

THE CORONA THEOREM

IAN HORNIK

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We will denote the space of all complex-valued, bounded, analytic functions on the unit disk \mathbb{D} as H^∞ . Equipped with the supremum norm $\|\cdot\|_\infty$ this space becomes a commutative Banach algebra. The space of all multiplicative, bounded, linear functionals on H^∞ not identically zero is denoted $\Delta(H^\infty)$ and is called the *Gelfand space* of H^∞ . We endow this space with the subspace topology of the weak-* topology on the topological dual $(H^\infty)'$, which we will refer to as the *Gelfand topology*. For each $z \in \mathbb{D}$ we consider the point-evaluation functional

$$\pi_z : H^\infty \rightarrow \mathbb{C}, f \mapsto f(z).$$

This is clearly multiplicative, bounded and linear and therefore belongs to $\Delta(H^\infty)$. The set of all such functionals $\pi_z, z \in \mathbb{D}$ will be denoted as Δ_0 . The *corona* is defined as the complement of the closure of Δ_0 in the Gelfand topology. The corona theorem now states:

Theorem 1 (L. Carleson). The corona is empty. In other words, Δ_0 is dense in $\Delta(H^\infty)$.

There is an equivalent version of the theorem, as given by the following proposition:

Proposition 2. Δ_0 is dense in $\Delta(H^\infty)$ if and only if for any $\delta > 0$ and $f_1, \dots, f_n \in H^\infty$ such that $\sum_{j=1}^n |f_j(z)| \geq \delta, z \in \mathbb{D}$, there exist $g_1, \dots, g_n \in H^\infty$ such that $\sum_{j=1}^n f_j g_j = 1$.

Proof. Assume Δ_0 is dense in $\Delta(H^\infty)$ and let $f_1, \dots, f_n \in H^\infty$, and $\delta > 0$ such that $\sum_{j=1}^n |f_j(z)| \geq \delta, z \in \mathbb{D}$. Denote by I the ideal in H^∞ generated by f_1, \dots, f_n . If $1 \in I$, then the assertion is established. Assume towards a contradiction that I is a proper ideal, then there exists a maximal ideal $J \supset I$. Since $\Delta(H^\infty)$ is a commutative Banach algebra, there exists a $\phi \in \Delta(H^\infty)$ such that $J = \ker \phi$. Therefore we have $\phi(f_j) = 0$ for $j = 1, \dots, n$. Since Δ_0 is dense, there is a net $(\pi_{z_m})_{m \in M}$ in Δ_0 such that $\pi_{z_m} \rightarrow \phi$ in the weak-* topology, that is the net converges pointwise. Therefore, for all $j = 1, \dots, n$ we have $f_j(z_m) = \pi_{z_m}(f_j) \rightarrow \phi(f_j) = 0$ and in particular

$$\lim_{m \in M} \sum_{j=1}^n |f_j(z_m)| = 0,$$

a contradiction.

For the other implication, assume towards a contradiction that Δ_0 is not dense in $\Delta(H^\infty)$. Then there exists some $\phi_0 \in \Delta(H^\infty)$ and an open neighbourhood U of ϕ_0 such that $\Delta_0 \cap U = \emptyset$. Since the sets of the form

$$\{\phi \in \Delta(H^\infty) : |(\phi - \phi_0)(f_j)| < \varepsilon, j = 1, \dots, n\},$$

for some $n \in \mathbb{N}$, $f_1, \dots, f_n \in H^\infty$ and $\varepsilon > 0$, form a neighbourhood basis of ϕ_0 in the weak-* topology, there exists a neighbourhood $V \subseteq U$ described by some $n \in \mathbb{N}$, $f_1, \dots, f_n \in H^\infty$ and $\delta > 0$. Define $\tilde{f}_j := f_j - \phi_0(f_j)$, for $j = 1, \dots, n$, then clearly $\phi_0(\tilde{f}_j) = 0$. Since $\Delta_0 \cap V = \emptyset$, for any $z \in \mathbb{D}$ we have $\pi_z \notin V$ and therefore there exists some $j_0 \in \{1, \dots, n\}$ such that,

$$\delta \leq |(\pi_z - \phi_0)(f_{j_0})| = |f_{j_0}(z) - \phi_0(f_{j_0})| = |\tilde{f}_{j_0}(z)|.$$

Since $\tilde{f}_j \in H^\infty$ for $j = 1, \dots, n$, and $\sum_{j=1}^n |\tilde{f}_j(z)| \geq \delta$, there exist $g_1, \dots, g_n \in H^\infty$ such that $\sum_{j=1}^n \tilde{f}_j g_j = 1$. But this yields

$$1 = \phi_0(1) = \phi_0\left(\sum_{j=1}^n \tilde{f}_j g_j\right) = \sum_{j=1}^n \phi_0(\tilde{f}_j) \phi_0(g_j) = 0,$$

a contradiction. □

1 First Steps

Over the following sections we will prove a stronger version of the right statement in Proposition 2:

Theorem 3. There exist constants $C_{n,\delta}$ only depending on $n \in \mathbb{N}$ and $\delta > 0$, such that if $f_1, \dots, f_n \in \text{Hol}(\mathbb{D})$ with

$$\|f_j\|_\infty \leq 1, \quad j = 1, \dots, n, \quad \text{and} \quad \sum_{j=1}^n |f_j(z)|^2 \geq \delta, \quad z \in \mathbb{D},$$

then there exist $g_1, \dots, g_n \in \text{Hol}(\mathbb{D})$ with

$$\|g_j\|_\infty \leq C_{n,\delta}, \quad j = 1, \dots, n, \quad \text{and} \quad \sum_{j=1}^n f_j g_j = 1.$$

Proof. We will give the proof in multiple steps. First, for a closed set $A \subset \mathbb{C}$ and a space of functions on an open sets $\Omega \supset A$, say $D(\Omega)$, we define

$$D(A) := \bigcup_{\Omega \supset A \text{ open}} T(D(\Omega)), \quad \text{where} \quad T(f) := f|_A.$$

We will make use of this to handle smooth or holomorphic functions on closed sets.

Step 1 (Reduction to $f_1, \dots, f_n \in \text{Hol}(\overline{\mathbb{D}})$): Assume that the statement of the theorem holds for all $\tilde{f}_1, \dots, \tilde{f}_n \in \text{Hol}(\overline{\mathbb{D}})$, we claim that it then also holds in its original form¹. For our given f_1, \dots, f_n satisfying the premise of the theorem and all $0 < s < 1$ we define

¹Note that this does **not** mean that we can assume $f_1, \dots, f_n \in \text{Hol}(\overline{\mathbb{D}})$ in the previous proposition.

$f_{j,s}(z) := f_j(sz)$, $j = 1, \dots, n$. Then for every $0 < s < 1$ and $j = 1, \dots, n$ the function $f_{j,s}$ is in $\text{Hol}(\overline{\mathbb{D}})$ and satisfies the premise of the theorem. By our assumption there exist $g_{j,s} \in H^\infty$, $j = 1, \dots, n$ such that

$$\|g_{j,s}\|_\infty \leq C_{n,\delta}, \quad j = 1, \dots, n, \quad \text{and} \quad \sum_{j=1}^n f_{j,s} g_{j,s} = 1.$$

For a fixed $j \in \{1, \dots, n\}$, the set $\{g_{j,s} : 0 < s < 1\}$ is uniformly bounded and therefore normal in $\text{Hol}(\overline{\mathbb{D}})$. By Montel's Theorem there exists a sequence $s_m \rightarrow 1$ and some $g_j \in \text{Hol}(\overline{\mathbb{D}})$ such that $g_{j,s_m} \rightarrow g_j$ compactly. In particular, we obtain

$$\|g_j\|_\infty = \lim_{m \rightarrow \infty} \|g_{j,s_m}\|_\infty \leq C_{n,\delta}, \quad j = 1, \dots, n,$$

and

$$1 = \lim_{m \rightarrow \infty} \sum_{j=1}^n f_{j,s_m} g_{j,s_m} = \sum_{j=1}^n f_j g_j,$$

concluding our claim. We may thus assume that our given f_1, \dots, f_n are holomorphic on $\overline{\mathbb{D}}$ instead.

Step 2 (Solve in $C^\infty(\overline{\mathbb{D}})$): For $j = 1, \dots, n$ we define

$$h_j := \frac{\bar{f}_j}{\sum_{k=1}^n |f_k|^2} \in C^\infty(\overline{\mathbb{D}}),$$

then clearly $\sum_{j=1}^n f_j h_j = 1$ and $\|h_j\|_\infty \leq \frac{1}{\delta}$. The real task now lies in changing the h_j to become holomorphic in \mathbb{D} , without losing control over the boundedness of the solutions.

2 Wirtinger Derivatives

Before we continue we want to briefly introduce a useful generalization of the complex derivative.

Definition 4. Let $\Omega \subseteq \mathbb{R}^2$ be open. Then the *Wirtinger derivatives* (or *Wirtinger operators*) are defined on $C^1(\Omega)$ by

$$\frac{\partial}{\partial z} := \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

We will also abbreviate these operators as ∂ and $\bar{\partial}$, respectively.

Note that by writing a complex number $z \in \mathbb{C}$ as $z = x + iy$ with $x, y \in \mathbb{R}$ we can identify $\mathbb{C} \cong \mathbb{R}^2$. Therefore we can also interpret the Wirtinger operators to act on $C^1(\Omega)$ with an open subset $\Omega \subseteq \mathbb{C}$.

Before listing properties of the Wirtinger operators we quickly want to recall that a function $f \in C^1(\Omega)$, $f = u + iv$ is holomorphic if and only if it satisfies the *Cauchy–Riemann equations*:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Remark 5. Let $\Omega \subseteq \mathbb{C}$ be open and $f \in C^1(\Omega)$.

1. The Wirtinger operators are \mathbb{C} -linear, satisfy the Leibniz rule² and

$$\overline{\left(\frac{\partial f}{\partial z}\right)} = \frac{\partial \bar{f}}{\partial \bar{z}}, \quad \overline{\left(\frac{\partial f}{\partial \bar{z}}\right)} = \frac{\partial \bar{f}}{\partial z}$$

2. If $f \in \text{Hol}(\Omega)$, $f = u + iv$, then

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) = \frac{1}{2} \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} - i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial f}{\partial x} = f'.$$

3. Since

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}} &= \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) = \frac{1}{2} \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} + i \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} \right) = \\ &= \frac{1}{2} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + \frac{i}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right), \end{aligned}$$

we have that³ $f \in \text{Hol}(\Omega)$ if and only if $\bar{\partial}f = 0$.

4. On $C^2(\Omega)$, the *Laplace operator* can be represented as

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}.$$

Proof (continued). Step 3 (The Koszul complex): We consider the spaces

$$C_0 := C^\infty(\overline{\mathbb{D}}), \quad C_1 := (C_0)^n, \quad C_2 := \{A \in (C_0)^{n \times n} : A = -A^T\}$$

and the linear maps

$$P_{1,0} : C_1 \rightarrow C_0, (g_j)_{j=1}^n \mapsto \sum_{j=1}^n g_j f_j, \quad P_{2,1} : C_2 \rightarrow C_1, (g_{jk})_{j,k=1}^n \mapsto \left(\sum_{k=1}^n g_{jk} f_k \right)_{j=1}^n.$$

We also consider the operator $\bar{\partial} : C_0 \rightarrow C_0$. It is well-defined, since if $f_1 \in C^\infty(\Omega_1)$, $f_2 \in C^\infty(\Omega_2)$ with $f_1 = f_2$ on $\overline{\mathbb{D}}$, then in particular $f_1 = f_2$ on \mathbb{D} and therefore $\bar{\partial}f_1 = \bar{\partial}f_2$ on \mathbb{D} . By continuity, we therefore also get $\bar{\partial}f_1 = \bar{\partial}f_2$ on $\overline{\mathbb{D}}$.

Applying $\bar{\partial}$ pointwise in C_1 and C_2 as well, the resulting connections are visualized in the diagram below, called the *Koszul complex*:

²This means that the Wirtinger operators are derivatives from an algebraic perspective.

³This can be interpreted as “ f is independant of \bar{z} ”.

$$\begin{array}{ccccc}
C_2 & \xrightarrow{P_{2,1}} & C_1 & \xrightarrow{P_{1,0}} & C_0 \\
\bar{\partial} \downarrow & & \bar{\partial} \downarrow & & \bar{\partial} \downarrow \\
C_2 & \xrightarrow{P_{2,1}} & C_1 & \xrightarrow{P_{1,0}} & C_0
\end{array}$$

Lemma 6. The Koszul complex has the following properties:

1. The diagram is commutative, that is we have $P_{j+1,j}\bar{\partial} = \bar{\partial}P_{j+1,j}$ for $j = 0, 1$.
2. The horizontal sequences are exact, that is $\text{ran } P_{2,1} = \ker P_{1,0}$.
3. The vertical maps $\bar{\partial} : C_j \rightarrow C_j$ for $j = 0, 1, 2$ are surjective.

Proof.

1. For $g \in C_0$ and $f \in \text{Hol}(\bar{\mathbb{D}})$ we have

$$\bar{\partial}(gf) = f\bar{\partial}g + g\bar{\partial}f = f\bar{\partial}g$$

and together with the linearity of $\bar{\partial}$ the statement follows.

2. “ \subseteq ”: Let $g \in C_2, g = (g_{jk})_{j,k=1}^n$, then

$$P_{1,0}P_{2,1}g = P_{1,0} \left[\left(\sum_{k=1}^n g_{jk} f_k \right)_{j=1}^n \right] = \sum_{j=1}^n \sum_{k=1}^n g_{jk} f_k f_j = 0$$

since g is skew-symmetric and therefore $g \in \ker P_{1,0}$.

“ \supseteq ” Let $g \in \ker P_{1,0} \subseteq C_1, g = (g_1, \dots, g_n)$. We define $p = (p_{jk})_{j,k=1}^n \in C_2$ by

$$p_{jk} := \frac{1}{\sum_{\ell=1}^n |f_\ell|^2} (g_j \bar{f}_k - g_k \bar{f}_j).$$

Then for any $j = 1, \dots, n$ we have

$$\begin{aligned}
(P_{2,1}p)_j &= \sum_{k=1}^n p_{jk} f_k = \frac{1}{\sum_{\ell=1}^n |f_\ell|^2} \sum_{k=1}^n (g_j |f_k|^2 - g_k \bar{f}_j f_k) = \\
&= g_j - \frac{1}{\sum_{\ell=1}^n |f_\ell|^2} \bar{f}_j \sum_{k=1}^n g_k f_k = g_j - \frac{1}{\sum_{\ell=1}^n |f_\ell|^2} \bar{f}_j P_{1,0}g = \\
&= g_j,
\end{aligned}$$

and therefore $g \in \text{ran } P_{2,1}$.

3. For given $v \in C^\infty(\overline{\mathbb{D}})$ we want to solve the partial differential equation

$$\frac{\partial u}{\partial \bar{z}} = v \quad (\text{on } \overline{\mathbb{D}})$$

for some $u \in C^\infty(\overline{\mathbb{D}})$. We will approach this using a fundamental solution of the differential operator $\bar{\partial}$. Recall that

$$\Gamma(z) := \frac{1}{2\pi} \log |z|$$

is a fundamental solution of the Laplace operator, that is we have $\Delta \Gamma = \delta_0$ distributionally, where δ_0 denotes the delta distribution at 0. We claim that $\frac{1}{\pi z}$ is a fundamental solution of $\bar{\partial}$, and verify this via

$$\begin{aligned} \bar{\partial} \frac{1}{z} &= \bar{\partial} \frac{\bar{z}}{|z|^2} = \frac{1}{2} (\partial_x + i\partial_y) \frac{x - iy}{x^2 + y^2} = \\ &= \frac{1}{2} \left[\partial_x \frac{x}{x^2 + y^2} - i\partial_x \frac{y}{x^2 + y^2} + i\partial_y \frac{x}{x^2 + y^2} + \partial_y \frac{y}{x^2 + y^2} \right] = \\ &= \frac{1}{2} \left[\partial_x^2 \log |z| + \partial_y^2 \log |z| + i \left(\frac{2xy}{x^2 + y^2} - \frac{2xy}{x^2 + y^2} \right) \right] = \\ &= \frac{1}{2} \Delta \log |z| = \frac{1}{2} 2\pi \delta_0 = \pi \delta_0. \end{aligned}$$

Now let $\Omega \supset \overline{\mathbb{D}}$ be open such that $v \in C^\infty(\Omega)$ and choose $\varphi \in C_c^\infty(\Omega)$ such that $\varphi|_{\overline{\mathbb{D}}} = 1$. Then $\varphi v \in C_c^\infty(\Omega)$, therefore

$$u(w) := \left(\frac{1}{\pi z} * \varphi v \right) (w) = \frac{1}{\pi} \int_{\Omega} \frac{\varphi(z)v(z)}{w - z} d\lambda^2(z)$$

is a classical solution of $\bar{\partial} u = \varphi v$ in Ω . Since $\varphi v = v$ on $\overline{\mathbb{D}}$, we get $\bar{\partial} u = v$ on $\overline{\mathbb{D}}$, as desired.

Arguing pointwise shows the surjectivity of the maps $\bar{\partial} : C_\ell \rightarrow C_\ell$ for $\ell = 0, 1$. For $\ell = 2$ and given $b = (b_{jk})_{j,k=1}^n \in C_2$ we first solve

$$\bar{\partial} a_{jk} = b_{jk}, \quad \text{for } 1 \leq j < k \leq n$$

and then set $a_{jj} = 0$ and $a_{jk} = -a_{kj}$ for $n \geq j > k \geq 1$.

□

Proof (continued). Step 4 (Apply to $h = (h_1, \dots, h_n) \in C_1$): In step 2 we constructed an element $h = (h_1, \dots, h_n) \in C_1$ by setting

$$h_j := \frac{\bar{f}_j}{\sum_{k=1}^n |f_k|^2}.$$

By our construction we have $P_{1,0}h = 1$ and therefore $0 = \bar{\partial} P_{1,0}h = P_{1,0}\bar{\partial}h$, thus $\bar{\partial}h \in \ker P_{1,0}$. By Lemma 6 there exists $b \in C_2$ such that $P_{2,1}b = \bar{\partial}h$ and $a \in C_2$ such that $\bar{\partial}a = b$. We now set $g := h - P_{2,1}a \in C_1$. Then

$$P_{1,0}g = P_{1,0}h - P_{1,0}P_{2,1}a = 1$$

and

$$\bar{\partial}g = \bar{\partial}h - \bar{\partial}P_{2,1}a = \bar{\partial}h - P_{2,1}b = 0.$$

Therefore g is a solution to

$$\sum_{k=1}^n f_k g_k = 1$$

in $\text{Hol}(\bar{\mathbb{D}})$. However, we do not have an estimate on $|g_j|$ yet. Recall that, explicitly,

$$g_j = h_j - \sum_{k=1}^n a_{jk} f_k,$$

where $a_{jk} \in C^\infty(\bar{\mathbb{D}})$ is a solution of the partial differential equation

$$\frac{\partial u}{\partial \bar{z}} = \left(\sum_{\ell=1}^n |f_\ell|^2 \right)^{-1} \left(\frac{\partial h_j}{\partial \bar{z}} \bar{f}_k - \frac{\partial h_k}{\partial \bar{z}} \bar{f}_j \right).$$

We want to show that the solution a_{jk} can be chosen in a way, that the resulting functions $g_j \in \text{Hol}(\bar{\mathbb{D}})$ are bounded in $\|\cdot\|_\infty$ by a constant depending only on n and δ , that is

$$\|g_j\|_\infty \leq C_{n,\delta}.$$

Denote by u_{jk} the right-hand side of the partial differential equation above. We fix a solution $\bar{\partial}v_{jk} = u_{jk}$ and notice that if $\bar{\partial}a_{jk} = u_{jk}$ is any bounded solution, then

$$\bar{\partial}(a_{jk} - v_{jk}) = \bar{\partial}a_{jk} - \bar{\partial}v_{jk} = 0,$$

that is the difference is bounded and holomorphic, thus in H^∞ . We can therefore write

$$a_{jk} = v_{jk} + p, \quad p \in H^\infty.$$

Kann man das noch irgendwie schöner machen?? We denote the restriction of v_{jk} to \mathbb{T} by v_{jk} as well, and can therefore view it as an element of $L^\infty(\mathbb{T})$. We can also consider $p^* \in L^\infty(\mathbb{T})$. If we therefore manage to bound

$$\|a_{jk}\|_\infty = \text{ess. sup}_{z \in \mathbb{T}} |a_{jk}(z)| \leq K_{n,\delta},$$

we immediately get

$$\|g_j\|_\infty = \|g_j^*\|_\infty \leq \frac{1}{\delta} + nK_{n,\delta},$$

resulting in the claim of the theorem.

Note that we can fluctuate $\|a_{jk}\|_\infty$ by choosing different $p \in H^\infty$. We therefore want to bound the quantity

$$\inf_{p \in H^\infty} \|v_{jk} + p^*\|_\infty,$$

which is precisely the norm of v_{jk} in the quotient space $L^\infty(\mathbb{T})/(H^\infty)^*$. To do this we will use the theory of *Hardy spaces*.

3 Hardy Spaces

Let μ denote the Lebesgue measure on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, i.e. the measure such that for a measurable function $f : \mathbb{T} \rightarrow \mathbb{C}$ it holds that

$$\int_{\mathbb{T}} f \, d\mu = \int_{-\pi}^{\pi} f(e^{i\vartheta}) \, d\vartheta.$$

We define the $L^p(\mathbb{T})$ -norms via the *normed* Lebesgue measure $\frac{1}{2\pi}\mu$:

$$\|f\|_p := \left(\frac{1}{2\pi} \int_{\mathbb{T}} |f|^p \, d\mu \right)^{1/p}, \quad \text{for } 1 \leq p < \infty, \quad \text{and} \quad \|f\|_{\infty} := \text{ess. sup } |f|.$$

For $f \in L^1(\mathbb{T})$ and $n \in \mathbb{N}$ we define the n -th *Fourier coefficient* by

$$\hat{f}(n) := \frac{1}{2\pi} \int_{\mathbb{T}} f(\xi) \xi^{-n} \, d\mu(\xi).$$

For $1 \leq p \leq \infty$ we define the *Hardy space* H^p as the set of all $f \in \text{Hol}(\mathbb{D})$ with $\|f\|_p < \infty$, where

$$\|f\|_p := \lim_{r \rightarrow 1} \left(\frac{1}{2\pi} \int_{\mathbb{T}} |f_r|^p \, d\mu \right)^{1/p} \quad \text{for } p < \infty, \quad \text{and} \quad \|f\|_{\infty} := \sup_{z \in \mathbb{D}} |f(z)|.$$

Equivalently one can interchange the limit in the definition above with the supremum over $0 < r < 1$. It is of note that convergence in the Hardy spaces implies compact convergence. We lastly define H_0^p as the (closed) subspace of all $f \in H^p$, for which $f(0) = 0$.

We summarize the characterisation of Hardy spaces:

Theorem 7. Let $1 \leq p \leq \infty$. Then:

1. H^p is a Banach space⁴.
2. For $p \leq q \leq \infty$ it holds that $H^p \supseteq H^q$.
3. Let $f \in H^p$, then for almost all $\xi \in \mathbb{T}$ the limit

$$\lim_{r \rightarrow 1} f(r\xi) =: f^*(\xi)$$

exists and defines a function in $L^p(\mathbb{T})$, also called the *boundary values* of f . If $p < \infty$, we also have $\lim_{r \rightarrow 1} \|f^* - f_r\|_p = 0$, where $f_r(\xi) := f(r\xi)$.

4. The map $*$: $f \mapsto f^*$ is an isometry from H^p onto

$$L_+^p(\mathbb{T}) := \{f \in L^p(\mathbb{T}) : \forall n < 0 : \hat{f}(n) = 0\},$$

which is a closed subspace of $L^p(\mathbb{T})$.

⁴In particular, H^{∞} is a Banach algebra, which we already used in the introduction.

5. Every $f \in H^p$ can be written as a Cauchy integral of its boundary values:

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f^*(\zeta)}{\zeta - z} d\zeta, \quad z \in \mathbb{D}$$

We will also use the following lemma:

Lemma 8. Let $f \in \text{Hol}(\mathbb{D})$, then there exist $g_1, g_2 \in \text{Hol}(\mathbb{D})$ such that

$$f = g_1 g_2, \quad \text{and} \quad \|g_1\|_2^2 = \|g_2\|_2^2 = \|f\|_1.$$

Returning to our proof, the following lemma allows us to translate the minimization problem into a maximization problem:

Lemma 9. The map

$$\Phi : L^\infty(\mathbb{T}) / (H^\infty)^* \rightarrow ((H_0^1)^*)', \quad f + (H^\infty)^* \mapsto \left[g \mapsto \frac{1}{2\pi} \int_{\mathbb{T}} f g d\mu \right]$$

is an isometric isomorphism.

Proof. We have $L^\infty(\mathbb{T}) \cong L^1(\mathbb{T})'$ via the duality

$$\langle f, g \rangle := \frac{1}{2\pi} \int_{\mathbb{T}} f g d\mu, \quad f \in L^\infty(\mathbb{T}), \quad g \in L^1(\mathbb{T}).$$

Since $(H_0^1)^* \leq L^1(\mathbb{T})$ we therefore have $((H_0^1)^*)' \cong L^\infty(\mathbb{T}) / ((H_0^1)^*)^\perp$ via

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{\mathbb{T}} f g d\mu, \quad f \in L^\infty(\mathbb{T}) / ((H_0^1)^*)^\perp, \quad g \in (H_0^1)^*.$$

It remains to show that $((H_0^1)^*)^\perp = (H^\infty)^*$. Let $w^* \in ((H_0^1)^*)^\perp \leq L^\infty(\mathbb{T})$, then for any $n \in \mathbb{N}$ we have

$$0 = \langle w^*, (z^n)^* \rangle = \langle w^*, z^n \rangle = \widehat{w^*}(-n).$$

Therefore $w^* \in L_+^\infty(\mathbb{T}) = (H^\infty)^*$. For the other inclusion let $w^* \in (H^\infty)^*$ and $h^* \in (H_0^1)^*$, then

$$\begin{aligned} \langle w^*, h^* \rangle &= \frac{1}{2\pi} \int_0^{2\pi} w^*(e^{i\vartheta}) h^*(e^{i\vartheta}) d\vartheta = \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} w(re^{i\vartheta}) h(re^{i\vartheta}) \frac{ie^{i\vartheta}}{ie^{i\vartheta}} d\vartheta = \\ &= \lim_{r \rightarrow 1} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{w_r(\zeta) h_r(\zeta)}{\zeta} d\zeta = \lim_{r \rightarrow 1} w_r(0) h_r(0) = 0, \end{aligned}$$

where the second-to-last equation follows since $w_r h_r \in \text{Hol}(\overline{\mathbb{D}})$ and the last one since $h(0) = 0$. Therefore $w^* \in ((H_0^1)^*)^\perp$, concluding the proof. \square

Proof (continued). Step 5 (Dualisation): **We need to explain what the previous situation is.** Applying the above lemma to our previous situation we can re-describe the norm of $v + (H^\infty)^*$ as

$$\|\Phi(v + (H^\infty)^*)\| = \sup_{\substack{f \in (H_0^1)^* \\ \|f\|_1 \leq 1}} |\Phi(v + (H^\infty)^*)(f)| = \sup_{\substack{f \in H_0^1 \\ \|f\|_1 \leq 1}} \left| \frac{1}{2\pi} \int_{\mathbb{T}} v f^* d\mu \right| =$$

$$= \sup_{\substack{f \in H_0^1 \\ \|f\|_1 \leq 1}} \left| \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_{\mathbb{T}} v f_r \, d\mu \right| = \sup_{\substack{f \in \text{Hol}(\overline{\mathbb{D}}) \\ f(0)=0, \|f\|_1 \leq 1}} \left| \frac{1}{2\pi} \int_{\mathbb{T}} v f \, d\mu \right|.$$

Proof (continued). Step 6: We want to redescribe the integral

$$\frac{1}{2\pi} \int_{\mathbb{T}} v f \, d\mu,$$

where $v \in C^\infty(\overline{\mathbb{D}})$ and $f \in \text{Hol}(\overline{\mathbb{D}})$, $f(0) = 0$, $\|f\|_1 \leq 1$. Let $\sigma := vf$ and

$$\varphi(z) := \frac{1}{2\pi} \log |z|, \quad z \neq 0.$$

For $\varepsilon > 0$ let $\mathbb{D}_\varepsilon := \mathbb{D} \setminus \overline{B_\varepsilon(0)}$. By Green's second identity we have

$$\int_{\mathbb{D}_\varepsilon} \sigma \Delta \varphi - \varphi \Delta \sigma \, d\lambda^2 = \int_{\mathbb{T}} \sigma \frac{\partial \varphi}{\partial r} - \varphi \frac{\partial \sigma}{\partial r} \, d\mu - \int_{\partial B_\varepsilon(0)} \sigma \frac{\partial \varphi}{\partial r} - \varphi \frac{\partial \sigma}{\partial r} \, d\mathcal{H}^1,$$

where $\frac{\partial}{\partial r}$ denotes the radial derivative. Simplifying results in

$$- \int_{\mathbb{D}_\varepsilon} \varphi \Delta \sigma \, d\lambda^2 = \frac{1}{2\pi} \int_{\mathbb{T}} \sigma \, d\mu - \frac{1}{2\pi} \int_{\partial B_\varepsilon(0)} \frac{\sigma}{\varepsilon} - \frac{\partial \sigma}{\partial r} \log \varepsilon \, d\mathcal{H}^1.$$

By the intermediate value theorem for integrals for any $\varepsilon > 0$ there exists a $\zeta_\varepsilon \in \partial B_\varepsilon(0)$ such that

$$\frac{1}{2\pi\varepsilon} \int_{\partial B_\varepsilon(0)} \sigma \, d\mathcal{H}^1 = \sigma(\zeta_\varepsilon) \rightarrow \sigma(0) = v(0)f(0) = 0.$$

Furthermore $\left| \frac{\partial \sigma}{\partial r} \right| \leq M$ on $\overline{\mathbb{D}}$ for some $M > 0$, thus

$$\frac{1}{2\pi} \int_{\partial B_\varepsilon(0)} \left| \frac{\partial \sigma}{\partial r} \log \varepsilon \right| \, d\mathcal{H}^1 \leq M\varepsilon \log \varepsilon \rightarrow 0.$$

Finally,

$$\Delta \sigma = \Delta(vf) = 4\partial\bar{\partial}(vf) = 4\partial(v\bar{\partial}f + f\bar{\partial}v) = 4\partial(fu) = 4(f\partial u + u\partial f').$$

With $\psi := -\varphi = \frac{1}{2\pi} \log \frac{1}{|z|}$ we thus obtain

$$\begin{aligned} \frac{1}{2\pi} \int_{\mathbb{T}} v f \, d\mu &= 4 \int_{\mathbb{D}} \psi (f\partial v + v\partial f') \, d\lambda^2 = \\ &= 4 \left(\int_{\mathbb{D}} f \partial v \psi \, d\lambda^2 + \int_{\mathbb{D}} v \partial f' \psi \, d\lambda^2 \right) =: 4(I_1 + I_2). \end{aligned}$$

Our goal is to show the existence of a constant $C_{n,\delta}$ such that

$$\left| \frac{1}{2\pi} \int_{\mathbb{T}} v f \, d\mu \right| \leq 4(|I_1| + |I_2|) \leq C_{n,\delta}.$$

We still need some reformulation at the end.

4 Integral estimates

Lemma 10. Let $f, g, u, v \in \text{Hol}(\overline{\mathbb{D}})$, then the following integral estimates hold:

1. $\int_{\mathbb{D}} |f'|^2 \psi \, d\lambda^2 \leq \frac{1}{4} \|f\|_2^2$
2. $\int_{\mathbb{D}} |fg'|^2 \psi \, d\lambda^2 \leq \|f\|_2^2 \|g\|_\infty^2$
3. $\int_{\mathbb{D}} |fgu'v'| \psi \, d\lambda^2 \leq \|f\|_2 \|g\|_2 \|u\|_\infty \|v\|_\infty$
4. $\int_{\mathbb{D}} |fu'v'| \psi \, d\lambda^2 \leq \|f\|_1 \|u\|_\infty \|v\|_\infty$
5. $\int_{\mathbb{D}} |fg'u'| \psi \, d\lambda^2 \leq \frac{1}{2} \|f\|_2 \|g\|_2 \|u\|_\infty$
6. $\int_{\mathbb{D}} |f'u'| \psi \, d\lambda^2 \leq \|f\|_1 \|u\|_\infty$

Proof.

1. Applying Green's formula on $f\bar{f}$ and ψ yields

$$\int_{\mathbb{D}_\varepsilon} \psi \Delta(f\bar{f}) - f\bar{f} \Delta \psi \, d\lambda^2 = \int_{\mathbb{T}} \psi \frac{\partial(f\bar{f})}{\partial r} - f\bar{f} \frac{\partial \psi}{\partial r} \, d\mu - \int_{\partial B_\varepsilon(0)} \psi \frac{\partial(f\bar{f})}{\partial r} - f\bar{f} \frac{\partial \psi}{\partial r} \, d\mathcal{H}^1,$$

and simplifying we obtain

$$\int_{\mathbb{D}_\varepsilon} \psi \Delta(f\bar{f}) \, d\lambda^2 = \frac{1}{2\pi} \int_{\mathbb{T}} |f|^2 \, d\mu + \frac{\log \varepsilon}{2\pi} \int_{\partial B_\varepsilon(0)} \frac{\partial(f\bar{f})}{\partial r} \, d\mathcal{H}^1 - \frac{1}{2\pi\varepsilon} \int_{\partial B_\varepsilon(0)} |f|^2 \, d\mathcal{H}^1$$

Arguing as before, taking $\varepsilon \rightarrow 0$ we get

$$\int_{\mathbb{D}} \psi \Delta(f\bar{f}) \, d\lambda^2 = \|f\|_2^2 - |f(0)|^2.$$

Since

$$\begin{aligned} \Delta(f\bar{f}) &= 4\partial\bar{\partial}(f\bar{f}) = 4\partial(\bar{f} \cdot \bar{\partial}f + f \cdot \bar{\partial}\bar{f}) = 4\partial(f \cdot \bar{\partial}\bar{f}) = 4(\partial f \cdot \bar{\partial}\bar{f} + f \cdot \bar{\partial}\partial\bar{f}) = \\ &= 4(\partial f \cdot \bar{\partial}\bar{f} + f \cdot \bar{\partial}\bar{\partial}\bar{f}) = 4f'\bar{f}' = 4|f'|^2 \end{aligned}$$

and $|f(0)|^2 \geq 0$ we obtain

$$\int_{\mathbb{D}} |f'|^2 \psi \, d\lambda^2 \leq \frac{1}{4} \|f\|_2^2.$$

2. We have $fg' = (fg)' - f'g$, therefore

$$|fg'|^2 \leq (|(fg)'| + |f'g|)^2 \leq 2(|(fg)'|^2 + |f'g|^2) \leq 2(|(fg)'|^2 + \|g\|_\infty^2 |f'|^2).$$

Integrating and using 1. yields

$$\begin{aligned} \int_{\mathbb{D}} |fg'|^2 \psi \, d\lambda^2 &\leq 2 \int_{\mathbb{D}} |(fg)'|^2 \psi \, d\lambda^2 + 2\|g\|_\infty^2 \int_{\mathbb{D}} |f'|^2 \psi \, d\lambda^2 \leq \\ &\leq \frac{1}{2} \|fg\|_2^2 + \frac{1}{2} \|f\|_2^2 \|g\|_\infty^2 \leq \|f\|_2^2 \|g\|_\infty^2. \end{aligned}$$

3. Consider the positive measure $\nu := \psi \, d\lambda^2$. Invoking the Cauchy-Schwarz inequality in $L^2(\nu)$ and using 2. we get

$$\begin{aligned} \int_{\mathbb{D}} |fgu'v'| \psi \, d\lambda^2 &\leq \left(\int_{\mathbb{D}} |fu'| \psi \, d\lambda^2 \right)^{1/2} \left(\int_{\mathbb{D}} |gv'| \psi \, d\lambda^2 \right)^{1/2} \leq \\ &\leq \|f\|_2 \|u\|_{\infty} \|g\|_2 \|v\|_{\infty}. \end{aligned}$$

4. By Lemma 8 we can write $f = g_1 g_2$ with $\|g_1\|_2^2 = \|g_2\|_2^2 = \|f\|_1$. Using 3. we then obtain

$$\begin{aligned} \int_{\mathbb{D}} |fu'v'| \psi \, d\lambda^2 &= \int_{\mathbb{D}} |g_1 g_2 u'v'| \psi \, d\lambda^2 \leq \\ &\leq \|g_1\|_2 \|g_2\|_2 \|u\|_{\infty} \|v\|_{\infty} = \|f\|_1 \|u\|_{\infty} \|v\|_{\infty}. \end{aligned}$$

5. Using the Cauchy-Schwarz inequality in $L^2(\nu)$, as well as 1. and 2. we obtain

$$\begin{aligned} \int_{\mathbb{D}} |fg'u'| \psi \, d\lambda^2 &\leq \left(\int_{\mathbb{D}} |fu'| \psi \, d\lambda^2 \right)^{1/2} \left(\int_{\mathbb{D}} |g'| \psi \, d\lambda^2 \right)^{1/2} \leq \\ &\leq \|f\|_2 \|u\|_{\infty} \cdot \frac{1}{2} \|g\|_2 = \frac{1}{2} \|f\|_2 \|g\|_2 \|u\|_{\infty}. \end{aligned}$$

6. We write $f = g_1 g_2$ as in 4. and use 5. to obtain

$$\begin{aligned} \int_{\mathbb{D}} |f'u'| \psi \, d\lambda^2 &\leq \int_{\mathbb{D}} |(g_1 g_2)'u'| \psi \, d\lambda^2 \leq \int_{\mathbb{D}} |g'_1 g_2 u'| \psi \, d\lambda^2 + \int_{\mathbb{D}} |g_1 g'_2 u'| \psi \, d\lambda^2 \leq \\ &\leq \frac{1}{2} \|g_2\|_2 \|g_1\|_2 \|u\|_{\infty} + \frac{1}{2} \|g_1\|_2 \|g_2\|_2 \|u\|_{\infty} = \|f\|_1 \|u\|_{\infty}. \end{aligned}$$

□