

§4 Mathematical Induction

Examine the propositions

$$\begin{aligned} 2^n &\geq n \text{ for all } n \in \mathbb{N}, \\ 1 + 2 + \dots + n &= \frac{n(n+1)}{2} \text{ for all } n \in \mathbb{N}, \\ n^{n+1} &\geq (n+1)^n \text{ for all } n \in \mathbb{N}. \end{aligned}$$

How do we prove them? They are statements involving a variable n running through the infinite set \mathbb{N} . Strictly speaking; each one of these propositions above is a collection of infinitely many, propositions. We can verify them for a finite number of cases where n assumes some specific values. Thus we might verify $2^n \geq n$ for $n=1,2,3, \dots, 1\,000\,000$ and convince ourselves -of the truth of this statement, but this is far from a proof. On the other hand, we cannot check the truth of infinitely many statements within finite time. So we must resort to some other means.

In order to prove propositions about all natural numbers, an axiom is introduced. It is the fifth Peano axiom about \mathbb{N} Giuseppe Peano (1858-1932), an Italian mathematician and logician). It is called the axiom of mathematical induction.

4.1 Axiom (of mathematical induction): If S is a subset of \mathbb{N} such that

- (i) $1 \in S$
- (ii) for all $k \in \mathbb{N}$, if $k \in S$, then $k+1 \in S$,

then S is the whole of \mathbb{N} , i.e., $S=\mathbb{N}$.

We can use this axiom to prove statements of the form ' p_n for all $n \in \mathbb{N}$ ' as follows. We let $S \subseteq \mathbb{N}$ be the set of all natural numbers n for which p_n is true. First we verify $1 \in S$, that is, we verify that p_1 is true. Second, we *assume* that $k \in S$ and under this hypothesis, which is called the *induction hypothesis*, we prove that p_{k+1} is true. So we show that $k \in S$ implies $k+1 \in S$. By the axiom of mathematical induction, $S = \mathbb{N}$, so the