IÔS FORMULA, KOLMOGOROV BACKWARD EQUATION AND BROWNIAN MOTIONS ON MARTINGALES

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Abstract. We prove that Kolmogorov Backward equation (KBE) combined with Infinitesimal Generator is able to construct martingales. Then we show a property of constructed martingale. Compared with the normal method, we show this property is a convenient tool to determine martingales through several examples.

1. Iôs Formula and Kolmogorov Backward equation

In this paper, we always define that $f : \star \to \mathbb{R}$ is in the C^2 space w.r.t any special illustration and P_t is a Markov Operator of a Markov Process X_t.

Meanwhile, we also need to mention that all the Brownian motions B_t in the paper satisfy two properties: $B_0 = 0$, $EB_t = 0$ and $EB_sB_t = s \wedge t$.

For a Markov Process X_t, we can construct right continuous σ-fields in this way: Define σ-field $\mathcal{F}_s^0 = \sigma(X_r : r \leqslant s)$, $\mathcal{F}_s^+ = \cap_{t>s}\mathcal{F}_t^0$. Then, we know $\mathcal{F}_s^+$ are right continuous. Right continuous σ-fields are useful to prove Brownian Motion's martingale property. We show this in the section 3.

Since we discussed Infinitesimal Generator quite well in the class, we omit the definition here. In this paper, we denote Infinitesimal Generator as

$$Af(x) = \lim_{h\to 0} \frac{E_x(f(X_h)) - f(x)}{h}, x \in \mathbb{R}, \tag{1}$$

corresponding to a Markov Process X_t.

In this paper, we also denote diffusions as X_t with drift $\mu(t, X_t)$ and diffusion coefficient $\sigma(t, X_t)$. We can denote it in this way since by the definition, a Markov Process can also be considered as a diffusion.

Theorem 1.1: *Suppose a function $f(t, x)$. The Infinitesimal Generators based on diffusion X_t can be found in this way:*

$$Af = \mu\frac{\partial f}{\partial x}(x) + \sigma^2\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x) \tag{2}$$

Proof: We have the properties of diffusion:

$$\begin{aligned} E[X_{t+h} - X_t|X_t = x] &= \mu(t, X_t)h + o(h) \\ E[(X_{t+h} - X_t|X_t = x)^2] &= \sigma^2(t, X_t)h + o(h). \end{aligned} \tag{3}$$

Since f is in C^2 space, by Taylor expansion, equations (3) and fixing t, we have

$$E[f(X_{t+h}) - f(X_t)|X_t = x] = E[\partial_x f(X_t)(X_{t+h} - X_t) + \frac{1}{2}\partial_x^2 f(X_t)(X_{t+h} - X_t)^2 + o((X_{t+h} - X_t)^2)|X_t] = \partial_x f(X_t)(\mu(t, X_t)h + o(h)) + \frac{1}{2}\partial_x^2 f(X_t)(\sigma^2(t, X_t)h + o(h)) + o(h). \tag{4}$$

Then we divide h on both side and let $h \to 0$, we have

$$Af(X_t) = \lim_{h\to 0} \frac{E[f(X_{t+h}) - f(X_t)|X_t = x]}{h} = \mu\frac{\partial f}{\partial x}(X_t) + \sigma^2 \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X_t).\blacksquare \tag{5}$$

Since f is defined in C^2 space, combined with Iôs Lemma ([5]) and Theorem 1.1, we have Iôs Formula in this way:

$$f(t, X_t) = f(0, X_0) + \int_0^t \partial_x f(s, X_s)dX_s + \int_0^t (\partial_t f + Af)(s, X_s)ds. \tag{6}$$

Theorem 1.2 (Kolmogorov Backward equation): *X_t is a Markov Process with Infinitesimal Generator A. $f(t, X_t)$ is a twice differential function. We have*

$$\frac{\partial f}{\partial t} + A(f) = 0 \tag{7}$$

Proof: Set $X_t = x$ then we have $P_t f(x) = f(t, X_t) = f(t, x)$. By the Tower Property, we have $f(t, x) = E_x[f(t + dt, X_{t+dt})]$. By Iôs Formula, we know that if dt is extremely small, we have

$$E_x[f(t + dt, X_{t+dt})] = E_x[f(t, x) + + \int_t^{t+dt} (\partial_t f + Af)(t, x)dt] = f(t, x) + E_x[\int_t^{t+dt} \partial_x f(t, X_s)dX_s] + (\partial_t f + Af)(t, x)dt. \tag{8}$$

Therefore, we are able to know

$$0 = -E_x[\int_t^{t+dt} \partial_x f(t, X_s)dX_s]/dt = (\partial_t f + Af)(t, x).\blacksquare \tag{9}$$

2. Iôs Formula and Martingales

Theorem 2.1: *Suppose $X(t)$ is a Markov Process with operator P_t and Infinitesimal Generator A. Then, $M(t)$ is a martingale defined as*

$$M(t) = f(t, X_t) - \int_0^t (\frac{\partial(f)}{\partial t} + Af)(s, X_t)ds \tag{10}$$

Proof: $f(0,X_0)+\int_0^t \partial_x f(s,X_s)dX_s$ is a martingale since

$$E[f(0,X_0)+\int_0^t \partial_x f(s,X_s)dX_s|\mathcal{F}_s^+] = f(0,X_0)+E[\int_0^t \partial_x f(s,X_s)dX_s|\mathcal{F}_s^+] \tag{11}$$
$$\text{(right continuous } \sigma\text{-field)} \Rightarrow = f(0,X_0)+\int_0^s \partial_x f(s,X_s)dX_s.$$

Thus, for Iôs Formula (6), if we move $\int_0^t(\partial_t f + Af)(s,X_s)ds$ to the left hind side, we have $f(t,X_t)-\int_0^t(\partial_t f+Af)(s,X_s)ds$ is a martingale. ■

Remark It also provides an interesting connection between the theorem and KBE. Since we know $f(0,X_0)+\int_0^t \partial_x f(s,X_s)dX_s$ is a martingale, if $\partial_t f + Af = 0$ (KBE), then $f(t,X_t) = f(0,X_0)+\int_0^t \partial_x f(s,X_s)dX_s$ is always a martingale.

It is also trivial to see:

Corollary 2.2: *Suppose f is a function w.r.t time t, which means it only depends on X_t. Then, $M(t)$ is a martingale defined as*

$$M(t) = f(X_t) - \int_0^t (Af)(s,X_t)ds \tag{12}$$

Example 1: We have a Poisson process $N(t)$. Then, $N(t)-\lambda t$ is a martingale.

Proof: Set $f(N(t)) = N(t)$. Corresponding to $N(t)$, $Af = \lambda(f(x+1)-f(x))$. By applying Corollary 2.2, we have a martingale $M(t) = N(t) - \int_0^t \lambda(N(t)+1-N(t))ds = N(t)-\lambda t$.

Example 2: Brownian Motion B_t is a martingale.

Proof: Set $f(B_t) = B_t$. Corresponding to Brownian Motion, we have a martingale $M(t) = B_t - \int_0^t 1/2 f''(B_t)ds = B_t$.

Example 3: $B_t^2 - t$ is a martingale.

Proof: Set $f(B_t) = B_t^2$. Then, we have a martingale $M(t) = B_t^2 - \int_0^t 1/2f''(B_t)ds = B_t^2 - \int_0^t (1/2)\cdot 2ds = B_t^2 - t$.

3. Brownian Motions

From Example 2 in the last section, we can trivially see that Brownian Motion is a martingale. However, here we will roughly prove this statement by checking its martingale property. By this proof, we can see how convenience Theorem 2.1 is to find a martingale.

Alternative proof of that Brownian Motion is a martingale:

By the definition of Brownian Motion, we have $E_y B_t = y, \forall t \geqslant 0$.

Combining Doob's Optional Stopping Time Theorem and Markov property, we have

$$E_x[B_t|\mathcal{F}_s^+] = E_{B_s}[B_{t-s}] = B_s, \tag{13}$$

which implies martingale property.

Compared with proof in section 2, we can see that in this way, we have to construct σ-fields firstly and to prove that martingales with right continuous martingale satisfy $EX_T = EX_0$, T is a bounded stopping time by using Doob's Optional Stopping Time Theorem.

We also introduce another more powerful tool to check the martingale property of a stochastic process based on Brownian Motion.

Theorem 3.1: *If $u(t,x)$, a twice differentiable function in t,x satisfy*

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{2\partial x^2} = 0, \tag{14}$$

then $u(t,B_t)$ is martingale

Proof: From previous proof, we know that when f satisfy KBE, then

$$f(t,Bt) = f(0,0) + \int_0^t f_x(s,B_s)dB_s \tag{15}$$

is martingale. Since corresponding to Brownian Motion,

$$Af = \frac{1}{2}f'' \Rightarrow \frac{\partial u}{\partial t} + \frac{\partial^2 u}{2\partial x^2} = 0 \tag{16}$$

according to Theorem 1.2. ■

Example 1: $\exp(\theta B_t - \frac{\theta^2 t}{2})$ is a martingale.

Proof I: By Theorem 3.1, suppose $u(t,B_t) = \exp(\theta B_t - \frac{\theta^2 t}{2})$. Then, we can finish the proof by following calculation:

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{2\partial x^2} = -\frac{\theta^2}{2}\exp(\theta x - \frac{\theta^2 t}{2}) + \frac{\theta^2}{2}\exp(\theta x - \frac{\theta^2 t}{2}) = 0. \tag{17}$$

Proof II: Actually, we still can prove the statement by normally checking the martingale property. Since $B_t - B_s$ is independent of $\mathcal{F}_s$, and

$$\begin{aligned} E[\exp(\theta B_t)] &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)\exp(\theta\sqrt{t}x)dx \\ &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp(-(x-\theta\sqrt{t})^2/2)\exp(\theta^2 t)dx = \exp(\theta^2 t). \end{aligned} \tag{18}$$

We have

$$\begin{aligned} E_x[\exp(\theta B_t)|\mathcal{F}_s] &= \exp(\theta B_s)E[\exp(\theta(B_t - B_s))|\mathcal{F}_s] \\ &= \exp(\theta B_s)\exp(\theta^2(t-s)). \end{aligned} \tag{19}$$

Then, we can prove its martingale property.

Example 2: $B_t^2 - t$ is a martingale.

We have proved this statement in section 2. If we use Theorem 3.1 to prove, the procedure will be easier. Suppose $u(t,x) = x^2 - t$ and then it is easy to see that $\frac{\partial u}{\partial t} + \frac{\partial^2 u}{2\partial x^2} = 0$.

4. Conclusion and discussion

We briefly introduce the most obvious connection among Kolmogrov Backward equation, Iôs formula and martingales. Specifically, we discuss the applications of those powerful tools to find a martingale based on Brownian Motions.

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