The Corona Theorem

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We will denote the space of all complex-valued, bounded, analytic functions on the unit disk $\mathbb{D}$ as H^∞. Equipped with the supremum norm $\|\cdot\|_\infty$ this space becomes a commutative Banach algebra. The space of all multiplicative, bounded, linear functionals on H^∞ not identically zero is denoted $\Delta(H^\infty)$ and is called the *Gelfand space* of H^∞. We endow this space with the subspace topology of the weak-* topology on the topological dual $(H^\infty)'$, which we will refer to as the *Gelfand topology*. For each $z \in \mathbb{D}$ we consider the point-evaluation functional

$$\pi_z : H^\infty \to \mathbb{C},\ f \mapsto f(z).$$

This is clearly multiplicative, bounded and linear and therefore belongs to $\Delta(H^\infty)$. The set of all such functionals $\pi_z, z \in \mathbb{D}$ will be denoted as Δ_0. The *corona* is defined as the complement of the closure of Δ_0 in the Gelfand topology. The corona theorem now states:

Theorem 1 (L. Carleson)**.** The corona is empty. In other words, Δ_0 is dense in $\Delta(H^\infty)$.

There is an equivalent version of the theorem, as given by the following proposition:

Proposition 2. Δ_0 is dense in $\Delta(H^\infty)$ if and only if for any $\delta > 0$ and $f_1, \dots, f_n \in H^\infty$ such that $\sum_{j=1}^n |f_j(z)| \geq \delta, z \in \mathbb{D}$, there exist $g_1, \dots g_n \in H^\infty$ such that $\sum_{j=1}^n f_j g_j = 1$.

Proof. Assume Δ_0 is dense in $\Delta(H^\infty)$ and let $f_1, \dots, f_n \in H^\infty$, and $\delta > 0$ such that $\sum_{j=1}^n |f_j(z)| \geq \delta, z \in \mathbb{D}$. Denote by I the ideal in H^∞ generated by $f_1, \dots, f_n$. If $1 \in I$, then the assertion is established. Assume towards a contradiction that I is a proper ideal, then there exists a maximal ideal $J \supset I$. Since $\Delta(H^\infty)$ is a commutative Banach algebra, there exists a $\phi \in \Delta(H^\infty)$ such that $J = \ker \phi$. Therefore we have $\phi(f_j) = 0$ for $j = 1, \dots, n$. Since Δ_0 is dense, there is a net $(\pi_{z_m})_{m \in M}$ in Δ_0 such that $\pi_{z_m} \to \phi$ in the weak-* topology, that is the net converges pointwise. Therefore, for all $j = 1, \dots, n$ we have $f_j(z_m) = \pi_{z_m}(f_j) \to \phi(f_j) = 0$ and in particular

$$\lim_{m \in M} \sum_{j=1}^n |f_j(z_m)| = 0,$$

a contradiction.

For the other implication, assume towards a contradiction that Δ_0 is not dense in $\Delta(H^\infty)$. Then there exists some $\phi_0 \in \Delta(H^\infty)$ and an open neighbourhood U of ϕ_0 such that $\Delta_0 \cap U = \emptyset$. Since the sets of the form

$$\{\phi \in \Delta(H^\infty) : |(\phi - \phi_0)(f_j)| < \varepsilon, j = 1, \dots, n\},$$

for some $n \in \mathbb{N}, f_1, \dots, f_n \in H^\infty$ and $\varepsilon > 0$, form a neighbourhood basis of ϕ_0 in the weak-* topology, there exists a neighbourhood $V \subseteq U$ described by some $n \in \mathbb{N}, f_1, \dots, f_n \in H^\infty$ and $\delta > 0$. Define $\widetilde{f_j} := f_j - \phi_0(f_j)$, for $j = 1, \dots, n$, then clearly $\phi_0(\widetilde{f_j}) = 0$. Since $\Delta_0 \cap V = \emptyset$, for any $z \in \mathbb{D}$ we have $\pi_z \notin V$ and therefore there exists some $j_0 \in \{1, \dots, n\}$ such that,

$$\delta \leq |(\pi_z - \phi_0)(f_{j_0})| = |f_{j_0}(z) - \phi_0(f_{j_0})| = |\widetilde{f_{j_0}}(z)|.$$

Since $\widetilde{f_j} \in H^\infty$ for $j = 1, \dots, n$, and $\sum_{j=1}^n |\widetilde{f_j}(z)| \geq \delta$, there exist $g_1, \dots, g_n \in H^\infty$ such that $\sum_{j=1}^n \widetilde{f_j} g_j = 1$. But this yields

$$1 = \phi_0(1) = \phi_0\left(\sum_{j=1}^n \widetilde{f_j} g_j\right) = \sum_{j=1}^n \phi_0(\widetilde{f_j})\phi_0(g_j) = 0,$$

a contradiction. $\square$

1 First Steps

Over the following sections we will prove a stronger version of the right statement in Proposition 2:

Theorem 3. There exist constants $C_{n,\delta}$ only depending on $n \in \mathbb{N}$ and $\delta > 0$, such that if $f_1, \dots f_n \in \mathrm{Hol}(\mathbb{D})$ with

$$\|f_j\|_\infty \leq 1, \;\; j = 1, \dots, n, \quad \text{and} \quad \sum_{j=1}^n |f_j(z)|^2 \geq \delta, \;\; z \in \mathbb{D},$$

then there exist $g_1, \dots, g_n \in \mathrm{Hol}(\mathbb{D})$ with

$$\|g_j\|_\infty \leq C_{n,\delta}, \;\; j = 1, \dots, n, \quad \text{and} \quad \sum_{j=1}^n f_j g_j = 1.$$

Proof. We will give the proof in multiple steps. First, for a closed set $A \subset \mathbb{C}$ and a space of functions on an open sets $\Omega \supset A$, say $D(\Omega)$, we define

$$D(A) := \bigcup_{\Omega \supset A \text{ open}} T(D(\Omega)), \quad \text{where} \quad T(f) := f\big|_A.$$

We will make use of this to handle smooth or holomorphic functions on closed sets.

Step 1 (Reduction to $f_1, \dots, f_n \in \mathrm{Hol}(\overline{\mathbb{D}})$): Assume that the statement of the theorem holds for all $\widetilde{f_1}, \dots, \widetilde{f_n} \in \mathrm{Hol}(\overline{\mathbb{D}})$, we claim that it then also holds in its original form[1]. For our given $f_1, \dots, f_n$ satisfying the premise of the theorem and all $0 < s < 1$ we define

[1]Note that this does **not** mean that we can assume $f_1, \dots, f_n \in \mathrm{Hol}(\overline{\mathbb{D}})$ in the previous proposition.

$f_{j,s}(z) := f_j(sz)$, $j = 1, \dots, n$. Then for every $0 < s < 1$ and $j = 1, \dots, n$ the function $f_{j,s}$ is in $\mathrm{Hol}(\overline{\mathbb{D}})$ and satisfies the premise of the theorem. By our assumption there exist $g_{j,s} \in H^\infty$, $j = 1, \dots, n$ such that

$$\|g_{j,s}\|_\infty \leq C_{n,\delta}, \ \ j = 1, \dots, n, \quad \text{and} \quad \sum_{j=1}^n f_{j,s} g_{j,s} = 1.$$

For a fixed $j \in \{1, \dots, n\}$, the set $\{g_{j,s} : 0 < s < 1\}$ is uniformly bounded and therefore normal in $\mathrm{Hol}(\mathbb{D})$. By Montel's Theorem there exists a sequence $s_m \to 1$ and some $g_j \in \mathrm{Hol}(\mathbb{D})$ such that $g_{j,s_m} \to g_j$ compactly. In particular, we obtain

$$\|g_j\|_\infty = \lim_{m\to\infty} \|g_{j,s_m}\|_\infty \leq C_{n,\delta}, \quad j = 1, \dots, n,$$

and

$$1 = \lim_{m\to\infty} \sum_{j=1}^n f_{j,s_m} g_{j,s_m} = \sum_{j=1}^n f_j g_j,$$

concluding our claim. We may thus assume that our given $f_1, \dots, f_n$ are holomorphic on $\overline{\mathbb{D}}$ instead.

Step 2 (Solve in $C^\infty(\overline{\mathbb{D}})$): For $j = 1, \dots, n$ we define

$$h_j := \frac{\bar{f}_j}{\sum_{k=1}^n |f_k|^2} \in C^\infty(\overline{\mathbb{D}}),$$

then clearly $\sum_{j=1}^n f_j h_j = 1$ and $\|h_j\|_\infty \leq \frac{1}{\delta}$. The real task now lies in changing the h_j to become holomorphic in $\mathbb{D}$, without losing control over the boundedness of the solutions.

2 Wirtinger Derivatives

Before we continue we want to briefly introduce a useful generalization of the complex derivative.

Definition 4. Let $\Omega \subseteq \mathbb{R}^2$ be open. Then the *Wirtinger derivatives* (or *Wirtinger operators*) are defined on $C^1(\Omega)$ by

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

We will also abbreviate these operators as ∂ and $\bar{\partial}$, respectively.

Note that by writing a complex number $z \in \mathbb{C}$ as $z = x + iy$ with $x, y \in \mathbb{R}$ we can identify $\mathbb{C} \cong \mathbb{R}^2$. Therefore we can also interpret the Wirtinger operators to act on $C^1(\Omega)$ with an open subset $\Omega \subseteq \mathbb{C}$.

Before listing properties of the Wirtinger operators we quickly want to recall that a function $f \in C^1(\Omega), f = u + iv$ is holomorphic if and only if it satisfies the *Cauchy–Riemann equations*:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Remark 5. Let $\Omega \subseteq \mathbb{C}$ be open and $f \in C^1(\Omega)$.

1. The Wirtinger operators are $\mathbb{C}$-linear, satisfy the Leibniz rule[2], the chain rule

$$\frac{\partial}{\partial z}(f \circ g) = \left(\frac{\partial f}{\partial z} \circ g\right) \frac{\partial g}{\partial z} + \left(\frac{\partial f}{\partial \bar{z}} \circ g\right) \frac{\partial \bar{g}}{\partial z},$$
$$\frac{\partial}{\partial \bar{z}}(f \circ g) = \left(\frac{\partial f}{\partial z} \circ g\right) \frac{\partial g}{\partial \bar{z}} + \left(\frac{\partial f}{\partial \bar{z}} \circ g\right) \frac{\partial \bar{g}}{\partial \bar{z}},$$

   where $g \in C^1(\Omega), g(\Omega) \subseteq \Omega$, and are compatible with complex conjugation, as in

$$\overline{\left(\frac{\partial f}{\partial z}\right)} = \frac{\partial \bar{f}}{\partial \bar{z}}, \quad \overline{\left(\frac{\partial f}{\partial \bar{z}}\right)} = \frac{\partial \bar{f}}{\partial z}$$

2. If $f \in \mathrm{Hol}(\Omega), f = u + iv$, then

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x} = f'.$$

3. Since

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) =$$
$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),$$

   we have that[3] $f \in \mathrm{Hol}(\Omega)$ if and only if $\overline{\partial} f = 0$.

4. On $C^2(\Omega)$, the *Laplace operator* can be represented as

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}.$$

Proof (continued). Step 3 (The Koszul complex): We consider the spaces

$$C_0 \coloneqq C^\infty(\overline{\mathbb{D}}), \quad C_1 \coloneqq (C_0)^n, \quad C_2 \coloneqq \{A \in (C_0)^{n\times n} : A = -A^T\}$$

and the linear maps

$$P_{1,0} : C_1 \to C_0,\ (g_j)_{j=1}^n \mapsto \sum_{j=1}^n g_j f_j, \quad P_{2,1} : C_2 \to C_1,\ (g_{jk})_{j,k=1}^n \mapsto \left(\sum_{k=1}^n g_{jk} f_k\right)_{j=1}^n.$$

[2]This means that the Wirtinger opeartors are derivatives from an algebraic perspective.
[3]This can be interpreted as "f is independant of $\overline{z}$".

We also consider the operator $\overline{\partial} : C_0 \to C_0$. It is well-defined, since if $f_1 \in C^\infty(\Omega_1)$, $f_2 \in C^\infty(\Omega_2)$ with $f_1 = f_2$ on $\overline{\mathbb{D}}$, then in particular $f_1 = f_2$ on $\mathbb{D}$ and therefore $\overline{\partial} f_1 = \overline{\partial} f_2$ on $\mathbb{D}$. By continuity, we therefore also get $\overline{\partial} f_1 = \overline{\partial} f_2$ on $\overline{\mathbb{D}}$.

Applying $\overline{\partial}$ pointwise in C_1 and C_2 as well, the resulting connections are visualized in the diagram below, called the *Koszul complex*:

$$\begin{tikzcd}
C_2 \arrow[r, "P_{2,1}"] \arrow[d, "\overline{\partial}"'] & C_1 \arrow[r, "P_{1,0}"] \arrow[d, "\overline{\partial}"'] & C_0 \arrow[d, "\overline{\partial}"'] \\
C_2 \arrow[r, "P_{2,1}"'] & C_1 \arrow[r, "P_{1,0}"'] & C_0
\end{tikzcd}$$

Lemma 6. The Koszul complex has the following properties:

1. The diagram is commutative, that is we have $P_{j+1,j}\overline{\partial} = \overline{\partial} P_{j+1,j}$ for $j = 0, 1$.
2. The horizontal sequences are exact, that is $\operatorname{ran} P_{2,1} = \ker P_{1,0}$.
3. The vertical maps $\overline{\partial} : C_j \to C_j$ for $j = 0, 1, 2$ are surjective.

Proof.

1. For $g \in C_0$ and $f \in \operatorname{Hol}(\overline{\mathbb{D}})$ we have

$$\overline{\partial}(gf) = f\overline{\partial} g + g\overline{\partial} f = f\overline{\partial} g$$

and together with the linearity of $\overline{\partial}$ the statement follows.

2. "$\subseteq$": Let $g \in C_2, g = (g_{jk})_{j,k=1}^n$, then

$$P_{1,0}P_{2,1}g = P_{1,0}\left[\left(\sum_{k=1}^n g_{jk}f_k\right)_{j=1}^n\right] = \sum_{j=1}^n\sum_{k=1}^n g_{jk}f_kf_j = 0$$

since g is skew-symmetric and therefore $g \in \ker P_{1,0}$.

"$\supseteq$" Let $g \in \ker P_{1,0} \subseteq C_1, g = (g_1, \dots, g_n)$. We define $p = (p_{jk})_{j,k=1}^n \in C_2$ by

$$p_{jk} := \frac{1}{\sum_{\ell=1}^n |f_\ell|^2}(g_j\overline{f_k} - g_k\overline{f_j}).$$

Then for any $j = 1, \dots, n$ we have

$$(P_{2,1}p)_j = \sum_{k=1}^n p_{jk}f_k = \frac{1}{\sum_{\ell=1}^n |f_\ell|^2}\sum_{k=1}^n(g_j|f_k|^2 - g_k\overline{f_j}f_k) =$$

$$
\begin{aligned}
&= g_j - \frac{1}{\sum_{\ell=1}^n |f_\ell|^2} \overline{f_j} \sum_{k=1}^n g_k f_k = g_j - \frac{1}{\sum_{\ell=1}^n |f_\ell|^2} \overline{f_j} P_{1,0} g = \\
&= g_j,
\end{aligned}
$$

and therefore $g \in \operatorname{ran} P_{2,1}$.

3. For given $v \in C^\infty(\overline{\mathbb{D}})$ we want to solve the partial differential equation

$$
\frac{\partial u}{\partial \bar{z}} = v \quad (\text{on } \overline{\mathbb{D}})
$$

for some $u \in C^\infty(\overline{\mathbb{D}})$. We will approach this using a fundamental solution of the differential operator $\overline{\partial}$. Recall that

$$
\Gamma(z) \coloneqq \frac{1}{2\pi} \log |z|
$$

is a fundamental solution of the Laplace operator, that is we have $\Delta\Gamma = \delta_0$ distributionally, where δ_0 denotes the delta distribution at 0. We claim that $\frac{1}{\pi z}$ is a fundamental solution of $\overline{\partial}$, and verify this via

$$
\begin{aligned}
\overline{\partial}\frac{1}{z} = \overline{\partial}\frac{\bar{z}}{|z|^2} &= \frac{1}{2}(\partial_x + i\partial_y)\frac{x-iy}{x^2+y^2} = \\
&= \frac{1}{2}\left[\partial_x \frac{x}{x^2+y^2} - i\partial_x \frac{y}{x^2+y^2} + i\partial_y \frac{x}{x^2+y^2} + \partial_y \frac{y}{x^2+y^2}\right] = \\
&= \frac{1}{2}\left[\partial_x^2 \log|z| + \partial_y^2 \log|z| + i\left(\frac{2xy}{x^2+y^2} - \frac{2xy}{x^2+y^2}\right)\right] = \\
&= \frac{1}{2}\Delta \log|z| = \frac{1}{2} 2\pi\delta_0 = \pi\delta_0.
\end{aligned}
$$

Now let $\Omega \supset \overline{\mathbb{D}}$ be open such that $v \in C^\infty(\Omega)$ and choose $\varphi \in C_c^\infty(\Omega)$ such that $\varphi|_{\overline{\mathbb{D}}} = 1$. Then $\varphi v \in C_c^\infty(\Omega)$, therefore

$$
u(w) \coloneqq \left(\frac{1}{\pi z} * \varphi v\right)(w) = \frac{1}{\pi}\int_\Omega \frac{\varphi(z)v(z)}{w-z} \,\mathrm{d}\lambda^2(z)
$$

is a classical solution of $\overline{\partial}u = \varphi v$ in Ω. Since $\varphi v = v$ on $\overline{\mathbb{D}}$, we get $\overline{\partial}u = v$ on $\overline{\mathbb{D}}$, as desired.

Arguing pointwise shows the surjectivity of the maps $\overline{\partial} : C_\ell \to C_\ell$ for $\ell = 0, 1$. For $\ell = 2$ and given $b = (b_{jk})_{j,k=1}^n \in C_2$ we first solve

$$
\overline{\partial} a_{jk} = b_{jk}, \quad \text{for} \quad 1 \le j < k \le n
$$

and then set $a_{jj} = 0$ and $a_{jk} = -a_{kj}$ for $n \ge j > k \ge 1$.

□

Proof (continued). Step 4 (Apply to $h = (h_1, \dots, h_n) \in C_1$): In step 2 we constructed an element $h = (h_1, \dots, h_n) \in C_1$ by setting

$$h_j := \frac{\bar{f}_j}{\sum_{k=1}^n |f_k|^2}.$$

By our construction we have $P_{1,0}h = 1$ and therefore $0 = \overline{\partial} P_{1,0} h = P_{1,0} \overline{\partial} h$, thus $\overline{\partial} h \in \ker P_{1,0}$. By Lemma 6 there exists $b \in C_2$ such that $P_{2,1} b = \overline{\partial} h$ and $a \in C_2$ such that $\overline{\partial} a = b$. We now set $g := h - P_{2,1} a \in C_1$. Then

$$P_{1,0} g = P_{1,0} h - P_{1,0} P_{2,1} a = 1$$

and

$$\overline{\partial} g = \overline{\partial} h - \overline{\partial} P_{2,1} a = \overline{\partial} h - P_{2,1} b = 0.$$

Therefore g is a solution to

$$\sum_{k=1}^n f_k g_k = 1$$

in $\mathrm{Hol}(\overline{\mathbb{D}})$. However, we do not have an estimate on $|g_j|$ yet.

3 Hardy Spaces

Let μ denote the Lebesgue measure on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, i.e. the measure such that for a measurable function $f : \mathbb{T} \to \mathbb{C}$ it holds that

$$\int_{\mathbb{T}} f \, \mathrm{d}\mu = \int_{-\pi}^{\pi} f(e^{i\vartheta}) \, \mathrm{d}\vartheta.$$

We define the $L^p(\mathbb{T})$-norms via the *normed* Lebesgue measure $\frac{1}{2\pi}\mu$:

$$\|f\|_p := \left(\frac{1}{2\pi} \int_{\mathbb{T}} |f|^p \, \mathrm{d}\mu \right)^{1/p}, \quad \text{for } 1 \le p < \infty, \quad \text{and} \quad \|f\|_\infty := \text{ess. sup} |f|.$$

For $f \in L^1(\mathbb{T})$ and $n \in \mathbb{N}$ we define the n-th *Fourier coefficient* by

$$\hat{f}(n) := \frac{1}{2\pi} \int_{\mathbb{T}} f(\xi) \xi^{-n} \, \mathrm{d}\mu(\xi).$$

For $1 \le p \le \infty$ we define the *Hardy space* H^p as the set of all $f \in \mathrm{Hol}(\mathbb{D})$ with $\|f\|_p < \infty$, where

$$\|f\|_p := \lim_{r \to 1} \left(\frac{1}{2\pi} \int_{\mathbb{T}} |f_r|^p \, \mathrm{d}\mu \right)^{1/p} \quad \text{for } p < \infty, \quad \text{and} \quad \|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)|.$$

Equivalently one can interchange the limit in the definition above with the supremum over $0 < r < 1$. It is of note that convergence in the Hardy spaces implies compact convergence. We lastly define H^p_0 as the (closed) subspace of all $f \in H^p$, for which $f(0) = 0$.

We summarize the characterisation of Hardy spaces:

Theorem 7. Let $1 \leq p \leq \infty$. Then:

1. H^p is a Banach space[4].
2. For $p \leq q \leq \infty$ it holds that $H^p \supseteq H^q$.
3. Let $f \in H^p$, then for almost all $\xi \in \mathbb{T}$ the limit
$$\lim_{r \to 1} f(r\xi) =: f^*(\xi)$$
exists and defines a function in $L^p(\mathbb{T})$, also called the *boundary values* of f. If $p < \infty$, we also have $\lim_{r\to 1} \|f^* - f_r\|_p = 0$, where $f_r(\xi) := f(r\xi)$.
4. The map $^* : f \mapsto f^*$ is an isometry from H^p onto
$$L^p_+(\mathbb{T}) := \{f \in L^p(\mathbb{T}) : \forall n < 0 : \hat{f}(n) = 0\},$$
which is a closed subspace of $L^p(\mathbb{T})$.
5. Every $f \in H^p$ can be written as a Cauchy integral of its boundary values:
$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f^*(\zeta)}{\zeta - z} \, \mathrm{d}\zeta, \quad z \in \mathbb{D}$$

We will also use the following lemma:

Lemma 8. Let $f \in \mathrm{Hol}(\mathbb{D})$, then there exist $g_1, g_2 \in \mathrm{Hol}(\mathbb{D})$ such that
$$f = g_1 g_2, \quad \text{and} \quad \|g_1\|_2^2 = \|g_2\|_2^2 = \|f\|_1.$$

Proof (continued). Returning to our proof, recall that we want to obtain a bound on the functions $\|g_j\|_\infty$, where
$$g_j = h_j - \sum_{k=1}^{n} a_{jk} f_k,$$
and $a_{jk} \in C^\infty(\overline{\mathbb{D}})$ is a solution of the partial differential equation
$$\frac{\partial y}{\partial \bar{z}} = \left(\sum_{\ell=1}^{n} |f_\ell|^2 \right)^{-1} \left(\frac{\partial h_j}{\partial \bar{z}} \bar{f}_k - \frac{\partial h_k}{\partial \bar{z}} \bar{f}_j \right).$$
We want to show that the solution a_{jk} can be chosen in a way, that the resulting functions $g_j \in \mathrm{Hol}(\overline{\mathbb{D}})$ are bounded in the H^∞-norm by a constant depending only on n and δ, that is
$$\|g_j\|_\infty \leq C_{n,\delta}.$$
Denote by u_{jk} the right-hand side of the partial differential equation above. We fix a solution $\overline{\partial} v_{jk} = u_{jk}$ and notice that if $\overline{\partial} a_{jk} = u_{jk}$ is another solution, then
$$\overline{\partial}(a_{jk} - v_{jk}) = \overline{\partial} a_{jk} - \overline{\partial} v_{jk} = 0,$$

[4] In particular, H^∞ is a Banach algebra, which we already used in the introduction.

that is the difference is bounded and holomorphic, thus in H^∞. We can therefore write

$$a_{jk} = v_{jk} + p, \quad p \in H^\infty.$$

We can view a_{jk} as an element of $L^\infty(\mathbb{T})$ by considering $v_{jk}|_{\mathbb{T}} \in L^\infty(\mathbb{T})$ and $p^* \in (H^\infty)^* \subset L^\infty(\mathbb{T})$. If we manage to bound

$$\|a_{jk}\|_{L^\infty(\mathbb{T})} = \operatorname{ess.}\sup_{z\in\mathbb{T}} |a_{jk}(z)| \leq K_{n,\delta},$$

we immediately get

$$\|g_j\|_{H^\infty} \leq \|h_j\|_{H^\infty} + \sum_{k=1}^n \|a_{jk} f_k\|_{H^\infty} \leq \frac{1}{\delta} + \sum_{k=1}^n \|a_{jk}\|_{L^\infty(\mathbb{T})} \|f_k\|_{H^\infty} \leq \frac{1}{\delta} + nK_{n,\delta},$$

resulting in the claim of the theorem.

Note that we can vary $\|a_{jk}\|_{L^\infty(\mathbb{T})}$ by choosing different functions $p \in H^\infty$. We therefore want to bound the quantity

$$\inf_{p\in H^\infty} \|v_{jk} + p^*\|_\infty,$$

which is precisely the norm of v_{jk} in the quotient space $L^\infty(\mathbb{T})/(H^\infty)^*$. The following lemma allows us to translate the minimization problem into a maximization problem.

Lemma 9. The map

$$\Phi : L^\infty(\mathbb{T})/(H^\infty)^* \to ((H_0^1)^*)',\ f + (H^\infty)^* \mapsto \left[g \mapsto \frac{1}{2\pi} \int_{\mathbb{T}} fg \,\mathrm{d}\mu \right]$$

is an isometric isomorphism.

Proof. We have $L^\infty(\mathbb{T}) \cong L^1(\mathbb{T})'$ via the duality

$$\langle f, g\rangle := \frac{1}{2\pi} \int_{\mathbb{T}} fg \,\mathrm{d}\mu, \quad f \in L^\infty(\mathbb{T}),\ g \in L^1(\mathbb{T}).$$

Since $(H_0^1)^* \leq L^1(\mathbb{T})$ we therefore have $((H_0^1)^*)' \cong L^\infty(\mathbb{T})/((H_0^1)^*)^\perp$ via

$$\langle f, g\rangle = \frac{1}{2\pi} \int_{\mathbb{T}} fg \,\mathrm{d}\mu, \quad f \in L^\infty(\mathbb{T})/((H_0^1)^*)^\perp,\ g \in (H_0^1)^*.$$

It remains to show that $((H_0^1)^*)^\perp = (H^\infty)^*$. Let $w^* \in ((H_0^1)^*)^\perp \leq L^\infty(\mathbb{T})$, then for any $n \in \mathbb{N}$ we have

$$0 = \langle w^*, (z^n)^*\rangle = \langle w^*, z^n\rangle = \widehat{w^*}(-n).$$

Therefore $w^* \in L_+^\infty(\mathbb{T}) = (H^\infty)^*$. For the other inclusion let $w^* \in (H^\infty)^*$ and $h^* \in (H_0^1)^*$, then

$$\begin{aligned} \langle w^*, h^*\rangle = \frac{1}{2\pi} \int_0^{2\pi} w^*(e^{i\vartheta}) h^*(e^{i\vartheta}) \,\mathrm{d}\vartheta = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} w(re^{i\vartheta}) h(re^{i\vartheta}) \frac{ie^{i\vartheta}}{ie^{i\vartheta}} \,\mathrm{d}\vartheta = \\ = \lim_{r\to 1} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{w_r(\zeta) h_r(\zeta)}{\zeta} \,\mathrm{d}\zeta = \lim_{r\to 1} w_r(0) h_r(0) = 0, \end{aligned}$$

where the second-to-last equation follows since $w_r h_r \in \operatorname{Hol}(\overline{\mathbb{D}})$ and the last one since $h(0) = 0$. Therefore $w^* \in ((H_0^1)^*)^\perp$, concluding the proof. $\square$

Proof (continued). Step 5 (Dualisation): Since we longer need the indices, we now simply write v instead of v_{jk}. Applying the above lemma to our previous situation we can re-describe the norm of $v+(H^\infty)^* \in L^\infty(\mathbb{T})/(H^\infty)^*$ as

$$\begin{aligned}\|\Phi(v+(H^\infty)^*)\| &= \sup_{\substack{F\in (H_0^1)^* \\ \|F\|_1\le 1}} |\Phi(v+(H^\infty)^*)(F)| = \sup_{\substack{F\in H_0^1 \\ \|F\|_1\le 1}} \left|\frac{1}{2\pi}\int_{\mathbb{T}} vF^*\,\mathrm{d}\mu\right| = \\ &= \sup_{\substack{F\in H_0^1 \\ \|F\|_1\le 1}} \left|\lim_{r\to 1}\frac{1}{2\pi}\int_{\mathbb{T}} vF_r\,\mathrm{d}\mu\right| = \sup_{\substack{F\in \mathrm{Hol}(\overline{\mathbb{D}}) \\ F(0)=0, \|F\|_1\le 1}} \left|\frac{1}{2\pi}\int_{\mathbb{T}} vF\,\mathrm{d}\mu\right|.\end{aligned}$$

We now want to bound this supremum, where, as before, $v\in C^\infty(\overline{\mathbb{D}})$.

Step 6: We want to redescribe the integral

$$\frac{1}{2\pi}\int_{\mathbb{T}} vF\,\mathrm{d}\mu,$$

where $v\in C^\infty(\overline{\mathbb{D}})$ and $F\in \mathrm{Hol}(\overline{\mathbb{D}}), F(0)=0, \|F\|_1\le 1$. Let $\sigma := vF$ and

$$\varphi(z) := \frac{1}{2\pi}\log|z|, \quad z\ne 0.$$

For $\varepsilon>0$ let $\mathbb{D}_\varepsilon := \mathbb{D}\setminus\overline{B_\varepsilon(0)}$. By Green's second identity we have

$$\int_{\mathbb{D}_\varepsilon} \sigma\Delta\varphi - \varphi\Delta\sigma\,\mathrm{d}\lambda^2 = \int_{\mathbb{T}} \sigma\frac{\partial\varphi}{\partial r} - \varphi\frac{\partial\sigma}{\partial r}\,\mathrm{d}\mu - \int_{\partial B_\varepsilon(0)} \sigma\frac{\partial\varphi}{\partial r} - \varphi\frac{\partial\sigma}{\partial r}\,\mathrm{d}\mathcal{H}^1,$$

where $\frac{\partial}{\partial r}$ denotes the radial derivative. Simplifying results in

$$-\int_{\mathbb{D}_\varepsilon} \varphi\Delta\sigma\,\mathrm{d}\lambda^2 = \frac{1}{2\pi}\int_{\mathbb{T}}\sigma\,\mathrm{d}\mu - \frac{1}{2\pi}\int_{\partial B_\varepsilon(0)} \frac{\sigma}{\varepsilon} - \frac{\partial\sigma}{\partial r}\log\varepsilon\,\mathrm{d}\mathcal{H}^1.$$

By the intermediate value theorem for integrals for any $\varepsilon>0$ there exists a $\zeta_\varepsilon\in\partial B_\varepsilon(0)$ such that

$$\frac{1}{2\pi\varepsilon}\int_{\partial B_\varepsilon(0)} \sigma\,\mathrm{d}\mathcal{H}^1 = \sigma(\zeta_\varepsilon)\to\sigma(0) = v(0)F(0) = 0.$$

Furthermore $\left|\frac{\partial\sigma}{\partial r}\right| \le M$ on $\overline{\mathbb{D}}$ for some $M>0$, thus

$$\frac{1}{2\pi}\int_{\partial B_\varepsilon(0)}\left|\frac{\partial\sigma}{\partial r}\log\varepsilon\right|\mathrm{d}\mathcal{H}^1 \le M\varepsilon\log\varepsilon\to 0.$$

Finally,

$$\Delta\sigma = \Delta(vF) = 4\partial\overline{\partial}(vF) = 4\partial(v\overline{\partial}F + F\overline{\partial}v) = 4\partial(Fu) = 4(F\partial u + uF').$$

With $\psi := -\varphi = \frac{1}{2\pi}\log\frac{1}{|z|}$, by letting $\varepsilon\to 0$ we thus obtain

$$\frac{1}{2\pi}\int_{\mathbb{T}} vF\,\mathrm{d}\mu = 4\int_{\mathbb{D}}\psi\left(F\partial u + uF'\right)\mathrm{d}\lambda^2 =$$

$$= 4\left(\int_{\mathbb{D}} F\,\partial u\,\psi\,\mathrm{d}\lambda^2 + \int_{\mathbb{D}} uF'\psi\,\mathrm{d}\lambda^2\right) =: 4(I_1 + I_2).$$

Our goal is to show the existence of a constant $K_{n,\delta}$ such that

$$\left|\frac{1}{2\pi}\int_{\mathbb{T}} vF\,\mathrm{d}\mu\right| \le 4(|I_1| + |I_2|) \le K_{n,\delta}.$$

Recall that

$$u = \tau(\bar f_k \overline{\partial} h_j - \bar f_j \overline{\partial} h_k), \quad \text{where} \quad \tau := \left(\sum_{\ell=1}^n |f_\ell|^2\right)^{-1}.$$

Since $h_\ell = \tau \bar f_\ell$ we first calculate want to calculate $\overline{\partial}\tau$. We first notice that with function $m(z) := z^{-1}$ and any nonvanishing function $\alpha \in C^\infty$ we have by the chain rule

$$\partial\alpha^{-1} = \partial(m \circ \alpha) = (\partial m \circ \alpha)\partial\alpha + (\overline{\partial} m \circ \alpha)\partial\bar g = -\alpha^{-2}\partial\alpha,$$

and analogously $\overline{\partial}\alpha^{-1} = -\alpha^{-2}\overline{\partial}\alpha$. Therefore

$$\begin{aligned}
\overline{\partial}\tau &= -\tau^2 \sum_{\ell=1}^n \overline{\partial}(f_\ell \bar f_\ell) = -\tau^2 \sum_{\ell=1}^n (f_\ell \overline{\partial} \bar f_\ell + \bar f_\ell \overline{\partial} f_\ell) = \\
&= -\tau^2 \sum_{\ell=1}^n f_\ell \bar f_\ell' =: -\tau^2 \eta, \\
\partial\tau &= \cdots = -\tau^2 \bar\eta.
\end{aligned}$$

We therefore obtain $\overline{\partial} h_\ell = \tau \overline{\partial} \bar f_\ell + \bar f_\ell \overline{\partial}\tau = \tau(\bar f_\ell' - \bar f_\ell \tau\eta)$ and by that the representations

$$\begin{aligned}
u &= \tau(\bar f_k \overline{\partial} h_j - \bar f_j \overline{\partial} h_k) = \tau^2(\bar f_k \bar f_j' - \bar f_k \bar f_j \tau\eta - \bar f_j \bar f_k' + \bar f_j \bar f_k \tau\eta) = \tau^2(\bar f_k \bar f_j' - \bar f_j \bar f_k'), \\
\partial u &= \tau^2 \partial(\bar f_k \bar f_j' - \bar f_j \bar f_k') + (\bar f_k \bar f_j' - \bar f_j \bar f_k')\partial\tau^2 = -2\tau^3\left(\sum_{\ell=1}^n \bar f_\ell f_\ell'\right)(\bar f_k \bar f_j' - \bar f_j \bar f_k').
\end{aligned}$$

4 Integral estimates

Lemma 10. Let $f, g, u, v \in \mathrm{Hol}(\overline{\mathbb{D}})$, then the following integral estimates hold:

1. $\displaystyle\int_{\mathbb{D}} |f'|^2 \psi\,\mathrm{d}\lambda^2 \le \frac{1}{4}\|f\|_2^2$
2. $\displaystyle\int_{\mathbb{D}} |fg'|^2 \psi\,\mathrm{d}\lambda^2 \le \|f\|_2^2 \|g\|_\infty^2$
3. $\displaystyle\int_{\mathbb{D}} |fgu'v'|\psi\,\mathrm{d}\lambda^2 \le \|f\|_2\|g\|_2\|u\|_\infty\|v\|_\infty$
4. $\displaystyle\int_{\mathbb{D}} |fu'v'|\psi\,\mathrm{d}\lambda^2 \le \|f\|_1\|u\|_\infty\|v\|_\infty$
5. $\displaystyle\int_{\mathbb{D}} |fg'u'|\psi\,\mathrm{d}\lambda^2 \le \frac{1}{2}\|f\|_2\|g\|_2\|u\|_\infty$

6. $\displaystyle\int_{\mathbb{D}} |f'u'|\psi \,\mathrm{d}\lambda^2 \leq \|f\|_1 \|u\|_\infty$

Proof.

1. Applying Green's formula on $f\bar{f}$ and ψ yields
$$\int_{\mathbb{D}_\varepsilon} \psi\Delta(f\bar{f}) - f\bar{f}\Delta\psi \,\mathrm{d}\lambda^2 = \int_{\mathbb{T}} \psi\frac{\partial(f\bar{f})}{\partial r} - f\bar{f}\frac{\partial\psi}{\partial r}\,\mathrm{d}\mu - \int_{\partial B_\varepsilon(0)} \psi\frac{\partial(f\bar{f})}{\partial r} - f\bar{f}\frac{\partial\psi}{\partial r}\,\mathrm{d}\mathcal{H}^1,$$
and simplifying we obtain
$$\int_{\mathbb{D}_\varepsilon} \psi\Delta(f\bar{f})\,\mathrm{d}\lambda^2 = \frac{1}{2\pi}\int_{\mathbb{T}} |f|^2\,\mathrm{d}\mu + \frac{\log\varepsilon}{2\pi}\int_{\partial B_\varepsilon(0)} \frac{\partial(f\bar{f})}{\partial r}\,\mathrm{d}\mathcal{H}^1 - \frac{1}{2\pi\varepsilon}\int_{\partial B_\varepsilon(0)} |f|^2\,\mathrm{d}\mathcal{H}^1$$
Arguing as before, taking $\varepsilon \to 0$ we get
$$\int_{\mathbb{D}} \psi\Delta(f\bar{f})\,\mathrm{d}\lambda^2 = \|f\|_2^2 - |f(0)|^2.$$
Since
$$\begin{aligned}\Delta(f\bar{f}) &= 4\partial\overline{\partial}(f\bar{f}) = 4\partial(\bar{f}\cdot\overline{\partial}f + f\cdot\overline{\partial}\bar{f}) = 4\partial(f\cdot\overline{\partial}\bar{f}) = 4(\partial f\cdot\overline{\partial}\bar{f} + f\cdot\overline{\partial}\partial\bar{f}) = \\ &= 4(\partial f\cdot\overline{\partial f} + f\cdot\overline{\partial}\,\overline{\overline{\partial} f}) = 4f'\bar{f}' = 4|f'|^2\end{aligned}$$
and $|f(0)|^2 \geq 0$ we obtain
$$\int_{\mathbb{D}} |f'|^2\psi\,\mathrm{d}\lambda^2 \leq \frac{1}{4}\|f\|_2^2.$$

2. We have $fg' = (fg)' - f'g$, therefore
$$|fg'|^2 \leq (|(fg)'| + |f'g|)^2 \leq 2(|(fg)'|^2 + |f'g|^2) \leq 2(|(fg)'|^2 + \|g\|_\infty^2|f'|^2).$$
Integrating and using 1. yields
$$\begin{aligned}\int_{\mathbb{D}} |fg'|^2\psi\,\mathrm{d}\lambda^2 &\leq 2\int_{\mathbb{D}} |(fg)'|^2\psi\,\mathrm{d}\lambda^2 + 2\|g\|_\infty^2\int_{\mathbb{D}} |f'|^2\psi\,\mathrm{d}\lambda^2 \leq \\ &\leq \frac{1}{2}\|fg\|_2^2 + \frac{1}{2}\|f\|_2^2\|g\|_\infty^2 \leq \|f\|_2^2\|g\|_\infty^2.\end{aligned}$$

3. Consider the positive measure $\nu := \psi\,\mathrm{d}\lambda^2$. Invoking the Cauchy-Schwarz inequality in $L^2(\nu)$ and using 2. we get
$$\begin{aligned}\int_{\mathbb{D}} |fgu'v'|\psi\,\mathrm{d}\lambda^2 &\leq \left(\int_{\mathbb{D}} |fu'|\psi\,\mathrm{d}\lambda^2\right)^{1/2}\left(\int_{\mathbb{D}} |gv'|\psi\,\mathrm{d}\lambda^2\right)^{1/2} \leq \\ &\leq \|f\|_2\|u\|_\infty\|g\|_2\|v\|_\infty.\end{aligned}$$

4. By Lemma 8 we can write $f = g_1g_2$ with $\|g_1\|_2^2 = \|g_2\|_2^2 = \|f\|_1$. Using 3. we then obtain
$$\begin{aligned}\int_{\mathbb{D}} |fu'v'|\psi\,\mathrm{d}\lambda^2 &= \int_{\mathbb{D}} |g_1g_2u'v'|\psi\,\mathrm{d}\lambda^2 \leq \\ &\leq \|g_1\|_2\|g_2\|_2\|u\|_\infty\|v\|_\infty = \|f\|_1\|u\|_\infty\|v\|_\infty.\end{aligned}$$

5. Using the Cauchy-Schwarz inequality in $L^2(\nu)$, as well as 1. and 2. we obtain

$$\begin{aligned}\int_{\mathbb{D}} |fg'u'|\psi \,\mathrm{d}\lambda^2 &\leq \left(\int_{\mathbb{D}} |fu'|\psi \,\mathrm{d}\lambda^2\right)^{1/2} \left(\int_{\mathbb{D}} |g'|\psi \,\mathrm{d}\lambda^2\right)^{1/2} \leq \\ &\leq \|f\|_2\|u\|_\infty \cdot \frac{1}{2}\|g\|_2 = \frac{1}{2}\|f\|_2\|g\|_2\|u\|_\infty.\end{aligned}$$

6. We write $f = g_1g_2$ as in 4. and use 5. to obtain

$$\begin{aligned}\int_{\mathbb{D}} |f'u'|\psi \,\mathrm{d}\lambda^2 &\leq \int_{\mathbb{D}} |(g_1g_2)'u'|\psi \,\mathrm{d}\lambda^2 \leq \int_{\mathbb{D}} |g_1'g_2u'|\psi \,\mathrm{d}\lambda^2 + \int_{\mathbb{D}} |g_1g_2'u'|\psi \,\mathrm{d}\lambda^2 \leq \\ &\leq \frac{1}{2}\|g_2\|_2\|g_1\|_2\|u\|_\infty + \frac{1}{2}\|g_1\|_2\|g_2\|_2\|u\|_\infty = \|f\|_1\|u\|_\infty.\end{aligned}$$

□

Proof (continued). Finish proof.