2. Mass, moments, center of grevity. moments of inertia

By the usual notations

$m = \iiint\limits_{\mathbb{R}} \delta(x,y,z)dV$

$M_{xy} = \iiint\limits_{\mathbb{R}} z\delta dV$, $M_{xz} = \iiint\limits_{\mathbb{R}} y\delta dV$, $M_{yz} = \iiint\limits_{\mathbb{R}} x\delta dV$

$G(M_{yz}/m, M_{xz}/m, M_{xy}/m)$

$I_{ox} = \iiint\limits_{\mathbb{R}} (y^2+z^2)\delta dV$, $I_{oy} = \iiint\limits_{\mathbb{R}} (x^2+z^2)\delta dV$, $I_{oz} = \iiint\limits_{\mathbb{R}} (x^2+y^2)\delta dV$

and in general

$M_{\pi} = \iiint\limits_{\mathbb{R}} d(P,\pi)\delta dV$, $M_{\ell} = \iiint\limits_{\mathbb{R}} d(P,\ell)\delta dV$, $M_A = \iiint\limits_{\mathbb{R}} d(P,A)\delta dV$

$I_{\pi} = \iiint\limits_{\mathbb{R}} d^2(P,\pi)\delta dV$, $I_{\ell} = \iiint\limits_{\mathbb{R}} d^2(P,\ell)\delta dV$, $I_A = \iiint\limits_{\mathbb{R}} d^2(P,A)\delta dV$

Example. Find the centroid (δ is constant) of the solid bounded by $x^2+y^2+z^2=a^2$, in the first octant.

Solution. From symmetry we have $\overline{x} = \overline{y} = \overline{z}$.

$m = \delta V = \frac{6}{\delta}\pi a^3$

$$\begin{aligned} m\overline{z} &= M_{xy} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \rho\cos\varphi\rho^2\sin\varphi d\rho d\varphi d\theta \\ &= \frac{\delta}{2}\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \rho^3 d\rho \sin 2\varphi d\varphi d\theta \\ &= \frac{\pi}{8}a^4\int_0^{\pi/2}\int_0^{\pi/2}\sin 2\varphi d\varphi d\theta \\ &= \frac{\delta}{8}a^4\frac{\pi}{2} = \frac{\delta}{16}\pi a^4 \\ & \quad \overline{x} = \overline{y} = \overline{z} = \frac{\delta}{16}\pi a^4/(\frac{\delta}{6}\pi a^3) = \frac{3}{8}a \end{aligned}$$