Proof. Let $F(x) = \int_a^x f(t)\,\mathrm{d}t$. Then

$$\begin{aligned}
F(x+h) - F(x) &= \int_a^{x+h} f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t \\
&= \int_a^{x+h} f(t)\,\mathrm{d}t + \int_x^a f(t)\,\mathrm{d}t \\
&= \int_x^{x+h} f(t)\,\mathrm{d}t \\
&= \Big((x+h) - x\Big) f(c), \qquad x < c < x+h
\end{aligned}$$

by the MVT for integral. Hence

$$\begin{aligned}
\frac{F(x+h) - F(x)}{h} &= f(c) = f(x + \theta h), \qquad 0 < \theta < 1 \\
\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} &= \lim_{h \to 0} f(x + \theta h) = f(x).
\end{aligned}$$

□

Theorem 0.1 (F.T. of integral calculus). *If $f(x) \epsilon C(a,b)$ and $F(x)$ is a primitive of $f(x)$, then*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

Proof. Since $D \int_a^x f(t)\,\mathrm{d}t = f(x)$ by previous theorem and $DF(x) = f(x)$, then $\int_a^x f(t)\,\mathrm{d}t$ differs from $F(x)$ by a constant:

$$\int_a^x f(t)\,\mathrm{d}t = F(x) + c$$

Now

$$\begin{aligned}
x = a \quad &\implies \quad 0 = F(a) + c \quad \implies \quad c = -F(a), \\
x = b \quad &\implies \quad \int_a^b f(t)\,\mathrm{d}t = F(b) + c = F(b) - F(a).
\end{aligned}$$

□

Notation. $\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a) = F(x) \Bigg|_{x=a}^{x=b} = F(x)\Bigg|_a^b$

In view of this theorem, evaluation of a definite integral reduces to that of an indefinite integral. It is to be noted that if the evaluation is done by substitution, the new limits