Lemma 0.1. *Let $(G, \circ)$ be a group and let e be a right identity element of G such that, for all $a \in G$, there exists a suitable x in G with $a \circ x = e$. The existence of e is assured by the group axioms 3 and 4.*

Axiom 1. *If $g \in G$ is such that $g \circ g = g$, then $g = e$.*

Axiom 2. *e is the unique right identity in G.*

Axiom 3. *A right inverse of an element in G is also a left inverse of the same element. In other words, if $a \circ x = e$, then $x \circ a = e$.*

Axiom 4. *e is a left identity in G. That is, $e \circ a = a$ for all $a \in G$.*

Axiom 5. *e is the unique left identity in G.*

Axiom 6. *Each element has a unique right inverse in G.*

Axiom 7. *Each element has a unique left inverse in G.*

Axiom 8. *The unique right inverse of any $a \in G$ is equal to the unique left inverse of a.*

Proof. (1) Let $g \in G$ be such that $g \circ g = g$. We choose a right inverse of g with respect to e. This is possible by the axiom 4. Let us call it h. Thus $g \circ h = e$. Then

$$(g \circ g) \circ h = g \circ h$$

$$g \circ (g \circ h) = g \circ h \text{ (by associativity)},$$

$$g \circ e = e \text{ (since } g \circ h = e),$$

$$g = e \text{ (since e is a right identity)}.$$

This proves part(1).

(2) The claim is that e is the unique right identity in G. This means: if $f \in G$ is a right identity, that is, if $a \circ f = a$ for all $a \in G$, then $f = e$. Suppose f is a right identity. Then $a \circ f = a$ for all $a \in G$. Writing f for a in particular, we see $f \circ f = f$. Hence $f = e$ by part (1).

(3) A right inverse x of an arbitrary element $a \in G$ is also a left inverse of a. This is what we are to prove. So we assume $a \circ x = e$ and try to derive $x \circ a = e$. We use part(1). If $a \circ x = e$, then

$$\begin{aligned}(x \circ a) \circ (x \circ a) &= [(x \circ a) \circ x] \circ a \text{ (by associativity)} \\ &= [x \circ (a \circ x)] \circ a \text{ (by associativity)} \\ &= [x \circ e] \circ a \\ &= x \circ a.\end{aligned}$$

So $g := (x \circ a)$ is such that $g \circ g = g$. By part (1), $g = e$. So $x \circ a = e$.

(4) We are to prove that e is a left identity. So we must show $e \circ a = a$ for all $a \in G$. Let $a \in G$ and let x be a right inverse of a. Then

$$a \circ x = e.$$

□