Proof. Let $A \neq 0$.Multiplying (1) by $4A$ and completing to (perfect) square:

$$(2Ax + By + D)^2 - (By + D)^2 + 4A(y^2 + 4AEy) + 4AF = 0$$
$$(2Ax + By + D)^2 - \Big((B^2 - 4AC)y^2 + 2(BD - AE)y + (D^2 - 4AF)\Big) = 0$$

To be factorable iff the bracket square.Then

$$\delta = (BD - 2AE)^2 - (B^2 - 4AC)(D^2 - 4AF) = 0$$
$$\delta = -4A(4ACF + BDE - AE^2 - CD^2 - FB^2) = -2AT \Rightarrow T = 0$$

When $A = 0$, T becomes: $2(BDE - CD^2 - FB^2)$: and (1) reduces to

$$Bxy + Cy^2 + Dx + Ey + F = 0$$

Multiplying it by $4C$ and completing to square we have:

$$(2Cy + Bx + E)^2 - \Big((Bx + E)^2 - 4CDx - 4CF\Big) = 0$$

where bracket is to be a perfect square implying $T = 0$.

□

The proof can be done considering the coefficient C instead of A, in a similar manner.

The following theorem states the cases where the second degree curve is real or imaginary.

Theorem 0.1. *A second degree curve given by*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0(A^2 + B^2 + C^2 \neq 0) \text{ (1)}$$

is imaginary (has no graph) iff

(1) *(Elliptic Case)* $\Delta < 0$ *and* $HT > 0(H = A + C)$
(2) *(Parabolic Case)*$\Delta = 0$ *and* $\psi < 0(\psi = \Delta' + \Delta'' = D^2 - 4AF + E^2 - 4CF)$

Proof. If the curve is real it contains at least one point in $\mathbb{R}^2$.Therefore the family $y = k(k \in \mathbb{R})$intersects it at one or more point. Setting $y = k$ in (1) one obtains the equation. □