(5) $Ha = Hb$ if and only if $a = hb$ for some $h \in H$, and there is a unique h with $a = hb$, namely $h = ab^{-1}$ (Lemma 7.5(2)); thus $a = hb$ for some $h \in H$ if and only if $ab^{-1} \in H$.

(6) $Ha = Hb$ if and only if $ab^{-1} \in H$ by (5), and $ab^{-1} \in H$ if and only if $Hab^{-1} = H$ by (2).

10.3 Lemma: *Let $H \leqslant G$. Then G is the union of the right cosets of H. The right cosets of H are mutually disjoint. Analogous statements hold for left cosets.*

Proof: As $Ha \subseteq G$ for any $a \in G$, we get $\bigcup_{a \in G} Ha \subseteq G$. Also, for any $g \in G$, we have $g \in Hg$, so $g \in \bigcup_{a \in G} Ha$, thus $G \subseteq \bigcup_{a \in G} Ha$. This proves $G = \bigcup_{a \in G} Ha$.

Now we prove that the right cosets of H are mutually disjoint. Assume $Ha \cap Hb \neq \emptyset$. We are to show $Ha = Hb$. Well, we take $c \in Ha \cap Hb$ if $Ha \cap Hb \neq \emptyset$. Then $c \in Ha$ and $c \in Hb$. So $Ha = Hc$ and $Hc = Hb$ by Lemma 10.2(4). We obtain $Ha = Hb$.

The left cosets are treated similarly.

In the terminology of Theorem 2,5, right cosets of H form a partition of G. Theorem 2.5 tells us that the right cosets are the equivalence classes of a certain equivalence relation on G. By the proof of Theorem 2.5, we see that this equivalence relation $\sim$ is given by

$$\text{for all } a, b \in G : a \sim b \text{ if and only if } Ha = Hb,$$

which we can read as

$$\text{for all } a, b \in G : a \sim b \text{ if and only if } ab^{-1} \in Hb,$$

It may be worth while to obtain Lemma 10.3 from this relation $\sim$ instead of obtaining the relation $\sim$ from Lemma 10.3.

10.4 Definition: Let $H \leqslant G$ and $a, b \in G$. We write $a \equiv_r b \pmod{H}$ and say *a is right congruent to b modulo H if* $ab^{-1} \in H$. Similarly, we write $a \equiv_l b \pmod{H}$ and say *a is left congruent to b modulo H* if $a^{-1}b \in H$.