(b) Prove that $2+2^2+2^3+\ldots+2^n = 2^{n+1}-2$ for all $n \in \mathbb{N}$.

I. We have $2 = 2^{1+1}-2$, which proves the assertion for $n=1$.

II. Assume $2+2^2+2^3+\ldots+2^k = 2^{k+1}-2$. Now we must prove $2+2^2+2^3+\ldots+2^k+2^{k+1} = 2^{(k+1)+1}-2$. We have

$$
\begin{aligned}
2+2^2+2^3+\ldots+2^k+2^{k+1} &= (2^{k+1}-2)+2^{k+1} \quad \text{(by inductive hyp.)}\\
&= 2(2^{k+1})-2\\
&= 2^{k+2}-2,
\end{aligned}
$$

so the assertion is true for $n=k+1$ if it is true for $n=k$. Thus

$$2+2^2+2^3+\ldots+2^n = 2^{n+1}-2 \text{ for all } n \in \mathbb{N}.$$

(c) Let $h > -1$ be a fixed real number. Prove that $1+h^n \geq 1+nh$ for all $n \in \mathbb{N}$.

I. We have $(1+h)^1 \geq 1+1h$, so the inequality is true for $n=1$.

II. Let us assume $(1+h)^k \geq 1+kh$. We want to prove that $(1+h)^{k+1} \geq 1+(k+1)h$. We have

$$
\begin{aligned}
(1+h)^{k+1} &= (1+h)^k(1+h)\\
&\geq (1+kh)(1+h) \qquad \text{(by inductive hyp. and } 1+h>0)\\
&= 1+h+kh+kh^2\\
&\geq 1+h+kh+0\\
&= 1+(k+1)h
\end{aligned}
$$

so the inequality is true for $n=k+1$ if it is true for $n=k$. By the principle of mathematical induction,

$$1+h^n \geq 1+nh \text{ for all } n \in \mathbb{N}.$$

Sometimes it is convenient to use the principle of mathematical induction in a slightly different form. We assume (not only q_k, but rather) each one of $q_1, q_2, q_3, \ldots, q_k$ is true and then conclude that q_{k+1} is true. This establishes the truth of q_n for all $n \in \mathbb{N}$, as the following lemma shows.

4.4 Lemma: *Let q_n be a statement involving a natural number n. Assume that*

i. *q_1 is true,*

ii. *for all $k \in \mathbb{N}$, if $q_1, q_2, q_3, \ldots, q_k$ are true, then q_{k+1} is true.*

Then q_n is true for all $n \in \mathbb{N}$.