(a) Let $4\mathbb{Z} = \{4z \in \mathbb{Z} : z \in \mathbb{Z}\} = \{u \in \mathbb{Z} : 4|u\} \subseteq \mathbb{Z}$. Now $\mathbb{Z}$ is a group under addition (Example 7.1(a)), and $4\mathbb{Z}$ is closed under addition and under the forming of inverses by Lemma 5.2(5) and Lemma5.2(1):

(i) if $x, y \in 4\mathbb{Z}$, then $4|x$ and $4|y$, then $4|x + y$, so $x + y \in 4\mathbb{Z}$,

(ii) if $x \in 4\mathbb{Z}$, then $4|x$,then $4| - x$, so $-x \in 4\mathbb{Z}$

Hence $4\mathbb{Z} \leqslant \mathbb{Z}$.

(b) The additive group $\mathbb{Z}$ is a subgroup of the additive group $\mathbb{Q}$. Also, we have $\mathbb{Q} \leqslant \mathbb{R} \leqslant \mathbb{C}$, where the group operation is ordinary addition.

(c) Under multiplication, $\mathbb{Q}^+ := \{x \in \mathbb{Q} : x > 0\}$ is a subgroup of $\mathbb{Q} \setminus \{0\}$, since

(i) the product of two positive rational numbers is a positive rational number, and

(ii) the reciprocal, that is, the multiplicative inverse $1/a$ of any positive rational number a is a positive rational number. ($\mathbb{Q} \setminus \{0\}$ is a group under multiplication by $\S 7, Ex.1(b)$.) Also, $\mathbb{Q}^+ \leqslant \mathbb{R}^+$(see Example 7.l(b)) and $\mathbb{Q} \setminus \{0\} \leqslant \mathbb{R} \setminus \{0\}$. We have in fact $\mathbb{Q}^+ = (\mathbb{Q} \setminus \{0\}) \cap \mathbb{R}^+$.

(d) If H_1 and H_2 are subgroups of G, then $H_1 \cap H_2$ is a subgroup of G. Indeed, $H_1 \cap H_2 \neq \emptyset$ since $1 \in H_1$ and $1 \in H_2$. Also

(i) $a, b \in H_1 \cap H_2 \Longrightarrow a, b \in H_1$ and $a, b \in H_2 \Longrightarrow ab \in H_1$ and $ab \in H_2 \Longrightarrow ab \in H_1 \cap H_2$,

(ii) $a \in H_1 \cap H_2 \Longrightarrow a \in H_1$ and $a \in H_2 \Longrightarrow a^{-1} \in H_1$ and $a^{-1} \in H_2 \Longrightarrow a^{-1} \in H_1 \cap H_2$,

Thus $H_1 \cap H_2 \leqslant G$. More generally, if $H_\imath$ are subgroups of G, where $\imath$ runs through an index set I, then $\bigcup\limits_{\imath \in I} H_\imath \leqslant G$. Indeed $\bigcup\limits_{\imath \in I} H_\imath \neq \emptyset$ since $1 \in H_\imath$ for all $\imath \in I$ and

(i) $a, b \in \bigcup\limits_{\imath \in I} H_\imath \Longrightarrow a, b \in H_\imath$ for all $\imath \in I \Longrightarrow ab \in H_\imath$ for all $\imath \in I \Longrightarrow ab \in \bigcup\limits_{\imath \in I} H_\imath$

(ii) $a \in \bigcup\limits_{\imath \in I} H_\imath \Longrightarrow a \in H_\imath$ for all $\imath \in I \Longrightarrow a^{-1} \in H_\imath$ for all $\imath \in I \Longrightarrow a^{-1} \in \bigcup\limits_{\imath \in I} H_\imath$

(e) Get $S_{[0,1]}$ to the set of all one-to-one mappings from $[0, 1]$ into $[0, 1]$, which is a group under the composition of mappings (Example 7.I(d)). Consider

$$T = \{\alpha \in S_{[0,1]} : 0\alpha = 0\}$$

Then T is a subgroup of $S_{[0,1]}$ for T is not empty (why?) and