The Corona Theorem

Ian Hornik

These are my notes for my talk in the winter semester of 2024, in the analysis seminar at the Technical University of Vienna.

We will denote the space of all complex-valued, bounded, analytic functions on the unit disk $\mathbb{D}$ as H^∞. Equipped with the supremum norm $\|\cdot\|_\infty$ this space becomes a commutative Banach algebra. The space of all multiplicative, bounded, linear functionals on H^∞ not identically zero is denoted $\Delta(H^\infty)$ and is called the *Gelfand space* of H^∞. We endow this space with the subspace topology of the weak-* topology on the topological dual $(H^\infty)'$, which we will refer to as the *Gelfand topology.* For each $z \in \mathbb{D}$ we consider the point-evaluation functional

$$\pi_z : H^\infty \to \mathbb{C},\ f \mapsto f(z).$$

This is clearly multiplicative, bounded and linear and therefore belongs to $\Delta(H^\infty)$. The set of all such functionals $\pi_z, z \in \mathbb{D}$ will be denoted as Δ_0. The *corona* is defined as the complement of the closure of Δ_0 in the Gelfand topology. The corona theorem now states:

Theorem 1 (L. Carleson)**.** The corona is empty. In other words, Δ_0 is dense in $\Delta(H^\infty)$.

There is an equivalent version of the theorem, as given by the following proposition:

Proposition 2. Δ_0 is dense in $\Delta(H^\infty)$ if and only if for any $\delta > 0$ and $f_1, \dots, f_n \in H^\infty$ such that $\sum_{j=1}^n |f_j(z)| \geq \delta, z \in \mathbb{D}$, there exist $g_1, \dots g_n \in H^\infty$ such that $\sum_{j=1}^n f_j g_j = 1$.

Proof. Assume Δ_0 is dense in $\Delta(H^\infty)$ and let $f_1, \dots, f_n \in H^\infty$, and $\delta > 0$ such that $\sum_{j=1}^n |f_j(z)| \geq \delta, z \in \mathbb{D}$. Denote by I the ideal in H^∞ generated by $f_1, \dots, f_n$. If $1 \in I$, then the assertion is established. Assume towards a contradiction that I is a proper ideal, then there exists a maximal ideal $J \supset I$. Since $\Delta(H^\infty)$ is a commutative Banach algebra, there exists a $\phi \in \Delta(H^\infty)$ such that $J = \ker \phi$. Therefore we have $\phi(f_j) = 0$ for $j = 1, \dots, n$. Since Δ_0 is dense, there is a net $(\pi_{z_m})_{m \in M}$ in Δ_0 such that $\pi_{z_m} \to \phi$ in the weak-* topology, that is the net converges pointwise. Therefore, for all $j = 1, \dots, n$ we have $f_j(z_m) = \pi_{z_m}(f_j) \to \phi(f_j) = 0$ and in particular

$$\lim_{m \in M} \sum_{j=1}^n |f_j(z_m)| = 0,$$

a contradiction.

For the other implication, assume towards a contradiction that Δ_0 is not dense in $\Delta(H^\infty)$. Then there exists some $\phi_0 \in \Delta(H^\infty)$ and an open neighbourhood U of ϕ_0 such that

$\Delta_0 \cap U = \emptyset$. Since the sets of the form

$$\bigcap_{j=1}^{n} \{\phi \in \Delta(H^\infty) : |(\phi - \phi_0)(f_j)| < \varepsilon\},$$

for some $n \in \mathbb{N}, f_1, \ldots, f_n \in H^\infty$ and $\varepsilon > 0$, form a neighbourhood basis of ϕ_0 in the weak-* topology, there exists a neighbourhood $V \subseteq U$ described by some $n \in \mathbb{N}, f_1, \ldots, f_n \in H^\infty$ and $\delta > 0$. Define $\widetilde{f_j} := f_j - \phi_0(f_j)$, for $j = 1, \ldots, n$, then clearly $\phi_0(\widetilde{f_j}) = 0$. Since $\Delta_0 \cap V = \emptyset$, for any $z \in \mathbb{D}$ we have $\pi_z \notin V$ and therefore there exists some $j_0 \in \{1, \ldots, n\}$ such that,

$$\delta \leq |(\pi_z - \phi_0)(f_{j_0})| = |f_{j_0}(z) - \phi_0(f_{j_0})| = |\widetilde{f_{j_0}}(z)|.$$

Since $\widetilde{f_j} \in H^\infty$ for $j = 1, \ldots, n$, and $\sum_{j=1}^n |\widetilde{f_j}(z)| \geq \delta$, there exist $g_1, \ldots, g_n \in H^\infty$ such that $\sum_{j=1}^n \widetilde{f_j} g_j = 1$. But this yields

$$1 = \phi_0(1) = \phi_0 \left(\sum_{j=1}^n \widetilde{f_j} g_j \right) = \sum_{j=1}^n \phi_0(\widetilde{f_j}) \phi_0(g_j) = 0,$$

a contradiction. □

1 First Steps

Over the following sections we will prove a stronger version of the right-hand statement in Proposition 2:

Theorem 3. There exist constants $C_{n,\delta}$ only depending on $n \in \mathbb{N}$ and $\delta > 0$, such that if $f_1, \ldots f_n \in \mathrm{Hol}(\mathbb{D})$ with

$$\|f_j\|_\infty \leq 1, \;\; j = 1, \ldots, n, \quad \text{and} \quad \sum_{j=1}^n |f_j(z)|^2 \geq \delta, \;\; z \in \mathbb{D},$$

then there exist $g_1, \ldots, g_n \in \mathrm{Hol}(\mathbb{D})$ with

$$\|g_j\|_\infty \leq C_{n,\delta}, \;\; j = 1, \ldots, n, \quad \text{and} \quad \sum_{j=1}^n f_j g_j = 1.$$

Proof. We will give the proof in multiple steps. First, for a closed set $A \subset \mathbb{C}$ and a space of functions on an open sets $\Omega \supset A$, say $D(\Omega)$, we define

$$D(A) := \bigcup_{\Omega \supset A \text{ open}} T(D(\Omega)), \quad \text{where} \quad T(f) := f\big|_A.$$

We will make use of this to handle smooth or holomorphic functions on closed sets.

Step 1 (Reduction to $f_1, \dots, f_n \in \mathrm{Hol}(\overline{\mathbb{D}})$): Assume that the statement of the theorem holds for all $\widetilde{f_1}, \dots, \widetilde{f_n} \in \mathrm{Hol}(\overline{\mathbb{D}})$, we claim that it then also holds in its original form[1]. For our given $f_1, \dots, f_n$ satisfying the premise of the theorem and all $0 < s < 1$ we define $f_{j,s}(z) \coloneqq f_j(sz)$, $j = 1, \dots, n$. Then for every $0 < s < 1$ the functions $f_{j,s}$ are in $\mathrm{Hol}(\overline{\mathbb{D}})$ and satisfy the premise of the theorem. By our assumption there exist $g_{j,s} \in H^\infty$, $j = 1, \dots, n$ such that

$$\|g_{j,s}\|_\infty \le C_{n,\delta}, \;\; j = 1, \dots, n, \quad \text{and} \quad \sum_{j=1}^n f_{j,s} g_{j,s} = 1.$$

For a fixed $j \in \{1, \dots, n\}$, the set $\{g_{j,s} : 0 < s < 1\}$ is uniformly bounded and therefore normal in $\mathrm{Hol}(\mathbb{D})$. By Montel's Theorem there exists a sequence $s_m \to 1$ and some $g_j \in \mathrm{Hol}(\mathbb{D})$ such that $g_{j,s_m} \to g_j$ compactly. In particular, we obtain

$$\|g_j\|_\infty = \lim_{m\to\infty} \|g_{j,s_m}\|_\infty \le C_{n,\delta}, \quad j = 1, \dots, n,$$

and

$$1 = \lim_{m\to\infty} \sum_{j=1}^n f_{j,s_m} g_{j,s_m} = \sum_{j=1}^n f_j g_j,$$

concluding our claim. We may thus assume that our given $f_1, \dots, f_n$ are holomorphic on $\overline{\mathbb{D}}$ instead.

Step 2 (Solve in $C^\infty(\overline{\mathbb{D}})$): For $j = 1, \dots, n$ we define

$$h_j \coloneqq \frac{\bar{f}_j}{\sum_{k=1}^n |f_k|^2} \in C^\infty(\overline{\mathbb{D}}),$$

then clearly $\sum_{j=1}^n f_j h_j = 1$ and $\|h_j\|_\infty \le \frac{1}{\delta}$. The real task now lies in changing the h_j to become holomorphic in $\mathbb{D}$, without losing control over the boundedness of the solutions.

2 Wirtinger Derivatives

Before we continue we want to briefly introduce a useful generalization of the complex derivative.

Definition 4. Let $\Omega \subseteq \mathbb{R}^2$ be open. Then the *Wirtinger derivatives* (or *Wirtinger operators*) are defined on $C^1(\Omega)$ by

$$\frac{\partial}{\partial z} \coloneqq \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} \coloneqq \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

We will also abbreviate these operators as ∂ and $\bar{\partial}$, respectively.

[1] Note that this does **not** mean that we can assume $f_1, \dots, f_n \in \mathrm{Hol}(\overline{\mathbb{D}})$ in the previous proposition.

Note that by writing a complex number $z \in \mathbb{C}$ as $z = x + iy$ with $x, y \in \mathbb{R}$ we can identify $\mathbb{C} \cong \mathbb{R}^2$. Therefore we can also interpret the Wirtinger operators to act on $C^1(\Omega)$ with an open subset $\Omega \subseteq \mathbb{C}$.

Before listing properties of the Wirtinger operators we quickly want to recall that a function $f \in C^1(\Omega), f = u + iv$ is holomorphic if and only if it satisfies the *Cauchy–Riemann equations*:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Remark 5. Let $\Omega \subseteq \mathbb{C}$ be open and $f \in C^1(\Omega)$.

1. The Wirtinger operators are $\mathbb{C}$-linear, satisfy the Leibniz rule[2], the chain rule

$$\begin{aligned}\frac{\partial}{\partial z}(f \circ g) &= \left(\frac{\partial f}{\partial z} \circ g\right) \frac{\partial g}{\partial z} + \left(\frac{\partial f}{\partial \bar{z}} \circ g\right) \frac{\partial \bar{g}}{\partial z}, \\ \frac{\partial}{\partial \bar{z}}(f \circ g) &= \left(\frac{\partial f}{\partial z} \circ g\right) \frac{\partial g}{\partial \bar{z}} + \left(\frac{\partial f}{\partial \bar{z}} \circ g\right) \frac{\partial \bar{g}}{\partial \bar{z}},\end{aligned}$$

   where $g \in C^1(\Omega), g(\Omega) \subseteq \Omega$, and are compatible with complex conjugation, as in

$$\overline{\left(\frac{\partial f}{\partial z}\right)} = \frac{\partial \bar{f}}{\partial \bar{z}}, \quad \overline{\left(\frac{\partial f}{\partial \bar{z}}\right)} = \frac{\partial \bar{f}}{\partial z}$$

2. If $f \in \mathrm{Hol}(\Omega), f = u + iv$, then

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x} = f'.$$

3. Since

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) = \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),\end{aligned}$$

   we have that[3] $f \in \mathrm{Hol}(\Omega)$ if and only if $\overline{\partial} f = 0$.

4. On $C^2(\Omega)$, the *Laplace operator* can be represented as

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}.$$

Proof (continued). Step 3 (The Koszul complex): We consider the spaces

$$C_0 := C^\infty(\overline{\mathbb{D}}), \quad C_1 := (C_0)^n, \quad C_2 := \{A \in (C_0)^{n \times n} : A = -A^T\}$$

[2] This means that the Wirtinger opeartors are derivatives from an algebraic perspective.
[3] This can be interpreted as "f is independant of $\overline{z}$".

and the linear maps

$$P_{1,0} : C_1 \to C_0,\ (g_j)_{j=1}^n \mapsto \sum_{j=1}^n g_j f_j, \quad P_{2,1} : C_2 \to C_1,\ (g_{jk})_{j,k=1}^n \mapsto \left(\sum_{k=1}^n g_{jk} f_k\right)_{j=1}^n.$$

We also consider the operator $\overline{\partial} : C_0 \to C_0$. It is well-defined, since if $f_1 \in C^\infty(\Omega_1)$, $f_2 \in C^\infty(\Omega_2)$ with $f_1 = f_2$ on $\overline{\mathbb{D}}$, then in particular $f_1 = f_2$ on $\mathbb{D}$ and therefore $\overline{\partial} f_1 = \overline{\partial} f_2$ on $\mathbb{D}$. By continuity, we therefore also get $\overline{\partial} f_1 = \overline{\partial} f_2$ on $\overline{\mathbb{D}}$.

Applying $\overline{\partial}$ pointwise in C_1 and C_2 as well, the resulting connections are visualized in the diagram below, called the *Koszul complex*:

$$\begin{tikzcd}
C_2 \arrow[r, "P_{2,1}"] \arrow[d, "\overline{\partial}"'] & C_1 \arrow[r, "P_{1,0}"] \arrow[d, "\overline{\partial}"'] & C_0 \arrow[d, "\overline{\partial}"'] \\
C_2 \arrow[r, "P_{2,1}"'] & C_1 \arrow[r, "P_{1,0}"'] & C_0
\end{tikzcd}$$

Lemma 6. The Koszul complex has the following properties:

1. The diagram is commutative, that is we have $P_{j+1,j}\overline{\partial} = \overline{\partial} P_{j+1,j}$ for $j = 0, 1$.
2. The horizontal sequences are exact, that is $\operatorname{ran} P_{2,1} = \ker P_{1,0}$.
3. The vertical maps $\overline{\partial} : C_j \to C_j$ for $j = 0, 1, 2$ are surjective.

Proof.

1. For $g \in C_0$ and $f \in \mathrm{Hol}(\overline{\mathbb{D}})$ we have
$$\overline{\partial}(gf) = f\overline{\partial} g + g\overline{\partial} f = f\overline{\partial} g$$
and together with the linearity of $\overline{\partial}$ the statement follows.
2. "$\subseteq$": Let $g \in C_2, g = (g_{jk})_{j,k=1}^n$, then
$$P_{1,0}P_{2,1}g = P_{1,0}\left[\left(\sum_{k=1}^n g_{jk} f_k\right)_{j=1}^n\right] = \sum_{j=1}^n \sum_{k=1}^n g_{jk} f_k f_j = 0$$
since g is skew-symmetric and therefore $g \in \ker P_{1,0}$.
"$\supseteq$" Let $g \in \ker P_{1,0} \subseteq C_1, g = (g_1, \dots, g_n)$. We define $p = (p_{jk})_{j,k=1}^n \in C_2$ by
$$p_{jk} := \frac{1}{\sum_{\ell=1}^n |f_\ell|^2}(g_j\overline{f_k} - g_k\overline{f_j}).$$

Then for any $j = 1, \dots, n$ we have

$$
\begin{aligned}
(P_{2,1}p)_j &= \sum_{k=1}^{n} p_{jk} f_k = \frac{1}{\sum_{\ell=1}^{n} |f_\ell|^2} \sum_{k=1}^{n} (g_j |f_k|^2 - g_k \overline{f_j} f_k) = \\
&= g_j - \frac{1}{\sum_{\ell=1}^{n} |f_\ell|^2} \overline{f_j} \sum_{k=1}^{n} g_k f_k = g_j - \frac{1}{\sum_{\ell=1}^{n} |f_\ell|^2} \overline{f_j} P_{1,0} g = \\
&= g_j,
\end{aligned}
$$

and therefore $g \in \operatorname{ran} P_{2,1}$.

3. For given $v \in C^\infty(\overline{\mathbb{D}})$ we want to solve the partial differential equation

$$
\frac{\partial u}{\partial \bar{z}} = v \quad (\text{on } \overline{\mathbb{D}})
$$

for some $u \in C^\infty(\overline{\mathbb{D}})$. We will approach this using a fundamental solution of the differential operator $\overline{\partial}$. Recall that

$$
\Gamma(z) := \frac{1}{2\pi} \log |z|
$$

is a fundamental solution of the Laplace operator, that is we have $\Delta \Gamma = \delta_0$ distributionally, where δ_0 denotes the delta distribution at 0. We claim that $\frac{1}{\pi z}$ is a fundamental solution of $\overline{\partial}$, and verify this via

$$
\begin{aligned}
\overline{\partial} \frac{1}{z} &= \overline{\partial} \frac{\bar{z}}{|z|^2} = \frac{1}{2} (\partial_x + i \partial_y) \frac{x - iy}{x^2 + y^2} = \\
&= \frac{1}{2} \left[\partial_x \frac{x}{x^2 + y^2} - i \partial_x \frac{y}{x^2 + y^2} + i \partial_y \frac{x}{x^2 + y^2} + \partial_y \frac{y}{x^2 + y^2} \right] = \\
&= \frac{1}{2} \left[\partial_x^2 \log |z| + \partial_y^2 \log |z| + i \left(\frac{2xy}{x^2 + y^2} - \frac{2xy}{x^2 + y^2} \right) \right] = \\
&= \frac{1}{2} \Delta \log |z| = \frac{1}{2} 2\pi \delta_0 = \pi \delta_0.
\end{aligned}
$$

Now let $\Omega \supset \overline{\mathbb{D}}$ be open such that $v \in C^\infty(\Omega)$ and choose $\varphi \in C_c^\infty(\Omega)$ such that $\varphi|_{\overline{\mathbb{D}}} = 1$. Then $\varphi v \in C_c^\infty(\Omega)$, therefore

$$
u(w) := \left(\frac{1}{\pi z} * \varphi v \right)(w) = \frac{1}{\pi} \int_\Omega \frac{\varphi(z) v(z)}{w - z} \, \mathrm{d}\lambda^2(z)
$$

is a classical solution of $\overline{\partial} u = \varphi v$ in Ω. Since $\varphi v = v$ on $\overline{\mathbb{D}}$, we get $\overline{\partial} u = v$ on $\overline{\mathbb{D}}$, as desired.

Arguing pointwise shows the surjectivity of the maps $\overline{\partial} : C_\ell \to C_\ell$ for $\ell = 0, 1$. For $\ell = 2$ and given $b = (b_{jk})_{j,k=1}^n \in C_2$ we first solve

$$
\overline{\partial} a_{jk} = b_{jk}, \quad \text{for} \quad 1 \le j < k \le n
$$

and then set $a_{jj} = 0$ and $a_{jk} = -a_{kj}$ for $n \ge j > k \ge 1$.

$\square$

Proof (continued). Step 4 (Apply to $h = (h_1, \ldots, h_n) \in C_1$): In step 2 we constructed an element $h = (h_1, \ldots, h_n) \in C_1$ by setting

$$h_j := \frac{\bar{f}_j}{\sum_{k=1}^n |f_k|^2}.$$

By our construction we have $P_{1,0}h = 1$ and therefore $0 = \overline{\partial} P_{1,0}h = P_{1,0}\overline{\partial}h$, thus $\overline{\partial}h \in \ker P_{1,0}$. By Lemma 6 there exists $b \in C_2$ such that $P_{2,1}b = \overline{\partial}h$ and $a \in C_2$ such that $\overline{\partial}a = b$. We now set $g := h - P_{2,1}a \in C_1$. Then

$$P_{1,0}g = P_{1,0}h - P_{1,0}P_{2,1}a = 1$$

and

$$\overline{\partial}g = \overline{\partial}h - \overline{\partial}P_{2,1}a = \overline{\partial}h - P_{2,1}b = 0.$$

Therefore g is a solution to

$$\sum_{k=1}^n f_k g_k = 1$$

in $\mathrm{Hol}(\overline{\mathbb{D}})$. However, we do not have an estimate on $|g_j|$ yet.

3 Hardy Spaces

Let μ denote the Lebesgue measure on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, i.e. the measure such that for a measurable function $f : \mathbb{T} \to \mathbb{C}$ it holds that

$$\int_{\mathbb{T}} f \,\mathrm{d}\mu = \int_{-\pi}^{\pi} f(e^{i\vartheta}) \,\mathrm{d}\vartheta.$$

We define the $L^p(\mathbb{T})$-norms via the *normed* Lebesgue measure $\frac{1}{2\pi}\mu$:

$$\|f\|_p := \left(\frac{1}{2\pi}\int_{\mathbb{T}} |f|^p \,\mathrm{d}\mu\right)^{1/p}, \quad \text{for } 1 \le p < \infty, \quad \text{and} \quad \|f\|_\infty := \text{ess. sup}\,|f|.$$

For $f \in L^1(\mathbb{T})$ and $n \in \mathbb{N}$ we define the n-th *Fourier coefficient* by

$$\hat{f}(n) := \frac{1}{2\pi}\int_{\mathbb{T}} f(\xi)\xi^{-n} \,\mathrm{d}\mu(\xi).$$

For $1 \le p \le \infty$ we define the *Hardy space* H^p as the set of all $f \in \mathrm{Hol}(\mathbb{D})$ with $\|f\|_p < \infty$, where

$$\|f\|_p := \lim_{r \to 1}\left(\frac{1}{2\pi}\int_{\mathbb{T}} |f_r|^p \,\mathrm{d}\mu\right)^{1/p} \quad \text{for } p < \infty, \quad \text{and} \quad \|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)|.$$

Equivalently one can interchange the limit in the definition above with the supremum over $0 < r < 1$. It is of note that convergence in the Hardy spaces implies compact convergence. We lastly define H_0^p as the (closed) subspace of all $f \in H^p$, for which $f(0) = 0$.

We summarize the characterisation of Hardy spaces:

Theorem 7. Let $1 \leq p \leq \infty$. Then:

1. H^p is a Banach space[4].
2. For $p \leq q \leq \infty$ it holds that $H^p \supseteq H^q$.
3. Let $f \in H^p$, then for almost all $\xi \in \mathbb{T}$ the limit
$$\lim_{r \to 1} f(r\xi) =: f^*(\xi)$$
exists and defines a function in $L^p(\mathbb{T})$, also called the *boundary values* of f. If $p < \infty$, we also have $\lim_{r \to 1} \|f^* - f_r\|_p = 0$, where $f_r(\xi) := f(r\xi)$.
4. The map $^* : f \mapsto f^*$ is an isometry from H^p onto
$$L_+^p(\mathbb{T}) := \{f \in L^p(\mathbb{T}) : \forall n < 0 : \hat{f}(n) = 0\},$$
which is a closed subspace of $L^p(\mathbb{T})$.

We will also use the following lemma:

Lemma 8. Let $f \in \mathrm{Hol}(\overline{\mathbb{D}})$, then there exist $g_1, g_2 \in \mathrm{Hol}(\overline{\mathbb{D}})$ such that
$$f = g_1 g_2, \quad \text{and} \quad \|g_1\|_2^2 = \|g_2\|_2^2 = \|f\|_1.$$

Proof (continued). Returning to our proof, recall that we want to obtain a bound on the functions $\|g_j\|_\infty$, where
$$g_j = h_j - \sum_{k=1}^{n} a_{jk} f_k,$$
and $a_{jk} \in C^\infty(\overline{\mathbb{D}})$ is a solution of the partial differential equation
$$\frac{\partial y}{\partial \bar{z}} = \left(\sum_{\ell=1}^{n} |f_\ell|^2 \right)^{-1} \left(\frac{\partial h_j}{\partial \bar{z}} \bar{f}_k - \frac{\partial h_k}{\partial \bar{z}} \bar{f}_j \right).$$
We want to show that the solution a_{jk} can be chosen in a way, that the resulting functions g_j are bounded in the H^∞-norm by a constant depending only on n and δ, that is
$$\|g_j\|_\infty \leq C_{n,\delta}.$$
Note that we only need $g_j \in H^\infty$, not necessarily $\in \mathrm{Hol}(\overline{\mathbb{D}})$. Denote by u_{jk} the right-hand side of the partial differential equation above. We fix a solution $\overline{\partial} v_{jk} = u_{jk}$ and notice that if $\overline{\partial} a_{jk} = u_{jk}$ is another solution bounded on $\mathbb{D}$, then
$$\overline{\partial}(a_{jk} - v_{jk}) = \overline{\partial} a_{jk} - \overline{\partial} v_{jk} = 0,$$

[4] In particular, H^∞ is a Banach algebra, which we already used in the introduction.

that is the difference is bounded and holomorphic, thus in H^∞. We can therefore write

$$a_{jk} = v_{jk} + p, \quad p \in H^\infty.$$

We can view a_{jk} as an element of $L^\infty(\mathbb{T})$ by considering $v_{jk}|_{\mathbb{T}} \in L^\infty(\mathbb{T})$ and $p^* \in (H^\infty)^* \subset L^\infty(\mathbb{T})$. If we manage to bound

$$\|a_{jk}\|_{L^\infty(\mathbb{T})} = \operatorname*{ess.\,sup}_{z\in\mathbb{T}} |a_{jk}(z)| \leq K_{n,\delta},$$

we immediately get

$$\|g_j\|_{H^\infty} \leq \|h_j\|_{H^\infty} + \sum_{k=1}^{n} \|a_{jk}f_k\|_{H^\infty} \leq \frac{1}{\delta} + \sum_{k=1}^{n} \|a_{jk}\|_{L^\infty(\mathbb{T})}\|f_k\|_{H^\infty} \leq \frac{1}{\delta} + nK_{n,\delta},$$

resulting in the claim of the theorem.

Note that we can vary $\|a_{jk}\|_{L^\infty(\mathbb{T})}$ by choosing different functions $p \in H^\infty$. We therefore want to bound the quantity

$$\inf_{p\in H^\infty} \|v_{jk} + p^*\|_\infty,$$

which is precisely the norm of v_{jk} in the quotient space $L^\infty(\mathbb{T})/(H^\infty)^*$. The following lemma allows us to translate the minimization problem into a maximization problem.

Lemma 9. The map

$$\Phi : L^\infty(\mathbb{T})/(H^\infty)^* \to ((H_0^1)^*)',\; f + (H^\infty)^* \mapsto \left[g \mapsto \frac{1}{2\pi}\int_{\mathbb{T}} fg \,\mathrm{d}\mu\right]$$

is an isometric isomorphism.

Proof. We have $L^\infty(\mathbb{T}) \cong L^1(\mathbb{T})'$ via the duality

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{\mathbb{T}} fg\,\mathrm{d}\mu, \quad f \in L^\infty(\mathbb{T}),\; g \in L^1(\mathbb{T}).$$

Since $(H_0^1)^* \leq L^1(\mathbb{T})$ we therefore have $((H_0^1)^*)' \cong L^\infty(\mathbb{T})/((H_0^1)^*)^\perp$ via

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{\mathbb{T}} fg\,\mathrm{d}\mu, \quad f \in L^\infty(\mathbb{T})/((H_0^1)^*)^\perp,\; g \in (H_0^1)^*.$$

It remains to show that $((H_0^1)^*)^\perp = (H^\infty)^*$. Let $w^* \in ((H_0^1)^*)^\perp \leq L^\infty(\mathbb{T})$, then for any $n \in \mathbb{N}$ we have

$$0 = \langle w^*, (z^n)^*\rangle = \langle w^*, z^n\rangle = \widehat{w^*}(-n).$$

Therefore $w^* \in L_+^\infty(\mathbb{T}) = (H^\infty)^*$. For the other inclusion let $w^* \in (H^\infty)^*$ and $h^* \in (H_0^1)^*$, then

$$\begin{aligned}\langle w^*, h^*\rangle = \frac{1}{2\pi}\int_0^{2\pi} w^*(e^{i\vartheta})h^*(e^{i\vartheta})\,\mathrm{d}\vartheta = \lim_{r\to1}\frac{1}{2\pi}\int_0^{2\pi} w(re^{i\vartheta})h(re^{i\vartheta})\frac{ie^{i\vartheta}}{ie^{i\vartheta}}\,\mathrm{d}\vartheta = \\ = \lim_{r\to1}\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{w_r(\zeta)h_r(\zeta)}{\zeta}\,\mathrm{d}\zeta = \lim_{r\to1} w_r(0)h_r(0) = 0,\end{aligned}$$

where the second-to-last equation follows since $w_r h_r \in \mathrm{Hol}(\overline{\mathbb{D}})$ and the last one since $h(0) = 0$. Therefore $w^* \in ((H_0^1)^*)^\perp$, concluding the proof. $\square$

Proof (continued). Step 5 (Dualisation): Since we longer need the indices, we now simply write v instead of v_{jk}. Applying the above lemma to our previous situation we can redescribe the norm of $v + (H^\infty)^* \in L^\infty(\mathbb{T})/(H^\infty)^*$ as

$$\begin{aligned}\|\Phi(v + (H^\infty)^*)\| &= \sup_{\substack{F \in (H_0^1)^* \\ \|F\|_1 \le 1}} |\Phi(v + (H^\infty)^*)(F)| = \sup_{\substack{F \in H_0^1 \\ \|F\|_1 \le 1}} \left| \frac{1}{2\pi} \int_{\mathbb{T}} vF^* \,\mathrm{d}\mu \right| = \\ &= \sup_{\substack{F \in H_0^1 \\ \|F\|_1 \le 1}} \left| \lim_{r \to 1} \frac{1}{2\pi} \int_{\mathbb{T}} vF_r \,\mathrm{d}\mu \right| = \sup_{\substack{F \in \mathrm{Hol}(\overline{\mathbb{D}}) \\ F(0)=0, \|F\|_1 \le 1}} \left| \frac{1}{2\pi} \int_{\mathbb{T}} vF \,\mathrm{d}\mu \right|.\end{aligned}$$

We now want to bound this supremum, where, as before, $v \in C^\infty(\overline{\mathbb{D}})$.

Step 6: We want to redescribe the integral

$$\frac{1}{2\pi} \int_{\mathbb{T}} vF \,\mathrm{d}\mu,$$

where $v \in C^\infty(\overline{\mathbb{D}})$ and $F \in \mathrm{Hol}(\overline{\mathbb{D}}), F(0) = 0, \|F\|_1 \le 1$. Let $\sigma \coloneqq vF$ and

$$\varphi(z) \coloneqq \frac{1}{2\pi} \log |z|, \quad z \neq 0.$$

For $\varepsilon > 0$ let $\mathbb{D}_\varepsilon \coloneqq \mathbb{D} \setminus \overline{B_\varepsilon(0)}$. By Green's second identity we have

$$\int_{\mathbb{D}_\varepsilon} \sigma \Delta \varphi - \varphi \Delta \sigma \,\mathrm{d}\lambda^2 = \int_{\mathbb{T}} \sigma \frac{\partial \varphi}{\partial r} - \varphi \frac{\partial \sigma}{\partial r} \,\mathrm{d}\mu - \int_{\partial B_\varepsilon(0)} \sigma \frac{\partial \varphi}{\partial r} - \varphi \frac{\partial \sigma}{\partial r} \,\mathrm{d}\mathcal{H}^1,$$

where $\frac{\partial}{\partial r}$ denotes the radial derivative. Simplifying results in

$$-\int_{\mathbb{D}_\varepsilon} \varphi \Delta \sigma \,\mathrm{d}\lambda^2 = \frac{1}{2\pi} \int_{\mathbb{T}} \sigma \,\mathrm{d}\mu - \frac{1}{2\pi} \int_{\partial B_\varepsilon(0)} \frac{\sigma}{\varepsilon} - \frac{\partial \sigma}{\partial r} \log \varepsilon \,\mathrm{d}\mathcal{H}^1.$$

By the intermediate value theorem for integrals for any $\varepsilon > 0$ there exists a $\zeta_\varepsilon \in \partial B_\varepsilon(0)$ such that

$$\frac{1}{2\pi\varepsilon} \int_{\partial B_\varepsilon(0)} \sigma \,\mathrm{d}\mathcal{H}^1 = \sigma(\zeta_\varepsilon) \to \sigma(0) = v(0)F(0) = 0.$$

Furthermore $\left|\frac{\partial \sigma}{\partial r}\right| \le M$ on $\overline{\mathbb{D}}$ for some $M > 0$, thus

$$\frac{1}{2\pi} \int_{\partial B_\varepsilon(0)} \left| \frac{\partial \sigma}{\partial r} \log \varepsilon \right| \mathrm{d}\mathcal{H}^1 \le M\varepsilon \log \varepsilon \to 0.$$

Finally,

$$\Delta \sigma = \Delta(vF) = 4\partial\overline{\partial}(vF) = 4\partial(v\overline{\partial}F + F\overline{\partial}v) = 4\partial(Fu) = 4(F\partial u + uF').$$

With $\psi \coloneqq -\varphi = \frac{1}{2\pi} \log \frac{1}{|z|}$, by letting $\varepsilon \to 0$ we thus obtain

$$\frac{1}{2\pi} \int_{\mathbb{T}} vF \,\mathrm{d}\mu = 4 \int_{\mathbb{D}} \psi \left(F\partial u + uF' \right) \mathrm{d}\lambda^2 =$$

$$= 4\left(\int_{\mathbb{D}} F\,\partial u\,\psi\,\mathrm{d}\lambda^2 + \int_{\mathbb{D}} uF'\psi\,\mathrm{d}\lambda^2\right) =: 4(I_1 + I_2).$$

Our goal is to show the existence of a constant $K_{n,\delta}$ such that

$$\left|\frac{1}{2\pi}\int_{\mathbb{T}} vF\,\mathrm{d}\mu\right| \le 4(|I_1| + |I_2|) \le K_{n,\delta}.$$

Recall that

$$u = \tau(\bar{f}_k\overline{\partial}h_j - \bar{f}_j\overline{\partial}h_k), \quad \text{where} \quad \tau := \left(\sum_{\ell=1}^n |f_\ell|^2\right)^{-1}.$$

Since $h_\ell = \tau\bar{f}_\ell$ we first calculate want to calculate $\overline{\partial}\tau$. We first notice that with function $m(z) := z^{-1}$ and any nonvanishing function $\alpha \in C^\infty$ we have by the chain rule

$$\partial\alpha^{-1} = \partial(m\circ\alpha) = (\partial m\circ\alpha)\partial\alpha + (\overline{\partial}m\circ\alpha)\partial\bar{\alpha} = -\alpha^{-2}\partial\alpha,$$

and analogously $\overline{\partial}\alpha^{-1} = -\alpha^{-2}\overline{\partial}\alpha$. Therefore

$$\begin{aligned}
\overline{\partial}\tau &= -\tau^2\sum_{\ell=1}^n \overline{\partial}(f_\ell\bar{f}_\ell) = -\tau^2\sum_{\ell=1}^n (f_\ell\overline{\partial}\bar{f}_\ell + \bar{f}_\ell\overline{\partial}f_\ell) = \\
&= -\tau^2\sum_{\ell=1}^n f_\ell\bar{f}'_\ell =: -\tau^2\eta, \\
\partial\tau &= \cdots = -\tau^2\bar{\eta}.
\end{aligned}$$

We therefore obtain $\overline{\partial}h_\ell = \tau\overline{\partial}\bar{f}_\ell + \bar{f}_\ell\overline{\partial}\tau = \tau(\bar{f}'_\ell - \bar{f}_\ell\tau\eta)$ and by that the representations

$$\begin{aligned}
u &= \tau(\bar{f}_k\overline{\partial}h_j - \bar{f}_j\overline{\partial}h_k) = \tau^2(\bar{f}_k\bar{f}'_j - \bar{f}_k\bar{f}_j\tau\eta - \bar{f}_j\bar{f}'_k + \bar{f}_j\bar{f}_k\tau\eta) = \tau^2(\bar{f}_k\bar{f}'_j - \bar{f}_j\bar{f}'_k), \\
\partial u &= \tau^2\partial(\bar{f}_k\bar{f}'_j - \bar{f}_j\bar{f}'_k) + (\bar{f}_k\bar{f}'_j - \bar{f}_j\bar{f}'_k)\partial\tau^2 = -2\tau^3\left(\sum_{\ell=1}^n \bar{f}_\ell f'_\ell\right)(\bar{f}_k\bar{f}'_j - \bar{f}_j\bar{f}'_k).
\end{aligned}$$

Inserting back we obtain (since $|f_\ell| \le 1$ and $|\tau| \le \delta^{-1}$)

$$\begin{aligned}
|I_1| &= \left|-2\int_{\mathbb{D}} \tau^3\left(\sum_{\ell=1}^n \bar{f}_\ell f'_\ell\bar{f}_k\bar{f}'_j - \bar{f}_\ell f'_\ell\bar{f}_j\bar{f}'_k\right)F\psi\,\mathrm{d}\lambda^2\right| \le \\
&\le \frac{2}{\delta^3}\sum_{\ell=1}^n\left(\int_{\mathbb{D}} |f'_\ell f'_j F|\psi\,\mathrm{d}\lambda^2 + \int_{\mathbb{D}} |f'_\ell f'_k F|\psi\,\mathrm{d}\lambda^2\right), \\
|I_2| &= \left|\int_{\mathbb{D}} \tau^2(\bar{f}_k\bar{f}'_j - \bar{f}_j\bar{f}'_k)F'\psi\,\mathrm{d}\lambda^2\right| \le \\
&\le \frac{1}{\delta^2}\left(\int_{\mathbb{D}} |f'_j F'|\psi\,\mathrm{d}\lambda^2 + \int_{\mathbb{D}} |f'_k F'|\psi\,\mathrm{d}\lambda^2\right).
\end{aligned}$$

It therefore suffices to bound the integrals

$$J_1 := \int_{\mathbb{D}} |f'_1 f'_2 F|\psi\,\mathrm{d}\lambda^2, \quad J_2 := \int_{\mathbb{D}} |f'F'|\psi\,\mathrm{d}\lambda^2,$$

where $f, f_1, f_2, F \in \mathrm{Hol}(\overline{\mathbb{D}})$ and $\|f\|_\infty, \|f_1\|_\infty, \|f_2\|_\infty, \|F\|_1 \le 1$. The issue is that a-priori we do not have bounds on the derivatives.

4 Integral estimates

Lemma 10. Let $f, g, u, v \in \mathrm{Hol}(\overline{\mathbb{D}})$, then the following integral estimates hold:

1. $\displaystyle\int_{\mathbb{D}} |f'|^2 \psi \, \mathrm{d}\lambda^2 \leq \frac{1}{4}\|f\|_2^2$
2. $\displaystyle\int_{\mathbb{D}} |fg'|^2 \psi \, \mathrm{d}\lambda^2 \leq \|f\|_2^2 \|g\|_\infty^2$
3. $\displaystyle\int_{\mathbb{D}} |fgu'v'| \psi \, \mathrm{d}\lambda^2 \leq \|f\|_2 \|g\|_2 \|u\|_\infty \|v\|_\infty$
4. $\displaystyle\int_{\mathbb{D}} |fu'v'| \psi \, \mathrm{d}\lambda^2 \leq \|f\|_1 \|u\|_\infty \|v\|_\infty$
5. $\displaystyle\int_{\mathbb{D}} |fg'u'| \psi \, \mathrm{d}\lambda^2 \leq \frac{1}{2}\|f\|_2 \|g\|_2 \|u\|_\infty$
6. $\displaystyle\int_{\mathbb{D}} |f'u'| \psi \, \mathrm{d}\lambda^2 \leq \|f\|_1 \|u\|_\infty$

Proof.

1. Applying Green's formula on $f\bar{f}$ and ψ yields

$$\int_{\mathbb{D}_\varepsilon} \psi \Delta(f\bar{f}) - f\bar{f}\Delta\psi \, \mathrm{d}\lambda^2 = \int_{\mathbb{T}} \psi \frac{\partial(f\bar{f})}{\partial r} - f\bar{f}\frac{\partial \psi}{\partial r} \, \mathrm{d}\mu - \int_{\partial B_\varepsilon(0)} \psi \frac{\partial(f\bar{f})}{\partial r} - f\bar{f}\frac{\partial \psi}{\partial r} \, \mathrm{d}\mathcal{H}^1,$$

and simplifying we obtain

$$\int_{\mathbb{D}_\varepsilon} \psi \Delta(f\bar{f}) \, \mathrm{d}\lambda^2 = \frac{1}{2\pi} \int_{\mathbb{T}} |f|^2 \, \mathrm{d}\mu + \frac{\log \varepsilon}{2\pi} \int_{\partial B_\varepsilon(0)} \frac{\partial(f\bar{f})}{\partial r} \, \mathrm{d}\mathcal{H}^1 - \frac{1}{2\pi\varepsilon} \int_{\partial B_\varepsilon(0)} |f|^2 \, \mathrm{d}\mathcal{H}^1$$

Arguing as before, taking $\varepsilon \to 0$ we get

$$\int_{\mathbb{D}} \psi \Delta(f\bar{f}) \, \mathrm{d}\lambda^2 = \|f\|_2^2 - |f(0)|^2.$$

Since

$$\begin{aligned}\Delta(f\bar{f}) &= 4\partial\overline{\partial}(f\bar{f}) = 4\partial(\bar{f}\cdot\overline{\partial}f + f\cdot\overline{\partial}\bar{f}) = 4\partial(f\cdot\overline{\partial}\bar{f}) = 4(\partial f\cdot\overline{\partial}\bar{f} + f\cdot\overline{\partial}\partial\bar{f}) = \\ &= 4(\partial f\cdot\overline{\partial f} + f\cdot\overline{\partial}\,\overline{\overline{\partial} f}) = 4f'\bar{f}' = 4|f'|^2\end{aligned}$$

and $|f(0)|^2 \geq 0$ we obtain

$$\int_{\mathbb{D}} |f'|^2 \psi \, \mathrm{d}\lambda^2 \leq \frac{1}{4}\|f\|_2^2.$$

2. We have $fg' = (fg)' - f'g$, therefore

$$|fg'|^2 \leq (|(fg)'| + |f'g|)^2 \leq 2(|(fg)'|^2 + |f'g|^2) \leq 2(|(fg)'|^2 + \|g\|_\infty^2 |f'|^2).$$

Integrating and using 1. yields

$$\begin{aligned}\int_{\mathbb{D}} |fg'|^2 \psi \, \mathrm{d}\lambda^2 &\leq 2\int_{\mathbb{D}} |(fg)'|^2 \psi \, \mathrm{d}\lambda^2 + 2\|g\|_\infty^2 \int_{\mathbb{D}} |f'|^2 \psi \, \mathrm{d}\lambda^2 \leq \\ &\leq \frac{1}{2}\|fg\|_2^2 + \frac{1}{2}\|f\|_2^2 \|g\|_\infty^2 \leq \|f\|_2^2 \|g\|_\infty^2.\end{aligned}$$

3. Consider the positive measure $\nu := \psi\, \mathrm{d}\lambda^2$. Invoking the Cauchy-Schwarz inequality in $L^2(\nu)$ and using 2. we get

$$\int_{\mathbb{D}} |fgu'v'|\psi\, \mathrm{d}\lambda^2 \leq \left(\int_{\mathbb{D}} |fu'|\psi\, \mathrm{d}\lambda^2\right)^{1/2} \left(\int_{\mathbb{D}} |gv'|\psi\, \mathrm{d}\lambda^2\right)^{1/2} \leq$$
$$\leq \|f\|_2\|u\|_\infty\|g\|_2\|v\|_\infty.$$

4. By Lemma 8 we can write $f = g_1g_2$ with $g_1, g_2 \in \mathrm{Hol}(\overline{\mathbb{D}})$ and $\|g_1\|_2^2 = \|g_2\|_2^2 = \|f\|_1$. Using 3. we then obtain

$$\int_{\mathbb{D}} |fu'v'|\psi\, \mathrm{d}\lambda^2 = \int_{\mathbb{D}} |g_1g_2u'v'|\psi\, \mathrm{d}\lambda^2 \leq$$
$$\leq \|g_1\|_2\|g_2\|_2\|u\|_\infty\|v\|_\infty = \|f\|_1\|u\|_\infty\|v\|_\infty.$$

5. Using the Cauchy-Schwarz inequality in $L^2(\nu)$, as well as 1. and 2. we obtain

$$\int_{\mathbb{D}} |fg'u'|\psi\, \mathrm{d}\lambda^2 \leq \left(\int_{\mathbb{D}} |fu'|\psi\, \mathrm{d}\lambda^2\right)^{1/2} \left(\int_{\mathbb{D}} |g'|\psi\, \mathrm{d}\lambda^2\right)^{1/2} \leq$$
$$\leq \|f\|_2\|u\|_\infty \cdot \frac{1}{2}\|g\|_2 = \frac{1}{2}\|f\|_2\|g\|_2\|u\|_\infty.$$

6. We write $f = g_1g_2$ as in 4. and use 5. to obtain

$$\int_{\mathbb{D}} |f'u'|\psi\, \mathrm{d}\lambda^2 \leq \int_{\mathbb{D}} |(g_1g_2)'u'|\psi\, \mathrm{d}\lambda^2 \leq \int_{\mathbb{D}} |g_1'g_2u'|\psi\, \mathrm{d}\lambda^2 + \int_{\mathbb{D}} |g_1g_2'u'|\psi\, \mathrm{d}\lambda^2 \leq$$
$$\leq \frac{1}{2}\|g_2\|_2\|g_1\|_2\|u\|_\infty + \frac{1}{2}\|g_1\|_2\|g_2\|_2\|u\|_\infty = \|f\|_1\|u\|_\infty. \qquad \square$$

Proof (continued). Step 7 (Conclusion): Using Lemma 10, specifically points 4. and 6., we obtain $|J_1| \leq 1$ and $|J_2| \leq 1$ concluding the proof. $\square$