BACHELORARBEIT

Picard's Great Theorem

AND

Growth, Zeros and Composition of Entire Functions

ausgeführt am

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Wien, am 22. Juli 2024

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1 Introduction

Some words why the subject is of interest.

A reference to the primary literature used.

Overview of used notation.

$B_r(a)$ open disk of radius r centered at a.

$C(G, S)$ space of continuous functions on G that map into S.

$H(G)$ space of holormphic functions on G.

In chapter 3 we will often utilize asymptotic notation:

2 Picard's Great Theorem

While the behaviour of holomorphic functions near their removable singularities and poles is fairly well-behaved, the same cannot be said for essential singularities.

Already an interesting result is given by the Casorati-Weierstrass Theorem: Let $G \subseteq \mathbb{C}$ be open, $w \in G$ and suppose $f \in H(G \setminus \{w\})$ such that f has an essential singularity at w. Then, for any punctured neighborhood $U \subseteq G$ of w, the set $f(U)$ is dense in $\mathbb{C}$.

Picard's Great Theorem will show that $f(U)$ is not only dense in $\mathbb{C}$, but that there is at most one value that is not taken on by f infinitely often, on any such punctured neighborhood.

To obtain the proof we first study Bloch's Theorem, which estimatates the size of disks in the image of a holomorphic map. We then immediately obtain Picard's Little Theorem, which states that an entire function which omits two values is constant.

One can see this as a motivation to more closely study (non-entire) holomorphic functions that omit (at least) two values. For such functions, Schottky's Theorem will give an upper bound on their modulus. We use this to obtain a sharped version of Montel's Theorem, which asserts normality of any family of holomorphic functions, which omit two fixed values.

Restricting the domain in the above to be the punctured unit disk, we use the above to obtain that such functions cannot have an essential singularity at the origin, which precisely yields Picard's Great Theorem.

2.1 Bloch's Theorem

If $G \subseteq \mathbb{C}$ is a domain and $f \in H(G)$ is non-constant, then $f(G)$ is a domain as well. In particular, $f(G)$ contains open disks of some, potentially very small, radius. Bloch's Theorem will show that for any $f \in H(D) \cap C(\overline{\mathbb{D}}; \mathbb{C})$ satisfying $f'(0) = 1$, the set $f(\mathbb{D})$ always contains a disk of fixed radius.

Note that $f(0)$ may not always be the center of such a disk; one must simply consider the sequence

$$f_n(z) := \frac{1 - e^{-nz}}{n} \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}}; \mathbb{C}), \quad n \in \mathbb{N},$$

which satisfies $f_n(0) = 0$ and $f_n'(0) = 1$, but omits the value $1/n$.

If f is as before, then f is an open mapping on G; we thus first observe a general criterion for the size of disks in image domains:

Lemma 2.1.1. *Let $G \subset \mathbb{C}$ be a bounded domain and $f \in C(\overline{G}; \mathbb{C})$ such that $f\big|_G$ is an* *open mapping. If there exists an $a \in G$ such that $s := \min_{z\in\partial G} |f(z) - f(a)| > 0$, then $B_s(f(a)) \subseteq f(G)$.*

Proof. Since G is bounded, $\overline{G}$ is compact and by continuity of f, so is $\overline{f(G)}$. The function $z \mapsto |z - f(a)|$ is continuous on the compact set $\partial f(G)$, hence it assumes its minimum m at some $w_* \in \partial f(G)$. Choose a sequence $(z_n)_{n\in\mathbb{N}}$ in G with $\lim_{n\to\infty} f(z_n) = w_*$ then, since $\overline{G}$ is compact, we can find a subsequence that converges to some $z_* \in \overline{G}$. By continuity of f we have $f(z_*) = w_*$.

If $z_* \in G$, since $f\big|_G$ is open, the image of any open set in G containing z_* under f is an open set in $f(G)$ containing w_*, which is impossible since $w_* \in \partial f(G)$.

Therefore $z_* \in \partial G$ and we have

$$d(f(a), \partial f(G)) = m = |w_* - f(a)| = |f(z_*) - f(a)| \geq s,$$

which implies $B_s(f(a)) \subseteq f(G)$. ■

Lemma 2.1.2. *Fix $a \in \mathbb{C}, r > 0$ and let $B := B_r(a)$. Suppose further that $f \in H(B) \cap C(\overline{B};\mathbb{C})$ such that $\|f'\|_B \leq 2|f'(a)|$. Then $B_R(f(a)) \subseteq f(B)$, where $R := (3-2\sqrt{2})r|f'(a)|$.*

Proof. We may assume $a = f(a) = 0$, otherwise we consider $f_1(z) := f(z+a) - f(a)$. The function

$$\alpha_f : \begin{cases} B \to \mathbb{C}, \\ z \mapsto f(z) - f'(0)z, \end{cases}$$

satifies, for all $z \in B$,

$$|\alpha_f(z)| = \left| \int_{[0,z]} f'(\zeta) - f'(0)\,\mathrm{d}\zeta \right| \leq \int_0^1 |f'(tz) - f'(0)||z|\,\mathrm{d}t. \tag{$*$}$$

We wish to further estimate the integrand. Let $w \in B$, then Cauchy's integral formula gives

$$\begin{aligned} |f'(w) - f'(0)| = \frac{1}{2\pi}\left| \int_{\partial B} \frac{f'(\zeta)}{\zeta - w} - \frac{f'(\zeta)}{\zeta}\,\mathrm{d}\zeta \right| = \frac{1}{2\pi}\left| \int_{\partial B} \frac{wf'(\zeta)}{\zeta(\zeta - w)}\,\mathrm{d}\zeta \right| \leq \\ \leq \frac{1}{2\pi}\int_{\partial B} \frac{|w|\|f'\|_B}{r(r-|w|)}\,\mathrm{d}\zeta = \frac{|w|}{r-|w|}\|f'\|_B. \end{aligned}$$

Combining the above with $(*)$ and our estimate on $\|f'\|_B$ yields

$$|\alpha_f(z)| \leq \int_0^1 \frac{|zt|\|f'\|_B}{r-|zt|}|z|\,\mathrm{d}t \leq \frac{|z|^2}{r-|z|}\|f'\|_B \int_0^1 t\,\mathrm{d}t \leq \frac{|z|^2}{r-|z|}|f'(0)|.$$

Let $0 < \rho < r$, then for $|z| = \rho$ we have

$$|f'(0)|\rho - |f(z)| \leq |\alpha_f(z)| \leq \frac{\rho^2}{r-\rho}|f'(0)|$$
$$\iff \quad |f(z)| \geq \left(\rho - \frac{\rho^2}{r-\rho}\right)|f'(0)|.$$

The function $\rho \mapsto \rho - \rho^2/(r-\rho)$ assumes its maximum value at $\rho_* := (1-\sqrt{2}/2)r \in (0,r)$, namely $(3-2\sqrt{2})r$. Therefore,

$$|f(z)| \geq (3-2\sqrt{2})r|f'(0)| = R, \quad \text{for all} \quad |z| = \rho_*.$$

In particular, $\min_{z\in\partial B_{\rho_*}} |f(z)| \geq R > 0$, thus invoking Lemma 2.1.1 with the domain $B_{\rho_*}(0)$ yields $B_R(0) \subseteq f(B_{\rho_*}(0)) \subseteq f(B)$. ■

Theorem 2.1.3. *Let $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}};\mathbb{C})$ be non-constant. Then there is a point $p \in \mathbb{D}$ and a constant $C_f > 0$ such that $B_R(f(p)) \subseteq f(\mathbb{D})$, where $R := (\frac{3}{2}-\sqrt{2})C_f \geq (\frac{3}{2}-\sqrt{2})|f'(0)|$.*

Proof. The function

$$\alpha_f : \begin{cases} \overline{\mathbb{D}} \to \mathbb{R} \\ z \mapsto |f'(z)|(1-|z|) \end{cases}$$

is continuous and assumes its maximum $C_f > 0$ at some point $p \in \overline{\mathbb{D}}$. Note that $C_f \geq |f'(0)|$ and since f is non-constant and $\alpha_f\big|_{\partial\mathbb{D}} = 0$ we even have $p \in \mathbb{D}$.

Set $t := \frac{1}{2}(1-|p|) > 0$, then we have $B_t(p) \subseteq \mathbb{D}$. Furthermore, for $z \in B_t(p)$, we have

$$1-|z| \geq 1-|z-p|-|p| \geq 1-t-|p| = t$$

and since $|f'(z)|(1-|z|) \leq C_f = 2t|f'(p)|$, this implies $|f'(z)| \leq 2|f'(p)|$ for all $z \in B_t(p)$. By Lemma 2.1.2, we get $B_R(f(p)) \subseteq f(\mathbb{D})$, where $R := (3-2\sqrt{2})t|f'(p)| = (\frac{3}{2}-\sqrt{2})C_f$, which establishes the assertion. ■

We now immediately obtain:

Theorem 2.1.4 (Bloch)**.** *Let $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}};\mathbb{C})$ and assume $f'(0) = 1$. Then $f(\mathbb{D})$ contains a disk of radius $\frac{3}{2}-\sqrt{2}$.*

In the following we will denote by $\beta > 0$ any constant less than or equal to the radius in Bloch's Theorem, for example $\beta = \frac{1}{12} < \frac{3}{2}-\sqrt{2}$.

Corollary 2.1.5. *Let $G \subseteq \mathbb{C}$ be a domain and $f \in H(G)$ with $f'(c) \neq 0$ for some $c \in G$. Then $f(G)$ contains a disk of every radius $\beta s|f'(c)|$, where $0 < s < d(c,\partial G)$.*

Proof. We may assume $c = 0$, otherwise we consider $f_1(z) := f(z+c)$. Let $0 < s < d(c,\partial G)$, then f is analytic on $\overline{B_s(0)} \subseteq G$, thus we have $g(z) := f(sz)/sf'(0) \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}};\mathbb{C})$.

Since $g'(0) = 1$, Bloch's Theorem yields a disk B of radius β with $B \subseteq g(\mathbb{D})$. Then $D := s|f'(0)|B$ is a disk of radius $\beta s|f'(0)|$ and we have

$$D = s|f'(0)|B \subseteq s|f'(0)|g(\mathbb{D}) = f(B_s(0)) \subseteq f(G).$$

■

Corollary 2.1.6. *If $f \in H(\mathbb{C})$ is non-constant, then $f(\mathbb{C})$ contains a disk of every radius.*

2.2 Schottky's Theorem

Holomorphic functions which omit the values 0 and 1 have a universal estimate on the growth of their modulus, which will be given by Schottky's Theorem.

For a domain $G \subseteq \mathbb{C}$ and a set $E \subseteq \mathbb{C}$ we define $H(G; E)$ as the set of all $f \in H(G)$ such that $f(G) \subseteq E$.

Lemma 2.2.1. *It holds that:*

i. If $a, b \in \mathbb{R}$ with $\cos \pi a = \cos \pi b$, then $b = \pm a + 2n$ for some $n \in \mathbb{Z}$.

ii. For every $w \in \mathbb{C}$ there exists a $v \in \mathbb{C}$ such that $\cos \pi v = w$ and $|v| \leq 1 + |w|$.

Proof. For the first part, it suffices to notice that

$$0 = \cos \pi a - \cos \pi b = -2 \sin \tfrac{\pi}{2}(a + b) \sin \tfrac{\pi}{2}(a - b).$$

Since the complex cosine function is surjective and $\mathbb{R}$-periodic, we can choose $v = a + ib$ with $w = \cos \pi v$ and $|a| \leq 1$. Now we have

$$\begin{aligned}
|w|^2 &= |\cos(\pi a + i\pi b)|^2 = |\cos \pi a \cos i\pi b + \sin \pi a \sin i\pi b|^2 = \\
&= |\cos \pi a \cosh \pi b - i \sin \pi a \sinh \pi b|^2 = \\
&= \cos^2 \pi a \cosh^2 \pi b + \sin^2 \pi a \sinh^2 \pi b = \\
&= \cos^2 \pi a + \cos^2 \pi a \sinh^2 \pi b + \sin^2 \pi a \sinh^2 \pi b = \\
&= \cos^2 \pi a + \sinh^2 \pi b \geq \sinh^2 \pi b \geq \pi^2 b^2,
\end{aligned}$$

where the last inequality holds since $\sinh x \geq x$ for $x \geq 0$. We conclude

$$|v| = \sqrt{a^2 + b^2} \leq \sqrt{1 + |w|^2/\pi^2} \leq 1 + |w|$$

■

We recall the following result: Let $G \subseteq \mathbb{C}$ be a simply connected domain and $f \in H(G)$ such that f vanishes nowhere on G. Then there is a $g \in H(G)$ such that $f = e^g$. This can also be used to obtain multiplicative n-th roots of such functions, by defining $\sqrt[n]{f} := e^{g/n}$.

Lemma 2.2.2. *Let $G \subseteq \mathbb{C}$ be a simply connected domain and $f \in H(G; \mathbb{C} \setminus \{-1, 1\})$. Then there exists an $F \in H(G)$ such that $f = \cos F$.*

Proof. Since $1 - f^2$ vanishes nowhere in G it has a square root $g \in H(G)$. It follows that

$$1 = f^2 + g^2 = (f + ig)(f - ig).$$

Thus $f + ig$ vanishes nowhere and there exists an $F \in H(G)$ with $f + ig = e^{iF}$. By the above we also have $f - ig = e^{-iF}$ and therefore

$$f = \tfrac{1}{2}(e^{iF} + e^{-iF}) = \cos F.$$

■

Lemma 2.2.3. *Let $G \subseteq \mathbb{C}$ be a simply connected domain and $f \in H(G; \mathbb{C} \setminus \{0, 1\})$. Then there exists a $g \in H(G)$ such that:*

i. $f = \frac{1}{2}(1 + \cos \pi(\cos \pi g))$.

ii. $|g(0)| \leq 3 + 2|f(0)|$.

iii. $g(G)$ *contains no disk of radius* 1.

iv. *If* $\mathbb{D} \subseteq G$ *then* $|g(z)| \leq |g(0)| + \theta/b(1 - \theta)$*, for all* $|z| \leq \theta$ *where* $0 < \theta < 1$.

Proof. By Lemma 2.2.2 there exists a $\widetilde{F} \in H(G)$ such that $2f - 1 = \cos \pi \widetilde{F}$ and by Lemma 2.2.1 there is a $b \in \mathbb{C}$ such that $\cos \pi b = 2f(0) - 1$ and $|b| \leq 1 + |2f(0) - 1| \leq 2 + 2|f(0)|$. Furthermore, since $\cos \pi b = \cos \pi \widetilde{F}(0)$, we have $b = \pm \widetilde{F}(0) + 2k$ for some $k \in \mathbb{Z}$. Then $F \coloneqq \pm \widetilde{F} + 2k \in H(G)$ satifies $F(0) = b$ and $2f - 1 = \cos \pi F$.

Since F must omit all integers, there exists a $\widetilde{g} \in H(G)$ such that $F = \cos \pi \widetilde{g}$. Similarly, there is an $a \in \mathbb{C}$ such that $\cos \pi a = b$ and $|a| \leq 1 + |b| \leq 3 + 2|f(0)|$. Like above, since $\cos \pi a = \cos \pi \widetilde{g}(0)$, we have $a = \pm \widetilde{g}(0) + 2\ell$ for some $\ell \in \mathbb{Z}$, thus $g \coloneqq \pm \widetilde{g} + 2\ell \in H(G)$ satisfies $g(0) = a$ and $F = \cos \pi g$. Ultimately, we obtain

$$f = \tfrac{1}{2}(1 + \cos \pi(\cos \pi g)), \quad \text{and} \quad |g(0)| = |a| \leq 3 + 2|f(0)|$$

and have thus shown (i) and (ii).

To show (iii) we consider the set

$$A \coloneqq \{m \pm i\pi^{-1} \log(n + \sqrt{n^2 - 1}) : m \in \mathbb{Z}, n \in \mathbb{N} \setminus \{0\}\},$$

the points of which can be considered the vertices of a rectangular grid in $\mathbb{C}$. The width of such a rectangular cell is 1, and since

$$\log((n+1)+\sqrt{(n+1)^2-1})-\log(n+\sqrt{n^2-1}) =$$
$$= \log\frac{1+\frac{1}{n}+\sqrt{1+\frac{2}{n}}}{1+\sqrt{1-\frac{1}{n^2}}} \leq \log(2+\sqrt{3}) < \pi$$

their height is bounded above by some $C<1$. Therefore, for all $z\in\mathbb{C}$ there is a $w_z\in A$ such that $|\operatorname{Re} z-\operatorname{Re} w_z|\leq\frac{1}{2}$ and $|\operatorname{Im} z-\operatorname{Im} w_z|\leq\frac{C}{2}$. Thus we have

$$|z-w_z|\leq|\operatorname{Re} z-\operatorname{Re} w_z|+|\operatorname{Im} z-\operatorname{Im} w_z|\leq\frac{1}{2}+\frac{C}{2}<1.$$

If we can show that $g(G)\cap A=\emptyset$, then $g(G)$ therefore cannot contain a disk of radius 1. Let $a=p+i\pi^{-1}\log(q+\sqrt{q^2-1})\in A$, then

$$\cos\pi a=\tfrac{1}{2}(e^{i\pi a}+e^{-i\pi a})=\tfrac{1}{2}(-1)^p((q+\sqrt{q^2-1})^{-1}+(q+\sqrt{q^2-1}))=$$
$$=(-1)^p\frac{1}{2}\frac{1+q^2+2q\sqrt{q^2-1}+q^2-1}{q+\sqrt{q^2-1}}=(-1)^p q$$

and thus $\cos\pi(\cos\pi a)=\pm1$. But $0,1\notin f(G)$, therefore $a\notin g(G)$ and $g(G)\cap A=\emptyset$, proving (iii).

For (iv), if $\mathbb{D}\subseteq G$, then $g\big|_{\mathbb{D}}\in H(\mathbb{D})$. Fix $0<\theta<1$, then for $|z|\leq\theta$ we have

$$d(z,\partial\mathbb{D})=\inf_{w\in\partial\mathbb{D}}|z-w|\geq\inf_{w\in\partial\mathbb{D}}(|w|-|z|)\geq1-\theta.$$

From (iii) it follows that $g\big|_{\mathbb{D}}(\mathbb{D})$ does not contain a disk of radius 1. Let $0<s<1-\theta$, then applying Corollary 2.1.5 to $g\big|_{\mathbb{D}}$ implies that $\beta s|g'(z)|<1$. Taking the supremum over s and rearranging yields $|g'(z)|\leq1/(\beta(1-\theta))$. Thus our desired estimate is shown by

$$|g(z)|\leq|g(0)|+|g(z)-g(0)|\leq|g(0)|+\int_{[0,z]}|g'(\zeta)|\,\mathrm{d}\zeta\leq|g(0)|+\frac{\theta}{\beta(1-\theta)}.$$

■

The result we just proved is quite powerful, as it contains not only Schottky's Theorem, but Picard's Little Theorem as well:

Theorem 2.2.4 (Picard's Little Theorem)**.** *Let $f\in H(\mathbb{C};\mathbb{C}\setminus\{a,b\})$ for distinct points $a,b\in\mathbb{C}$. Then f is constant.*

Proof. Consider $f_1(z):=\frac{f(z)-a}{b-a}\in H(\mathbb{C};\mathbb{C}\setminus\{0,1\})$. By Lemma 2.2.3 there is some $g\in H(\mathbb{C})$ such that $f_1=\frac{1}{2}(1+\cos\pi(\cos\pi g))$ and $g(\mathbb{C})$ does not contain a disk of radius 1. By Corollary 2.1.6 we thereby have that g must be constant and therefore so are f_1 and f. ■

Theorem 2.2.5 (Schottky). *There exists a function $\psi : (0,1) \times (0,\infty) \to (0,\infty)$ such that for any $f \in H(\mathbb{D}; \mathbb{C} \setminus \{0,1\})$ with $|f(0)| \leq \omega$ it holds that*

$$|f(z)| \leq \psi(\theta,\omega), \quad |z| \leq \theta. \tag{2.1}$$

Proof. Note that for all $w \in \mathbb{C}$ we have $|\cos w| \leq e^{|w|}$ and $\frac{1}{2}|1+\cos w| \leq e^{|w|}$. Hence, from Lemma 2.2.3, we get

$$\begin{aligned} |f(z)| = |\tfrac{1}{2}(1+\cos\pi(\cos\pi g(z)))| &\leq \exp(\pi \exp \pi |g(z)|) \leq \\ &\leq \exp(\pi \exp \pi(|g(0)| + \theta/\beta(1-\theta))) \leq \\ &\leq \exp(\pi \exp \pi(3 + 2\omega + \theta/\beta(1-\theta))), \end{aligned}$$

and defining $\psi(\theta,\omega)$ as the final term establishes the assertion. ∎

2.3 Normal families

We first recall a generalized form of uniform convergence:

Definition 2.3.1. Let $G \subseteq \mathbb{C}$ be a domain, $f \in H(G)$ and $(f_n)_{n\in\mathbb{N}}$ a sequence in $H(G)$. We say that f_n *converges compactly in G to f*, or f_n *converges compactly in G to ∞* as $n \to \infty$, if for every compact set $K \subset G$

$$\lim_{n\to\infty} \sup_{z\in K} |f_n(z) - f(z)| = 0, \quad \text{or} \quad \lim_{n\to\infty} \inf_{z\in K} |f_n(z)| = \infty.$$

Remark 2.3.2. One can define compact convergence generally for functions from a topological space $(X, \mathcal{T})$ into a metric space (Y, d_Y). Compact convergence to ∞ can then be seen as compact convergence to a constant function with value ∞, where $d_Y(y,\infty)$ is appropriately defined for $y \in Y$. This is precisely the case when considering the chordal metric of the Riemann sphere, but we shall not elaborate further on this.

A well-known theorem on compact convervence is:

Theorem 2.3.3 (Hurwitz). *Let $G \subseteq \mathbb{C}$ be a domain and $(f_n)_{n\in\mathbb{N}}$ a sequence in $H(G)$ that converges compactly to $f \in H(G)$. If for every $n \in \mathbb{N}$ the number of a-points of f_n is bounded by some $m \in \mathbb{N}_0$, then either the number of a-points of f are also bounded by m, or $f \equiv a$.*

As an immediate consequence we obtain that compact convergence is, in some ways, compatible with reciprocals:

Lemma 2.3.4. *Let $G \subseteq \mathbb{C}$ be a domain and $(f_n)_{n\in\mathbb{N}}$ a sequence in $H(G; \mathbb{C} \setminus \{0\})$. If it converges compactly to some $f \in H(G)$, then it holds that either:*

- *$0 \notin f(G)$ and $1/f_n \to 1/f$ compactly in G, or*

- *$f \equiv 0$ and $1/f_n \to \infty$ compactly in G.*

If it converges compactly to ∞, then $1/f_n \to 0$ compactly in G.

Proof. For the equivalence "$f_n \to 0$ if and only if $1/f_n \to \infty$" it suffices to notice that for any compact $K \subset G$ we have

$$\frac{1}{\sup_{z\in K} |f_n(z)|} = \inf_{z\in K} \left| \frac{1}{f_n(z)} \right|.$$

Since the f_n vanish nowhere, by Hurwitz' Theorem we either have $0 \notin f(G)$ or $f \equiv 0$. In the latter case we have just shown that $1/f_n \to \infty$ compactly.

In the former case we have, again for any compact $K \subset G$, that $m := \min_{z\in K} |f(z)| > 0$ and $\sup_{z\in K} |f_n(z) - f(z)| < \frac{m}{2}$ for sufficiently large $n \in \mathbb{N}$. Thus for all $z \in K$

$$\frac{m}{2} > |f(z) - f_n(z)| \geq |f(z)| - |f_n(z)| \geq m - |f_n(z)|$$

and $|f_n(z)| \geq \frac{m}{2}$. We obtain, for large $n \in \mathbb{N}$,

$$\sup_{z\in K} \left| \frac{1}{f_n(z)} - \frac{1}{f(z)} \right| = \sup_{z\in K} \left| \frac{f(z) - f_n(z)}{f(z) f_n(z)} \right| \leq \sup_{z\in K} |f_n(z) - f(z)| \cdot \frac{1}{m} \cdot \frac{2}{m}$$

and therefore, after letting $n \to \infty$, that $1/f_n \to 1/f$ compactly in G. ∎

Definition 2.3.5. Let $G \subseteq \mathbb{C}$ be a domain and $\mathscr{F} \subseteq H(G)$. Then $\mathscr{F}$ is called:

- *locally bounded* if for every $w \in G$ there is a neighborhood U of w and a constant $C > 0$ such that $|f(z)| \leq C$ for all $f \in \mathscr{F}$ and $z \in U$.
- *normal* in G if every sequence in $\mathscr{F}$ has a subsequence which converges compactly in G to some $f \in H(G)$. If the limit ∞ is also permitted it is instead called $*$-*normal*.

The former two concepts are equivalent by the following well-known theorem:

Theorem 2.3.6 (Montel)**.** *Let $G \subseteq \mathbb{C}$ be a domain. Then a family $\mathscr{F} \subseteq H(G)$ is normal if and only if it is locally bounded.*

The following theorem can be interpreted as a sharpened version of Montel's Theorem and is sometimes referred to as the *fundamental normality test*.

Theorem 2.3.7. *Let $G \subseteq \mathbb{C}$ be a domain. Then any family $\mathscr{F} \subseteq H(G, \mathbb{C} \setminus \{0, 1\})$ is $*$-normal in G.*

Proof. We give the proof in three steps:

1. Let $w \in G$, $c > 0$ and $\mathscr{F}_* \subseteq \mathscr{F}$ such that $|f(w)| \leq c$ for all $f \in \mathscr{F}_*$. We aim to show that there is an open disk at w in which $\mathscr{F}_*$ is bounded. Select $t > 0$ such that

$B_t(w) \subseteq G$. Let $f \in \mathscr{F}_*$, then $g(z) := f(tz + w) \in H(\mathbb{D})$. By the maximum modulus principle and Schottky's Theorem we obtain

$$\sup_{z \in B_{t/2}(w)} |f(z)| \leq \sup_{z \in B_{1/2}(0)} |g(z)| \leq \sup_{|z|=1/2} |g(z)| \leq \psi(1/2, c)$$

and f is bounded on the disk $B_{t/2}(w)$. Since f was arbitrary, $\mathscr{F}_*$ is bounded aswell.

2. Fix some $w_* \in G$ and set $\mathscr{F}_1 := \{f \in \mathscr{F} : |f(w_*)| \leq 1\}$. We aim to show that $\mathscr{F}_1$ is locally bounded in G. Consider the set

$$U := \{w \in G : \mathscr{F}_1 \text{ is bounded in a neighborhood of } w\},$$

by (1) we have that $w_* \in U$. Note that U is open in G, since if $\mathscr{F}_1$ is bounded in a disk $B_r(w)$, then for any $w' \in B_r(w)$ there is a disk $B_{r'}(w') \subseteq B_r(w)$, on which $\mathscr{F}_1$ is bounded as well.

For sake of contradiction, suppose that $U \neq G$. Then there exists some $w \in \partial U \cap G$ such $\mathscr{F}_1$ is unbounded in every neighborhood of w.

If there were some $c > 0$ such that $|f(w)| \leq c$ for all $f \in \mathscr{F}_1$, then by (1) there would exist an open disk centered at w on which $\mathscr{F}_1$ would be bounded – contradicting our assumption on w. Thus for every $n \in \mathbb{N}$ we can find some $f_n \in \mathscr{F}_1$ such that $|f_n(w)| \geq n$ and we obtain that $\lim_{n\to\infty} |f_n(w)| = \infty$.

Set $g_n := 1/f_n \in \mathscr{F}$, then $\lim_{n\to\infty} |g_n(w)| = 0$. In particular, the family $(g_n)_{n\in\mathbb{N}}$ is bounded at w by some constant, thus by (1) the family is bounded in some disk B around w. By Montel's Theorem it is therefore normal in B, and there exists a subsequence $(g_{n_k})_{k\in\mathbb{N}}$ which converges compactly to a $g \in H(B)$. The functions g_{n_k} have no zeros, but $g(w) = 0$; by Hurwitz's Theorem we therefore have $g \equiv 0$. Then for any $z \in B \cap U$ we have

$$\lim_{k\to\infty} f_{n_k}(z) = \lim_{k\to\infty} 1/g_{n_k}(z) = \infty,$$

contradicting the assumption that $\mathscr{F}_1$ is bounded in a neighborhood of such z. We thus have $U = G$, therefore $\mathscr{F}_1$ is locally bounded and by Montel's Theorem therefore normal.

3. We can now conclude the proof. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{F}$, we claim that it has some subsequence which converges compactly to some function in $H(G)$ or to ∞.

If infinitely many f_n lie in $\mathscr{F}_1$, then there is a subsequence $(f_{n_m})_{m\in\mathbb{N}}$ in $\mathscr{F}_1$, which by (2) has a subsequence $(f_{n_{m_k}})_{k\in\mathbb{N}}$ in $\mathscr{F}_1$ which convergences compactly in G to some $f \in H(G)$. This sequence is also a subsequence of $(f_n)_{n\in\mathbb{N}}$, concluding the claim in this case.

On the other hand, if there are only finitely many f_n in $\mathscr{F}_1$, then infinitely many $1/f_n$ lie in $\mathscr{F}_1$. As above, we thus obtain some subsequence in $\mathscr{F}_1$, say $(g_n)_{n\in\mathbb{N}}$, converging compactly in G to some $g \in H(G)$. The sequence $(1/g_n)_{n\in\mathbb{N}}$ is a subsequence of $(f_n)_{n\in\mathbb{N}}$, which – by Lemma 2.3.4 – converges compactly to $1/g$ if $0 \notin g(G)$, and to

∞ otherwise.

■

2.4 Picard's Great Theorem

The following lemma is integral in showing that functions which omit two values on the punctured unit disk cannot have an essential singularity at 0.

Lemma 2.4.1. *Let $f \in H(\mathbb{D}^\times; \mathbb{C} \setminus \{0, 1\})$. Then f or $1/f$ is bounded in a punctured neighborhood of zero.*

Proof. For $n \in \mathbb{N}$ set $f_n(z) := f(z/n) \in H(\mathbb{D}^\times; \mathbb{C} \setminus \{0, 1\})$. By Theorem 2.3.7 the sequence $(f_n)_{n\in\mathbb{N}}$ has a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ that converges compactly to a $f \in H(\mathbb{D}^\times)$ or to ∞.

Assume the former case. Then there is some $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have $\|f_{n_k} - f\|_{\partial B_{1/2}(0)} < 1$ and thus

$$\|f\|_{\partial B_{1/2n_k}(0)} = \|f_{n_k}\|_{\partial B_{1/2}(0)} \leq \|f_{n_k} - f\|_{\partial B_{1/2}(0)} + \|f\|_{\partial B_{1/2}(0)} \leq 1 + \|f\|_{\partial B_{1/2}(0)} =: C.$$

By the maximum modulus principle, f must therefore be bounded on every annulus $1/(2n_{k+1}) \leq |z| \leq 1/(2n_k)$, for $k \geq k_0$. Thus f is bounded on

$$V := \bigcup_{k\geq k_0} \left\{ z \in \mathbb{C} : \frac{1}{2n_{k+1}} \leq |z| \leq \frac{1}{2n_k} \right\},$$

which is a punctured neighborhood of zero.

In the latter case, $(1/f_{n_k})_{k\in\mathbb{N}}$ converges compactly to 0 by Lemma 2.3.4. Replacing f_{n_k} with $1/f_{n_k}$ and f with 0 in the above we likewise obtain that $1/f$ is bounded in a punctured neighborhood of zero. ■

Theorem 2.4.2 (Picard's Great Theorem)**.** *Let $G \subseteq \mathbb{C}$ be open, $w \in G$ and suppose $f \in H(G \setminus \{w\})$ such that f has an essential singularity at w. Then f assumes all values in $\mathbb{C}$, with at most one exception, infinitely often in any punctured neighborhood of w.*

Proof. Aiming for contradiction, assume that f only takes on $z_0, z_1 \in \mathbb{C}$ finitely often in some punctured neighborhood W of w. Then W contains a punctured disk of radius $t > 0$ around w, on which f does not assume z_0 or z_1, and thus

$$g(z) := \frac{f(tz + w) - z_0}{z_1 - z_0} \in H(\mathbb{D}^\times; \mathbb{C} \setminus \{0, 1\}),$$

where g has an essential singularity at zero. By Lemma 2.4.1, we have that either g or $1/g$ must be bounded in a punctured neighborhood of zero. By the classification of isolated singularities, in the former case the singularity must therefore be removable, whereas in the latter case it must be a pole, yielding a contradiction. ■

The following corollary is also referred to as Picard's Great Theorem:

Corollary 2.4.3. *Every entire transcendental function assumes every value in* $\mathbb{C}$ *infinitely often, with at most one exception.*

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be transcendental and entire, and consider $g(z) := f(1/z) \in H(\mathbb{C}^\times)$. Then the Laurent series expansion of g at 0 has infinite principal part, therefore g has an essential singularity at zero. By Picard's Great Theorem g assumes all values in $\mathbb{C}$ on $B_1(0) \setminus \{0\}$ infinitely often, except at most one, and so f does the same on $\mathbb{C} \setminus B_1(0)$. ■

Remark 2.4.4. Picard's Little Theorem is contained in Picard's Great Theorem:

A non-constant $f \in H(\mathbb{C})$ is either a non-constant polynomial or transcendental. In the latter case, f omits at most one value by Corollary 2.4.3.

In the former case let $w \in \mathbb{C}$, then $f(z) - w$ has a zero in $\mathbb{C}$ by the Fundamental Theorem of Algebra and hence f assumes all values.

3 Growth and Zeros Distribution

3.1 Order and Type

To study the rate of growth of an entire function we first introduce the following:

Definition 3.1.1. Let $f \in H(\mathbb{C})$, then for $r \geq 0$ we define

$$M_f(r) := \max_{|z|=r} |f(z)|.$$

By the maximum modulus principle, M_f is monotonically increasing. We shall show it is continuous: Let $\varepsilon > 0$ and choose $\delta_\varepsilon > 0$ such that for $|z_1 - z_2| < \delta_\varepsilon$ we have $||f(z_1)| - |f(z_2)|| < \varepsilon$. Now let $r_1 < r_2$ and choose θ such that $M_f(r_2) = |f(r_2 e^{i\theta})|$. Then

$$0 \leq M_f(r_2) - M_f(r_1) \leq |f(r_2 e^{i\theta})| - |f(r_1 e^{i\theta})| < \varepsilon,$$

for $r_2 - r_1 < \delta_\varepsilon$.

Definition 3.1.2. Let $f \in H(\mathbb{C})$. The *order* of f is defined by

$$\rho_f := \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}. \tag{3.1}$$

Constant functions, by convention, have order 0.

Note that, for any entire function f, we have $0 \leq \rho_f \leq \infty$.

For functions of finite order, we have an equivalent characterization:

Proposition 3.1.3. *Let $f \in H(\mathbb{C})$, then f is of finite order if and only if*

$$\rho := \inf\{s > 0 : M_f(r) = O(\exp r^s) \text{ as } r \to \infty\} \tag{3.2}$$

is finite, and in either case we have $\rho_f = \rho$.

Proof. Suppose $0 \leq \rho < \infty$, then for all $\varepsilon > 0$ we have $M_f(r) = O(\exp r^{\rho+\varepsilon})$ as $r \to \infty$. Thus there exists a constant $K_\varepsilon > 0$ (we may assume $K_\varepsilon > 1$) and an $r_0 > 0$ such that for all $r \geq r_0$ we have $M_f(r) \leq K_\varepsilon \exp r^{\rho+\varepsilon}$. Using the fact that $\log(a+b) = \log a + \log(1 + \frac{b}{a})$

we get

$$\frac{\log\log M_f(r)}{\log r} \leq \frac{\log(r^{\rho+\varepsilon} + \log K_\varepsilon)}{\log r} = \rho + \varepsilon + \frac{\log(1 + \log(K_\varepsilon)/r^{\rho+\varepsilon})}{\log r}. \qquad (*)$$

Since

$$0 \leq \frac{\log(1 + \log(K_\varepsilon)/r^{\rho+\varepsilon})}{\log r} \leq \frac{\log(K_\varepsilon)/r^{\rho+\varepsilon}}{\log r} = \frac{\log K_\varepsilon}{r^{\rho+\varepsilon}\log r} \xrightarrow{r\to\infty} 0,$$

by taking the limit superior as $r \to \infty$ in $(*)$ and then letting $\varepsilon \to 0$ we obtain $\rho_f \leq \rho$.

Now suppose $0 \leq \rho_f < \infty$ and let $\varepsilon > 0$. Then, by definition of the limit superior, there is an $r_0 > 0$ such that for all $r \geq r_0$ we have $\log\log M_f(r) \leq (\rho_f + \varepsilon)\log r$ and therefore $M_f(r) \leq \exp(r^{\rho_f+\varepsilon})$. Thus $M_f(r) = O(\exp r^{\rho_f+\varepsilon})$, thereby $\rho \leq \rho_f + \varepsilon$ and letting $\varepsilon \to 0$ yields $\rho \leq \rho_f$. ∎

Remark 3.1.4. Any polynomial is of order zero. Indeed, let $P(z) = \sum_{k=0}^n a_k z^k$ be a polynomial and $m \in \mathbb{N}$. Then for any $r > 0$ we have

$$\exp r^{1/m} = \sum_{k=0}^\infty \frac{r^{k/m}}{k!} > \frac{r^n}{(mn)!} \geq \frac{r^n}{(mn)!}$$

and therefore

$$M_P(r) = \max_{|z|=r} |P(z)| \leq n \max_{k=0,\dots,n} |a_k| r^n \leq \left(n \max_{k=0,\dots,n} |a_k|(mn)!\right) \exp r^{1/m}.$$

Since $m \in \mathbb{N}$ was arbitrary, Proposition 3.1.3 gives $\rho_P = 0$.

There are however non-polynomial functions of order zero; one must simply construct an entire function of sufficiently rapid decay. Consider the entire function

$$f(z) := \sum_{k=0}^\infty \frac{z^k}{(k^2)!}$$

and let $m \in \mathbb{N}$. By the above we have $\sum_{k=0}^{m-1} \frac{r^k}{(k^2)!} \leq K \exp r^{1/m}$ for some constant $K_m > 0$ depending only on m and all $r > 0$. Thus

$$\sum_{k=0}^\infty \frac{r^k}{(k^2)!} \leq K_m \exp r^{1/m} + \sum_{k=m}^\infty \frac{r^k}{(km)!} \leq K_m + \sum_{k=0}^\infty \frac{r^{k/m}}{k!} \leq (K_m + 1)\exp r^{1/m}$$

and we obtain $\rho_f = 0$, as above.

Remark 3.1.5. If $Q(z) = \sum_{k=0}^n a_k z^k$ is a polynomial we claim that $f(z) := \exp Q(z)$ is of order n. Let $\varepsilon > 0$, then since $\lim_{r\to\infty}(\sum_{k=0}^n |a_k| r^k)/r^{n+\varepsilon} = 0$ there is some r_0 such that for all $r > r_0$ we have $(\sum_{k=0}^{n} |a_k| r^k)/r^{n+\varepsilon} < 1$ and thus

$$M_f(r) \leq \exp \sum_{k=0}^{n} |a_n| r^k \leq \exp(r^{n+\varepsilon}),$$

showing $\rho_f \leq n + \varepsilon$. On the other hand, since

Proposition 3.1.6. *Let $f, g \in H(\mathbb{C})$ be of finite order. Then it holds that:*

i. $\rho_{f+g} \leq \max\{\rho_f, \rho_g\}$*, with equality holding if* $\rho_f \neq \rho_g$.

ii. $\rho_{fg} \leq \max\{\rho_f, \rho_g\}$.

Proof. Let $\varepsilon > 0$, then by Proposition 3.1.3 there are constants $C_f, C_g, r_f, r_g > 0$ such that for $r > r_f$ we have $M_f(r) \leq A \exp(r^{\rho_f + \varepsilon})$ and for $r > r_g$ we have $M_g(r) \leq C_g \exp(r^{\rho_g + \varepsilon})$. Thus, for $r > \max\{r_f, r_g\}$, we obtain

$$\begin{aligned} M_{f+g}(r) &\leq 2 \max\{M_f(r), M_g(r)\} \leq 2 \max\{C_f \exp(r^{\rho_f+\varepsilon}), C_g \exp(r^{\rho_g+\varepsilon})\} \leq \\ &\leq 2 \max\{C_f, C_g\} \exp(r^{\max\{\rho_f, \rho_g\}+\varepsilon}), \end{aligned}$$

thus $M_{f+g}(r) = O(\exp r^{\max\{\rho_f,\rho_g\}+\varepsilon})$ for all $\varepsilon > 0$, proving $\rho_{f+g} \leq \max\{\rho_f, \rho_g\}$. Similarly, for $r > \max\{r_f, r_g, 2^{1/\varepsilon}\}$, we have

$$\begin{aligned} M_{fg}(r) &\leq \max\{M_f(r), M_g(r)\}^2 \leq \max\{C_f \exp(r^{\rho_f+\varepsilon}), C_g \exp(r^{\rho_g+\varepsilon})\}^2 \leq \\ &\leq \max\{C_f, C_g\}^2 \exp(2r^{\max\{\rho_f,\rho_g\}+\varepsilon}) \max\{C_f, C_g\}^2 \exp(r^{\max\{\rho_f,\rho_g\}+2\varepsilon}) \end{aligned}$$

which shows $\rho_{fg} \leq \max\{\rho_f, \rho_g\}$.

Now assume $\rho_f \neq \rho_g$, without loss of generality $\rho_f < \rho_g$. By the above we have

$$\rho_{f+g} \leq \max\{\rho_f, \rho_g\} = \rho_g, \quad \text{and} \quad \rho_g = \rho_{f+g+(-f)} \leq \max\{\rho_f, \rho_{f+g}\} = \rho_{f+g},$$

since if $\rho_f > \rho_{f+g}$ then $\rho_f > \rho_{f+g} \geq \rho_g$ is a contradiction. Thus $\rho_{f+g} = \rho_f$, proving the assertion.

■

It is of note that Proposition 3.1.6 implies that the order of an entire function of finite order remains unchanged when adding a polynomial of any degree to it.

A slightly deeper result is that, for entire functions of finite order, the order is invariant under derivatives:

Proposition 3.1.7. *If $f \in H(\mathbb{C})$ is of finite order, then $\rho_{f'} = \rho_f$.*

Proof. Without loss of generality we may assume $f(0) = 0$. If f is a polynomial the assertion is clear. Otherwise for any $r > 0$ we have, using the maximum modulus principle,

$$M_f(r) = \max_{|z|=r} |f(z)| = \max_{|z|=r} \left| \int_{[0,z]} f'(\zeta)\, \mathrm{d}\zeta \right| \leq$$
$$\leq \max_{|z|=r} \left(|z| \max_{w \in \overline{B_r(0)}} |f'(w)| \right) + \leq r M_{f'}(r).$$

By Cauchy's integral formula we have

$$M_{f'}(r) = \max_{z \in \partial B_r(0)} |f'(z)| = \max_{z \in \partial B_r(0)} \left| \frac{1}{2\pi i} \int_{\partial B_r(z)} \frac{f(\zeta)}{(\zeta - z)^2}\, \mathrm{d}\zeta \right| \leq$$
$$\leq \frac{1}{2\pi} 2\pi r \max_{\substack{z \in \partial B_r(0) \\ w \in \partial B_r(z)}} |f(w)| \frac{1}{r^2} \leq \frac{M_f(2r)}{r}.$$

Combining the above we get

$$M_f(r) \leq r M_{f'}(r) \leq M_f(2r).$$

Applying the logarithm twice and dividing by $\log r$ thus yields, for sufficiently large r,

$$\frac{\log\log M_f(r)}{\log r} \leq \frac{\log\log(r M_{f'}(r))}{\log r} \leq \frac{\log\log M_f(2r)}{\log 2r} \frac{\log 2r}{\log r}. \tag{$*$}$$

But we have

$$\frac{\log\log(r M_{f'}(r))}{\log r} = \frac{\log(\log M_{f'}(r) + \log r)}{\log r} = \frac{\log\log M_{f'}(r)}{\log r} + \frac{\log(1 + \log r / \log M_{f'}(r))}{\log r}$$

and

$$0 \leq \frac{\log(1 + \log r / \log M_{f'}(r))}{\log r} \leq \frac{\log r / \log M_{f'}(r)}{\log r} = \frac{1}{\log M_{f'}(r)} \xrightarrow{r \to \infty} 0,$$

therefore by taking the limit superior as $r \to \infty$ in $(*)$ we obtain

$$\rho_f = \limsup_{r \to \infty} \frac{\log\log(r M_{f'}(r))}{\log r} = \limsup_{r \to \infty} \frac{\log\log M_{f'}(r)}{\log r} = \rho_{f'},$$

concluding the claim. ■

For functions of finite and positive order, we can obtain a natural refinement of the concept of order:

Definition 3.1.8. Let $f \in H(\mathbb{C})$ be of order $0 < \rho_f < \infty$. The *type* of f is defined by

$$\tau_f := \limsup_{r \to \infty} \frac{\log M_f(r)}{r^{\rho_f}} \tag{3.3}$$

For any $f \in H(\mathbb{C})$ with $0 < \rho_f < \infty$, we have $0 \leq \tau_f \leq \infty$.

Once again, we have an equivalent characterization for functions of finite order and type:

Proposition 3.1.9. *Let $f \in H(\mathbb{C})$ be of finite order, then f is of finite type if and only if*

$$\tau := \inf\{t > 0 : M_f(r) = O(\exp(tr^{\rho_f})) \text{ as } r \to \infty\} \tag{3.4}$$

is finite, and in either case we have $\tau_f = \tau$.

Proof. Suppose $0 \leq \tau < \infty$, then for all $\varepsilon > 0$ we have $M_f(r) = O(\exp(\tau + \varepsilon)r^{\rho_f})$ as $r \to \infty$. Thus there exists a constant $K_\varepsilon > 0$ and an $r_0 > 0$ such that for all $r \geq r_0$ we have $M_f(r) \leq K_\varepsilon \exp(\tau + \varepsilon)r^{\rho_f}$. Therefore

$$\frac{\log M_f(r)}{r^{\rho_f}} \leq \frac{\log K_\varepsilon + (\tau + \varepsilon)r^{\rho_f}}{r^{\rho_f}} = \frac{\log K_\varepsilon}{r^{\rho_f}} + \tau + \varepsilon$$

and taking the limit superior as $r \to \infty$ and letting $\varepsilon \to 0$ afterwards yields $\tau_f \leq \tau$.

Now suppose $0 \leq \tau_f < \infty$ and let $\varepsilon > 0$. Then, by definition of the limit superior, there is an $r_0 > 0$ such that for all $r \geq r_0$ we have $\log M_f(r) \leq (\tau_f + \varepsilon)r^{\rho_f}$ and therefore $M_f(r) \leq \exp((\tau_f + \varepsilon)r^{\rho_f})$. Thus $M_f(r) = O(\exp((\tau_f + \varepsilon)r^{\rho_f}))$, thereby $\tau \leq \tau_f + \varepsilon$ and letting $\varepsilon \to 0$ yields $\tau \leq \tau_f$. ■

Definition 3.1.10. Let $f \in H(\mathbb{C})$. Then f is said to be of *growth* (a, b) if

- $\rho_f < a$, or
- $\rho_f = a$ and $\tau_f \leq b$.

Example 3.1.11. For $\rho, \tau \in (0, \infty)$, the function

$$f(z) := \exp \tau z^\rho$$

is of order ρ and type τ.

We have already seen in Remark 3.1.4 that polynomials are of order zero.

The function

$$f(z) := \exp \exp z$$

is of infinite order, as seen by Todo.

3.2 Hadamard's Theorem

Theorem 3.2.1 (Weierstrass [1, 4])**.** *Let $(z_j)_{j \in \mathbb{N}}$ be a sequence in $\mathbb{C}$ without accumulation points. Then there exists an $E \in H(\mathbb{C})$ (called the* Weierstrass canonical product *formed from said sequence) that has zeros precisely at $(z_j)_{j \in \mathbb{N}}$, and z_j has multiplicity equal to how often z_j occurs in the sequence.*

In particular, we have

$$E(z) = z^k \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{R_n(z/z_n)}, \tag{3.5}$$

where k is the order of the zero at $z = 0$ and $R_n(z/z_n)$ is a polynomial, namely a truncation of the power series for $-\log(1 - \frac{z}{z_n})$ chosen of smallest degree to ensure convergence of the product [3].

Weierstrass' Theorem is also known as the *Weierstrass Factorization Theorem*, due to the following corollary:

Corollary 3.2.2. *Let $f \in H(\mathbb{C})$ have zeros $(z_j)_{j\in\mathbb{N}}$. Then there exists a $g \in H(\mathbb{C})$ such that*

$$f(z) = e^{g(z)} E(z),$$

where $E \in H(\mathbb{C})$ is a Weierstrass canonical product formed from $(z_j)_{j\in\mathbb{N}}$.

Proof. Since f/E has removable singularities at all $(z_j)_{j\in\mathbb{N}}$, we have that f/E is entire and nowhere zero. Thus there exists an entire function g with $f/E = e^g$, which yields $f = e^g E$. ■

Hadamard's Theorem will show that, for functions of finite order ρ, the function g in Corollary 3.2.2 can be taken to be a polynomial of degree less than ρ and the degree of the polynomials R_n in (3.5) can be taken to be independant of n. To prove this we want to study the connection between the order and zeros of an entire function.

We first recall a rather explicit connection between the moduli of the zeros of an analytic function and the modulus of the function itself:

Theorem 3.2.3 (Jensen [4])**.** *Let $f \in H(B_R(0))$ with $f(0) \neq 0$ and let $r_1, r_2, \ldots$ denote the moduli of the zeros of f in $B_R(0)$ arranged in a non-decreasing sequence. Then, for $r_n < r < r_{n+1}$, we have*

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \,\mathrm{d}\theta = \log |f(0)| + \log \frac{r^n}{r_1 \ldots r_n}. \tag{3.6}$$

Definition 3.2.4. Let $f \in H(B_R(0))$. Then, for $0 < r < R$, we denote by $n_f(r)$ the number of zeros of f in $\overline{B_r(0)}$.

Jensen's Theorem has a useful equivalent form:

Corollary 3.2.5. *Let $f \in H(B_R(0))$ with $f(0) \neq 0$. Then, for $0 < r < R$, we have*

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \,\mathrm{d}\theta = \log |f(0)| + \int_0^r \frac{n_f(s)}{s} \,\mathrm{d}s \tag{3.7}$$

Proof. Let $r_1, r_2, \dots$ denote the moduli of the zeros of f in $B_R(0)$ arranged in a non-decreasing sequence. Then, for any $r_n < r < r_{n+1}$, we obtain

$$\begin{aligned}\log \frac{r^n}{r_1 \dots r_n} &= \sum_{k=1}^{n} \log \frac{r}{r_k} = \sum_{k=1}^{n} \int_{r_k}^{r} \frac{1}{s} \, \mathrm{d}s = \\ &= \sum_{k=1}^{n} \int_{0}^{r} \mathbb{1}_{(r_k,\infty)}(s) \frac{1}{s} \, \mathrm{d}s = \int_0^r \left(\sum_{k=1}^{n} \mathbb{1}_{(r_k,\infty)}(s) \right) \frac{1}{s} \, \mathrm{d}s = \\ &= \int_0^r \frac{n_f(s)}{s} \, \mathrm{d}s\end{aligned}$$

and Theorem 3.2.3 concludes the claim. ■

This gives an immediate connection between the zeros and the growth of the modulus of an entire function:

Lemma 3.2.6. *Let $f \in H(\mathbb{C})$ be of finite order. Then for all $\varepsilon > 0$ we have*

$$n_f(r) = O(r^{\rho_f + \varepsilon}), \quad \textit{as} \quad r \to \infty.$$

Proof. We may assume $f(0) \neq 0$. Let $r > 0$, then since n_f is non-negative and non-decreasing we have

$$\int_0^{2r} \frac{n_f(s)}{s} \, \mathrm{d}s \geq \int_r^{2r} \frac{n_f(s)}{s} \, \mathrm{d}s \geq n_f(r) \int_r^{2r} \frac{1}{s} \, \mathrm{d}s = n_f(r)(\log 2r - \log r) = n_f(r) \log 2.$$

Corollary 3.2.5 now yields

$$n_f(r) \log 2 \leq \int_0^{2r} \frac{n_f(s)}{s} \, \mathrm{d}s = \log |f(0)| + \frac{1}{2\pi} \int_0^{2\pi} \log |f(2re^{i\theta})| \, \mathrm{d}\theta \leq \log |f(0)| + \log M_f(2r)$$

and holds for all $r > 0$ since f is entire. Now, f is of finite order, thus for any $\varepsilon > 0$ there is a constant K (we may assume $K > 1$) such that for sufficiently large r we have $M_f(r) \leq K \exp r^{\rho_f + \varepsilon}$. Thus

$$n_f(r) \log 2 \leq \log |f(0)| + \log K + (2r)^{\rho_f + \varepsilon}.$$

and thus $n_f(r) \leq K' r^{\rho_f + \varepsilon}$ for some constant $K' > 0$ and sufficiently large r. ■

We observe that the more zeros a function $f(z)$ has, the faster $M_f(r)$ must grow as $r \to \infty$. The converse is naturally false, as seen by iterated exponentials [3].

Definition 3.2.7. Let $f \in H(\mathbb{C})$ and denote with $(r_j)_{j \in \mathbb{N}}$ the non-zero moduli of its zeros, arranged in non-decreasing order. Then

$$\lambda_f := \inf \left\{ \lambda > 0 : \sum_{n=1}^{\infty} \frac{1}{r_n^{\lambda}} < \infty \right\}$$

is called the *exponent of convergence of the zeros of f.* If f has finitely many zeros, then we set $\lambda_f = 0$ by convention.

Furthermore, for any $a \in \mathbb{C}$, the *exponent of convergence of the a-points of f* is defined as exponent of convergence of zeros of $f(z) - a$.

Theorem 3.2.8. *If $f \in H(\mathbb{C})$ is of finite order, then $\lambda_f \leq \rho_f$.*

Proof. We may assume $f(0) \neq 0$. Let $\varepsilon = 0$ then Lemma 3.2.6 yields a constant $K_1 > 0$ such that for sufficiently large r we have $n_f(r) \leq K_1 r^{\rho_f + \varepsilon}$. If $(r_j)_{j \in \mathbb{N}}$ denote the non-zero moduli of its zeros, arranged in non-decreasing order, then for any $m \in \mathbb{N}$ we have $m = n_f(r_m) \leq K_1 r_m^{\rho_f + \varepsilon}$. Let $\delta > 0$ then this implies

$$\left(\frac{1}{r_m}\right)^{\lambda_f - \delta} \leq K_2 m^{-\frac{\lambda_f - \delta}{\rho_f + \varepsilon}}.$$

for some constant $K_2 > 0$. Therefore

$$\sum_{m=1}^{\infty} \left(\frac{1}{r_m}\right)^{\lambda_f - \delta} \leq K_2 \sum_{m=1}^{\infty} m^{-\frac{\lambda_f - \delta}{\rho_f + \varepsilon}}.$$

The left series is divergent by Definition 3.2.7, thus so is the right one. But the right one is divergent if and only if $\frac{\lambda_f - \delta}{\rho_f + \varepsilon} \leq 1$, therefore letting $\delta \to 0$ and then $\varepsilon \to 0$ yields $\lambda_f \leq \rho_f$. ■

Remark 3.2.9. The function $f(z) := \exp z$ is of order one and has no zeros. Thus we observe that we may have $\lambda_f < \rho_f$ in some cases.

Example 3.2.10. Consider $f(z) := \sin(z) \in H(\mathbb{C})$, we want to calculate λ_f and ρ_f. First, let $\lambda > 0$ and recall that f has zeros at $(n\pi)_{n \in \mathbb{Z}}$. Since

$$\sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{|n\pi|^{\lambda}} = \frac{2}{\pi^{\lambda}} \sum_{n=1}^{\infty} \frac{1}{|n|^{\lambda}}$$

is finite if and only if $\lambda > 1$, we obtain $\lambda_f = 1$. Furthermore, we have $|\sin(z)| \leq e^{|z|}$ and therefore $M_f(r) \leq e^r$, whereby $\rho_f \leq 1$. Finally, Theorem 3.2.8 concludes $\rho_f = 1$.

The final key to proving Hadamard's Theorem is the following lemma, which can be considered as a version of the maximum modulus principle applied to the real part of a holomorphic function.

Lemma 3.2.11 (Borel-Carathéodory)**.** *Let $f \in H(\overline{\mathbb{D}})$ and define*

$$A_f(r) := \max_{|z|=r} \operatorname{Re} f(z).$$

Then, for $0 < r < R$,

$$M_f(r) \leq \frac{2r}{R-r} A_f(R) + \frac{R+r}{R-r} |f(0)|$$

and, if additionally $A_f(R) \geq 0$, *then for* $n \in \mathbb{N}$

$$\max_{|z|=r} |f^{(n)}(z)| \leq \frac{2^{n+2} n! R}{(R-r)^{n+1}} (A_f(R) + |f(0)|).$$

Proof. TODO. ∎

Theorem 3.2.12 (Hadamard)**.** *Let* $f \in H(\mathbb{C})$ *be of finite order with zeros* $(z_j)_{j \in \mathbb{N}}$. *Then there exists a polynomial* Q *with* $\deg Q \leq \rho_f$, *such that*

$$f(z) = e^{Q(z)} E(z),$$

where E *is a Weierstrass canonical product formed from* $(z_j)_{j \in \mathbb{N}}$.

Proof. TODO. ∎

3.3 Zeros Distribution

For Weierstrass canonical products the exponent of convergence of zeros and the order are even more closely related:

Theorem 3.3.1. *Let* $E \in H(\mathbb{C})$ *be a Weierstrass canonical product formed from* $(z_j)_{j \in \mathbb{N}}$. *If* E *is of finite order, then* $\lambda_E = \rho_E$.

Proof. TODO. ∎

This result allows us to prove easily prove two results regarding functions of finite, non-integer order.

Theorem 3.3.2. *Let* $f \in H(\mathbb{C})$ *be of finite, non-integer order. Then* $\rho_f = \lambda_f$.

Proof. By Theorem 3.2.8 we have $\lambda_f \leq \rho_f$. Invoking Hadamard's Theorem we can write $f = e^Q E$ for a polynomial Q with $\deg Q \leq \rho_f$. Since ρ_f is not an integer, this implies $\deg Q \leq \lfloor \rho_f \rfloor < \rho_f$. As seen in Remark 3.1.5 e^Q has order $\deg Q$ and by Theorem 3.3.1 E has order λ_f. Using Proposition 3.1.6 we obtain

$$\rho_f \leq \max\{\deg Q, \lambda_f\} = \lambda_f \leq \rho_f,$$

since $\rho_f \leq \max\{\deg Q, \lambda_f\} = \deg Q < \rho_f$ is a contradiction, and we get $\rho_f = \lambda_f$. ∎

Theorem 3.3.3. *Let $f \in H(\mathbb{C})$ be of finite, non-integer order. Then f has infinitely many zeros.*

Proof. By Theorem 3.3.2 we have $\rho_f = \lambda_f$. Since ρ_f is not an integer, $\lambda_f > 0$, which implies that f has infinitely many zeros. ∎

Maybe introduce Borel exceptional values as a definition? But then again, I will never need them again. Maybe also add a remark on the relation to lacunary values (Picard).

Theorem 3.3.4 (Borel)**.** *Existence of Borel exceptional values.*

Proof. TODO. ∎

Zeros distribution (upper density, etc.).

4 Composition of Entire Functions

As seen by Proposition 3.1.6, the sum of two entire functions of finite order is again of finite order. One can similarly show that this is also true for the product of such functions. However it is no longer the case when composition is involved; indeed, consider the function $f(z) := \exp\exp z \in H(\mathbb{C})$, which is of infinite order, yet is the composition of two functions of order 1.

4.1 Pólya's Theorem

Necessary conditions for the order of a composition to be finite will be given by Pòlya's Theorem, the proof of which relies on a result that was first proven by Harald Bohr. While Bloch's Theorem dealt with disks contained in the image of holomorphic functions, we now focus on circles contained in such images.

Proposition 4.1.1. *Let $G \subset \mathbb{C}$ be a bounded domain with $0 \in G$. Then the set*

$$S := \{r \geq 0 : \partial B_r(0) \subseteq \overline{G}\} \tag{4.1}$$

has a positive maximum; that is, $\overline{G}$ contains a circle of positive, maximal radius[1].

Proof. Since G is open and contains 0 there is some $t > 0$ such that $B_t(0) \subseteq G$. Since $\partial B_t(0) \subset \overline{B_t(0)} \subseteq \overline{G}$ we thus have $t \in S$ and S is non-empty. It now suffices to show that S is compact, since then it would contain its maximum $m \geq t$. Clearly S is bounded from below by 0 and since $\overline{G}$ is bounded it must also be bounded from above. Thus it remains to show that S is closed.

Let $(r_n)_{n\in\mathbb{N}}$ be a convergent sequence in S with limit $r \in [0, \infty)$. If $r = 0$, then $r \in S$. Otherwise we claim that $\partial B_r(0) \subseteq \overline{G}$. Let $z \in \partial B_r(0)$, and set $z_n := r_n z/r$ for $n \in \mathbb{N}$. Now $|z_n| = r_n$, thus $z_n \in \partial B_{r_n}(0) \subseteq \overline{G}$ and therefore $(z_n)_{n\in\mathbb{N}}$ is a sequence in $\overline{G}$. Clearly $z_n \to z$ and since $\overline{G}$ is closed we therefore have $z \in \overline{G}$. Since z was arbitrary, this concludes the claim, we have $r \in S$ and thus S is closed. ■

If $G \subset \mathbb{C}$ is a bounded domain and $f \in H(G) \cap C(\overline{G}; \mathbb{C})$ is non-constant with $f(0) = 0$, then $f(G)$ is a bounded domain with $0 \in f(G)$. This, together with the above remark, justifies the following definition:

[1] The circle of radius 0 is taken to be the singleton $\{0\}$.

Definition 4.1.2. Let $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}}; \mathbb{C})$ be non-constant with $f(0) = 0$, then we define r_f as the positive maximum of the set S in (4.1), where $G = f(\mathbb{D})$.

Bohr's Theorem now asserts that r_f does (almost) not depend on f itself:

Theorem 4.1.3 (Bohr)**.** *There exists a function $\phi : (0,1)\times \to (0,\infty)$ such that for any $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}}; \mathbb{C})$ with $f(0) = 0$ and $M_f(\theta) = 1$ it holds that $r_f \geq \phi(\theta)$.*

Proof. Suppose f satisfies the hypothesis of the theorem, let $\varepsilon > 0$ and set $R_\varepsilon := r_f + \varepsilon$. By definition of r_f, for all $r \geq R_\varepsilon$ there exists some point $w_r \in \partial B_r(0)$ with $w_r \notin f(\overline{\mathbb{D}})$. Choose such points $w_{R_\varepsilon}, w_{2R_\varepsilon}$ and define

$$h(z) := \frac{f(z) - w_{R_\varepsilon}}{w_{2R_\varepsilon} - w_{R_\varepsilon}} \in H(\mathbb{D}, \mathbb{C} \setminus \{0,1\}).$$

Since

$$|h(0)| = \left| \frac{f(0) - w_{R_\varepsilon}}{w_{2R_\varepsilon} - w_{R_\varepsilon}} \right| \leq \frac{|w_{R_\varepsilon}|}{||w_{2R_\varepsilon}| - |w_{R_\varepsilon}||} = \frac{R_\varepsilon}{2R_\varepsilon - R_\varepsilon} = 1,$$

by Schottky's Theorem we have $|h(z)| \leq \psi(\theta, 1)$ for all $|z| \leq \theta$. Therefore

$$|f(z)| - R_\varepsilon \leq |g(z) - w_{R_\varepsilon}| \leq |w_{2R_\varepsilon} - w_{R_\varepsilon}|\psi(\theta, 1) \leq 3R_\varepsilon \psi(\theta, 1)$$

and thus $|f(z)| \leq R_\varepsilon + 3R_\varepsilon\psi(\theta, 1)$ for all $|z| \leq \theta$. Using the hypothesis that $M_f(\theta) = 1$ and the maximum modulus principle we obtain $1 \leq R_\varepsilon + 3R_\varepsilon\psi(\theta, 1)$, and letting $\varepsilon \to 0$ we have $1 \leq r_f + 3r_f\psi(\theta, 1)$. Thus

$$r_f \geq \frac{1}{1 + 3\psi(\theta, 1)}$$

and defining $\phi(\theta)$ as the right-hand side establishes the assertion. ■

Theorem 4.1.4 (Pólya)**.** *Let $g, h \in H(\mathbb{C})$ be non-constant. For the order of $g \circ h$ to be finite, it must hold that either*

i. h is a polynomial and $\rho_g < \infty$, or

ii. h is not a polynomial, $\rho_h < \infty$ and $\rho_g = 0$.

Proof. Without loss of generality we can assume $h(0) = 0$; otherwise we just consider $h_0(z) := h(z) - h(0)$ and $g_0(w) := g(w + h(0))$. Set $f := g \circ h$ and define

$$k_r(z) := \frac{h(rz)}{M_h(r/2)} \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}}; \mathbb{C}), \quad \text{for } r > 0.$$

Note that by definition we have $M_{k_r}(1/2) = 1$ and $k_r(0) = 0$, thus by Bohr's Theorem there is some constant $C > 0$ and an $R/M_h(r/2) > C$ such that $\partial B_{R/M_h(r/2)}(0) \subseteq k_r(\overline{\mathbb{D}})$ and thus $\partial B_R(0) \subseteq h(\overline{B_r(0)})$. By the maximum modulus principle, $|g|$ assumes its maximum over $\overline{B_R(0)}$ at some $w_0 \in \partial B_R(0)$. By the above there is a $z_0 \in \overline{B_r(0)}$ with $h(z_0) = w_0$. Thus we get

$$M_g(CM_h(r/2)) < M_g(R) = |g(w_0)| = |g(h(z_0))| = |f(z_0)| \leq M_f(r).$$

Assuming $\rho_f < \infty$, we have $M_f(r) < K\exp(r^\alpha)$ for some $\alpha > \rho_f$. Consider the power series expansion $h(z) = \sum_{n=0}^{\infty} a_n z^n$ and let a_m denote any non-zero coefficient; note that since $h(0) = 0$ we have $m \geq 1$. By Cauchy's integral formula we have, for all $s > 0$,

$$|a_m| = \left|\frac{h^{(m)}(0)}{m!}\right| = \frac{1}{2\pi}\left|\int_{\partial B_s(0)} \frac{h(\zeta)}{\zeta^{n+1}}\,\mathrm{d}\zeta\right| \leq \frac{M_h(s)}{s^m} \qquad (*)$$

and thus

$$M_g(C|a_m|(r/2)^m) \leq M_g(CM_h(r/2)) < M_f(r) < K\exp(r^\alpha), \quad \text{for all } r > 0.$$

Replacing $(r/2)^m$ with r we obtain $\rho_g \leq \alpha/m$. If h is not a polynomial we may let $m \to \infty$, thus $\rho_g = 0$.

Now consider $g(z) = \sum_{n=0}^{\infty} b_n z^n$. Replacing h with g in $(*)$ we obtain $|b_n|s^n \leq M_g(s)$ for all $s > 0$ and $n \geq 1$ and thus

$$|b_n|(CM_h(r/2))^n \leq M_g(CM_h(r/2)) < M_f(r) < K\exp(r^\alpha),$$

which implies $\rho_h \leq \alpha < \infty$. ■

Theorem 4.1.5 (Thron). *Let $g \in H(\mathbb{C})$ be transcendental, $\rho_g < \infty$ and suppose that g assumes some value $w \in \mathbb{C}$ only finitely often. Then there exists no $f \in H(\mathbb{C})$ with $f \circ f = g$.*

Proof. Seeking contradiction, suppose there were such a $f \in H(\mathbb{C})$. Since g is not a polynomial, f is not a polynomial either. Thus Pólya's Theorem implies $\rho_f = 0$.

Consider the sets

$$Z := f^{-1}(\{w\}), \quad Z' := \bigcup_{z \in Z} f^{-1}(\{z\}).$$

By definition, for each $z' \in Z'$ there is some $z \in Z$ with $z' \in f^{-1}(\{z\})$, thus

$$g(z') = f(f(z')) = f(z) = w.$$

Our hypothesis on g implies that Z' must be finite. Since pre-images of singletons are disjoint, Z' is a disjoint union, therefore

$$\sum_{z \in Z} |f^{-1}(\{z\})| = \left|\bigcup_{z \in Z} f^{-1}(\{z\})\right| = |Z'| < \infty$$

and thus all points in Z are only assumed finitely often by f. But by Corollary 2.4.3, f

assumes at most one value only finitely often; therefore $|Z| \leq 1$.

If $Z = \emptyset$, then $h(z) := f(z) - w$ is entire, of order 0 and nowhere 0. By Hadamard's Theorem we have $h = e^Q$ for some constant polynomial Q. Therefore h is constant and consequently so is f, a contradiction.

If $Z = \{z_0\}$, then $h(z) := f(z) - w$ has a single zero of finite order $n \in \mathbb{N}$ at z_0. Therefore we can write $h(z) = (z - z_0)^n p(z)$, where p is entire, of order 0 and nowhere 0. Again, this implies that p is constant, and therefore f a polynomial, a contradiction. ■

Example 4.1.6. A natural application of Thron's Theorem is taking g to be e^z, which never assumes zero as a value. Indeed, this implies that there is no entire function f satisfying

$$f(f(z)) = e^z.$$

On the other hand, there does exist a real-analytic function satisfying the above, as demonstrated by H. Kneser. I probably still need a citation here.

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