BACHELORARBEIT

Growth, order and zeros of entire functions

ausgeführt am

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1 Introduction

Some words why the subject is of interest.

A reference to the primary literature used.

Overview of used notation.

$M_f(r) \coloneqq \max_{|z|=r} |f(z)|$

2 Order

Definition 2.1. Let f be an entire function. The *order* of f is defined by

$$\rho_f := \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}. \tag{2.1}$$

Constant functions, by convention, have order 0.

Remark 2.2. Initial explanation and intuition of the order. Make sure to note the possible values of the order ($0 \le \rho \le \infty$). And that ρ can also be seen as the infimum over all ρ that satisfy $|f(z)| \le Ae^{B|z|^\rho}$ for suitable $A, B > 0$ and sufficiently large $|z|$.

Maybe this should not be a remark, but rather an equivalent characterization?

Proposition 2.3. *Let f, g be entire functions of finite order. Then it holds that:*

i. $\rho_{f+g} \le \max\{\rho_f, \rho_g\}$

ii. $\rho_{fg} \le \max\{\rho_f, \rho_g\}$

Proof. To prove (i), note that

$$\begin{aligned} M_{f+g}(r) &= \max_{|z|=r} |f(z)+g(z)| \le \max_{|z|=r} |f(z)| + |g(z)| \le \max_{|z|=r} |f(z)| + \max_{|z|=r} |g(z)| \le \\ &= M_f(r) + M_g(r) \le 2\max\{M_f(r), M_g(r)\} \end{aligned}$$

thus

$$\log M_{f+g}(r) \le \log 2 + \log\max\{M_f(r), M_g(r)\} = \log 2 + \max\{\log M_f(r), \log M_g(r)\}.$$

If $M_f(r)$ and $M_g(r)$ are bounded, then applying the above in eq. (2.1) implies that f, g and $f+g$ all have order 0. If either one is not, then $\max\{\log M_f(r), \log M_g(r)\}$ necessarily dominates $\log 2$ and we obtain

$$\begin{aligned}\rho_{f+g} &= \limsup_{r\to\infty} \frac{\log\log M_{f+g}(r)}{\log r} \le \limsup_{r\to\infty} \frac{\log(\log 2 + \max\{\log M_f(r), \log M_g(r)\})}{\log r} = \\ &= \limsup_{r\to\infty} \frac{\log \max\{\log M_f(r), \log M_g(r)\}}{\log r} = \\ &= \limsup_{r\to\infty} \max\left\{\frac{\log\log M_f(r)}{\log r}, \frac{\log\log M_g(r)}{\log r}\right\} = \\ &= \max\left\{\limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}, \limsup_{r\to\infty} \frac{\log\log M_g(r)}{\log r}\right\} = \max\{\rho_f, \rho_g\}.\end{aligned}$$

To prove (ii), we similarly note that

$$\begin{aligned}\log\log M_{fg}(r) &\le \log\log(M_f(r)M_g(r)) = \log(\log M_f(r) + \log M_g(r)) \le \\ &\le \log(2\max\{\log M_f(r), \log M_g(r)\}) = \\ &= \log 2 + \max\{\log\log M_f(r), \log\log M_g(r)\},\end{aligned}$$

from where we can proceed as in (i). ∎

I could remark that if $\rho_f \ne \rho_g$ we can actually achieve equality in the above, but the proof for the multiplication case is difficult.

For entire functions of finite order, we can obtain a representation of the order via the coefficients in their power series expansion.

Theorem 2.4. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. Then f is of finite order ρ if and only if*

$$\mu := \limsup_{n\to\infty} \frac{n \log n}{\log \frac{1}{|a_n|}}$$

is finite, where we take the quotient to be zero if $a_n = 0$. In either case we have $\rho = \mu$.

Proof. We shall first show that $\mu \le \rho$. If $\mu = 0$ there is nothing to show. Suppose we have $0 < \varepsilon < \mu$. Then, for infinitely many n, we have

$$n \log n \ge \kappa \log \frac{1}{|a_n|}, \quad \text{where } \kappa := \begin{cases} \mu - \varepsilon, & \mu < \infty, \\ \varepsilon, & \text{otherwise.} \end{cases}$$

By Cauchy's formula we have, for all $r > 0$,

$$|a_n| = \left|\frac{f^{(n)}(0)}{n!}\right| = \left|\frac{1}{2\pi i}\int_{|z|=r} \frac{f(\zeta)}{\zeta^{n+1}}\,\mathrm{d}\zeta\right| \le \left|\frac{2\pi r}{2\pi i}\frac{M_f(r)}{r^{n+1}}\right| = \frac{M_f(r)}{r^n}.$$

Thus we obtain, by combining the above,

$$\begin{aligned} n \log n &\geq \kappa \log \frac{r^n}{M_f(r)} \\ \Leftrightarrow \quad \log M_f(r) &\geq n \log r - \frac{n}{\kappa} \log n. \end{aligned} \tag{$*$}$$

Differentiating the right hand side with respect to n yields

$$\begin{aligned} 0 &\overset{!}{=} \log r - \frac{1}{\kappa} \log n - \frac{1}{\kappa} \\ \Leftrightarrow \quad \log n &= \kappa \log r - 1 \\ \Leftrightarrow \quad n &= \frac{r^\kappa}{e}, \end{aligned}$$

thus the right hand side of $(*)$ becomes maximal for such n. Since the above holds for infinitely many n, we obtain an unbounded sequence of values for r satisfying

$$\begin{aligned} \log M_f(r) &\geq \frac{r^\kappa}{e} \log r - \frac{r^\kappa}{e\kappa} \log \frac{r^\kappa}{e} = \frac{r^\kappa}{e\kappa} \\ \Leftrightarrow \quad \log \log M_f(r) &\geq \kappa \log r - (1 + \log \kappa) \\ \Leftrightarrow \quad \frac{\log \log M_f(r)}{\log r} &\geq \kappa - \frac{1 + \log \kappa}{\log r}. \end{aligned}$$

Taking the limit superior as $r \to \infty$ yields $\rho \geq \kappa$. Finally, if $\mu < \infty$, then letting $\varepsilon \to 0$, $\varepsilon \to \infty$ otherwise, concludes $\rho \geq \mu$.

TODO: $\rho \leq \mu$. ■

Example 2.5. We shall apply Theorem 2.4 to obtain the order of certain entire functions via their power series expansion $\sum_{n=0}^{\infty} a_n z^n$:

i. It is immediately apparent that polynomials have order zero. Non-polynomial functions of zero order do exist, given that their coefficients decrease sufficiently rapidly, as in the example of $a_n := n^{-(n^2)}$.

ii. By Stirling's approximation we have $\log n! = n \log n + O(n)$, from which we conclude that the exponential function, sine and cosine all have order 1.

iii. For any given $\rho > 0$ the coefficients $a_n := n^{-\rho n}$ define an entire function of order ρ.

iv. For $n \geq 2$, the coefficients $a_n := n^{-\frac{n}{\sqrt{\log n}}}$ define an entire function of infinite order. Note that by Remark 2.2, we can also conclude that e^{e^z} is of infinite order.

Proposition 2.6. *Let f be an entire function of finite order with derivative f'. Then $\rho_{f'} = \rho_f$.*

Proof. Given $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we have $f'(z) = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$.

Since $n \log n \sim (n+1)\log(n+1)$ we have

$$\begin{aligned}
\limsup_{n\to\infty} \frac{n\log n}{\log\frac{1}{|(n+1)a_{n+1}|}} &= \liminf_{n\to\infty}\left(\frac{-\log(n+1)+\log\frac{1}{|a_{n+1}|}}{n\log n}\right)^{-1} = \\
&= \liminf_{n\to\infty}\left(\frac{\log\frac{1}{|a_{n+1}|}}{n\log n}\right)^{-1} \cdot \lim_{m\to\infty}\left(\frac{m\log m}{(m+1)\log(m+1)}\right)^{-1} = \\
&= \liminf_{n\to\infty}\left(\frac{\log\frac{1}{|a_{n+1}|}}{n\log n}\cdot\frac{n\log n}{(n+1)\log(n+1)}\right)^{-1} = \\
&= \liminf_{n\to\infty}\left(\frac{\log\frac{1}{|a_{n+1}|}}{(n+1)\log(n+1)}\right)^{-1} = \limsup_{n\to\infty}\frac{(n+1)\log(n+1)}{\log\frac{1}{|a_{n+1}|}} = \\
&= \limsup_{n\to\infty}\frac{n\log n}{\log\frac{1}{|a_n|}}
\end{aligned}$$

and, since $\rho_f < \infty$, Theorem 2.4 concludes $\rho_{f'} = \rho_f$. ∎

3 Factorization

Theorem 3.1 (Weierstrass [1, 3])**.** *Let* $(z_j)_{j\in\mathbb{N}}$ *be a sequence in* $\mathbb{C}$ *without accumulation points. Then there exists an entire function* E *(called the* Weierstrass canonical product *formed from said sequence) that has zeros precisely at* $(z_j)_{j\in\mathbb{N}}$*, with multiplicities equal to how often* z_j *occurs in the sequence.*

Furthermore, if f *is any other entire function satisfying the above, then there exists an entire function* g *such that*

$$f(z) = e^{g(z)}E(z).$$

In particular we have

$$E(z) = z^k \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{R_n(z/z_n)},$$

where k is the order of the zero at $z = 0$ and R_n is a polynomial, namely a truncation of the power series for $-\log(1 - \frac{z}{z_n})$ chosen of smallest degree to ensure convergence of the product [2].

Hadamard's Theorem will show that, for functions of finite order ρ, the function g in Theorem 3.1 can be taken to be a polynomial of degree less than ρ and the degree of the polynomials R_n can be taken to be independant of n. To prove this we require the following lemma, which can be interpreted as a version of the maximum modulus theorem applied to the real part of an analytic function.

Lemma 3.2 (Borel-Carathéodory)**.** *Let* f *be analytic in* $\mathrm{cl}(B(0,R))$ *and let*

$$A_f(r) = \max_{|z|=r} \Re f(z).$$

Then, for $0 < r < R$*,*

$$M_f(r) \leq \frac{2r}{R-r} A_f(R) + \frac{R+r}{R-r}|f(0)|$$

and, if additionally $A_f(R) \geq 0$*, then for* $n \in \mathbb{N}$

$$\max_{|z|=r} |f^{(n)}(z)| \leq \frac{2^{n+2} n! R}{(R-r)^{n+1}} (A_f(R) + |f(0)|).$$

Proof. TODO. ■

Theorem 3.3 (Hadamard)**.** *Let f be an entire function of finite order with zeros $(z_j)_{j\in\mathbb{N}}$ and $f(0) \neq 0$. Then there exists a polynomial Q with $\deg Q \leq \rho_f$, such that*

$$f(z) = e^{Q(z)} E(z),$$

where E is a Weierstrass canonical product formed from the zeros of f.

Proof. TODO. ■

4 Zeros

We recall a rather explicit connection between the moduli of the zeros of an analytic function and the modulus of the function itself:

Theorem 4.1 (Jensen [3])**.** *Let f be analytic on $B(0,R)$ with $f(0) \neq 0$ and let $r_1, r_2, \ldots$ denote the moduli of the zeros of f in $B(0,R)$ arranged in a non-decreasing sequence. Then, for $r_n < r < r_{n+1}$, we have*

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\vartheta})|\,\mathrm{d}\vartheta = \log|f(0)| + \log\frac{r^n}{r_1 \ldots r_n}.$$

Definition 4.2. Let f be analytic on $B(0,R)$. Then, for $0 < r < R$, we denote by $n_f(r)$ the number of zeros of f in $\mathrm{cl}(B(0,r))$.

Corollary 4.3. *Let f be analytic on $B(0,R)$ with $f(0) \neq 0$. Then, for $0 < r < R$, we have*

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\vartheta})|\,\mathrm{d}\vartheta = \log|f(0)| + \int_0^r \frac{n_f(s)}{s}\,\mathrm{d}s$$

Proof. Let $r_1, r_2, \ldots$ denote the moduli of the zeros of f in $B(0,R)$ arranged in a non-decreasing sequence. Then, for any $r_n < r < r_{n+1}$, we obtain

$$\begin{aligned}
\log\frac{r^n}{r_1 \ldots r_n} &= \sum_{k=1}^n \log\frac{r}{r_k} = \sum_{k=1}^n \int_{r_k}^r \frac{1}{s}\,\mathrm{d}s = \\
&= \sum_{k=1}^n \int_0^r \mathbb{1}_{(r_k,\infty)}(s)\frac{1}{s}\,\mathrm{d}s = \int_0^r \left(\sum_{k=1}^n \mathbb{1}_{(r_k,\infty)}(s)\right)\frac{1}{s}\,\mathrm{d}s = \\
&= \int_0^r \frac{n_f(s)}{s}\,\mathrm{d}s
\end{aligned}$$

and Theorem 4.1 concludes the claim. ∎

In particular, we observe that the more zeros a function $f(z)$ has, the faster its modulus must grow as $|z| \to \infty$. The converse is naturally false, as seen by iterated exponentials [2].

Definition 4.4. Let f be an entire function satisfying $f(0) \neq 0$. Let $(r_j)_{j\in\mathbb{N}}$ denote the moduli of the zeros of f (if any) arranged in non-decreasing order. Then

$$\lambda_f := \inf\left\{\lambda > 0 : \sum_{n=1}^{\infty} \frac{1}{r_n^{\lambda}} < \infty\right\}$$

is called the *exponent of convergence of the zeros of f*. If f has finitely many zeros, then we set $\lambda_f = 0$ by convention.

Furthermore, the *exponent of convergence of the a-points of f* is defined as exponent of convergence of zeros of $f(z) - a$.

Theorem 4.5. *Let f be an entire function of finite order. Then $\lambda_f \le \rho_f$.*

Proof. TODO. ■

Example 4.6. An example of a series where we obtain convergence for some appropriate function using the above.

Theorem 4.7. *Let E be a Weierstrass canonical product of finite order. Then $\lambda_E = \rho_E$.*

Proof. TODO. ■

Theorem 4.8. *Let f be an entire function of finite, non-integer order. Then $\rho_f = \lambda_f$.*

Proof. By Theorem 4.5 we have $\lambda_f \le \rho_f$. Invoking Hadamard's Theorem we can write $f = e^Q E$ for a polynomial Q with $\deg Q \le \rho_f$. Since ρ_f is not an integer, this implies $\deg Q \le \lfloor \rho_f \rfloor < \rho_f$. Now, again by Hadamard's Theorem, e^Q has order $\deg Q$ and by Theorem 4.7 E has order λ_f. Using Proposition 2.3 we obtain

$$\rho_f \le \max\{\deg Q, \lambda_f\} = \lambda_f \le \rho_f,$$

implying $\rho_f = \lambda_f$. ■

Theorem 4.9. *Let f be an entire function of finite, non-integer order. Then f has infinitely many zeros.*

Proof. By Theorem 4.8 we have $\rho_f = \lambda_f$. Since ρ_f is not an integer, $\lambda_f > 0$, which implies that f has infinitely many zeros. ■

Maybe introduce Borel exceptional values as a definition? But then again, I will never need them again. Maybe also add a remark on the relation to lacunary values (Picard).

Theorem 4.10 (Borel)**.** *Existence of Borel exceptional values.*

Proof. TODO. ■

5 Composition

As seen by Proposition 2.3, the order of the sum or product of two entire functions is reasonably bounded by the order of the functions involved. This is not the case when composition is involved. Indeed, consider e^{e^z}, which has infinite order, yet is the composition of two functions of order 1. Necessary conditions for the order of a composition to be finite will be illustrated by Pòlya's Theorem, the proof of which relies on the following result:

Lemma 5.1 (Bohr). *Let $0 < R < 1$ and suppose f is analytic on $\mathrm{cl}(B(0,1))$, such that $f(0) = 0$ and $M_f(R) = 1$. Let r_f denote the largest $r \geq 0$ such that $C_r \subseteq f(\mathrm{cl}(B(0,1)))$. Then we have $r_f > C > 0$, where C is a constant depending only on R.*

Proof. TODO. Note that the proof relies on the strong form of Schottky's Theorem. ∎

Theorem 5.2 (Pólya). *Let g, h be entire. For the order of $g \circ h$ to be finite, it must hold that either*

i. h is a polynomial and g of finite order, or

ii. h is of finite order, not a polynomial, and g is of order zero.

Proof. TODO. ∎

Theorem 5.3 (Thron). *Let g be an entire function of finite order, not a polynomial, which takes some value w only finitely often. Suppose further that there exists f such that $f \circ f = g$. Then f is not entire.*

Proof. Seeking contradiction, suppose f were entire. Since g is not a polynomial, Theorem 5.2 implies that f is of order 0 and not a polynomial. Let $(z_j)_{j \in J}$ denote the points where f equals w. For each $m \in J$ we additionally denote by $(z_{j,m})_{j \in J_m}$ the points where f equals z_m. Thus, for each $m \in J$ and $n \in J_m$ we have

$$g(z_{n,m}) = f(f(z_{n,m})) = f(z_m) = w.$$

By our assumption on g, there must only be finitely many distinct points among the $(z_{n,m})_{m \in J, n \in J_m}$. Thus, each point in $(z_j)_{j \in J}$ is only taken on by f finitely often.

Do I need a citation for Picard's Great Theorem?

Since f is entire and not a polynomial, it has an essential singularity at ∞. By Picard's Great Theorem, f therefore attains all values in the complex plane infinitely often, with at most one exception. This implies that that there is at most one z_0 in $(z_j)_{j\in J}$ that is only taken on finitely often by f.

If there is no such z_0, then $h(z) \coloneqq f(z) - z_0$ is entire, of order 0 and nowhere 0. Thus, by Hadamard's Theorem, h must be constant, and therefore f aswell, a contradiction.

If such a z_0 exists, then $h(z) \coloneqq f(z) - z_0$ has a zero of finite order $n \in \mathbb{N}$ at z_0. Therefore we can write $h(z) = (z - z_0)^n p(z)$, where p is entire, of order 0 and nowhere 0. Again, this implies that p is constant, and therefore f a polynomial, a contradiction. ■

Example 5.4. A natural application of Theorem 5.3 is taking g to be e^z, which never takes on 0 as a value. Indeed, this implies that there is no entire function f satisfying

$$(f \circ f)(z) = e^z.$$

On the other hand, there does exist a real-analytic function satisfying the above, as demonstrated by H. Kneser. I probably still need a citation here.

Bibliography

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