Example. Evaluate

$$A = \int Sh^2 t\, dt, \qquad B = \int \frac{e^t - e^{-t}}{e^t + e^t}\, dt$$

Solution.

$$\begin{aligned}
A &= \int (e^t - e^{-t})^2\, dt = \frac{1}{4}\int (e^{2t} - 2 + e^{-2t})\, dt \\
&= \frac{1}{4}[\frac{1}{2}e^{2t} - 2t - \frac{1}{2}e^{-2t}] + c = \frac{1}{4}Sh\, 2t - \frac{1}{2}t + c \\
B &= \int \frac{Sh\, t}{Ch\, t}\, dt = \ln Ch\, t + c
\end{aligned}$$

Integral evaluated by recurrence formulas

$$1.\, c_n = \int \cos^n\theta\, d\theta \qquad 1'.\, C_n = \int Ch^n\theta\, d\theta$$

$$c_n = \int \sin^n\theta\, d\theta \qquad S_n = \int Sh^n\theta\, d\theta$$

We establish the formula for c_n; the others we obtained similarly:

$$\begin{aligned}
c_n &= \int \cos^n\theta\, d\theta = \int \cos^{n-1}\theta \cdot \cos\theta\, d\theta \qquad (n \geq 2) \\
&= \cos^{n-1}\theta\, \sin\theta - \int \sin\theta (n-1)\, \cos^{n-2}\theta\, (-\sin\theta)\, d\theta \\
&= \cos^{n-1}\theta\, \sin\theta + (n-1)\, \cos^{n-2}\theta\, (1 - \cos^2\theta)\, d\theta \\
&= \cos^{n-1}\theta\, \sin\theta + (n-1)c_{n-2} - (n-1)c_n \\
n\, C_n &= (n-1)c_{n-2} + \cos^{n-1}\theta\, \sin\theta \\
c_n &= \frac{n-1}{n}c_{n-2} + \frac{1}{n}\cos^{n-1}\theta \cdot \sin\theta \\
s_n &= -\frac{n-1}{n}s_{n-2} + \frac{1}{n}\sin^{n-1}\theta \cdot \cos\theta \\
C_n &= \frac{n-1}{n}C_{n-2} + \frac{1}{n}\cosh^{n-1}\theta \cdot Sh\,\theta \\
S_n &= -\frac{n-1}{n}S_{n-2} + \frac{1}{n}\sinh^{n-1}\theta \cdot Ch\,\theta
\end{aligned}$$

$$2.\, t_n = \int \tan^n\theta\, d\theta \qquad 2'.\, T_n = \int \tanh^n\theta\, d\theta$$

$$t_n = \int \cot^n\theta\, d\theta \qquad T_n = \int \coth^n\theta\, d\theta$$