B. THE FUNDAMENTAL THEOREMS

We state two fundamental theorems (F.T.) the proofs of which are based on the following mean value theorem for integrals:

THEOREM (MVT for integrals). If $f(x) \in C(a, b)$, then there exists an interior point $c \in (a, b)$ such that

$$\int_a^b f(x)dx = (b-a)f(c)$$

PROOF. If the function is constant, say $f(x) = y_0$, then

$$\int_a^b f(x)dx = \int_a^b y_0 dx = y_0 \int_a^b dx = (b-a)y_0 = (b-a)f(c)$$

for any $c \in (a, b)$.

Let then $f(x)$ be a non constant function. By its continuty it attains $m = min\ f(x),\ M = max\ f(x)$ on (a, b) so that

$$\int_a^b m dx \le \int_a^b f(x)dx \le \int_a^b M dx$$

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

$$m \le \frac{\int_a^b f(x)dx}{b-a} \le M.$$

Again from continuity of $f(x)$ the intermediate value

$$\overline{y} = \frac{\int_a^b f(x)dx}{b-a}$$

is attained at a point c which is certainly between a and b, so that

(a) $$\overline{y} = \frac{\int_a^b f(x)dx}{b-a} = f(c)$$

The value $\overline{y}$ defined by (a) or by

$$\overline{y} = \frac{\int\limits_a^b f(x)dx}{\int\limits_a^b dx}$$