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B A C H E L O R A R B E I T

Growth, order and zeros of entire functions

ausgeführt am

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1 Introduction

Some words why the subject is of interest.

A reference to the primary literature used.

Overview of used notation.

2 Order

Definition 2.1. Let f be an entire function. The *order* of f is defined by

$$\rho := \limsup_{r \rightarrow \infty} \frac{\log \log M(r)}{\log r},$$

where $M(r) := \max_{|z|=r} |f(z)|$. Constant functions, by convention, have order 0.

Remark 2.2. Initial explanation and intuition of the order. Make sure to note the possible values of the order ($0 \leq \rho \leq \infty$). And that ρ can also be seen as the infimum over all ρ that satisfy $|f(z)| \leq Ae^{B|z|^\rho}$ for suitable $A, B > 0$.

Theorem 2.3. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. Then f is of finite order ρ if and only if

$$\mu := \limsup_{n \rightarrow \infty} \frac{n \log n}{\log \frac{1}{|a_n|}} < \infty,$$

where we take the quotient to be zero if $a_n = 0$. In either case we have $\rho = \mu$.

Proof. **TODO.** ■

Example 2.4. Some examples for functions of specific order. Specifically order 0 (polynomials, rapidly converging sum), order 1 (exp, sin), arbitrary order ρ and order ∞ .

Proposition 2.5. Let f, g be entire functions of respective finite orders ρ_f, ρ_g . Then it holds that

- i. $f + g$ has order at most $\max\{\rho_f, \rho_g\}$.
- ii. fg has order at most $\max\{\rho_f, \rho_g\}$.
- iii. f' has order ρ_f .

Proof. **TODO.** ■

3 Factorization

Short introduction.

Do I need a citation for this?

Theorem 3.1 (Weierstrass). *Let $(z_j)_{j \in \mathbb{N}}$ be a sequence in \mathbb{C} without accumulation points. Then there exists an entire function P (called the Weierstrass canonical product formed from said sequence) that has zeros precisely at $z_j, j \in \mathbb{N}$, with multiplicities equal to how often z_j occurs in the sequence.*

Furthermore, if f is any other entire function satisfying the above, then there exists an entire function g such that

$$f = e^g P.$$

Short remark on how Hadamard refines Weierstrass (for functions of finite order).

Lemma 3.2 (Borel-Carathéodory). *Let f be analytic in $\text{cl}(B(0, R))$ and let*

$$M(r) = \max_{|z|=r} |f(z)|, \quad A(r) = \max_{|z|=r} \Re f(z).$$

Then, for $0 < r < R$,

$$M(r) \leq \frac{2r}{R-r} A(R) + \frac{R+r}{R-r} |f(0)|$$

and, if additionally $A(R) \geq 0$, then for $n \in \mathbb{N}$

$$\max_{|z|=r} |f^{(n)}(z)| \leq \frac{2^{n+2} n! R}{(R-r)^{n+1}} (A(R) + |f(0)|).$$

Proof. TODO. ■

Theorem 3.3 (Hadamard). *Let f be an entire function of finite order ρ with zeros $(z_j)_{j \in \mathbb{N}}$ and $f(0) \neq 0$. Then there exists a polynomial Q with $\deg Q \leq \rho$, such that*

$$f = e^Q P,$$

where P is the Weierstrass canonical product formed from the zeros of f .

Proof. TODO. ■

4 Zeros

Maybe Jensen? Not sure if I need it earlier. Anyhow, include the equivalent version using the zero counting function.

Definition 4.1. Let f be an entire function satisfying $f(0) \neq 0$. Let $(r_j)_{j \in \mathbb{N}}$ denote the moduli of the zeros of f (if any) arranged in non-decreasing order. Then

$$\rho_1 := \inf\{\alpha > 0 : \sum_{n=1}^{\infty} \frac{1}{r_n^\alpha} < \infty\}$$

is called the *exponent of convergence of the zeros of f* . If f has finitely many zeros, then we set $\rho_1 = 0$ by convention. I am not sure about this – the paper only establishes this convention if f has no zeros at all.

Furthermore, the *exponent of convergence of the a -points of f* is defined as exponent of convergence of $f(z) - a$.

Theorem 4.2. Let f be an entire function of finite order ρ and exponent of convergence of zeros ρ_1 . Then $\rho_1 \leq \rho$.

Proof. TODO. ■

Example 4.3. An example of a series where we see some convergence for some appropriate function using the above.

Theorem 4.4. Let P be a Weierstrass canonical product of finite order ρ and exponent of convergence of zeros ρ_1 . Then $\rho_1 = \rho$.

Proof. TODO. ■

Theorem 4.5. Let f be an entire function of finite order ρ and exponent of convergence of zeros ρ_1 . If ρ is not an integer, then $\rho = \rho_1$.

Proof. TODO. ■

Theorem 4.6. Let f be an entire function of finite, non-integer order. Then f has infinitely many zeros.

Proof. TODO. ■

Maybe introduce Borel exceptional values as a definition? But then again, I will never need them again. Maybe also add a remark on the relation to lacunary values (Picard).

Theorem 4.7 (Borel). *Existence of Borel exceptional values.*

Proof. TODO. ■

5 Composition

Theorem 5.1 (Pólya). *Let g, h be entire. For the order of $g \circ h$ to be finite, it must hold that either*

- i. h is a polynomial and g of finite order, or*
- ii. h is of finite order, not a polynomial, and g is of order zero.*

Proof. **TODO.** ■

Theorem 5.2. *Let g be an entire function of finite order, not a polynomial, which takes some value w only finitely often. Suppose further that there exists f such that $f \circ f = g$. Then f is not entire.*

Proof. **TODO.** ■

Example 5.3. **The example with $f(f(z)) = e^z$.**

Bibliography

- [1] S. L. Segal. *Nine introductions in complex analysis*, volume 208 of *North-Holland Mathematics Studies*. Elsevier Science B.V., Amsterdam, revised edition, 2008.