a) $f(1,1) = f(2,3) + (1-2)f_x(x^* + y^*) + (1-3)f_y(x^*, y^*)$
$\Rightarrow 4 = 59 - (8x^{*3} + 3y^{*2}) - 2(6x^* y^* - 3y^{*2})$
$\Rightarrow 55 = 8x^{*3} + 12x^* y^*$ (1)

Since (x^*, y^*) lies on AB , we have

$$y^* = 2x^* - 1(2)$$

and (1), (2) give

$$\varphi(x^*) = 8x^{*3} + 24x^{*2} - 12x^* - 55 = 0$$

Since $\varphi(1) = -35 < 0$, $\varphi(2) = 81 > 0$ hold , x^* must lie between 1 and 2.

EXAMPLE 0.1. Given

$$f(x,y) = \arctan \frac{y}{x}$$

a) obtain TAYLOR's Formula with R_3 at $A(1, \sqrt{3})$
b) evaluate $f(2,3)$
c) Show existence of (x^*, y^*) on the open line segment joining $A(1, \sqrt{3})$ to $B(\sqrt{3}, 1)$

Solution.

a) $f(x,y) = f(1,\sqrt{3}) + [(x-1)f_x(A) + (y-\sqrt{3})f_y(B)] + \frac{1}{2}[(x-1)^2 f_{xx}(A) + 2(x-1)(y-\sqrt{3})f_{xy}(A) + (y-\sqrt{3})^2 f_{yy}(A)] + R_3$
where
$f(1,\sqrt{3}) = \frac{\pi}{3}, f_x(A) = -\frac{\sqrt{3}}{4}, f_y(A) = \frac{1}{4}, f_{xx}(A) = \frac{\sqrt{3}}{8}, f_{xy}(A) = \frac{1}{8}$, $f_{yy}(A) = -\frac{\sqrt{3}}{8}$
$f(x,y) = \frac{\pi}{3} + [-\frac{\sqrt{3}}{4}(x-1) + \frac{1}{4}(y-\sqrt{3})] + \frac{1}{2}[\frac{\sqrt{3}}{8}(x-1)^2 + \frac{1}{4}(x-1)(y-\sqrt{3}) - \frac{\sqrt{3}}{8}(y-\sqrt{3})^2] + R_3$