BACHELORARBEIT

Growth, order and zeros of entire functions

ausgeführt am

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Contents

1 Introduction

Some words why the subject is of interest.

A reference to the primary literature used.

Overview of used notation.

$M_f(r) \coloneqq \max_{|z|=r} |f(z)|$

2 Order

Definition 2.1. Let f be an entire function. The *order* of f is defined by

$$\rho_f := \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}. \tag{2.1}$$

Constant functions, by convention, have order 0.

Remark 2.2. Initial explanation and intuition of the order. Make sure to note the possible values of the order ($0 \leq \rho \leq \infty$). And that ρ can also be seen as the infimum over all ρ that satisfy $|f(z)| \leq Ae^{B|z|^\rho}$ for suitable $A, B > 0$.

Proposition 2.3. *Let f, g be entire functions of finite order. Then it holds that:*

i. $\rho_{f+g} \leq \max\{\rho_f, \rho_g\}$

ii. $\rho_{fg} \leq \max\{\rho_f, \rho_g\}$

Proof. To prove (i), note that

$$\begin{aligned} M_{f+g}(r) &= \max_{|z|=r} |f(z)+g(z)| \leq \max_{|z|=r} |f(z)| + |g(z)| \leq \max_{|z|=r} |f(z)| + \max_{|z|=r} |g(z)| \leq \\ &= M_f(r) + M_g(r) \leq 2\max\{M_f(r), M_g(r)\} \end{aligned}$$

thus

$$\log M_{f+g}(r) \leq \log 2 + \log\max\{M_f(r), M_g(r)\} = \log 2 + \max\{\log M_f(r), \log M_g(r)\}.$$

If $M_f(r)$ and $M_g(r)$ are bounded, then applying the above in eq. (2.1) implies that f, g and $f+g$ all have order 0. If either one is not, then $\max\{\log M_f(r), \log M_g(r)\}$ necessarily outgrows $\log 2$ and we obtain

$$\begin{aligned} \rho_{f+g} &= \limsup_{r\to\infty} \frac{\log\log M_{f+g}(r)}{\log r} \leq \limsup_{r\to\infty} \frac{\log(\log 2 + \max\{\log M_f(r), \log M_g(r)\})}{\log r} = \\ &= \limsup_{r\to\infty} \frac{\log\max\{\log M_f(r), \log M_g(r)\}}{\log r} = \\ &= \limsup_{r\to\infty} \max\left\{\frac{\log\log M_f(r)}{\log r}, \frac{\log\log M_g(r)}{\log r}\right\} = \\ &= \max\left\{\limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}, \limsup_{r\to\infty} \frac{\log\log M_g(r)}{\log r}\right\} = \max\{\rho_f, \rho_g\}. \end{aligned}$$

To prove (ii), we similarly note that

$$\begin{aligned}\log\log M_{fg}(r) \leq \log\log(M_f(r)M_g(r)) = \log(\log M_f(r) + \log M_g(r)) \leq \\ \leq \log(2\max\{\log M_f(r), \log M_g(r)\}) = \\ = \log 2 + \max\{\log\log M_f(r), \log\log M_g(r)\},\end{aligned}$$

from where we can proceed as in (i). ■

For entire functions of finite order we can obtain an alternative representation of the order via the power series coefficients.

Theorem 2.4. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. Then f is of finite order ρ if and only if*

$$\mu := \limsup_{n\to\infty} \frac{n\log n}{\log\frac{1}{|a_n|}} < \infty,$$

where we take the quotient to be zero if $a_n = 0$. In either case we have $\rho = \mu$.

Proof. TODO. ■

Example 2.5. Some examples for functions of specific order. Specifically order 0 (polynomials, rapidly convering sum), order 1 (exp, sin), arbitrary order ρ and order ∞.

We shall apply Theorem 2.4 to provide examples for entire functions of specific order.

It is immediately apparent that polynomials have order zero. Non-polynomial functions of zero order do exist, provided their coefficients decrease rapidly enough, for example $\sum_{n=0}^{\infty} n^{-(n^2)} z^n$.

By using Stirling's approximation $\log n! = n\log n - n + O(\log n)$, we obtain that the exponential function, sine and cosine all have order 1.

For a given $\rho \in \mathbb{R}$ we can construct an entire function f of that order by defining

$$f(z) := \sum_{n=0}^{\infty} n^{-\rho n} z^n.$$

$$\sum_{n=3}^{\infty} n^{-\frac{n}{\sqrt{\log n}}} z^n$$

Proposition 2.6. *Let f be an entire function of finite order with derivative f'. Then $\rho_{f'} = \rho_f$.*

Proof. Given $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we have $f'(z) = \sum_{n=0}^{\infty} (n+1)a_{n+1} z^n$. Since

$$\lim_{n\to\infty} \left(\frac{n\log n}{(n+1)\log(n+1)}\right)^{-1} = 1$$

we have

$$\begin{aligned}
\limsup_{n\to\infty} \frac{n\log n}{\log \frac{1}{|(n+1)a_{n+1}|}} &= \liminf_{n\to\infty} \left(\frac{-\log(n+1) + \log \frac{1}{|a_{n+1}|}}{n\log n} \right)^{-1} = \\
&= \liminf_{n\to\infty} \left(\frac{-\log|a_{n+1}|}{n\log n} \right)^{-1} \cdot \lim_{m\to\infty} \left(\frac{m\log m}{(m+1)\log(m+1)} \right)^{-1} = \\
&= \liminf_{n\to\infty} \left(\frac{-\log|a_{n+1}|}{n\log n} \cdot \frac{n\log n}{(n+1)\log(n+1)} \right)^{-1} = \\
&= \liminf_{n\to\infty} \left(\frac{-\log|a_{n+1}|}{(n+1)\log(n+1)} \right)^{-1} = \limsup_{n\to\infty} \frac{(n+1)\log(n+1)}{\log \frac{1}{|a_{n+1}|}} = \\
&= \limsup_{n\to\infty} \frac{n\log n}{\log \frac{1}{|a_n|}}
\end{aligned}$$

and since $\rho_f < \infty$ Theorem 2.4 concludes $\rho_{f'} = \rho_f$. ■

3 Factorization

Short introduction.

Do I need a citation for this?

Theorem 3.1 (Weierstrass)**.** *Let $(z_j)_{j\in\mathbb{N}}$ be a sequence in $\mathbb{C}$ without accumulation points. Then there exists an entire function P (called the* Weierstrass canonical product *formed from said sequence) that has zeros precisely at $z_j, j \in \mathbb{N}$, with multiplicities equal to how often z_j occurs in the sequence.*

Furthermore, if f is any other entire function satisfying the above, then there exists an entire function g such that

$$f = e^g P.$$

Short remark on how Hadamard refines Weierstrass (for functions of finite order).

Lemma 3.2 (Borel-Carathéodory)**.** *Let f be analytic in $\operatorname{cl}(B(0,R))$ and let*

$$M(r) = \max_{|z|=r} |f(z)|, \quad A(r) = \max_{|z|=r} \Re f(z).$$

Then, for $0 < r < R$,

$$M(r) \leq \frac{2r}{R-r} A(R) + \frac{R+r}{R-r} |f(0)|$$

and, if additionally $A(R) \geq 0$, then for $n \in \mathbb{N}$

$$\max_{|z|=r} |f^{(n)}(z)| \leq \frac{2^{n+2} n! R}{(R-r)^{n+1}} (A(R) + |f(0)|).$$

Proof. TODO. ■

Theorem 3.3 (Hadamard)**.** *Let f be an entire function of finite order ρ with zeros $(z_j)_{j\in\mathbb{N}}$ and $f(0) \neq 0$. Then there exists a polynomial Q with $\deg Q \leq \rho$, such that*

$$f = e^Q P,$$

where P is the Weierstrass canonical product formed from the zeros of f.

Proof. TODO. ■

4 Zeros

Maybe Jensen? Not sure if I need it earlier. Anyhow, include the equivalent version using the zero counting function.

Definition 4.1. Let f be an entire function satisfying $f(0) \neq 0$. Let $(r_j)_{j \in \mathbb{N}}$ denote the moduli of the zeros of f (if any) arranged in non-decreasing order. Then

$$\rho_1 := \inf\{\alpha > 0 : \sum_{n=1}^{\infty} \frac{1}{r_n^{\alpha}} < \infty\}$$

is called the *exponent of convergence of the zeros of f*. If f has finitely many zeros, then we set $\rho_1 = 0$ by convention. I am not sure about this – the paper only establishes this convention if f has no zeros at all.

Furthermore, the *exponent of convergence of the a-points of f* is defined as exponent of convergence of $f(z) - a$.

Theorem 4.2. *Let f be an entire function of finite order ρ and exponent of convergence of zeros ρ_1. Then $\rho_1 \leq \rho$.*

Proof. TODO. ∎

Example 4.3. An example of a series where we see some convergence for some appropriate function using the above.

Theorem 4.4. *Let P be a Weierstrass canonical product of finite order ρ and exponent of convergence of zeros ρ_1. Then $\rho_1 = \rho$.*

Proof. TODO. ∎

Theorem 4.5. *Let f be an entire function of finite order ρ and exponent of convergence of zeros ρ_1. If ρ is not an integer, then $\rho = \rho_1$.*

Proof. TODO. ∎

Theorem 4.6. *Let f be an entire function of finite, non-integer order. Then f has infinitely many zeros.*

Proof. TODO. ■

Maybe introduce Borel exceptional values as a definition? But then again, I will never need them again. Maybe also add a remark on the relation to lacunary values (Picard).

Theorem 4.7 (Borel)**.** *Existence of Borel exceptional values.*

Proof. TODO. ■

5 Composition

Theorem 5.1 (Pólya)**.** *Let g, h be entire. For the order of $g \circ h$ to be finite, it must hold that either*

i. h is a polynomial and g of finite order, or

ii. h is of finite order, not a polynomial, and g is of order zero.

Proof. TODO. ■

Theorem 5.2. *Let g be an entire function of finite order, not a polynomial, which takes some value w only finitely often. Suppose further that there exists f such that $f \circ f = g$. Then f is not entire.*

Proof. TODO. ■

Example 5.3. The example with $f(f(z)) = e^z$.

Bibliography

[1] S. L. Segal. *Nine introductions in complex analysis*, volume 208 of *North-Holland Mathematics Studies*. Elsevier Science B.V., Amsterdam, revised edition, 2008.