$n:$	0	1	2	3	4
$a_n:$	0	1	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{4}$
$b_n:$	0	1	0	$-\frac{1}{6}$	0
$p_n:$	0	0	1	$-\frac{1}{2}$	$\frac{1}{6}$

$$[\ln(1+x)]\sin x = x^2 - \tfrac{1}{2}x^3 + \tfrac{1}{6}x^4 + \dots$$

where the general term is omitted since it is unnecessary, and we have all properties of the series, because, those of $\ln(x+1)$ and $\sin x$ are known.

Example 4. Obtain power series expansions of the following rational functions at the indicated points:

a) $\frac{3+x}{1-x}, x=0$ b) $\frac{3+x}{x}, x=1$

Solution.

a) Direct division gives (since $b_0 = 1 \neq 0$)

$$\frac{3+x}{1-x} = 3 + 4x + 4x^2 + \dots + 4x^n + \dots .$$

Observe that it involves a geometric series with common ratio x. Then it is convergent for $|x| < 1$.

Obtain the same series by performing

$$(3+x)(1+x+\dots+x^n+\dots),$$

and also by differentiating $(3+x)(1-x)$ successively at $x=0$.

b) The series being in powers of $x-1$, use substitution $x-1=t$ or $x=1+t$. Then

$$\frac{3+x}{x} = \frac{3+(1+t)}{1+t} = \frac{4+t}{1+t} = 4 - 3t + 3t^2 - \dots + (-1)^n 3t^n - \dots$$
$$= 4 - 3(x-1) + 3(x-1)^2 - \dots + (-1)^n 3(x-1)^n - \dots$$

convergent for $|x-1| < 1$.