BACHELORARBEIT

Growth, order and zeros of entire functions

ausgeführt am

Institut für
Analysis und Scientific Computing
TU Wien

unter der Anleitung von

Ao.Univ.Prof. Dipl.-Ing. Dr.techn. Harald Woracek

durch

Ian Hornik
Matrikelnummer: 12115564
Haizingergasse 18/4
1180 Wien

Wien, am Datum

Contents

1 Introduction

Some words why the subject is of interest.

A reference to the primary literature used.

Overview of used notation.

$M_f(r) \coloneqq \max_{|z|=r} |f(z)|$

2 Order

Definition 2.1. Let f be an entire function. The *order* of f is defined by

$$\rho_f := \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}. \tag{2.1}$$

Constant functions, by convention, have order 0.

Remark 2.2. Initial explanation and intuition of the order. Make sure to note the possible values of the order ($0 \le \rho \le \infty$). And that ρ can also be seen as the infimum over all ρ that satisfy $|f(z)| \le Ae^{B|z|^\rho}$ for suitable $A, B > 0$.

Proposition 2.3. *Let f, g be entire functions of finite order. Then it holds that:*

i. $\rho_{f+g} \le \max\{\rho_f, \rho_g\}$

ii. $\rho_{fg} \le \max\{\rho_f, \rho_g\}$

Proof. To prove (i), note that

$$\begin{aligned} M_{f+g}(r) &= \max_{|z|=r} |f(z)+g(z)| \le \max_{|z|=r} |f(z)| + |g(z)| \le \max_{|z|=r} |f(z)| + \max_{|z|=r} |g(z)| \le \\ &= M_f(r) + M_g(r) \le 2\max\{M_f(r), M_g(r)\} \end{aligned}$$

thus

$$\log M_{f+g}(r) \le \log 2 + \log\max\{M_f(r), M_g(r)\} = \log 2 + \max\{\log M_f(r), \log M_g(r)\}.$$

If $M_f(r)$ and $M_g(r)$ are bounded, then applying the above in eq. (2.1) implies that f, g and $f+g$ all have order 0. If either one is not, then $\max\{\log M_f(r), \log M_g(r)\}$ necessarily outgrows $\log 2$ and we obtain

$$\begin{aligned} \rho_{f+g} &= \limsup_{r\to\infty} \frac{\log\log M_{f+g}(r)}{\log r} \le \limsup_{r\to\infty} \frac{\log(\log 2 + \max\{\log M_f(r), \log M_g(r)\})}{\log r} = \\ &= \limsup_{r\to\infty} \frac{\log\max\{\log M_f(r), \log M_g(r)\}}{\log r} = \\ &= \limsup_{r\to\infty} \max\left\{\frac{\log\log M_f(r)}{\log r}, \frac{\log\log M_g(r)}{\log r}\right\} = \\ &= \max\left\{\limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}, \limsup_{r\to\infty} \frac{\log\log M_g(r)}{\log r}\right\} = \max\{\rho_f, \rho_g\}. \end{aligned}$$

To prove (ii), we similarly note that

$$\begin{aligned}\log\log M_{fg}(r) &\leq \log\log(M_f(r)M_g(r)) = \log(\log M_f(r) + \log M_g(r)) \leq \\ &\leq \log(2\max\{\log M_f(r), \log M_g(r)\}) = \\ &= \log 2 + \max\{\log\log M_f(r), \log\log M_g(r)\},\end{aligned}$$

from where we can proceed as in (i). ∎

In the above, if we additionally demand $\rho_f \neq \rho_g$, one can actually obtain equality in both cases, as seen in find citation (the $+$ case is easy enough, the $\cdot$ case is tedious and not worth it).

For entire functions of finite order, we can obtain a representation of the order via the coefficients in their power series expansion.

Theorem 2.4. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. Then f is of finite order ρ if and only if*

$$\mu := \limsup_{n\to\infty} \frac{n\log n}{\log\frac{1}{|a_n|}} < \infty,$$

where we take the quotient to be zero if $a_n = 0$. In either case we have $\rho = \mu$.

Proof. TODO. ∎

Example 2.5. We shall apply Theorem 2.4 to obtain the order of certain entire functions via their power series expansion $\sum_{n=0}^{\infty} a_n z^n$:

i. It is immediately apparent that polynomials have order zero. Non-polynomial functions of zero order do exist, given that their coefficients decrease sufficiently rapidly, as in the example of $a_n := n^{-(n^2)}$.

ii. By Stirling's approximation we have $\log n! = n\log n + O(n)$, from which we conclude that the exponential function, sine and cosine all have order 1.

iii. For any given $\rho > 0$ the coefficients $a_n := n^{-\rho n}$ define an entire function of order ρ.

iv. For $n \geq 2$, the coefficients $a_n := n^{-\frac{n}{\sqrt{\log n}}}$ define an entire function of infinite order. Note that by Remark 2.2, we can also conclude that e^{e^z} is of infinite order.

Proposition 2.6. *Let f be an entire function of finite order with derivative f'. Then $\rho_{f'} = \rho_f$.*

Proof. Given $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we have $f'(z) = \sum_{n=0}^{\infty} (n+1)a_{n+1} z^n$. Since

$$\lim_{n\to\infty} \left(\frac{n\log n}{(n+1)\log(n+1)}\right)^{-1} = 1$$

we have

$$
\begin{aligned}
\limsup_{n\to\infty} \frac{n\log n}{\log \frac{1}{|(n+1)a_{n+1}|}} &= \liminf_{n\to\infty} \left(\frac{-\log(n+1) + \log\frac{1}{|a_{n+1}|}}{n\log n} \right)^{-1} = \\
&= \liminf_{n\to\infty} \left(\frac{\frac{1}{|(n+1)a_{n+1}|}}{n\log n} \right)^{-1} \cdot \lim_{m\to\infty} \left(\frac{m\log m}{(m+1)\log(m+1)} \right)^{-1} = \\
&= \liminf_{n\to\infty} \left(\frac{-\log|a_{n+1}|}{n\log n} \cdot \frac{n\log n}{(n+1)\log(n+1)} \right)^{-1} = \\
&= \liminf_{n\to\infty} \left(\frac{-\log|a_{n+1}|}{(n+1)\log(n+1)} \right)^{-1} = \limsup_{n\to\infty} \frac{(n+1)\log(n+1)}{\log\frac{1}{|a_{n+1}|}} = \\
&= \limsup_{n\to\infty} \frac{n\log n}{\log\frac{1}{|a_n|}}
\end{aligned}
$$

and since $\rho_f < \infty$ Theorem 2.4 concludes $\rho_{f'} = \rho_f$. ■

3 Factorization

Short introduction.

Do I need a citation for this?

Theorem 3.1 (Weierstrass). *Let $(z_j)_{j\in\mathbb{N}}$ be a sequence in $\mathbb{C}$ without accumulation points. Then there exists an entire function E (called the* Weierstrass canonical product *formed from said sequence) that has zeros precisely at $(z_j)_{j\in\mathbb{N}}$, with multiplicities equal to how often z_j occurs in the sequence.*

Furthermore, if f is any other entire function satisfying the above, then there exists an entire function g such that

$$f = e^g E.$$

Short remark on how Hadamard refines Weierstrass (for functions of finite order).

Lemma 3.2 (Borel-Carathéodory). *Let f be analytic in $\operatorname{cl}(B(0,R))$ and let*

$$M(r) = \max_{|z|=r} |f(z)|, \quad A(r) = \max_{|z|=r} \Re f(z).$$

Then, for $0 < r < R$,

$$M(r) \leq \frac{2r}{R-r} A(R) + \frac{R+r}{R-r} |f(0)|$$

and, if additionally $A(R) \geq 0$, then for $n \in \mathbb{N}$

$$\max_{|z|=r} |f^{(n)}(z)| \leq \frac{2^{n+2} n! R}{(R-r)^{n+1}} (A(R) + |f(0)|).$$

Proof. TODO. ■

Theorem 3.3 (Hadamard). *Let f be an entire function of finite order with zeros $(z_j)_{j\in\mathbb{N}}$ and $f(0) \neq 0$. Then there exists a polynomial Q with $\deg Q \leq \rho_f$, such that*

$$f = e^Q E,$$

where E is the Weierstrass canonical product formed from the zeros of f.

Proof. TODO. ■

4 Zeros

We recall a rather explicit connection between the moduli of the zeros of an analytic function and the modulus of the function itself:

Theorem 4.1 (Jensen). *Let f be analytic on $B(0,R)$ with $f(0) \neq 0$ and let $(r_j)_{j=1}^n$ denote the moduli of the zeros of f in $B(0,R)$ arranged in a non-decreasing sequence. Then, for $r_m < r < r_{m+1}$, we have*

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\vartheta})|\,\mathrm{d}\vartheta = \log|f(0)| + \log\frac{r^n}{r_1 \dots r_m}.$$

Definition 4.2. Let f be analytic on $B(0,R)$. Then for $0 < r < R$ we denote by $n_f(r)$ the number of zeros of f in $\mathrm{cl}(B(0,r))$.

Corollary 4.3. *Let f be analytic on $B(0,R)$ with $f(0) \neq 0$. Then for $0 < r < R$ we have*

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\vartheta})|\,\mathrm{d}\vartheta = \log|f(0)| + \int_0^r \frac{n_f(s)}{s}\,\mathrm{d}s$$

Proof. TODO. ■

In particular, we observe that the more zeros a function $f(z)$ has, the faster its modulus must grow as $|z| \to \infty$. The converse is naturally false, as seen by iterated exponentials.

Definition 4.4. Let f be an entire function satisfying $f(0) \neq 0$. Let $(r_j)_{j\in\mathbb{N}}$ denote the moduli of the zeros of f (if any) arranged in non-decreasing order. Then

$$\lambda_f := \inf\left\{\lambda > 0 : \sum_{n=1}^{\infty} \frac{1}{r_n^\lambda} < \infty\right\}$$

is called the *exponent of convergence of the zeros of f*. If f has finitely many zeros, then we set $\lambda_f = 0$ by convention. I am not sure about this – the paper only establishes this convention if f has no zeros at all.

Furthermore, the *exponent of convergence of the a-points of f* is defined as exponent of convergence of $f(z) - a$.

Theorem 4.5. *Let f be an entire function of finite order. Then $\lambda_f \leq \rho_f$.*

Proof. TODO. ■

Example 4.6. An example of a series where we see some convergence for some appropriate function using the above.

Theorem 4.7. *Let E be a Weierstrass canonical product of finite order. Then $\lambda_E = \rho_E$.*

Proof. TODO. ■

Theorem 4.8. *Let f be an entire function of finite, non-integer order. Then $\rho_f = \lambda_f$.*

Proof. By Theorem 4.5 we have $\lambda_f \leq \rho_f$. Invoking Hadamard's Theorem we can write $f = e^Q E$ for a polynomial Q with $\deg Q \leq \rho_f$. Since ρ_f is not an integer, this implies $\deg Q \leq \lfloor \rho_f \rfloor < \rho_f$. Now, again by Hadamard's Theorem, e^Q has order $\deg Q$ and by Theorem 4.7 E has order λ_f. Using Proposition 2.3 we obtain

$$\rho_f \leq \max\{\deg Q, \lambda_f\} = \lambda_f \leq \rho_f,$$

implying $\rho_f = \lambda_f$. ■

Theorem 4.9. *Let f be an entire function of finite, non-integer order. Then f has infinitely many zeros.*

Proof. By Theorem 4.8 we have $\rho_f = \lambda_f$. Since ρ_f is not an integer, $\lambda_f > 0$, which implies that f has infinitely many zeros. ■

Maybe introduce Borel exceptional values as a definition? But then again, I will never need them again. Maybe also add a remark on the relation to lacunary values (Picard).

Theorem 4.10 (Borel)**.** *Existence of Borel exceptional values.*

Proof. TODO. ■

5 Composition

Theorem 5.1 (Pólya). *Let g, h be entire. For the order of $g \circ h$ to be finite, it must hold that either*

i. h is a polynomial and g of finite order, or

ii. h is of finite order, not a polynomial, and g is of order zero.

Proof. TODO. ∎

Theorem 5.2 (Thron). *Let g be an entire function of finite order, not a polynomial, which takes some value w only finitely often. Suppose further that there exists f such that $f \circ f = g$. Then f is not entire.*

Proof. Seeking contradiction, suppose f were entire. Since f is not a polynomial, Theorem 5.1 implies that f is of order 0 and not a polynomial. Let $(z_j)_{j\in J}$ denote the points where f equals w. For each $m \in J$ we additionally denote by $(z_{j,m})_{j\in J_m}$ the points where f equals z_m. Thus, for each $m \in J$ and $n \in J_m$ we have

$$g(z_{n,m}) = f(f(z_{n,m})) = f(z_m) = w.$$

By our assumption on g, there must only be finitely many distinct points among the $(z_{j,m})_{m\in J, j\in J_m}$. Thus, each point in $(z_j)_{j\in J}$ is only taken on by f finitely often.

Do I need a citation for Picard's Big Theorem?

Since f is entire and not a polynomial, it has an essential singularity at ∞. By Picard's Big Theorem, f therefore attains all values in the complex plane infinitely often, with at most one exception. This implies that that there is at most one z_0 that is only taken on finitely often.

If there is no such z_0, then $h(z) := f(z) - z_0$ is entire, of order 0 and nowhere 0. Thus, by Hadamard's Theorem, h must be constant, and therefore f aswell, a contradiction.

If such a z_0 exists, then $h(z) := f(z) - z_0$ has a zero of finite order $n \in \mathbb{N}$ at z_0. Therefore we can write $h(z) = (z - z_0)^n p(z)$, where p is entire, of order 0 and nowhere 0. Again, this implies that p is constant, and therefore f a polynomial, a contradiction. ∎

Example 5.3. The example with $f(f(z)) = e^z$.

Bibliography

[1] S. L. Segal. *Nine introductions in complex analysis*, volume 208 of *North-Holland Mathematics Studies*. Elsevier Science B.V., Amsterdam, revised edition, 2008.