Proof. We prove the lemma by the principle of mathematical induction.
We put
$p_1 = q_1$
$p_k = q_1$ and q_2 and $\cdots$ and q_k (for all k $\in \mathbb{N}, k \geq 2$).
Now induction.

I. p_1 is true (by the hypothesis i.)

II. Make the inductive hypothesis that p_k is true. Then q_1 and q_2 and $\cdots$ and q_k is true. (definition of p_k)
$q_1, q_2, \cdots, q_k$ are all true. (truth value of conjunction)
q_{k+1} is true. (by the hypothesis ii.)
$q_1, q_2, \cdots, q_k, q_{k+1}$ are all true.
q_1 and q_2 and $\cdots$ and q_k and q_{k+1} is true.
p_{k+1} is true.

Hence, for all $k \in \mathbb{N}$, if p_k is true, then p_{k+1} is true. By the principal of mathematical induction, p_n is true for all $n \in \mathbb{N}$. So
q_1 and q_2 and $\cdots$ and q_n is true for all $n \in \mathbb{N}$.
In particular, q_n is true for all $n \in \mathbb{N}$. This completes the proof. □

We can now formulate a new form of the principle of mathematical induction. This form will be used many times in the sequel.

0.1 Principle of mathematical induction:

Let q_n be a statement-involving a natural number n. We can prove the proposition for all $n \in \mathbb{N}, q_n$
by establishing that

i. q_1 is true

ii. for all $k \in \mathbb{N}$, if $q_1, q_2, \cdots, q_k$ are true then, q_{k+1} is true.

The statement $2^n \geq n^2$ is not true for all natural numbers n, but true for all natural numbers $n \geq 5$. The principal of mathematical induction can be used to prove this and similar propositions. Let a be a fixed integer (positive, negative or zero) and let p_n be a statement involving an integer $n \geq a$. We prove the truth of p_n for all $n \geq a$ by showing that