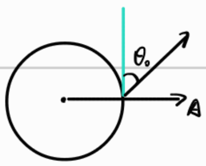
§1.



比奈公式有:

$$E=\frac{mc^2}{2}\left(\left(\frac{du}{d\varphi}\right)^2+u^2\right)+V\left(\frac{1}{u}\right)$$

其中 $c=\frac{\dot{\varphi}}{u^2}$ 为守恒量.

引力场中 $V\left(\frac{1}{u}\right)=-GMmu.$

即有 $$\frac{du}{d\varphi}=\sqrt{\frac{2}{mc^2}\left(E+GMmu\right)-u^2}$$

$$d\varphi=\frac{du}{\sqrt{\frac{2}{mc^2}\left(E+GMmu\right)-u^2}}$$

由初始条件. 有

$$\varphi-\varphi_0=\int_{\frac{1}{R}}^{u}\frac{du}{\sqrt{-\left(u-\frac{GM}{c^2}\right)^2+\left(\frac{GM}{c^2}\right)^2+\frac{2E}{mc^2}}}$$

设 $A^2=\frac{G^2M^2}{C^4}+\frac{2E}{mc^2}$ $\quad B=\frac{GM}{C^2}$

即有 $$\varphi=\int_{\frac{1}{R}}^{u}\frac{du}{\sqrt{A^2-(u-B)^2}}=\int_{\frac{1}{R}-B}^{u-B}\frac{ds}{\sqrt{A^2-s^2}}$$

$$=\arcsin\frac{s}{A}\Big|_{\frac{1}{R}-B}^{u-B}=\arcsin\frac{u-B}{A}-\varphi_0$$

$$\varphi_0=\arcsin\left(\tfrac{1}{R}-B\right)/A$$

$$\cos\left(\varphi+\varphi_0-\frac{\pi}{2}\right)=\frac{u-B}{A}. \qquad u=A\cos\left(\varphi+\varphi_0-\frac{\pi}{2}\right)+B$$

轨道方程 $$r=\frac{1}{B+A\cos\left(\varphi+\varphi_0-\frac{\pi}{2}\right)}$$

$$c=R\cdot\frac{1}{\sqrt{2}}V=R\cdot\frac{1}{\sqrt{2}}\sqrt{\frac{GM}{4R}}=\sqrt{\frac{GMR}{8}}$$

$$E=\frac{1}{2}mV^2-\frac{GMm}{R}=\frac{GMm}{8R}-\frac{GMm}{R}=-\frac{7GMm}{8R}$$

$$B=\frac{GM}{C^2}=\frac{8}{R}$$

$A=\frac{5\sqrt{2}}{R}$ $\quad$ 因此有 $$r=\frac{1}{\frac{8}{R}+\frac{5\sqrt{2}}{R}\cos\left(\varphi+\varphi_0-\frac{\pi}{2}\right)}$$

整理. $$r=\frac{R/8}{1+\frac{5\sqrt{2}}{8}\cos\left(\varphi+\varphi_0-\frac{\pi}{2}\right)}$$

其中 $\varphi_0=-\arcsin\frac{7}{5\sqrt{2}}$ 化为直角即

$$r=\frac{R}{8-\sin\varphi-7\cos\varphi}$$

§2. 飞离地球后. 设 $V_\perp$ 和 $V_\parallel$. 有 $\frac{1}{2}mV_\perp^2=\frac{1}{2}mV_3^2-\frac{GMem}{Re}$ $\quad V_\perp=\sqrt{V_3^2-\frac{2GMe}{Re}}$

$$V_\parallel=\sqrt{\frac{GMs}{Rs}}$$

$$V_3=\sqrt{(3-2\sqrt{2})\frac{GMs}{Rs}+\frac{2GMe}{Re}}\Rightarrow V_\perp=\sqrt{(3-2\sqrt{2})\frac{GMs}{Rs}}$$

$$E=\frac{1}{2}m(4-2\sqrt{2})\frac{MsG}{Rs}-\frac{GMsm}{Rs}=(1-\sqrt{2})\frac{GMsm}{Rs}$$

$L=mRsV_\parallel$ $\qquad C=V_\parallel Rs=\sqrt{GMsRs}.$

$$B=\frac{GMs}{C^2}=\frac{1}{Rs}\qquad A=\sqrt{B^2+\frac{2E}{mC^2}}=\sqrt{\frac{1}{Rs^2}+\frac{(2-2\sqrt{2})m\frac{GMs}{Rs}}{mGMsRs}}=\sqrt{\frac{1}{Rs^2}+(2-2\sqrt{2})\frac{1}{Rs^2}}$$

$$=\frac{1}{Rs}\sqrt{3-2\sqrt{2}}=\frac{(\sqrt{2}-1)}{Rs}$$

得出轨道方程为 $$r=\frac{1}{\frac{1}{Rs}\left(1+(\sqrt{2}-1)\cos(\varphi\pm\varphi_0)\right)}.$$

$$=\frac{Rs}{1+(\sqrt{2}-1)\cos(\varphi\pm\varphi_0)}\qquad\varphi_0=\frac{\pi}{2}$$

即 $$r=\frac{Rs}{1+(\sqrt{2}-1)\sin\varphi}$$

§3.

设飞出地球系后，相对地球切向速度 $V_{\parallel}$，径向速度 $V_\perp$，故有

$$C=(V_{\parallel}+V_1)R. \quad E=\frac12 m\left(V_\perp^2+(V_\parallel+V_1)^2\right)-mV_1^2$$
$$=\frac12 m\left(V_\perp^2+V_\parallel^2+2V_\parallel V_1-V_1^2\right)$$

$$B=\frac{GM}{C^2}=\frac{V_1^2R}{(V_\parallel+V_1)^2R^2}=\frac1R\left(\frac{V_1}{V_1+V_\parallel}\right)^2$$

$$A^2=B^2+\frac{2E}{mC^2}=\frac{1}{R^2}\left(\frac{V_1}{V_1+V_\parallel}\right)^4+\frac{m(V_\perp^2+V_\parallel^2+2V_\parallel V_1-V_1^2)}{m(V_\parallel+V_1)^2R^2}$$

$$A=\frac{1}{R(V_1+V_\parallel)}\sqrt{V_\perp^2+V_\parallel^2+2V_\parallel V_1-V_1^2+\frac{V_1^4}{(V_1+V_\parallel)^2}}$$

近日点 $r_{min}=\frac{1}{B+A}$

$$=\frac{R}{\frac{V_1^2}{(V_\parallel+V_1)^2}+\frac{1}{(V_1+V_\parallel)^2}\sqrt{(V_1+V_\parallel)^2(V_\perp^2+V_\parallel^2+2V_\parallel V_1-V_1^2)+V_1^4}}$$

设 $V_\parallel=V\cos\theta$，$V_\perp=V\sin\theta$

$$r_{min}=\frac{R(V_1+V\cos\theta)^2}{V_1^2+\sqrt{(V_1+V\cos\theta)^2(V^2+2V_1V\cos\theta-V_1^2)+V_1^4}}$$

设 $\frac{V}{V_1}=y$，则有

$$r_{min}=\frac{R(1+y\cos\theta)^2}{1+\sqrt{(1+y\cos\theta)^2(y^2+2y\cos\theta-1)+1}}=0.7R.$$

要使 y 最小，即 $\theta=\pi$ 此时，沿切向向后发射。同此，轨道形状确定。

发射器轨道能量为 $E_0=-\frac{GM_Sm}{1.7R}$

脱离地球引力后，发射器动能为（太阳系下）

$$\frac12 mV_S^2=E_0+\frac{GM_Sm}{R}\Rightarrow V_S^2=\frac{2E_0}{m}+\frac{2GM_S}{R}$$

地系下，脱离后的速度可以最小为 $V_{min}=V_D-V_0$

则即，地面上的发射速度满足

$$\frac12 mV_f^2-\frac{GM_Em}{r}=\frac12 mV_{min}^2$$

$$V_f=\sqrt{V_{min}^2+\frac{2GM_E}{r}}=\sqrt{\left(\sqrt{\frac{GM_S}{R}}-\sqrt{\frac{2E_0}{m}+\frac{2GM_S}{R}}\right)^2+\frac{2GM_E}{r}}$$

$$=\sqrt{\left(\frac{GM_S}{R}+\left(\frac{2GM_S}{R}-\frac{20}{17}\frac{GM_S}{R}\right)-2\sqrt{2-\frac{20}{17}}\frac{GM}{R}\right)+\frac{2GM_E}{r}}$$

$$=\sqrt{\frac{GM_S}{R}\left(1+2-\frac{20}{17}-2\sqrt{2-\frac{20}{17}}\right)+\frac{2GM_E}{r}}$$

$$V_f=\sqrt{\frac{2GM_E}{r}+\frac{GM_S}{R}\left(\frac{31}{17}-2\sqrt{\frac{14}{17}}\right)}$$ 为最小发射速度

代入数据得为 $V_f=11.512\ \text{km/s}$

§4. 自由度为3. θ. φ. x.

$$T=\frac12 m\dot x^2+\frac12 mx^2\dot\theta^2+\frac12 M(l-x)^2\dot\varphi^2+\frac12 M\dot x^2$$

广义动量守恒：$\begin{cases}\dot\varphi=0\\ x^2\dot\theta=C_0\end{cases}\Rightarrow \frac{d}{dt}=\frac{d}{d\theta}\frac{d\theta}{dt}=\frac{d}{d\theta}\cdot\frac{C_0}{x^2}$

能量守恒，$\frac12 m\dot x^2+\frac12 M\dot x^2+\frac12 mx^2\dot\theta^2=C_1$

得 $\frac12(M+m)\left(\frac{dx}{d\theta}\frac{C_0}{x^2}\right)^2+\frac12 mx^2\left(\frac{C_0}{x^2}\right)^2=C_1$

$$\frac12(M+m)\frac{C_0^2}{x^4}\left(\frac{dx}{d\theta}\right)^2=C_1-\frac12 m\frac{C_0^2}{x^2}\Leftrightarrow\left(\frac{dx}{d\theta}\right)^2=\frac{2x^4}{C_0^2(m+M)}\left(C_1-\frac12 m\frac{C_0^2}{x^2}\right)$$

$$\frac{dx}{d\theta}=\frac{x^2}{C_0}\sqrt{\frac{2}{m+M}\left(C_1-\frac12\frac{mC_0^2}{x^2}\right)}\Leftrightarrow-\frac{d\frac1x}{\sqrt{\frac{2}{m+M}\left(C_1-\frac12\frac{mC_0^2}{x^2}\right)}}=\frac{d\theta}{C_0}$$

$$-\frac{d\frac{1}{x}}{\sqrt{\frac{2}{m+M}\left(C_1-\frac{1}{2}\frac{mC_0^2}{x^2}\right)}}=\frac{d\theta}{C_0}\quad 即\quad \frac{d\frac{1}{x}}{\sqrt{1-\frac{1}{2}\frac{mC_0^2}{C_1x^2}}}=-\sqrt{\frac{2C_1}{m+M}}\frac{d\theta}{C_0}$$

设 $u=\frac{1}{x}$. 即 $\frac{du}{\sqrt{1-Au^2}}=-k\,d\theta \qquad A=\frac{1}{2}\frac{mC_0^2}{C_1}\qquad k=\sqrt{\frac{2C_1}{m+M}}\frac{1}{C_0}$

设 $\sqrt{A}u=\cos\varphi$. $\frac{\frac{1}{\sqrt{A}}d\cos\varphi}{\sin\varphi}=\pm k\,d\theta$

$d\theta=\pm\frac{1}{k\sqrt{A}}d\varphi \qquad \theta+\theta_0=\frac{1}{\pm k\sqrt{A}}\arccos\sqrt{A}u$

$\sqrt{A}u=\cos[k\sqrt{A}(\theta+\theta_0)]. \qquad r=\frac{\sqrt{A}}{\cos[k\sqrt{A}(\theta+\theta_0)]}$

即 $r=\sqrt{\frac{1}{2}\frac{mC_0^2}{C_1}}\sec\left[\frac{1}{C_0}\sqrt{\frac{mC_0^2}{2C_1}}\sqrt{\frac{2C_1}{m+M}}(\theta+\theta_0)\right]$

$C_0=r_0v_0 \quad C_1=\frac{1}{2}mv_0^2$

$$r=\sqrt{\frac{\frac{1}{2}mv_0^2r_0^2}{\frac{1}{2}mv_0^2}}\sec\left[\sqrt{\frac{m}{m+M}}(\theta+\theta_0)\right]$$

$=r_0\sec\left[\sqrt{\frac{m}{m+M}}(\theta+\theta_0)\right]$ 代入初始条件有 $\theta_0=0$.

即 $r=r_0\sec\left(\sqrt{\frac{m}{m+M}}\theta\right)$.

85. $E_{eff}=V(r)+\frac{J^2}{2mr^2}\qquad m\ddot{r}=-\frac{\partial E_{eff}}{\partial r}=-\frac{k}{r^2}e^{-r/a}-\frac{J^2}{mr^3}$

即 $m\ddot{\varepsilon}+\frac{J^2}{m(r_0+\varepsilon)^3}=F(r_0+\varepsilon)$

$m\ddot{\varepsilon}+\frac{J^2}{mr_0^3}\left(1-\frac{3\varepsilon}{r_0}\right)=F(r_0)+F'(r_0)\varepsilon$

$m\ddot{\varepsilon}-\left(\frac{3J^2}{mr_0^4}+F'(r_0)\right)\varepsilon=0.$

$3F(r_0)+r_0F'(r_0)<0$ 时稳定.

即 $-\frac{3k}{r_0^2}e^{-\frac{r_0}{a}}+r_0\left(\frac{2k}{r_0^3}e^{-\frac{r_0}{a}}-\frac{k}{r_0^2}e^{-\frac{r_0}{a}}\left(-\frac{1}{a}\right)\right)<0$

即 $-\frac{3k}{r_0^2}e^{-\frac{r_0}{a}}+\frac{2k}{r_0^2}e^{-\frac{r_0}{a}}+\frac{k}{ar_0}e^{-\frac{r_0}{a}}<0.$

即 $\frac{k}{r_0}\left(-\frac{1}{r_0}+\frac{1}{a}\right)<0$ 即 $r_0<a$ 时.圆轨道稳定.

86. 地球半径 a

$$\begin{cases} \frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 - \frac{GM_em}{a} \\ bv_0 = va \end{cases} \Rightarrow \quad v_0^2 = \frac{v_0^2b^2}{a^2} - \frac{2GM_e}{a} \quad \Leftrightarrow \quad v_0^2 + u^2 = \frac{v_0^2b^2}{a^2}$$

即碰撞截面的半径 $b = \frac{a}{v_0}\sqrt{v_0^2+u^2}$ 其中 u 为第二宇宙速度.

截面面积为 $S = \pi b^2 = \pi a^2\left(1+\frac{u^2}{v_0^2}\right)$