The Corona Theorem

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These are my notes for my talk in the winter semester of 2024, in the analysis seminar at the Technical University of Vienna.

We will denote the space of all complex-valued, bounded, analytic functions on the unit disk $\mathbb{D}$ as H^∞. Equipped with the supremum norm $\|\cdot\|_\infty$ this space becomes a commutative Banach algebra. The space of all multiplicative, bounded, linear functionals on H^∞ not identically zero is denoted $\Delta(H^\infty)$ and is called the *Gelfand space* of H^∞. We endow this space with the subspace topology of the weak-* topology on the dual $(H^\infty)'$, which we will refer to as the *Gelfand topology.* For each $z \in \mathbb{D}$ we consider the point-evaluation functional

$$\pi_z : H^\infty \to \mathbb{C},\ f \mapsto f(z).$$

This is clearly multiplicative, bounded and linear and therefore belongs to $\Delta(H^\infty)$. The set of all such functionals $\pi_z, z \in \mathbb{D}$ will be denoted as Δ_0. The *corona* is defined as the complement of closure of Δ_0 in the Gelfand topology. The corona theorem now states:

Theorem 1 (L. Carleson)**.** The corona is empty. In other words, Δ_0 is dense in $\Delta(H^\infty)$.

There is an equivalent version of the theorem, as given by the following proposition:

Proposition 2. Δ_0 is dense in $\Delta(H^\infty)$ if and only if for any $\delta > 0$ and $f_1, \dots, f_n \in H^\infty$ such that $\sum_{j=1}^n |f_j(z)| \geq \delta, z \in \mathbb{D}$, there exist $g_1, \dots g_n \in H^\infty$ such that $\sum_{j=1}^n f_j g_j = 1$.

Proof. Assume Δ_0 is dense in $\Delta(H^\infty)$ and let $f_1, \dots, f_n \in H^\infty$, and $\delta > 0$ such that $\sum_{j=1}^n |f_j(z)| \geq \delta, z \in \mathbb{D}$. Denote by I the ideal in H^∞ generated by $f_1, \dots, f_n$. If $1 \in I$, then the assertion is established. Assume towards a contradiction that I is a proper ideal, then there exists a maximal ideal $J \supset I$. Since $\Delta(H^\infty)$ is a commutative Banach algebra, there exists a $\phi \in \Delta(H^\infty)$ such that $J = \ker \phi$. Therefore we have $\phi(f_j) = 0$ for $j = 1, \dots, n$. Since Δ_0 is dense, there is a net $(\pi_{z_m})_{m \in M}$ in Δ_0 such that $\pi_{z_m} \to \phi$ in the weak-* topology, that is the net converges pointwise. Therefore, for all $j = 1, \dots, n$ we have $f_j(z_m) = \pi_{z_m}(f_j) \to \phi(f_j) = 0$ and in particular

$$\lim_{m \in M} \sum_{j=1}^n |f_j(z_m)| = 0,$$

a contradiction.

For the other implication, assume towards a contradiction that Δ_0 is not dense in $\Delta(H^\infty)$. Then there exists some $\phi_0 \in \Delta(H^\infty)$ and an open neighbourhood U of ϕ_0 such that $\Delta_0 \cap U = \emptyset$. Since the sets of the form

$$\{\phi \in \Delta(H^\infty) : |(\phi - \phi_0)(f_j)| < \varepsilon, j = 1, \dots, n\},$$

for some $n \in \mathbb{N}, f_1, \ldots, f_n \in H^\infty$ and $\varepsilon > 0$, form a neighbourhood basis of ϕ_0 in the weak-* topology, there exists a neighbourhood $V \subseteq U$ described by some $n \in \mathbb{N}, f_1, \ldots, f_n \in H^\infty$ and $\delta > 0$. Define $\widetilde{f_j} \coloneqq f_j - \phi_0(f_j)$, for $j = 1, \ldots, n$, then clearly $\phi_0(\widetilde{f_j}) = 0$. Since $\Delta_0 \cap V = \emptyset$, for any $z \in \mathbb{D}$ we have $\pi_z \notin V$ and therefore there exists some $j_0 \in \{1, \ldots, n\}$ such that,

$$\delta \leq |(\pi_z - \phi_0)(f_{j_0})| = |f_{j_0}(z) - \phi_0(f_{j_0})| = |\widetilde{f_{j_0}}(z)|.$$

Since $\widetilde{f_j} \in H^\infty$ for $j = 1, \ldots, n$, and $\sum_{j=1}^n |\widetilde{f_j}(z)| \geq \delta$, there exist $g_1, \ldots, g_n \in H^\infty$ such that $\sum_{j=1}^n \widetilde{f_j} g_j = 1$. But this yields

$$1 = \phi_0(1) = \phi_0\left(\sum_{j=1}^n \widetilde{f_j} g_j\right) = \sum_{j=1}^n \phi_0(\widetilde{f_j})\phi_0(g_j) = 0,$$

a contradiction. □

1 First Steps

Over the following sections we will prove a stronger version of the right statement in Proposition 2:

Theorem 3. There exist constants $C_{n,\delta}$ only depending on $n \in \mathbb{N}$ and $\delta > 0$, such that if $f_1, \ldots f_n \in \operatorname{Hol}(\mathbb{D})$ with

$$\|f_j\|_\infty \leq 1, \;\; j = 1, \ldots, n, \quad \text{and} \quad \sum_{j=1}^n |f_j(z)|^2 \geq \delta, \;\; z \in \mathbb{D},$$

then there exist $g_1, \ldots, g_n \in \operatorname{Hol}(\mathbb{D})$ with

$$\|g_j\|_\infty \leq C_{n,\delta}, \;\; j = 1, \ldots, n, \quad \text{and} \quad \sum_{j=1}^n f_j g_j = 1.$$

Proof. We will give the proof in multiple steps.

Step 1 (Reduction to $f_1, \ldots, f_n \in \operatorname{Hol}(\overline{\mathbb{D}})$): Assume that the statement of the theorem holds for all $\widetilde{f_1}, \ldots, \widetilde{f_n} \in \operatorname{Hol}(\overline{\mathbb{D}})$, we claim that it then also holds in its original form[1]. For our given $f_1, \ldots, f_n$ satisfying the premise of the theorem and all $0 < s < 1$ we define $f_{j,s}(z) \coloneqq f_j(sz)$, $j = 1, \ldots, n$. Then for every $0 < s < 1$ and $j = 1, \ldots, n$ the function $f_{j,s}$ is in $\operatorname{Hol}(\overline{\mathbb{D}})$ and satisfies the premise of the theorem. By our assumption there exist $g_{j,s} \in H^\infty$, $j = 1, \ldots, n$ such that

$$\|g_{j,s}\|_\infty \leq C_{n,\delta}, \;\; j = 1, \ldots, n, \quad \text{and} \quad \sum_{j=1}^n f_{j,s} g_{j,s} = 1.$$

[1]Note that this does **not** mean that we can assume $f_1, \ldots, f_n \in \operatorname{Hol}(\overline{\mathbb{D}})$ in the previous proposition.

For a fixed $j \in \{1, \ldots, n\}$, the set $\{g_{j,s} : 0 < s < 1\}$ is uniformly bounded and therefore normal in $\mathrm{Hol}(\mathbb{D})$. By Montel's Theorem there exists a sequence $s_m \to 1$ and some $g_j \in \mathrm{Hol}(\mathbb{D})$ such that $g_{j,s_m} \to g_j$ compactly. In particular, we obtain

$$\|g_j\|_\infty = \lim_{m\to\infty} \|g_{j,s_m}\|_\infty \le C_{n,\delta}, \quad j = 1, \ldots, n,$$

and

$$1 = \lim_{m\to\infty} \sum_{j=1}^{n} f_{j,s_m} g_{j,s_m} = \sum_{j=1}^{n} f_j g_j,$$

concluding our claim. We may thus assume that our given $f_1, \ldots, f_n$ are holomorphic on $\overline{\mathbb{D}}$ instead.

Step 2 (Solve with $g_1, \ldots, g_n \in C^\infty(\overline{\mathbb{D}})$): For $j = 1, \ldots, n$ we define

$$h_j := \frac{\bar{f}_j}{\sum_{k=1}^{n} |f_k|^2},$$

then clearly $h_j \in C^\infty(\overline{\mathbb{D}})$, $\sum_{j=1}^{n} f_j h_j = 1$ and $\|h_j\| \le \frac{1}{\delta}$. The real task now lies in changing the h_j to become holomorphic in $\mathbb{D}$, without losing control over the boundedness of the solutions.

2 Wirtinger Derivatives

Before we continue we want to briefly introduce a useful generalization of the complex derivative.

Definition 4. Let $\Omega \subseteq \mathbb{R}^2$ be open. Then the *Wirtinger derivatives* (or *Wirtinger operators*) are defined on $C^1(\Omega)$ by

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

We will also abbreviate these operators as ∂ and $\bar{\partial}$, respectively.

Note that by writing a complex number $z \in \mathbb{C}$ as $z = x + iy$ with $x, y \in \mathbb{R}$ we can identify $\mathbb{C} \cong \mathbb{R}^2$. Therefore we can also reasonably interpret the Wirtinger operators to act on $C^1(\Omega)$ with an open subset $\Omega \subseteq \mathbb{C}$.

Before listing properties of the Wirtinger operators we quickly want to recall that a function $f \in C^1(\Omega), f = u + iv$ is holomorphic if and only if it satisfies the *Cauchy–Riemann equations*:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Remark 5. Let $\Omega \subseteq \mathbb{C}$ be open and $f \in C^1(\Omega)$.

1. The Wirtinger operators are $\mathbb{C}$-linear, satisfy the Leibniz rule[2] and

$$\overline{\left(\frac{\partial f}{\partial z}\right)} = \frac{\partial \bar{f}}{\partial \bar{z}}, \quad \overline{\left(\frac{\partial f}{\partial \bar{z}}\right)} = \frac{\partial \bar{f}}{\partial z}$$

2. If $f \in \mathrm{Hol}(\Omega)$, then

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x} = f'.$$

3. Since

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) = \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),\end{aligned}$$

we have that[3] $f \in \mathrm{Hol}(\Omega)$ if and only if $\overline{\partial} f = 0$.

4. On $C^2(\Omega)$, the *Laplace operator* can be represented as

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}.$$

Proof. Step 3 (The Koszul complex): We consider the spaces

$$C_0 := C^\infty(\overline{\mathbb{D}}), \quad C_1 := (C_0)^n, \quad C_2 := \{A \in (C_0)^{n\times n} : A = -A^T\}$$

and the maps

$$P_{1,0} : C_1 \to C_0,\ (g_j)_{j=1}^n \mapsto \sum_{j=1}^n g_j f_j, \quad P_{2,1} : C_2 \to C_1,\ (g_{jk})_{j,k=1}^n \mapsto \left(\sum_{k=1}^n g_{jk} f_k\right)_{j=1}^n.$$

Applying $\overline{\partial}$ pointwise in $C_j, j = 0, 1, 2$, the resulting connections are visualized in the diagram below, called the *Koszul complex*:

$$\begin{tikzcd} C_2 \arrow[r, "P_{2,1}"] \arrow[d, "\overline{\partial}"'] & C_1 \arrow[r, "P_{1,0}"] \arrow[d, "\overline{\partial}"'] & C_0 \arrow[d, "\overline{\partial}"'] \\ C_2 \arrow[r, "P_{2,1}"'] & C_1 \arrow[r, "P_{1,0}"'] & C_0 \end{tikzcd}$$

Lemma 6. The Koszul complex has the following properties:

1. The diagram is commutative, that is we have $P_{j+1,j}\overline{\partial} = \overline{\partial} P_{j+1,j}$ for $j = 0, 1$.
2. The horizontal sequences are exact, that is $\operatorname{ran} P_{2,1} = \ker P_{1,0}$.

[2] This means that the Wirtinger opeartors are derivatives from an algebraic perspective.
[3] This can be interpreted as "f is independant of $\bar{z}$".

3. The maps $\overline{\partial} : C_j \to C_j$ for $j = 0, 1, 2$ are surjective.

Proof.

1. For $g \in C_0$ and $f \in \mathrm{Hol}(\mathbb{D})$ we have

$$\frac{\partial(gf)}{\partial \bar{z}} = \frac{\partial g}{\partial \bar{z}} f + g \frac{\partial f}{\partial \bar{z}} = \frac{\partial g}{\partial \bar{z}} f$$

and together with the linearity of $\overline{\partial}$ the statement follows.

2. "$\subseteq$": Let $g \in C_2, g = (g_{jk})_{j,k=1}^n$, then

$$P_{1,0}P_{2,1}g = P_{1,0}\left[\left(\sum_{k=1}^n g_{jk} f_k\right)_{j=1}^n\right] = \sum_{j=1}^n \sum_{k=1}^n g_{jk} f_k = 0$$

since g is skew-symmetric and therefore $g \in \ker P_{1,0}$.

"$\supseteq$" Let $g \in \ker P_{1,0} \subseteq C_1, g = (g_1, \ldots, g_n)$. We define $p = (p_{jk})_{j,k=1}^n \in C_2$ by

$$p_{jk} \coloneqq \frac{1}{\sum_{\ell=1}^n |f_\ell|^2}(g_j \overline{f_k} - g_k \overline{f_j}).$$

Then for any $j = 1, \ldots, n$ we have

$$\begin{aligned} P_{2,1}p = \sum_{k=1}^n p_{jk} f_k &= \frac{1}{\sum_{\ell=1}^n |f_\ell|^2} \sum_{k=1}^n (g_j |f_k|^2 - g_k \overline{f_j} f_k) = \\ &= g_j - \frac{1}{\sum_{\ell=1}^n |f_\ell|^2} \overline{f_j} \sum_{k=1}^n g_k f_k = g_j - \frac{1}{\sum_{\ell=1}^n |f_\ell|^2} \overline{f_j} P_{1,0} g = \\ &= g_j, \end{aligned}$$

and therefore $g_j \in \operatorname{ran} P_{2,1}$.

3. For given $v \in C^\infty(\mathbb{D})$ we want to solve the partial differential equation

$$\frac{\partial u}{\partial \bar{z}} = v$$

for some $u \in C^\infty(\mathbb{D})$.

...

Arguing pointwise shows the surjectivity of the maps $\overline{\partial} : C_\ell \to C_\ell$ for $\ell = 0, 1$. For $\ell = 2$ and given $b = (b_{jk})_{j,k=1}^n \in C_2$ we first solve

$$\overline{\partial} a_{jk} = b_{jk}, \quad \text{for} \quad 1 \leq j < k \leq n$$

and then set $a_{jj} = 0$ and $a_{jk} = -a_{kj}$ for $n \geq j > k \geq 1$.

$\square$

Proof (continued). Step 4 (Apply to $h = (h_1, \ldots, h_n) \in C_1$): todo

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3 Hardy Spaces

For $1 \leq p \leq \infty$ we define the *Hardy space* H^p as the set of all $f \in \mathrm{Hol}(\mathbb{D})$ with $\|f\|_p < \infty$, where

$$\|f\|_p := \lim_{r\to 1}\left[\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\vartheta})|^p \,\mathrm{d}\vartheta\right]^{1/p} \quad \text{for } p < \infty, \quad \text{and} \quad \|f\|_\infty := \sup_{z\in\mathbb{D}} |f(z)|.$$

For $f \in L^1(\mathbb{T})$ and $n \in \mathbb{N}$ we define the n-th *Fourier coefficient* by

$$\hat{f}(n) := \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{inx}\,\mathrm{d}x.$$

We define H_0^1 as the (closed) subspace of all $f \in H^1$, for which $f(0) = 0$.

We summarize the characterisation of Hardy spaces:

Theorem 7. Let $1 \leq p \leq \infty$. Then:

1. H^p is a Banach space[4].
2. Let $f \in H^p$, then for almost all $e^{i\vartheta} \in \mathbb{T}$ the limit
$$\lim_{r\to 1} f(re^{i\vartheta}) =: f^*(e^{i\vartheta})$$
exists and defines a function in $L^p(\mathbb{T})$, also called the *boundary values* of f.
3. The map $^* : f \mapsto f^*$ is an isometry from H^p onto
$$L_+^p(\mathbb{T}) := \{f \in L^p(\mathbb{T}) : \forall n < 0 : \hat{f}(n) = 0\},$$
which is a closed subspace of $L^p(\mathbb{T})$.
4. Every $f \in H^p$ can be written as a Cauchy integral of its boundary values:
$$f(z) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f^*(\zeta)}{\zeta - z}\,\mathrm{d}\mu(\zeta), \quad z \in \mathbb{D}$$

Lemma 8. The map

$$\Phi : L^\infty(\mathbb{T})/(H^\infty)^* \to ((H_0^1)^*)',\ f + (H^\infty)^* \mapsto \left[g \mapsto \frac{1}{2\pi}\int_{-\pi}^{\pi} fg\,\mathrm{d}x\right]$$

is an isometric isomorphism.

[4]In particular, H^∞ is a Banach algebra, which we already used in the introduction.

Proof. We have $L^\infty(\mathbb{T}) \cong L^1(\mathbb{T})'$ via the duality

$$\langle f, g\rangle \coloneqq \frac{1}{2\pi}\int_{\mathbb{T}} fg \,\mathrm{d}\mu, \quad f \in L^\infty(\mathbb{T}),\ g \in L^1(\mathbb{T}).$$

Since $(H_0^1)^* \le L^1(\mathbb{T})$ we therefore have $((H_0^1)^*)' \cong L^\infty(\mathbb{T})/(H_0^1)^\perp$ via

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{\mathbb{T}} fg \,\mathrm{d}\mu, \quad f \in L^\infty(\mathbb{T})/(H_0^1)^\perp,\ g \in (H_0^1)^*.$$

It remains to show that $(H_0^1)^\perp = (H^\infty)^*$. Let $w \in (H_0^1)^\perp \le L^\infty(\mathbb{T})$, then for any $n \in \mathbb{N}$ we have

$$0 = \langle w, \bar{z}^n\rangle = \langle w, e^{-int}\rangle.$$

Therefore $w \in L_+^\infty(\mathbb{T}) = (H^\infty)^*$. □

Proof (continued). Step 5 (Dualisation): Applying the above lemma to our previous situation we can re-describe the norm of $v + (H^\infty)^*$:

$$\begin{aligned}
\|v + (H^\infty)^*\|_\infty &= \|\Phi(f + (H^\infty)^*)\| = \\
&= \sup\{|\Phi(f + (H^\infty)^*)(g)| : g \in (H_0^1)^*, \|g\|_1 \le 1\} = \\
&= \sup\left\{\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} fg \,\mathrm{d}x\right| : g \in H_0^1, \|g\|_1 \le 1\right\}
\end{aligned}$$

Proof. □

Proof (continued). Step 6: todo

4 Integral estimates

Lemma 9. Let $f, g, u, v \in \mathrm{Hol}(\overline{\mathbb{D}})$, then the following integral estimates hold:

1. $\displaystyle\int_{\mathbb{D}} |f'|^2 \log\frac{1}{|z|}\,\mathrm{d}\lambda^2 \le \frac{\pi}{2}\|f\|_2^2$
2. $\displaystyle\int_{\mathbb{D}} |fg'| \log\frac{1}{|z|}\,\mathrm{d}\lambda^2 \le 2\pi\|f\|_2^2\|g\|_\infty$
3. $\displaystyle\int_{\mathbb{D}} |fgu'v'| \log\frac{1}{|z|}\,\mathrm{d}\lambda^2 \le 2\pi\|f\|_2\|g\|_2\|u\|_\infty\|v\|_\infty$
4. $\displaystyle\int_{\mathbb{D}} |fu'v'| \log\frac{1}{|z|}\,\mathrm{d}\lambda^2 \le 2\pi\|f\|_1\|u\|_\infty\|v\|_\infty$
5. $\displaystyle\int_{\mathbb{D}} |fg'u'| \log\frac{1}{|z|}\,\mathrm{d}\lambda^2 \le \pi\|f\|_2\|g\|_2\|u\|_\infty$
6. $\displaystyle\int_{\mathbb{D}} |f'u'| \log\frac{1}{|z|}\,\mathrm{d}\lambda^2 \le 2\pi\|f\|_1\|u\|_\infty$

Proof.

1. Applying Green's formula on $f\bar{f}$ yields

$$\int_{\mathbb{D}} (\log \frac{1}{|z|} \Delta(f\bar{f}) - f\bar{f})$$

$$|f(0)|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f^*(e^{i\vartheta})|^2 \,\mathrm{d}\vartheta - \frac{1}{2\pi} \int_{\mathbb{D}} \Delta(f\bar{f}) \log \frac{1}{|z|} \,\mathrm{d}\lambda^2$$

Since

$$\begin{aligned}\Delta(f\bar{f}) &= 4\partial\overline{\partial}(f\bar{f}) = 4\partial(\bar{f} \cdot \overline{\partial} f + f \cdot \overline{\partial}\bar{f}) = 4\partial(f \cdot \overline{\partial}\bar{f}) = 4(\partial f \cdot \overline{\partial}\bar{f} + f \cdot \overline{\partial}\partial\bar{f}) = \\ &= 4(\partial f \cdot \overline{\partial f} + f \cdot \overline{\partial}\,\overline{\overline{\partial} f}) = 4\partial(f\overline{\partial}\bar{f}) = 4|f'|^2\end{aligned}$$

and $|f(0)|^2 \geq 0$ we obtain

$$\frac{2}{\pi} \int_{\mathbb{D}} |f'|^2 \log \frac{1}{|z|} \,\mathrm{d}\lambda^2 \leq \|f\|_2^2$$

and rearranging yields the desired inequality.

□