Marker Code Trace Reconstruction

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1 Forward-Backward Algorithm

1.1 Forward Messages

$$\begin{aligned} f^M(i,j) &= P(s_1...s_i, \pi^M(i,j)|\mathbf{t}) \\ f^I\ (i,j) &= P(s_1...s_i, \pi^I\ (i,j)|\mathbf{t}) \\ f^D\ (i,j) &= P(s_1...s_i, \pi^D\ (i,j)|\mathbf{t}) \end{aligned} \tag{1}$$

$$\begin{aligned} f^M(i,j) &= P(s_1...s_i, \pi^M(i,j)|\mathbf{t}) = \sum\nolimits_{\pi_{prev}} P(s_1...s_i, \pi^M(i,j), \pi_{prev}|\mathbf{t}) \\ &= P(s_1...s_i, \pi^M(i,j), \pi^M(i-1,j-1)|\mathbf{t}) + P(s_1...s_i, \pi^M(i,j), \pi^I(i-1,j-1)|\mathbf{t}) \\ &\quad + P(s_1...s_i, \pi^M(i,j), \pi^D(i-1,j-1)|\mathbf{t}) \end{aligned} \tag{2}$$

First, we examine the first term of (2)

$$\begin{aligned} &P(s_1...s_i, \pi^M(i,j), \pi^M(i-1,j-1)|\mathbf{t}) \\ &= P(s_1...s_{i-1}|s_i, \pi^M(i,j), \pi^M(i-1,j-1)\mathbf{t}) \times P(s_i|\pi^M(i-1,j-1), \pi^M(i,j), \mathbf{t}) \\ &\quad \times P(\pi^M(i,j)|\pi^M(i-1,j-1), \mathbf{t}) \times P(\pi^M(i-1,j-1)|\mathbf{t}) \\ &= P(s_1...s_{i-1}|\pi^M(i-1,j-1)\mathbf{t}) \times P(s_i|\pi^M(i,j), \mathbf{t}) \times P(\pi^M(i,j)|\pi^M(i-1,j-1)) \\ &\quad \times P(\pi^M(i-1,j-1)|\mathbf{t}) \\ &= P(s_1...s_{i-1}|\pi^M(i-1,j-1)\mathbf{t}) \times P(\pi^M(i-1,j-1)|\mathbf{t}) \times P(\pi^M(i,j)|\pi^M(i-1,j-1)) \\ &\quad \times P(s_i|\pi^M(i,j), \mathbf{t}) \\ &= P(s_1...s_{i-1}, \pi^M(i-1,j-1)|\mathbf{t}) \times P(\pi^M(i,j)|\pi^M(i-1,j-1)) \times\ P(s_i|\pi^M(i,j), \mathbf{t}) \\ &= f^M(i-1,j-1) \times T_{MM} \times Memission[j](s_i) \end{aligned} \tag{3}$$

Similar procedures can be applied to the second and the third term of (2)

$$P(\pi^I(i-1,j-1), \pi^M(i,j), s_1...s_i|\mathbf{t}) = f^I(i-1,j-1) \times T_{IM} \times Memission[j](s_i) \tag{4}$$

$$P(\pi^D(i-1,j-1), \pi^M(i,j), s_1...s_i|\mathbf{t}) = f^D(i-1,j-1) \times T_{DM} \times Memission[j](s_i) \tag{5}$$

Put (3) (4) and (5) into (2), and we can obtain the recursion of f^M

$$f^M(i,j) = (f^M(i-1,j-1) \times T_{MM} + f^I(i-1,j-1) \times T_{IM} + f^D(i-1,j-1) \times T_{DM}) \times Memission[j](s_i) \tag{6}$$

Similarly, we can obtain the recursion formula of f^I and f^D

$$\begin{aligned} f^I\ (i,j) &= \sum\nolimits_{\pi_{prev}} P(s_1...s_i, \pi^I(i,j), \pi_{prev}|\mathbf{t}) \\ &= (f^M(i-1,j) \times T_{MI} + f^I\ \ (i-1,j) \times T_{II} + f^D(i-1,j) \times T_{DI}) \times Iemission(s_i) \end{aligned} \tag{7}$$

$$\begin{aligned} f^D(i,j) &= \sum\nolimits_{\pi_{prev}} P(s_1...s_i, \pi^I(i,j), \pi_{prev}|\mathbf{t}) \\ &= f^M(i,j-1) \times T_{MD} + f^I(i,j-1) \times T_{ID} + f^D(i,j-1) \times T_{DD} \end{aligned} \tag{8}$$

1.2 Backward Messages

$$\begin{aligned} b^M(i,j) &= P(s_{i+1}...s_M|\pi^M(i,j), \mathbf{t}) \\ b^I\ (i,j) &= P(s_{i+1}...s_M|\pi^I\ (i,j), \mathbf{t}) \\ b^D\ (i,j) &= P(s_{i+1}...s_M|\pi^D\ (i,j), \mathbf{t}) \end{aligned} \tag{9}$$

$$\begin{aligned} b^M(i,j) &= P(s_{i+1}...s_M|\pi^M(i,j), \mathbf{t}) = \sum\nolimits_{\pi_{next}} P(s_{i+1}...s_M, \pi_{next}|\pi^M(i,j), \mathbf{t}) \\ &= P(s_{i+1}...s_M, \pi^M(i+1,j+1)|\pi^M(i,j), \mathbf{t}) + P(s_{i+1}...s_M, \pi^I(i+1,j)|\pi^M(i,j), \mathbf{t}) \\ &\quad + P(s_{i+1}...s_M, \pi^D(i,j+1))|\pi^M(i,j), \mathbf{t}) \end{aligned} \tag{10}$$

Again, $b^M(i,j)$ is the sum of three terms which correspond to three possible next states. Examine the first term of (10).

$$\begin{aligned}
&P(s_{i+1}...s_M, \pi^M(i+1,j+1)|\pi^M(i,j),\mathbf{t}) \\
&\quad = P(s_{i+2}...s_M|s_{i+1},\pi^M(i+1,j+1),\pi^M(i,j),\mathbf{t}) \ \times P(s_{i+1},\pi^M(i+1,j+1)|\pi^M(i,j),\mathbf{t}) \\
&\quad = P(s_{i+2}...s_M|s_{i+1},\pi^M(i+1,j+1),\pi^M(i,j),\mathbf{t}) \ \times P(s_{i+1}|\pi^M(i+1,j+1)\pi^M(i,j),\mathbf{t}) \\
&\qquad \times P(\pi^M(i+1,j+1)|\pi^M(i,j),\mathbf{t}) \\
&\quad = P(s_{i+2}...s_M|\pi^M(i+1,j+1),\mathbf{t}) \ \times P(s_{i+1}|\pi^M(i+1,j+1),\mathbf{t}) \ \times P(\pi^M(i+1,j+1)|\pi^M(i,j)) \\
&\quad = b^M(i+1,j+1) \times Memission[j+1](s_{i+1}) \times T_{MM}
\end{aligned} \tag{11}$$

Similar procedures can be applied to the second and the third term of (10)

$$P(s_{i+1}...s_M, \pi^I\ (i+1,j)|\pi^M(i,j),\mathbf{t}) \ = b^I\ (i+1,j) \times Iemission(s_{i+1}) \times T_{MI} \tag{12}$$
$$P(s_{i+1}...s_M, \pi^D(i,j+1))|\pi^M(i,j),\mathbf{t}) \ = b^D(i,j+1) \times T_{MD} \tag{13}$$

Put (11) (12) and (13) into (10), and we can obtain the recursion of b^M

$$\begin{aligned}
b^M(i,j) \ &= b^M(i+1,j+1) \times Memission[j+1](s_{i+1}) \times T_{MM} \ + b^I(i+1,j) \times Iemission(s_{i+1}) \times T_{MI} \\
&\quad + b^D(i,j+1) \times T_{MD}
\end{aligned} \tag{14}$$

Similarly, we can obtain the recursion formula of b^I and b^D

$$\begin{aligned}
b^I\ (i,j) \ &= b^M(i+1,j+1) \times Memission[j+1](s_{i+1}) \times T_{IM} \ + b^I\ (i+1,j) \times Iemission(s_{i+1}) \times T_{II} \\
&\quad + b^D(i,j+1) \times T_{ID}
\end{aligned} \tag{15}$$

$$\begin{aligned}
b^D(i,j) \ &= b^M(i+1,j+1) \times Memission[j+1](s_{i+1}) \times T_{DM} \ + b^I\ (i+1,j) \times Iemission(s_{i+1}) \times T_{DI} \\
&\quad + b^D(i,j+1) \times T_{DD}
\end{aligned} \tag{16}$$

1.3 Scaling Factors

$$\begin{aligned}
\widehat{f}^M(i,j) \ &= \ P(\pi^M(i,j)|s_1...s_i,\mathbf{t}) = \frac{f^M(i,j)}{p(s_1...s_i|\mathbf{t})} \\
\widehat{f}^I\ \ (i,j) \ &= \ P(\pi^I\ \ (i,j)|s_1...s_i,\mathbf{t}) = \frac{f^I(i,j)}{p(s_1...s_i|\mathbf{t})} \\
\widehat{f}^D\ (i,j) \ &= \ P(\pi^D\ \ (i,j)|s_1...s_i,\mathbf{t}) = \frac{f^D(i,j)}{p(s_1...s_i|\mathbf{t})}
\end{aligned} \tag{17}$$

$$\begin{aligned}
\widehat{b}^M(i,j) \ &= \ \frac{P(s_{i+1}...s_M|\pi^M(i,j),\mathbf{t})}{p(s_{i+1}...s_M|s_1...s_i,\mathbf{t})} = \frac{b^M(i,j)}{p(s_{i+1}...s_M|s_1...s_i,\mathbf{t})} \\
\widehat{b}^I\ \ (i,j) \ &= \ \frac{P(s_{i+1}...s_M|\pi^I(i,j),\mathbf{t})}{p(s_{i+1}...s_M|s_1...s_i,\mathbf{t})} = \frac{b^I(i,j)}{p(s_{i+1}...s_M|s_1...s_i,\mathbf{t})} \\
\widehat{b}^D\ (i,j) \ &= \ \frac{P(s_{i+1}...s_M|\pi^D(i,j),\mathbf{t})}{p(s_{i+1}...s_M|s_1...s_i,\mathbf{t})} = \frac{b^D(i,j)}{p(s_{i+1}...s_M|s_1...s_i,\mathbf{t})}
\end{aligned} \tag{18}$$

$$c_i = P(s_i|s_1...s_{i-1},\mathbf{t}) \tag{19}$$
$$P(s_1...s_m|\mathbf{t}) = \prod\nolimits_{i=1}^{m} c_i \tag{20}$$

$$\begin{aligned}
c_i\widehat{f}^M(i,j) &= (\widehat{f}^M(i-1,j-1) \times T_{MM} + \widehat{f}^I\ \ (i-1,j-1) \times T_{IM} + \widehat{f}^D\ (i-1,j-1) \times T_{DM}) \\
&\qquad \times Memission[j](s_i)
\end{aligned} \tag{21}$$
$$c_i\widehat{f}^I\ \ (i,j) = (\widehat{f}^M(i-1,j) \times T_{MI} + \widehat{f}^I\ \ (i-1,j) \times T_{II} + \widehat{f}^D\ (i-1,j) \times T_{DI}) \times Iemission(s_i) \tag{22}$$
$$\widehat{f}^D\ (i,j) = \widehat{f}^M((i,j-1) \times T_{MD} + \widehat{f}^I\ \ ((i,j-1) \times T_{ID} + \widehat{f}^D\ ((i,j-1) \times T_{DD} \tag{23}$$

$$\sum\nolimits_{j=0}^{N} \left(\pi^M(i,j) + \pi^I\ (i,j)\right) = 1 \tag{24}$$

$$\sum\nolimits_{j=0}^{N}\left(\widehat{f}^{M}(i,j)+\widehat{f}^{I}\ (i,j)\right)=1 \tag{25}$$

$$c_i=\sum\nolimits_{j=0}^{N}\left(\widetilde{f}^{M}(i,j)+\widetilde{f}^{I}\ (i,j)\right) \tag{26}$$

$$\begin{aligned}
c_{i+1}\widehat{b}^{M}(i,j) &= \widehat{b}^{M}(i+1,j+1)\times Memission[j+1](s_{i+1})\ \times T_{MM}+\widehat{b}^{I}(i+1,j)\times Iemission(s_{i+1})\times T_{MI}\\
&\quad +c_{i+1}\times\widehat{b}^{D}(i,j+1)\times T_{MD}\\
c_{i+1}\widehat{b}^{I}\ (i,j) &= \widehat{b}^{M}(i+1,j+1)\times Memission[j+1](s_{i+1})\ \times T_{IM}+\widehat{b}^{I}(i+1,j)\times Iemission(s_{i+1})\times T_{II}\\
&\quad +c_{i+1}\times\widehat{b}^{D}(i,j+1)\times T_{ID}\\
c_{i+1}\widehat{b}^{D}\ (i,j) &= \widehat{b}^{M}(i+1,j+1)\times Memission[j+1](s_{i+1})\ \times T_{DM}+\widehat{b}^{I}(i+1,j)\times Iemission(s_{i+1})\times T_{DI}\\
&\quad +c_{i+1}\times\widehat{b}^{D}(i,j+1)\times T_{DD}
\end{aligned} \tag{27}$$