

THE CORONA THEOREM

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We will denote the space of all complex-valued, bounded, analytic functions on the unit disk \mathbb{D} as H^∞ . Equipped with the supremum norm $\|\cdot\|_\infty$ this space becomes a commutative Banach algebra. The space of all multiplicative, bounded, linear functionals on H^∞ not identically zero is denoted $\Delta(H^\infty)$ and is called the *Gelfand space* of H^∞ . We endow this space with the subspace topology of the weak-* topology on the dual $(H^\infty)'$, which we will refer to as the *Gelfand topology*. For each $z \in \mathbb{D}$ we consider the point-evaluation functional

$$\pi_z : H^\infty \rightarrow \mathbb{C}, f \mapsto f(z).$$

This is clearly multiplicative, bounded and linear and therefore belongs to $\Delta(H^\infty)$. The set of all such functionals $\pi_z, z \in \mathbb{D}$ will be denoted as Δ_0 . The *corona* is defined as the complement of closure of Δ_0 in the Gelfand topology. The corona theorem now states:

Theorem 1 (L. Carleson). The corona is empty. In other words, Δ_0 is dense in $\Delta(H^\infty)$.

There is an equivalent version of the theorem, as given by the following proposition:

Proposition 2. Δ_0 is dense in $\Delta(H^\infty)$ if and only if for any $\delta > 0$ and $f_1, \dots, f_n \in H^\infty$ such that $\sum_{j=1}^n |f_j(z)| \geq \delta, z \in \mathbb{D}$, there exist $g_1, \dots, g_n \in H^\infty$ such that $\sum_{j=1}^n f_j g_j = 1$.

Proof. Assume Δ_0 is dense in $\Delta(H^\infty)$ and let $f_1, \dots, f_n \in H^\infty$, and $\delta > 0$ such that $\sum_{j=1}^n |f_j(z)| \geq \delta, z \in \mathbb{D}$. Denote by I the ideal in H^∞ generated by f_1, \dots, f_n . If $1 \in I$, then the assertion is established. Assume towards a contradiction that I is a proper ideal, then there exists a maximal ideal $J \supset I$. Since $\Delta(H^\infty)$ is a commutative Banach algebra, there exists a $\phi \in \Delta(H^\infty)$ such that $J = \ker \phi$. Therefore we have $\phi(f_j) = 0$ for $j = 1, \dots, n$. Since Δ_0 is dense, there is a net $(\pi_{z_m})_{m \in M}$ in Δ_0 such that $\pi_{z_m} \rightarrow \phi$ in the weak-* topology, that is the net converges pointwise. Therefore, for all $j = 1, \dots, n$ we have $f_j(z_m) = \pi_{z_m}(f_j) \rightarrow \phi(f_j) = 0$ and in particular

$$\lim_{m \in M} \sum_{j=1}^n |f_j(z_m)| = 0,$$

a contradiction.

For the other implication, assume towards a contradiction that Δ_0 is not dense in $\Delta(H^\infty)$. Then there exists some $\phi_0 \in \Delta(H^\infty)$ and an open neighbourhood U of ϕ_0 such that $\Delta_0 \cap U = \emptyset$. Since the sets of the form

$$\{\phi \in \Delta(H^\infty) : |(\phi - \phi_0)(f_j)| < \varepsilon, j = 1, \dots, n\},$$

for some $n \in \mathbb{N}$, $f_1, \dots, f_n \in H^\infty$ and $\varepsilon > 0$, form a neighbourhood basis of ϕ_0 in the weak-* topology, there exists a neighbourhood $V \subseteq U$ described by some $n \in \mathbb{N}$, $f_1, \dots, f_n \in H^\infty$ and $\delta > 0$. Define $\tilde{f}_j := f_j - \phi_0(f_j)$, for $j = 1, \dots, n$, then clearly $\phi_0(\tilde{f}_j) = 0$. Since $\Delta_0 \cap V = \emptyset$, for any $z \in \mathbb{D}$ we have $\pi_z \notin V$ and therefore there exists some $j_0 \in \{1, \dots, n\}$ such that,

$$\delta \leq |(\pi_z - \phi_0)(f_{j_0})| = |f_{j_0}(z) - \phi_0(f_{j_0})| = |\tilde{f}_{j_0}(z)|.$$

Since $\tilde{f}_j \in H^\infty$ for $j = 1, \dots, n$, and $\sum_{j=1}^n |\tilde{f}_j(z)| \geq \delta$, there exist $g_1, \dots, g_n \in H^\infty$ such that $\sum_{j=1}^n \tilde{f}_j g_j = 1$. But this yields

$$1 = \phi_0(1) = \phi_0\left(\sum_{j=1}^n \tilde{f}_j g_j\right) = \sum_{j=1}^n \phi_0(\tilde{f}_j) \phi_0(g_j) = 0,$$

a contradiction. □

1 First Steps

Over the following sections we will prove a stronger version of the right statement in Proposition 2:

Theorem 3. There exist constants $C_{n,\delta}$ only depending on $n \in \mathbb{N}$ and $\delta > 0$, such that if $f_1, \dots, f_n \in \text{Hol}(\mathbb{D})$ with

$$\|f_j\|_\infty \leq 1, \quad j = 1, \dots, n, \quad \text{and} \quad \sum_{j=1}^n |f_j(z)|^2 \geq \delta, \quad z \in \mathbb{D},$$

then there exist $g_1, \dots, g_n \in \text{Hol}(\mathbb{D})$ with

$$\|g_j\|_\infty \leq C_{n,\delta}, \quad j = 1, \dots, n, \quad \text{and} \quad \sum_{j=1}^n f_j g_j = 1.$$

Proof. We will give the proof in multiple steps.

Step 1 (Reduction to $f_1, \dots, f_n \in \text{Hol}(\overline{\mathbb{D}})$): Assume that the statement of the theorem holds for all $\tilde{f}_1, \dots, \tilde{f}_n \in \text{Hol}(\overline{\mathbb{D}})$, we claim that it then also holds in its original form¹. For our given f_1, \dots, f_n satisfying the premise of the theorem and all $0 < s < 1$ we define $f_{j,s}(z) := f_j(sz)$, $j = 1, \dots, n$. Then for every $0 < s < 1$ and $j = 1, \dots, n$ the function $f_{j,s}$ is in $\text{Hol}(\overline{\mathbb{D}})$ and satisfies the premise of the theorem. By our assumption there exist $g_{j,s} \in H^\infty$, $j = 1, \dots, n$ such that

$$\|g_{j,s}\|_\infty \leq C_{n,\delta}, \quad j = 1, \dots, n, \quad \text{and} \quad \sum_{j=1}^n f_{j,s} g_{j,s} = 1.$$

¹Note that this does **not** mean that we can assume $f_1, \dots, f_n \in \text{Hol}(\overline{\mathbb{D}})$ in the previous proposition.

For a fixed $j \in \{1, \dots, n\}$, the set $\{g_{j,s} : 0 < s < 1\}$ is uniformly bounded and therefore normal in $\text{Hol}(\mathbb{D})$. By Montel's Theorem there exists a sequence $s_m \rightarrow 1$ and some $g_j \in \text{Hol}(\mathbb{D})$ such that $g_{j,s_m} \rightarrow g_j$ compactly. In particular, we obtain

$$\|g_j\|_\infty = \lim_{m \rightarrow \infty} \|g_{j,s_m}\|_\infty \leq C_{n,\delta}, \quad j = 1, \dots, n,$$

and

$$1 = \lim_{m \rightarrow \infty} \sum_{j=1}^n f_{j,s_m} g_{j,s_m} = \sum_{j=1}^n f_j g_j,$$

concluding our claim. We may thus assume that our given f_1, \dots, f_n are holomorphic on $\bar{\mathbb{D}}$ instead.

Step 2 (Solve with $g_1, \dots, g_n \in C^\infty(\bar{\mathbb{D}})$): For $j = 1, \dots, n$ we define

$$h_j := \frac{\bar{f}_j}{\sum_{k=1}^n |f_k|^2},$$

then clearly $h_j \in C^\infty(\bar{\mathbb{D}})$, $\sum_{j=1}^n f_j h_j = 1$ and $\|h_j\| \leq \frac{1}{\delta}$. The real task now lies in changing the h_j to become holomorphic in \mathbb{D} , without losing control over the boundedness of the solutions.

2 Wirtinger Derivatives

Before we continue we want to briefly introduce a useful generalization of the complex derivative.

Definition 4. Let $\Omega \subseteq \mathbb{R}^2$ be open. Then the *Wirtinger derivatives* (or *Wirtinger operators*) are defined on $C^1(\Omega)$ by

$$\frac{\partial}{\partial z} := \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

We will also abbreviate these operators as ∂ and $\bar{\partial}$, respectively.

Note that by writing a complex number $z \in \mathbb{C}$ as $z = x + iy$ with $x, y \in \mathbb{R}$ we can identify $\mathbb{C} \cong \mathbb{R}^2$. Therefore we can also reasonably interpret the Wirtinger operators to act on $C^1(\Omega)$ with an open subset $\Omega \subseteq \mathbb{C}$.

Before listing properties of the Wirtinger operators we quickly want to recall that a function $f \in C^1(\Omega)$, $f = u + iv$ is holomorphic if and only if it satisfies the *Cauchy-Riemann equations*:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Remark 5. Let $\Omega \subseteq \mathbb{C}$ be open and $f \in C^1(\Omega)$.

1. The Wirtinger operators are \mathbb{C} -linear, satisfy the Leibniz rule² and

$$\overline{\left(\frac{\partial f}{\partial z}\right)} = \frac{\partial \bar{f}}{\partial \bar{z}}, \quad \overline{\left(\frac{\partial f}{\partial \bar{z}}\right)} = \frac{\partial \bar{f}}{\partial z}$$

2. If $f \in \text{Hol}(\Omega)$, then

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) = \frac{1}{2} \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} - i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial f}{\partial x} = f'.$$

3. Since

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}} &= \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) = \frac{1}{2} \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} + i \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} \right) = \\ &= \frac{1}{2} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + \frac{i}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right), \end{aligned}$$

we have that³ $f \in \text{Hol}(\Omega)$ if and only if $\bar{\partial}f = 0$.

4. On $C^2(\Omega)$, the *Laplace operator* can be represented as

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}.$$

Proof. Step 3 (The Koszul complex): We consider the spaces

$$C_0 := C^\infty(\overline{\mathbb{D}}), \quad C_1 := (C_0)^n, \quad C_2 := \{A \in (C_0)^{n \times n} : A = -A^T\}$$

and the maps

$$P_{1,0} : C_1 \rightarrow C_0, (g_j)_{j=1}^n \mapsto \sum_{j=1}^n g_j f_j, \quad P_{2,1} : C_2 \rightarrow C_1, (g_{jk})_{j,k=1}^n \mapsto \left(\sum_{k=1}^n g_{jk} f_k \right)_{j=1}^n.$$

Applying $\bar{\partial}$ pointwise in $C_j, j = 0, 1, 2$, the resulting connections are visualized in the diagram below, called the *Koszul complex*:

$$\begin{array}{ccccc} C_2 & \xrightarrow{P_{2,1}} & C_1 & \xrightarrow{P_{1,0}} & C_0 \\ \bar{\partial} \downarrow & & \bar{\partial} \downarrow & & \bar{\partial} \downarrow \\ C_2 & \xrightarrow{P_{2,1}} & C_1 & \xrightarrow{P_{1,0}} & C_0 \end{array}$$

Lemma 6. The Koszul complex has the following properties:

1. The diagram is commutative, that is we have $P_{j+1,j} \bar{\partial} = \bar{\partial} P_{j+1,j}$ for $j = 0, 1$.
2. The horizontal sequences are exact, that is $\text{ran } P_{2,1} = \ker P_{1,0}$.

²This means that the Wirtinger operators are derivatives from an algebraic perspective.

³This can be interpreted as “ f is independant of \bar{z} ”.

3. The maps $\bar{\partial} : C_j \rightarrow C_j$ for $j = 0, 1, 2$ are surjective.

Proof.

1. For $g \in C_0$ and $f \in \text{Hol}(\mathbb{D})$ we have

$$\frac{\partial(gf)}{\partial \bar{z}} = \frac{\partial g}{\partial \bar{z}} \cdot f + g \cdot \frac{\partial f}{\partial \bar{z}} = \frac{\partial g}{\partial \bar{z}} \cdot f$$

and together with the linearity of $\bar{\partial}$ the statement follows.

2. “ \subseteq ”: Let $g \in C_2, g = (g_{jk})_{j,k=1}^n$, then

$$P_{1,0}P_{2,1}g = P_{1,0} \left[\left(\sum_{k=1}^n g_{jk} f_k \right)_{j=1}^n \right] = \sum_{j=1}^n \sum_{k=1}^n g_{jk} f_k f_j = 0$$

since g is skew-symmetric and therefore $g \in \ker P_{1,0}$.

“ \supseteq ” Let $g \in \ker P_{1,0} \subseteq C_1, g = (g_1, \dots, g_n)$. We define $p = (p_{jk})_{j,k=1}^n \in C_2$ by

$$p_{jk} := \frac{1}{\sum_{\ell=1}^n |f_\ell|^2} (g_j \bar{f}_k - g_k \bar{f}_j).$$

Then for any $j = 1, \dots, n$ we have

$$\begin{aligned} P_{2,1}p &= \sum_{k=1}^n p_{jk} f_k = \frac{1}{\sum_{\ell=1}^n |f_\ell|^2} \sum_{k=1}^n (g_j |f_k|^2 - g_k \bar{f}_j f_k) = \\ &= g_j - \frac{1}{\sum_{\ell=1}^n |f_\ell|^2} \bar{f}_j \sum_{k=1}^n g_k f_k = g_j - \frac{1}{\sum_{\ell=1}^n |f_\ell|^2} \bar{f}_j P_{1,0}g = \\ &= g_j, \end{aligned}$$

and therefore $g_j \in \text{ran } P_{2,1}$.

3. For given $v \in C^\infty(\mathbb{D})$ we want to solve the partial differential equation

$$\frac{\partial u}{\partial \bar{z}} = v$$

for some $u \in C^\infty(\mathbb{D})$.

...

Arguing pointwise shows the surjectivity of the maps $\bar{\partial} : C_\ell \rightarrow C_\ell$ for $\ell = 0, 1$. For $\ell = 2$ and given $b = (b_{jk})_{j,k=1}^n \in C_2$ we first solve

$$\bar{\partial} a_{jk} = b_{jk}, \quad \text{for } 1 \leq j < k \leq n$$

and then set $a_{jj} = 0$ and $a_{jk} = -a_{kj}$ for $n \geq j > k \geq 1$.

□

Proof (continued). Step 4 (Apply to $h = (h_1, \dots, h_n) \in C_1$): todo

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3 Hardy Spaces

For $1 \leq p \leq \infty$ we define the *Hardy space* H^p as the set of all $f \in \text{Hol}(\mathbb{D})$ with $\|f\|_p < \infty$, where

$$\|f\|_p := \lim_{r \rightarrow 1} \left[\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\vartheta})|^p d\vartheta \right]^{1/p} \quad \text{for } p < \infty, \quad \text{and} \quad \|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)|.$$

For $f \in L^1(\mathbb{T})$ and $n \in \mathbb{N}$ we define the n -th *Fourier coefficient* by

$$\hat{f}(n) := \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{inx} d\mu(x).$$

We define H_0^1 as the (closed) subspace of all $f \in H^1$, for which $f(0) = 0$.

We summarize the characterisation of Hardy spaces:

Theorem 7. Let $1 \leq p \leq \infty$. Then:

1. H^p is a Banach space⁴.
2. For $p \leq q \leq \infty$ it holds that $H^p \supseteq H^q$.
3. Let $f \in H^p$, then for almost all $e^{i\vartheta} \in \mathbb{T}$ the limit

$$\lim_{r \rightarrow 1} f(re^{i\vartheta}) =: f^*(e^{i\vartheta})$$

exists and defines a function in $L^p(\mathbb{T})$, also called the *boundary values* of f .

4. The map $*$: $f \mapsto f^*$ is an isometry from H^p onto

$$L_+^p(\mathbb{T}) := \{f \in L^p(\mathbb{T}) : \forall n < 0 : \hat{f}(n) = 0\},$$

which is a closed subspace of $L^p(\mathbb{T})$.

5. Every $f \in H^p$ can be written as a Cauchy integral of its boundary values:

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f^*(\zeta)}{\zeta - z} d\mu(\zeta), \quad z \in \mathbb{D}$$

Lemma 8. The map

$$\Phi : L^\infty(\mathbb{T}) / (H^\infty)^* \rightarrow ((H_0^1)^*)', f + (H^\infty)^* \mapsto \left[g \mapsto \frac{1}{2\pi} \int_{\mathbb{T}} fg d\mu \right]$$

is an isometric isomorphism.

⁴In particular, H^∞ is a Banach algebra, which we already used in the introduction.

Proof. We have $L^\infty(\mathbb{T}) \cong L^1(\mathbb{T})'$ via the duality

$$\langle f, g \rangle := \frac{1}{2\pi} \int_{\mathbb{T}} f g \, d\mu, \quad f \in L^\infty(\mathbb{T}), \, g \in L^1(\mathbb{T}).$$

Since $(H_0^1)^* \leq L^1(\mathbb{T})$ we therefore have $((H_0^1)^*)' \cong L^\infty(\mathbb{T})/((H_0^1)^*)^\perp$ via

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{\mathbb{T}} f g \, d\mu, \quad f \in L^\infty(\mathbb{T})/((H_0^1)^*)^\perp, \, g \in (H_0^1)^*.$$

It remains to show that $((H_0^1)^*)^\perp = (H^\infty)^*$. Let $w^* \in ((H_0^1)^*)^\perp \leq L^\infty(\mathbb{T})$, then for any $n \in \mathbb{N}$ we have

$$0 = \langle w^*, (z^n)^* \rangle = \langle w^*, e^{int} \rangle = \widehat{w^*}(-n).$$

Therefore $w^* \in L_+^\infty(\mathbb{T}) = (H^\infty)^*$. For the other inclusion let $w^* \in (H^\infty)^*$ and $h^* \in (H_0^1)^*$, then

$$\begin{aligned} \langle w^*, h^* \rangle &= \frac{1}{2\pi} \int_0^{2\pi} w^*(e^{i\vartheta}) h^*(e^{i\vartheta}) \, d\vartheta = \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} w(re^{i\vartheta}) h(re^{i\vartheta}) \, d\vartheta = \\ &= \lim_{r \rightarrow 1} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{w_r(\zeta) h_r(\zeta)}{\zeta} \, d\zeta = \lim_{r \rightarrow 1} w_r(0) h_r(0) = 0, \end{aligned}$$

where the second-to-last equation follows since $w_r h_r \in \text{Hol}(\overline{\mathbb{D}})$ and the last one since $h(0) = 0$. Therefore $w^* \in ((H_0^1)^*)^\perp$, concluding the proof. \square

Proof (continued). Step 5 (Dualisation): Applying the above lemma to our previous situation we can re-describe the norm of $v + (H^\infty)^*$:

$$\begin{aligned} \|v + (H^\infty)^*\|_\infty &= \|\Phi(f + (H^\infty)^*)\| = \sup\{|\Phi(f + (H^\infty)^*)(g)| : g \in (H_0^1)^*, \|g\|_1 \leq 1\} = \\ &= \sup\left\{\left|\frac{1}{2\pi} \int_{-\pi}^{\pi} f g \, dx\right| : g \in H_0^1, \|g\|_1 \leq 1\right\} \end{aligned}$$

Proof. \square

Proof (continued). Step 6: todo

4 Integral estimates

Lemma 9. Let $f, g, u, v \in \text{Hol}(\overline{\mathbb{D}})$, then the following integral estimates hold:

1. $\int_{\mathbb{D}} |f'|^2 \log \frac{1}{|z|} \, d\lambda^2 \leq \frac{\pi}{2} \|f\|_2^2$
2. $\int_{\mathbb{D}} |f g'| \log \frac{1}{|z|} \, d\lambda^2 \leq 2\pi \|f\|_2^2 \|g\|_\infty$
3. $\int_{\mathbb{D}} |f g u' v'| \log \frac{1}{|z|} \, d\lambda^2 \leq 2\pi \|f\|_2 \|g\|_2 \|u\|_\infty \|v\|_\infty$

4. $\int_{\mathbb{D}} |fu'v'| \log \frac{1}{|z|} d\lambda^2 \leq 2\pi \|f\|_1 \|u\|_{\infty} \|v\|_{\infty}$
5. $\int_{\mathbb{D}} |fg'u'| \log \frac{1}{|z|} d\lambda^2 \leq \pi \|f\|_2 \|g\|_2 \|u\|_{\infty}$
6. $\int_{\mathbb{D}} |f'u'| \log \frac{1}{|z|} d\lambda^2 \leq 2\pi \|f\|_1 \|u\|_{\infty}$

Proof.

1. Applying Green's formula on $f\bar{f}$ yields

$$\int_{\mathbb{D}} (\log \frac{1}{|z|} \Delta(f\bar{f}) - f\bar{f})$$

$$|f(0)|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f^*(e^{i\vartheta})|^2 d\vartheta - \frac{1}{2\pi} \int_{\mathbb{D}} \Delta(f\bar{f}) \log \frac{1}{|z|} d\lambda^2$$

Since

$$\begin{aligned} \Delta(f\bar{f}) &= 4\partial\bar{\partial}(f\bar{f}) = 4\partial(\bar{f} \cdot \bar{\partial}f + f \cdot \bar{\partial}\bar{f}) = 4\partial(f \cdot \bar{\partial}\bar{f}) = 4(\partial f \cdot \bar{\partial}\bar{f} + f \cdot \bar{\partial}\partial\bar{f}) = \\ &= 4(\partial f \cdot \bar{\partial}\bar{f} + f \cdot \bar{\partial}\bar{\partial}\bar{f}) = 4\partial(f\bar{\partial}\bar{f}) = 4|f'|^2 \end{aligned}$$

and $|f(0)|^2 \geq 0$ we obtain

$$\frac{2}{\pi} \int_{\mathbb{D}} |f'|^2 \log \frac{1}{|z|} d\lambda^2 \leq \|f\|_2^2$$

and rearranging yields the desired inequality.

□