page=b1p1/009

The positive square root of $a(>0)$ is denoted by $\sqrt{a}$ and the negative one by $-\sqrt{a}$. Thus,

$$\sqrt{4}=2, \quad -\sqrt{4}=-2, \quad \sqrt{(-3)^2}=\sqrt{9}=3$$

The number 0 which is neither positive nor negative has only one square root, namely 0, as a double root of $x^2=0$.

Absolute value

The absolute value of a real number "a" is a non negative real number, denoted by $|a|$ and defined by

$$|a|=\sqrt{a}^2 \quad (\geq 0)$$

or equivalently, by

$$|a|=\begin{cases} a \quad if \quad a>0 \\ 0 \quad if \quad a=0 \\ -a \quad if \quad a<0 \end{cases}$$

The equivalency of two definitions can be seen by considering three cases $a>0, a=0, a<0$ seperately.

$|5|=\sqrt{5^2}=5, \qquad |-3|=\sqrt{(-3)^2}=\sqrt{9}=3$

$|-2|=-(-2)=2, \qquad |2|=2$

As an immediate corollary we have

Corollary

1. $|a|^2=a^2$ 2. $-|a|\leq a\leq |a|$

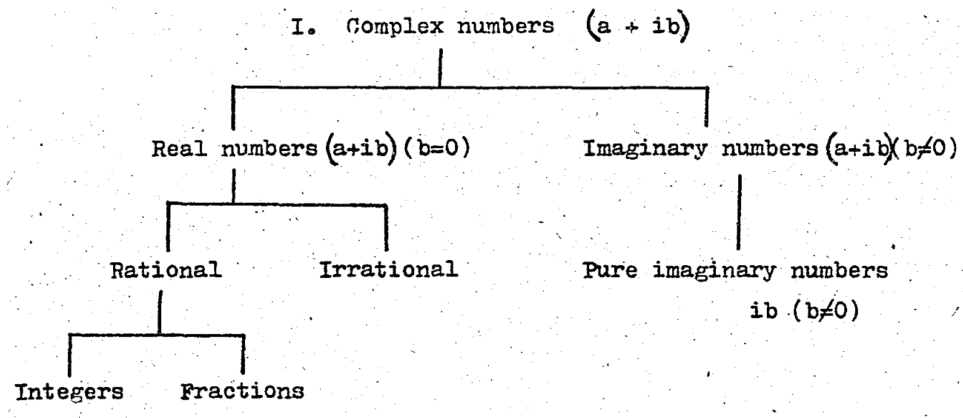
Some other properties are stated in the next theorem.

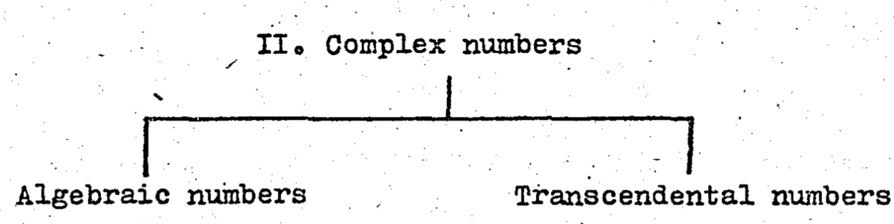
Theorem. If a, b are real numbers, then

page=b1p1/15

will be treated in Chapter.

We conclude this secftion by two classification of numbers:





0.1. EXERCISES.

.

1. Construct the following numbers on the number axis:

   a) $3/5$ b) $-7/3$ *(use Thales Theorem).*

   c) $\sqrt{8}$ d) $\sqrt{12}$ *(use Pythagorean Theorem).*

.

2. Give examples of two irrational numbers such that their.

page=b1p1/19

The meanings of the symbols $\{x : p(x) \text{ and } q(x)\}$ and $\{x : p(x) \text{ or } q(x)\}$ are clear.

Example 0.1. (for finite sets):
1. $D = \{n : n \text{ is a digit}\} = \{0, 1, 2, \ldots, 9\}$
2. $\{n : n \in D, n \text{ is prime}\} = \{2, 3, 5, 7\}$
3. $\{n : n \in D, 1 \leq n < 7\} = \{1, 2, 3, 4, 5, 6\}$

Example 0.2. The following infinite sets of numbers are used frequently in mathematics:
1. $\mathbb{N} = \{n : n \text{ is a natural number}\} = \{1, 2, \ldots, n, \ldots\}$
2. $\mathbb{Z} = \{n : n \text{ is an integer}\} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
3. $\mathbb{Q} = \{r : r \text{ is a rational number}\} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$
4. $\mathbb{Q}' = \{r' : r' \text{ is an irrational number}\}$
5. $\mathbb{R} = \{x : x \text{ is a real number}\} = \{x : x \in \mathbb{Q} \text{ or } x \in \mathbb{Q}'\}$

6. $\mathbb{C} = \{z : z \text{ is a complex number}\} = \{a + ib : a, b \in \mathbb{R}, i^2 = -1\}$

A set worth of mentioning is the one having no element at all. It is called the *empty set* (*null set*) and denoted by $\emptyset$, so that $n(\emptyset) = 0$.

EXAMPLE 0.3. Each one of the following is the null set $\emptyset$:
1. $\{x : x^2 + 1 = 0, x \in \mathbb{R}\}$
2. $\{x : |\,x\,| < 0, x \in \mathbb{R}\}$
3. $\{x : x \text{ is a box, } x \text{ is open and } x \text{ is closed}\}$

In any particular discussion, a set that contains all the objects that enter into that discussion is called the *uni-*

page=b1p1/53

function given by the rule $y = |\,x - 1\,| - 2\,|\,x\,| + x$ to be
a) a constant function,
b) an invertible function.

Solution.
The given function is the piecewisely defined function:

$$y = \begin{cases} 1 + 2x & \text{if } x \in (-\infty, 0] \\ 1 - 2x & \text{if } x \in (0, 1] \\ -1 & \text{if } x \in (1, \infty). \end{cases}$$

1 2

a) A domain of restriction is $(1, \infty)$,
b) A domain of restriction is $(-\infty, 0]$ on which the function is increasing, or (0, 1] on which it is decreasing.

0.2. Operation with functions. Let

$$f \colon I \to \mathbb{R}, \quad y = f(x)$$

be a function with domain I. If $c \in \mathbb{R}$, then the function

$$cf \colon I \to \mathbb{R}, \quad y = (cf)(x) = cf(x) \tag{0.1}$$

is called a *scalar multiple* of f.

Let now be given two functions

$$f \colon I \to \mathbb{R}, \quad y = f(x)$$
$$g \colon J \to \mathbb{R}, \quad y = g(x)$$

with non disjoint domain I and J, then f+g, f-g, fg,

page=b1p1/17

10. Find the distance between the given points. First express them as absolute value, and then compute.

[1]Corrected: "0)" in the first case of the equation must be closed interval "0]"
[2]Corrected: Missing element of sign in the third case

a) 2,72 and 5,16 b) 3,86 and -7,28
c) -3,86 and 7,28 d) -1,23 and -12,35
11. $(i+3)^3$=? Ans. 18+26i
12. $\frac{2+i}{3-2i}$ =? *Ans.* $\frac{4+7i}{13}$
13. Write a polynomial of least degree with real coefficients having the roots 3, 1-2i.[$x^3 - 5x^2 + 11x - 15$]
14. Solve for real X and y :

$$\frac{2-i}{3+iy} = \frac{2x-3iy}{2+i} \qquad \text{Ans.} \qquad x=5/6,\ y=0$$

15. If z=5+4i find $z^2 - 2z + z\bar{z}$ Ans. 60+32i

0.3. SETS.

0.3.1. *Definitions.* Any collection of objects (concrete or abstract) is called a set, and the objects in the set are its elements or members.

The sets are usually represented by capital letters A, B, ... Two sets formed by the same elements are said to be equal.

The set A consisting of elements, say, 2, a, Ankara, -7, is denoted either by listing the elements within two braces, or by a diagram (Venn diagram) in which the elements

page=b1p1/21

$$A \subseteq B \text{ and } B \subseteq A \iff A = B$$

This implication can be used to prove equality of sets.

Some subsets of R are in so frequent use that they bear special symbols, namely:

$$R^+ = \{x:\ x>0, x\in R\},\ R^- = \{x:\ x<0, x\in R\},\ R^* = \{x:\ x\in R, x\neq 0\}$$

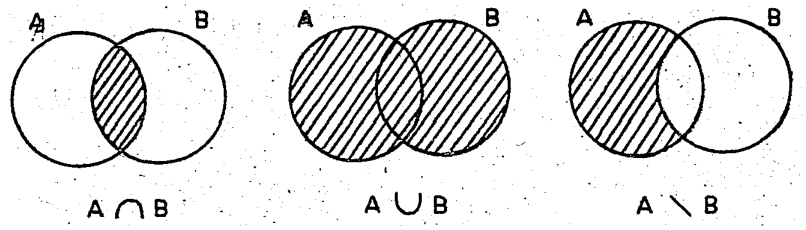
In the same way we may talk about the subsets of $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ (Why $\mathbb{N}^- = \emptyset$? However some authors use $\mathbb{N}^-$ for $\mathbb{Z}^-$. In our notation, $\mathbb{N}^-$ is the set of all negative elements of $\mathbb{N}$, which is the empty set.)

C. Operations with sets

Given two sets A and B, by means of three operations "$\cap$", "$\cup$" and "$\setminus$" we define the three sets, namely

(1) $A \cap B = \{x : x \in A \text{ and } x \in B\}$ "A intersection B"
(2) $A \cup B = \{x : x \in A \text{ or } x \in B\}$ "A union B"
(3) $A \setminus B = \{x : x \in A,\ x \notin B\}$ "A minus B"

Venn diagrams of these sets are indicated by shaded sets given below:



page=b1p1/055

EXAMPLE 0.4. [3]

(1) condition is jointly written with the rule.

(2)
$$(\frac{f}{g})(x) = \frac{f(x)}{g(x)} = \frac{|\,x\,|}{x^2\sqrt{1-x}}$$
As to the composition $g \circ f$ and $f \circ g$ we have.

(3) [4]
$$(g \circ f)(x) = g(f(x)) = g(\frac{|\,x\,|}{x}) = \frac{|\,x\,|}{x}\sqrt{1 - \frac{|\,x\,|}{x}}$$
$$(f \circ g)(x) = f(g(x)) = f(x\sqrt{1-x}) = \frac{|\,x\sqrt{1-x}\,|}{x\sqrt{1-x}} = \frac{|\,x\,|\sqrt{1-x}}{x\sqrt{1-x}}$$
$$= \frac{|\,x\,|}{x} \quad (x \neq 1)$$
and
$$D_{g\circ f} = (-\infty, 1] - \{\,0\,\}\,,\ D_{f\circ g} = (-\infty, 1] - \{\,0, 1\,\}$$
$$= (-\infty, 1) - \{\,0\,\} = (-\infty, 1)$$

EXAMPLE 0.5. Given the functions
$$f:\ \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{x}{x-2}; \quad g:\ \mathbb{R} \to \mathbb{R}, \quad g(x) = x^2 - x$$
find the rules for the composite functions $g \circ f$ and $f \circ g$, and then determine their domains.

Solution.

$$1.\ (g \circ f)(x) = g(f(x)) = f^2(x) - f(x) = \frac{x^2}{(x-2)^2} - \frac{x}{x-2}$$
$$= \frac{x^2 - x(x-2)}{(x-2)^2} = \frac{2x}{(x-2)^2}$$

[3] example enumeration continues from the previous page.

[4] g(f(x) written for the first equation here in the original book, so I corrected them.

$$2.\ (f\circ g)(x) = f(g(x)) = \frac{g(x)}{g(x)-2} = \frac{x(x-1)}{(x+1)(x-2)}$$

$$D_{g\circ f} = \mathbb{R} - \{\,2\,\}\,, \qquad D_{f\circ g} = \mathbb{R} - \{\,-1, 2\,\}$$

page=b1p1/53

function given by the rule $y = |\,x-1\,| - 2\,|\,x\,| + x$ to be
a) a constant function,
b) an invertible function.

Solution.
The given function is the piecewisely defined function:

$$y = \begin{cases} 1+2x & \text{if } x \in (-\infty, 0] \\ 1-2x & \text{if } x \in (0, 1] \\ -1 & \text{if } x \in (1, \infty). \end{cases}$$

[5] [6]

a) A domain of restriction is $(1, \infty)$,
b) A domain of restriction is $(-\infty, 0]$ on which the function is increasing, or (0, 1] on which it is decreasing.

0.4. Operation with functions. Let

$$f\colon I \to \mathbb{R}, \quad y = f(x)$$

be a function with domain I. If $c \in \mathbb{R}$, then the function

$$cf\colon I \to \mathbb{R}, \quad y = (cf)(x) = cf(x) \tag{0.2}$$

is called a *scalar multiple* of f.

Let now be given two functions

$$f\colon I \to \mathbb{R}, \quad y = f(x)$$
$$g\colon J \to \mathbb{R}, \quad y = g(x)$$

with non disjoint domain I and J, then f+g, f-g, fg,

page=b1p1/124

Theorem 0.1. [7]

Proof. exists at x_0 and it is the slope of the *unique* tangent line at x_0. Consequently, if $f(x)$ has derivative at x_0, (then the curve represented by this function has (unique) tangent line with slope $f'(x_0)$, and normal

[5] Corrected: "0)" in the first case of the equation must be closed interval "0]"
[6] Corrected: Missing element of sign in the third case
[7] The theorem and the proof starts in the previous page. Begin tags should be removed.

line with slope $-1/f'(x_0)$. Then we have the equations of tangent and the normal lines through $(x_0, f(x_0))$ with known slope:

$$y - f(x_0) = f'(x_0) \cdot (x - x_0)$$

and

$$y - f(x_0) = -\frac{1}{f'(x_0)} \cdot (x - x_0)$$

□

The slope of the tangent line at a point is the *slope of the curve* $y = f(x)$ at the same point. Let α be the angle between the positive x-axis and the tangent line at x_0. Then $\tan\alpha = f'(x_0)$ where $\alpha = \arctan f'(x_0)$ is the *slope angle* of the curves at x_0.

The *angle between two curves* at a certain common point is defined to be the angle between the tangent lines at this point. Two curves intersect each other *orthogonally* at a point if the angle between them is $90°$ at that point, and they are tangent at x_0 if the angle of intersection is zero at x_0.

Example 0.6. Find the equations of tangent and normal lines at the points of intersects of the following curves of $y = f(x) = x^2$,
$y = g(x) = \sqrt{8x}$, and the angles between their point of intersection.

Solution.
Equating y's we have $x^4 = 8x \Rightarrow x_1 = 0$, $x_2 = 2 \Rightarrow y_1 = 0$, $y_2 = 4$ and the points of intersection are $O = (0, 0)$, $A = (2, 4)$.

page=b1p1/225

181. Find m, M of the function $f(x) = (x/x + 1)$
182. Same question for $f(x) = \sqrt[3]{x/(x^2 + 1)}$
183. Same question for $f(x) = \sin(x) + \sqrt{3}\cos(x)$
184. If m, n are positive integers with $m > n$, prove.
    a) $\frac{x^m-1}{m} > \frac{x^n-1}{n}$, for $x > 1$
    b) $\frac{x^m-1}{m} > \frac{x^n-1}{n}$, for $0 < x < 1$
185. Prove the inequalities is Exercise 184 for $m, n \in Q$.
186. If $f(x) = x + 2$ and $g(f(x)) = x^2 - 3x$, find $g(x)$.
187. Defining

$$\max\{f(x), g(x)\} = \begin{cases} f(x) \text{ when } f(x) \geq g(x) \\ g(x) \text{ when } g(x) \geq f(x), \end{cases}$$

$$\min\{f(x), g(x)\} = \begin{cases} f(x) \text{ when } f(x) \leq g(x) \\ g(x) \text{ when } g(x) \leq f(x), \end{cases}$$

prove that if $f(x), g(x)$ are continuous, then

a) $\max\{f(x), g(x)\}$ b) $\min\{f(x), g(x)\}$

are continuous.

188. Sketch $\max\{f(x), g(x)\}$ where

a) $f(x) = x^2$, $g(x) = x$ b) $f(x) = 1$, $g(x) = x^2$

c) $f(x) = x^2$, $g(x) = x^2 + 1$ d) $f(x) = x^3 - x$, $g(x) = 2x - 2$

189. For the given functions in Exercise 188, sketch $\min\{f(x), g(x)\}$

190. Sketch:

a) $\{(x, y) : max\{x, y\} = 2\}$ b) $\{(x, y) : min\{x, y\} = 2\}$

208. a) 1, b) -1

210. $r = (r_1 + \ldots + r_n)/n$.