Theorem 0.1. *The area* $|\,R\,|$ *of the plane region* R *bounded by the curve* $y = f(x)$*, x-axis and the vertical lines* $x = a,\ x = b$ *is given by*

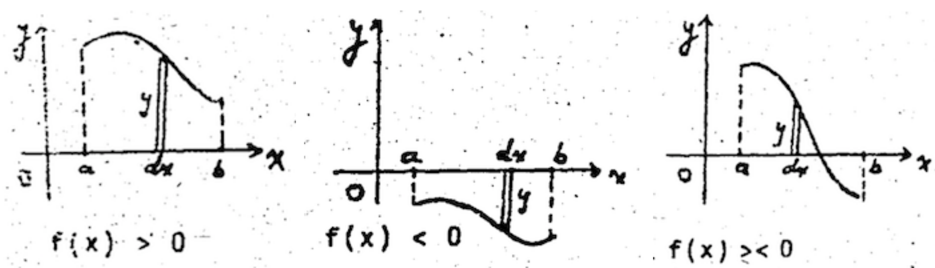
$$|\,R\,| = \int_a^b |\,f(x)\,|\,\mathrm{d}x \quad (f(x) \in C(a,\ b))$$

Proof. The statement is trivially true if $f(x) > 0$ on $(a,\ b)$, since the RIEMANN sum

$$\sum_{i=1}^{n} f(t_i)\Delta x_i$$

is an approximation of the area under the curve and the limit is the area $|\,R\,|$. (See left fig.)

□



If $f(x) < 0$ on $(a,\ b)$, we have

$$|\,R\,| = \int_a^b (-f(x))\,\mathrm{d}x = \int_a^b |\,f(x)\,|\,\mathrm{d}x$$

If $f(x)$ is positive and negative on $(a,\ b)$, say positive on $(a,\ x_0)$, and negative on $(x_0,\ b)$, then one gets

$$\begin{aligned} |\,R\,| &= \int_a^{x_0} f(x)\,\mathrm{d}x + \int_{x_0}^b (-f(x))\,\mathrm{d}x \\ &= \int_a^{x_0} |\,f(x)\,|\,\mathrm{d}x + \int_{x_0}^b |\,f(x)\,|\,\mathrm{d}x = \int_a^b |\,f(x)\,|\,\mathrm{d}x \end{aligned}$$

Corollary 0.2. *The area of a plane region bounded by the curve* $y = f(x)$*, the y-axis and the horizontal lines* $y = c,\ y = d$ *is given by*