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Proof. Writing (e') with a^{-1} in place of a and using (2), we get $(a^m)^{-n}{=}[(a^m)^{-1}]^n{=}(a^{-m})^n{=}[(a^{-1})^m]^n{=}(a^{-1})^{mn}{=}a^{-(mn)}{=}a^{m(-n)}$. This proves (i). We also get $(a^{-m})^n{=}a^{-(mn)}{=}a^{-(m)n}$. This proves (ii). Finally, we have $(a^{-m})^{-n}{=}[(a^m)^{-1}]^{-n}{=}([(a^m)^{-1}]^{-1})^n{=}(a^m)^n{=}a^{mn}{=}a^{(-m)(-n)}$. This proves (iii).

Thus $(a^m)^n{=}a^{mn}$ for all $m, n \in \mathbb{Z}$.

The proof is complete. $\square$

Lemma 0.1. *Let G be a group and $a_1, a_2, \ldots, a_n \in G$. Then*

$$(a_1 a_2 \ldots a_n)^{-1}{=}a_n^{-1} \ldots a_2^{-1} a_1^{-1}$$

Proof. By induction on n. If $n = 2$, the assertion is true by Lemma 8.2(2). Suppose now $n \in \mathbb{N}$, $n \geq 3$ and $(a_1 a_2 \ldots a_{n-1})^{-1}{=}a_{n-1}^{-1} \ldots a_2^{-1} a_1^{-1}$. Then

$$\begin{aligned}(a_1 a_2 \ldots a_{n-1} a_n)^{-1} &= ((a_1 a_2 \ldots a_{n-1}) a_n)^{-1} \\ &= a_n^{-1} (a_1 a_2 \ldots a_{n-1})^{-1} \\ &= a_n^{-1} (a_{n-1}^{-1} \ldots a_2^{-1} a_1^{-1}) \\ &= a_n^{-1} a_{n-1}^{-1} \ldots a_2^{-1} a_1^{-1}\end{aligned}$$

as was to be proved. $\square$

Lemma 8.8 gives an alternative proof of $(a^{-1})^m{=}(a^m)^{-1}$. When our group is commutative, we have additional results for example $a^m b^n{=}b^n a^m$.

Lemma 0.2. *Let G be a group and $a, b \in G$. If $ab = ba$ then*

$(1) ab^n = b^n a$;

$(2) a^m b^n = b^n a^m$;

for all $m, n \in \mathbb{Z}$.

Proof. (1) We prove $ab^n{=}b^n a$: The case n=0 is trivial. Also $ab^1 = ab = ba = b^1 a$ by hypothesis and the claim is true for $n = 1$. Suppose now $n \in \mathbb{N}$, $n \geq 2$ and the claim is proved for $n - 1$, so that $ab^{n-1}{=}b^{n-1}a$. Then $ab^n = a(b^{n-1}b) = (ab^{n-1})b = (b^{n-1}a)b = b^{n-1}(ab) = b^{n-1}(ba) = (b^{n-1}b)a = b^n a$. By induction, $ab^n = b^n a$ for all $n \in \mathbb{N}$. $\square$