

# THE CORONA THEOREM

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These are my notes for my talk in the winter semester of 2024, in the analysis seminar at the Technical University of Vienna.

We will denote the space of all complex-valued, bounded, analytic functions on the unit disk  $\mathbb{D}$  as  $H^\infty$ . Equipped with the supremum norm  $\|\cdot\|_\infty$  this space becomes a commutative Banach algebra. The space of all multiplicative, bounded, linear functionals on  $H^\infty$  not identically zero is denoted  $\Delta(H^\infty)$  and is called the *Gelfand space* of  $H^\infty$ . We endow this space with the subspace topology of the weak-\* topology on the dual  $(H^\infty)'$ , which we will refer to as the *Gelfand topology*. For each  $z \in \mathbb{D}$  we consider the point-evaluation functional

$$\pi_z : H^\infty \rightarrow \mathbb{C}, f \mapsto f(z).$$

This is clearly multiplicative, bounded and linear and therefore belongs to  $\Delta(H^\infty)$ . The set of all such functionals  $\pi_z, z \in \mathbb{D}$  will be denoted as  $\Delta_0$ . The *corona* is defined as the complement of closure of  $\Delta_0$  in the Gelfand topology. The corona theorem now states:

**Theorem 1** (L. Carleson). The corona is empty. In other words,  $\Delta_0$  is dense in  $\Delta(H^\infty)$ .

There is an equivalent version of the theorem, as given by the following proposition:

**Proposition 2.**  $\Delta_0$  is dense in  $\Delta(H^\infty)$  if and only if for any  $\delta > 0$  and  $f_1, \dots, f_n \in H^\infty$  such that  $\sum_{j=1}^n |f_j(z)| \geq \delta, z \in \mathbb{D}$ , there exist  $g_1, \dots, g_n \in H^\infty$  such that  $\sum_{j=1}^n f_j g_j = 1$ .

*Proof.* Assume  $\Delta_0$  is dense in  $\Delta(H^\infty)$  and let  $f_1, \dots, f_n \in H^\infty$ , and  $\delta > 0$  such that  $\sum_{j=1}^n |f_j(z)| \geq \delta, z \in \mathbb{D}$ . Denote by  $I$  the ideal in  $H^\infty$  generated by  $f_1, \dots, f_n$ . If  $1 \in I$ , then the assertion is established. Assume towards a contradiction that  $I$  is a proper ideal, then there exists a maximal ideal  $J \supset I$ . Since  $\Delta(H^\infty)$  is a commutative Banach algebra, there exists a  $\phi \in \Delta(H^\infty)$  such that  $J = \ker \phi$ . Therefore we have  $\phi(f_j) = 0$  for  $j = 1, \dots, n$ . Since  $\Delta_0$  is dense, there is a net  $(\pi_{z_m})_{m \in M}$  in  $\Delta_0$  such that  $\pi_{z_m} \rightarrow \phi$  in the weak-\* topology, that is the net converges pointwise. Therefore, for all  $j = 1, \dots, n$  we have  $f_j(z_m) = \pi_{z_m}(f_j) \rightarrow \phi(f_j) = 0$  and in particular

$$\lim_{m \in M} \sum_{j=1}^n |f_j(z_m)| = 0,$$

a contradiction.

For the other implication, assume towards a contradiction that  $\Delta_0$  is not dense in  $\Delta(H^\infty)$ . Then there exists some  $\phi_0 \in \Delta(H^\infty)$  and an open neighbourhood  $U$  of  $\phi_0$  such that  $\Delta_0 \cap U = \emptyset$ . Since the sets of the form

$$\{\phi \in \Delta(H^\infty) : |(\phi - \phi_0)(f_j)| < \varepsilon, j = 1, \dots, n\},$$

for some  $n \in \mathbb{N}$ ,  $f_1, \dots, f_n \in H^\infty$  and  $\varepsilon > 0$ , form a neighbourhood basis of  $\phi_0$  in the weak-\* topology, there exists a neighbourhood  $V \subseteq U$  described by some  $n \in \mathbb{N}$ ,  $f_1, \dots, f_n \in H^\infty$  and  $\delta > 0$ . Define  $\widetilde{f}_j := f_j - \phi_0(f_j)$ , for  $j = 1, \dots, n$ , then clearly  $\phi_0(\widetilde{f}_j) = 0$ . Since  $\Delta_0 \cap V = \emptyset$ , for any  $z \in \mathbb{D}$  we have  $\pi_z \notin V$  and therefore there exists some  $j_0 \in \{1, \dots, n\}$  such that,

$$\delta \leq |(\pi_z - \phi_0)(f_{j_0})| = |f_{j_0}(z) - \phi_0(f_{j_0})| = |\widetilde{f}_{j_0}(z)|.$$

Since  $\widetilde{f}_j \in H^\infty$  for  $j = 1, \dots, n$ , and  $\sum_{j=1}^n |\widetilde{f}_j(z)| \geq \delta$ , there exist  $g_1, \dots, g_n \in H^\infty$  such that  $\sum_{j=1}^n \widetilde{f}_j g_j = 1$ . But this yields

$$1 = \phi_0(1) = \phi_0 \left( \sum_{j=1}^n \widetilde{f}_j g_j \right) = \sum_{j=1}^n \phi_0(\widetilde{f}_j) \phi_0(g_j) = 0,$$

a contradiction. □

## 1 First Steps

Over the following sections we will prove a stronger version of the right statement in Proposition 2:

**Theorem 3.** There exist constants  $C_{n,\delta}$  only depending on  $n \in \mathbb{N}$  and  $\delta > 0$ , such that if  $f_1, \dots, f_n \in \text{Hol}(\mathbb{D})$  with

$$\|f_j\|_\infty \leq 1, \quad j = 1, \dots, n, \quad \text{and} \quad \sum_{j=1}^n |f_j(z)|^2 \geq \delta, \quad z \in \mathbb{D},$$

then there exist  $g_1, \dots, g_n \in \text{Hol}(\mathbb{D})$  with

$$\|g_j\|_\infty \leq C_{n,\delta}, \quad j = 1, \dots, n, \quad \text{and} \quad \sum_{j=1}^n f_j g_j = 1.$$

*Proof.* We will give the proof in multiple steps.

*Step 1 (Reduction to  $f_1, \dots, f_n \in \text{Hol}(\overline{\mathbb{D}})$ ):* Assume that the statement of the theorem holds for all  $\widetilde{f}_1, \dots, \widetilde{f}_n \in \text{Hol}(\overline{\mathbb{D}})$ , we claim that it then also holds in its original form<sup>1</sup>. For our given  $f_1, \dots, f_n$  satisfying the premise of the theorem and all  $0 < s < 1$  we define  $f_{j,s}(z) := f_j(sz)$ ,  $j = 1, \dots, n$ . Then for every  $0 < s < 1$  and  $j = 1, \dots, n$  the function  $f_{j,s}$  is in  $\text{Hol}(\overline{\mathbb{D}})$  and satisfies the premise of the theorem. By our assumption there exist  $g_{j,s} \in H^\infty$ ,  $j = 1, \dots, n$  such that

$$\|g_{j,s}\|_\infty \leq C_{n,\delta}, \quad j = 1, \dots, n, \quad \text{and} \quad \sum_{j=1}^n f_{j,s} g_{j,s} = 1.$$

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<sup>1</sup>Note that this does **not** mean that we can assume  $f_1, \dots, f_n \in \text{Hol}(\overline{\mathbb{D}})$  in the previous proposition.

For a fixed  $j \in \{1, \dots, n\}$ , the set  $\{g_{j,s} : 0 < s < 1\}$  is uniformly bounded and therefore normal in  $\text{Hol}(\mathbb{D})$ . By Montel's Theorem there exists a sequence  $s_m \rightarrow 1$  and some  $g_j \in \text{Hol}(\mathbb{D})$  such that  $g_{j,s_m} \rightarrow g_j$  compactly. In particular, we obtain

$$\|g_j\|_\infty = \lim_{m \rightarrow \infty} \|g_{j,s_m}\|_\infty \leq C_{n,\delta}, \quad j = 1, \dots, n,$$

and

$$1 = \lim_{m \rightarrow \infty} \sum_{j=1}^n f_{j,s_m} g_{j,s_m} = \sum_{j=1}^n f_j g_j,$$

concluding our claim. We may thus assume that our given  $f_1, \dots, f_n$  are holomorphic on  $\overline{\mathbb{D}}$  instead.

*Step 2 (Solve with  $g_1, \dots, g_n \in C^\infty(\overline{\mathbb{D}})$ ):* For  $j = 1, \dots, n$  we define

$$h_j := \frac{\bar{f}_j}{\sum_{k=1}^n |f_k|^2},$$

then clearly  $h_j \in C^\infty(\overline{\mathbb{D}})$ ,  $\sum_{j=1}^n f_j h_j = 1$  and  $\|h_j\| \leq \frac{1}{\delta}$ . The real task now lies in changing the  $h_j$  to become holomorphic in  $\mathbb{D}$ , without losing control over the boundedness of the solutions.

## 2 Wirtinger Derivatives

Before we continue we want to briefly introduce a useful generalization of the complex derivative.

**Definition 4.** Let  $\Omega \subseteq \mathbb{R}^2$  be open. Then the *Wirtinger derivatives* (or *Wirtinger operators*) are defined on  $C^1(\Omega)$  by

$$\frac{\partial}{\partial z} := \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

We will also abbreviate these operators as  $\partial$  and  $\bar{\partial}$ , respectively.

Note that by writing a complex number  $z \in \mathbb{C}$  as  $z = x + iy$  with  $x, y \in \mathbb{R}$  we can identify  $\mathbb{C} \cong \mathbb{R}^2$ . Therefore we can also reasonably interpret the Wirtinger operators to act on  $C^1(\Omega)$  with an open subset  $\Omega \subseteq \mathbb{C}$ .

Before listing properties of the Wirtinger operators we quickly want to recall that a function  $f : \mathbb{C} \rightarrow \mathbb{C}$ ,  $f = u + iv$  is holomorphic if and only if it satisfies the *Cauchy-Riemann equations*:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

**Remark 5.** Let  $\Omega \subseteq \mathbb{C}$  be open and  $f \in C^1(\Omega)$ .

1. The Wirtinger operators are  $\mathbb{C}$ -linear and satisfy the Leibniz rule<sup>2</sup>.

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<sup>2</sup>This means that the Wirtinger operators are derivatives from an algebraic perspective.

2. If  $f \in \text{Hol}(\Omega)$ , then

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) = \frac{1}{2} \left( \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} - i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial f}{\partial x} = f'.$$

3. Since

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}} &= \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) = \frac{1}{2} \left( \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} + i \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} \right) = \\ &= \frac{1}{2} \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + \frac{i}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right), \end{aligned}$$

we have that<sup>3</sup>  $f \in \text{Hol}(\Omega)$  if and only if  $\bar{\partial}f = 0$ .

4. On  $C^2(\Omega)$ , the *Laplace operator* can be represented as

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}.$$

*Proof. Step 3 (The Koszul complex):* We consider the spaces

$$C_0 := C^\infty(\bar{\mathbb{D}}), \quad C_1 := (C_0)^n, \quad C_2 := \{A \in (C_0)^{n \times n} : A = -A^T\}$$

and the maps

$$P_{1,0} : C_1 \rightarrow C_0, (g_j)_{j=1}^n \mapsto \sum_{j=1}^n g_j f_j, \quad P_{2,1} : C_2 \rightarrow C_1, (g_{jk})_{j,k=1}^n \mapsto \left( \sum_{k=1}^n g_{jk} f_k \right)_{j=1}^n.$$

The resulting connections are visualized in the diagram below, called the *Koszul complex*:

$$\begin{array}{ccccc} C_2 & \xrightarrow{P_{2,1}} & C_1 & \xrightarrow{P_{1,0}} & C_0 \\ \bar{\partial} \downarrow & & \bar{\partial} \downarrow & & \bar{\partial} \downarrow \\ C_2 & \xrightarrow{P_{2,1}} & C_1 & \xrightarrow{P_{1,0}} & C_0 \end{array}$$

**Lemma 6.** The Koszul complex has the following properties:

1. The diagram is commutative, that is we have  $P_{j+1,j} \bar{\partial} = \bar{\partial} P_{j+1,j}$  for  $j = 0, 1$ .
2. The horizontal sequences are exact, that is  $\text{ran } P_{2,1} = \ker P_{1,0}$ .
3. The maps  $\bar{\partial} : C_j \rightarrow C_j$  for  $j = 0, 1, 2$  are surjective.

*Proof.* Lorem ipsum dolor sit amet, consectetur adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetur id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac

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<sup>3</sup>This can be interpreted as “ $f$  is independant of  $\bar{z}$ ”.

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*Proof (continued). Step 4 (Apply to  $h = (h_1, \dots, h_n) \in C_1$ ):* todo

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### 3 Hardy Spaces

For  $1 \leq p \leq \infty$  we define the *Hardy space*  $H^p$  as the set of all  $f \in \text{Hol}(\mathbb{D})$  with  $\|f\|_p < \infty$ , where

$$\|f\|_p := \lim_{r \rightarrow 1} \left[ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\vartheta})|^p d\vartheta \right]^{1/p} \quad \text{for } p < \infty, \quad \text{and} \quad \|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)|.$$

We define  $H_0^1$  as the (closed) subspace of all  $f \in H^1$ , for which  $f(0) = 0$ .

We summarize the characterisation of Hardy spaces:

**Theorem 7.** Let  $1 \leq p \leq \infty$ . Then:

1.  $H^p$  is a Banach space.
2. For  $f \in H^p$ , then for almost all  $e^{i\vartheta} \in \mathbb{T}$  the limit

$$\lim_{r \rightarrow 1} f(re^{i\vartheta}) =: f^*(e^{i\vartheta})$$

exists and defines a function in  $L^p(\mathbb{T})$ .

3. The map  $f \mapsto f^*$  is an isometry from  $H^p$  onto

$$L_+^p(\mathbb{T}) := \{f \in L^p(\mathbb{T}) : \widehat{f}(n) = 0, n < 0\}$$

**Lemma 8.** The map

$$\Phi : L^\infty(\mathbb{T})/H^\infty \rightarrow (H_0^1)', f + H^\infty \mapsto \left[ g \mapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} fg dx \right]$$

is an isometric isomorphism.

*Proof.* From functional analysis we know that  $(H_0^1)' \cong (L^1(\mathbb{T}))'/(H_0^1)^\perp$  via

$$\sigma : \begin{cases} (L^1(\mathbb{T}))'/(H_0^1)^\perp & \rightarrow & (H_0^1)' \\ x' + (H_0^1)^\perp & \mapsto & x'|_{(H_0^1)}. \end{cases}$$

We also know  $(L^1(\mathbb{T}))' \cong L^\infty(\mathbb{T})$  via

$$\Psi : L^\infty(\mathbb{T}) \rightarrow (L^1(\mathbb{T}))', f \mapsto \left[ g \mapsto \langle f, \bar{g} \rangle := \frac{1}{2\pi} \int_0^{2\pi} f \bar{g} d\lambda \right].$$

We can identify  $H_0^1$  as a closed subspace of  $L^1(\mathbb{T})$  via the isometry  $\rho : f \mapsto f^*$ . We want to show  $(\iota(H_0^1))^\perp \cong L^\infty$ . Let  $w^* \in (\iota(H_0^1))^\perp$ , then for any  $n \in \mathbb{N}$  we have

$$0 = \langle w^*, \bar{z}^n \rangle = \langle w^*, e^{-int} \rangle.$$

Therefore  $w^* \in L_+^\infty = \iota(H^\infty)$  and thus  $w \in H^\infty$ .

Let

$$\Psi(f + H^\infty) := \sigma()$$

□

*Proof (continued). Step 5 (Dualisation):* todo

*Proof.* Lorem ipsum dolor sit amet, consectetur adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetur id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

□

*Proof (continued). Step 6:* todo

## 4 Integral estimates

**Lemma 9.** Let  $f, g, u, v \in \text{Hol}(\overline{\mathbb{D}})$ , then the following integral estimates hold:

1.  $\int_{\mathbb{D}} |f'| \log \frac{1}{|z|} d\lambda^2 \leq \frac{\pi}{2} \|f\|_2^2$
2.  $\int_{\mathbb{D}} |fg'| \log \frac{1}{|z|} d\lambda^2 \leq 2\pi \|f\|_2^2 \|g\|_\infty$
3.  $\int_{\mathbb{D}} |fgu'v'| \log \frac{1}{|z|} d\lambda^2 \leq 2\pi \|f\|_2 \|g\|_2 \|u\|_\infty \|v\|_\infty$
4.  $\int_{\mathbb{D}} |fu'v'| \log \frac{1}{|z|} d\lambda^2 \leq 2\pi \|f\|_1 \|u\|_\infty \|v\|_\infty$

$$5. \int_{\mathbb{D}} |fg'u'| \log \frac{1}{|z|} d\lambda^2 \leq \pi \|f\|_2 \|g\|_2 \|u\|_\infty$$

$$6. \int_{\mathbb{D}} |f'u'| \log \frac{1}{|z|} d\lambda^2 \leq 2\pi \|f\|_1 \|u\|_\infty$$

*Proof.*

1. d

$$\Delta(ff) = 4\partial\bar{\partial}(ff) = 4\partial(f\bar{\partial}f) = 4(\partial f\bar{\partial}f) = 4|f'|$$

□