The Corona Theorem

These are my notes for my talk in the winter semester of 2024, in the analysis seminar at the Technical University of Vienna.

We will denote the space of all complex-valued, bounded, analytic functions on the unit disk $\mathbb{D}$ as H^∞. The space of all multiplicative, bounded, linear functionals on H^∞ not identically zero is denoted $\Delta(H^\infty)$ and is called the *Gelfand space* of H^∞. We endow this space with the subspace topology of the weak-* topology on the dual $(H^\infty)'$ and refer to this as the *Gelfand topology.* For each $z \in \mathbb{D}$ we consider the point-evaluation functional

$$\pi_z : H^\infty \to \mathbb{C},\ f \mapsto f(z).$$

This is clearly multiplicative, bounded and linear and therefore belongs to $\Delta(H^\infty)$. The set of all such functionals $\pi_z, z \in \mathbb{D}$ will be denoted as Δ_0. The *corona* is defined as the complement of closure of Δ_0 in the Gelfand topology. The corona theorem now states:

Theorem 1 (L. Carleson)**.** The corona is empty. In other words, Δ_0 is dense in $\Delta(H^\infty)$.

There is an equivalent version of the theorem, as given by the following proposition:

Proposition 2. Δ_0 is dense in $\Delta(H^\infty)$ if and only if for any $\delta > 0$ and $f_1, \dots, f_n \in H^\infty$ such that $\sum_{j=1}^n |f_j(z)| \geq \delta, z \in \mathbb{D}$, there exist $g_1, \dots g_n \in H^\infty$ such that $\sum_{j=1}^n f_j g_j = 1$.

Proof. Assume Δ_0 is dense in $\Delta(H^\infty)$ and let $f_1, \dots, f_n \in H^\infty$, and $\delta > 0$ such that $\sum_{j=1}^n |f_j(z)| \geq \delta, z \in \mathbb{D}$. Denote by I the ideal in H^∞ generated by $f_1, \dots, f_n$. If $1 \in I$, then the assertion is established. Assume towards a contradiction that I is a proper ideal, then there exists a maximal ideal $J \supset I$. Since $\Delta(H^\infty)$ forms a commutative Banach algebra, there exists a $\phi \in \Delta(H^\infty)$ such that $J = \ker \phi$. Therefore we have $\phi(f_j) = 0$ for $j = 1, \dots, n$. Since Δ_0 is dense, there is a net $(\pi_{z_m})_{m \in M}$ in Δ_0 such that $\pi_{z_m} \to \phi$ in the weak-* topology, that is the net converges pointwise. Therefore, for all $j = 1, \dots, n$ we have $f_j(z_m) = \pi_{z_m}(f_j) \to \phi(f_j) = 0$ and in particular

$$\lim_{m \in M} \sum_{j=1}^n |f_j(z_m)| = 0,$$

a contradiction.

For the other implication, assume towards a contradiction that Δ_0 is not dense in $\Delta(H^\infty)$. Then there exists some $\phi_0 \in \Delta(H^\infty)$ and an open neighbourhood U of ϕ_0 such that $\Delta_0 \cap U = \emptyset$. Since the sets of the form

$$\{\phi \in \Delta(H^\infty) : |(\phi - \phi_0)(f_j)| < \varepsilon, j = 1, \dots, n\},$$

for some $n \in \mathbb{N}, f_1, \dots, f_n \in H^\infty$ and $\varepsilon > 0$, form a neighbourhood basis of ϕ_0 in the weak-* topology, there exists a neighbourhood $V \subseteq U$ described by some $n \in$

$\mathbb{N}, f_1, \ldots, f_n \in H^\infty$ and $\delta > 0$. Define $\widetilde{f_j} := f_j - \phi_0(f_j)$, for $j = 1, \ldots, n$, then clearly $\phi_0(\widetilde{f_j}) = 0$. Since $\Delta_0 \cap V = \emptyset$, for any $z \in \mathbb{D}$ we have $\pi_z \notin V$ and therefore there exists some $j_0 \in \{1, \ldots, n\}$ such that,

$$\delta \leq |(\pi_z - \phi_0)(f_{j_0})| = |f_{j_0}(z) - \phi_0(f_{j_0})| = |\widetilde{f_{j_0}}(z)|.$$

Since $\widetilde{f_j} \in H^\infty$ for $j = 1, \ldots, n$, and $\sum_{j=1}^n |\widetilde{f_j}(z)| \geq \delta$, there exist $g_1, \ldots, g_n \in H^\infty$ such that $\sum_{j=1}^n \widetilde{f_j} g_j = 1$. But this yields

$$1 = \phi_0(1) = \phi_0 \left(\sum_{j=1}^n \widetilde{f_j} g_j \right) = \sum_{j=1}^n \phi_0(\widetilde{f_j}) \phi_0(g_j) = 0,$$

a contradiction. $\square$

We will prove a stronger version of the right statement in the previous proposition:

Theorem 3. There exist constants $C_{n,\delta}$ only depending on $n \in \mathbb{N}$ and $\delta > 0$, such that if $f_1, \ldots f_n \in \mathrm{Hol}(\mathbb{D})$ with

$$\|f_j\|_\infty \leq 1, \;\; j = 1, \ldots, n, \quad \text{and} \quad \sum_{j=1}^n |f_j(z)|^2 \geq \delta, \;\; z \in \mathbb{D},$$

then there exist $g_1, \ldots, g_n \in \mathrm{Hol}(\mathbb{D})$ with

$$\|g_j\|_\infty \leq C_{n,\delta}, \;\; j = 1, \ldots, n, \quad \text{and} \quad \sum_{j=1}^n f_j g_j = 1.$$

Proof. We will give the proof in multiple steps.

Step 1 (Reduction to $f_1, \ldots, f_n \in \mathrm{Hol}(\overline{\mathbb{D}})$): Assume that the statement of the theorem holds for all $\widetilde{f_1}, \ldots, \widetilde{f_n} \in \mathrm{Hol}(\overline{\mathbb{D}})$, we claim that it then also holds in its original form[1]. For our given $f_1, \ldots, f_n$ satisfying the premise of the theorem and all $0 < s < 1$ we define $f_{j,s}(z) := f_j(sz)$, $j = 1, \ldots, n$. Then for every $0 < s < 1$ and $j = 1, \ldots, n$ the function $f_{j,s}$ is in $\mathrm{Hol}(\overline{\mathbb{D}})$ and satisfies the premise of the theorem. By our assumption there exist $g_{j,s} \in H^\infty$, $j = 1, \ldots, n$ such that

$$\|g_{j,s}\|_\infty \leq C_{n,\delta}, \;\; j = 1, \ldots, n, \quad \text{and} \quad \sum_{j=1}^n f_{j,s} g_{j,s} = 1.$$

For a fixed $j \in \{1, \ldots, n\}$, the set $\{g_{j,s} : 0 < s < 1\}$ is uniformly bounded and therefore normal in $\mathrm{Hol}(\mathbb{D})$. By Montel's Theorem there exists a sequence $s_m \to 1$ and some $g_j \in \mathrm{Hol}(\mathbb{D})$ such that $g_{j,s_m} \to g_j$ compactly. In particular, we obtain

$$\|g_j\|_\infty = \lim_{m \to \infty} \|g_{j,s_m}\|_\infty \leq C_{n,\delta}, \quad j = 1, \ldots, n,$$

[1]Note that this does **not** mean that we can assume $f_1, \ldots, f_n \in \mathrm{Hol}(\overline{\mathbb{D}})$ in the previous proposition.

and

$$1 = \lim_{m\to\infty} \sum_{j=1}^{n} f_{j,s_m} g_{j,s_m} = \sum_{j=1}^{n} f_j g_j,$$

concluding our claim. We may thus assume that our given $f_1, \dots, f_n$ are holomorphic on $\overline{\mathbb{D}}$ instead.

Step 2 (Solve with $g_1, \dots, g_n \in C^\infty(\overline{\mathbb{D}})$): For $j = 1, \dots, n$ we define

$$h_j := \frac{\bar{f}_j}{\sum_{k=1}^{n} |f_k|^2},$$

then clearly $h_j \in C^\infty(\overline{\mathbb{D}})$, $\sum_{j=1}^{n} f_j h_j = 1$ and $\|h_j\| \leq \frac{1}{\delta}$. The real task now lies in changing the h_j to become holomorphic in $\mathbb{D}$, without losing control over the boundedness of the solutions.

Before we continue we want to briefly introduce a "generalization" of the complex derivative. By writing a complex number $z \in \mathbb{C}$ as $z = x + iy$ for $x, y \in \mathbb{R}$ we can identify $\mathbb{C} \cong \mathbb{R}^2$. The *Wirtinger derivatives* are now defined as the following linear partial differential operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

We will also abbreviate these operators as ∂ and $\bar{\partial}$, respectively.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Assume that f is holomorphic and write $f = u + iv$ for real-valued functions u, v. Then combining the Wirtinger derivatives with the Cauchy-Riemann equations yields

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x} = f',$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) = 0,$$

that is the Wirtinger derivative with respect to z coincides with the complex derivative, and holomorphic functions are "independant" of $\bar{z}$. Conversely we also obtain that a function g with $\frac{\partial g}{\partial \bar{z}} = 0$ also satisfies the Cauchy-Riemann equations and is therefore holomorphic.

We also obtain a representation of the *Laplace operator*:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}$$

Proof. Step 3 (The Koszul complex): Consider the spaces $C_0 := C^\infty(\overline{\mathbb{D}}), C_1 := (C_0)^n$ and $C_2 := \{(a_{jk})_{j,k=1}^{n} : a_{jk} \in C_0, a_{jk} = -a_{kj}\}$.

$$
\begin{aligned}
P_{1,0} &: C_1 \to C_0,\ (g_j)_{j=1}^n \mapsto \sum_{j=1}^n g_j f_j, \\
P_{2,1} &: C_2 \to C_1,\ (g_{jk})_{j,k=1}^n \mapsto \left(\sum_{k=1}^n g_{jk} f_k\right)_{j=1}^n
\end{aligned}
$$

$$
\begin{tikzcd}
C_2 \arrow[r, "P_{2,1}"] \arrow[d, "\overline{\partial}"'] & C_1 \arrow[r, "P_{1,0}"] \arrow[d, "\overline{\partial}"'] & C_0 \arrow[d, "\overline{\partial}"'] \\
C_2 \arrow[r, "P_{2,1}"'] & C_1 \arrow[r, "P_{1,0}"'] & C_0
\end{tikzcd}
$$

Lemma 4. The Koszul complex has the following properties:

1. The diagram is commutative, that is we have $P_{j+1,j}\overline{\partial} = \overline{\partial}P_{j+1,j}$ for $j = 0, 1$.
2. The horizontal sequences are exact, that is $\operatorname{ran} P_{2,1} = \ker P_{1,0}$.
3. The maps $\overline{\partial} : C_j \to C_j$ for $j = 0, 1, 2$ are surjective.

Proof. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum. ☐

Proof (continued). Step 4 (Apply to $h = (h_1, \dots, h_n) \in C_1$): todo

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For $1 \le p \le \infty$ we define the *Hardy space* H^p as the set of all $f \in \operatorname{Hol}(\mathbb{D})$ with $\|f\|_p < \infty$, where

$$
\|f\|_p \coloneqq \lim_{r\to 1} \left[\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\vartheta})|^p \,\mathrm{d}\vartheta\right]^{1/p} \quad \text{for } p < \infty, \quad \text{and} \quad \|f\|_\infty \coloneqq \sup_{z\in\mathbb{D}} |f(z)|.
$$

For real- or complex-valued functions defined on $\mathbb{T}$ we define an "inner product"

$$\langle f, g\rangle := \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\vartheta})\overline{g(e^{i\vartheta})}\,\mathrm{d}\vartheta.$$

For $f \in L^1(\mathbb{T})$ and $n \in \mathbb{N}$ we define the n-th *Fourier coefficient* by

$$\hat{f}(n) := \langle f, e^{in\vartheta}\rangle.$$

We define H_0^1 as the (closed) subspace of all $f \in H^1$, for which $f(0) = 0$.

We summarize the characterisation of Hardy spaces:

Theorem 5. Let $1 \le p \le \infty$. Then:

1. H^p is a Banach space[2].
2. Let $f \in H^p$, then for almost all $e^{i\vartheta} \in \mathbb{T}$ the limit

$$\lim_{r\to 1} f(re^{i\vartheta}) =: f^*(e^{i\vartheta})$$

exists and defines a function in $L^p(\mathbb{T})$, also called the *boundary values* of f.

3. The map $f \mapsto f^*$ is an isometry from H^p onto

$$L^p_+(\mathbb{T}) := \{f \in L^p(\mathbb{T}) : \forall n < 0 : \hat{f}(n) = 0\},$$

which is a closed subspace of $L^p(\mathbb{T})$.

4. Every $f \in H^p$ can be written as a Cauchy integral of its boundary values:

$$f(z) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f^*(\zeta)}{\zeta - z}, \quad z \in \mathbb{D}$$

Lemma 6. The map

$$\Phi : L^\infty(\mathbb{T})/H^\infty \to (H_0^1)',\ f + H^\infty \mapsto \left[g \mapsto \frac{1}{2\pi}\int_{-\pi}^{\pi} fg^*\,\mathrm{d}x\right]$$

is an isometric isomorphism.

Proof. From functional analysis we know that $(H_0^1)' \cong (L^1(\mathbb{T}))'/(H_0^1)^\perp$ via

$$\sigma : \begin{cases} (L^1(\mathbb{T}))'/(H_0^1)^\perp & \to & (H_0^1)' \\ x' + (H_0^1)^\perp & \mapsto & x'|_{(H_0^1)}. \end{cases}$$

We also know $(L^1(\mathbb{T}))' \cong L^\infty(\mathbb{T})$ via

$$\Psi : L^\infty(\mathbb{T}) \to (L^1(\mathbb{T}))',\ f \mapsto \left[g \mapsto \langle f, \bar{g}\rangle := \frac{1}{2\pi}\int_0^{2\pi} f\bar{g}\,\mathrm{d}\lambda\right].$$

[2] In particular, H^∞ is a Banach algebra, which we already used in the introduction.

We can identify H_0^1 as a closed subspace of $L^1(\mathbb{T})$ via the isometry $\rho : f \mapsto f^*$. We want to show $(\iota(H_0^1))^\perp \cong L^\infty$. Let $w^* \in (\iota(H_0^1))^\perp$, then for any $n \in \mathbb{N}$ we have

$$0 = \langle w^*, \bar{z}^n \rangle = \langle w^*, e^{-int} \rangle.$$

Therefore $w^* \in L_+^\infty = \iota(H^\infty)$ and thus $w \in H^\infty$.

Let

$$\Psi(f + H^\infty) := \sigma()$$

□

Proof (continued). Step 5 (Dualisation): todo

Proof. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum. □

Proof (continued). Step 6: todo

Lemma 7. Let $f, g, u, v \in \mathrm{Hol}(\overline{\mathbb{D}})$, then the following integral estimates hold:

1. $\displaystyle\int_{\mathbb{D}} |f'|^2 \log \frac{1}{|z|} \,\mathrm{d}\lambda^2 \leq \frac{\pi}{2} \|f\|_2^2$
2. $\displaystyle\int_{\mathbb{D}} |fg'| \log \frac{1}{|z|} \,\mathrm{d}\lambda^2 \leq 2\pi \|f\|_2^2 \|g\|_\infty$
3. $\displaystyle\int_{\mathbb{D}} |fgu'v'| \log \frac{1}{|z|} \,\mathrm{d}\lambda^2 \leq 2\pi \|f\|_2 \|g\|_2 \|u\|_\infty \|v\|_\infty$
4. $\displaystyle\int_{\mathbb{D}} |fu'v'| \log \frac{1}{|z|} \,\mathrm{d}\lambda^2 \leq 2\pi \|f\|_1 \|u\|_\infty \|v\|_\infty$
5. $\displaystyle\int_{\mathbb{D}} |fg'u'| \log \frac{1}{|z|} \,\mathrm{d}\lambda^2 \leq \pi \|f\|_2 \|g\|_2 \|u\|_\infty$
6. $\displaystyle\int_{\mathbb{D}} |f'u'| \log \frac{1}{|z|} \,\mathrm{d}\lambda^2 \leq 2\pi \|f\|_1 \|u\|_\infty$

Proof.

1. Applying Green's formula on $f\bar{f}$ yields

$$|f(0)|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f^*(e^{i\vartheta})|^2 \,\mathrm{d}\vartheta - \frac{1}{2\pi}\int_{\mathbb{D}} \Delta(f\bar{f}) \log\frac{1}{|z|}\,\mathrm{d}\lambda^2$$

Sine $\Delta(f\bar{f}) = 4\partial\overline{\partial}(f\bar{f}) = 4\partial(f\overline{\partial}\bar{f}) = 4(\partial f\overline{\partial}\bar{f}) = 4|f'|^2$ and $|f(0)|^2 \geq 0$ we obtain

$$\frac{2}{\pi}\int_{\mathbb{D}} |f'|^2 \log\frac{1}{|z|}\,\mathrm{d}\lambda^2 \leq \|f\|_2^2$$

and rearranging yields the desired inequality.

$\square$