategy f.

e.g. coin toss $\to$ one person with $\sigma = \text{Bernoulli}(\frac{1}{2})$ $\quad E_{s\sim\sigma}(C(s)) = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot(-1) = 0.$

e.g. two people $S_1=S_2=\{H,T\}$. $\to \sigma=(\sigma_1,\sigma_2) = \text{Bernoulli}(\frac{1}{2}) \times \text{Bernoulli}(\frac{1}{2}) = ((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2}))$
(matching pennies game)

$\underbrace{E_{s\sim\sigma}(C_1(s))}_{= C_1(\sigma)} = \frac{1}{4}(C_1(H,H)+C_1(H,T)+C_1(T,H)+C_1(T,T)) = 0.$

$s=(s_i,s_{-i})$

Def: "mixed" Distributions $\sigma_1,\cdots,\sigma_k$ over strategy sets $S_1,\cdots,S_k$ of a cost minimization game constitute a Mixed Nash Equilibrium (MNE) if for every agent i and every unilateral deviation $s_i' \in S_i$, $E_{s\sim\sigma}(C_i(s)) \le E_{s\sim\sigma}(C_i(s_i',s_{-i}))$ (i.e. $C_i(\sigma) \le C_i(s_i',\sigma_{-i})$)

to pure strategies. $\uparrow$ must be product distribution.

Example: Show $\sigma=((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2}))$ is a MNE of the matching pennis game.

$C_1(\sigma)=0.$

$C_1(H,\sigma_2) = \frac{1}{2}C_1(H,H) + \frac{1}{2}C_2(H,T) = 0 \ge 0.$

$C_1(T,\sigma_2) = \frac{1}{2}C_1(T,H) + \frac{1}{2}C_2(T,T) = 0 \ge 0$

Exercise 13.1. prove MNE $= E_{s\sim\sigma}[C_i(s)] \le E_{\underset{\text{before } s_i' \in S_i}{s_i' \sim \sigma_i'},\, s_{-i}\sim\sigma_{-i}}[C_i(s_i', s_{-i})]$

for the matching pennis game. let σ be the MNE. $p+q=1$.

$C_1(\sigma) \le C_1(H, \sigma_2) \Longleftrightarrow pC_1(\sigma) \le pC_1(H, \sigma_2)$

$C_1(\sigma) \le C_1(T, \sigma_2) \Longleftrightarrow qC_1(\sigma) \le qC_1(T, \sigma_2)$

$+ \Longrightarrow C_1(\sigma) \le C_1(\underbrace{pH+qT}_{\sigma_1'}, \sigma_2)$

more generally, let σ be a MNE and player i have n pure strategies $s_1, \cdots, s_n$.

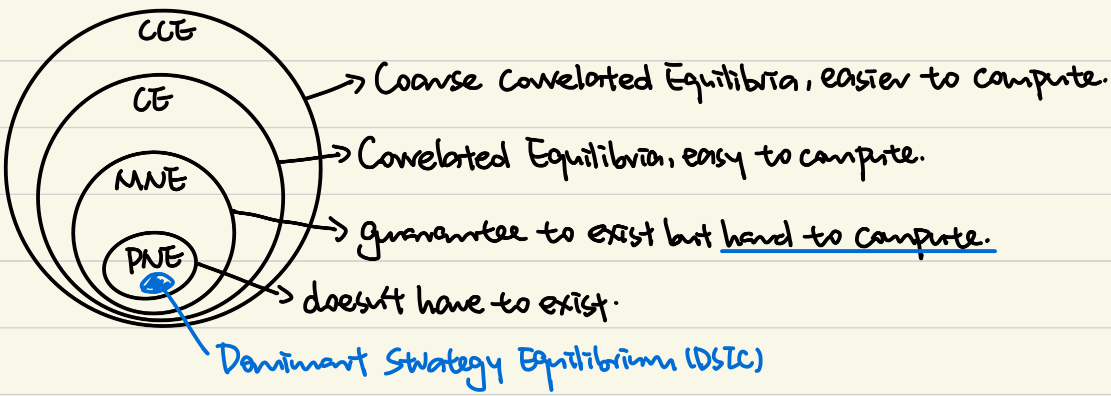
$C_i(\sigma) \le C_i(s', \sigma_{-i})$ let $\sum_{i=1}^{n} a_i = 1$.

$a_1 C_i(\sigma) \le a_1 C_i(s_1, \sigma_{-i})$

$\vdots$

$a_n C_i(\sigma) \le a_n C_i(s_n, \sigma_{-i})$

$\Longrightarrow C_i(\sigma) \le C_i(\sum_{i=1}^{n} a_i s_i, \sigma_{-i})$



Nash's theorem: Every cost-minimization game has at least one MNE.

$$\text{PoA} = \frac{\text{expected cost (fn) of worst MNE}}{\text{minimal cost (fn)}}$$

prop: if $\sigma = \sigma_i \times \sigma_{-i}$ is a MNE and both s_i and s_i' are in the support of σ_i (i.e. $\sigma_i(s_i) > 0$ and $\sigma_i(s_i') > 0$), then $C_i(s_i, \sigma_{-i}) = C_i(s_i', \sigma_{-i})$.

Example: Matching Pennis Game.

Suppose $\sigma_2 = (q, 1-q)$. Assume $\sigma_1 = (\overset{H}{p}, \overset{T}{1-p})$ with $0 < p < 1$. $\Longrightarrow C_1(H, \sigma_2) = C_1(T, \sigma_2)$

$$\begin{cases} C_1(H, \sigma_2) = qC_1(H,H) + (1-q)C_1(H,T) = 2q-1 \\ C_1(T, \sigma_2) = qC_1(T,H) + (1-q)C_1(T,T) = -2q+1 \end{cases} \Longrightarrow q = \tfrac{1}{2}$$

Similarly $p = \frac{1}{2} \Longrightarrow ((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$ is an MNE.

Proof: Suppose $C_i(s_i, \sigma_{-i}) < C_i(s_i', \sigma_{-i})$, then player i would have a strictly lower expected cost by not using s_i' in its mixed strategy and won't use s_i'. This contradicts the assumption that s_i' is in the support of σ_i.

04/04/2022 Monday.

Example:

		q stop	$1-q$ go
p	stop	1,1	1,0
$1-p$	go	0,1	5,5

(cost minimization).

PNE: (stop, go), (go, stop).

MNE: $C_1(\text{stop}, \sigma_2) = C_1(\text{go}, \sigma_2)$

$$\begin{cases} qC_1(\text{stop},\text{stop}) + (1-q)C_1(\text{stop},\text{go}) = 1 \\ qC_1(\text{go},\text{stop}) + (1-q)C_1(\text{go},\text{go}) = 5-5q \end{cases} \Rightarrow q = \frac{4}{5}$$

Similarly $p = \frac{4}{5}$, $((\frac{4}{5},\frac{1}{5}),(\frac{4}{5},\frac{1}{5}))$ is a MNE.

Example: three firms are considering entering a new market. payoff $= \frac{150}{n}$, cost $= 62$.

PNE: no firm in the market → one firm enters → another firm enters → no more deviations

$\{(E,E,D),(E,D,E),(D,E,E)\}$.

MNE: $\sigma_1 = \sigma_2 = \sigma_3 = p(1-p)$ $0<p<1$.

$$\begin{cases} C_1(E,\sigma_2,\sigma_3) = p^2C_1(E,E,E) + p(1-p)C_1(E,E,D) + p(1-p)C_1(E,D,E) + (1-p)^2C_1(E,D,D) = 25p^2 - 75p + 44 \\ C_1(D,\sigma_2,\sigma_3) = 0 \end{cases}$$

so $p = \frac{4}{5}$, $\{(\frac{4}{5},\frac{1}{5}),(\frac{4}{5},\frac{1}{5}),(\frac{4}{5},\frac{1}{5})\}$.

Correlated Equilibrium (CE)

Example:

	stop	go
stop	1,1	1,0 $\frac{1}{2}$
go	0,1 $\frac{1}{2}$	5,5

the traffic light is a trusted third party that samples an outcome s according to σ (publically known) and privately shares s_i with player i.

$\sigma = \underset{(p_{11},p_{12},p_{21},p_{22})}{(0,\frac{1}{2},\frac{1}{2},0)}$ is a CE.

		p_1 stop	p_2 go
p_1	stop	$p_{11}=p_1p_1$	$p_{12}=p_1p_2$
p_2	go	$p_{21}=p_2p_1$	$p_{22}=p_2p_2$

if MNE, σ is the product distribution of outcomes.

Def: A distribution σ on the set $S_1 \times \cdots \times S_k$ of outcomes of a cost minimization game is a CE if for every agent $i \in \{1,2,\cdots,k\}$, strategy $s_i \in S_i$ and deviation $s_i' \in S_i$

$$E_{s\sim\sigma}[C_i(s) \mid \underset{\text{fixed}}{s_i}] \le E_{s\sim\sigma}[C_i(\underset{\text{fixed}}{s_i'}, s_{-i}) \mid \underset{\text{fixed}}{s_i}]$$

$$\sum_{a\in S_i}\sum_{s_{-i}} \underbrace{p(a,s_{-i}\mid s_i)}_{=0 \text{ unless } a=s_i} C_i(a,s_{-i}) \le \sum_{a\in S_i}\sum_{s_{-i}} p(a,s_{-i}\mid s_i)C_i(s_i',s_{-i})$$

$$\sum_{s_{-i}} p(s_i,s_{-i})C_i(s_i,s_{-i}) \le \sum_{s_{-i}} p(s_i,s_{-i})C_i(s_i',s_{-i}) \qquad p(s_i,s_{-i}\mid s_i) = \frac{p(s_i,s_{-i})}{p(s_i)}$$

$$\sum_{s_{-i}} [C_i(s_i,s_{-i}) - C_i(s_i',s_{-i})]p(s_i,s_{-i}) \le 0.$$

04/06/2022 Wednesday.

Example: Show the PNE (go, stop) $\xrightarrow{} (0,0,1,0)$ is CE.

① we assume the other player does what the traffic light tells him.
The traffic light always shows GO to player 1 and STOP to player 2.
Player 1 has no incentive not to follow traffic light.
Similarly player 2 has no incentive to GO knowing player 1 will Go.

② $\sigma = (go, stop)$ let $s_1' = stop$.

$$E_{s\sim\sigma}(C_1(s)\mid s_1=go) = \underbrace{P(s_1=go, s_2=stop \mid s_1=go)}_{=P(s_2=stop)=1}\, \underbrace{C_1(go,stop)}_{=0} = 0.$$

$$\leq$$

$$E_{s\sim\sigma}(C_1(s_1',s_2)\mid s_1=go) = \underbrace{P(s_1=go, s_2=stop \mid s_1=go)}_{=1}\, \underbrace{C_1(stop,stop)}_{=1} = 1.$$

Also $\underbrace{P(s_1=go, s_2=stop \mid s_2=stop)}_{=1}\underbrace{C_2(go,stop)}_{=1} \leq \underbrace{P(s_1=go,s_2=stop \mid s_2=stop)}_{=1}\underbrace{C_2(go,go)}_{=5}$

③ Alternatively, since only $P(go,stop) \neq 0$,
check $(C_1(go,stop) - C_1(stop,stop))P(go,stop) \leq 0$. $(C_2(go,stop) - C_2(go,go))P(go,stop) \leq 0$.

Cost at PNEs: $C(stop,go) = (1,0)$ $C(go,stop) = (0,1)$

Cost at MNE $((\frac{4}{5},\frac{1}{5}),(\frac{4}{5},\frac{1}{5}))$: $C = (1,1) > C(PNE)$

Cost at CE $(0,p,q,0)$ with $p+q=1$: $C=(p,q)$. $< C(MNE)$
both better off.

	stop	go
stop	1,1	1,0
go	0,1	5,5

Consider $\sigma = (\frac{1}{3},\frac{1}{3},\frac{1}{3},0)$.
Show that this is a CE.

① if light tells play 1 to go, she knows player 2 was told to stop. ⇒ no incentive to deviate.
if light tells player 1 to stop, she knows (stop, go) and (stop, stop) are equally likely. crucial to give info privately.
if she follows, cost $= \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 1 = 1$. if she obeys, cost $= \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 5 = \frac{5}{2}$ ⇒ no incentive to deviate.
Similarly for player 2.

② player 1: $s_1 = s,\ s_1' = g$, $(C_1(s,s) - C_1(g,s))P(s,s) + (C_1(s,g) - C_1(g,g))P(s,g) < 0$. $(1-0)\frac{1}{3} + (1-5)\cdot\frac{1}{3} < 0$

$s_1 = g,\ s_1' = s$, $(C_1(g,s) - C_1(s,s))P(g,s) + (C_1(g,g) - C_1(s,g))P(g,g) < 0$. $(0-1)\frac{1}{3} + (5-1)\cdot 0 < 0$

player 2: $s_2 = s,\ s_2' = g$, $(C_2(s,s) - C_2(s,g))P(s,s) + (C_2(g,s) - C_2(g,g))P(g,s) < 0$. $(1-0)\frac{1}{3} + (1-5)\frac{1}{3} < 0$.

$s_2 = g,\ s_2' = s$, $(C_2(s,g) - C_2(s,s))P(s,g) + (C_2(g,g) - C_2(g,s))P(g,g) < 0$. $(0-1)\frac{1}{3} + (5-1)0 < 0$.

convex combination.

prop: Given any two CE, you can combine them to get another.

e.g. $a\sigma_1 + (1-a)\sigma_2$ is a CE for $0 \le a \le 1$.

proof: $\sum_{s_{-i}} [C_i(s_i, s_{-i}) - C_i(s_i', s_{-i})]\, p(s_i, s_{-i}) \le 0$

$\sum_{s_{-i}} [C_i(s_i, s_{-i}) - C_i(s_i', s_{-i})]\, \hat{p}(s_i, s_{-i}) \le 0.$

$\Rightarrow \sum_{s_{-i}} [C_i(s_i, s_{-i}) - C_i(s_i', s_{-i})]\big(a p(s_i, s_{-i}) + (1-a)\hat{p}(s_i, s_{-i})\big) \le 0.$

$C(\text{combined}) = aC(\sigma_1) + (1-a)C(\sigma_2)$. $\Rightarrow$ we can get any cost profile between $C(\sigma_1)$ and $C(\sigma_2)$

Find all CE $(p_{11}, p_{12}, p_{21}, p_{22})$ in the traffic light game.

$(1-0)p_{11} + (1-5)p_{12} \le 0.$ All linear constraints $\Rightarrow$ can be solved efficiently by linear programming.

$(0-1)p_{21} + (5-1)p_{22} \le 0.$ $p_{11}, p_{12}, p_{21}, p_{22} \ge 0$, $p_{11} + p_{12} + p_{21} + p_{22} = 0.$

$(1-0)p_{11} + (1-5)p_{21} \le 0.$

$(0-1)p_{12} + (5-1)p_{22} \le 0.$

PNE and MNE are always corners of CE polytope. 3D #CE $\Rightarrow$ #PNE, #MNE.

04/08/2022 Friday.

Example:

	A	B	C
A	(-1,-1)	(1,1)	(0,0)
B	(1,1)	(-1,-1)	(0,0)
C	(0,0)	(0,0)	(1.1,1.1)

Find a CE that is Not MNE.

$\sigma = (\overset{AA}{\tfrac{1}{2}}, 0, 0, 0, \overset{BB}{\tfrac{1}{2}}, 0, 0, 0, 0)$

MNE is a case of CE.

Prop: if σ is a product distribution, CE iff MNE.

σ is not a product distribution since if so, player 1 and player 2 must have posterior probability on A, B. $P(AB) \neq 0$, $P(BA) \neq 0$.

$P(A|B) = P(AB)/P(B) = P(B|A) \times P(A)/P(B)$

(Why CE?) ① if the traffic light tells player 1 A, it must tell player 2 A. player 1 has no incentive to make a unilateral deviation. if tells player 1 B, similarly.

② $S_i = A$ $S_i' = C$. $[C_1(A,A) - C_1(C,A)] \cdot \frac{1}{2} < 0$, etc.

In CE we can achieve equilibria that can never result from PNE since players can communicate via a third party.

PNE are examples of CE where players' actions are drawn from an independent distribution, and hence conditioning on S_i provides no additional info about S_{-i}.

doesn't have to be a product distribution.

Def: A distribution σ on the set $S_1 \times \cdots \times S_k$ of outcomes of a cost minimization game is a Coarse Correlated Equilibria (CCE) if for every agent $i \in \{1, 2, \cdots, k\}$ and every unilateral deviation $S_i' \in S_i$, $E_{s \sim \sigma}(C_i(s)) \le E_{s \sim \sigma}(C_i(S_i', S_{-i}))$.

OR $\sum_s P(s) C_i(s) \le \sum_s P(s) C_i(S_i', S_{-i})$ OR $\sum_s [C_i(s) - C_i(S_i', S_{-i})] P(s) \le 0$.

$\sum_{S_i} \sum_{S_{-i}} [C_i(S_i, S_{-i}) - C_i(S_i', S_{-i})] P(S_i, S_{-i}) \le 0$.

$\le 0 \Rightarrow$ condition for CE.

no regret learning algorithms of repeat games.

Example of CCE but Not CE: $\sigma = (\frac{1}{3}, 0, 0, 0, \frac{1}{3}, 0, 0, 0, \frac{1}{3})$.

(Why CCE?) ① $C_1 = \frac{1}{3}(-1) + \frac{1}{3}(-1) + \frac{1}{3}(1.1) = -0.3$.

deviate to A: $C_1 = \frac{1}{3}(-1) + \frac{1}{3}(1) + \frac{1}{3}(0) = 0 > -0.3$.

deviate to B: $C_1 = \frac{1}{3}(1) + \frac{1}{3}(-1) + \frac{1}{3}(0) = 0 > -0.3$.

deviate to C: $C_1 = \frac{1}{3}(0) + \frac{1}{3}(0) + \frac{1}{3}(1.1) = \frac{11}{30} > -0.3$.

So no unilateral deviation.

② $s_i' = A$:

$$i=1\left\{\begin{aligned}&(c_1(B,A)-c_1(A,A))\,p(\cancel{B,A}) + (c_1(B,B)-c_1(A,B))\,p(B,B) + (c_1(B,C)-c_1(A,C))\,p(\cancel{B,C})\\ &+ (c_1(C,A)-c_1(A,A))\,p(\cancel{C,A}) + (c_1(C,B)-c_1(A,B))\,p(\cancel{C,B}) + (c_1(C,C)-c_1(A,C))\,p(C,C)\end{aligned}\right.$$

$$i=2\left\{\begin{aligned}&+ (c_2(A,B)-c_2(A,A))\,p(\cancel{A,B}) + (c_2(B,B)-c_2(B,A))\,p(B,B) + (c_2(C,B)-c_2(C,A))\,p(\cancel{C,B}).\\ &+ (c_2(A,C)-c_2(A,A))\,p(\cancel{A,C}) + (c_2(B,C)-c_2(B,A))\,p(\cancel{B,C}) + (c_2(C,C)-c_2(C,A))\,p(C,C).\end{aligned}\right.$$

$$(-1-1)\tfrac{1}{3} + (1.1-0)\tfrac{1}{3} + (-1-1)\tfrac{1}{3} + (1.1-0)\tfrac{1}{3} \le 0.$$

etc.

04/11/2022 Monday.

Example: Show $\sigma=(\frac{1}{6},\frac{1}{6},0,0,\frac{1}{3},0,0,0,\frac{1}{3})$ is Not CCE.

let $\sigma_1=(\frac{1}{3},\frac{1}{3},\frac{1}{3})$ be the marginal distribution of σ for player 1. let $s_1'=B$.

$P(A_2|A_1)=\frac{1}{2}$ $P(B_2|A_1)=\frac{1}{2}$.

If traffic light says A $(p=\frac{1}{3})$, avg cost$(A)=\frac{1}{2}(-1)+\frac{1}{2}(1)=0$ avg cost$(B)=\frac{1}{2}(1)+\frac{1}{2}(-1)=0$.

If traffic light says B $(p=\frac{1}{3})$, avg cost$(B)=1(-1)=-1$ avg cost$(B)=1(-1)=-1$.

If traffic light says C $(p=\frac{1}{3})$, avg cost$(C)=1(1.1)=1.1$ avg cost$(B)=1(0)=0$.

change in avg cost of unilateral switch to $B=\frac{1}{3}(0-0)+\frac{1}{3}(-1-(-1))+\frac{1}{3}(0-1.1)=-\frac{11}{30}<0$

has the incentive to switch $\Rightarrow$ Not CCE.

Since $E_{s\sim\sigma}(C_i(s))=E_{s_i\in S_i}(E_{s\sim\sigma}(C_i(s)|s_i])$ $E(Y)=E(E(X|Y))$

for all $s_i'\in S_i$, $E_{s\sim\sigma}(C_i(s))\le E_{s\sim\sigma}(C_i(s_i',s_{-i}))$.

$\Leftrightarrow$ let σ_i be the marginal distribution of σ for agent i. for all $s_i'\in S_i$,

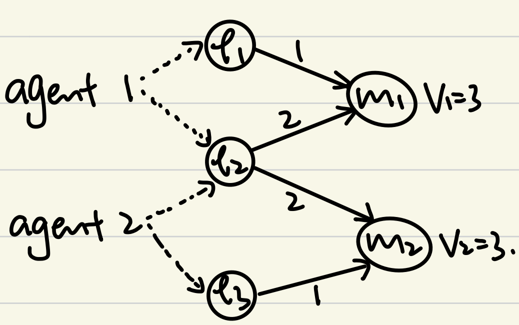
$E_{s_i\sim\sigma_i}(E_{s\sim\sigma}[C_i(s)|s_i])\le E_{s_i\sim\sigma_i}(E_{s\sim\sigma}C_i(s_i',s_{-i})|s_i))$.

σ is a CCE iff when the light gives agent i instructions, agent i is best (on average) to listen.

A Location game.

a payoff maximization game $\Leftrightarrow$ cost minimization game (atomic selfish routing.).

- A set L of locations. locations of a power plant.
- A set of k agents. Each chooses one location $L_i\in L$. electricity companies.
- A set M of markets. Each market $j\in M$ has a value V_j (publically known.) cities.
- For each location $\ell\in L$ and market $j\in M$, there is a fixed cost $C_{\ell j}$ of serving j from ℓ.



Agent i's payoff $=\pi_i(s)=\sum_{j\in M}\pi_{ij}(s)$

agent 1 location. agent 2 location.

e.g. $\pi_{11}(\ell_1,\ell_3)=2$ $\pi_{12}(\ell_1,\ell_3)=0 \Rightarrow \pi_1(\ell_1,\ell_3)=2$.

agent 1 market 1

$\pi_{12}(\ell_2,\ell_3)=0$ $\pi_{22}(\ell_2,\ell_3)=2-1=1$.

both competing M_2, agent 2 will lower price to agent 1's cost to prevent it from entering the market.

second lowest.

Is strategy profile (ℓ_1, ℓ_3) a PNE?

$\pi_1(\ell_1, \ell_3) = 2$ $\pi_2(\ell_1, \ell_3) = 2$.

$1 * \ell_3$
$2 * \ell_1$

$\pi_1(\ell_2, \ell_3) = 1+0 = 1$ $\pi_2(\ell_1, \ell_2) = 0+1 = 1$ $\Rightarrow$ PNE.

Another PNE? No.

$* \pi_1(\ell_2, \ell_2) = 0$ $* \pi_2(\ell_2, \ell_2) = 0$

04/18/2022 Monday.

let S be the location profile.

The payoff of agent i serving market j with valuation V_j, from location ℓ, with cost $C_{\ell j}$ is $\underset{\geq 0}{\Pi_{ij}(S)} = \begin{cases} 0 & \text{if agent } i \text{ isn't the closest competitor to market } j, \text{ or if } C_{\ell j} \geq V_j. \\ V_j - C_{\ell j} & \text{if market } j \text{ has no competition.} \\ d_j^{(2)}(S) - C_{\ell j} & \text{if market } j \text{ has competition and he is the closest.} \end{cases}$

where $d_j^{(2)}(S) = \min(V_j$, second smallest cost among those competing for market j).

$\Pi_i = \sum_{j \in M} \Pi_{ij}$.

$\downarrow$ markets.

(objective fn)

Social welfare $\underset{\geq 0}{W(S)} = \sum_{j \in M} (V_j - d_j(S))$. where $d_j(S) = \min(V_j$, smallest cost)

A strategy profile, S, has a high social welfare if markets with the highest valuations are efficiently served.

well defined for $T \subseteq L$, i.e. $W(T) = \sum_{j \in M} (V_j - d_j(T))$ where $d_j(T) = \min(V_j$, smallest cost among T).

Thm 14.1 The PoA bound for location game is at least 1/2.

$$\text{PoA} = \frac{\text{worst objective of equilibrium}}{\text{objective of optimal}}$$

Property 1: $\forall S, \sum_{i=1}^{k} \Pi_i(S) \leq W(S)$. ($k \to$ #agents)

proof: $d_j^{(2)}(S) - d_j(S) \leq V_j - d_j(S)$

Property 2: $\forall S, \Pi_i(S) = W(S) - W(S_{-i})$ payoff = extra welfare she brings. ($W(S_{-i}) \to$ welfare without agent i.)

proof: if agent i joins and he might be the closet one.

$$[V_j - d_j^{new}] - [V_j - d_j^{old}] = d_j^{old} - d_j^{new} = d_j^{(2)} - d_j.$$

04/20/2022 Wednesday.

Corollary: every location game is a potential game, and hence has at least one PNE.

proof: show $w(s)$ is a potential fn. $W(s) - W(s_i', s_{-i}) = \pi_i(s) - \pi_i(s_i', s_{-i})$

by P2, $\pi_i(s) = W(s) - W(s_{-i})$ $\quad \pi_i(s_i', s_{-i}) = W(s_i', s_{-i}) - W(s_{-i})$

Lemma: if $T_1 \subseteq T_2 \subseteq L$, $W(T_1) \le W(T_2)$. $W(s)$ is monotone.

proof: adding a location introduces a potentially closest location to the market or makes a new market available.

Def: for all locations and the sets $T_1 \subseteq T_2 \subseteq L$, $W(T_2 \cup \ell) - W(T_2) \le W(T_1 \cup \ell) - W(T_1)$.
$W(s)$ is submodular.

(diminishing returns in increased welfare adding a location to a bigger set.).

Proof: your payoff is greater when you enter a less competitive market.

Proof of Thm 14.1:

$S = $ PNE, $S^* = $ OPT. $\pi_i(s) \ge \pi_i(s_i^*, s_{-i})$. $\Rightarrow \sum_{i=1}^{k} \pi_i(s) \ge \sum_{i=1}^{k} \pi_i(s_i^*, s_{-i})$. $\Rightarrow W(s) \overset{P_1}{\ge} \sum_{i=1}^{k} \pi_i(s_i^*, s_{-i})$.

claim: $\underbrace{\sum_{i=1}^{k} \pi_i(s_i^*, s_{-i}) \ge W(s^*) - W(s)}_{\text{disentangle.}}$

$\sum_{i=1}^{k} \pi_i(s_i^*, s_{-i}) \overset{P2}{=} \sum_{i=1}^{k} (W(\underbrace{s_i^*, s_{-i}}_{T_2}) - W(\underbrace{s_{-i}}_{T_1})) \overset{\text{submodular}}{\ge} \sum_{i=1}^{k} [W(\underbrace{s_1^*, \dots, s_{i-1}^*}_{\ell}, s_i^*, s_{-i}) - W(\underbrace{s_1^*, \dots, s_{i-1}^*}_{\ell}, s_{-i})]$

$\overset{\text{submodular}}{\ge} \sum_{i=1}^{k} [W(s_1^*, \dots, s_{i-1}^*, s_i^*, \underbrace{s_i}_{\ell}, s_{-i}) - W(s_1^*, \dots, s_{i-1}^*, \underbrace{s_i}_{\ell}, s_{-i})]$.

$= \sum_{i=1}^{k} [\underbrace{W(s_1^*, \dots, s_i^*, s)}_{A_i} - \underbrace{W(s_1^*, \dots, s_{i-1}^*, s)}_{A_{i-1}}] = W(s_1^*, \dots, s_k^*, s) - W(s) = W(s^*, s) - W(s)$.

telescoping sum, $\sum_{i=1}^{k} (A_i - A_{i-1}) = (A_1 - A_0) + (A_2 - A_1) + \dots + (A_k - A_{k-1}) = A_k - A_0$.

$\overset{\text{monotone}}{\ge} W(s^*) - W(s)$

So $W(s) \ge W(s^*) - W(s) \Rightarrow \text{PoA} = \frac{W(s)}{W(s^*)} \ge \frac{1}{2}$.

Def: A cost-minimization game is (λ, μ)-smooth if $\sum_{i=1}^{k} C_i(s_i^*, s_{-i}) \le \lambda \operatorname{cost}(s^*) + \mu \operatorname{cost}(s)$ for all strategy profiles s, s^*. Here $\operatorname{cost}(s)$ is an objective fn s.t. $\operatorname{cost}(s) \le \sum_{i=1}^{k} C_i(s)$ for all s.

payoff-maximization. ($\ge$, $-$, $\ge$)

games that allow 3rd disentanglement are called smooth games and are easy to get PoA bound.

e.g. $W(s) \ge 1 W(s^*) - 1 W(s) \Rightarrow (1,1)$ smooth game. $\quad W(s) \ge \sum_{i=1}^{k} \pi_i(s)$.

selfish routing games. $C(f) \ge \frac{5}{3} C(f^*) + \frac{1}{3} C(f) \Rightarrow (\frac{5}{3}, \frac{1}{3})$ smooth game.

$C(f) = \sum_{i=1}^{k} \sum_{e \in P_i} C_e(f_e) = \sum_{i=1}^{k} C_i(f)$

$PNE \subseteq MNE \subseteq CE \subseteq CCE$. $PoA = \dfrac{\text{worst objective of equilibria } (PNE, ME, CE, CCE)}{\text{OPT objective}}$.

payoff-maximization

for cost-minimization games: $PoA(PNE) \overset{\geq}{\leq} PoA(ME) \overset{\geq}{\leq} PoA(CE) \overset{\geq}{\leq} PoA(CCE)$.

Smooth games, same upper bond for min: $\frac{5}{2} \geq PoA(PNE) \Rightarrow \frac{5}{2} \geq PoA(CCE)$.

same lower bond for max: $\frac{1}{2} \leq PoA(PNE) \Rightarrow \frac{1}{2} \leq PoA(CCE)$.

04/22/2022 Friday.

*small λ, μ not too close to 1.

Proposition: In a (λ,μ) smooth cost minimization game with $\mu<1$, $PoA(PNE) \le \frac{\lambda}{1-\mu}$.

Proof: $cost(s) \le \sum_{i=1}^{k} C_i(s) \overset{PNE}{\le} \sum_{i=1}^{k} C_i(s_i^*, s_{-i}) \le \lambda cost(s^*) + \mu cost(s)$. $\Rightarrow PoA(PNE) = \frac{cost(s)}{cost(s^*)} \le \frac{\lambda}{1-\mu}$.

Thm: In a (λ,μ) smooth cost minimization game with $\mu<1$, $PoA(CCE) \le \frac{\lambda}{1-\mu}$.

Proof: let σ be a CCE of (λ,μ) smooth game.

$$\mathbb{E}_{s\sim\sigma}[cost(s)] \le \mathbb{E}_{s\sim\sigma}\left[\sum_{i=1}^{k} C_i(s)\right] = \sum_{i=1}^{k} \mathbb{E}_{s\sim\sigma}[C_i(s)] \overset{CCE}{\le} \sum_{i=1}^{k} \mathbb{E}_{s\sim\sigma}[C_i(s_i^*, s_{-i})] = \mathbb{E}_{s\sim\sigma}\left[\sum_{i=1}^{k} C_i(s_i^*, s_{-i})\right]$$

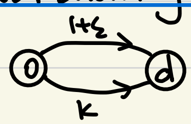
$$\le \mathbb{E}_{s\sim\sigma}[\lambda cost(s^*) + \mu cost(s)] = \lambda \underbrace{cost(s^*)}_{\text{constant}} + \mu \mathbb{E}_{s\sim\sigma}[cost(s)]. \Rightarrow PoA(CCE) = \frac{\mathbb{E}_{s\sim\sigma}[cost(s)]}{cost(s^*)} \le \frac{\lambda}{1-\mu}.$$

In CCE, no particular strategy profile is Nash, rather a distribution of outcome profiles is Nash on average.

the PoA bound for a (λ,μ) smooth game is Robust.

Thm: In a (λ,μ) smooth payoff maximization game, $PoA(PNE) \ge \frac{\lambda}{1+\mu}$, $PoA(CCE) \ge \frac{\lambda}{1+\mu}$.

Cost sharing game.



$PNE = \{(top, top, \cdots, top), (bot, bot, \cdots, bot)\}$.

$OPT = (top, top, \cdots, top)$.

$PoA = \frac{\text{worst PNE}}{OPT} = \frac{k}{1+\varepsilon} \approx k$ bad, can grow linearly with #agents.

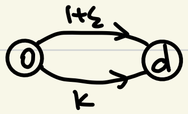
PoA is misleading.

possible solutions:

① use Price of Stability (PoS) instead, which involves best case PNE.

② limit PNE to a subset of PNE called Strong Nash (no coalition of agents has a beneficial deviation).

04/25/2022 Monday.



Externality = difference in objective fn you impose on others

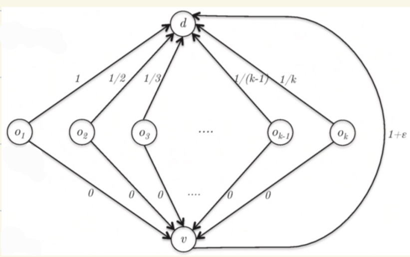
$= (k-1)\left(\frac{1+\varepsilon}{k-1}\right) - (k-1)\frac{1+\varepsilon}{k} = \frac{1+\varepsilon}{k} \geq 0.$

for a cost-minimization game, positive externality ⟹ people are happy to have you.

A network cost sharing game has graph (V,E), directed or undirected, and each edge $e \in E$ carries a fixed cost $\gamma_e > 0$. k agents with origin $\{o_i\}_{i=1}^{k}$ and destinations $\{d_i\}_{i=1}^{k}$.

$P=(P_1,\cdots,P_k)$. $C_i(P)=\sum_{e\in P_i}\frac{\gamma_e}{f_e}$. fe↑, Ce↓. Not nondecreasing ⟹ Not atomic selfish routing game. negative externality.

Objective fn: $Cost(P)=\sum_{i=1}^{k} C_i(P)$.



Example: k agents have option for meeting at pick up point for ride share. Each agent can opt out with cost $= \frac{1}{i}$.

OPT $= 1+\varepsilon$.

PNE $= \sum_{i=1}^{k}\frac{1}{i}$ (agent k has a dominant strategy to opt out. ⟹ All opt out.)

PoA $= \sum_{i=1}^{k}\frac{1}{i} / 1+\varepsilon \approx H_k$. kth harmonic number. $H_k = \sum_{i=1}^{k}\frac{1}{i} \approx \ln k + \gamma$ ⟹ Euler Constant ≈ 0.57. for large k.

Thm: In every network cost sharing game with k agents, ∃ PNE with cost ≤ $H_k \cdot$ OPT.

⇕

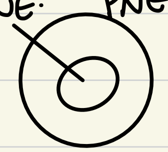
PoS ≤ H_k. SNE: since the person with smallest cost in the coalition will always be worse off by deviating.

It implies there always exists a PNE.

04/27/2022 Wednesday.

Def: let s be an outcome of a cost-minimization game. Strategies $s'_A \in \prod_{i \in A} S_i$ (≥ 2.) are a beneficial deviation for a subset A of agents if $C_i(s'_A, s_{-A}) \leq C_i(s)$ for every agent $i \in A$, with the inequality holding strictly for at least one agent of A.

The outcome s is a strong Nash Equilibrium if there is no coalition of agents with a beneficial deviation.



Def: Price of Stability (PoS) $= \dfrac{\text{cost of best equilibria}}{\text{cost of optimal outcome}}$. (PoS bound is much weaker than PoA bound).

prove ⭐: by potential function $\Phi(f) = \sum_{e \in E} \sum_{i=1}^{f_e} c_e(i)$ which applies to any (decreasing) c_e,

$$\Phi(p) = \sum_{e \in E} \sum_{i=1}^{f_e} \frac{\gamma_e}{i} = \sum_{e \in E} \gamma_e \sum_{i=1}^{f_e} \frac{1}{i} \leq \sum_{e \in E} \gamma_e \sum_{i=1}^{k} \frac{1}{i} = \sum_{i=1}^{k} C_i(p) H_k = Cost(p) H_k.$$

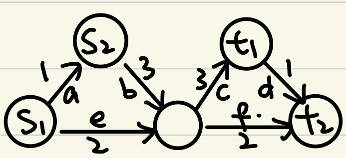
$$Cost(p) = \sum_{e \in E: f_e \geq 1} \gamma_e \leq \sum_{e \in E: f_e \geq 1} \gamma_e \sum_{i=1}^{f_e} \frac{1}{i} = \Phi(p).$$

So $\frac{\Phi(p)}{H_k} \leq Cost(p) \leq \Phi(p)$. $\Phi(p)$ is a good estimate of $Cost(p)$.

let p be the PNE minimizing Φ, $Cost(p) \leq \Phi(p) \leq \Phi(p^*) \leq H_k Cost(p^*)$.

Example:

(No SNE).



only PNE: (ec, bf). both cost = 5. Not SNE (deviate to OPT).

OPT: (abc, bcd). both cost = 4. Not PNE. (-> 3, 5).

04/29/2022 Friday.

Thm: In every network cost sharing game with k agents, every strong Nash equilibrium has cost at most H_k times that of an optimal outcome.

* SNE Not always exist.

Proof: Fix a cost sharing game and SNE p.

by SNE, $\exists i$ s.t. $C_i(p) \le C_i(p^*)$ otherwise, all agents can deviate to p^*. Reorder $\Rightarrow C_k(p) \le C_k(p^*)$.

by SNE, $\exists i \in \{1,2,\cdots,k-1\}$ s.t. $C_i(p) \le C_i(p_1^*,\cdots,p_{k-1}^*,p_k)$. Reorder $\Rightarrow C_{k-1}(p) \le C_{k-1}(p_1^*,\cdots,p_{k-1}^*,p_k)$.

$\cdots\cdots$ we get $C_1(p) \le C_1(p_1^*,p_2,\cdots,p_k)$. Every agent's cost has an entangled upper bound.

by positive externality, $C_i(p_1^*,\cdots,p_i^*,p_{i+1},\cdots,p_k) \le C_i(p_1^*,p_2^*,\cdots,p_i^*)$.

$\mathrm{Cost}(p) = \sum_{i=1}^{k} C_i(p) \le \sum_{i=1}^{k} C_i(p_1^*,p_2^*,\cdots,p_i^*,p_{i+1},\cdots,p_k) \le \sum_{i=1}^{k} C_i(p_1^*,p_2^*,\cdots,p_i^*)$.

$\Phi(p_1^*,p_2^*,\cdots,p_{i-1}^*) = \sum_{e\in E} \gamma_e \sum_{j=1}^{f_e \le i-1} \frac{1}{j}$ when we add agent i:

for $e \notin p_i^*$. $\Phi(p_1^*,\cdots,p_i^*) - \Phi(p_1^*,\cdots,p_{i-1}^*) = 0$.

for $e \in p_i^*$, $\Phi(p_1^*,\cdots,p_i^*) - \Phi(p_1^*,\cdots,p_{i-1}^*) = \sum_{e\in p_i^*}\left(\gamma_e \sum_{j=1}^{f_e+1}\frac{1}{j} - \gamma_e\sum_{j=1}^{f_e}\frac{1}{j}\right) = \sum_{e\in p_i^*}\left(\frac{\gamma_e}{f_e+1}\right) = f_e^i$.

$C_i(p) = \sum_{e\in p_i}\frac{\gamma_e}{f_e}$ $\quad C_i(p_1^*,p_2^*,\cdots,p_i^*) = \sum_{e\in p_i^*}\frac{\gamma_e}{f_e^i} = \Phi(p_1^*,\cdots,p_i^*) - \Phi(p_1^*,\cdots,p_{i-1}^*)$.

$\mathrm{Cost}(p) \le \sum_{i=1}^{k} C_i(p_1^*,\cdots,p_i^*) = \sum_{i=1}^{k}[\Phi(p_1^*,\cdots,p_i^*) - \Phi(p_1^*,\cdots,p_{i-1}^*)] \overset{\text{telescoping}}{=} \Phi(p_1^*,\cdots,p_k^*) - \Phi(\phi) = \Phi(p^*) \le H_k \mathrm{Cost}(p^*)$.