$$F_x \mathrm{d}x + F_y \mathrm{d}y + F_z \mathrm{d}z + F_\lambda \mathrm{d}\lambda + F_\mu \mathrm{d}\mu = 0$$

holds, where the sum of the first three terms is zero, since $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$ are direction numbers of a tangent line to E at C, lying on the tangent plane.

Hence $F_\lambda \mathrm{d}\lambda + F_\mu \mathrm{d}\mu = 0$ holds for every λ, μ implying $F_\lambda = 0, F_\mu = 0$.

Corollary. The envelope of

$$F(x, y, z, \lambda, \mu, \gamma) = 0, \quad \varphi(\lambda, \mu, \gamma) = 0$$

is given by

$$F = 0, \varphi = 0, \frac{F_\lambda}{\varphi_\lambda} = \frac{F_\mu}{\varphi_\mu} = \frac{F_\gamma}{\varphi_\gamma}$$

Example. Find the envelope of the family

a) of spheres with centers on z-axis and radii are half the third coordinate of the centers.

b) of plane $4ux + 8vy - z - (2u^2 + 4v^2) = 0$

Solution.

a) $S_\lambda : x^2 + y^2 + (z - \lambda)^2 = (\frac{\lambda}{2})^2$

Differentiating every term with respect to λ one gets

$$0 + 0 - 2(z - \lambda) = \frac{1}{2}\lambda \Longrightarrow \lambda = 4z/3$$
$$x^2 + y^2 + (z - \frac{4}{3}z)^2 = (\frac{2}{3})^2 = z^2$$
$$x^2 + y^2 - \frac{z^2}{3} = 0 \text{(a cone)}$$

b) Diferentiating every term with respect to u and v one gets

$$\begin{aligned} & 4x - 4u = 0, \qquad && 8y - 8v = 0 \\ \Longrightarrow\ & u = x, && v = y \end{aligned}$$