
Marker Code Trace Reconstruction

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1 Forward-Backward Algorithm

1.1 Forward Messages

$$\begin{aligned} f^M(i, j) &= P(s_1 \dots s_j, \pi^M(i, j) | \mathbf{t}) \\ f^I(i, j) &= P(s_1 \dots s_j, \pi^I(i, j) | \mathbf{t}) \\ f^D(i, j) &= P(s_1 \dots s_j, \pi^D(i, j) | \mathbf{t}) \end{aligned} \quad (1)$$

$$\begin{aligned} f^M(i, j) &= P(\pi^M(i, j), s_1 \dots s_j | \mathbf{t}) = \sum_{\pi_{prev}} P(s_1 \dots s_j, \pi^M(i, j), \pi_{prev} | \mathbf{t}) \\ &= P(s_1 \dots s_j, \pi^M(i, j), \pi^M(i-1, j-1) | \mathbf{t}) + P(\pi^I(i-1, j-1), \pi^M(i, j), s_1 \dots s_j | \mathbf{t}) \\ &\quad + P(s_1 \dots s_j, \pi^M(i, j), \pi^D(i-1, j-1) | \mathbf{t}) \end{aligned} \quad (2)$$

First, we examine the first term of (2)

$$\begin{aligned} &P(\pi^M(i-1, j-1), \pi^M(i, j), s_1 \dots s_j | \mathbf{t}) \\ &= P(s_1 \dots s_{j-1} | s_j, \pi^M(i-1, j-1), \pi^M(i, j), \mathbf{t}) \times P(s_j | \pi^M(i-1, j-1), \pi^M(i, j), \mathbf{t}) \\ &\quad \times P(\pi^M(i, j) | \pi^M(i-1, j-1), \mathbf{t}) \times P(\pi^M(i-1, j-1) | \mathbf{t}) \\ &= P(s_1 \dots s_{j-1} | \pi^M(i-1, j-1) \mathbf{t}) \times P(s_j | \pi^M(i, j), \mathbf{t}) \times P(\pi^M(i, j) | \pi^M(i-1, j-1), \mathbf{t}) \\ &\quad \times P(\pi^M(i-1, j-1) | \mathbf{t}) \\ &= P(s_1 \dots s_{j-1} | \pi^M(i-1, j-1) \mathbf{t}) \times P(\pi^M(i-1, j-1) | \mathbf{t}) \times P(\pi^M(i, j) | \pi^M(i-1, j-1), \mathbf{t}) \\ &\quad \times P(s_j | \pi^M(i, j), \mathbf{t}) \\ &= P(s_1 \dots s_{j-1}, \pi^M(i-1, j-1) | \mathbf{t}) \times P(\pi^M(i, j) | \pi^M(i-1, j-1)) \times P(s_j | \pi^M(i, j), \mathbf{t}) \\ &= f^M(i-1, j-1) \times T_{MM} \times M_{emission}[i](s_j) \end{aligned} \quad (3)$$

Similar procedures can be applied to the second and the third term of (2)

$$P(\pi^I(i-1, j-1), \pi^M(i, j), s_1 \dots s_j | \mathbf{t}) = f^I(i-1, j-1) \times T_{IM} \times M_{emission}[i](s_j) \quad (4)$$

$$P(\pi^D(i-1, j-1), \pi^M(i, j), s_1 \dots s_j | \mathbf{t}) = f^D(i-1, j-1) \times T_{DM} \times M_{emission}[i](s_j) \quad (5)$$

Put (3) (4) and (5) into (2), and we can obtain the recursion of f^M

$$f^M(i, j) = (f^M(i-1, j-1) \times T_{MM} + f^I(i-1, j-1) \times T_{IM} + f^D(i-1, j-1) \times T_{DM}) \times M_{emission}[i](s_j) \quad (6)$$

Similarly, we can obtain the recursion formula of f^I and f^D

$$\begin{aligned} f^I(i, j) &= \sum_{\pi_{prev}} P(\pi_{prev}, \pi^I(i, j), s_1 \dots s_j | \mathbf{t}) \\ &= (f^M(i, j-1) \times T_{MI} + f^I(i, j-1) \times T_{II} + f^D(i, j-1) \times T_{DI}) \times I_{emission}(s_j) \end{aligned} \quad (7)$$

$$\begin{aligned} f^D(i, j) &= \sum_{\pi_{prev}} P(\pi_{prev}, \pi^D(i, j), s_1 \dots s_j | \mathbf{t}) \\ &= f^M(i-1, j) \times T_{MD} + f^I(i-1, j) \times T_{ID} + f^D(i-1, j) \times T_{DD} \end{aligned} \quad (8)$$

1.2 Backward Messages

$$\begin{aligned} b^M(i, j) &= P(s_{j+1} \dots s_N | \pi^M(i, j), \mathbf{t}) \\ b^I(i, j) &= P(s_{j+1} \dots s_N | \pi^I(i, j), \mathbf{t}) \\ b^D(i, j) &= P(s_{j+1} \dots s_N | \pi^D(i, j), \mathbf{t}) \end{aligned} \quad (9)$$

$$\begin{aligned} b^M(i, j) &= P(s_{j+1} \dots s_N | \pi^M(i, j), \mathbf{t}) = \sum_{\pi_{next}} P(s_{j+1} \dots s_N, \pi_{next} | \pi^M(i, j), \mathbf{t}) \\ &= P(s_{j+1} \dots s_N, \pi^M(i+1, j+1) | \pi^M(i, j), \mathbf{t}) + P(s_{j+1} \dots s_N, \pi^I(i, j+1) | \pi^M(i, j), \mathbf{t}) \\ &\quad + P(s_{j+1} \dots s_N, \pi^D(i+1, j)) | \pi^M(i, j), \mathbf{t}) \end{aligned} \quad (10)$$

Again, $b^M(i, j)$ is the sum of three terms which correspond to three possible next states. Examine the first term of (10).

$$\begin{aligned}
& P(s_{j+1} \dots s_N, \pi^M(i+1, j+1) | \pi^M(i, j), \mathbf{t}) \\
&= P(s_{j+2} \dots s_N | s_{j+1}, \pi^M(i+1, j+1), \pi^M(i, j), \mathbf{t}) \times P(s_{j+1}, \pi^M(i+1, j+1) | \pi^M(i, j), \mathbf{t}) \\
&= P(s_{j+2} \dots s_N | s_{j+1}, \pi^M(i+1, j+1), \pi^M(i, j), \mathbf{t}) \times P(s_{j+1} | \pi^M(i+1, j+1) \pi^M(i, j), \mathbf{t}) \\
&\quad \times P(\pi^M(i+1, j+1) | \pi^M(i, j), \mathbf{t}) \\
&= P(s_{j+2} \dots s_N | \pi^M(i+1, j+1), \mathbf{t}) \times P(s_{j+1} | \pi^M(i+1, j+1), \mathbf{t}) \times P(\pi^M(i+1, j+1) | \pi^M(i, j)) \\
&= b^M(i+1, j+1) \times Memission[i+1][s_{j+1}] \times T_{MM}
\end{aligned} \tag{11}$$

Similar procedures can be applied to the second and the third term of (10)

$$P(s_{j+1} \dots s_N, \pi^I(i, j+1) | \pi^M(i, j), \mathbf{t}) = b^I(i, j+1) \times Iemission[s_{j+1}] \times T_{MI} \tag{12}$$

$$P(s_{j+1} \dots s_N, \pi^D(i+1, j) | \pi^M(i, j), \mathbf{t}) = b^D(i+1, j) \times T_{MD} \tag{13}$$

Put (11) (12) and (13) into (10), and we can obtain the recursion of b^M

$$\begin{aligned}
b^M(i, j) &= b^M(i+1, j+1) \times Memission[i+1][s_{j+1}] \times T_{MM} + b^I(i, j+1) \times Iemission[s_{j+1}] \times T_{MI} \\
&\quad + b^D(i+1, j) \times T_{MD}
\end{aligned} \tag{14}$$

Similarly, we can obtain the recursion formula of b^I and b^D

$$\begin{aligned}
b^I(i, j) &= b^M(i+1, j+1) \times Memission[i+1][s_{j+1}] \times T_{IM} + b^I(i, j+1) \times Iemission[s_{j+1}] \times T_{II} \\
&\quad + b^D(i+1, j) \times T_{ID}
\end{aligned} \tag{15}$$

$$\begin{aligned}
b^D(i, j) &= b^M(i+1, j+1) \times Memission[i+1][s_{j+1}] \times T_{DM} + b^I(i, j+1) \times Iemission[s_{j+1}] \times T_{DI} \\
&\quad + b^D(i+1, j) \times T_{DD}
\end{aligned} \tag{16}$$

1.3 Scaling Factors

$$\begin{aligned}
\hat{f}^M(i, j) &= P(\pi^M(i, j) | s_1 \dots s_j, \mathbf{t}) = \frac{f^M(i, j)}{p(s_1 \dots s_j | \mathbf{t})} \\
\hat{f}^I(i, j) &= P(\pi^I(i, j) | s_1 \dots s_j, \mathbf{t}) = \frac{f^I(i, j)}{p(s_1 \dots s_j | \mathbf{t})} \\
\hat{f}^D(i, j) &= P(\pi^D(i, j) | s_1 \dots s_j, \mathbf{t}) = \frac{f^D(i, j)}{p(s_1 \dots s_j | \mathbf{t})}
\end{aligned} \tag{17}$$

$$\begin{aligned}
\hat{b}^M(i, j) &= \frac{P(s_{j+1} \dots s_N | \pi^M(i, j), \mathbf{t})}{p(s_{j+1} \dots s_N | s_1 \dots s_j, \mathbf{t})} = \frac{b^M(i, j)}{p(s_{j+1} \dots s_N | s_1 \dots s_j, \mathbf{t})} \\
\hat{b}^I(i, j) &= \frac{P(s_{j+1} \dots s_N | \pi^I(i, j), \mathbf{t})}{p(s_{j+1} \dots s_N | s_1 \dots s_j, \mathbf{t})} = \frac{b^I(i, j)}{p(s_{j+1} \dots s_N | s_1 \dots s_j, \mathbf{t})} \\
\hat{b}^D(i, j) &= \frac{P(s_{j+1} \dots s_N | \pi^D(i, j), \mathbf{t})}{p(s_{j+1} \dots s_N | s_1 \dots s_j, \mathbf{t})} = \frac{b^D(i, j)}{p(s_{j+1} \dots s_N | s_1 \dots s_j, \mathbf{t})}
\end{aligned} \tag{18}$$

$$c_j = P(s_j | s_1 \dots s_{j-1}, \mathbf{t}) \tag{19}$$

$$P(s_1 \dots s_m | \mathbf{t}) = \prod_{i=1}^m c_i \tag{20}$$

$$c_j \hat{f}^M(i, j) = (\hat{f}^M(i-1, j-1) \times T_{MM} + \hat{f}^I(i-1, j-1) \times T_{IM} + \hat{f}^D(i-1, j-1) \times T_{DM}) \times Memission[i](s_j) \tag{21}$$

$$c_j \hat{f}^I(i, j) = (\hat{f}^M(i, j-1) \times T_{MI} + \hat{f}^I(i, j-1) \times T_{II} + \hat{f}^D(i, j-1) \times T_{DI}) \times Iemission(s_j) \tag{22}$$

$$c_j \hat{f}^D(i, j) = \hat{f}^M(i-1, j) \times T_{MD} + \hat{f}^I(i-1, j) \times T_{ID} + \hat{f}^D(i-1, j) \times T_{DD} \tag{23}$$

$$\sum_{i=0}^M (\pi^M(i, j) + \pi^I(i, j)) = 1 \tag{24}$$

$$\sum_{i=0}^M \left(\hat{f}^M(i, j) + \hat{f}^I(i, j) \right) = 1 \quad (25)$$

$$c_j = \sum_{i=0}^M \left(\tilde{f}^M(i, j) + \tilde{f}^I(i, j) \right) \quad (26)$$

$$\begin{aligned} c_{j+1} \hat{b}^M(i, j) &= \hat{b}^M(i+1, j+1) \times M_{\text{emission}}[i+1][s_{j+1}] \times T_{MM} + \hat{b}^I(i, j+1) \times I_{\text{emission}}[s_{j+1}] \times T_{MI} \\ &\quad + \hat{b}^D(i+1, j) \times T_{MD} \\ c_{j+1} \hat{b}^I(i, j) &= \hat{b}^M(i+1, j+1) \times M_{\text{emission}}[i+1][s_{j+1}] \times T_{IM} + \hat{b}^I(i, j+1) \times I_{\text{emission}}[s_{j+1}] \times T_{II} \\ &\quad + \hat{b}^D(i+1, j) \times T_{ID} \\ c_{j+1} \hat{b}^D(i, j) &= \hat{b}^M(i+1, j+1) \times M_{\text{emission}}[i+1][s_{j+1}] \times T_{DM} + \hat{b}^I(i, j+1) \times I_{\text{emission}}[s_{j+1}] \times T_{DI} \\ &\quad + \hat{b}^D(i+1, j) \times T_{DD} \end{aligned} \quad (27)$$