

- (a) Let $4\mathbb{Z} = \{4z \in \mathbb{Z} : z \in \mathbb{Z}\} = \{u \in \mathbb{Z} : 4|u\} \subseteq \mathbb{Z}$. Now \mathbb{Z} is a group under addition (Example 7.1(a)), and $4\mathbb{Z}$ is closed under addition and under the forming of inverses by Lemma 5.2(5) and Lemma 5.2(1):

- (i) if $x, y \in 4\mathbb{Z}$, then $4|x$ and $4|y$, then $4|x + y$, so $x + y \in 4\mathbb{Z}$,
- (ii) if $x \in 4\mathbb{Z}$, then $4|x$, then $4|-x$, so $-x \in 4\mathbb{Z}$

Hence $4\mathbb{Z} \leq \mathbb{Z}$.

- (b) The additive group \mathbb{Z} is a subgroup of the additive group \mathbb{Q} . Also, we have $\mathbb{Q} \leq \mathbb{R} \leq \mathbb{C}$, where the group operation is ordinary addition.
- (c) Under multiplication, $\mathbb{Q}^+ := \{x \in \mathbb{Q} : x > 0\}$ is a subgroup of $\mathbb{Q} \setminus \{0\}$, since
- (i) the product of two positive rational numbers is a positive rational number, and
 - (ii) the reciprocal, that is, the multiplicative inverse $1/a$ of any positive rational number a is a positive rational number. ($\mathbb{Q} \setminus \{0\}$ is a group under multiplication by §7, Ex.1(b).) Also, $\mathbb{Q}^+ \leq \mathbb{R}^+$ (see Example 7.1(b)) and $\mathbb{Q} \setminus \{0\} \leq \mathbb{R} \setminus \{0\}$. We have in fact $\mathbb{Q}^+ = (\mathbb{Q} \setminus \{0\}) \cap \mathbb{R}^+$.
- (d) If H_1 and H_2 are subgroups of G , then $H_1 \cap H_2$ is a subgroup of G . Indeed, $H_1 \cap H_2 \neq \emptyset$ since $1 \in H_1$ and $1 \in H_2$. Also
- (i) $a, b \in H_1 \cap H_2 \implies a, b \in H_1$ and $a, b \in H_2 \implies ab \in H_1$ and $ab \in H_2 \implies ab \in H_1 \cap H_2$,
 - (ii) $a \in H_1 \cap H_2 \implies a \in H_1$ and $a \in H_2 \implies a^{-1} \in H_1$ and $a^{-1} \in H_2 \implies a^{-1} \in H_1 \cap H_2$,

Thus $H_1 \cap H_2 \leq G$. More generally, if H_i are subgroups of G , where i runs through an index set I , then $\bigcup_{i \in I} H_i \leq G$. Indeed $\bigcup_{i \in I} H_i \neq \emptyset$ since $1 \in H_i$ for all $i \in I$ and

- (i) $a, b \in \bigcup_{i \in I} H_i \implies a, b \in H_i$ for all $i \in I \implies ab \in H_i$ for all $i \in I \implies ab \in \bigcup_{i \in I} H_i$
- (ii) $a \in \bigcup_{i \in I} H_i \implies a \in H_i$ for all $i \in I \implies a^{-1} \in H_i$ for all $i \in I \implies a^{-1} \in \bigcup_{i \in I} H_i$

- (e) Get $S_{[0,1]}$ to the set of all one-to-one mappings from $[0, 1]$ into $[0, 1]$, which is a group under the composition of mappings (Example 7.1(d)). Consider

$$T = \{\alpha \in S_{[0,1]} : 0\alpha = 0\}$$

Then T is a subgroup of $S_{[0,1]}$ for T is not empty (why?) and