INTRODUCTION TO RANDOM WALK AND PÓLYA'S RECURRENCE THEOREM

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Abstract. We introduce the basic definition of random walk, a branch in probability theory. The definitions of recurrence and transience are given, which are the preliminaries of Pólya's Recurrence Theorem. We present two proofs of Pólya's Recurrence Theorem. One is based on a pure combinatorial enumeration of walks. The other one is based on a Fourier transformation computation of the Green function.

1. Introduction

The term **random walk** was first introduced by Karl Pearson in 1905. A random walk is a mathematical object which describes a path that consists of a succession of random steps. For example, the path traced by a molecule as it travels in a liquid or a gas, the search path of a foraging animal and the price of a fluctuating stock can all be approximated by random walk models, even though they may not be truly random in reality. As illustrated by those examples, random walks have applications to many scientific fields including ecology, psychology, computer science, physics, chemistry, biology, and economics. Random walks explain the observed behaviors of many processes in these fields, and thus serve as a fundamental model for the recorded stochastic activity.

2. Preliminaries

Notations:

- $X_1, X_2, ..., X_n, ...$ are i.i.d taking values in $\mathbb{R}^d$. d here is the dimension of the space. We define that the measure of $X_i, i = 1, 2, ...$ is μ.
- $\Omega = \{(w_1, w_2, ..., w_d)\}$ is our "event space".
- $P = \mu \times \mu \times ... \times \mu \times ...$ where μ is the measure of X_i.

Definition 2.1: (Random Walk) *Let $S_n = X_1 + X_2 + ... + X_n$. We call S_n is a* ***random walk****.*

Since in this paper we only consider simple random walk which will be introduced next, we suppose X_i takes value in $\mathbb{Z}^d$ in this paper for the convenience of statement. In this way, S_n also takes value in $\mathbb{Z}^d$.

We say a random walk S_n is a **simple random walk** if each step taken has equal probability of going to the its neighbors. For instance, when $d = 1$, mathematically if $P(X_i = 1) = P(X_i = -1) = 1/2$, S_n is a simple random walk and S_n is defined on $\mathbb{Z}$. When $d = 3$, then there are 6 options for next steps and the probability of each option is $1/6$.

In order to avoid unnecessary confusion, the random walks in the rest of this paper are all considered as simple random walk. For the purpose of simplification, We also suppose the random walks in this paper are starting at the origin, which means $S_0 = 0$.

Definition 2.2 (Recurrence & Transience): *A simple random walk $\{Sn : n \geqslant 0\}$ on $\mathbb{Z}^d$ is said to be* ***recurrent*** *if almost surely it visits every value in $\mathbb{Z}^d$ infinitely many times i.e.*

$$P(S_n = x, x \in \mathbb{Z}_d, i.o.) = 1 \tag{1}$$

If this is not the case then we say the walk is ***transient****.*

According to the definition, it suffices to show a random walk is recurrent by showing that almost all paths return to the origin for infinitely many n. Meanwhile, it also suffices to show a random walk is transient by showing that there is a positive probability that the path never returns to the origin.

3. Pólya's Recurrence Theorem and proof

We firstly give the statement of Pólya's Recurrence Theorem. The proof I is based on enumeration while the proof II is based on analysis.

Theorem 3.1 (Pólya's Recurrence Theorem): *a simple random walk on a d-dimensional space is recurrent for d = 1, 2 and transient for $d > 2$.*

Proof I: Let $p_n^{(d)} = P(S_n = 0)$. Then it is trivial to see that n must be an even number. Thus, suppose $p_{2n}^{(d)} = P(S_{2n} = 0)$. By Borel-Cantelli lemma, $\sum_{n=0}^{\infty} p_{2n}^{(d)} = \infty$ implies $P(S_{2n} = 0, i.o.) = 1$.

Lemma 3.2 (Stirling Formula): $n! \sim n^n e^{-n}\sqrt{2\pi n}$ **as** $n \to \infty$.

Stirling formula is a useful common tool in mathematic. The exact proof is presented in reference [2].

When $d = 1$, if it is able to return to 0, S_{2n} contains equal steps of "right" and "left". By enumeration of those steps, we know that

$$p_{2n}^{(1)} = \frac{1}{2^{2n}}\binom{2n}{n} = \frac{1}{2^{2n}}\frac{2n!}{n!n!} \sim \frac{1}{2^{2n}}\frac{(2n)^{2n}e^{-2n}\sqrt{4\pi n}}{n^n e^{-n}\sqrt{2\pi n} n^n e^{-n}\sqrt{2\pi n}} = (\pi n)^{-1/2}. \tag{2}$$

Then, $\sum_{n=0}^{\infty} p_{2n}^{(1)} = \infty$ so that S_{2n} is recurrent when $d = 1$.

When $d = 2$, we must have m "up" steps, m "down" steps for some $0 \leqslant m \leqslant 2n$ and $2n - m$ "left" steps and $2n - m$ "right" steps. Hence, we know that

$$p_{2n}^{(2)} = 4^{-2n}\sum_{m=0}^{n}\frac{2n!}{m!m!(n-m)!(n-m)!} = 4^{-2n}\sum_{m=0}^{n}\binom{n}{m}\binom{n}{n-m} = 4^{-2n}\binom{2n}{n}^2 \tag{3}$$

We can see that $p_{2n}^{(2)} = (p_{2n}^{(1)})^2$ which implies $p_{2n}^{(2)} \sim (\pi n)^{-1}$. And we know that $\sum_{n=1}^{\infty} p_{2n}^{(2)} = \infty \Rightarrow S_{2n}$ is recurrent when $d = 2$.

When $d = 3$, since the process in a sense is similar as above, we can write the formula directly. Suppose $0 \leqslant j, k \leqslant n$ and $j + k \leqslant n$. Then,

(4)
$$p_{2n}^{(3)} = 6^{-2n} \sum_{j,k} \frac{2n!}{(j!k!(n-j-k)!)^2} = 2^{-2n} \binom{2n}{n} \sum_{j,k} (3^{-n} \frac{n!}{(j!k!(n-j-k)!)^2})^2.$$
$$\sum_{j,k} \frac{n!}{(j!k!(n-j-k)!)^2} = 3^n \Rightarrow \sum_{j,k} (3^{-n} \frac{n!}{(j!k!(n-j-k)!)^2}) = 1$$
$$\sum_{j,k} (3^{-n} \frac{n!}{(j!k!(n-j-k)!)^2})^2 \leqslant \max_{j,k} 3^{-n} \frac{n!}{(j!k!(n-j-k)!)}$$

The last inequality is based on Cauchy Schwarz Inequality. Then, by Stirling Formula, there exists some constant C determined by n such that

$$\frac{n!}{(j!k!(n-j-k)!)} \sim \frac{1}{2\pi} \frac{n^n}{j^j k^k (n-j-k)^{n-j-k}} \sqrt{\frac{n}{jk(n-j-k)}} \leqslant C3^n \tag{5}$$

Generalize above results, we have $p_{2n}^{(3)} \sim Cn^{-3/2}$ so that $\sum_{n=1}^{\infty} p_{2n}^{(3)}$ converges so that in this case S_{2n} is not recurrent.

Proof II: Suppose $\xi \in \mathbb{R}^d$ and $\xi = (\tilde{\xi_1}, \tilde{\xi_2}, ..., \tilde{\xi_d})$. Let $\xi_1 = (\tilde{\xi_1}, 0, ..., 0), \xi_2 = (0, \tilde{\xi_2}, ..., 0), ..., \xi_d = (0, 0, ..., \tilde{\xi_d})$. Then we know that $\xi = \sum_{j=1}^{d} \xi_j$.

Different from the 1st proof, we start from the characteristics function. Suppose $\varphi(\xi) = \sum_{k \in \mathbb{Z}^d} P(\{w : X_i = k\}) e^{-2\pi i k\xi}$ the characteristics function of $X_i (i = 1, 2, ...)$. $\xi = (\xi_1, \xi_2, ..., \xi_d)$. Since we just consider simple random walk, $k \in \{(\pm1, 0, ..., 0), (0, \pm1, ..., 0), ..., (0, 0, ..., \pm1)\}$ and for each k, $P(X_i = k) = \frac{1}{2d}$.

Hence, we can deduce that

$$\begin{aligned} \varphi(\xi) &= \sum_{k \in \mathbb{Z}^d} P(\{w : X_i = k\}) e^{-2\pi i k\xi} = \frac{1}{2d} \sum_{k} \{\cos(2\pi k\xi) - i \sin(2\pi k\xi)\} \\ &= \frac{1}{2d} (2\cos(2\pi\xi_1) + 2\cos(2\pi\xi_2) + ... + 2\cos(2\pi\xi_d)). \end{aligned} \tag{6}$$

Hence, $\varphi(\xi)$ is periodic with periods $e_1 = (1, 0, ..., 0), e_2 = (0, 1, ..., 0), ..., e_d = (0, 0, ..., 1)$.

Corresponding to S_n, $\varphi_n(\xi) = \sum_{k \in \mathbb{Z}^d} P(\{w : S_n = k\}) e^{-2\pi i k\xi}$ is the characteristics function of S_n. Since $S_n = \sum_{i=1}^{n} X_i$ and X_i are i.i.d., from the properties of characteristics function, we know that $\varphi_n(\xi) = \varphi(\xi)^n$. Thus, $\varphi_n(\xi)$ is also periodic with periods $e_1, e_2, ..., e_d$.

Then, we know that

$$P(\{w : S_n = 0\}) = \int_Q \varphi_n(\xi) d\xi = \int_Q \varphi(\xi)^n d\xi, \tag{7}$$

where Q is a cube defined by $Q = \{\xi : -1/2 \leqslant \tilde{\xi_j} \leqslant 1/2, j = 1, 2, ..., d\}$.

If we multiply both sides of equation (7) by $r^n, 0 < r < 1$ and get the sum then we have

$$\begin{aligned}\sum_{n=0}^{\infty} r^n P(\{w : S_n = 0\}) &= \sum_{n=0}^{\infty} \int_Q (r\varphi(\xi))^n d\xi \\ &= \int_Q \sum_{n=0}^{\infty} (r\varphi(\xi))^n d\xi = \int_Q \frac{1}{1 - r\varphi(\xi)} d\xi.\end{aligned} \tag{8}$$

and let $r \to 1$. Then, it yields

$$\sum_{n=0}^{\infty} P(\{w : S_n = 0\}) = \int_Q \frac{1}{1 - \varphi(\xi)} d\xi. \tag{9}$$

Firstly, we notice that $\int_Q \frac{1}{1-\varphi(\xi)} d\xi$ is always non-negative (or ∞) since $\varphi(\xi)$ is a characteristics function so that it is always less than 1. Secondly, according to our deduction, $\sum_{n=0}^{\infty} P(\{w : S_n = 0\})$ and $\int_Q \frac{1}{1-\varphi(\xi)} d\xi$ are simultaneously infinite or finite and equal.

Now, by Taylor expansion, we know

$$\begin{aligned}1 - \varphi(\xi) &= 1 - \frac{1}{d}[\cos(2\pi\xi_1) + \cos(2\pi\xi_2) + ... + \cos(2\pi\xi_d)] \\ &= 1 - \frac{1}{d}[1 - \frac{(2\pi\xi_1)^2}{2!} + O(\xi_1^4) + ... + 1 - \frac{(2\pi\xi_d)^2}{2!} + O(\xi_d^4)] \\ &= \frac{2\pi^2}{d}|\xi|^2 + O(|\xi|^4), \text{as} \xi \to 0.\end{aligned} \tag{10}$$

We also have $1 - \varphi(\xi) \geqslant c_1$ if $|\xi| \geqslant c_2$ for some suitable c_1 and c_2.

Thus, based on above analysis of $1 - \varphi(\xi)$ around origin, we can conclude that $\int_Q \frac{1}{1-\varphi(\xi)} d\xi$ diverges when $d \leqslant 2$ and converges when $d \geqslant 3$.

Finally, in the same way of 1st proof, we can deduce the recurrence or transience according to the convergence or divergence of $\sum_{n=0}^{\infty} P(\{w : S_n = 0\})$.

4. Conclusion and discussion

Pólya's Recurrence Theorem is important and simple for us to determine whether a simple random walk is recurrent or transient. It has many applications in the study of probability theory. For example, when we study Brownian Motion, we can estimate the recurrence of the motion by Pólya's Recurrence Theorem directly.

On the other hand, there are several ways to prove Pólya's Recurrence Theorem. For example, besides of two proofs in this paper, there is an interesting way to prove the theorem by constructing an unit flow of finite energy from origin to infinity ([3, 4]). [1] [5] [6]

References

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[6] Wikipedia. Random walk — wikipedia, the free encyclopedia, 2016.