$$c_1u_1 + \cdots + c_nu_n = 0, (c_i \epsilon \mathbb{R})$$

one has

$$c_1(u_n, \cdots, u_{1n}) + \cdots + c_n(u_{n1}, \cdots, u_{nn}) = 0$$

or

$$(c_1u_{11} + \cdots + c_nu_{n1}, \cdots, c_nu_{11} + \cdots + c_nu_{nn}) = 0$$

implying the homogeneous square system

$$\begin{array}{lclcl} u_{11}c_1 & + \cdots + & u_{n1}c_n & = & 0 \\ \vdots & & \vdots & & \vdots \\ u_{1n}c_n1 & + \cdots + & u_{nn}c_n & = & 0 \end{array}$$

of linear equations in the unknowns $c_1, \cdots, c_n$ of which the determinant is $D = det|u_{ij}|$.

If $D \neq 0$, the system admits solution other tfian the trivial one, meaning that not all c's are zero, and the vectors are linearly dependent.

2. $\mathbb{R}^n$ contains the unit vectors

$$e_1 = (1, 0, \cdots, 0), \cdots, e_n = (0, \cdots, 0, 1)$$

which are linearly independent since $det|u_{ij}|$ is $|I_n| = 1 \neq 0$.

3. Let $u_1, \cdots, u_n, u_{n+1}$ be non zero vectors in $\mathbb{R}^n$.

The theorem is proved if n of them, say $u_1, \cdots, u_n$ are linearly dependent, in which case $u_1, \cdots, u_n, u_{n+1}$ are linearly dependent.

Let the $u_1, \cdots, u_n$ be linearly independent. It will then suffice to prove that $u = u_{n+1}$ is a linear combination of $u_1, \cdots, u_n$:

$$c_1u_1 + \cdots + c_nu_n = u$$