Bachelorarbeit

Picard's Great Theorem
and
Growth, Zeros, and Composition of Entire Functions

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1 Introduction

In this thesis, we explore the theory of entire functions, that is, complex-valued functions that are holomorphic in the entire complex plane.

Our early focus lies on Picard's Great Theorem, an important corollary of which states that an entire function assumes every complex value infinitely often, with at most one exception. A natural example for this behaviour is the complex exponential function, which does not assume zero, but does assume all other values infinitely often. Chapter 2 is dedicated to deriving this theorem and providing groundwork for the results in the later chapters.

Section 3.1 then briefly introduces a scale of asymptotic growth for the modulus of entire functions, in the form of order $\rho \in [0, \infty]$ and type $\tau \in [0, \infty]$. Roughly speaking, if f is of finite order, then f grows not faster than a function of the form $e^{Q(z)}$, where Q is a polynomial, as $|z| \to \infty$. Results introduced here will frequently be utilized towards the end of Chapter 3.

Sections 3.2 to 3.4 explore the nature of zeros of entire functions. As is well-known, a complex polynomial can be represented solely by its zeros and some scaling factor. In Section 3.3, this property is generalized for entire functions by the Weierstraß Factorization Theorem: For any entire function f there exists an entire function g and a so-called "canonical product" Π, which depends on the zeros of f, such that

$$f(z) = z^m e^{g(z)} \Pi(z), \quad \text{for all } z \in \mathbb{C},$$

where $m \in \mathbb{N}_0$ denotes the order of the zero of f at the origin. Hadamard's Theorem, which is the focus of Section 3.4, then provides an important refinement for entire functions of finite order ρ, showing that g can be taken to be a polynomial of degree at most ρ. This has immediate corollaries, in particular if the order is not an integer, which we explore towards the end of the section.

Finally, Chapter 4 studies when the composition of functions of finite order is, again, of finite order. Pólya's Theorem will provide necessary conditions for this to hold. A consequence is Thron's Theorem, which states that for any entire function g of finite order, which assumes some complex value only finitely often, there does not exist an entire function f for which $f \circ f = g$ holds on the entire complex plane.

Introductory results on complex analysis are assumed as prerequisites and will be used without further notice. These include Cauchy's integral formula, the maximum modulus principle, the theorems of Montel, Hurwitz and Jensen, the Schwarz lemma and standard results on power series. Some standard references are [1, 4, 6].

The following is an overview of the notation used:

- $\rightsquigarrow$ The sets $\mathbb{N}$ and $\mathbb{Z}$ denote, respectively, the natural numbers (starting with 1) and the integers. Furthermore we define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
- $\rightsquigarrow$ For a set A, its indicator function $\mathbb{1}_A(a)$ is defined as 1 if $a \in A$ and 0 otherwise.
- $\rightsquigarrow$ The restriction of a function $f : M \to N$ to a set $D \subseteq M$ will be denoted $f\big|_D$.
- $\rightsquigarrow$ The sets $\mathbb{R}$ and $\mathbb{C}$ denote, respectively, the real line and the complex plane, which will both be endowed with the standard Euclidean metric and topology.
- $\rightsquigarrow$ The closure of a subset A of $\mathbb{R}$ or $\mathbb{C}$ (with respect to the appropriate topology) will be denoted as $c\ell(A)$ and the boundary as ∂A.
- $\rightsquigarrow$ For subsets M, N of $\mathbb{R}$ or $\mathbb{C}$ we denote by $C(M, N)$ the space of continuous maps from M into N.
- $\rightsquigarrow$ For a real- or complex-valued function f defined on some set M we define $\|f\|_M := \sup_{x\in M} |f(x)|$.
- $\rightsquigarrow$ If a function f is differentiable, in the real or complex sense, then f' and f'' denote the respective first and second derivatives of f. For $n \in \mathbb{N}$, the n-th derivative will be denoted $f^{(n)}$.
- $\rightsquigarrow$ Let $a \in \mathbb{R}$ and let f, g be functions on $[a, \infty)$ such that f is either real- or complex-valued and g is real-valued. Then we write $f(x) = \mathcal{O}(g(x))$ as $x \to \infty$ if there exist constants $C > 0$ and $x_0 \geq a$ such that for $x \geq x_0$ we have $|f(x)| \leq Cg(x)$.
- $\rightsquigarrow$ For $a, b \in \mathbb{C}$ we denote by $[a, b]$ the line segment connecting a and b, parametrized by $\gamma : [0, 1] \to \mathbb{C}, t \mapsto (1 - t)a + tb$.
- $\rightsquigarrow$ If $G \subseteq \mathbb{C}$, then G is said to be a *domain* if G is non-empty, open and connected.
- $\rightsquigarrow$ For a point $a \in \mathbb{C}$ and $r > 0$ we denote by $B_r(a)$ the open disk of radius r centered at a. The open unit disk $B_1(0)$ will be denoted $\mathbb{D}$. Furthermore we define $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$ and $\mathbb{D}^\times := \mathbb{D} \setminus \{0\}$.
- $\rightsquigarrow$ Let $G \subseteq \mathbb{C}$ be non-empty and open, then a function $f : G \to \mathbb{C}$ is said to be *holomorphic* at $z \in G$ if f is complex differentiable at z. If f is holomorphic at all $z \in G$, then f is said to be holomorphic on G. The space of all such holomorphic functions on G will be denoted as $H(G)$. For a closed connected set $M \subseteq \mathbb{C}$ we denote by $H(M)$ the space of all functions holomorphic in a domain $G \supset M$. If $f \in H(\mathbb{C})$, then f is called *entire*.
- $\rightsquigarrow$ Let $G \subseteq \mathbb{C}$ be a domain and $D \subseteq \mathbb{C}$ a set with $c\ell(D) \subseteq G$ and a boundary ∂D that can be described by a directed smooth curve. Then the contour integral of a function $f \in H(G)$ along the boundary of D will be denoted

  $$\oint_{\partial D} f(z)\,\mathrm{d}z$$

  and is, unless otherwise indicated, always parametrized in mathematically positive (counter-clockwise) orientation.

- $\rightsquigarrow$ The (complex) exponential function will be denoted $\exp : \mathbb{C} \to \mathbb{C}^\times$. The (local) inverse will be denoted log, where we, unless specified otherwise, always choose the principle branch, i.e., the branch where $\log 1 = 0$.

2 Picard's Great Theorem

An isolated singularity of a holomorphic function which is not removable nor a pole, is called an *essential singularity*. While holomorphic functions are fairly well-behaved near their removable singularities and poles, the same cannot be said for their essential singularities.

An interesting result is already given by the Casorati–Weierstraß Theorem: Let $G \subseteq \mathbb{C}$ be open, $w \in G$ and suppose $f \in H(G \setminus \{w\})$ such that f has an essential singularity at w. Then, for any punctured neighborhood $U \subseteq G \setminus \{w\}$ of w, the set $f(U)$ is dense in $\mathbb{C}$.

Picard's Great Theorem will show that $f(U)$ is not only dense in $\mathbb{C}$, but there is at most one value which is not taken on by f infinitely often on any such punctured neighborhood.

We will reach the proof the same way as in Remmert [4]. We first study Bloch's Theorem, which estimates the size of disks in the image of a holomorphic map. As a corollary of the proof we obtain Picard's Little Theorem, which states that a non-constant entire function omits at most one value.

One can see this as a motivation to intently study (non-entire) holomorphic functions that omit (at least) two values. For such functions, Schottky's Theorem will give an upper bound on their modulus. We use this to obtain the Fundamental Normality Test, which asserts normality of any family of holomorphic functions omitting two fixed values.

Restricting the domain in the above to be the punctured unit disk, we obtain that such functions cannot have an essential singularity at the origin, which will imply Picard's Great Theorem.

2.1 Bloch's Theorem

If $G \subseteq \mathbb{C}$ is a domain and $f \in H(G)$ is non-constant, then $f(G)$ is a domain as well. In particular, $f(G)$ contains open disks of some, potentially very small, radius. Bloch's Theorem asserts that for any $f \in H(c\ell(\mathbb{D}))$ satisfying $f'(0) = 1$, the set $f(\mathbb{D})$ always contains a disk of fixed radius.

Note that $f(0)$ may not always be the center of such a disk; consider the sequence

$$f_n(z) := \frac{1 - e^{-nz}}{n} \in H(c\ell(\mathbb{D})), \quad n \in \mathbb{N},$$

which satisfies $f_n(0) = 0$ and $f_n'(0) = 1$, but omits the value $1/n$.

We first observe a more general criterion, not relying on the holomorphic property, for the size of disks in the image domain of open mappings. An open mapping is a function which maps open sets to open sets.

Lemma 2.1.1. *Let $G \subset \mathbb{C}$ be a bounded domain and $f \in C(c\ell(G); \mathbb{C})$ such that $f\big|_G$ is an open mapping. Let $a \in G$ and set $s := d(f(a), f(\partial G))$. If $s > 0$, then $B_s(f(a)) \subseteq f(G)$.*

Proof. Since G is bounded, $c\ell(G)$ is compact and, by continuity of f, so is $c\ell(f(G))$. The function $z \mapsto |z - f(a)|$ is continuous on the compact set $\partial f(G)$, hence it assumes its minimum m at some $w_* \in \partial f(G)$. Choose a sequence $(z_n)_{n\in\mathbb{N}}$ in G with $\lim_{n\to\infty} f(z_n) = w_*$, then, since $c\ell(G)$ is compact, we can find a subsequence that converges to some $z_* \in c\ell(G)$. By continuity of f, we have $f(z_*) = w_*$.

If $z_* \in G$, since $f\big|_G$ is open, the image of any open set in G containing z_* under f is an open set in $f(G)$ containing w_*, which is impossible since $w_* \in \partial f(G)$.

Therefore, $z_* \in \partial G$ and we have

$$d(f(a), \partial f(G)) = m = |w_* - f(a)| = |f(z_*) - f(a)| \geq s,$$

which implies $B_s(f(a)) \subseteq f(G)$ if $s > 0$. ■

Lemma 2.1.2. *Fix $a \in \mathbb{C}, r > 0$ and let $B := B_r(a)$. Suppose further $G \subseteq \mathbb{C}$ is a domain such that $c\ell(B) \subset G$ and $f \in H(G)$ such that $\|f'\|_B \leq 2|f'(a)|$. Then $B_R(f(a)) \subseteq f(B)$, where $R := (3 - 2\sqrt{2})r|f'(a)|$.*

Proof. We may assume $a = f(a) = 0$, otherwise we consider $f_1(z) := f(z+a) - f(a)$. The function

$$\alpha_f : \begin{cases} B \to \mathbb{C}, \\ z \mapsto f(z) - f'(0)z, \end{cases}$$

satisfies, for all $z \in B$,

$$|\alpha_f(z)| = \left| \int_{[0,z]} f'(\zeta) - f'(0)\,\mathrm{d}\zeta \right| \leq \int_0^1 |f'(tz) - f'(0)||z|\,\mathrm{d}t. \tag{$*$}$$

We wish to further estimate the integrand. Let $w \in B$, then Cauchy's integral formula gives

$$\begin{aligned} |f'(w) - f'(0)| &= \frac{1}{2\pi} \left| \oint_{\partial B} \frac{f'(\zeta)}{\zeta - w} - \frac{f'(\zeta)}{\zeta}\,\mathrm{d}\zeta \right| = \frac{1}{2\pi} \left| \oint_{\partial B} \frac{w f'(\zeta)}{\zeta(\zeta - w)}\,\mathrm{d}\zeta \right| \leq \\ &\leq \frac{1}{2\pi} \oint_{\partial B} \frac{|w|\|f'\|_B}{r(r - |w|)}\,\mathrm{d}\zeta = \frac{|w|}{r - |w|}\|f'\|_B. \end{aligned}$$

Combining the above with $(*)$ and our estimate on $\|f'\|_B$ yields

$$|\alpha_f(z)| \leq \int_0^1 \frac{|zt|\|f'\|_B}{r - |zt|}|z|\,\mathrm{d}t \leq \frac{|z|^2}{r - |z|}\|f'\|_B \int_0^1 t\,\mathrm{d}t \leq \frac{|z|^2}{r - |z|}|f'(0)|.$$

Let $0 < \rho < r$, then for $|z| = \rho$ we have

$$\begin{aligned} |f'(0)|\rho - |f(z)| \le |\alpha_f(z)| &\le \frac{\rho^2}{r-\rho}|f'(0)| \\ \iff \quad |f(z)| &\ge \left(\rho - \frac{\rho^2}{r-\rho}\right)|f'(0)|. \end{aligned}$$

The function $\rho \mapsto \rho - \rho^2/(r-\rho)$ assumes its maximum value at $\rho_* := (1 - 1/\sqrt{2})r \in (0, r)$, namely $(3 - 2\sqrt{2})r$. Therefore,

$$|f(z)| \ge (3 - 2\sqrt{2})r|f'(0)| = R, \quad \text{for all} \quad |z| = \rho_*.$$

In particular, $\min_{z \in \partial B_{\rho_*}} |f(z)| \ge R > 0$, thus invoking Lemma 2.1.1 with the domain $B_{\rho_*}(0)$ yields $B_R(0) \subseteq f(B_{\rho_*}(0)) \subseteq f(B)$. ∎

Theorem 2.1.3. *Let $f \in H(c\ell(\mathbb{D}))$ be non-constant. Then there is a point $p \in \mathbb{D}$ and a constant $C_f > 0$ such that $B_R(f(p)) \subseteq f(\mathbb{D})$, where $R := (\frac{3}{2} - \sqrt{2})C_f \ge (\frac{3}{2} - \sqrt{2})|f'(0)|$.*

Proof. The function

$$\alpha_f : \begin{cases} c\ell(\mathbb{D}) \to \mathbb{R} \\ \quad z \quad \mapsto |f'(z)|(1 - |z|) \end{cases}$$

is continuous and assumes its maximum $C_f > 0$ at some point $p \in c\ell(\mathbb{D})$. Note that $C_f \ge |f'(0)|$ and, since f is non-constant and $\alpha_f\big|_{\partial\mathbb{D}} = 0$, we even have $p \in \mathbb{D}$.

Set $t := \frac{1}{2}(1 - |p|) > 0$, then we have $B_t(p) \subseteq \mathbb{D}$. Furthermore, for $z \in B_t(p)$, we have

$$1 - |z| \ge 1 - |z - p| - |p| \ge 1 - t - |p| = t.$$

Since $|f'(z)|(1 - |z|) \le C_f = 2t|f'(p)|$, this implies $|f'(z)| \le 2|f'(p)|$ for all $z \in B_t(p)$. By Lemma 2.1.2, we get $B_R(f(p)) \subseteq f(\mathbb{D})$, where $R := (3 - 2\sqrt{2})t|f'(p)| = (\frac{3}{2} - \sqrt{2})C_f$, which establishes the assertion. ∎

We now immediately obtain:

Theorem 2.1.4 (Bloch)**.** *Let $f \in H(c\ell(\mathbb{D}))$ and assume $f'(0) = 1$. Then $f(\mathbb{D})$ contains a disk of radius $\frac{3}{2} - \sqrt{2}$.*

In the following we will denote by $\beta > 0$ any constant less than or equal to the radius in Bloch's Theorem, for example $\beta = \frac{1}{12} < \frac{3}{2} - \sqrt{2}$.

Corollary 2.1.5. *Let $G \subseteq \mathbb{C}$ be a domain and $f \in H(G)$ with $f'(c) \ne 0$ for some $c \in G$. Then $f(G)$ contains a disk of every radius $\beta s|f'(c)|$, where $0 < s < d(c, \partial G)$.*

Proof. We may assume $c = 0$, otherwise we consider $f_1(z) := f(z+c)$. Let $0 < s < d(c, \partial G)$, then

$$g(z) := \frac{f(sz)}{sf'(0)} \in H(c\ell(\mathbb{D})).$$

Since $g'(0) = 1$, Bloch's Theorem yields a disk B of radius β with $B \subseteq g(\mathbb{D})$. Then $D := s|f'(0)|B$ is a disk of radius $\beta s|f'(0)|$ and we have

$$D = s|f'(0)|B \subseteq s|f'(0)|g(\mathbb{D}) = f(B_s(0)) \subseteq f(G). \qquad \blacksquare$$

Corollary 2.1.6. *If $f \in H(\mathbb{C})$ is non-constant, then $f(\mathbb{C})$ contains a disk of every radius.*

2.2 Schottky's Theorem

Holomorphic functions which omit the values 0 and 1 have a universal estimate on the growth of their modulus, which will be given by Schottky's Theorem.

For a domain $G \subseteq \mathbb{C}$ and a set $E \subseteq \mathbb{C}$ we define $H(G; E)$ as the set[1] of all $f \in H(G)$ such that $f(G) \subseteq E$.

Lemma 2.2.1. *It holds that:*

i. If $a, b \in \mathbb{R}$ with $\cos \pi a = \cos \pi b$, then $b = \pm a + 2n$ for some $n \in \mathbb{Z}$.

ii. For every $w \in \mathbb{C}$ there exists $v \in \mathbb{C}$ such that $\cos \pi v = w$ and $|v| \leq 1 + |w|$.

Proof. For the first part it suffices to note that

$$0 = \cos \pi a - \cos \pi b = -2 \sin \tfrac{\pi}{2}(a + b) \sin \tfrac{\pi}{2}(a - b).$$

Since the complex cosine function is surjective and 2π-periodic, we can choose $v = a + ib$ with $w = \cos \pi v$ and $|a| \leq 1$. Now we have

$$\begin{aligned} |w|^2 &= |\cos(\pi a + i\pi b)|^2 = |\cos \pi a \cos i\pi b + \sin \pi a \sin i\pi b|^2 = \\ &= |\cos \pi a \cosh \pi b - i \sin \pi a \sinh \pi b|^2 = \\ &= \cos^2 \pi a \cosh^2 \pi b + \sin^2 \pi a \sinh^2 \pi b = \\ &= \cos^2 \pi a + \cos^2 \pi a \sinh^2 \pi b + \sin^2 \pi a \sinh^2 \pi b = \\ &= \cos^2 \pi a + \sinh^2 \pi b \geq \sinh^2 \pi b \geq \pi^2 b^2, \end{aligned}$$

where the last inequality holds since $\sinh^2 x \geq x^2$ for $x \in \mathbb{R}$. We conclude

$$|v| = \sqrt{a^2 + b^2} \leq \sqrt{1 + |w|^2/\pi^2} \leq 1 + |w|. \qquad \blacksquare$$

We recall the following result: Let $G \subseteq \mathbb{C}$ be a simply connected domain and $f \in H(G)$, such that f vanishes nowhere on G. There is a function[2] $g \in H(G)$ such that $f = e^g$, which, for $n \in \mathbb{N}$, can also be used to obtain n-th roots of such functions by defining $\sqrt[n]{f} := e^{g/n}$.

[1] Note that, unlike $H(G)$, the set $H(G; E)$ is usually not a linear space.
[2] g is also sometimes called a *logarithm* of f.

Lemma 2.2.2. *Let $G \subseteq \mathbb{C}$ be a simply connected domain and $f \in H(G; \mathbb{C} \setminus \{-1, 1\})$, then there exists $F \in H(G)$ such that $f = \cos F$.*

Proof. Since $1 - f^2$ vanishes nowhere in G, it has a square root $g \in H(G)$, therefore

$$1 = f^2 + g^2 = (f + ig)(f - ig).$$

Thus $f + ig$ vanishes nowhere and there exists an $F \in H(G)$ with $f + ig = e^{iF}$. By the above we also have $f - ig = e^{-iF}$ and therefore

$$f = \tfrac{1}{2}(e^{iF} + e^{-iF}) = \cos F. \qquad \blacksquare$$

Lemma 2.2.3. *Let $G \subseteq \mathbb{C}$ be a simply connected domain and $f \in H(G; \mathbb{C} \setminus \{0, 1\})$. Then there exists $g \in H(G)$ such that:*

i. $f = \frac{1}{2}(1 + \cos \pi(\cos \pi g))$.

ii. $|g(0)| \leq 3 + 2|f(0)|$.

iii. $g(G)$ *contains no disk of radius* 1.

iv. If $\mathbb{D} \subseteq G$, *then* $|g(z)| \leq |g(0)| + \frac{\theta}{\beta(1-\theta)}$ *for all* $|z| \leq \theta$, *where* $0 < \theta < 1$.

Proof. By Lemma 2.2.2, there exists $\widetilde{F} \in H(G)$ such that $2f - 1 = \cos \pi \widetilde{F}$ and, by Lemma 2.2.1, there is $b \in \mathbb{C}$ with $\cos \pi b = 2f(0) - 1$ and $|b| \leq 1 + |2f(0) - 1| \leq 2 + 2|f(0)|$. Furthermore, since $\cos \pi b = \cos \pi \widetilde{F}(0)$, we have $b = \pm \widetilde{F}(0) + 2k$ for some $k \in \mathbb{Z}$. Then $F := \pm \widetilde{F} + 2k \in H(G)$ satisfies $F(0) = b$ and $2f - 1 = \cos \pi F$.

Since F must omit all integers, there exists $\widetilde{g} \in H(G)$ such that $F = \cos \pi \widetilde{g}$. Similarly, there is $a \in \mathbb{C}$ such that $\cos \pi a = b$ and $|a| \leq 1 + |b| \leq 3 + 2|f(0)|$. Like in the above, since $\cos \pi a = \cos \pi \widetilde{g}(0)$, we have $a = \pm \widetilde{g}(0) + 2\ell$ for some $\ell \in \mathbb{Z}$, thus $g := \pm \widetilde{g} + 2\ell \in H(G)$ satisfies $g(0) = a$ and $F = \cos \pi g$. Together we obtain

$$f = \tfrac{1}{2}(1 + \cos \pi(\cos \pi g)), \quad \text{and} \quad |g(0)| = |a| \leq 3 + 2|f(0)|$$

and thus have shown i. and ii.

To show iii. we consider the set

$$A := \{m \pm i\pi^{-1} \log(n + \sqrt{n^2 - 1}) : m \in \mathbb{Z}, n \in \mathbb{N} \setminus \{0\}\},$$

the points of which can be considered the vertices of a rectangular grid in $\mathbb{C}$, the cells of which are of fixed width and variable height. The width of such a rectangular cell is 1, and since

$$\begin{aligned}\log((n+1)+\sqrt{(n+1)^2-1})-\log(n+\sqrt{n^2-1}) &= \\ = \log\frac{1+\frac{1}{n}+\sqrt{1+\frac{2}{n}}}{1+\sqrt{1-\frac{1}{n^2}}} &\le \log(2+\sqrt{3}) < \pi,\end{aligned}$$

their height is bounded above by some $C < 1$. Therefore, for all $z \in \mathbb{C}$ there is $w_z \in A$ such that $|\operatorname{Re} z - \operatorname{Re} w_z| \le \frac{1}{2}$ and $|\operatorname{Im} z - \operatorname{Im} w_z| \le \frac{C}{2}$. Thus, we have

$$|z - w_z| \le |\operatorname{Re} z - \operatorname{Re} w_z| + |\operatorname{Im} z - \operatorname{Im} w_z| \le \frac{1}{2} + \frac{C}{2} < 1.$$

If we can show that $g(G) \cap A = \emptyset$, then $g(G)$ cannot contain a disk of radius 1. Let $a = p + i\pi^{-1}\log(q+\sqrt{q^2-1}) \in A$, then

$$\begin{aligned}\cos \pi a = \tfrac{1}{2}(e^{i\pi a}+e^{-i\pi a}) &= \tfrac{1}{2}(-1)^p((q+\sqrt{q^2-1})^{-1}+(q+\sqrt{q^2-1})) = \\ = \frac{(-1)^p}{2}\,\frac{1+q^2+2q\sqrt{q^2-1}+q^2-1}{q+\sqrt{q^2-1}} &= (-1)^p q\end{aligned}$$

and thus $\cos\pi(\cos\pi a) = \pm 1$. But $0, 1 \notin f(G)$, therefore $a \notin g(G)$ and $g(G) \cap A = \emptyset$, proving (iii).

For (iv), if $\mathbb{D} \subseteq G$, then $g\big|_{\mathbb{D}} \in H(\mathbb{D})$. Fix $0 < \theta < 1$, then for $|z| \le \theta$ we have

$$d(z, \partial\mathbb{D}) = \inf_{w\in\partial\mathbb{D}} |z-w| \ge \inf_{w\in\partial\mathbb{D}} (|w|-|z|) \ge 1-\theta.$$

From (iii) it follows that $g\big|_{\mathbb{D}}(\mathbb{D})$ does not contain a disk of radius 1. Let $0 < s < 1-\theta$, then applying Corollary 2.1.5 to $g\big|_{\mathbb{D}}$ implies that $\beta s|g'(z)| < 1$, for any $z \in \mathbb{D}$. Taking the supremum over s and rearranging yields $|g'(z)| \le 1/(\beta(1-\theta))$, thus our desired estimate is shown by

$$|g(z)| \le |g(0)| + |g(z)-g(0)| \le |g(0)| + \int_{[0,z]} |g'(\zeta)|\,\mathrm{d}\zeta \le |g(0)| + \frac{\theta}{\beta(1-\theta)}. \qquad \blacksquare$$

Theorem 2.2.4 (Schottky). *There exists a function $\psi : (0,1)\times(0,\infty) \to (0,\infty)$ such that for any $f \in H(\mathbb{D}; \mathbb{C}\setminus\{0,1\})$ with $|f(0)| \le \omega$ it holds that*

$$|f(z)| \le \psi(\theta,\omega), \quad |z| \le \theta. \tag{2.1}$$

Proof. Note that for all $w \in \mathbb{C}$ we have $|\cos w| \le e^{|w|}$ and $\frac{1}{2}|1+\cos w| \le e^{|w|}$. Hence, from

Lemma 2.2.3, we get

$$\begin{aligned}|f(z)| = |\tfrac{1}{2}(1+\cos\pi(\cos\pi g(z)))| &\leq \exp(\pi\exp\pi|g(z)|) \leq \\ &\leq \exp\left(\pi\exp\pi\left(|g(0)| + \frac{\theta}{\beta(1-\theta)}\right)\right) \leq \\ &\leq \exp\left(\pi\exp\pi\left(3+2\omega+\frac{\theta}{\beta(1-\theta)}\right)\right),\end{aligned}$$

and defining $\psi(\theta,\omega)$ as the final term establishes the assertion. ■

Lemma 2.2.3 is quite powerful, as it contains not only Schottky's Theorem, but Picard's Little Theorem as well:

Theorem 2.2.5 (Picard's Little Theorem)**.** *Let $f \in H(\mathbb{C};\mathbb{C}\setminus\{a,b\})$ for distinct points $a,b\in\mathbb{C}$. Then f is constant.*

Proof. Consider $f_1(z) := \frac{f(z)-a}{b-a} \in H(\mathbb{C};\mathbb{C}\setminus\{0,1\})$. By Lemma 2.2.3 there is some $g\in H(\mathbb{C})$ such that $f_1 = \frac{1}{2}(1+\cos\pi(\cos\pi g))$ and $g(\mathbb{C})$ does not contain a disk of radius 1. By Corollary 2.1.6, we thereby have that g must be constant, and therefore so are f_1 and f. ■

2.3 Normal Families

First, we recall a generalized form of locally uniform convergence:

Definition 2.3.1. Let $G\subseteq\mathbb{C}$ be a domain, $f\in C(G,\mathbb{C})$ and $(f_n)_{n\in\mathbb{N}}$ a sequence in $C(G,\mathbb{C})$. We say that f_n *converges compactly in G to f*, or f_n *converges compactly in G to ∞* as $n\to\infty$, if for every compact set $K\subset G$

$$\lim_{n\to\infty}\sup_{z\in K}|f_n(z)-f(z)| = 0, \quad \text{or} \quad \lim_{n\to\infty}\inf_{z\in K}|f_n(z)| = \infty, \tag{2.2}$$

where the supremum and infimum over empty sets are defined as 0 and ∞, respectively. //

Definition 2.3.2. Let $f:\mathbb{C}\to\mathbb{C}$ and $a\in\mathbb{C}$, then the *a-points of f* are defined as the zeros of $f(z)-a$, that is the set of all points $w\in\mathbb{C}$ with $f(w)=a$. //

A well-known theorem on compact convergence is:

Theorem 2.3.3 (Hurwitz). *Let $G \subseteq \mathbb{C}$ be a domain, $a \in \mathbb{C}$ and $(f_n)_{n\in\mathbb{N}}$ a sequence in $H(G)$ that converges compactly to $f \in H(G)$. If for every $n \in \mathbb{N}$ the number of a-points of f_n is bounded by some $m \in \mathbb{N}_0$, then either*

- *the number of a-points of f are also bounded by m or*
- *$f \equiv a$.*

As an immediate consequence we obtain that compact convergence is, in some ways, compatible with reciprocals:

Lemma 2.3.4. *Let $G \subseteq \mathbb{C}$ be a domain and $(f_n)_{n\in\mathbb{N}}$ a sequence in $H(G; \mathbb{C} \setminus \{0\})$. If $(f_n)_{n\in\mathbb{N}}$ converges compactly to some $f \in H(G)$, then it holds that either:*

- *$0 \notin f(G)$ and $1/f_n \to 1/f$ compactly in G, or*
- *$f \equiv 0$ and $1/f_n \to \infty$ compactly in G.*

If, on the other hand, $(f_n)_{n\in\mathbb{N}}$ converges compactly to ∞, then $1/f_n \to 0$ compactly in G.

Proof. For the equivalence "$f_n \to 0$ if and only if $1/f_n \to \infty$" it suffices to notice that for any compact $K \subset G$ we have

$$\frac{1}{\sup_{z\in K} |f_n(z)|} = \inf_{z\in K} \left| \frac{1}{f_n(z)} \right|.$$

Since the functions $(f_n)_{n\in\mathbb{N}}$ vanish nowhere, by Hurwitz' Theorem we either have $0 \notin f(G)$ or $f \equiv 0$. In the latter case we have just shown that $1/f_n \to \infty$ compactly.

In the former case, we have, again for any compact $K \subset G$, that $m := \min_{z\in K} |f(z)| > 0$ and $\sup_{z\in K} |f_n(z) - f(z)| < \frac{m}{2}$ for all sufficiently large $n \in \mathbb{N}$. Thus, for all $z \in K$

$$\frac{m}{2} > |f(z) - f_n(z)| \geq |f(z)| - |f_n(z)| \geq m - |f_n(z)|$$

and $|f_n(z)| \geq \frac{m}{2}$. We obtain, for large $n \in \mathbb{N}$,

$$\sup_{z\in K} \left| \frac{1}{f_n(z)} - \frac{1}{f(z)} \right| = \sup_{z\in K} \left| \frac{f(z) - f_n(z)}{f(z) f_n(z)} \right| \leq \sup_{z\in K} |f_n(z) - f(z)| \cdot \frac{1}{m} \cdot \frac{2}{m}$$

and therefore, after letting $n \to \infty$, that $1/f_n \to 1/f$ compactly in G. ■

Definition 2.3.5. Let $G \subseteq \mathbb{C}$ be a domain and $\mathscr{F} \subseteq H(G)$, then $\mathscr{F}$ is called:

- *locally bounded*, if for every $w \in G$ there is a neighborhood U of w and a constant $C > 0$ such that $|f(z)| \leq C$ for all $f \in \mathscr{F}$ and $z \in U$.
- *normal* in G, if every sequence in $\mathscr{F}$ has a subsequence which converges compactly in G to some $f \in H(G)$. If the limit ∞ is also permitted, it is instead called $*$-*normal.*

//

These two concepts are equivalent by the following well-known theorem:

Theorem 2.3.6 (Montel)**.** *Let $G \subseteq \mathbb{C}$ be a domain, then a family $\mathscr{F} \subseteq H(G)$ is normal if and only if it is locally bounded.*

The following theorem can be interpreted as a sharpened version of the above:

Theorem 2.3.7 (Fundamental Normality Test)**.** *Let $G \subseteq \mathbb{C}$ be a domain, then any family $\mathscr{F} \subseteq H(G, \mathbb{C} \setminus \{0,1\})$ is $*$-normal in G.*

Proof. We present the proof in three steps:

1. Let $w \in G$, $c > 0$ and $\mathscr{F}_* \subseteq \mathscr{F}$ such that $|f(w)| \leq c$ for all $f \in \mathscr{F}_*$. We aim to show that there is an open disk around w in which $\mathscr{F}_*$ is bounded. Select $t > 0$ such that $B_t(w) \subseteq G$. Let $f \in \mathscr{F}_*$, then $g(z) \coloneqq f(tz + w) \in H(\mathbb{D})$. By the maximum modulus principle and ψ from Schottky's Theorem we obtain

$$\sup_{z \in B_{t/2}(w)} |f(z)| = \sup_{z \in B_{1/2}(0)} |g(z)| \leq \sup_{|z|=1/2} |g(z)| \leq \psi(1/2, c)$$

   and f is bounded on the disk $B_{t/2}(w)$. Since f was arbitrary, $\mathscr{F}_*$ is bounded as well.

2. Fix some $w_* \in G$ and set $\mathscr{F}_1 \coloneqq \{f \in \mathscr{F} : |f(w_*)| \leq 1\}$. We aim to show that $\mathscr{F}_1$ is locally bounded in G. Consider the set

$$U \coloneqq \{w \in G : \mathscr{F}_1 \text{ is bounded in a neighborhood of } w\},$$

   by 1. we have that $w_* \in U$. Note that U is open in G, since if $\mathscr{F}_1$ is bounded in a disk $B_r(w)$, then for any $w' \in B_r(w)$ there is a disk $B_{r'}(w') \subseteq B_r(w)$, on which $\mathscr{F}_1$ is bounded as well.

   Assume towards a contradiction that $U \neq G$, then there exists some $w \in \partial U \cap G$ such that $\mathscr{F}_1$ is unbounded in every neighborhood of w.

   If there were some $c > 0$ such that $|f(w)| \leq c$ for all $f \in \mathscr{F}_1$, then by 1. there would exist an open disk centered at w on which $\mathscr{F}_1$ would be bounded – contradicting our assumption on w. Thus, for every $n \in \mathbb{N}$ we can find some $f_n \in \mathscr{F}_1$ such that $|f_n(w)| \geq n$ and we obtain that $\lim_{n\to\infty} |f_n(w)| = \infty$.

   Since the functions $(f_n)_{n\in\mathbb{N}}$ do not assume 0 or 1 as values, the functions $g_n \coloneqq 1/f_n, n \in \mathbb{N}$ are well-defined, do not assume 0 or 1 as values, and satisfy

$$\lim_{n\to\infty} |g_n(w)| = 0.$$

   In particular, the family $(g_n)_{n\in\mathbb{N}}$ is bounded at w by some constant, thus by 1. the family is bounded in some disk B around w. By Montel's Theorem it is therefore normal in B, and there exists a subsequence $(g_{n_k})_{k\in\mathbb{N}}$ which converges compactly to some $g \in H(B)$. The functions g_{n_k} have no zeros, but $g(w) = 0$; by Hurwitz's

Theorem we therefore have $g \equiv 0$. Then for any $z \in B \cap U$ we have

$$\lim_{k\to\infty} |f_{n_k}(z)| = \lim_{k\to\infty} 1/|g_{n_k}(z)| = \infty,$$

contradicting the assumption that $\mathscr{F}_1$ is bounded in a neighborhood of such z. We thus have $U = G$, therefore $\mathscr{F}_1$ is locally bounded and by Montel's Theorem therefore normal.

3. We can now conclude the proof. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{F}$, we claim that it has some subsequence which converges compactly to some function in $H(G)$ or to ∞.

   If infinitely many f_n lie in $\mathscr{F}_1$, then there is a subsequence $(f_{n_m})_{m\in\mathbb{N}}$ in $\mathscr{F}_1$, which by 2. has a subsequence $(f_{n_{m_k}})_{k\in\mathbb{N}}$ in $\mathscr{F}_1$ which converges compactly in G to some $f \in H(G)$. This sequence is also a subsequence of $(f_n)_{n\in\mathbb{N}}$, concluding the claim in this case.

   On the other hand, if there are only finitely many f_n in $\mathscr{F}_1$, then infinitely many $1/f_n$ lie in $\mathscr{F}_1$. As above, we thus obtain some subsequence in $\mathscr{F}_1$, say $(g_n)_{n\in\mathbb{N}}$, converging compactly in G to some $g \in H(G)$. The sequence $(1/g_n)_{n\in\mathbb{N}}$ is a subsequence of $(f_n)_{n\in\mathbb{N}}$, which – by Lemma 2.3.4 – converges compactly to $1/g$ if $0 \notin g(G)$, and to ∞ otherwise. ■

2.4 Picard's Great Theorem

We are now ready to prove Picard's Great Theorem, a fundamental result in complex analysis. It is not immediately clear how the nature of essential singularities can influence the value distribution of a holomorphic function. The following lemma, however, shows that functions, which omit two values on the punctured unit disk, cannot have an essential singularity at 0.

Lemma 2.4.1. *Let $f \in H(\mathbb{D}^\times; \mathbb{C} \setminus \{0,1\})$. Then f or $1/f$ is bounded in a punctured neighborhood of zero.*

Proof. For $n \in \mathbb{N}$ set $f_n(z) := f(z/n) \in H(\mathbb{D}^\times; \mathbb{C} \setminus \{0,1\})$. By Theorem 2.3.7 the sequence $(f_n)_{n\in\mathbb{N}}$ has a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ that converges compactly to some $f \in H(\mathbb{D}^\times)$ or to ∞.

Assume the former case, then there is some $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have $\|f_{n_k} - f\|_{\partial B_{1/2}(0)} < 1$ and thus

$$\|f\|_{\partial B_{1/(2n_k)}(0)} = \|f_{n_k}\|_{\partial B_{1/2}(0)} \leq \|f_{n_k} - f\|_{\partial B_{1/2}(0)} + \|f\|_{\partial B_{1/2}(0)} \leq 1 + \|f\|_{\partial B_{1/2}(0)},$$

whereby f is bounded on $\partial B_{1/(2n_k)}(0)$ uniformly, with respect to k. By the maximum modulus principle, f therefore must be bounded uniformly on every annulus $1/(2n_{k+1}) \leq$

$|z| \leq 1/(2n_k)$ for $k \geq k_0$. Thus, f is bounded on

$$\bigcup_{k \geq k_0} \left\{ z \in \mathbb{C} : \frac{1}{2n_{k+1}} \leq |z| \leq \frac{1}{2n_k} \right\} = c\ell(B_{1/(2n_k)}(0)) \setminus \{0\},$$

which is a punctured neighborhood of zero.

In the latter case, $(1/f_{n_k})_{k \in \mathbb{N}}$ converges compactly to 0 by Lemma 2.3.4. Replacing f_{n_k} with $1/f_{n_k}$ and f with 0 in the above, we similarly obtain that $1/f$ is bounded in a punctured neighborhood of zero. ∎

Theorem 2.4.2 (Picard's Great Theorem)**.** *Let $G \subseteq \mathbb{C}$ be open, $w \in G$, and suppose $f \in H(G \setminus \{w\})$ such that f has an essential singularity at w. Then f assumes all values in $\mathbb{C}$, with at most one exception, infinitely often in any punctured neighborhood of w.*

Such an exceptional value is also referred to as a *Picard exceptional value.*

Proof. Assume towards a contradiction that f takes on two distinct values $a, b \in \mathbb{C}$ only finitely often in some punctured neighborhood W of w, then W contains a punctured disk of radius $t > 0$ around w, on which f does not assume a or b, and thus

$$g(z) := \frac{f(tz + w) - a}{b - a} \in H(\mathbb{D}^\times; \mathbb{C} \setminus \{0, 1\}),$$

where g has an essential singularity at zero. By Lemma 2.4.1, we have that either g or $1/g$ must be bounded in a punctured neighborhood of zero. Using the classification of isolated singularities, assuming the former case we have that the singularity must be removable, whereas in the latter case it must be a pole, yielding a contradiction in both cases. ∎

The following corollary is also referred to as Picard's Great Theorem. Here an entire transcendental function is an entire function which is not a polynomial.

Corollary 2.4.3. *Every entire transcendental function assumes every value in $\mathbb{C}$ infinitely often, with at most one exception.*

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be transcendental and entire, and consider $g(z) := f(1/z) \in H(\mathbb{C}^\times)$, then the Laurent series expansion of g at $z = 0$ has infinite principal part, therefore g has an essential singularity at zero. By Picard's Great Theorem, g assumes all values in $\mathbb{C}$ on $B_1(0) \setminus \{0\}$ infinitely often, except at most one, and so f does the same on $\mathbb{C} \setminus B_1(0)$. ∎

Remark 2.4.4. Picard's Little Theorem is contained in Picard's Great Theorem:

A non-constant $f \in H(\mathbb{C})$ is either a non-constant polynomial or transcendental. In the latter case, f omits at most one value by Corollary 2.4.3.

In the former case, let $w \in \mathbb{C}$, then $f(z) - w$ has a zero in $\mathbb{C}$ by the Fundamental Theorem of Algebra and hence f even assumes all values. //

3 Growth and Zeros of Entire Functions

Entire functions admit an interesting relationship between the asymptotic growth of their modulus and the number of their zeros. Consider a complex polynomial

$$p(z) = a_0 + a_1 z + \ldots + a_n z^n.$$

The asymptotic growth of p is determined by its degree n, which also corresponds to the number of its zeros. Thus, the higher the number of zeros, the faster the asymptotic growth of p. Another interesting property is that p is uniquely determined by its zeros and some scaling factor, as shown by the Fundamental Theorem of Algebra. Thus, for complex polynomials, their rate of growth and their zeros are closely related.

A function $f \in H(\mathbb{C})$ can be represented as an everywhere convergent power series

$$f(z) = a_0 + a_1 z + \ldots + a_n z^n + \ldots,$$

and thus the entire functions form a natural generalization of the complex polynomials. Motivated by the properties mentioned above, we want to study if similar results hold for entire functions.

3.1 Order and Type of Entire Functions

The first goal is to introduce a growth scale for the asymptotic modulus of an entire function. To characterize this scale we introduce a "maximum modulus function".

Definition 3.1.1. Let $f \in H(\mathbb{C})$, then for $r \geq 0$ we define

$$M_f(r) \coloneqq \max_{|z|=r} |f(z)|. \qquad /\!/$$

Remark 3.1.2. By the maximum modulus principle, M_f is either strictly increasing (if f is non-constant) or constant (otherwise), and we have

$$M_f(r) = \max_{|z|\leq r} |f(z)|.$$

We shall show that M_f is continuous: Let $\varepsilon > 0$ and choose $\delta_\varepsilon > 0$ such that for $|z_1 - z_2| < \delta_\varepsilon$ we have $|f(z_1) - f(z_2)| < \varepsilon$. Now let $r_1 < r_2$ and choose θ such that $M_f(r_2) =$

$|f(r_2e^{i\theta})|$. Then

$$0 \le M_f(r_2) - M_f(r_1) \le |f(r_2e^{i\theta})| - |f(r_1e^{i\theta})| \le |f(r_2e^{i\theta}) - f(r_1e^{i\theta})| < \varepsilon,$$

for $r_2 - r_1 < \delta_\varepsilon$. //

Definition 3.1.3. Let $f \in H(\mathbb{C})$. The *order* of f is defined by

$$\rho_f := \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}. \tag{3.1}$$

Constant functions, by convention, have order 0. //

Note that for $f \in H(\mathbb{C})$ we have $0 \le \rho_f \le \infty$.

Proposition 3.1.4. *If Q is a polynomial of degree $n \in \mathbb{N}_0$, then $\exp Q$ has order n.*

Proof. Write $Q(z) = \sum_{k=0}^n a_k z^k$ and let ζ be an n-th root of $\overline{a_n}/|a_n|$. Then

$$|a_n| - \sum_{k=0}^{n-1} |a_k| r^{k-n} \le \operatorname{Re} \sum_{k=0}^{n} a_k \zeta^k r^{k-n} = \operatorname{Re} \frac{Q(r\zeta)}{r^n} \le \max_{|z|=r} \operatorname{Re} \frac{Q(z)}{r^n} \le |a_n| + \sum_{k=0}^{n-1} |a_k| r^{k-n}$$

and taking limits as $r \to \infty$ we get $\lim_{r\to\infty} \max_{|z|=r} \operatorname{Re} \frac{Q(z)}{r^n} = |a_n|$. Set $f(z) := \exp Q(z)$, then

$$\begin{aligned}\frac{\log\log M_f(r)}{\log r} &= \frac{\log\log \max_{|z|=r} |\exp Q(z)|}{\log r} = \frac{\log\log \max_{|z|=r} \exp \operatorname{Re} Q(z)}{\log r} = \\ &= \frac{\log \max_{|z|=r} \operatorname{Re} Q(z)}{\log r} = \frac{\log \max_{|z|=r} \operatorname{Re} Q(z)/r^n}{\log r} + n\end{aligned}$$

and taking the limit superior as $r \to \infty$ yields $\rho_f = n$. ∎

We have an equivalent characterization.

Proposition 3.1.5. *Let $f \in H(\mathbb{C})$, then*

$$\rho_f = \inf\{s > 0 : M_f(r) = \mathcal{O}(\exp r^s) \text{ as } r \to \infty\}, \tag{3.2}$$

where $\inf \emptyset := \infty$.

Proof. Define ρ as the right-hand side of (3.2), then $\rho \in [0, \infty]$. We claim that if $\rho < \infty$ then $\rho_f \le \rho$ and if $\rho_f < \infty$ then $\rho < \rho_f$. From this it follows immediately that $\rho = \infty$ if and only if $\rho_f = \infty$.

Suppose $0 \le \rho < \infty$, then for all $s > \rho$ we have $M_f(r) = \mathcal{O}(\exp r^s)$ as $r \to \infty$. Thus there exists a constant $K > 0$ (we may assume $K > 1$) and such that for all sufficiently large

$r > 0$ we have $M_f(r) \leq K \exp r^s$. Using the fact that $\log(a+b) = \log a + \log(1 + \frac{b}{a})$ we get

$$\frac{\log\log M_f(r)}{\log r} \leq \frac{\log(r^s + \log K)}{\log r} = s + \frac{\log(1 + (\log K)/r^s)}{\log r}. \tag{$*$}$$

Since

$$0 \leq \frac{\log(1 + (\log K)/r^s)}{\log r} \leq \frac{(\log K)/r^s}{\log r} = \frac{\log K}{r^s \log r} \xrightarrow{r\to\infty} 0,$$

by taking the limit superior as $r \to \infty$ in ($*$) and then letting $s \to \rho$ we obtain $\rho_f \leq \rho$.

Now suppose $0 \leq \rho_f < \infty$ and let $s > \rho_f$. Then, by definition of the limit superior, for all sufficiently large $r > 0$ we have $\log\log M_f(r) \leq s \log r$ and therefore $M_f(r) \leq \exp r^s$. Thus $M_f(r) = \mathcal{O}(\exp r^s)$, thereby $\rho \leq s$ and letting $s \to \rho_f$ yields $\rho \leq \rho_f$, concluding our claim. ■

Remark 3.1.6. Any polynomial is of order zero. Indeed, let $Q(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial and $m \in \mathbb{N}$. Then for any $r > 1$ we have

$$\exp r^{1/m} = \sum_{k=0}^{\infty} \frac{r^{k/m}}{k!} > \frac{r^n}{(mn)!}$$

and therefore

$$M_Q(r) = \max_{|z|=r} |Q(z)| \leq \left(\sum_{k=0}^{n} |a_k|\right) r^n \leq \left((mn)! \sum_{k=0}^{n} |a_k|\right) \exp r^{1/m}.$$

Since $m \in \mathbb{N}$ was arbitrary, Proposition 3.1.5 gives $\rho_Q = 0$.

There are, however, non-polynomial functions of order zero; one may simply construct an entire function with power series coefficients of sufficiently rapid decay. Consider

$$f(z) := \sum_{k=0}^{\infty} \frac{z^k}{(k^2)!}$$

and let $m \in \mathbb{N}$. By the above, we have $\sum_{k=0}^{m-1} \frac{r^k}{(k^2)!} \leq K \exp r^{1/m}$ for some constant $K > 0$ depending only on m, holding for all $r > 1$. Thus,

$$\sum_{k=0}^{\infty} \frac{r^k}{(k^2)!} \leq K \exp r^{1/m} + \sum_{k=m}^{\infty} \frac{r^k}{(km)!} \leq K \exp r^{1/m} + \sum_{k=0}^{\infty} \frac{r^{k/m}}{k!} = (K+1) \exp r^{1/m}$$

and we obtain $\rho_f = 0$, as above. //

Remark 3.1.7. If $f \in H(\mathbb{C})$ then the order of f can determined via the power series coefficients. Namely, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$\rho_f = \limsup_{n\to\infty} \frac{n \log n}{\log \frac{1}{|a_n|}},$$

where in the case of $a_n = 0$ the entire quotient is taken to be zero. This result can be found in Segal [5] as Theorem 3.2.1. //

From Proposition 3.1.5, together with some rough estimates, it follows that the order of the sum or product of two entire functions is bounded above by the larger of the two:

Proposition 3.1.8. *Let $f, g \in H(\mathbb{C})$ be of finite order. Then it holds that:*

i. $\rho_{f+g} \le \max\{\rho_f, \rho_g\}$.

ii. $\rho_{fg} \le \max\{\rho_f, \rho_g\}$.

For two entire functions of different, finite order, their sum has the order of the higher of the two:

Proposition 3.1.9. *Let $f, g \in H(\mathbb{C})$ be of finite order. If $\rho_f < \rho_g$, then $\rho_{f+g} = \rho_g$.*

Proof. By Proposition 3.1.8 we have

$$\rho_{f+g} \le \max\{\rho_f, \rho_g\} = \rho_g, \quad \text{and} \quad \rho_g = \rho_{f+g+(-f)} \le \max\{\rho_f, \rho_{f+g}\} = \rho_{f+g},$$

since if $\rho_f > \rho_{f+g}$ then $\rho_f > \rho_{f+g} \ge \rho_g$ would be a contradiction. Thus $\rho_{f+g} = \rho_g$, proving the assertion. ■

Proposition 3.1.9 implies that the order of an entire function of finite order remains unchanged when adding a polynomial of arbitrary degree to it.

There is a connection between the order of an entire function and the order of its derivative:

Proposition 3.1.10. *If $f \in H(\mathbb{C})$ is of finite order, then $\rho_{f'} = \rho_f$.*

Proof. Without loss of generality we may assume $f(0) = 0$. If f is a polynomial the assertion is clear. Otherwise for any $r > 0$ we have,

$$\begin{aligned} M_f(r) = \max_{|z|=r} |f(z)| = \max_{|z|=r} \left| \int_{[0,z]} f'(\zeta)\, \mathrm{d}\zeta \right| \le \\ \le \max_{|z|=r} \left(|z| \max_{w \in c\ell(B_r(0))} |f'(w)| \right) \le r M_{f'}(r). \end{aligned}$$

Thus

$$\begin{aligned}\frac{\log\log M_f(r)}{\log r} &\leq \frac{\log\log rM_{f'}(r)}{\log r} = \frac{\log\log M_{f'}(r)}{\log r} + \frac{\log(1+\log r/\log M_{f'}(r))}{\log r} \leq \\ &\leq \frac{\log\log M_{f'}(r)}{\log r} + \frac{\log r/\log M_{f'}(r)}{\log r} = \frac{\log\log M_{f'}(r)}{\log r} + \frac{1}{\log M_{f'}(r)}\end{aligned}$$

and taking limits superior as $r \to \infty$ yields $\rho_f \leq \rho_{f'}$.

On the other hand, Cauchy's integral formula gives

$$\begin{aligned}M_{f'}(r) &= \max_{z\in\partial B_r(0)} |f'(z)| = \max_{z\in\partial B_r(0)} \left|\frac{1}{2\pi i}\oint_{\partial B_1(z)} \frac{f(\zeta)}{(\zeta-z)^2}\,\mathrm{d}\zeta\right| \leq \\ &\leq \max_{\substack{z\in\partial B_r(0)\\ w\in\partial B_1(z)}} |f(w)| \leq \max_{z\in c\ell(B_{r+1}(0))} |f(z)| = M_f(r+1).\end{aligned}$$

Therefore

$$\frac{\log\log M_{f'}(r)}{\log r} \leq \frac{\log\log M_f(r+1)}{\log(r+1)}\frac{\log(r+1)}{\log r}$$

and, again, taking limits superior as $r \to \infty$ we get $\rho_{f'} \leq \rho_f$. ■

For functions of finite and positive order, we can obtain a natural refinement of the concept of order:

Definition 3.1.11. Let $f \in H(\mathbb{C})$ be of order $0 < \rho_f < \infty$. The *type* of f is defined by

$$\tau_f \coloneqq \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}}. \tag{3.3}$$

If $\tau_f = 0$, then f is said to be of *minimal* type, if $0 < \tau_f < \infty$, of *normal* type, and if $\tau_f = \infty$, of *maximal* type. //

Once again, we have an equivalent characterization for functions of finite type:

Proposition 3.1.12. *Let $f \in H(\mathbb{C})$ be of finite, positive order, then*

$$\tau_f = \inf\{t > 0 : M_f(r) = \mathcal{O}(\exp(tr^{\rho_f})) \text{ as } r \to \infty\}, \tag{3.4}$$

where $\inf\emptyset \coloneqq \infty$.

Proof. We follow the same strategy as in the proof of Proposition 3.1.5: Define τ as the right-hand side of (3.2), then $\tau \in [0,\infty]$. We claim that if $\tau < \infty$ then $\tau_f \leq \tau$ and if $\tau_f < \infty$ then $\tau < \tau_f$. Again, from this it follows immediately that $\tau = \infty$ if and only if $\tau_f = \infty$.

Suppose $0 \leq \tau < \infty$, then for all $t > \tau$ we have $M_f(r) = \mathcal{O}(\exp tr^{\rho_f})$ as $r \to \infty$. Thus there exists a constant $K > 0$ such that for all sufficiently large $r > 0$ we have $M_f(r) \leq K \exp tr^{\rho_f}$. Therefore

$$\frac{\log M_f(r)}{r^{\rho_f}} \leq \frac{\log K + tr^{\rho_f}}{r^{\rho_f}} = \frac{\log K}{r^{\rho_f}} + t$$

and taking limits superior as $r \to \infty$ and letting $t \to \tau$ afterwards yields $\tau_f \leq \tau$.

Now suppose $0 \leq \tau_f < \infty$ and let $t > \tau_f$. Then, by definition of the limit superior, for all sufficiently large $r > 0$ we have $\log M_f(r) \leq tr^{\rho_f}$ and therefore $M_f(r) \leq \exp(tr^{\rho_f})$. Thus $M_f(r) = \mathcal{O}(\exp(tr^{\rho_f}))$, thereby $\tau \leq t$ and letting $t \to \tau_f$ yields $\tau \leq \tau_f$, concluding the claim. ∎

If $f \in H(\mathbb{C})$ is of finite, positive order, then f and f' not only have the same order, they have the same type:

Proposition 3.1.13. *If $f \in H(\mathbb{C})$ is of finite, positive order and finite type, then $\tau_{f'} = \tau_f$.*

Proof. Reusing the inequalities obtained in the proof of Proposition 3.1.10 we have, for all $r > 0$,

$$M_f(r) \leq rM_{f'}(r), \quad \text{and} \quad M_{f'}(r) \leq M_f(r+1).$$

Therefore

$$\frac{\log M_f(r)}{r^{\rho_f}} \leq \frac{\log r + \log M_{f'}(r)}{r^{\rho_f}}, \quad \text{and} \quad \frac{\log M_{f'}(r)}{r^{\rho_f}} \leq \frac{\log M_f(r+1)}{(r+1)^{\rho_f}} \frac{(r+1)^{\rho_f}}{r^{\rho_f}}$$

and, since by Proposition 3.1.10 $\rho_f = \rho_{f'}$, taking limits superior as $r \to \infty$ yields $\tau_f \leq \tau_{f'}$ and $\tau_{f'} \leq \tau_f$. ∎

Example 3.1.14.

- We have already seen in Remark 3.1.6 that polynomials are of order zero.
- For $\rho, \tau \in (0, \infty)$, the function

  $$f(z) := \exp(\tau z^\rho)$$

  is of order ρ and type τ.
- The function

  $$f(z) := \exp \exp z$$

  is of infinite order. For any $m \in \mathbb{N}$

  $$M_f(r) \geq \exp \exp r > \exp(r^m/m!),$$

  therefore $\rho_f \geq m$, and since m was arbitrary it follows that $\rho_f = \infty$.

$\rightsquigarrow$ The functions defined by

$$\sum_{n=2}^{\infty} \left(\frac{\log n}{n} \right)^n z^n, \quad \text{and} \quad \sum_{n=2}^{\infty} \left(\frac{1}{n \log n} \right)^n z^n$$

are of order 1 and of maximal and minimal type, respectively. This can be seen via Theorem 2.1 in Chapter 3 of Segal [5]. //

3.2 Order, Density and Exponent of Convergence of Zeros

We first recall a rather explicit connection between the moduli of the zeros of a holomorphic function and the modulus of the function itself:

Theorem 3.2.1 (Jensen). *Let $R > 0$, $f \in H(B_R(0))$ with $f(0) \neq 0$ and let $r_1, r_2, \ldots$ denote the moduli of the zeros of f in $B_R(0)$, repeated by multiplicity, arranged in a non-decreasing sequence. Then, for $r_n < r < r_{n+1}$, we have*

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, \mathrm{d}\theta = \log |f(0)| + \log \frac{r^n}{r_1 \ldots r_n}. \tag{3.5}$$

Definition 3.2.2. Let $f \in H(B_R(0))$ be not identically zero. Then, for $0 < r < R$, we define

$$n_f(r) := |\{z \in c\ell(B_r(0)) : f(z) = 0\}|, \tag{3.6}$$

that is, the number of zeros of f in $c\ell(B_r(0))$. //

This zero-counting function is, in some scenarios, simpler to work with than the sequence of zeros. For instance, it can be used to obtain an equivalent version of Jensen's Theorem:

Corollary 3.2.3. *Let $f \in H(B_R(0))$ with $f(0) \neq 0$. Then, for $0 < r < R$, we have*

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, \mathrm{d}\theta = \log |f(0)| + \int_0^r \frac{n_f(s)}{s} \, \mathrm{d}s \tag{3.7}$$

Proof. Let $r_1, r_2, \ldots$ denote the moduli of the zeros of f in $B_R(0)$, repeated by multiplicity, arranged in a non-decreasing sequence. Then, for any $r_n < r < r_{n+1}$, we obtain

$$\begin{aligned} \log \frac{r^n}{r_1 \ldots r_n} &= \sum_{k=1}^{n} \log \frac{r}{r_k} = \sum_{k=1}^{n} \int_{r_k}^{r} \frac{1}{s} \, \mathrm{d}s = \\ &= \sum_{k=1}^{n} \int_0^r \mathbb{1}_{(r_k, \infty)}(s) \frac{1}{s} \, \mathrm{d}s = \int_0^r \left(\sum_{k=1}^{n} \mathbb{1}_{(r_k, \infty)}(s) \right) \frac{1}{s} \, \mathrm{d}s = \\ &= \int_0^r \frac{n_f(s)}{s} \, \mathrm{d}s \end{aligned}$$

and Theorem 3.2.1 establishes the assertion. ■

This gives an immediate connection between the zeros and the growth of the modulus of an entire function:

Lemma 3.2.4. *If $f \in H(\mathbb{C})$, then*

$$n_f(r) = \mathcal{O}(\log M_f(er)).$$

Proof. We may assume $|f(0)| = 1$. Let $r > 0$, then since n_f is non-negative and non-decreasing we have

$$n_f(r) \leq \int_r^{er} \frac{n_f(t)}{t} \,\mathrm{d}t \leq \frac{1}{2\pi} \log|f(ere^{i\theta})| \,\mathrm{d}\theta \leq \log M_f(er),$$

establishing the assertion. ■

Lemma 3.2.4 shows that the more zeros a function f has, the faster M_f must grow as $r \to \infty$. The converse is naturally false; by composing exponentials one can obtain an entire function f for which M_f grow arbitrarily fast, yet f has no zeros.

Definition 3.2.5. Let $M \subseteq \mathbb{N}$ and $\mathbf{z} = (z_m)_{m \in M}$ be a family of points[1] in $\mathbb{C}^\times$. Then

$$\lambda_{\mathbf{z}} := \inf \left\{ \lambda > 0 : \sum_{m \in M} \frac{1}{|z_m|^\lambda} < \infty \right\}$$

is called the *exponent of convergence* of the family $\mathbf{z}$.

Let $f \in H(\mathbb{C})$, then the *exponent of convergence of the zeros of f* is defined as the exponent of convergence of the non-zero roots of f, repeated by multiplicity, and is denoted λ_f. If f is constant, we set $\lambda_f = 0$ by convention.

Furthermore, for any $a \in \mathbb{C}$, the *exponent of convergence of the a-points of f* is defined as exponent of convergence of the zeros of $f(z) - a$ and will be denoted $\lambda_f^{(a)}$. //

If $f \in H(\mathbb{C})$ assumes some value $a \in \mathbb{C}$ only finitely often, then clearly $\lambda_f^{(a)} = 0$.

Theorem 3.2.6. *If $f \in H(\mathbb{C})$ is of finite order, then $\lambda_f \leq \rho_f$.*

Proof. We may assume $f(0) \neq 0$. If $\lambda_f = 0$ there is nothing to show. Let $\rho > \rho_f$ then by Lemma 3.2.4 and Proposition 3.1.5 we have $n_f(r) = \mathcal{O}(\log M_f(er))$ and $M_f(r) = \mathcal{O}(\exp r^\rho)$, respectively. Therefore $n_f(r) = \mathcal{O}(r^\rho)$, thus there exists a constant $K_1 > 0$ such that for sufficiently large $r > 0$ we have $n_f(r) \leq K_1 r^\rho$.

[1] Note that the exponent of convergence is only affected by their moduli, not by their arguments.

If $(r_j)_{j\in\mathbb{N}}$ denote the non-zero moduli of the zeros of f, repeated by multiplicity, arranged in non-decreasing order, then for any $m \in \mathbb{N}$ we have $m \leq n_f(r_m) \leq K_1 r_m^\rho$. Let $0 < \lambda < \lambda_f$ then this implies

$$\left(\frac{1}{r_m}\right)^\lambda \leq K_2 \left(\frac{1}{m}\right)^{\lambda/\rho}.$$

for some constant $K_2 > 0$. Therefore

$$\sum_{m=1}^{\infty} \left(\frac{1}{r_m}\right)^\lambda \leq K_2 \sum_{m=1}^{\infty} \left(\frac{1}{m}\right)^{\lambda/\rho}.$$

The left-hand side diverges by Definition 3.2.5, thus so does the right-hand side. But the latter diverges if and only if $\frac{\lambda}{\rho} \leq 1$, therefore letting $\lambda \nearrow \lambda_f$ and then $\rho \searrow \rho_f$ yields $\lambda_f \leq \rho_f$. ■

Remark 3.2.7. The function $f(z) := \exp z$ is of order one and has no zeros – thus we observe that we may have $\lambda_f < \rho_f$ in some cases. //

A more precise analysis of the density of the zeros of an entire function f is contained in the growth of the zero-counting function n_f. We therefore introduce the following definitions:

Definition 3.2.8. Let $f \in H(\mathbb{C})$. As the *order* of the zero-counting function n_f we define

$$\nu_f := \limsup_{r\to\infty} \frac{\log n_f(r)}{\log r}. \tag{3.8}$$

If ν_f is finite, we additionally define the *upper density* of the zeros of f as

$$\Delta_f := \limsup_{r\to\infty} \frac{n_f(r)}{r^{\nu_f}}. \tag{3.9}$$

If the limit exists, then Δ_f is simply called the *density.* //

The following lemma provides great utility in further analyzing the function n_f.

Lemma 3.2.9. *Let $f \in H(\mathbb{C})$ and denote by $(r_m)_{m\in M}$ the moduli of the zeros of f, repeated by multiplicity, arranged in non-decreasing order. Then f induces a unique Lebesgue-Stieltjes measure ω_f on $(0,\infty)$, such that, for any non-negative, measureable function φ on $(0,\infty)$, it holds that*

$$\sum_{m\in M} \varphi(r_m) = \int_{(0,\infty)} \varphi \,\mathrm{d}\omega_f.$$

If in addition φ is monotone and continuously differentiable on $(0,\infty)$, then for all $r > 0$ it holds that

$$\int_{(0,r)} \varphi \,\mathrm{d}\omega_f = n_f(r)\varphi(r) - \int_0^r n_f(t)\varphi'(t)\,\mathrm{d}t.$$

Proof. From the representation

$$n_f(r) = \sum_{m \in M} \mathbb{1}_{[r_m,\infty)}(r)$$

we observe that n_f is right-continuous, non-negative and non-decreasing. Therefore n_f induces a unique Lebesgue–Stieltjes measure ω_f. Since an indicator function of the form $\mathbb{1}_{[a,\infty)}$, for $a \in \mathbb{R}$, corresponds to a Dirac measure $\delta_{\{a\}}$, and the correspondence between Lebesgue-Stieltjes measures and their distribution functions is linear in nature, we have

$$\omega = \sum_{m \in M} \delta_{\{r_m\}}.$$

From this it follows that

$$\sum_{m \in M} \varphi(r_m) = \sum_{m \in M} \int_{(0,\infty)} \varphi \, \mathrm{d}\delta_{\{r_m\}} = \int_{(0,\infty)} \varphi \, \mathrm{d} \left(\sum_{m \in M} \delta_{\{r_m\}} \right) = \int_{(0,\infty)} \varphi \, \mathrm{d}\omega_f.$$

Assume φ is monotone and continuously differentiable on $(0, \infty)$. Then φ also induces a Lebesgue–Stieltjes measure η_φ, with density φ' with respect to the Lebesgue measure. Thus, for any $0 < \varepsilon < r < \infty$, integration by parts gives

$$\int_{(\varepsilon,r)} \varphi \, \mathrm{d}\omega_f + \int_{(\varepsilon,r)} n_f \, \mathrm{d}\eta_\varphi = n_f(r)\varphi(r) - n_f(\varepsilon)\varphi(\varepsilon).$$

Since $\kappa := \min_{m \in M} r_m > 0$ we have $n_f(s) = 0$ for $0 < s < \kappa$. Letting $\varepsilon \to 0$ and using the density of η_φ therefore yields

$$\int_{(0,r)} \varphi \, \mathrm{d}\omega_f + \int_0^r n_f(t)\varphi'(t) \, \mathrm{d}t = n_f(r)\varphi(r)$$

and rearranging terms provides the desired result. ■

Proposition 3.2.10. *Let $f \in H(\mathbb{C})$, then λ_f is finite if and only if ν_f is finite and in either case we have $\lambda_f = \nu_f$.*

Proof. If f only has finitely many zeros the assertion is clear. Let $\lambda > \lambda_f$ and $r > 0$. Invoking Lemma 3.2.9 with $\varphi(t) := t^{-\lambda}$ we get

$$\int_{(0,r)} \frac{1}{t^\lambda} \, \mathrm{d}\omega_f(t) = \frac{n_f(r)}{r^\lambda} + \lambda \int_0^r \frac{n_f(t)}{t^{\lambda+1}} \, \mathrm{d}t.$$

Since $\lambda > \lambda_f$, the same lemma implies that the left integral converges as $r \to \infty$. In particular, the right-hand side must be bounded. Since the right integrand is non-negative, the right-most term is increasing in r and must therefore converge as $r \to \infty$. Therefore the right-most term in

$$0 \leq \frac{n_f(r)}{r^\lambda} = \lambda n_f(r) \int_r^\infty \frac{1}{t^{\lambda+1}} \,\mathrm{d}t \leq \lambda \int_r^\infty \frac{n_f(t)}{t^{\lambda+1}} \,\mathrm{d}t$$

converges to 0 as $r \to \infty$, from which we get

$$\lim_{r\to\infty} \frac{n_f(r)}{r^\lambda} = 0. \tag{3.10}$$

Taking limits superior as $r \to \infty$ in

$$\frac{\log n_f(r)}{\log r} = \left(\log \frac{n_f(r)}{r^\lambda} + \log r^\lambda\right) /\log r = \log \frac{n_f(r)}{r^\lambda} / \log r + \lambda$$

and then letting $\lambda \searrow \lambda_f$ yields $\nu_f \leq \lambda_f$.

Conversely suppose $\nu_f < \infty$ and let $\varepsilon > 0$. By the definition of ν_f and the nature of the limit superior there is an $r_0 > 0$ such that

$$n_f(r) \leq r^{\nu_f+\varepsilon}$$

for all $r \geq r_0$. Set $\lambda \coloneqq \nu_f + 2\varepsilon$, then

$$\int_{r_0}^r \frac{n_f(t)}{t^{\lambda+1}} \,\mathrm{d}t \leq \int_{r_0}^r \frac{t^{\nu_f+\varepsilon}}{t^{\nu_f+2\varepsilon+1}} \,\mathrm{d}t = \int_{r_0}^r t^{-\varepsilon-1} \,\mathrm{d}t = \frac{1}{\varepsilon r_0^\varepsilon} - \frac{1}{\varepsilon r^\varepsilon}$$

and since the right-hand side remains finite as $r \to \infty$ therefore the leftmost integral converges as $r \to \infty$. Clearly (3.10) holds for our λ, therefore Lemma 3.2.9 implies the convergence of

$$\int_{(0,\infty)} \frac{1}{t^\lambda} \,\mathrm{d}\omega_f(t) = \sum_{m\in M} \frac{1}{r_m^\lambda},$$

yielding $\lambda_f \leq \lambda$, and letting $\varepsilon \to 0$ results in $\lambda_f \leq \nu_f$. ∎

3.3 Canonical Products

By the Fundamental Theorem of Algebra every complex polynomial can be written as a product of linear factors involving only its zeros and some scaling factor. Conversely, for a finite sequence of points $z_1, \dots, z_n$ the polynomial $p(z) \coloneqq \prod_{k=1}^n (z - z_k)$ vanishes only at said points.

The question arises if one can construct an entire function that vanishes precisely at the points of some infinite sequence $(z_k)_{k\in\mathbb{N}}$ and nowhere else. Clearly the product $\prod_{k=1}^\infty (z - z_k)$ need not be convergent in the entire complex plane. This issue is solved by introducing additional exponential factors, which improve convergence without introducing new zeros. We first study sufficient conditions for convergence of infinite products of numbers and holomorphic functions.

The early parts of the section will closely follow Stein and Shakarchi [6], while the later estimates and results will follow Levin [3].

Lemma 3.3.1. *Suppose $G \subseteq \mathbb{C}$ is open and let $(f_k)_{k\in\mathbb{N}}$ be a sequence of non-vanishing functions in $H(G)$. If $\sum_{k=1}^{\infty}(1-f_k(z))$ converges absolutely compactly in G, then $\prod_{k=1}^{\infty} f_k(z)$ converges compactly in G to a non-vanishing $f \in H(G)$.*

Proof. Let $K \subset G$ be compact, then the absolute, compact convergence of the sum implies that there is a k_0 such that $|1 - f_k(z)| \leq 1/2$ for $z \in K, k > k_0$. For $|w| \leq 1/2$ we have

$$|\log(1-w)| \leq \sum_{k=1}^{\infty} \left|\frac{1}{k} w^k\right| \leq |w| \sum_{k=0}^{\infty} |w|^k = \frac{|w|}{1-|w|} \leq 2|w|.$$

Therefore

$$\sum_{k=1}^{\infty} |\log f_k(z)| = \sum_{k=1}^{\infty} |\log(1-(1-f_k(z)))| \leq \sum_{k=1}^{k_0-1} |\log f_k(z)| + 2 \sum_{k=k_0}^{\infty} |1-f_k(z)|$$

and since the rightmost sum converges compactly it follows that $\sum_{k=1}^{\infty} \log f_k(z)$ converges absolutely compactly in G. Since the exponential function is uniformly continuous, the sequence

$$\exp\left(\sum_{k=k_0}^{n} \log f_k(z)\right) = \prod_{k=k_0}^{n} f_k(z), \quad n \in \mathbb{N}$$

converges absolutely compactly to some $f \in H(G)$. Again by continuity of the exponential function we have

$$f(z) = \prod_{k=1}^{\infty} f_k(z) = \exp\left(\sum_{k=1}^{\infty} \log f_k(z)\right) \neq 0$$

and thus f is non-vanishing. ■

Definition 3.3.2. The *canonical factors* are defined at $z \in \mathbb{C}$ by

$$W(z;0) := 1-z, \quad \text{and} \quad W(z;n) := (1-z)\exp\left(\sum_{k=1}^{n} \frac{z^k}{k}\right), \quad n \in \mathbb{N}.$$

The number n is called the *degree* of the canonical factor. //

Lemma 3.3.3. *Let $n \in \mathbb{N}_0$ and $|z| \leq 1/2$, then $|1 - W(z;n)| \leq 2e|z|^{n+1}$.*

Proof. For $n = 0$ there is nothing to show. Note that for such given z we can expand $\log(1-z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}$, therefore

$$W(z;n) = \exp\left(\log(1-z) + \sum_{k=1}^{n} \frac{z^k}{k}\right) = \exp\zeta,$$

where $\zeta := -\sum_{k=n+1}^{\infty} z^k/k$. Furthermore, we have that

$$|\zeta| \le |z|^{n+1} \sum_{k=n+1}^{\infty} |z|^{k-n-1}/k \le |z|^{n+1} \sum_{j=0}^{\infty} 2^{-j} \le 2|z|^{n+1}$$

and in particular $|\zeta| \le 1$. We conclude

$$|1 - W(z;n)| = |\exp \zeta - 1| \le \sum_{k=1}^{\infty} \frac{|\zeta|^k}{k!} \le |\zeta| \sum_{k=0}^{\infty} \frac{1}{k!} \le 2e|z|^{n+1}.$$

■

The next theorem – which is also known as the *Weierstraß Product Theorem* – now constructively asserts the existence of entire function with prescribed zeros.

Theorem 3.3.4. *Let $M \subseteq \mathbb{N}$ and $\mathbf{z} = (z_m)_{m\in M}$ be a family in $\mathbb{C}^\times$ without accumulation points. Then there is a family $(p_m)_{m\in M}$ in $\mathbb{N}_0$ such that*

$$\Pi(z) := \prod_{m\in M} W\left(\frac{z}{z_m}; p_m\right) \tag{3.11}$$

is an entire function such that if $z_m, m \in M$ occurs in $\mathbf{z}$ exactly n times, then Π has a zero of order n at z_m. If

$$\mu := \min\left\{p \in \mathbb{N}_0 : \sum_{m\in M} \frac{1}{|z_m|^{p+1}} < \infty\right\} \tag{3.12}$$

is finite, then one can take $p_m = \mu, m \in M$.

Proof. If M is finite there is nothing to show. We may thus assume that M is infinite and without loss of generality $M = \mathbb{N}$. Thus $\mathbf{z}$ is a sequence, which we can assume to be arranged in non-decreasing order according to the moduli its elements.

We claim that there exists a sequence $(p_n)_{n\in\mathbb{N}}$ in $\mathbb{N}_0$ such that

$$\sum_{n=1}^{\infty} \left|\frac{z}{z_n}\right|^{p_n+1} \tag{3.13}$$

converges compactly in $\mathbb{C}$. Let $K \subset B_R(0) \subset \mathbb{C}$ be compact. Note that since $\mathbf{z}$ has no accumulation points we necessarily have $\lim_{n\to\infty} |z_n| = \infty$, thus there is some n_0 such that for $n > n_0$ we have $|z_n| \ge R + 1$. Therefore, for any $z \in K$,

$$\sum_{n=1}^{\infty} \left|\frac{z}{z_n}\right|^{p_n+1} \le \sum_{n=1}^{n_0-1} \left|\frac{z}{z_n}\right|^{p_n+1} + \sum_{n=n_0}^{\infty} \left(\frac{R}{R+1}\right)^{p_n+1},$$

and the rightmost series is finite if, for example, $p_n = n, n \in \mathbb{N}$. Therefore the series is uniformly bounded and thus compactly convergent, concluding our claim. Note that if μ

is finite then we can take $p_n = \mu, n \in \mathbb{N}$, since for $z \in K$

$$\sum_{n=1}^{\infty} \left| \frac{z}{z_n} \right|^{\mu+1} \leq |z|^{\mu} \sum_{n=1}^{\infty} \frac{1}{|z_n|^{\mu+1}} \leq R^{\mu} \sum_{n=1}^{\infty} \frac{1}{|z_n|^{\mu+1}},$$

where the rightmost sum is a finite constant by the definition of μ.

It remains to show that is Π is an entire function that satisfies the desired properties.Let $R > 0$, then it suffices to show that Π has is holomorphic in $B_R(0)$ and vanishes only at points of $\mathbf{z}$ that are also in $B_R(0)$. We write

$$\Pi(z) = \left(\prod_{\substack{n=1 \\ |z_n|<2R}}^{\infty} W\left(\frac{z}{z_n}; p_n\right) \right) \left(\prod_{\substack{n=1 \\ |z_n|\geq 2R}}^{\infty} W\left(\frac{z}{z_n}; p_n\right) \right) =: \Pi_1(z)\Pi_2(z).$$

Clearly Π_1 is a finite product which vanishes at points of $\mathbf{z}$ in $B_R(0)$ and nowhere else. Note that for $|z_m| \geq 2R$ we have $|z/z_m| \leq R/|z_m| \leq 1/2$, hence by Lemma 3.3.3 we have

$$\sum_{\substack{n=1 \\ |z_n|\geq 2R}}^{\infty} \left| 1 - W\left(\frac{z}{z_n}; p_n\right) \right| \leq 2e \sum_{\substack{n=1 \\ |z_n|\geq 2R}}^{\infty} \left| \frac{z}{z_n} \right|^{p_n+1}$$

and the compact convergence of the right sum, which stems from our choice of the $p_n, n \in \mathbb{N}$, implies compact convergence of the left sum. By Lemma 3.3.1 we thereby have that $\Pi_2 \in H(B_R(0))$ and vanishes nowhere. Therefore Π has the desired properties and, since R was arbitrary, this concludes the proof. ∎

Definition 3.3.5. With the notation of Theorem 3.3.4, Π is called a *Weierstraß product*[2] formed from the family $\mathbf{z}$. If $\mu < \infty$, then

$$\Pi(z) := \prod_{m\in M} W\left(\frac{z}{z_m}; \mu\right) \tag{3.14}$$

is called a *(Weierstraß) canonical product* and μ is called the *genus* of the canonical product, which will also be denoted as μ_Π. Otherwise the genus of Π is said to be infinite. //

Remark 3.3.6. By comparing Definition 3.2.5 and Definition 3.3.5 we immediately observe that if $\lambda_\Pi < \infty$, then $\mu_\Pi \leq \lambda_\Pi \leq \mu_\Pi + 1$. If therefore λ_Π is an integer, and Π has infinitely many zeros, the series

$$\sum_{m\in M} \frac{1}{|z_m|^{\lambda}},$$

will converge if $\lambda_\Pi = \mu_\Pi + 1$ and diverge if $\lambda_\Pi = \mu_\Pi$. //

Just as any polynomial decomposes into linear factors involving their zeros by the Funda-

[2]Note that this product is not uniquely determined, since the sequence $(p_m)_{m\in M}$ is not unique.

mental Theorem of Algebra, Weierstraß products allow us to obtain a similar factorization result for arbitrary entire functions, known as the *Weierstraß Factorization Theorem.*

Theorem 3.3.7 (Weierstraß)**.** *Let $f \in H(\mathbb{C})$ be not identically zero and let $m \in \mathbb{N}_0$ be the order of the zero of f at the origin. Then there exists $g \in H(\mathbb{C})$ such that*

$$f(z) = z^m e^{g(z)} \Pi(z), \tag{3.15}$$

where Π denotes a Weierstraß product formed from the zeros of f.

Proof. Since $f(z)$ and $z^m \Pi(z)$ have the same zeros with identical multiplicities the function

$$\varphi(z) := \frac{f(z)}{z^m \Pi(z)}$$

is entire and vanishes nowhere. Therefore we can write

$$\varphi(z) = e^{g(z)}$$

for some $g \in H(\mathbb{C})$ and rearranging terms yields the desired representation. ■

Definition 3.3.8. In the context of Weierstraß' Theorem, if Π is of finite genus and g is a polynomial, then the *genus of the entire function f* is defined as

$$\mu_f := \max\{\mu_\Pi, \deg g\}. \tag{3.16}$$

If g is not a polynomial or if Π is of infinite genus, then f is said to be of infinite genus. //

We wish to further estimate canonical products. To do this we first focus on the canonical factors.

Lemma 3.3.9. *For $p \in \mathbb{N}$ and all $z \in \mathbb{C}$ we have*

$$\log|W(z;p)| < A_p \frac{|z|^{p+1}}{1+|z|}, \quad \textit{where} \quad A_p := 3e(2 + \log p). \tag{3.17}$$

For $p = 0$ we have

$$\log|W(z;0)| \le \log(1+|z|). \tag{3.18}$$

Proof. The second assertion is clear since

$$\log|W(z;0)| = \log|1-z| \le \log(1+|z|).$$

If $|z| \leq \frac{p}{p+1}$ then in particular we have $|z| < 1$. Since furthermore $1 + |z| < 2 < A_p$ we get

$$\begin{aligned}
\log|W(z;p)| &= \log\left|(1-u)\exp\left(\sum_{k=1}^{p}\frac{z^k}{k}\right)\right| = \operatorname{Re}\left(\log(1-u) + \sum_{k=1}^{p}\frac{z^k}{k}\right) = \\
&= -\operatorname{Re}\sum_{k=p+1}^{\infty}\frac{z^k}{k} \leq \sum_{k=p+1}^{\infty}\frac{|z|^k}{k} < \frac{|z|^{p+1}}{(p+1)(1-|z|)} \leq |z|^{p+1} \leq \\
&\leq A_p\frac{|z|^{p+1}}{1+|z|}.
\end{aligned}$$

If otherwise $|z| > \frac{p}{p+1}$ then we have $\frac{1}{|z|} < 1 + \frac{1}{p}$ and $2|z| > 1$. From there we conclude

$$\begin{aligned}
\log|W(z;p)| &= \log\left|(1-z)\exp\left(\sum_{k=1}^{p}\frac{z^k}{k}\right)\right| \leq \log(1+|z|) + \sum_{k=1}^{p}\frac{|z|^k}{k} \leq \\
&\leq 2|z| + \sum_{k=2}^{p}\frac{|z|^k}{k} = |z|^p\left(2\left|\frac{1}{z}\right|^{p-1} + \sum_{k=2}^{p}\frac{1}{k}\left|\frac{1}{z}\right|^{p-k}\right) < \\
&< |z|^p\left(2\left(1+\frac{1}{p}\right)^{p-1} + \sum_{k=2}^{p}\frac{1}{k}\left(1+\frac{1}{p}\right)^{p-k}\right) < \\
&< |z|^p\left(1+\frac{1}{p}\right)^p\left(2 + \sum_{k=1}^{p}\frac{1}{k}\right) < |z|^p e\left(2 + \int_1^p \frac{1}{t}\,\mathrm{d}t\right) = \\
&= |z|^p e\,(2+\log p) = \frac{1+|z|}{1+|z|}|z|^p e\,(2+\log p) < \\
&< \frac{3|z|}{1+|z|}|z|^p e\,(2+\log p) = A_p\frac{|z|^{p+1}}{1+|z|}.
\end{aligned}$$

■

Lemma 3.3.10. *Let $M \subseteq \mathbb{N}$ and $(z_m)_{m\in M}$ be a family in $\mathbb{C}^\times$ without accumulation points. If for given $p \in \mathbb{N}_0$ it holds that*

$$\sum_{m\in M}\frac{1}{|z_m|^{p+1}} < \infty,$$

then the entire function

$$\Pi(z) := \prod_{m\in M} W\left(\frac{z}{z_m};p\right)$$

satisfies

$$\log|\Pi(z)| < k_p r^p\left(\int_0^r \frac{n_\Pi(t)}{t^{p+1}}\,\mathrm{d}t + r\int_r^\infty \frac{n_\Pi(t)}{t^{p+2}}\,\mathrm{d}t\right), \tag{3.19}$$

for all $z \in \mathbb{C}^\times$ and $r = |z|$, where

$$k_0 := 1, \quad \textit{and} \quad k_p := 3e(p+1)(2+\log p), \quad p \in \mathbb{N}.$$

Proof. Assuming $p > 0$, combining Lemmas 3.2.9 and 3.3.9 gives

$$\begin{aligned}\log|\Pi(z)| &= \prod_{m\in M} W\left(\frac{z}{z_m};p\right) = \sum_{m\in M}\log\left|W\left(\frac{z}{z_m};p\right)\right| < \\ &< A_p\sum_{m\in M}\frac{\left|\frac{z}{z_m}\right|^{p+1}}{1+\left|\frac{z}{z_m}\right|} = A_p r^{p+1}\sum_{m\in M}\frac{1}{|z_m|^p(|z_m|+r)} = \\ &= A_p r^{p+1}\int_{(0,\infty)}\frac{1}{t^p(t+r)}\,\mathrm{d}\omega_\Pi(t),\end{aligned}$$

Lemma 3.2.9 also yields, for $s > 0$,

$$\int_{(0,s)}\frac{1}{t^p(t+r)}\,\mathrm{d}\omega_\Pi(t) = \frac{n_\Pi(s)}{s^p(s+r)} + \int_{(0,s)} n_\Pi(t)\frac{(p+1)t+pr}{t^{p+1}(t+r)^2}\,\mathrm{d}t.$$

As seen in the proof of Proposition 3.2.10, the left term on the right-hand side tends to 0 as $s\to\infty$. Therefore we obtain

$$\begin{aligned}\log|\Pi(z)| &< A_p r^{p+1}\int_{(0,\infty)}\frac{1}{t^p(t+r)}\,\mathrm{d}\omega_\Pi(t) = A_p r^{p+1}\int_0^\infty n_\Pi(t)\frac{(p+1)t+pr}{t^{p+1}(t+r)^2}\,\mathrm{d}t < \\ &< A_p(p+1)r^{p+1}\int_0^\infty\frac{n_\Pi(t)}{t^{p+1}(t+r)}\,\mathrm{d}t < \\ &< A_p(p+1)r^{p+1}\left(\frac{1}{r}\int_0^r\frac{n_\Pi(t)}{t^{p+1}}\,\mathrm{d}t + \int_r^\infty\frac{n_\Pi(t)}{t^{p+2}}\,\mathrm{d}t\right) = \\ &= k_p r^p\left(\int_0^r\frac{n_\Pi(t)}{t^{p+1}}\,\mathrm{d}t + r\int_r^\infty\frac{n_\Pi(t)}{t^{p+2}}\,\mathrm{d}t\right).\end{aligned}$$

The case $p = 0$ is verified in the same fashion:

$$\begin{aligned}\log|\Pi(z)| &= \sum_{m\in M}\log\left|W\left(\frac{z}{z_m};0\right)\right| \le \sum_{m\in M}\log\left(1+\left|\frac{z}{z_m}\right|\right) \le \\ &\le \sum_{m\in M}\left|\frac{z}{z_m}\right| = r\sum_{m\in M}\frac{1}{|z_m|} = r\int_{(0,\infty)}\frac{1}{t}\,\mathrm{d}\omega(t) = -r\int_0^\infty -\frac{n_\Pi(t)}{t^2}\,\mathrm{d}t < \\ &< r\left(\int_0^r\frac{n_\Pi(t)}{tr}\,\mathrm{d}t + \int_r^\infty\frac{n_\Pi(t)}{t^2}\,\mathrm{d}t\right) = \int_0^r\frac{n_\Pi(t)}{t}\,\mathrm{d}t + r\int_r^\infty\frac{n_\Pi(t)}{t^2}\,\mathrm{d}t.\end{aligned}$$ ■

Theorem 3.3.11. *Let $\Pi\in H(\mathbb{C})$ be a canonical product, then $\lambda_\Pi = \rho_\Pi$.*

Proof. By Theorem 3.2.6 we already have $\lambda_\Pi \le \rho_\Pi$, it remains to show that $\rho_\Pi \le \lambda_\Pi$. We first recall that the convergence exponent always satisfies $\mu \le \lambda_\Pi \le \mu + 1$, where $\mu := \mu_\Pi$.

Suppose $\lambda_\Pi < \mu + 1$ and let $\lambda_\Pi < \lambda < \mu + 1$. By Proposition 3.2.10 we have

$$\lambda_\Pi = \nu_\Pi = \limsup_{r\to\infty} \frac{\log n_\Pi(r)}{\log r} < \lambda.$$

Therefore there exists some $r_0 > 0$ such that for all $r \geq r_0$ we have $n_\Pi(r) \leq r^\lambda$. Choose $\varepsilon > 0$ small enough such that $n_\Pi(\varepsilon) = 0$, then

$$n_\Pi(r) = \frac{n_\Pi(r)}{r^\lambda} r^\lambda \leq \frac{n_\Pi(r_0)}{\varepsilon^\lambda} r^\lambda$$

and choosing $C_\lambda > \max\{1, n_\Pi(r_0)/\varepsilon^\lambda\}$ yields $n_\Pi(r) \leq C_\lambda r^\lambda$ for all $r > 0$. Applying this inequality to Lemma 3.3.10 we get, for $r = |z|$,

$$\begin{aligned}
\log|\Pi(z)| &< k_\mu r^\mu \left(\int_0^r \frac{n_\Pi(t)}{t^{\mu+1}} \,\mathrm{d}t + r \int_r^\infty \frac{n_\Pi(t)}{t^{\mu+2}} \,\mathrm{d}t \right) \leq \\
&\leq k_\mu r^\mu C_\lambda \left(\int_0^r t^{\lambda-\mu-1} \,\mathrm{d}t + r \int_r^\infty t^{\lambda-\mu-2} \,\mathrm{d}t \right) = \\
&= k_\mu r^\mu C_\lambda \left(\frac{r^{\lambda-\mu}}{\lambda-\mu} - \frac{r^{\lambda-\mu}}{\lambda-\mu-1} \right) = k_\mu C_\lambda \left(\frac{1}{\lambda-\mu} - \frac{1}{\lambda-\mu-1} \right) r^\lambda.
\end{aligned}$$

From this it follows $M_\Pi(r) = \mathcal{O}(\exp r^\lambda)$ and thus that $\rho_\Pi \leq \lambda$. Letting $\lambda \searrow \lambda_\Pi$ yields $\rho_\Pi \leq \lambda_\Pi$.

If we otherwise assume $\lambda_\Pi = \mu + 1$, then by Lemma 3.2.9

$$\int_{(0,r)} \frac{1}{t^{\mu+1}} \,\mathrm{d}\omega_\Pi(t) = \frac{n_\Pi(r)}{r^{\mu+1}} + (\mu+1) \int_0^r \frac{n_\Pi(t)}{t^{\mu+2}} \,\mathrm{d}t,$$

wherein the leftmost integral converges as $r \to \infty$. Imitating the proof of Proposition 3.2.10 we see that the left term on the right-hand side converges to 0 as $r \to \infty$, thus the rightmost integral converges as well. Let $\varepsilon > 0$. These two convergences imply that there is an $r_0 > 0$ such that for all $r \geq r_0$ we have

$$\int_{r_0}^\infty \frac{n_\Pi(t)}{t^{\mu+2}} \,\mathrm{d}t < \varepsilon, \quad \text{and} \quad \frac{n_\Pi(r)}{r^{\mu+1}} < \varepsilon.$$

Inserting these results into Lemma 3.3.10 yields, for $r = |z| \geq r_0$,

$$\begin{aligned}
\log|\Pi(z)| &< k_\mu r^\mu \left(\int_0^r \frac{n_\Pi(t)}{t^{\mu+1}} \,\mathrm{d}t + r \int_r^\infty \frac{n_\Pi(t)}{t^{\mu+2}} \,\mathrm{d}t \right) = \\
&= k_\mu r^\mu \left(\int_0^{r_1} \frac{n_\Pi(t)}{t^{\mu+1}} \,\mathrm{d}t + \int_{r_1}^r \frac{n_\Pi(t)}{t^{\mu+1}} \,\mathrm{d}t + r \int_r^\infty \frac{n_\Pi(t)}{t^{\mu+2}} \,\mathrm{d}t \right) \leq \\
&\leq k_\mu r^\mu \left(C + (r - r_1)\varepsilon + r\varepsilon \right) \leq k_\mu (C r^\mu + 2\varepsilon r^{\mu+1}).
\end{aligned}$$

The right-most term asymptotically dominates, therefore $M_\Pi(r) = \mathcal{O}(\exp(\varepsilon r^{\mu+1}))$ and thus Π is at most of order $\mu + 1 = \lambda_\Pi$ and, since ε was arbitrary, minimal type. ∎

Just as the order of a canonical product is determined by the order of growth of its zeros, its type is (to some extent) determined by the upper density of its zeros. A peculiarity arises when considering canonical products of integral order, in which case the type may also depend on the argument of the zeros [3]. However, for non-integral order we can obtain the following result:

Theorem 3.3.12. *Let $\Pi \in H(\mathbb{C})$ be a canonical product. If λ_Π is not an integer, then Π is of maximal, minimal or normal type according to whether Δ_Π is equal to infinity, zero, or a positive, real number.*

Proof. by Lemma 3.2.4 there is a constant $K > 0$ such that for all sufficiently large r we have $n_\Pi(r) \leq K \log M_\Pi(er)$ and by Proposition 3.2.10 and Theorem 3.3.11 we have $\nu_\Pi = \lambda_\Pi = \rho_\Pi$. Therefore we have, for large r,

$$\frac{n_\Pi(r)}{r^{\nu_\Pi}} = \frac{n_\Pi(r)}{r^{\rho_\Pi}} \leq K e^{\rho_\Pi} \frac{\log M_\Pi(er)}{(er)^{\rho_\Pi}}. \tag{3.20}$$

We now consider two cases. If Δ_Π is infinite then taking limits superior as $r \to \infty$ in the above shows than Π must be of maximal type.

If Δ_Π is finite then let $\Delta > \Delta_\Pi$. By definition of the limit superior there is an $r_0 > 0$ such that, for all $r \geq r_0$, we have

$$n_\Pi(r) \leq \Delta r^{\nu_\Pi} = \Delta r^{\rho_\Pi}.$$

As demonstrated in the proof of Theorem 3.3.11, we can find a constant $C_{\lambda_\Pi} > 0$ such that for all $r > 0$ we have

$$n_\Pi(r) \leq C_{\lambda_\Pi} \Delta r^{\rho_\Pi}$$

The genus μ of the canonical product satisfies $\mu < \lambda_\Pi < \mu + 1$, since λ_Π is not an integer. Therefore, by Lemma 3.3.10, for all $r > 0$ and $|z| = r$ we have

$$\begin{aligned}
\log|\Pi(z)| &< k_\mu r^\mu \left(\int_0^r \frac{n_\Pi(t)}{t^{\mu+1}} \,\mathrm{d}t + r \int_r^\infty \frac{n_\Pi(t)}{t^{\mu+2}} \,\mathrm{d}t \right) \leq \\
&\leq k_\mu r^\mu C_{\lambda_\Pi} \Delta \left(\int_0^r t^{\rho_\Pi - \mu - 1} \,\mathrm{d}t + r \int_r^\infty t^{\rho_\Pi - \mu - 2} \,\mathrm{d}t \right) = \\
&= k_\mu r^\mu C_{\lambda_\Pi} \Delta \left(\frac{r^{\rho_\Pi - \mu}}{\rho_\Pi - \mu} - \frac{r^{\rho_\Pi - \mu}}{\rho_\Pi - \mu - 1} \right) = k_\mu C_{\lambda_\Pi} \Delta \left(\frac{1}{\rho_\Pi - \mu} - \frac{1}{\rho_\Pi - \mu - 1} \right) r^{\rho_\Pi}.
\end{aligned}$$

Dividing by r^{ρ_Π}, taking limits superior as $r \to \infty$ and letting $\Delta \searrow \Delta_\Pi$ therefore yields $\tau_\Pi \leq C \Delta_\Pi$ for a constant $C > 0$. Combining with (3.20), we get

$$C_1 \tau_\Pi \leq \Delta_\Pi \leq C_2 \tau_\Pi$$

for constants $C_1, C_2 > 0$. This shows that, if $\Delta_\Pi \in (0, \infty)$, then Π is of normal type, and if $\Delta_\Pi = 0$, then Π is of minimal type. ∎

3.4 Hadamard's Theorem

Our main goal in this section will be to refine the factorization given by the Weierstraß Factorization Theorem for entire functions of finite order. Such a refinement is given by Hadamard's Theorem, the proof of which relies on the following lemma, which can be considered as a version of the maximum modulus principle applied to the real part of a holomorphic function.

The contents of this section closely follow the results found in Segal [5].

Lemma 3.4.1 (Borel–Carathéodory)**.** *Let $G \subseteq \mathbb{C}$ be a domain, $R > 0$ and suppose $c\ell(B_R(0)) \subseteq G$ and $f \in H(G)$. Define $A_f(r) := \max_{|z|=r} \operatorname{Re} f(z)$, then, for $0 < r < R$,*

$$M_f(r) \leq \frac{2r}{R-r} A_f(R) + \frac{R+r}{R-r} |f(0)| \tag{3.21}$$

and, if additionally $A_f(R) \geq 0$, then for any $n \in \mathbb{N}$

$$M_{f^{(n)}}(r) \leq \frac{2^{n+2} n! R}{(R-r)^{n+1}} (A_f(R) + |f(0)|). \tag{3.22}$$

Proof. If f is constant, there is nothing to show.

We assume f being non-constant. First, for $r > 0$ we have

$$A_f(r) = \max_{|z|=r} \operatorname{Re} f(z) = \log \max_{|z|=r} \exp \operatorname{Re} f(z) = \log \max_{|z|=r} |\exp f(z)| = \log M_{\exp f}(r)$$

and since $M_{\exp f}$ is strictly increasing and continuous therefore so is A_f.

We will show (3.21) by considering two cases. First assume f is non-constant and $f(0) = 0$. Then by the above we have $A_f(R) > A_f(0) = 0$. Define

$$\phi(z) := \frac{f(z)}{2A_f(R) - f(z)} \in H(B_R(0)).$$

Note that we have $\phi(0) = 0$ and

$$\begin{aligned} |\phi(z)|^2 = \phi(z)\overline{\phi(z)} &= \frac{|f(z)|^2}{(2A_f(R) - f(z))(2A_f(R) - \overline{f(z)})} = \\ &= \frac{|f(z)|^2}{(2A_f(R) - \operatorname{Re} f(z))^2 + |f(z)|^2 - (\operatorname{Re} f(z))^2} \leq 1, \end{aligned}$$

since clearly $2A_f(R) - \operatorname{Re} f(z) \geq \operatorname{Re} f(z)$. Now, since $\phi(zR) \in H(\mathbb{D})$ and $|\phi(zR)| \leq 1$ for $z \in \mathbb{D}$, Schwartz' Lemma implies $|\phi(zR)| \leq |z|$ for $z \in \mathbb{D}$. Therefore, for $z \in B_R(0)$ we have $|\phi(z)| \leq \frac{|z|}{R}$, and for $z \in B_r(0)$ we have $|\phi(z)| < \frac{r}{R} < 1$. Since

$$f(z) = 2A_f(R) \frac{\phi(z)}{1 + \phi(z)}$$

we obtain

$$|f(z)| = 2A_f(R)\left|\sum_{n=1}^{\infty}(-1)^n\phi(z)^n\right| \leq 2A_f(R)\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n = \frac{2r}{R-r}A_f(R).$$

Now assume that f is non-constant and $f(0) \neq 0$. Set $g(z) := f(z) - f(0)$, then by the above we have, for $w \in B_r(0)$,

$$|f(w)| - |f(0)| \leq |g(w)| \leq M_g(r) \leq \frac{2r}{R-r}A_g(R) \leq \frac{2r}{R-r}A_f(R) - \frac{2r}{R-r}\operatorname{Re} f(0).$$

Since $-\operatorname{Re} f(0) \leq |f(0)|$, this implies

$$|f(w)| \leq \frac{2r}{R-r}A_f(R) + \frac{2r}{R-r}|f(0)| + |f(0)| \leq \frac{2r}{R-r}A_f(R) + \frac{R+r}{R-r}|f(0)|,$$

thus proving (3.21).

To show (3.22), let $z \in \partial B_r(0)$ and set $s := \frac{R-r}{2}$, then by Cauchy's integral formula

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_{\partial B_s(z)} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,\mathrm{d}\zeta.$$

Replacing r with $\frac{R+r}{2} < R$ in (3.21) we get

$$\begin{aligned}
|f^{(n)}(z)| &\leq \frac{n!}{2\pi}2\pi\frac{R-r}{2}\left(\frac{2}{R-r}\right)^{n+1}\left(\frac{2(R+r)/2}{R-(R+r)/2}A_f(R) + \frac{R+(R+r)/2}{R-(R+r)/2}|f(0)|\right) \leq \\
&\leq \frac{2^{n+1}n!}{(R-r)^{n+1}}\frac{R-r}{2}\left(2\frac{R+r}{R-r}A_f(R) + \frac{3R+r}{R-r}|f(0)|\right) \leq \\
&\leq \frac{2^{n+1}n!}{(R-r)^{n+1}}\left((R+r)A_f(R) + \frac{3R+r}{2}|f(0)|\right) \leq \frac{2^{n+2}n!R}{(R-r)^{n+1}}(A_f(R) + |f(0)|).
\end{aligned}$$

∎

Theorem 3.4.2 (Hadamard)**.** *Let $f \in H(\mathbb{C})$ be not identically zero, of finite order and let $m \in \mathbb{N}_0$ be the order of the zero of f at the origin. Then there exists a polynomial Q with $\deg Q \leq \rho_f$ such that*

$$f(z) = z^m e^{Q(z)}\Pi(z), \tag{3.23}$$

where Π denotes the canonical product formed from the zeros of f.

Hadamard's Theorem can also be formulated more concisely: Let $f \in H(\mathbb{C})$ be of finite order, then $\mu_f \leq \rho_f$, that is the genus of f does not exceed its order.

Proof. We may assume $f(0) \neq 0$. By the Weierstraß Factorization Theorem we have

$f(z) = e^{Q(z)}\Pi(z)$ for some $Q \in H(\mathbb{C})$ and a Weierstraß product Π. Since

$$\mu_\Pi \leq \lambda_\Pi = \lambda_f \leq \rho_f$$

we may take Π to be a canonical product. Set $\nu := \lfloor \rho_f \rfloor$, then taking logarithms in the above factorization and differentiating $\nu + 1$ times we get

$$\begin{aligned}
\frac{\mathrm{d}^\nu}{\mathrm{d}z^\nu}\left(\frac{f'(z)}{f(z)}\right) &= Q^{(\nu+1)}(z) + \frac{\mathrm{d}^{\nu+1}}{\mathrm{d}z^{\nu+1}} \log \prod_{m\in M} W\left(\frac{z}{z_m}; \mu_\Pi\right) = \\
&= Q^{(\nu+1)}(z) + \frac{\mathrm{d}^{\nu+1}}{\mathrm{d}z^{\nu+1}} \sum_{m\in M}\left(\log\left(1 - \frac{z}{z_m}\right) + \sum_{k=1}^{\mu_\Pi} \frac{z^k}{k}\right) = \\
&= Q^{(\nu+1)}(z) - \sum_{m\in M} \frac{1}{z_m} \frac{\mathrm{d}^\nu}{\mathrm{d}z^\nu}\left(1 - \frac{z}{z_m}\right)^{-1} = \\
&= Q^{(\nu+1)}(z) - \nu! \sum_{m\in M} \frac{1}{(z_m - z)^{\nu+1}}.
\end{aligned}$$

We now aim to show that $Q^{(\nu+1)}$ is identically zero. For $R > 0$ we set

$$g_R(z) := \frac{f(z)}{f(0)} \prod_{\substack{m\in M \\ |z_m|\leq R}} \left(1 - \frac{z}{z_m}\right)^{-1} \in H(\mathbb{C}).$$

Note that $g_R(0) = 1$. For $|z| = 2R$ and $|z_m| \leq R$ we have $|1 - z/z_m| \geq |z|/|z_m| - 1 \geq 2R/R - 1 = 1$, therefore

$$M_{g_R}(2R) \leq \max_{|z|=2R} \left|\frac{f(z)}{f(0)}\right| = \mathcal{O}(\exp(2R)^\alpha), \quad \text{as } R \to \infty$$

for all $\alpha > \rho_f$. By the maximum modulus theorem this also holds for $|z| \leq 2R$, and in particular we have

$$\log M_{g_R}(R) = \mathcal{O}(R^\alpha).$$

Note that g_R has no zeros in $c\ell(B_R(0))$, therefore we can define $h_R(z) := \log g_R(z)$, choosing the branch of the logarithm for which $h_R(0) = 0$, then h_R is holomorphic on some domain containing $c\ell(B_R(0))$. We have

$$A_{h_R}(R) = \max_{|z|=R} \operatorname{Re} h_R(z) = \max_{|z|=R} \log|g_R(z)| = \log M_{g_R}(R) = \mathcal{O}(R^\alpha).$$

Since by the maximum modulus theorem $|g_R(z)| \geq |g_R(0)| = 1$ we have $\operatorname{Re} h_R(z) \geq 0$. Invoking the Borel-Carathéodory lemma we therefore get, for $0 < r < R$ and sufficiently large R,

$$M_{h_R^{(\nu+1)}}(R) \leq \frac{2^{\nu+3}(\nu+1)!R}{(R-r)^{\nu+2}} A_{h_R}(R) \leq \frac{K_0 2^{\nu+3}(\nu+1)!R^{\alpha+1}}{(R-r)^{\nu+2}},$$

where $K_0 > 0$ is some constant depending only on α. With $r := R/2$ it follows that

$$M_{h_R^{(\nu+1)}}(R) \leq K_1 R^{\alpha-(\nu+1)},$$

where $K_1 > 0$ is some constant depending only on α and ν. Note that we have

$$h_R^{(\nu+1)}(z) = \frac{\mathrm{d}^\nu}{\mathrm{d}z^\nu}\left(\frac{f'(z)}{f(z)}\right) + \nu! \sum_{\substack{m\in M \\ |z_m|\leq R}} \frac{1}{(z_m - z)^{\nu+1}} = Q^{(\nu+1)}(z) - \nu! \sum_{\substack{m\in M \\ |z_m|>R}} \frac{1}{(z_m - z)^{\nu+1}}.$$

For $|z| = R/2$ and $z_m > R$ we have $|z_m - z| \geq |z_m| - R/2 \geq |z_m|/2$, therefore

$$|Q^{(\nu+1)}(z)| \leq \nu! \sum_{\substack{m\in M \\ |z_m|>R}} \frac{1}{|z_m - z|^{\nu+1}} + |h_R^{(\nu+1)}(z)| \leq K_2 \sum_{\substack{m\in M \\ |z_m|>R}} \frac{1}{|z_m|^{\nu+1}} + K_1 R^{\alpha-(\nu+1)},$$

where K_2 is some constant depending only on ν. By the maximum modulus principle this holds for all $|z| \leq R/2$. Since $\nu + 1 > \rho_f \geq \lambda_f$ both terms on the right side tend to zero as $R \to \infty$, if $\rho_f < \alpha < \nu + 1$. Therefore $Q^{(\nu+1)}$ vanishes identically, thus Q is a polynomial of degree at most $\nu \leq \rho_f$. ■

Remark 3.4.3. An immediate consequence of Hadamard's Theorem is that entire functions of order $\rho < 1$ are of genus 0, and are therefore – like polynomials – solely determined by their zeros and some scaling factor. Indeed, in this case the polynomial Q in the factorization is constant and the canonical product Π is of genus 0 as well. In particular, any non-vanishing function of order zero is constant.

Furthermore, we can deduce that non-polynomial functions of order zero have no Picard exceptional value: Let f denote such a function and assume towards a contradiction that f assumes some value $a \in \mathbb{C}$ only finitely often. Then, by Hadamard's Theorem, we can write

$$f(z) - a = Cz^m\Pi(z)$$

for some constant $C > 0, m \in \mathbb{N}_0$ and a canonical product Π. Since a is assumed only finitely often, the canonical product Π must be finite and therefore f a polynomial, a contradiction.

As shown in Remark 3.1.6 the entire function

$$f(z) := \sum_{k=0}^{\infty} \frac{z^k}{(k^2)!}$$

is an example of a non-polynomial, entire function of order zero and must therefore, by the above reasoning, assume all values in $\mathbb{C}$ infinitely often. //

Hadamard's Theorem has a few consequences for entire functions of finite order, which we will explore further. Combining it with Theorem 3.3.11 allows us to prove two results regarding functions of finite, non-integer order.

Theorem 3.4.4. *Let $f \in H(\mathbb{C})$ be of finite, non-integer order. Then $\rho_f = \lambda_f$.*

Proof. By Theorem 3.2.6 we have $\lambda_f \leq \rho_f$. Invoking Hadamard's Theorem we can write

$$f(z) = z^m e^{Q(z)} \Pi(z)$$

for some $m \in \mathbb{N}_0$, a polynomial Q with $\deg Q \leq \rho_f$ and a canonical product Π. Since ρ_f is not an integer we have $\deg Q \leq \lfloor \rho_f \rfloor < \rho_f$. By Proposition 3.1.4 e^Q has order at most $\deg Q$ and by Theorem 3.3.11 Π has order λ_f, therefore using Proposition 3.1.8 we obtain

$$\rho_f \leq \max\{0, \deg Q, \lambda_f\} = \lambda_f \leq \rho_f,$$

since $\rho_f \leq \max\{\deg Q, \lambda_f\} = \deg Q < \rho_f$ would be a contradiction, and we get $\rho_f = \lambda_f$. ■

Theorem 3.4.5. *Let $f \in H(\mathbb{C})$ be of finite, non-integer order. Then f has infinitely many zeros.*

Proof. By Theorem 3.4.4 we have $\rho_f = \lambda_f$. Since ρ_f is not an integer it follows that $\lambda_f > 0$, which implies that f has infinitely many zeros. ■

Theorem 3.4.6 (Borel)**.** *Let $f \in H(\mathbb{C})$ be of integer order $\rho_f > 0$. Then for any $a \in \mathbb{C}$ we have $\lambda_f^{(a)} = \rho_f$, except possibly for one value of a.*

Proof. Note that for any $w \in \mathbb{C}$ we have

$$\lambda_f^{(w)} = \lambda_{f-w} \leq \rho_{f-w} = \rho_f.$$

Assume towards a contradiction that there exist $a, b \in \mathbb{C}$ such that $\lambda_f^{(a)} < \rho_f$ and $\lambda_f^{(b)} < \rho_f$. By Hadamard's Theorem we can write

$$\alpha(z) := f(z) - a = z^{m_1} e^{Q_1(z)} \Pi_1(z), \quad \beta(z) := f(z) - b = z^{m_2} e^{Q_2(z)} \Pi_2(z),$$

where $m_1, m_2 \in \mathbb{N}_0$, Q_1, Q_2 are polynomials of degree at most ρ_f and Π_1, Π_2 denote appropriate canonical products. We have

$$\deg Q_1 \leq \rho_f = \rho_\alpha \leq \max\{0, \deg Q_1, \rho_{\Pi_1}\},$$

and since by Theorem 3.3.11 $\rho_{\Pi_1} = \lambda_f^{(a)} < \rho_f$, it follows that $\deg Q_1 = \rho_f$. The analogous argument for Q_2 yields $\deg Q_2 = \rho_f$. Subtracting β from α yields

$$\begin{aligned} && b - a &= z^{m_1} e^{Q_1(z)} \Pi_1(z) - z^{m_2} e^{Q_2(z)} \Pi_2(z) \\ \Leftrightarrow && (b-a) e^{-Q_2(z)} &= z^{m_1} e^{Q_1(z) - Q_2(z)} \Pi_1(z) - z^{m_2} \Pi_2(z) \qquad (3.24) \\ \Leftrightarrow && (b-a) e^{-Q_2(z)} + z^{m_2} \Pi_2(z) &= z^{m_1} e^{Q_1(z) - Q_2(z)} \Pi_1(z) \end{aligned}$$

Since $\rho_{\Pi_2} = \lambda_f^{(b)} < \rho_f$ and $-Q_2$ is of degree ρ_f, the left-hand side is of order ρ_f and thus so is the right-hand side. Now similarly, since $\rho_{\Pi_1} = \lambda_f^{(b)} < \rho_f$ and the right-hand side has order ρ_f, we have that $Q_1 - Q_2$ is of degree ρ_f.

Differentiating the first equation in (3.24) gives

$$\begin{aligned} 0 &= m_1 z^{m_1-1} e^{Q_1(z)} \Pi_1(z) + z^{m_1} Q_1'(z) e^{Q_1(z)} \Pi_1(z) + z^{m_1} e^{Q_1}(z) \Pi_1'(z) \\ &\quad - m_2 z^{m_2-1} e^{Q_2(z)} \Pi_2(z) - z^{m_2} Q_2'(z) e^{Q_2(z)} \Pi_2(z) - z^{m_2} e^{Q_2(z)} \Pi_2'(z) \\ \Leftrightarrow \quad 0 &= e^{Q_1(z)} (m_1 z^{m_1-1} \Pi_1(z) + z^{m_1} Q_1'(z) \Pi_1(z) + z^{m_1} \Pi_1'(z)) \\ &\quad - e^{Q_2(z)} (m_2 z^{m_2-1} \Pi_2(z) + z^{m_2} Q_2'(z) \Pi_2(z) + z^{m_2} \Pi_2'(z)) \end{aligned}$$

By Proposition 3.1.10 we have $\rho_{\Pi_1'} = \rho_{\Pi_1}$ and $\rho_{\Pi_2'} = \rho_{\Pi_2}$, therefore the coefficients of e^{Q_1} and e^{Q_2} have order less than ρ_f. By Hadamard's Theorem we therefore get

$$e^{Q_1(z)} z^{m_3} e^{Q_3(z)} \Pi_3(z) = e^{Q_2(z)} z^{m_4} e^{Q_4(z)} \Pi_4(z)$$

where $m_3, m_4 \in \mathbb{N}_0, Q_3, Q_4$ are polynomials of degree less than or equal to $\rho_f - 1$ (since ρ_f is an integer) and P_3, P_4 denote canonical products. Rewriting the above we get

$$z^{m_3} e^{Q_1(z)+Q_3(z)} \Pi_3(z) = z^{m_4} e^{Q_2(z)+Q_4(z)} \Pi_4(z)$$

Since both sides are equal, in particular they must share the same zeros and multiplicities, therefore we have $m_3 = m_4$ and $\Pi_3 = \Pi_4$. But this implies

$$\begin{aligned} Q_1(z) + Q_3(z) &= Q_2(z) + Q_4(z) \\ \Leftrightarrow \quad Q_2(z) - Q_1(z) &= Q_3(z) - Q_4(z) \end{aligned}$$

and by the above the left side is of degree ρ_f, whereas the right side is of degree less than ρ_f, a contradiction. ■

In the context of Borel's Theorem, we refer to such an exceptional value as a *Borel exceptional value.* The theorem thus shows that entire functions of integer order have at most one Borel exceptional value.

Borel's Theorem together with Theorem 3.4.5 allow us to obtain a stronger version of Picard's Little Theorem for functions of finite, positive order:

Corollary 3.4.7. *Let $f \in H(\mathbb{C})$ with $\rho_f \in (0, \infty)$. Then f assumes at most one value only finitely often.*

Proof. If ρ_f is an integer then and f takes on some value $a \in \mathbb{C}$ only on finitely often, then $\lambda_f^{(a)} = 0 < \rho_f$ and by Borel's Theorem there is at most one value of a for which this holds.

If ρ_f is not an integer, then for any $w \in \mathbb{C}$ the function $f(z) - w$ is also of finite, non-integer order and therefore has infinitely many zeros by Theorem 3.4.5. Thus f assumes all values infinitely often. ■

Hadamard's Theorem can also be used to obtain a generalization of Theorem 3.3.12 which holds for general entire functions of non-integer order, not just canonical products. This requires the following result, the proof of which relies on finding a lower bound on the modulus of arbitrary entire functions and can be found in Levin [3] as Theorem 12(a).

Proposition 3.4.8. *Let $f, g \in H(\mathbb{C})$ be of finite order. If $\rho_f < \rho_g$, then it holds that $\rho_{fg} = \rho_g$ and*[3] *$\tau_{fg} = \tau_g$.*

Theorem 3.4.9. *Let $f \in H(\mathbb{C})$ be of finite, non-integer order. Then f is of maximal, minimal or normal type according to whether Δ_f is equal to infinity, zero, or a positive, real number.*

Proof. By Hadamard's Theorem we can write

$$f(z) = z^m e^{Q(z)} \Pi(z).$$

The first factor is of order zero, the second is at most of order $\lfloor \rho_f \rfloor < \rho_f$ and the last is of order ρ_f. Therefore, by Proposition 3.4.8 we have $\tau_f = \tau_\Pi$. Since the order of Π is not an integer and clearly $\Delta_f = \Delta_\Pi$, the assertion now follows from Theorem 3.3.12. ■

In general, if $f \in H(\mathbb{C})$ is of finite order with representation

$$f(z) = e^{Q(z)} \Pi(z),$$

where Q is a polynomial and Π is a canonical product, the order and type of f may be determined by either of the factors. However, Theorems 3.4.4 and 3.4.9 show that if ρ_f is not an integer, then the growth of f is determined solely by the canonical product Π, and therefore by the zeros of f.

[3] This holds in the sense that if g is of maximal or minimal type, then fg is of maximal or minimal type respectively, and, if g is of normal type, then $\tau_{fg} = \tau_g$.

4 Composition of Entire Functions

As seen by Proposition 3.1.8, the order of the sum or product of two entire functions of finite order is bounded above by the higher of the respective two orders. This is no longer the case when composition is involved; indeed, as shown in Example 3.1.14, the function

$$f(z) := \exp\exp z \in H(\mathbb{C})$$

is of infinite order, yet is the composition of two functions of order 1.

The results of this section are based on Segal [5].

4.1 Pólya's Theorem

Necessary conditions for the order of a composition to be finite will be given by Pólya's Theorem, the proof of which relies on a result that was first proven by Harald Bohr [5]. While Bloch's Theorem dealt with disks contained in the image of holomorphic functions, we now focus on circles contained in such images.

Proposition 4.1.1. *Let $G \subset \mathbb{C}$ be a bounded domain with $0 \in G$. Then the set*

$$S := \{r \geq 0 : \partial B_r(0) \subseteq c\ell(G)\} \tag{4.1}$$

has a positive maximum; that is, $c\ell(G)$ contains a circle of positive, maximal radius[1]*.*

Proof. Since G is open and contains 0 there is some $t > 0$ such that $B_t(0) \subseteq G$. Since $\partial B_t(0) \subset c\ell(B_t(0)) \subseteq c\ell(G)$ we thus have $t \in S$ and S is non-empty. It now suffices to show that S is compact, since then it would contain its maximum $m \geq t$. Clearly S is bounded from below by 0 and since $c\ell(G)$ is bounded it must also be bounded from above. Thus it remains to show that S is closed.

Let $(r_n)_{n\in\mathbb{N}}$ be a convergent sequence in S with limit $r \in [0, \infty)$. If $r = 0$, then $r \in S$. Otherwise we claim that $\partial B_r(0) \subseteq c\ell(G)$. Let $z \in \partial B_r(0)$, and set $z_n := r_n z/r$ for $n \in \mathbb{N}$. Now $|z_n| = r_n$, thus $z_n \in \partial B_{r_n}(0) \subseteq c\ell(G)$ and therefore $(z_n)_{n\in\mathbb{N}}$ is a sequence in $\overline{G}$. Clearly $z_n \to z$ and since $c\ell(G)$ is closed, we therefore have $z \in \overline{G}$. Since z was arbitrary, this concludes the claim, we have $r \in S$ and thus S is closed. ∎

[1] The circle of radius 0 is understood as the singleton $\{0\}$.

If $G \subset \mathbb{C}$ is a bounded domain and $f \in H(c\ell(G))$ is non-constant with $f(0) = 0$, then $f(G)$ is a bounded domain with $0 \in f(G)$. This, together with Proposition 4.1.1, justifies the following definition:

Definition 4.1.2. Let $f \in H(c\ell(\mathbb{D}))$ be non-constant with $f(0) = 0$, then we define

$$r_f := \max\{r \geq 0 : \partial B_r(0) \subseteq c\ell(f(\mathbb{D}))\}. \qquad /\!/$$

Bohr's Theorem now asserts that r_f (almost) does not depend on f itself:

Theorem 4.1.3 (Bohr)**.** *There exists a function $\phi : (0,1) \to (0,\infty)$ such that for any $\theta \in (0,1)$ and $f \in H(c\ell(\mathbb{D}))$ with $f(0) = 0$ and $M_f(\theta) = 1$ it holds that $r_f \geq \phi(\theta)$.*

Proof. Suppose f satisfies the hypothesis of the theorem, let $\varepsilon > 0$ and set $R_\varepsilon := r_f + \varepsilon$. By definition of r_f, for all $r \geq R_\varepsilon$ there exists some point $w_r \in \partial B_r(0)$ with $w_r \notin f(c\ell(\mathbb{D}))$. Choose such points $w_{R_\varepsilon}, w_{2R_\varepsilon}$ and define

$$h(z) := \frac{f(z) - w_{R_\varepsilon}}{w_{2R_\varepsilon} - w_{R_\varepsilon}} \in H(\mathbb{D}; \mathbb{C} \setminus \{0,1\}).$$

Since

$$|h(0)| = \left|\frac{f(0) - w_{R_\varepsilon}}{w_{2R_\varepsilon} - w_{R_\varepsilon}}\right| \leq \frac{|w_{R_\varepsilon}|}{||w_{2R_\varepsilon}| - |w_{R_\varepsilon}||} = \frac{R_\varepsilon}{2R_\varepsilon - R_\varepsilon} = 1,$$

by Schottky's Theorem we have $|h(z)| \leq \psi(\theta, 1)$ for all $|z| \leq \theta$. Therefore

$$|f(z)| - R_\varepsilon \leq |g(z) - w_{R_\varepsilon}| \leq |w_{2R_\varepsilon} - w_{R_\varepsilon}|\psi(\theta,1) \leq 3R_\varepsilon\psi(\theta,1)$$

and thus $|f(z)| \leq R_\varepsilon + 3R_\varepsilon\psi(\theta,1)$ for all $|z| \leq \theta$. Using the hypothesis that $M_f(\theta) = 1$ and the maximum modulus principle we obtain $1 \leq R_\varepsilon + 3R_\varepsilon\psi(\theta,1)$, and letting $\varepsilon \to 0$ we have $1 \leq r_f + 3r_f\psi(\theta,1)$. Thus

$$r_f \geq \frac{1}{1 + 3\psi(\theta,1)}$$

and defining $\phi(\theta)$ as the right-hand side establishes the assertion. ∎

For $g, h \in H(\mathbb{C})$ the following theorem now gives necessary conditions for the order of $g \circ h$ to be finite. Clearly the order of both functions needs to be finite, but in fact it even must hold that g is of order zero, or h a polynomial.

Theorem 4.1.4 (Pólya)**.** *Let $g, h \in H(\mathbb{C})$ be non-constant. For the order of $g \circ h$ to be finite, it must hold that either*

i. h is a polynomial and $\rho_g < \infty$, or

ii. h is not a polynomial, $\rho_h < \infty$ and $\rho_g = 0$.

Proof. Without loss of generality we can assume $h(0) = 0$; otherwise we just consider $h_0(z) := h(z) - h(0)$ and $g_0(w) := g(w + h(0))$. Set $f := g \circ h$ and define

$$k_r(z) := \frac{h(rz)}{M_h(r/2)} \in H(c\ell(\mathbb{D})), \quad \text{for } r > 0.$$

Note that by definition we have $M_{k_r}(1/2) = 1$ and $k_r(0) = 0$, thus by Bohr's Theorem there is some constant $C > 0$ and an $R > CM_h(r/2)$ such that $\partial B_{R/M_h(r/2)}(0) \subseteq k_r(c\ell(\mathbb{D}))$ and thus $\partial B_R(0) \subseteq h(c\ell(B_r(0)))$. By the maximum modulus principle, g assumes its maximum modulus over $c\ell(B_R(0))$ at some $w_0 \in \partial B_R(0)$. By the above there is a $z_0 \in c\ell(B_r(0))$ with $h(z_0) = w_0$. Thus we get

$$M_g(CM_h(r/2)) < M_g(R) = |g(w_0)| = |g(h(z_0))| = |f(z_0)| \leq M_f(r).$$

Assuming $\rho_f < \infty$, we have $M_f(r) < K \exp(r^\alpha)$ for every $\alpha > \rho_f$, with suitable $K > 0$ and all $r > r_0$. Consider the power series expansion $h(z) = \sum_{n=0}^{\infty} a_n z^n$. Let a_m denote any non-zero coefficient; note that since $h(0) = 0$ we have $m \geq 1$. By Cauchy's integral formula we have, for all $s > 0$,

$$|a_m| = \left| \frac{h^{(m)}(0)}{m!} \right| = \frac{1}{2\pi} \left| \oint_{\partial B_s(0)} \frac{h(\zeta)}{\zeta^{n+1}} \,\mathrm{d}\zeta \right| \leq \frac{M_h(s)}{s^m} \tag{$*$}$$

and thus

$$M_g(C|a_m|(r/2)^m) \leq M_g(CM_h(r/2)) < M_f(r) < K \exp(r^\alpha), \quad \text{for all } r > r_0.$$

Replacing $(r/2)^m$ with r we obtain $\rho_g \leq \alpha/m$. If h is not a polynomial we may let $m \to \infty$, thus $\rho_g = 0$.

Now consider $g(z) = \sum_{n=0}^{\infty} b_n z^n$. Replacing h with g in $(*)$ we obtain $|b_n| s^n \leq M_g(s)$ for all $s > 0$ and $n \geq 1$ and thus

$$|b_n|(CM_h(r/2))^n \leq M_g(CM_h(r/2)) < M_f(r) < K \exp(r^\alpha), \quad \text{for all } r > r_0,$$

which implies $\rho_h \leq \alpha < \infty$. ∎

An interesting consequence of Pólya's Theorem is Thron's Theorem, which asserts that a necessary condition for a transcendental entire function $g \in H(\mathbb{C})$ to decompose as $g = f \circ f$, for an entire function $f \in H(\mathbb{C})$, is that g must assume all complex values infinitely often. Once again, a transcendental entire function is an entire function that is not a polynomial.

Theorem 4.1.5 (Thron)**.** *Let $g \in H(\mathbb{C})$ be transcendental, $\rho_g < \infty$ and suppose that g assumes some value $w \in \mathbb{C}$ only finitely often. Then there exists no $f \in H(\mathbb{C})$ with $f \circ f = g$.*

Proof. Assume towards a contradiction that $f \in H(\mathbb{C})$ with $f \circ f = g$ exists. Since g is not a polynomial, f is not a polynomial either. Thus Pólya's Theorem implies $\rho_f = 0$.

Consider the sets

$$Z := f^{-1}(\{w\}), \quad Z' := \bigcup_{z \in Z} f^{-1}(\{z\}).$$

By definition, for each $z' \in Z'$ there is some $z \in Z$ with $z' \in f^{-1}(\{z\})$, thus

$$g(z') = f(f(z')) = f(z) = w.$$

Our hypothesis on g implies that Z' must be finite. Since pre-images of singletons are disjoint, Z' is a disjoint union, therefore

$$\sum_{z \in Z} |f^{-1}(\{z\})| = \left| \bigcup_{z \in Z} f^{-1}(\{z\}) \right| = |Z'| < \infty$$

and thus all points in Z are only assumed finitely often by f. But by Corollary 2.4.3, f assumes at most one value only finitely often; therefore $|Z| \leq 1$.

If $Z = \emptyset$, then $h(z) := f(z) - w$ is entire, of order 0 and nowhere 0. As discussed in Remark 3.4.3, as a consequence of Hadamard's Theorem we have that h is constant and consequently so is f, a contradiction.

If $Z = \{z_0\}$, then $h(z) := f(z) - w$ has a single zero of finite order $n \in \mathbb{N}$ at z_0. Therefore we can write $h(z) = (z - z_0)^n p(z)$, where p is entire, of order 0 and nowhere 0. Again, this implies that p is constant, and therefore f a polynomial, a contradiction. ■

Example 4.1.6. A natural application of Thron's Theorem is to set $g(z) := \exp z$, which never assumes zero as a value. Indeed, this implies that there is no $f \in H(\mathbb{C})$ satisfying

$$f(f(z)) = \exp z.$$

On the other hand, there does exist a *real*-analytic function satisfying the above. Constructing such a function, however, is difficult, but was demonstrated by H. Kneser [2]. //

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