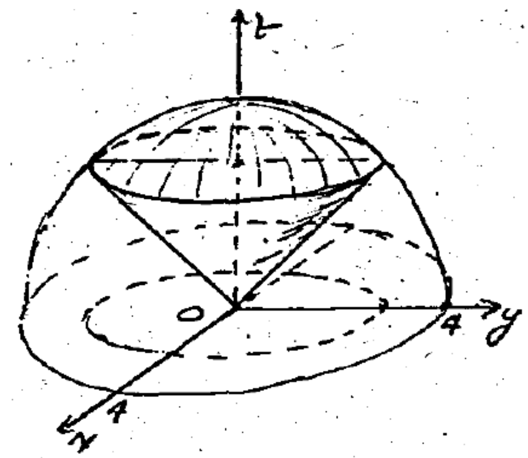
with

$$J = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = \text{abc}$$

$\Rightarrow |\, V \,| = \int\int_{R'} \int abc \; \mathrm{d}V' = abc \,|\, R' \,| = abc.\frac{4}{3}\pi = \frac{4}{3}\pi abc.$

Example 0.1. Find the volme of the solid defined by

$$x^2 + y^2 + z^2 \leq 16, x^2 + y^2 \leq z^2, z \geq 0$$



Solution.

$|\, R \,| = 4 \int_0^{2\sqrt{2}} \int_0^{\sqrt{8-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} \mathrm{d}z \mathrm{d}y \mathrm{d}x$

Transforming it into spherical coordinates, we have the transformation

$$\begin{aligned} x =& \rho sin\varphi cos\theta \\ y =& \rho sin\varphi sin\theta \\ z =& \rho cos\varphi \end{aligned}$$

with Jacobian

$$\begin{aligned} J =& \frac{\partial(x,y,z)}{\partial(\theta,\varphi,\rho)} \\ =& \begin{vmatrix} -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta & \sin\varphi\cos\theta \\ \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta & \sin\varphi\sin\theta \\ 0 & -\rho\sin\varphi & cos\varphi \end{vmatrix} = -\rho^2\sin\varphi \\ |\, R \,| =& 4\int_0^{\pi/2}\int_0^{\pi/4}\int_0^4 \rho^2\sin\varphi.\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta = \frac{16}{3}\sqrt{2}\pi \end{aligned}$$