The Corona Theorem

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We will denote the space of all complex-valued, bounded, analytic functions on the unit disk $\mathbb{D}$ as H^∞. Equipped with the supremum norm $\|\cdot\|_\infty$ this space becomes a commutative Banach algebra. The space of all multiplicative, bounded, linear functionals on H^∞ not identically zero is denoted $\Delta(H^\infty)$ and is called the *Gelfand space* of H^∞. We endow this space with the subspace topology of the weak-* topology on the dual $(H^\infty)'$, which we will refer to as the *Gelfand topology*. For each $z \in \mathbb{D}$ we consider the point-evaluation functional

$$\pi_z : H^\infty \to \mathbb{C},\ f \mapsto f(z).$$

This is clearly multiplicative, bounded and linear and therefore belongs to $\Delta(H^\infty)$. The set of all such functionals $\pi_z, z \in \mathbb{D}$ will be denoted as Δ_0. The *corona* is defined as the complement of closure of Δ_0 in the Gelfand topology. The corona theorem now states:

Theorem 1 (L. Carleson)**.** The corona is empty. In other words, Δ_0 is dense in $\Delta(H^\infty)$.

There is an equivalent version of the theorem, as given by the following proposition:

Proposition 2. Δ_0 is dense in $\Delta(H^\infty)$ if and only if for any $\delta > 0$ and $f_1, \dots, f_n \in H^\infty$ such that $\sum_{j=1}^n |f_j(z)| \geq \delta, z \in \mathbb{D}$, there exist $g_1, \dots g_n \in H^\infty$ such that $\sum_{j=1}^n f_j g_j = 1$.

Proof. Assume Δ_0 is dense in $\Delta(H^\infty)$ and let $f_1, \dots, f_n \in H^\infty$, and $\delta > 0$ such that $\sum_{j=1}^n |f_j(z)| \geq \delta, z \in \mathbb{D}$. Denote by I the ideal in H^∞ generated by $f_1, \dots, f_n$. If $1 \in I$, then the assertion is established. Assume towards a contradiction that I is a proper ideal, then there exists a maximal ideal $J \supset I$. Since $\Delta(H^\infty)$ is a commutative Banach algebra, there exists a $\phi \in \Delta(H^\infty)$ such that $J = \ker \phi$. Therefore we have $\phi(f_j) = 0$ for $j = 1, \dots, n$. Since Δ_0 is dense, there is a net $(\pi_{z_m})_{m \in M}$ in Δ_0 such that $\pi_{z_m} \to \phi$ in the weak-* topology, that is the net converges pointwise. Therefore, for all $j = 1, \dots, n$ we have $f_j(z_m) = \pi_{z_m}(f_j) \to \phi(f_j) = 0$ and in particular

$$\lim_{m \in M} \sum_{j=1}^n |f_j(z_m)| = 0,$$

a contradiction.

For the other implication, assume towards a contradiction that Δ_0 is not dense in $\Delta(H^\infty)$. Then there exists some $\phi_0 \in \Delta(H^\infty)$ and an open neighbourhood U of ϕ_0 such that $\Delta_0 \cap U = \emptyset$. Since the sets of the form

$$\{\phi \in \Delta(H^\infty) : |(\phi - \phi_0)(f_j)| < \varepsilon, j = 1, \dots, n\},$$

for some $n \in \mathbb{N}, f_1, \ldots, f_n \in H^\infty$ and $\varepsilon > 0$, form a neighbourhood basis of ϕ_0 in the weak-* topology, there exists a neighbourhood $V \subseteq U$ described by some $n \in \mathbb{N}, f_1, \ldots, f_n \in H^\infty$ and $\delta > 0$. Define $\widetilde{f_j} := f_j - \phi_0(f_j)$, for $j = 1, \ldots, n$, then clearly $\phi_0(\widetilde{f_j}) = 0$. Since $\Delta_0 \cap V = \emptyset$, for any $z \in \mathbb{D}$ we have $\pi_z \notin V$ and therefore there exists some $j_0 \in \{1, \ldots, n\}$ such that,

$$\delta \leq |(\pi_z - \phi_0)(f_{j_0})| = |f_{j_0}(z) - \phi_0(f_{j_0})| = |\widetilde{f_{j_0}}(z)|.$$

Since $\widetilde{f_j} \in H^\infty$ for $j = 1, \ldots, n$, and $\sum_{j=1}^n |\widetilde{f_j}(z)| \geq \delta$, there exist $g_1, \ldots, g_n \in H^\infty$ such that $\sum_{j=1}^n \widetilde{f_j} g_j = 1$. But this yields

$$1 = \phi_0(1) = \phi_0\left(\sum_{j=1}^n \widetilde{f_j} g_j\right) = \sum_{j=1}^n \phi_0(\widetilde{f_j})\phi_0(g_j) = 0,$$

a contradiction. □

1 First Steps

Over the following sections we will prove a stronger version of the right statement in Proposition 2:

Theorem 3. There exist constants $C_{n,\delta}$ only depending on $n \in \mathbb{N}$ and $\delta > 0$, such that if $f_1, \ldots f_n \in \mathrm{Hol}(\mathbb{D})$ with

$$\|f_j\|_\infty \leq 1, \;\; j = 1, \ldots, n, \quad \text{and} \quad \sum_{j=1}^n |f_j(z)|^2 \geq \delta, \;\; z \in \mathbb{D},$$

then there exist $g_1, \ldots, g_n \in \mathrm{Hol}(\mathbb{D})$ with

$$\|g_j\|_\infty \leq C_{n,\delta}, \;\; j = 1, \ldots, n, \quad \text{and} \quad \sum_{j=1}^n f_j g_j = 1.$$

Proof. We will give the proof in multiple steps.

Step 1 (Reduction to $f_1, \ldots, f_n \in \mathrm{Hol}(\overline{\mathbb{D}})$): Assume that the statement of the theorem holds for all $\widetilde{f_1}, \ldots, \widetilde{f_n} \in \mathrm{Hol}(\overline{\mathbb{D}})$, we claim that it then also holds in its original form[1]. For our given $f_1, \ldots, f_n$ satisfying the premise of the theorem and all $0 < s < 1$ we define $f_{j,s}(z) := f_j(sz)$, $j = 1, \ldots, n$. Then for every $0 < s < 1$ and $j = 1, \ldots, n$ the function $f_{j,s}$ is in $\mathrm{Hol}(\overline{\mathbb{D}})$ and satisfies the premise of the theorem. By our assumption there exist $g_{j,s} \in H^\infty$, $j = 1, \ldots, n$ such that

$$\|g_{j,s}\|_\infty \leq C_{n,\delta}, \;\; j = 1, \ldots, n, \quad \text{and} \quad \sum_{j=1}^n f_{j,s} g_{j,s} = 1.$$

[1]Note that this does **not** mean that we can assume $f_1, \ldots, f_n \in \mathrm{Hol}(\overline{\mathbb{D}})$ in the previous proposition.

For a fixed $j \in \{1, \dots, n\}$, the set $\{g_{j,s} : 0 < s < 1\}$ is uniformly bounded and therefore normal in $\mathrm{Hol}(\mathbb{D})$. By Montel's Theorem there exists a sequence $s_m \to 1$ and some $g_j \in \mathrm{Hol}(\mathbb{D})$ such that $g_{j,s_m} \to g_j$ compactly. In particular, we obtain

$$\|g_j\|_\infty = \lim_{m\to\infty} \|g_{j,s_m}\|_\infty \le C_{n,\delta}, \quad j = 1, \dots, n,$$

and

$$1 = \lim_{m\to\infty} \sum_{j=1}^{n} f_{j,s_m} g_{j,s_m} = \sum_{j=1}^{n} f_j g_j,$$

concluding our claim. We may thus assume that our given $f_1, \dots, f_n$ are holomorphic on $\overline{\mathbb{D}}$ instead.

Step 2 (Solve with $g_1, \dots, g_n \in C^\infty(\overline{\mathbb{D}})$): For $j = 1, \dots, n$ we define

$$h_j \coloneqq \frac{\bar{f}_j}{\sum_{k=1}^{n} |f_k|^2},$$

then clearly $h_j \in C^\infty(\overline{\mathbb{D}})$, $\sum_{j=1}^{n} f_j h_j = 1$ and $\|h_j\| \le \frac{1}{\delta}$. The real task now lies in changing the h_j to become holomorphic in $\mathbb{D}$, without losing control over the boundedness of the solutions.

2 Wirtinger Derivatives

Before we continue we want to briefly introduce a useful generalization of the complex derivative.

Definition 4. Let $\Omega \subseteq \mathbb{R}^2$ be open. Then the *Wirtinger derivatives* (or *Wirtinger operators*) are defined on $C^1(\Omega)$ by

$$\frac{\partial}{\partial z} \coloneqq \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} \coloneqq \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

We will also abbreviate these operators as ∂ and $\bar{\partial}$, respectively.

Note that by writing a complex number $z \in \mathbb{C}$ as $z = x + iy$ with $x, y \in \mathbb{R}$ we can identify $\mathbb{C} \cong \mathbb{R}^2$. Therefore we can also reasonably interpret the Wirtinger operators to act on $C^1(\Omega)$ with an open subset $\Omega \subseteq \mathbb{C}$.

Before listing properties of the Wirtinger operators we quickly want to recall that a function $f \in C^1(\Omega), f = u + iv$ is holomorphic if and only if it satisfies the *Cauchy–Riemann equations*:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Remark 5. Let $\Omega \subseteq \mathbb{C}$ be open and $f \in C^1(\Omega)$.

1. The Wirtinger operators are $\mathbb{C}$-linear, satisfy the Leibniz rule[2] and

$$\overline{\left(\frac{\partial f}{\partial z}\right)} = \frac{\partial \bar{f}}{\partial \bar{z}}, \quad \overline{\left(\frac{\partial f}{\partial \bar{z}}\right)} = \frac{\partial \bar{f}}{\partial z}$$

2. If $f \in \mathrm{Hol}(\Omega)$, then

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x} = f'.$$

3. Since

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) = \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),\end{aligned}$$

   we have that[3] $f \in \mathrm{Hol}(\Omega)$ if and only if $\overline{\partial} f = 0$.

4. On $C^2(\Omega)$, the *Laplace operator* can be represented as

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}.$$

Proof. Step 3 (The Koszul complex): We consider the spaces

$$C_0 := C^\infty(\overline{\mathbb{D}}), \quad C_1 := (C_0)^n, \quad C_2 := \{A \in (C_0)^{n\times n} : A = -A^T\}$$

and the maps

$$P_{1,0} : C_1 \to C_0,\ (g_j)_{j=1}^n \mapsto \sum_{j=1}^n g_j f_j, \quad P_{2,1} : C_2 \to C_1,\ (g_{jk})_{j,k=1}^n \mapsto \left(\sum_{k=1}^n g_{jk} f_k\right)_{j=1}^n.$$

Applying $\overline{\partial}$ pointwise in $C_j, j = 0, 1, 2$, the resulting connections are visualized in the diagram below, called the *Koszul complex*:

$$\begin{tikzcd} C_2 \arrow[r, "P_{2,1}"] \arrow[d, "\overline{\partial}"'] & C_1 \arrow[r, "P_{1,0}"] \arrow[d, "\overline{\partial}"'] & C_0 \arrow[d, "\overline{\partial}"'] \\ C_2 \arrow[r, "P_{2,1}"'] & C_1 \arrow[r, "P_{1,0}"'] & C_0 \end{tikzcd}$$

Lemma 6. The Koszul complex has the following properties:

1. The diagram is commutative, that is we have $P_{j+1,j}\overline{\partial} = \overline{\partial}P_{j+1,j}$ for $j = 0, 1$.
2. The horizontal sequences are exact, that is $\operatorname{ran} P_{2,1} = \ker P_{1,0}$.

[2]This means that the Wirtinger opeartors are derivatives from an algebraic perspective.
[3]This can be interpreted as "f is independant of $\overline{z}$".

3. The maps $\overline{\partial} : C_j \to C_j$ for $j = 0, 1, 2$ are surjective.

Proof.

1. For $g \in C_0$ and $f \in \mathrm{Hol}(\mathbb{D})$ we have

$$\frac{\partial (gf)}{\partial \bar{z}} = \frac{\partial g}{\partial \bar{z}} \cdot f + g \cdot \frac{\partial f}{\partial \bar{z}} = \frac{\partial g}{\partial \bar{z}} \cdot f$$

and together with the linearity of $\overline{\partial}$ the statement follows.

2. "$\subseteq$": Let $g \in C_2, g = (g_{jk})_{j,k=1}^n$, then

$$P_{1,0}P_{2,1}g = P_{1,0}\left[\left(\sum_{k=1}^n g_{jk}f_k\right)_{j=1}^n\right] = \sum_{j=1}^n\sum_{k=1}^n g_{jk}f_kf_j = 0$$

since g is skew-symmetric and therefore $g \in \ker P_{1,0}$.

"$\supseteq$" Let $g \in \ker P_{1,0} \subseteq C_1, g = (g_1, \ldots, g_n)$. We define $p = (p_{jk})_{j,k=1}^n \in C_2$ by

$$p_{jk} \coloneqq \frac{1}{\sum_{\ell=1}^n |f_\ell|^2}(g_j\overline{f_k} - g_k\overline{f_j}).$$

Then for any $j = 1, \ldots, n$ we have

$$\begin{aligned} P_{2,1}p = \sum_{k=1}^n p_{jk}f_k &= \frac{1}{\sum_{\ell=1}^n |f_\ell|^2}\sum_{k=1}^n (g_j|f_k|^2 - g_k\overline{f_j}f_k) = \\ &= g_j - \frac{1}{\sum_{\ell=1}^n |f_\ell|^2}\overline{f_j}\sum_{k=1}^n g_kf_k = g_j - \frac{1}{\sum_{\ell=1}^n |f_\ell|^2}\overline{f_j}P_{1,0}g = \\ &= g_j, \end{aligned}$$

and therefore $g_j \in \operatorname{ran} P_{2,1}$.

3. For given $v \in C^\infty(\mathbb{D})$ we want to solve the partial differential equation

$$\frac{\partial u}{\partial \bar{z}} = v$$

for some $u \in C^\infty(\mathbb{D})$.

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Arguing pointwise shows the surjectivity of the maps $\overline{\partial} : C_\ell \to C_\ell$ for $\ell = 0, 1$. For $\ell = 2$ and given $b = (b_{jk})_{j,k=1}^n \in C_2$ we first solve

$$\overline{\partial} a_{jk} = b_{jk}, \quad \text{for} \quad 1 \le j < k \le n$$

and then set $a_{jj} = 0$ and $a_{jk} = -a_{kj}$ for $n \ge j > k \ge 1$.

$\square$

Proof (continued). Step 4 (Apply to $h = (h_1, \dots, h_n) \in C_1$): todo

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3 Hardy Spaces

For $1 \le p \le \infty$ we define the *Hardy space* H^p as the set of all $f \in \mathrm{Hol}(\mathbb{D})$ with $\|f\|_p < \infty$, where

$$\|f\|_p := \lim_{r \to 1} \left[\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\vartheta})|^p \, \mathrm{d}\vartheta \right]^{1/p} \quad \text{for } p < \infty, \quad \text{and} \quad \|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)|.$$

For $f \in L^1(\mathbb{T})$ and $n \in \mathbb{N}$ we define the n-th *Fourier coefficient* by

$$\hat{f}(n) := \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{inx} \, \mathrm{d}x.$$

We define H_0^1 as the (closed) subspace of all $f \in H^1$, for which $f(0) = 0$.

We summarize the characterisation of Hardy spaces:

Theorem 7. Let $1 \le p \le \infty$. Then:

1. H^p is a Banach space[4].
2. Let $f \in H^p$, then for almost all $e^{i\vartheta} \in \mathbb{T}$ the limit

   $$\lim_{r \to 1} f(re^{i\vartheta}) =: f^*(e^{i\vartheta})$$

   exists and defines a function in $L^p(\mathbb{T})$, also called the *boundary values* of f.
3. The map $^* : f \mapsto f^*$ is an isometry from H^p onto

   $$L_+^p(\mathbb{T}) := \{ f \in L^p(\mathbb{T}) : \forall n < 0 : \hat{f}(n) = 0 \},$$

   which is a closed subspace of $L^p(\mathbb{T})$.
4. Every $f \in H^p$ can be written as a Cauchy integral of its boundary values:

   $$f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f^*(\zeta)}{\zeta - z} \, \mathrm{d}\mu(\zeta), \quad z \in \mathbb{D}$$

Lemma 8. The map

$$\Phi : L^\infty(\mathbb{T}) / (H^\infty)^* \to ((H_0^1)^*)', \; f + (H^\infty)^* \mapsto \left[g \mapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} fg \, \mathrm{d}x \right]$$

is an isometric isomorphism.

[4]In particular, H^∞ is a Banach algebra, which we already used in the introduction.

Proof. We have $L^\infty(\mathbb{T}) \cong L^1(\mathbb{T})'$ via the duality

$$\langle f, g\rangle \coloneqq \frac{1}{2\pi}\int_{\mathbb{T}} fg \,\mathrm{d}\mu, \quad f \in L^\infty(\mathbb{T}),\ g \in L^1(\mathbb{T}).$$

Since $(H_0^1)^* \le L^1(\mathbb{T})$ we therefore have $((H_0^1)^*)' \cong L^\infty(\mathbb{T})/(H_0^1)^\perp$ via

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{\mathbb{T}} fg \,\mathrm{d}\mu, \quad f \in L^\infty(\mathbb{T})/(H_0^1)^\perp,\ g \in (H_0^1)^*.$$

It remains to show that $(H_0^1)^\perp = (H^\infty)^*$. Let $w \in (H_0^1)^\perp \le L^\infty(\mathbb{T})$, then for any $n \in \mathbb{N}$ we have

$$0 = \langle w, \bar{z}^n\rangle = \langle w, e^{-int}\rangle.$$

Therefore $w \in L_+^\infty(\mathbb{T}) = (H^\infty)^*$. □

Proof (continued). Step 5 (Dualisation): Applying the above lemma to our previous situation we can re-describe the norm of $v + (H^\infty)^*$:

$$\begin{aligned}\|v + (H^\infty)^*\|_\infty &= \|\Phi(f + (H^\infty)^*)\| = \\ &= \sup\{|\Phi(f + (H^\infty)^*)(g)| : g \in (H_0^1)^*, \|g\|_1 \le 1\} = \\ &= \sup\left\{\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} fg\,\mathrm{d}x\right| : g \in H_0^1, \|g\|_1 \le 1\right\}\end{aligned}$$

Proof. □

Proof (continued). Step 6: todo

4 Integral estimates

Lemma 9. Let $f, g, u, v \in \mathrm{Hol}(\overline{\mathbb{D}})$, then the following integral estimates hold:

1. $\displaystyle\int_{\mathbb{D}} |f'|^2 \log\frac{1}{|z|}\,\mathrm{d}\lambda^2 \le \frac{\pi}{2}\|f\|_2^2$
2. $\displaystyle\int_{\mathbb{D}} |fg'| \log\frac{1}{|z|}\,\mathrm{d}\lambda^2 \le 2\pi\|f\|_2^2\|g\|_\infty$
3. $\displaystyle\int_{\mathbb{D}} |fgu'v'| \log\frac{1}{|z|}\,\mathrm{d}\lambda^2 \le 2\pi\|f\|_2\|g\|_2\|u\|_\infty\|v\|_\infty$
4. $\displaystyle\int_{\mathbb{D}} |fu'v'| \log\frac{1}{|z|}\,\mathrm{d}\lambda^2 \le 2\pi\|f\|_1\|u\|_\infty\|v\|_\infty$
5. $\displaystyle\int_{\mathbb{D}} |fg'u'| \log\frac{1}{|z|}\,\mathrm{d}\lambda^2 \le \pi\|f\|_2\|g\|_2\|u\|_\infty$
6. $\displaystyle\int_{\mathbb{D}} |f'u'| \log\frac{1}{|z|}\,\mathrm{d}\lambda^2 \le 2\pi\|f\|_1\|u\|_\infty$

Proof.

1. Applying Green's formula on $f\bar{f}$ yields

$$\int_{\mathbb{D}} (\log \frac{1}{|z|} \Delta(f\bar{f}) - f\bar{f})$$

$$|f(0)|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f^*(e^{i\vartheta})|^2 \,\mathrm{d}\vartheta - \frac{1}{2\pi} \int_{\mathbb{D}} \Delta(f\bar{f}) \log \frac{1}{|z|} \,\mathrm{d}\lambda^2$$

Since

$$\begin{aligned}\Delta(f\bar{f}) &= 4\partial\overline{\partial}(f\bar{f}) = 4\partial(\bar{f} \cdot \overline{\partial} f + f \cdot \overline{\partial}\bar{f}) = 4\partial(f \cdot \overline{\partial}\bar{f}) = 4(\partial f \cdot \overline{\partial}\bar{f} + f \cdot \overline{\partial}\partial\bar{f}) = \\ &= 4(\partial f \cdot \overline{\partial f} + f \cdot \overline{\partial}\,\overline{\overline{\partial} f}) = 4\partial(f\overline{\partial}\bar{f}) = 4|f'|^2\end{aligned}$$

and $|f(0)|^2 \geq 0$ we obtain

$$\frac{2}{\pi} \int_{\mathbb{D}} |f'|^2 \log \frac{1}{|z|} \,\mathrm{d}\lambda^2 \leq \|f\|_2^2$$

and rearranging yields the desired inequality.

$\square$