The proof of a theorem

$$\text{“}p(n),\ for\ all\ n \in \mathbb{Z}_m = \{m,\ m+1,\ m+2,\ \dots\}\text{”};\ m \in \mathbb{Z}$$

by induction is done in four steps:

1. Verifying the truth of $p(m)$, or verifying $p(n)$ for the first integer m in $\mathbb{Z}_m$,
2. Assuming the truth of $p(k)$ for a number $k \in \mathbb{Z}_m$,
3. Proving $p(k+1)$ using (2),
4. Arguing as follows:
   $p(m)$ is true by (1). Since $p(m)$ is true, then $p(m+1)$ must be true by (3). Since $p(m+1)$ is true, then $p(m+2)$ must be true again by (3). Continuing this way $p(n)$ must be true for all $n \in \mathbb{Z}_m$.

Example 0.1. Prove by induction:

$$p(n) : \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}\ ,\ for\ n \in \mathbb{Z}_1$$

Proof. Here $\mathbb{Z}_m$ is $\mathbb{Z}_1$ since 1 is the least value taken by n.

1)

$$p(1) : \sum_{i=1}^{1} i^2 = \frac{1(1+1)(2+1)}{6} \iff 1 = 1 \text{ (true)}$$

(In case $p(m)$ is false the statement is disproved and hence there is no need to go further.)

2) Suppose $p(k)$ is true for some $k \in \mathbb{Z}_1$, that is, suppose

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

□