The Corona Theorem

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We will denote the space of all complex-valued, bounded, analytic functions on the unit disk $\mathbb{D}$ as H^∞. Equipped with the supremum norm $\|\cdot\|_\infty$ this space becomes a commutative Banach algebra. The space of all multiplicative, bounded, linear functionals on H^∞ not identically zero is denoted $\Delta(H^\infty)$ and is called the *Gelfand space* of H^∞. We endow this space with the subspace topology of the weak-* topology on the topological dual $(H^\infty)'$, which we will refer to as the *Gelfand topology*. For each $z \in \mathbb{D}$ we consider the point-evaluation functional

$$\pi_z : H^\infty \to \mathbb{C},\ f \mapsto f(z).$$

This is clearly multiplicative, bounded and linear and therefore belongs to $\Delta(H^\infty)$. The set of all such functionals $\pi_z, z \in \mathbb{D}$ will be denoted as Δ_0. The *corona* is defined as the complement of closure of Δ_0 in the Gelfand topology. The corona theorem now states:

Theorem 1 (L. Carleson)**.** The corona is empty. In other words, Δ_0 is dense in $\Delta(H^\infty)$.

There is an equivalent version of the theorem, as given by the following proposition:

Proposition 2. Δ_0 is dense in $\Delta(H^\infty)$ if and only if for any $\delta > 0$ and $f_1, \dots, f_n \in H^\infty$ such that $\sum_{j=1}^n |f_j(z)| \geq \delta, z \in \mathbb{D}$, there exist $g_1, \dots g_n \in H^\infty$ such that $\sum_{j=1}^n f_j g_j = 1$.

Proof. Assume Δ_0 is dense in $\Delta(H^\infty)$ and let $f_1, \dots, f_n \in H^\infty$, and $\delta > 0$ such that $\sum_{j=1}^n |f_j(z)| \geq \delta, z \in \mathbb{D}$. Denote by I the ideal in H^∞ generated by $f_1, \dots, f_n$. If $1 \in I$, then the assertion is established. Assume towards a contradiction that I is a proper ideal, then there exists a maximal ideal $J \supset I$. Since $\Delta(H^\infty)$ is a commutative Banach algebra, there exists a $\phi \in \Delta(H^\infty)$ such that $J = \ker \phi$. Therefore we have $\phi(f_j) = 0$ for $j = 1, \dots, n$. Since Δ_0 is dense, there is a net $(\pi_{z_m})_{m \in M}$ in Δ_0 such that $\pi_{z_m} \to \phi$ in the weak-* topology, that is the net converges pointwise. Therefore, for all $j = 1, \dots, n$ we have $f_j(z_m) = \pi_{z_m}(f_j) \to \phi(f_j) = 0$ and in particular

$$\lim_{m \in M} \sum_{j=1}^n |f_j(z_m)| = 0,$$

a contradiction.

For the other implication, assume towards a contradiction that Δ_0 is not dense in $\Delta(H^\infty)$. Then there exists some $\phi_0 \in \Delta(H^\infty)$ and an open neighbourhood U of ϕ_0 such that $\Delta_0 \cap U = \emptyset$. Since the sets of the form

$$\{\phi \in \Delta(H^\infty) : |(\phi - \phi_0)(f_j)| < \varepsilon, j = 1, \dots, n\},$$

for some $n \in \mathbb{N}, f_1, \ldots, f_n \in H^\infty$ and $\varepsilon > 0$, form a neighbourhood basis of ϕ_0 in the weak-* topology, there exists a neighbourhood $V \subseteq U$ described by some $n \in \mathbb{N}, f_1, \ldots, f_n \in H^\infty$ and $\delta > 0$. Define $\widetilde{f_j} := f_j - \phi_0(f_j)$, for $j = 1, \ldots, n$, then clearly $\phi_0(\widetilde{f_j}) = 0$. Since $\Delta_0 \cap V = \emptyset$, for any $z \in \mathbb{D}$ we have $\pi_z \notin V$ and therefore there exists some $j_0 \in \{1, \ldots, n\}$ such that,

$$\delta \leq |(\pi_z - \phi_0)(f_{j_0})| = |f_{j_0}(z) - \phi_0(f_{j_0})| = |\widetilde{f_{j_0}}(z)|.$$

Since $\widetilde{f_j} \in H^\infty$ for $j = 1, \ldots, n$, and $\sum_{j=1}^n |\widetilde{f_j}(z)| \geq \delta$, there exist $g_1, \ldots, g_n \in H^\infty$ such that $\sum_{j=1}^n \widetilde{f_j} g_j = 1$. But this yields

$$1 = \phi_0(1) = \phi_0 \left(\sum_{j=1}^n \widetilde{f_j} g_j \right) = \sum_{j=1}^n \phi_0(\widetilde{f_j}) \phi_0(g_j) = 0,$$

a contradiction. □

1 First Steps

Over the following sections we will prove a stronger version of the right statement in Proposition 2:

Theorem 3. There exist constants $C_{n,\delta}$ only depending on $n \in \mathbb{N}$ and $\delta > 0$, such that if $f_1, \ldots f_n \in \mathrm{Hol}(\mathbb{D})$ with

$$\|f_j\|_\infty \leq 1, \;\; j = 1, \ldots, n, \quad \text{and} \quad \sum_{j=1}^n |f_j(z)|^2 \geq \delta, \;\; z \in \mathbb{D},$$

then there exist $g_1, \ldots, g_n \in \mathrm{Hol}(\mathbb{D})$ with

$$\|g_j\|_\infty \leq C_{n,\delta}, \;\; j = 1, \ldots, n, \quad \text{and} \quad \sum_{j=1}^n f_j g_j = 1.$$

Proof. We will give the proof in multiple steps.

Step 1 (Reduction to $f_1, \ldots, f_n \in \mathrm{Hol}(\overline{\mathbb{D}})$): Assume that the statement of the theorem holds for all $\widetilde{f_1}, \ldots, \widetilde{f_n} \in \mathrm{Hol}(\overline{\mathbb{D}})$, we claim that it then also holds in its original form[1]. For our given $f_1, \ldots, f_n$ satisfying the premise of the theorem and all $0 < s < 1$ we define $f_{j,s}(z) := f_j(sz)$, $j = 1, \ldots, n$. Then for every $0 < s < 1$ and $j = 1, \ldots, n$ the function $f_{j,s}$ is in $\mathrm{Hol}(\overline{\mathbb{D}})$ and satisfies the premise of the theorem. By our assumption there exist $g_{j,s} \in H^\infty$, $j = 1, \ldots, n$ such that

$$\|g_{j,s}\|_\infty \leq C_{n,\delta}, \;\; j = 1, \ldots, n, \quad \text{and} \quad \sum_{j=1}^n f_{j,s} g_{j,s} = 1.$$

[1]Note that this does **not** mean that we can assume $f_1, \ldots, f_n \in \mathrm{Hol}(\overline{\mathbb{D}})$ in the previous proposition.

For a fixed $j \in \{1, \dots, n\}$, the set $\{g_{j,s} : 0 < s < 1\}$ is uniformly bounded and therefore normal in $\mathrm{Hol}(\mathbb{D})$. By Montel's Theorem there exists a sequence $s_m \to 1$ and some $g_j \in \mathrm{Hol}(\mathbb{D})$ such that $g_{j,s_m} \to g_j$ compactly. In particular, we obtain

$$\|g_j\|_\infty = \lim_{m\to\infty} \|g_{j,s_m}\|_\infty \leq C_{n,\delta}, \quad j = 1, \dots, n,$$

and

$$1 = \lim_{m\to\infty} \sum_{j=1}^{n} f_{j,s_m} g_{j,s_m} = \sum_{j=1}^{n} f_j g_j,$$

concluding our claim. We may thus assume that our given $f_1, \dots, f_n$ are holomorphic on $\overline{\mathbb{D}}$ instead.

Step 2 (Solve in $C^\infty(\overline{\mathbb{D}})$): For $j = 1, \dots, n$ we define

$$h_j := \frac{\bar{f}_j}{\sum_{k=1}^{n} |f_k|^2} \in C^\infty(\overline{\mathbb{D}}),$$

then clearly $\sum_{j=1}^{n} f_j h_j = 1$ and $\|h_j\|_\infty \leq \frac{1}{\delta}$. The real task now lies in changing the h_j to become holomorphic in $\mathbb{D}$, without losing control over the boundedness of the solutions.

2 Wirtinger Derivatives

Before we continue we want to briefly introduce a useful generalization of the complex derivative.

Definition 4. Let $\Omega \subseteq \mathbb{R}^2$ be open. Then the *Wirtinger derivatives* (or *Wirtinger operators*) are defined on $C^1(\Omega)$ by

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

We will also abbreviate these operators as ∂ and $\bar{\partial}$, respectively.

Note that by writing a complex number $z \in \mathbb{C}$ as $z = x + iy$ with $x, y \in \mathbb{R}$ we can identify $\mathbb{C} \cong \mathbb{R}^2$. Therefore we can also interpret the Wirtinger operators to act on $C^1(\Omega)$ with an open subset $\Omega \subseteq \mathbb{C}$.

Before listing properties of the Wirtinger operators we quickly want to recall that a function $f \in C^1(\Omega), f = u + iv$ is holomorphic if and only if it satisfies the *Cauchy–Riemann equations*:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Remark 5. Let $\Omega \subseteq \mathbb{C}$ be open and $f \in C^1(\Omega)$.

1. The Wirtinger operators are $\mathbb{C}$-linear, satisfy the Leibniz rule[2] and
$$\overline{\left(\frac{\partial f}{\partial z}\right)} = \frac{\partial \bar{f}}{\partial \bar{z}}, \quad \overline{\left(\frac{\partial f}{\partial \bar{z}}\right)} = \frac{\partial \bar{f}}{\partial z}$$
2. If $f \in \operatorname{Hol}(\Omega)$, $f = u + iv$, then
$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x} = f'.$$
3. Since
$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) = \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),\end{aligned}$$
we have that[3] $f \in \operatorname{Hol}(\Omega)$ if and only if $\overline{\partial} f = 0$.
4. On $C^2(\Omega)$, the *Laplace operator* can be represented as
$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}.$$

Proof (continued). Step 3 (The Koszul complex): We consider the spaces
$$C_0 \coloneqq C^\infty(\overline{\mathbb{D}}), \quad C_1 \coloneqq (C_0)^n, \quad C_2 \coloneqq \{A \in (C_0)^{n\times n} : A = -A^T\}$$
and the linear maps
$$P_{1,0} : C_1 \to C_0,\ (g_j)_{j=1}^n \mapsto \sum_{j=1}^n g_j f_j, \quad P_{2,1} : C_2 \to C_1,\ (g_{jk})_{j,k=1}^n \mapsto \left(\sum_{k=1}^n g_{jk} f_k\right)_{j=1}^n.$$
Applying $\overline{\partial}$ pointwise in C_j, $j = 0, 1, 2$, the resulting connections are visualized in the diagram below, called the *Koszul complex*:
$$\begin{tikzcd} C_2 \arrow[r, "P_{2,1}"] \arrow[d, "\overline{\partial}"'] & C_1 \arrow[r, "P_{1,0}"] \arrow[d, "\overline{\partial}"'] & C_0 \arrow[d, "\overline{\partial}"'] \\ C_2 \arrow[r, "P_{2,1}"'] & C_1 \arrow[r, "P_{1,0}"'] & C_0 \end{tikzcd}$$

Lemma 6. The Koszul complex has the following properties:

1. The diagram is commutative, that is we have $P_{j+1,j}\overline{\partial} = \overline{\partial} P_{j+1,j}$ for $j = 0, 1$.
2. The horizontal sequences are exact, that is $\operatorname{ran} P_{2,1} = \ker P_{1,0}$.

[2] This means that the Wirtinger opeartors are derivatives from an algebraic perspective.
[3] This can be interpreted as "f is independant of $\overline{z}$".

3. The maps $\overline{\partial} : C_j \to C_j$ for $j = 0, 1, 2$ are surjective.

Proof.

1. For $g \in C_0$ and $f \in \mathrm{Hol}(\overline{\mathbb{D}})$ we have

$$\frac{\partial(gf)}{\partial \bar{z}} = \frac{\partial g}{\partial \bar{z}} \cdot f + g \cdot \frac{\partial f}{\partial \bar{z}} = \frac{\partial g}{\partial \bar{z}} \cdot f$$

   and together with the linearity of $\overline{\partial}$ the statement follows.

2. "$\subseteq$": Let $g \in C_2, g = (g_{jk})_{j,k=1}^n$, then

$$P_{1,0}P_{2,1}g = P_{1,0}\left[\left(\sum_{k=1}^n g_{jk} f_k\right)_{j=1}^n\right] = \sum_{j=1}^n \sum_{k=1}^n g_{jk} f_k f_j = 0$$

   since g is skew-symmetric and therefore $g \in \ker P_{1,0}$.

   "$\supseteq$" Let $g \in \ker P_{1,0} \subseteq C_1, g = (g_1, \dots, g_n)$. We define $p = (p_{jk})_{j,k=1}^n \in C_2$ by

$$p_{jk} := \frac{1}{\sum_{\ell=1}^n |f_\ell|^2}(g_j \overline{f_k} - g_k \overline{f_j}).$$

   Then for any $j = 1, \dots, n$ we have

$$\begin{aligned}(P_{2,1}p)_j &= \sum_{k=1}^n p_{jk} f_k = \frac{1}{\sum_{\ell=1}^n |f_\ell|^2} \sum_{k=1}^n (g_j |f_k|^2 - g_k \overline{f_j} f_k) = \\ &= g_j - \frac{1}{\sum_{\ell=1}^n |f_\ell|^2} \overline{f_j} \sum_{k=1}^n g_k f_k = g_j - \frac{1}{\sum_{\ell=1}^n |f_\ell|^2} \overline{f_j} P_{1,0} g = \\ &= g_j,\end{aligned}$$

   and therefore $g_j \in \operatorname{ran} P_{2,1}$.

3. For given $v \in C^\infty(\overline{\mathbb{D}})$ we want to solve the partial differential equation

$$\frac{\partial u}{\partial \bar{z}} = v$$

   for some $u \in C^\infty(\overline{\mathbb{D}})$.

   ...

   Arguing pointwise shows the surjectivity of the maps $\overline{\partial} : C_\ell \to C_\ell$ for $\ell = 0, 1$. For $\ell = 2$ and given $b = (b_{jk})_{j,k=1}^n \in C_2$ we first solve

$$\overline{\partial} a_{jk} = b_{jk}, \quad \text{for} \quad 1 \le j < k \le n$$

   and then set $a_{jj} = 0$ and $a_{jk} = -a_{kj}$ for $n \ge j > k \ge 1$.

$\square$

Proof (continued). Step 4 (Apply to $h = (h_1, \dots, h_n) \in C_1$): In step 2 we constructed an element $h = (h_1, \dots, h_n) \in C_1$ by setting

$$h_j \coloneqq \frac{\bar{f}_j}{\sum_{k=1}^n |f_k|^2}.$$

By our construction we have $P_{1,0}h = 1$ and therefore $0 = \overline{\partial} P_{1,0} h = P_{1,0} \overline{\partial} h$, thus $\overline{\partial} h \in \ker P_{1,0}$. By Lemma 6 there exists $b \in C_2$ such that $P_{2,1} b = \overline{\partial} h$ and $a \in C_2$ such that $\overline{\partial} a = b$. We now set $g \coloneqq h - P_{2,1} a \in C_1$. Then

$$P_{1,0} g = P_{1,0} h - P_{1,0} P_{2,1} a = 1$$

and

$$\overline{\partial} g = \overline{\partial} h - \overline{\partial} P_{2,1} a = \overline{\partial} h - P_{2,1} b = 0.$$

Therefore g is a solution to

$$\sum_{k=1}^n f_k g_k = 1$$

in $\mathrm{Hol}(\overline{\mathbb{D}})$. However, we do not have an estimate on $|g_j|$ yet.

3 Hardy Spaces

Let μ denote the Lebesgue measure on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, i.e. the measure such that for a function $f : \mathbb{T} \to \mathbb{C}$ it holds that

$$\int_{\mathbb{T}} f \, \mathrm{d}\mu = \int_{-\pi}^{\pi} f(e^{i\vartheta}) \, \mathrm{d}\vartheta.$$

We define the $L^p(\mathbb{T})$-norms via the *normed* Lebesgue measure $\frac{1}{2\pi}\mu$:

$$\|f\|_p \coloneqq \left(\frac{1}{2\pi} \int_{\mathbb{T}} |f|^p \, \mathrm{d}\mu \right)^{1/p}, \quad 1 \le p < \infty,$$
$$\|f\|_\infty \coloneqq \operatorname{ess\,sup} |f|.$$

For $1 \le p \le \infty$ we define the *Hardy space* H^p as the set of all $f \in \mathrm{Hol}(\mathbb{D})$ with $\|f\|_p < \infty$, where

$$\|f\|_p \coloneqq \lim_{r \to 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\vartheta})|^p \, \mathrm{d}\vartheta \right)^{1/p} \quad \text{for } p < \infty, \quad \text{and} \quad \|f\|_\infty \coloneqq \sup_{z \in \mathbb{D}} |f(z)|.$$

For $f \in L^1(\mathbb{T})$ and $n \in \mathbb{N}$ we define the n-th *Fourier coefficient* by

$$\hat{f}(n) \coloneqq \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\vartheta}) e^{-in\vartheta} \, \mathrm{d}\vartheta.$$

We define H_0^1 as the (closed) subspace of all $f \in H^1$, for which $f(0) = 0$.

We summarize the characterisation of Hardy spaces:

Theorem 7. Let $1 \le p \le \infty$. Then:

1. H^p is a Banach space[4].
2. For $p \le q \le \infty$ it holds that $H^p \supseteq H^q$.
3. Let $f \in H^p$, then for almost all $e^{i\vartheta} \in \mathbb{T}$ the limit
$$\lim_{r\to 1} f(re^{i\vartheta}) =: f^*(e^{i\vartheta})$$
exists and defines a function in $L^p(\mathbb{T})$, also called the *boundary values* of f.
4. The map $^* : f \mapsto f^*$ is an isometry from H^p onto
$$L^p_+(\mathbb{T}) := \{f \in L^p(\mathbb{T}) : \forall n < 0 : \hat{f}(n) = 0\},$$
which is a closed subspace of $L^p(\mathbb{T})$.
5. Every $f \in H^p$ can be written as a Cauchy integral of its boundary values:
$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f^*(\zeta)}{\zeta - z} \,\mathrm{d}\mu(\zeta), \quad z \in \mathbb{D}$$

Lemma 8. The map

$$\Phi : L^\infty(\mathbb{T})/(H^\infty)^* \to ((H^1_0)^*)',\ f + (H^\infty)^* \mapsto \left[g \mapsto \frac{1}{2\pi}\int_{\mathbb{T}} fg \,\mathrm{d}\mu\right]$$

is an isometric isomorphism.

Proof. We have $L^\infty(\mathbb{T}) \cong L^1(\mathbb{T})'$ via the duality

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{\mathbb{T}} fg \,\mathrm{d}\mu, \quad f \in L^\infty(\mathbb{T}),\ g \in L^1(\mathbb{T}).$$

Since $(H^1_0)^* \le L^1(\mathbb{T})$ we therefore have $((H^1_0)^*)' \cong L^\infty(\mathbb{T})/((H^1_0)^*)^\perp$ via

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{\mathbb{T}} fg \,\mathrm{d}\mu, \quad f \in L^\infty(\mathbb{T})/((H^1_0)^*)^\perp,\ g \in (H^1_0)^*.$$

It remains to show that $((H^1_0)^*)^\perp = (H^\infty)^*$. Let $w^* \in ((H^1_0)^*)^\perp \le L^\infty(\mathbb{T})$, then for any $n \in \mathbb{N}$ we have

$$0 = \langle w^*, (z^n)^*\rangle = \langle w^*, e^{int}\rangle = \widehat{w^*}(-n).$$

Therefore $w^* \in L^\infty_+(\mathbb{T}) = (H^\infty)^*$. For the other inclusion let $w^* \in (H^\infty)^*$ and $h^* \in (H^1_0)^*$, then

$$\begin{aligned}\langle w^*, h^*\rangle &= \frac{1}{2\pi}\int_0^{2\pi} w^*(e^{i\vartheta})h^*(e^{i\vartheta}) \,\mathrm{d}\vartheta = \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} w(re^{i\vartheta})h(re^{i\vartheta}) \,\mathrm{d}\vartheta = \\ &= \lim_{r\to 1} \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{w_r(\zeta)h_r(\zeta)}{\zeta} \,\mathrm{d}\zeta = \lim_{r\to 1} w_r(0)h_r(0) = 0,\end{aligned}$$

where the second-to-last equation follows since $w_r h_r \in \mathrm{Hol}(\overline{\mathbb{D}})$ and the last one since $h(0) = 0$. Therefore $w^* \in ((H^1_0)^*)^\perp$, concluding the proof. □

[4] In particular, H^∞ is a Banach algebra, which we already used in the introduction.

Proof (continued). Step 5 (Dualisation): Applying the above lemma to our previous situation we can re-describe the norm of $v + (H^\infty)^*$:

$$
\begin{aligned}
\|v + (H^\infty)^*\|_\infty = \|\Phi(v + (H^\infty)^*)\| = \\
&= \sup\{|\Phi(v + (H^\infty)^*)(g)| : f \in (H_0^1)^*, \|f\|_1 \leq 1\} = \\
&= \sup\left\{\left|\frac{1}{2\pi}\int_{\mathbb{T}} vf^* \, \mathrm{d}\mu\right| : f \in H_0^1, \|f\|_1 \leq 1\right\} = \\
&= \sup\left\{\left|\frac{1}{2\pi}\int_{\mathbb{T}} vf \, \mathrm{d}\mu\right| : f \in \mathrm{Hol}(\overline{\mathbb{D}}), f(0) = 0, \|f\|_1 \leq 1\right\}
\end{aligned}
$$

Proof. □

Proof (continued). Step 6: todo

We want to redescribe the integral

$$\frac{1}{2\pi}\int_{\mathbb{T}} vf \, \mathrm{d}\mu,$$

where $v \in L^\infty(\mathbb{T})$ and $f \in \mathrm{Hol}(\overline{\mathbb{D}}), f(0) = 0, \|f\|_1 \leq 1$. Let $\psi := vf$ and

$$\Gamma(z) := \frac{1}{2\pi}\log|z|, \quad z \neq 0.$$

By Green's second identity we have

$$\int_{\mathbb{D}} \Gamma\Delta\psi - \psi\Delta\Gamma \, \mathrm{d}\lambda^2 = \int_{\mathbb{T}} \Gamma\frac{\partial\psi}{\partial r} - \psi\frac{\partial\Gamma}{\partial r} \, \mathrm{d}\mu,$$

where $\frac{\partial}{\partial r}$ denotes the radial derivative. Since $\Gamma|_{\mathbb{T}} = 0$ and $\frac{\partial\Gamma}{\partial r}\big|_{\mathbb{T}} = \frac{1}{2\pi}$ the right integral equals

$$-\frac{1}{2\pi}\int_{\mathbb{T}} vf \, \mathrm{d}\mu$$

Since distributionally $\Delta\Gamma$ equals δ_0, the delta distribution at 0, we have

$$\int_{\mathbb{D}} \psi\Delta\Gamma \, \mathrm{d}\lambda^2 = \psi(0) = v(0)f(0) = 0.$$

Finally,

$$\Delta\psi = \Delta(vf) = 4\partial\overline{\partial}(vf) = 4\partial(v\overline{\partial}f + f\overline{\partial}v) = 4\partial(fu) = 4(f\partial u + uf').$$

We thus obtain

$$
\begin{aligned}
\frac{1}{2\pi}\int_{\mathbb{T}} vf \, \mathrm{d}\mu = -4\int_{\mathbb{D}} \Gamma\left(f\partial u + uf'\right) \mathrm{d}\lambda^2 = \\
&= -4\left(\int_{\mathbb{D}} f\,\partial u\,\Gamma \, \mathrm{d}\lambda^2 + \int_{\mathbb{D}} uf'\Gamma \, \mathrm{d}\lambda^2\right) =: -4(I_1 + I_2).
\end{aligned}
$$

Our goal is to show the existence of a constant $C_{n,\delta}$ such that

$$\left|\frac{1}{2\pi}\int_{\mathbb{T}} vf \, \mathrm{d}\mu\right| \leq 4(|I_1| + |I_2|) \leq C_{n,\delta}.$$

4 Integral estimates

Lemma 9. Let $f, g, u, v \in \mathrm{Hol}(\overline{\mathbb{D}})$, then the following integral estimates hold:

1. $\displaystyle\int_{\mathbb{D}} |f'|^2 \Gamma \, \mathrm{d}\lambda^2 \leq ? \|f\|_2^2$
2. $\displaystyle\int_{\mathbb{D}} |fg'| \log \frac{1}{|z|} \, \mathrm{d}\lambda^2 \leq 2\pi \|f\|_2^2 \|g\|_\infty$
3. $\displaystyle\int_{\mathbb{D}} |fgu'v'| \log \frac{1}{|z|} \, \mathrm{d}\lambda^2 \leq 2\pi \|f\|_2 \|g\|_2 \|u\|_\infty \|v\|_\infty$
4. $\displaystyle\int_{\mathbb{D}} |fu'v'| \log \frac{1}{|z|} \, \mathrm{d}\lambda^2 \leq 2\pi \|f\|_1 \|u\|_\infty \|v\|_\infty$
5. $\displaystyle\int_{\mathbb{D}} |fg'u'| \log \frac{1}{|z|} \, \mathrm{d}\lambda^2 \leq \pi \|f\|_2 \|g\|_2 \|u\|_\infty$
6. $\displaystyle\int_{\mathbb{D}} |f'u'| \log \frac{1}{|z|} \, \mathrm{d}\lambda^2 \leq 2\pi \|f\|_1 \|u\|_\infty$

Proof.

1. Applying Green's formula on $f\bar{f}$ and Γ yields

$$\int_{\mathbb{D}} \Gamma \Delta(f\bar{f}) - f\bar{f}\Delta\Gamma \, \mathrm{d}\lambda^2 = \int_{\mathbb{T}} \Gamma \frac{\partial}{\partial r}(f\bar{f}) - f\bar{f}\frac{\partial}{\partial r}\Gamma \, \mathrm{d}\mu,$$

or, simplified,

$$\int_{\mathbb{D}} \Gamma \Delta(f\bar{f}) \, \mathrm{d}\lambda^2 - |f(0)|^2 = \int_{\mathbb{T}} |f|^2 \, \mathrm{d}\mu = \|f\|_2^2$$

Since

$$\begin{aligned}\Delta(f\bar{f}) &= 4\partial\overline{\partial}(f\bar{f}) = 4\partial(\bar{f} \cdot \overline{\partial} f + f \cdot \overline{\partial}\bar{f}) = 4\partial(f \cdot \overline{\partial}\bar{f}) = 4(\partial f \cdot \overline{\partial}\bar{f} + f \cdot \overline{\partial}\partial\bar{f}) = \\ &= 4(\partial f \cdot \overline{\partial f} + f \cdot \overline{\partial}\,\overline{\overline{\partial} f}) = 4\partial(f\overline{\partial}\bar{f}) = 4|f'|^2\end{aligned}$$

and $|f(0)|^2 \geq 0$ we obtain

$$\frac{2}{\pi} \int_{\mathbb{D}} |f'|^2 \log \frac{1}{|z|} \, \mathrm{d}\lambda^2 \leq \|f\|_2^2$$

and rearranging yields the desired inequality.

□