BACHELORARBEIT

Picard's Great Theorem

AND

Growth, Zeros and Composition of Entire Functions

ausgeführt am

Institut für
Analysis und Scientific Computing
TU Wien

unter der Anleitung von

Ao.Univ.Prof. Dipl.-Ing. Dr.techn. Harald Woracek

durch

Ian Hornik
Haizingergasse 18/4
1180 Wien

Wien, am 13. August 2024

Contents

1 Introduction

Some words why the subject is of interest.

A reference to the primary literature used.

Overview of used notation.

Some results from complex analysis are assumed as pre-requisites, such as Cauchy's integral formula, the maximum modulus principle, the Weierstrass theorem on canonical products, the theorems of Montel, Hurwitz and Jensen, the Schwarz lemma and standard results on power series.

The sets $\mathbb{N}$ and $\mathbb{Z}$ denote, respectively, the natural numbers (starting with 1) and the integers. Furthermore we define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The sets $\mathbb{R}$ and $\mathbb{C}$ denote, respectively, the real line and the complex plane, which will both be endowed with the standard Euclidean metric and topology. The closure of a subset A of $\mathbb{R}$ or $\mathbb{C}$ (with respect to the appropriate topology) will be denoted as $\overline{A}$ and the boundary as ∂A.

When integrating over the boundary of a set in $\mathbb{C}$ the boundary is, unless otherwise indicated, parametrized with mathematically positive (counter-clockwise) orientation.

If A is some set with $0 \in A$ we define $A^\times := A \setminus \{0\}$.

For $a, b \in \mathbb{C}$ we denote by $[a, b]$ the straight line connecting a to b, parametrized by the function $\gamma : [0, 1] \to \mathbb{C}, t \mapsto (1 - t)a + tb$.

For subsets M, N of $\mathbb{R}$ or $\mathbb{C}$ we denote by $C(M, N)$ the space of continuous functions which map from M into N.

If $G \subseteq \mathbb{C}$, then G is said to be a domain if G is non-empty, open and connected.

For a point $a \in \mathbb{C}$ and $r > 0$ we denote by $B_r(a)$ the open disk of radius r centered at a. The unit disk $B_1(0)$ will be denoted $\mathbb{D}$.

Let $G \subseteq \mathbb{C}$ be open, then a function $f : G \to \mathbb{C}$ is said to be holomorphic at $z \in G$ if f is complex differentiable at z. If f is holomorphic at all $z \in G$ then f is said to be holomorphic on G. The space of all such holomorphic functions on G will be denoted as $H(G)$.

If f is (real or complex) differentiable, then f' and f'' denote the respective first and second derivatives of f. For $n \in \mathbb{N}$, the n-th derivative will be denoted $f^{(n)}$.

For a real- or complex-valued function f defined on some set M we define $\|f\|_M := \sup_{x \in M} |f(x)|$.

The exponential function will be denoted $\exp : \mathbb{R} \to (0, \infty)$, with inverse $\log : (0, \infty) \to \mathbb{R}$.

For a set A, the indicator function $\mathbb{1}_A(a)$ is defined as 1 if $a \in A$ and 0 otherwise.

To simplify the asymptotic analysis in chapter 3 we shall employ the following notation: Let $a \in \mathbb{R}$ and let f, g be functions on $[a, \infty)$ such that f is either real- or complex-valued and g is real-valued. Then we write $f(x) = \mathcal{O}(g(x))$ as $x \to \infty$ if there exist constants $C > 0$ and $x_0 \geq a$ such that for $x \geq x_0$ we have $|f(x)| \leq Cg(x)$.

2 Picard's Great Theorem

An isolated singularity of a holomorphic function that is not removable or a pole is called an *essential singularity.* While the behaviour of holomorphic functions near their removable singularities and poles is fairly well-behaved, the same cannot be said for their essential singularities.

An interesting result is already given by the Casorati-Weierstrass Theorem: Let $G \subseteq \mathbb{C}$ be open, $w \in G$ and suppose $f \in H(G \setminus \{w\})$ such that f has an essential singularity at w. Then, for any punctured neighborhood $U \subseteq G$ of w, the set $f(U)$ is dense in $\mathbb{C}$.

Picard's Great Theorem will show that $f(U)$ is not only dense in $\mathbb{C}$, but that there is at most one value that is not taken on by f infinitely often, on any such punctured neighborhood.

To obtain the proof we first study Bloch's Theorem, which estimatates the size of disks in the image of a holomorphic map. As a corollary of the proof we obtain Picard's Little Theorem, which states that an entire function which omits two values is constant.

One can see this as a motivation to more closely study (non-entire) holomorphic functions that omit (at least) two values. For such functions, Schottky's Theorem will give an upper bound on their modulus. We use this to obtain the Fundamental Normality Test, which asserts normality of any family of holomorphic functions, which omit two fixed values.

Restricting the domain in the above to be the punctured unit disk, we obtain that such functions cannot have an essential singularity at the origin, which will imply Picard's Great Theorem.

2.1 Bloch's Theorem

If $G \subseteq \mathbb{C}$ is a domain and $f \in H(G)$ is non-constant, then $f(G)$ is a domain as well. In particular, $f(G)$ contains open disks of some, potentially very small, radius. Bloch's Theorem asserts that for any $f \in H(D) \cap C(\overline{\mathbb{D}}; \mathbb{C})$ satisfying $f'(0) = 1$, the set $f(\mathbb{D})$ always contains a disk of fixed radius.

Note that $f(0)$ may not always be the center of such a disk; consider the sequence

$$f_n(z) := \frac{1 - e^{-nz}}{n} \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}}; \mathbb{C}), \quad n \in \mathbb{N},$$

which satisfies $f_n(0) = 0$ and $f_n'(0) = 1$, but omits the value $1/n$.

If f is as before, then f is an open mapping on G, that is f maps open sets to open sets.

We thus first observe a general criterion for the size of disks in the image domain of such functions:

Lemma 2.1.1. *Let $G \subset \mathbb{C}$ be a bounded domain and $f \in C(\overline{G}; \mathbb{C})$ such that $f\big|_G$ is an open mapping. Let $a \in G$ and set $s := d(f(a), f(\partial G))$. If $s > 0$, then $B_s(f(a)) \subseteq f(G)$.*

Proof. Since G is bounded, $\overline{G}$ is compact and by continuity of f, so is $\overline{f(G)}$. The function $z \mapsto |z - f(a)|$ is continuous on the compact set $\partial f(G)$, hence it assumes its minimum m at some $w_* \in \partial f(G)$. Choose a sequence $(z_n)_{n\in\mathbb{N}}$ in G with $\lim_{n\to\infty} f(z_n) = w_*$ then, since $\overline{G}$ is compact, we can find a subsequence that converges to some $z_* \in \overline{G}$. By continuity of f we have $f(z_*) = w_*$.

If $z_* \in G$, since $f\big|_G$ is open, the image of any open set in G containing z_* under f is an open set in $f(G)$ containing w_*, which is impossible since $w_* \in \partial f(G)$.

Therefore $z_* \in \partial G$ and we have

$$d(f(a), \partial f(G)) = m = |w_* - f(a)| = |f(z_*) - f(a)| \geq s,$$

which implies $B_s(f(a)) \subseteq f(G)$ if $s > 0$. ∎

Lemma 2.1.2. *Fix $a \in \mathbb{C}, r > 0$ and let $B := B_r(a)$. Suppose further $G \subseteq \mathbb{C}$ is a domain such that $\overline{B} \subset G$ and $f \in H(G)$ such that $\|f'\|_B \leq 2|f'(a)|$. Then $B_R(f(a)) \subseteq f(B)$, where $R := (3 - 2\sqrt{2})r|f'(a)|$.*

Proof. We may assume $a = f(a) = 0$, otherwise we consider $f_1(z) := f(z+a) - f(a)$. The function

$$\alpha_f : \begin{cases} B \to \mathbb{C}, \\ z \mapsto f(z) - f'(0)z, \end{cases}$$

satisfies, for all $z \in B$,

$$|\alpha_f(z)| = \left| \int_{[0,z]} f'(\zeta) - f'(0)\,\mathrm{d}\zeta \right| \leq \int_0^1 |f'(tz) - f'(0)||z|\,\mathrm{d}t. \tag{$*$}$$

We wish to further estimate the integrand. Let $w \in B$, then Cauchy's integral formula gives

$$\begin{aligned} |f'(w) - f'(0)| &= \frac{1}{2\pi}\left| \oint_{\partial B} \frac{f'(\zeta)}{\zeta - w} - \frac{f'(\zeta)}{\zeta}\,\mathrm{d}\zeta \right| = \frac{1}{2\pi}\left| \oint_{\partial B} \frac{wf'(\zeta)}{\zeta(\zeta - w)}\,\mathrm{d}\zeta \right| \leq \\ &\leq \frac{1}{2\pi} \oint_{\partial B} \frac{|w|\|f'\|_B}{r(r - |w|)}\,\mathrm{d}\zeta = \frac{|w|}{r - |w|}\|f'\|_B. \end{aligned}$$

Combining the above with $(*)$ and our estimate on $\|f'\|_B$ yields

$$|\alpha_f(z)| \leq \int_0^1 \frac{|zt|\|f'\|_B}{r - |zt|}|z|\,\mathrm{d}t \leq \frac{|z|^2}{r - |z|}\|f'\|_B \int_0^1 t\,\mathrm{d}t \leq \frac{|z|^2}{r - |z|}|f'(0)|.$$

Let $0 < \rho < r$, then for $|z| = \rho$ we have

$$\begin{aligned} |f'(0)|\rho - |f(z)| \le |\alpha_f(z)| &\le \frac{\rho^2}{r-\rho}|f'(0)| \\ \iff \quad |f(z)| &\ge \left(\rho - \frac{\rho^2}{r-\rho}\right)|f'(0)|. \end{aligned}$$

The function $\rho \mapsto \rho - \rho^2/(r-\rho)$ assumes its maximum value at $\rho_* := (1-\sqrt{2}/2)r \in (0,r)$, namely $(3-2\sqrt{2})r$. Therefore,

$$|f(z)| \ge (3-2\sqrt{2})r|f'(0)| = R, \quad \text{for all} \quad |z| = \rho_*.$$

In particular, $\min_{z\in\partial B_{\rho_*}} |f(z)| \ge R > 0$, thus invoking Lemma 2.1.1 with the domain $B_{\rho_*}(0)$ yields $B_R(0) \subseteq f(B_{\rho_*}(0)) \subseteq f(B)$. ■

Theorem 2.1.3. *Let $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}};\mathbb{C})$ be non-constant. Then there is a point $p \in \mathbb{D}$ and a constant $C_f > 0$ such that $B_R(f(p)) \subseteq f(\mathbb{D})$, where $R := (\frac{3}{2} - \sqrt{2})C_f \ge (\frac{3}{2} - \sqrt{2})|f'(0)|$.*

Proof. The function

$$\alpha_f : \begin{cases} \overline{\mathbb{D}} \to \mathbb{R} \\ z \mapsto |f'(z)|(1-|z|) \end{cases}$$

is continuous and assumes its maximum $C_f > 0$ at some point $p \in \overline{\mathbb{D}}$. Note that $C_f \ge |f'(0)|$ and since f is non-constant and $\alpha_f\big|_{\partial\mathbb{D}} = 0$ we even have $p \in \mathbb{D}$.

Set $t := \frac{1}{2}(1-|p|) > 0$, then we have $B_t(p) \subseteq \mathbb{D}$. Furthermore, for $z \in B_t(p)$, we have

$$1 - |z| \ge 1 - |z-p| - |p| \ge 1 - t - |p| = t$$

and since $|f'(z)|(1-|z|) \le C_f = 2t|f'(p)|$, this implies $|f'(z)| \le 2|f'(p)|$ for all $z \in B_t(p)$. By Lemma 2.1.2, we get $B_R(f(p)) \subseteq f(\mathbb{D})$, where $R := (3-2\sqrt{2})t|f'(p)| = (\frac{3}{2} - \sqrt{2})C_f$, which establishes the assertion. ■

We now immediately obtain:

Theorem 2.1.4 (Bloch)**.** *Let $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}};\mathbb{C})$ and assume $f'(0) = 1$. Then $f(\mathbb{D})$ contains a disk of radius $\frac{3}{2} - \sqrt{2}$.*

In the following we will denote by $\beta > 0$ any constant less than or equal to the radius in Bloch's Theorem, for example $\beta = \frac{1}{12} < \frac{3}{2} - \sqrt{2}$.

Corollary 2.1.5. *Let $G \subseteq \mathbb{C}$ be a domain and $f \in H(G)$ with $f'(c) \neq 0$ for some $c \in G$. Then $f(G)$ contains a disk of every radius $\beta s|f'(c)|$, where $0 < s < d(c, \partial G)$.*

Proof. We may assume $c = 0$, otherwise we consider $f_1(z) := f(z+c)$. Let $0 < s < d(c,\partial G)$, then f is holomorphic on $\overline{B_s(0)} \subseteq G$, thus we have $g(z) := f(sz)/sf'(0) \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}};\mathbb{C})$.

Since $g'(0) = 1$, Bloch's Theorem yields a disk B of radius β with $B \subseteq g(\mathbb{D})$. Then $D := s|f'(0)|B$ is a disk of radius $\beta s|f'(0)|$ and we have

$$D = s|f'(0)|B \subseteq s|f'(0)|g(\mathbb{D}) = f(B_s(0)) \subseteq f(G).$$

■

Corollary 2.1.6. *If $f \in H(\mathbb{C})$ is non-constant, then $f(\mathbb{C})$ contains a disk of every radius.*

2.2 Schottky's Theorem

Holomorphic functions which omit the values 0 and 1 have a universal estimate on the growth of their modulus, which will be given by Schottky's Theorem.

For a domain $G \subseteq \mathbb{C}$ and a set $E \subseteq \mathbb{C}$ we define $H(G; E)$ as the set[1] of all $f \in H(G)$ such that $f(G) \subseteq E$.

Lemma 2.2.1. *It holds that:*

i. If $a, b \in \mathbb{R}$ with $\cos \pi a = \cos \pi b$, then $b = \pm a + 2n$ for some $n \in \mathbb{Z}$.

ii. For every $w \in \mathbb{C}$ there exists a $v \in \mathbb{C}$ such that $\cos \pi v = w$ and $|v| \leq 1 + |w|$.

Proof. For the first part, it suffices to notice that

$$0 = \cos \pi a - \cos \pi b = -2 \sin \tfrac{\pi}{2}(a + b) \sin \tfrac{\pi}{2}(a - b).$$

Since the complex cosine function is surjective and 2π-periodic, we can choose $v = a + ib$ with $w = \cos \pi v$ and $|a| \leq 1$. Now we have

$$\begin{aligned} |w|^2 &= |\cos(\pi a + i\pi b)|^2 = |\cos \pi a \cos i\pi b + \sin \pi a \sin i\pi b|^2 = \\ &= |\cos \pi a \cosh \pi b - i \sin \pi a \sinh \pi b|^2 = \\ &= \cos^2 \pi a \cosh^2 \pi b + \sin^2 \pi a \sinh^2 \pi b = \\ &= \cos^2 \pi a + \cos^2 \pi a \sinh^2 \pi b + \sin^2 \pi a \sinh^2 \pi b = \\ &= \cos^2 \pi a + \sinh^2 \pi b \geq \sinh^2 \pi b \geq \pi^2 b^2, \end{aligned}$$

where the last inequality holds since $\sinh x \geq x$ for $x \geq 0$. We conclude

$$|v| = \sqrt{a^2 + b^2} \leq \sqrt{1 + |w|^2/\pi^2} \leq 1 + |w|$$

■

We recall the following result: Let $G \subseteq \mathbb{C}$ be a simply connected domain and $f \in H(G)$

[1] Note that, unlike $H(G)$, the set $H(G; E)$ is usually not a linear space.

such that f vanishes nowhere on G. Then there is a $g \in H(G)$ such that $f = e^g$. This can also be used to obtain multiplicative n-th roots of such functions, by defining $\sqrt[n]{f} := e^{g/n}$.

Lemma 2.2.2. *Let $G \subseteq \mathbb{C}$ be a simply connected domain and $f \in H(G; \mathbb{C} \setminus \{-1, 1\})$. Then there exists an $F \in H(G)$ such that $f = \cos F$.*

Proof. Since $1 - f^2$ vanishes nowhere in G it has a square root $g \in H(G)$. It follows that

$$1 = f^2 + g^2 = (f + ig)(f - ig).$$

Thus $f + ig$ vanishes nowhere and there exists an $F \in H(G)$ with $f + ig = e^{iF}$. By the above we also have $f - ig = e^{-iF}$ and therefore

$$f = \tfrac{1}{2}(e^{iF} + e^{-iF}) = \cos F.$$

∎

Lemma 2.2.3. *Let $G \subseteq \mathbb{C}$ be a simply connected domain and $f \in H(G; \mathbb{C} \setminus \{0, 1\})$. Then there exists a $g \in H(G)$ such that:*

i. $f = \frac{1}{2}(1 + \cos \pi(\cos \pi g))$.

ii. $|g(0)| \leq 3 + 2|f(0)|$.

iii. *$g(G)$ contains no disk of radius* 1.

iv. *If $\mathbb{D} \subseteq G$ then $|g(z)| \leq |g(0)| + \frac{\theta}{\beta(1-\theta)}$, for all $|z| \leq \theta$ where $0 < \theta < 1$.*

Proof. By Lemma 2.2.2 there exists a $\widetilde{F} \in H(G)$ such that $2f - 1 = \cos \pi \widetilde{F}$ and by Lemma 2.2.1 there is a $b \in \mathbb{C}$ such that $\cos \pi b = 2f(0) - 1$ and $|b| \leq 1 + |2f(0) - 1| \leq 2 + 2|f(0)|$. Furthermore, since $\cos \pi b = \cos \pi \widetilde{F}(0)$, we have $b = \pm \widetilde{F}(0) + 2k$ for some $k \in \mathbb{Z}$. Then $F := \pm \widetilde{F} + 2k \in H(G)$ satisfies $F(0) = b$ and $2f - 1 = \cos \pi F$.

Since F must omit all integers, there exists a $\widetilde{g} \in H(G)$ such that $F = \cos \pi \widetilde{g}$. Similarly, there is an $a \in \mathbb{C}$ such that $\cos \pi a = b$ and $|a| \leq 1 + |b| \leq 3 + 2|f(0)|$. Like above, since $\cos \pi a = \cos \pi \widetilde{g}(0)$, we have $a = \pm \widetilde{g}(0) + 2\ell$ for some $\ell \in \mathbb{Z}$, thus $g := \pm \widetilde{g} + 2\ell \in H(G)$ satisfies $g(0) = a$ and $F = \cos \pi g$. Together we obtain

$$f = \tfrac{1}{2}(1 + \cos \pi(\cos \pi g)), \quad \text{and} \quad |g(0)| = |a| \leq 3 + 2|f(0)|$$

and have thus shown (i) and (ii).

To show (iii) we consider the set

$$A := \{m \pm i\pi^{-1} \log(n + \sqrt{n^2 - 1}) : m \in \mathbb{Z}, n \in \mathbb{N} \setminus \{0\}\},$$

the points of which can be considered the vertices of a rectangular grid in $\mathbb{C}$. The width of such a rectangular cell is 1, and since

$$\log((n+1)+\sqrt{(n+1)^2-1})-\log(n+\sqrt{n^2-1}) = \\ = \log\frac{1+\frac{1}{n}+\sqrt{1+\frac{2}{n}}}{1+\sqrt{1-\frac{1}{n^2}}} \leq \log(2+\sqrt{3}) < \pi$$

their height is bounded above by some $C < 1$. Therefore, for all $z \in \mathbb{C}$ there is a $w_z \in A$ such that $|\operatorname{Re} z - \operatorname{Re} w_z| \leq \frac{1}{2}$ and $|\operatorname{Im} z - \operatorname{Im} w_z| \leq \frac{C}{2}$. Thus we have

$$|z-w_z| \leq |\operatorname{Re} z - \operatorname{Re} w_z| + |\operatorname{Im} z - \operatorname{Im} w_z| \leq \frac{1}{2}+\frac{C}{2} < 1.$$

If we can show that $g(G)\cap A = \emptyset$, then $g(G)$ therefore cannot contain a disk of radius 1. Let $a = p + i\pi^{-1}\log(q+\sqrt{q^2-1}) \in A$, then

$$\cos\pi a = \tfrac{1}{2}(e^{i\pi a}+e^{-i\pi a}) = \tfrac{1}{2}(-1)^p((q+\sqrt{q^2-1})^{-1}+(q+\sqrt{q^2-1})) = \\ = (-1)^p\frac{1}{2}\frac{1+q^2+2q\sqrt{q^2-1}+q^2-1}{q+\sqrt{q^2-1}} = (-1)^p q$$

and thus $\cos\pi(\cos\pi a) = \pm 1$. But $0,1 \notin f(G)$, therefore $a \notin g(G)$ and $g(G)\cap A = \emptyset$, proving (iii).

For (iv), if $\mathbb{D} \subseteq G$, then $g\big|_{\mathbb{D}} \in H(\mathbb{D})$. Fix $0<\theta<1$, then for $|z| \leq \theta$ we have

$$d(z,\partial\mathbb{D}) = \inf_{w\in\partial\mathbb{D}} |z-w| \geq \inf_{w\in\partial\mathbb{D}} (|w|-|z|) \geq 1-\theta.$$

From (iii) it follows that $g\big|_{\mathbb{D}}(\mathbb{D})$ does not contain a disk of radius 1. Let $0 < s < 1-\theta$, then applying Corollary 2.1.5 to $g\big|_{\mathbb{D}}$ implies that $\beta s|g'(z)| < 1$. Taking the supremum over s and rearranging yields $|g'(z)| \leq 1/(\beta(1-\theta))$. Thus our desired estimate is shown by

$$|g(z)| \leq |g(0)| + |g(z)-g(0)| \leq |g(0)| + \int_{[0,z]} |g'(\zeta)|\,\mathrm{d}\zeta \leq |g(0)| + \frac{\theta}{\beta(1-\theta)}.$$

■

Theorem 2.2.4 (Schottky). *There exists a function $\psi : (0,1)\times(0,\infty) \to (0,\infty)$ such that for any $f \in H(\mathbb{D};\mathbb{C}\setminus\{0,1\})$ with $|f(0)| \leq \omega$ it holds that*

$$|f(z)| \leq \psi(\theta,\omega), \quad |z| \leq \theta. \tag{2.1}$$

Proof. Note that for all $w \in \mathbb{C}$ we have $|\cos w| \leq e^{|w|}$ and $\frac{1}{2}|1+\cos w| \leq e^{|w|}$. Hence, from

Lemma 2.2.3, we get

$$\begin{aligned}|f(z)| = |\tfrac{1}{2}(1+\cos\pi(\cos\pi g(z)))| &\leq \exp(\pi\exp\pi|g(z)|) \leq \\ &\leq \exp(\pi\exp\pi(|g(0)| + \theta/\beta(1-\theta))) \leq \\ &\leq \exp(\pi\exp\pi(3+2\omega+\theta/\beta(1-\theta))),\end{aligned}$$

and defining $\psi(\theta,\omega)$ as the final term establishes the assertion. ∎

Lemma 2.2.3 is quite powerful, as it contains not only Schottky's Theorem, but Picard's Little Theorem as well:

Theorem 2.2.5 (Picard's Little Theorem)**.** *Let $f \in H(\mathbb{C}; \mathbb{C} \setminus \{a,b\})$ for distinct points $a, b \in \mathbb{C}$. Then f is constant.*

Proof. Consider $f_1(z) := \frac{f(z)-a}{b-a} \in H(\mathbb{C}; \mathbb{C}\setminus\{0,1\})$. By Lemma 2.2.3 there is some $g \in H(\mathbb{C})$ such that $f_1 = \frac{1}{2}(1+\cos\pi(\cos\pi g))$ and $g(\mathbb{C})$ does not contain a disk of radius 1. By Corollary 2.1.6 we thereby have that g must be constant and therefore so are f_1 and f. ∎

2.3 Normal families

We first recall a generalized form of locally uniform convergence:

Definition 2.3.1. Let $G \subseteq \mathbb{C}$ be a domain, $f \in H(G)$ and $(f_n)_{n\in\mathbb{N}}$ a sequence in $H(G)$. We say that f_n *converges compactly in G to f*, or f_n *converges compactly in G to ∞* as $n \to \infty$, if for every compact set $K \subset G$

$$\lim_{n\to\infty} \sup_{z\in K} |f_n(z) - f(z)| = 0, \quad \text{or} \quad \lim_{n\to\infty} \inf_{z\in K} |f_n(z)| = \infty. \tag{2.2}$$

//

Remark 2.3.2. One can define compact convergence generally for functions from a topological space $(X, \mathcal{T})$ into a metric space (Y, d_Y). Compact convergence to ∞ can then be seen as compact convergence to a constant function with value ∞, where $d_Y(y,\infty)$ is appropriately defined for $y \in Y$. This is precisely the case when considering the chordal metric of the Riemann sphere, but we shall not elaborate further on this. //

Definition 2.3.3. Let $f : \mathbb{C} \to \mathbb{C}$ and $a \in \mathbb{C}$. Then the *a-points of f* are defined as the zeros of $f(z) - a$, that is the set of all points $w \in \mathbb{C}$ with $f(w) = a$. //

A well-known theorem on compact convervence is:

Theorem 2.3.4 (Hurwitz)**.** *Let $G \subseteq \mathbb{C}$ be a domain and $(f_n)_{n\in\mathbb{N}}$ a sequence in $H(G)$ that converges compactly to $f \in H(G)$. If for every $n \in \mathbb{N}$ the number of a-points of f_n is* *bounded by some $m \in \mathbb{N}_0$, then either the number of a-points of f are also bounded by m, or $f \equiv a$.*

As an immediate consequence we obtain that compact convergence is, in some ways, compatible with reciprocals:

Lemma 2.3.5. *Let $G \subseteq \mathbb{C}$ be a domain and $(f_n)_{n\in\mathbb{N}}$ a sequence in $H(G; \mathbb{C} \setminus \{0\})$. If it converges compactly to some $f \in H(G)$, then it holds that either:*

- *$0 \notin f(G)$ and $1/f_n \to 1/f$ compactly in G, or*
- *$f \equiv 0$ and $1/f_n \to \infty$ compactly in G.*

If it converges compactly to ∞, then $1/f_n \to 0$ compactly in G.

Proof. For the equivalence "$f_n \to 0$ if and only if $1/f_n \to \infty$" it suffices to notice that for any compact $K \subset G$ we have

$$\frac{1}{\sup_{z\in K} |f_n(z)|} = \inf_{z\in K} \left| \frac{1}{f_n(z)} \right|.$$

Since the f_n vanish nowhere, by Hurwitz' Theorem we either have $0 \notin f(G)$ or $f \equiv 0$. In the latter case we have just shown that $1/f_n \to \infty$ compactly.

In the former case we have, again for any compact $K \subset G$, that $m := \min_{z\in K} |f(z)| > 0$ and $\sup_{z\in K} |f_n(z) - f(z)| < \frac{m}{2}$ for sufficiently large $n \in \mathbb{N}$. Thus for all $z \in K$

$$\frac{m}{2} > |f(z) - f_n(z)| \geq |f(z)| - |f_n(z)| \geq m - |f_n(z)|$$

and $|f_n(z)| \geq \frac{m}{2}$. We obtain, for large $n \in \mathbb{N}$,

$$\sup_{z\in K} \left| \frac{1}{f_n(z)} - \frac{1}{f(z)} \right| = \sup_{z\in K} \left| \frac{f(z) - f_n(z)}{f(z)f_n(z)} \right| \leq \sup_{z\in K} |f_n(z) - f(z)| \cdot \frac{1}{m} \cdot \frac{2}{m}$$

and therefore, after letting $n \to \infty$, that $1/f_n \to 1/f$ compactly in G. ■

Definition 2.3.6. Let $G \subseteq \mathbb{C}$ be a domain and $\mathscr{F} \subseteq H(G)$. Then $\mathscr{F}$ is called:

- *locally bounded* if for every $w \in G$ there is a neighborhood U of w and a constant $C > 0$ such that $|f(z)| \leq C$ for all $f \in \mathscr{F}$ and $z \in U$.
- *normal* in G if every sequence in $\mathscr{F}$ has a subsequence which converges compactly in G to some $f \in H(G)$. If the limit ∞ is also permitted it is instead called $*$-*normal.*

//

The former two concepts are equivalent by the following well-known theorem:

Theorem 2.3.7 (Montel)**.** *Let $G \subseteq \mathbb{C}$ be a domain. Then a family $\mathscr{F} \subseteq H(G)$ is normal if and only if it is locally bounded.*

The following theorem can be interpreted as a sharpened version of Montel's Theorem and is sometimes referred to as the *fundamental normality test.*

Theorem 2.3.8. *Let $G \subseteq \mathbb{C}$ be a domain. Then any family $\mathscr{F} \subseteq H(G, \mathbb{C} \setminus \{0,1\})$ is $*$-normal in G.*

Proof. We give the proof in three steps:

1. Let $w \in G$, $c > 0$ and $\mathscr{F}_* \subseteq \mathscr{F}$ such that $|f(w)| \leq c$ for all $f \in \mathscr{F}_*$. We aim to show that there is an open disk at w in which $\mathscr{F}_*$ is bounded. Select $t > 0$ such that $B_t(w) \subseteq G$. Let $f \in \mathscr{F}_*$, then $g(z) := f(tz + w) \in H(\mathbb{D})$. By the maximum modulus principle and Schottky's Theorem we obtain

$$\sup_{z \in B_{t/2}(w)} |f(z)| \leq \sup_{z \in B_{1/2}(0)} |g(z)| \leq \sup_{|z|=1/2} |g(z)| \leq \psi(1/2, c)$$

   and f is bounded on the disk $B_{t/2}(w)$. Since f was arbitrary, $\mathscr{F}_*$ is bounded as well.

2. Fix some $w_* \in G$ and set $\mathscr{F}_1 := \{f \in \mathscr{F} : |f(w_*)| \leq 1\}$. We aim to show that $\mathscr{F}_1$ is locally bounded in G. Consider the set

$$U := \{w \in G : \mathscr{F}_1 \text{ is bounded in a neighborhood of } w\},$$

   by (1) we have that $w_* \in U$. Note that U is open in G, since if $\mathscr{F}_1$ is bounded in a disk $B_r(w)$, then for any $w' \in B_r(w)$ there is a disk $B_{r'}(w') \subseteq B_r(w)$, on which $\mathscr{F}_1$ is bounded as well.

   Assume towards a contradiction that $U \neq G$. Then there exists some $w \in \partial U \cap G$ such that $\mathscr{F}_1$ is unbounded in every neighborhood of w.

   If there were some $c > 0$ such that $|f(w)| \leq c$ for all $f \in \mathscr{F}_1$, then by (1) there would exist an open disk centered at w on which $\mathscr{F}_1$ would be bounded – contradicting our assumption on w. Thus for every $n \in \mathbb{N}$ we can find some $f_n \in \mathscr{F}_1$ such that $|f_n(w)| \geq n$ and we obtain that $\lim_{n\to\infty} |f_n(w)| = \infty$.

   Set $g_n := 1/f_n \in \mathscr{F}$, then $\lim_{n\to\infty} |g_n(w)| = 0$. In particular, the family $(g_n)_{n\in\mathbb{N}}$ is bounded at w by some constant, thus by (1) the family is bounded in some disk B around w. By Montel's Theorem it is therefore normal in B, and there exists a subsequence $(g_{n_k})_{k\in\mathbb{N}}$ which converges compactly to a $g \in H(B)$. The functions g_{n_k} have no zeros, but $g(w) = 0$; by Hurwitz's Theorem we therefore have $g \equiv 0$. Then for any $z \in B \cap U$ we have

$$\lim_{k\to\infty} |f_{n_k}(z)| = \lim_{k\to\infty} 1/|g_{n_k}(z)| = \infty,$$

   contradicting the assumption that $\mathscr{F}_1$ is bounded in a neighborhood of such z. We thus have $U = G$, therefore $\mathscr{F}_1$ is locally bounded and by Montel's Theorem therefore normal.

3. We can now conclude the proof. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{F}$, we claim that it

has some subsequence which converges compactly to some function in $H(G)$ or to ∞.

If infinitely many f_n lie in $\mathscr{F}_1$, then there is a subsequence $(f_{n_m})_{m\in\mathbb{N}}$ in $\mathscr{F}_1$, which by (2) has a subsequence $(f_{n_{m_k}})_{k\in\mathbb{N}}$ in $\mathscr{F}_1$ which convergences compactly in G to some $f \in H(G)$. This sequence is also a subsequence of $(f_n)_{n\in\mathbb{N}}$, concluding the claim in this case.

On the other hand, if there are only finitely many f_n in $\mathscr{F}_1$, then infinitely many $1/f_n$ lie in $\mathscr{F}_1$. As above, we thus obtain some subsequence in $\mathscr{F}_1$, say $(g_n)_{n\in\mathbb{N}}$, converging compactly in G to some $g \in H(G)$. The sequence $(1/g_n)_{n\in\mathbb{N}}$ is a subsequence of $(f_n)_{n\in\mathbb{N}}$, which – by Lemma 2.3.5 – converges compactly to $1/g$ if $0 \notin g(G)$, and to ∞ otherwise.

■

2.4 Picard's Great Theorem

The following lemma is essential in showing that functions which omit two values on the punctured unit disk cannot have an essential singularity at 0.

Lemma 2.4.1. *Let $f \in H(\mathbb{D}^\times; \mathbb{C} \setminus \{0,1\})$. Then f or $1/f$ is bounded in a punctured neighborhood of zero.*

Proof. For $n \in \mathbb{N}$ set $f_n(z) := f(z/n) \in H(\mathbb{D}^\times; \mathbb{C} \setminus \{0,1\})$. By Theorem 2.3.8 the sequence $(f_n)_{n\in\mathbb{N}}$ has a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ that converges compactly to a $f \in H(\mathbb{D}^\times)$ or to ∞.

Assume the former case. Then there is some $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have $\|f_{n_k} - f\|_{\partial B_{1/2}(0)} < 1$ and thus

$$\|f\|_{\partial B_{1/2n_k}(0)} = \|f_{n_k}\|_{\partial B_{1/2}(0)} \leq \|f_{n_k} - f\|_{\partial B_{1/2}(0)} + \|f\|_{\partial B_{1/2}(0)} \leq 1 + \|f\|_{\partial B_{1/2}(0)},$$

whereby f is bounded on $\partial B_{1/2n_k}(0)$. By the maximum modulus principle, f must therefore be bounded on every annulus $1/(2n_{k+1}) \leq |z| \leq 1/(2n_k)$, for $k \geq k_0$. Thus f is bounded on

$$\bigcup_{k \geq k_0} \left\{ z \in \mathbb{C} : \frac{1}{2n_{k+1}} \leq |z| \leq \frac{1}{2n_k} \right\} = \overline{B_{1/2n_k}(0)} \setminus \{0\},$$

which is a punctured neighborhood of zero.

In the latter case, $(1/f_{n_k})_{k\in\mathbb{N}}$ converges compactly to 0 by Lemma 2.3.5. Replacing f_{n_k} with $1/f_{n_k}$ and f with 0 in the above we likewise obtain that $1/f$ is bounded in a punctured neighborhood of zero. ■

Theorem 2.4.2 (Picard's Great Theorem)**.** *Let $G \subseteq \mathbb{C}$ be open, $w \in G$ and suppose $f \in H(G \setminus \{w\})$ such that f has an essential singularity at w. Then f assumes all values in $\mathbb{C}$, with at most one exception, infinitely often in any punctured neighborhood of w.*

Proof. Assume towards a contradiction that f takes on two distinct values $a, b \in \mathbb{C}$ only finitely often in some punctured neighborhood W of w. Then W contains a punctured disk of radius $t > 0$ around w, on which f does not assume a or b, and thus

$$g(z) := \frac{f(tz+w)-a}{b-a} \in H(\mathbb{D}^\times; \mathbb{C} \setminus \{0,1\}),$$

where g has an essential singularity at zero. By Lemma 2.4.1, we have that either g or $1/g$ must be bounded in a punctured neighborhood of zero. By the classification of isolated singularities, in the former case the singularity must therefore be removable, whereas in the latter case it must be a pole, yielding a contradiction. ∎

The following corollary is also referred to as Picard's Great Theorem. Here an entire transcendental function is an entire function which is not a polynomial.

Corollary 2.4.3. *Every entire transcendental function assumes every value in $\mathbb{C}$ infinitely often, with at most one exception.*

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be transcendental and entire, and consider $g(z) := f(1/z) \in H(\mathbb{C}^\times)$. Then the Laurent series expansion of g at $z = 0$ has infinite principal part, therefore g has an essential singularity at zero. By Picard's Great Theorem g assumes all values in $\mathbb{C}$ on $B_1(0) \setminus \{0\}$ infinitely often, except at most one, and so f does the same on $\mathbb{C} \setminus B_1(0)$. ∎

Remark 2.4.4. Picard's Little Theorem is contained in Picard's Great Theorem:

A non-constant $f \in H(\mathbb{C})$ is either a non-constant polynomial or transcendental. In the latter case, f omits at most one value by Corollary 2.4.3.

In the former case let $w \in \mathbb{C}$, then $f(z) - w$ has a zero in $\mathbb{C}$ by the Fundamental Theorem of Algebra and hence f assumes all values. //

3 Growth and Zeros of Entire Functions

3.1 Order, Type and Growth

To study the rate of growth of an entire function we first introduce the following:

Definition 3.1.1. Let $f \in H(\mathbb{C})$, then for $r \geq 0$ we define

$$M_f(r) := \max_{|z|=r} |f(z)|.$$

//

Remark 3.1.2. By the maximum modulus principle, M_f is either strictly increasing (if f is non-constant) or constant (otherwise), and we have

$$M_f(r) = \max_{|z|\leq r} |f(z)|.$$

We shall show that M_f is continuous: Let $\varepsilon > 0$ and choose $\delta_\varepsilon > 0$ such that for $|z_1 - z_2| < \delta_\varepsilon$ we have $|f(z_1) - f(z_2)| < \varepsilon$. Now let $r_1 < r_2$ and choose θ such that $M_f(r_2) = |f(r_2 e^{i\theta})|$. Then

$$0 \leq M_f(r_2) - M_f(r_1) \leq |f(r_2 e^{i\theta})| - |f(r_1 e^{i\theta})| \leq |f(z_1) - f(z_2)| < \varepsilon,$$

for $r_2 - r_1 < \delta_\varepsilon$. //

3.1.1 Order

Definition 3.1.3. Let $f \in H(\mathbb{C})$. The *order* of f is defined by

$$\rho_f := \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}. \tag{3.1}$$

Constant functions, by convention, have order 0. //

Note that, for $f \in H(\mathbb{C})$, we have $0 \leq \rho_f \leq \infty$.

Proposition 3.1.4. *If Q is a polynomial of degree $n \in \mathbb{N}$, then $\exp Q$ has order n.*

Proof. Write $Q(z) = \sum_{k=0}^{n} a_k z^k$ and let ζ be an n-th root of $\overline{a_n}/|a_n|$. Then

$$|a_n| - \sum_{k=0}^{n-1} |a_k| r^{k-n} \le \operatorname{Re} \sum_{k=0}^{n} a_k \zeta^k r^{k-n} = \operatorname{Re} \frac{Q(r\zeta)}{r^n} \le \max_{|z|=r} \operatorname{Re} \frac{Q(z)}{r^n} \le |a_n| + \sum_{k=0}^{n-1} |a_k| r^{k-n}$$

and taking limits as $r \to \infty$ we get $\lim_{r\to\infty} \max_{|z|=r} \operatorname{Re} \frac{Q(z)}{r^n} = |a_n|$. Set $f(z) \coloneqq \exp Q(z)$, then

$$\begin{aligned}\frac{\log\log M_f(r)}{\log r} &= \frac{\log\log \max_{|z|=r} |\exp Q(z)|}{\log r} = \frac{\log\log \max_{|z|=r} \exp \operatorname{Re} Q(z)}{\log r} = \\ &= \frac{\log \max_{|z|=r} \operatorname{Re} Q(z)}{\log r} = \frac{\log \max_{|z|=r} \operatorname{Re} Q(z)/r^n}{\log r} + n\end{aligned}$$

and taking the limit superior as $r \to \infty$ yields $\rho_f = n$. ∎

If the order of an entire function is finite we have an equivalent characterization.

Proposition 3.1.5. *Let $f \in H(\mathbb{C})$, then f is of finite order if and only if*

$$\rho \coloneqq \inf\{s > 0 : M_f(r) = \mathcal{O}(\exp r^s) \text{ as } r \to \infty\} \tag{3.2}$$

is finite, and in either case we have $\rho_f = \rho$.

Proof. Suppose $0 \le \rho < \infty$, then for all $s > \rho$ we have $M_f(r) = \mathcal{O}(\exp r^s)$ as $r \to \infty$. Thus there exists a constant $K > 0$ (we may assume $K > 1$) and such that for all sufficiently large $r > 0$ we have $M_f(r) \le K \exp r^s$. Using the fact that $\log(a+b) = \log a + \log(1 + \frac{b}{a})$ we get

$$\frac{\log\log M_f(r)}{\log r} \le \frac{\log(r^s + \log K)}{\log r} = s + \frac{\log(1 + (\log K)/r^s)}{\log r}. \tag{$*$}$$

Since

$$0 \le \frac{\log(1 + (\log K)/r^s)}{\log r} \le \frac{(\log K)/r^s}{\log r} = \frac{\log K}{r^s \log r} \xrightarrow{r\to\infty} 0,$$

by taking the limit superior as $r \to \infty$ in $(*)$ and then letting $s \to \rho$ we obtain $\rho_f \le \rho$.

Now suppose $0 \le \rho_f < \infty$ and let $s > \rho_f$. Then, by definition of the limit superior, for all sufficiently large $r > 0$ we have $\log\log M_f(r) \le s \log r$ and therefore $M_f(r) \le \exp r^s$. Thus $M_f(r) = \mathcal{O}(\exp r^s)$, thereby $\rho \le s$ and letting $s \to \rho_f$ yields $\rho \le \rho_f$. ∎

Remark 3.1.6. Any polynomial is of order zero. Indeed, let $Q(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial and $m \in \mathbb{N}$. Then for any $r > 1$ we have

$$\exp r^{1/m} = \sum_{k=0}^{\infty} \frac{r^{k/m}}{k!} > \frac{r^n}{(mn)!}$$

and therefore

$$M_Q(r) = \max_{|z|=r} |Q(z)| \leq \left(\sum_{k=0}^{n} |a_k| \right) r^n \leq \left((mn)! \sum_{k=0}^{n} |a_k| \right) \exp r^{1/m}.$$

Since $m \in \mathbb{N}$ was arbitrary, Proposition 3.1.5 gives $\rho_Q = 0$.

There are however non-polynomial functions of order zero; one must simply construct an entire function with power series coefficients of sufficiently rapid decay. Consider

$$f(z) := \sum_{k=0}^{\infty} \frac{z^k}{(k^2)!}$$

and let $m \in \mathbb{N}$. By the above we have $\sum_{k=0}^{m-1} \frac{r^k}{(k^2)!} \leq K \exp r^{1/m}$ for some constant $K > 0$ depending only on m, holding for all $r > 1$. Thus

$$\sum_{k=0}^{\infty} \frac{r^k}{(k^2)!} \leq K \exp r^{1/m} + \sum_{k=m}^{\infty} \frac{r^k}{(km)!} \leq K \exp r^{1/m} + \sum_{k=0}^{\infty} \frac{r^{k/m}}{k!} = (K+1) \exp r^{1/m}$$

and we obtain $\rho_f = 0$, as above. //

Proposition 3.1.7. *Let $f, g \in H(\mathbb{C})$ be of finite order. Then it holds that:*

i. $\rho_{f+g} \leq \max\{\rho_f, \rho_g\}$, *with equality holding if* $\rho_f \neq \rho_g$.

ii. $\rho_{fg} \leq \max\{\rho_f, \rho_g\}$.

Proof. That the order of the sum or product is bounded above by the larger of the respective two orders is immediately obtained through Proposition 3.1.5.

Assume $\rho_f \neq \rho_g$, without loss of generality $\rho_f < \rho_g$. By the above we have

$$\rho_{f+g} \leq \max\{\rho_f, \rho_g\} = \rho_g, \quad \text{and} \quad \rho_g = \rho_{f+g+(-f)} \leq \max\{\rho_f, \rho_{f+g}\} = \rho_{f+g},$$

since if $\rho_f > \rho_{f+g}$ then $\rho_f > \rho_{f+g} \geq \rho_g$ would be a contradiction. Thus $\rho_{f+g} = \rho_g$, proving the assertion. ∎

It is of note that Proposition 3.1.7 implies that the order of an entire function of finite order remains unchanged when adding a polynomial of arbitrary degree to it.

There is a, perhaps surprising, connection between the order of an entire function and the order of its derivative:

Proposition 3.1.8. *If $f \in H(\mathbb{C})$ is of finite order, then $\rho_{f'} = \rho_f$.*

Proof. Without loss of generality we may assume $f(0) = 0$. If f is a polynomial the assertion is clear. Otherwise for any $r > 0$ we have,

$$M_f(r) = \max_{|z|=r} |f(z)| = \max_{|z|=r} \left| \int_{[0,z]} f'(\zeta)\,\mathrm{d}\zeta \right| \leq$$
$$\leq \max_{|z|=r} \left(|z| \max_{w \in \overline{B_r(0)}} |f'(w)| \right) \leq r M_{f'}(r).$$

Thus

$$\frac{\log\log M_f(r)}{\log r} \leq \frac{\log\log r M_{f'}(r)}{\log r} = \frac{\log\log M_{f'}(r)}{\log r} + \frac{\log(1 + \log r/\log M_{f'}(r))}{\log r} \leq$$
$$\leq \frac{\log\log M_{f'}(r)}{\log r} + \frac{\log r/\log M_{f'}(r)}{\log r} = \frac{\log\log M_{f'}(r)}{\log r} + \frac{1}{\log M_{f'}(r)}$$

and taking limits superior as $r \to \infty$ yields $\rho_f \leq \rho_{f'}$.

On the other hand, Cauchy's integral formula gives

$$M_{f'}(r) = \max_{z \in \partial B_r(0)} |f'(z)| = \max_{z \in \partial B_r(0)} \left| \frac{1}{2\pi i} \oint_{\partial B_1(z)} \frac{f(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\zeta \right| \leq$$
$$\leq \max_{\substack{z \in \partial B_r(0) \\ w \in \partial B_1(z)}} |f(w)| \leq \max_{z \in \overline{B_{r+1}(0)}} |f(z)| = M_f(r+1).$$

Therefore

$$\frac{\log\log M_{f'}(r)}{\log r} \leq \frac{\log\log M_f(r+1)}{\log(r+1)} \frac{\log(r+1)}{\log r}$$

and, again, taking limits superior as $r \to \infty$ we get $\rho_{f'} \leq \rho_f$. ■

3.1.2 Type

For functions of finite and positive order, we can obtain a natural refinement of the concept of order:

Definition 3.1.9. Let $f \in H(\mathbb{C})$ be of order $0 < \rho_f < \infty$. The *type* of f is defined by

$$\tau_f := \limsup_{r \to \infty} \frac{\log M_f(r)}{r^{\rho_f}} \tag{3.3}$$

If $\tau_f = 0$, then f is said to be of *minimal* type, if $0 < \tau_f < \infty$ of *normal* type, and if $\tau_f = \infty$ of *maximal* type. //

Once again, we have an equivalent characterization for functions of finite type:

Proposition 3.1.10. *Let $f \in H(\mathbb{C})$ be of finite, positive order, then f is of finite type if and only if*

$$\tau := \inf\{t > 0 : M_f(r) = \mathcal{O}(\exp(tr^{\rho_f})) \text{ as } r \to \infty\} \tag{3.4}$$

is finite, and in either case we have $\tau_f = \tau$.

Proof. Suppose $0 \leq \tau < \infty$, then for all $t > \tau$ we have $M_f(r) = \mathcal{O}(\exp tr^{\rho_f})$ as $r \to \infty$. Thus there exists a constant $K > 0$ such that for all sufficiently large $r > 0$ we have $M_f(r) \leq K \exp tr^{\rho_f}$. Therefore

$$\frac{\log M_f(r)}{r^{\rho_f}} \leq \frac{\log K + tr^{\rho_f}}{r^{\rho_f}} = \frac{\log K}{r^{\rho_f}} + t$$

and taking limits superior as $r \to \infty$ and letting $t \to \tau$ afterwards yields $\tau_f \leq \tau$.

Now suppose $0 \leq \tau_f < \infty$ and let $t > \tau_f$. Then, by definition of the limit superior, for all sufficiently large $r > 0$ we have $\log M_f(r) \leq tr^{\rho_f}$ and therefore $M_f(r) \leq \exp(tr^{\rho_f})$. Thus $M_f(r) = \mathcal{O}(\exp(tr^{\rho_f}))$, thereby $\tau \leq t$ and letting $t \to \tau_f$ yields $\tau \leq \tau_f$. ∎

If $f \in H(\mathbb{C})$ is of finite, positive order, then f and f' not only have the same order, they have the same type:

Proposition 3.1.11. *If $f \in H(\mathbb{C})$ is of finite, positive order and finite type, then $\tau_{f'} = \tau_f$.*

Proof. Reusing the inequalities obtained in Proposition 3.1.8 we have, for all $r > 0$,

$$M_f(r) \leq rM_{f'}(r), \quad \text{and} \quad M_{f'}(r) \leq M_f(r+1).$$

Therefore

$$\frac{\log M_f(r)}{r^{\rho_f}} \leq \frac{\log r + \log M_{f'}(r)}{r^{\rho_f}}, \quad \text{and} \quad \frac{\log M_{f'}(r)}{r^{\rho_f}} \leq \frac{\log M_f(r+1)}{(r+1)^{\rho_f}} \frac{(r+1)^{\rho_f}}{r^{\rho_f}}$$

and taking limits superior as $r \to \infty$ yields $\tau_f \leq \tau_{f'}$ and $\tau_{f'} \leq \tau_f$. ∎

3.1.3 Growth

Definition 3.1.12. Let $f \in H(\mathbb{C})$. Then f is said to be of *growth* (a, b) if

- $\rho_f < a$, or
- $\rho_f = a$ and $\tau_f \leq b$.

//

Example 3.1.13. For $\rho, \tau \in (0, \infty)$, the function

$$f(z) := \exp \tau z^{\rho}$$

is of order ρ and type τ.

We have already seen in Remark 3.1.6 that polynomials are of order zero.

The function

$$f(z) := \exp\exp z$$

is of infinite order, since for any $m \in M$

$$M_f(r) \geq \exp\exp r \geq \exp(m!r^m),$$

therefore $\rho_f \geq m$, and thus $\rho_f = \infty$. //

3.2 Zeros

3.2.1 Order, Density and Exponent of Convergence

We first recall a rather explicit connection between the moduli of the zeros of a holomorphic function and the modulus of the function itself:

Theorem 3.2.1 (Jensen)**.** *Let $f \in H(B_R(0))$ with $f(0) \neq 0$ and let $r_1, r_2, \ldots$ denote the moduli of the zeros of f in $B_R(0)$ arranged in a non-decreasing sequence. Then, for $r_n < r < r_{n+1}$, we have*

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,\mathrm{d}\theta = \log|f(0)| + \log\frac{r^n}{r_1 \ldots r_n}. \tag{3.5}$$

Definition 3.2.2. Let $f \in H(B_R(0))$. Then, for $0 < r < R$, we define

$$n_f(r) := |\{z \in \overline{B_r(0)} : f(z) = 0\}|, \tag{3.6}$$

that is, the number of zeros of f in $\overline{B_r(0)}$. //

This zero-counting function is, in some scenarios, simpler to work with than the sequence of zeros. For instance, it can be used to obtain an equivalent version of Jensen's Theorem:

Corollary 3.2.3. *Let $f \in H(B_R(0))$ with $f(0) \neq 0$. Then, for $0 < r < R$, we have*

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,\mathrm{d}\theta = \log|f(0)| + \int_0^r \frac{n_f(s)}{s}\,\mathrm{d}s \tag{3.7}$$

Proof. Let $r_1, r_2, \ldots$ denote the moduli of the zeros of f in $B_R(0)$ arranged in a non-decreasing sequence. Then, for any $r_n < r < r_{n+1}$, we obtain

$$\begin{aligned}\log \frac{r^n}{r_1 \dots r_n} &= \sum_{k=1}^{n} \log \frac{r}{r_k} = \sum_{k=1}^{n} \int_{r_k}^{r} \frac{1}{s} \,\mathrm{d}s = \\ &= \sum_{k=1}^{n} \int_{0}^{r} \mathbb{1}_{(r_k,\infty)}(s) \frac{1}{s} \,\mathrm{d}s = \int_{0}^{r} \left(\sum_{k=1}^{n} \mathbb{1}_{(r_k,\infty)}(s) \right) \frac{1}{s} \,\mathrm{d}s = \\ &= \int_{0}^{r} \frac{n_f(s)}{s} \,\mathrm{d}s\end{aligned}$$

and Theorem 3.2.1 establishes the assertion. ■

This gives an immediate connection between the zeros and the growth of the modulus of an entire function:

Proposition 3.2.4. *Let $f \in H(\mathbb{C})$ be of finite order. Then for any $\rho > \rho_f$ we have*

$$n_f(r) = \mathcal{O}(r^{\rho}), \quad \textit{as} \quad r \to \infty.$$

Proof. We may assume $f(0) \neq 0$. Let $r > 0$, then since n_f is non-negative and non-decreasing we have

$$\int_{0}^{2r} \frac{n_f(s)}{s} \,\mathrm{d}s \geq \int_{r}^{2r} \frac{n_f(s)}{s} \,\mathrm{d}s \geq n_f(r) \int_{r}^{2r} \frac{1}{s} \,\mathrm{d}s = n_f(r)(\log 2r - \log r) = n_f(r) \log 2.$$

Corollary 3.2.3 now yields

$$n_f(r) \log 2 \leq \int_{0}^{2r} \frac{n_f(s)}{s} \,\mathrm{d}s = \log |f(0)| + \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(2re^{i\theta})| \,\mathrm{d}\theta \leq \log |f(0)| + \log M_f(2r)$$

and holds for all $r > 0$ since f is entire. Now, f is of finite order, thus for any $\varepsilon > 0$ there is a constant $K_1 > 0$ such that for sufficiently large r we have $M_f(r) \leq K_1 \exp r^{\rho_f + \varepsilon}$. Thus

$$n_f(r) \log 2 \leq \log |f(0)| + \log K_1 + (2r)^{\rho_f + \varepsilon}.$$

and therefore $n_f(r) \leq K_2 r^{\rho_f + \varepsilon}$ for some constant $K_2 > 0$ and sufficiently large r. ■

Proposition 3.2.4 shows that the more zeros a function f has, the faster M_f must grow as $r \to \infty$. The converse is naturally false; by composing exponentials one can obtain an entire function f for which M_f grow arbitrarily fast, yet f has no zeros.

Definition 3.2.5. Let $M \subseteq \mathbb{N}$ and $\mathbf{z} = (z_m)_{m \in M}$ be a family of points in $\mathbb{C}^{\times}$. Then

$$\lambda_{\mathbf{z}} := \inf \left\{ \lambda > 0 : \sum_{m \in M} \frac{1}{|z_m|^{\lambda}} < \infty \right\}$$

is called the *exponent of convergence* of the family $\mathbf{z}$.

Let $f \in H(\mathbb{C})$, then the *exponent of convergence of the zeros of* f is defined as the exponent of convergence of the non-zero roots of f and is denoted λ_f.

Furthermore, for any $a \in \mathbb{C}$, the *exponent of convergence of the a-points of* f is defined as exponent of convergence of the zeros of $f(z) - a$ and will be denoted $\lambda_f^{(a)}$. //

If $f \in H(\mathbb{C})$ assumes some value $a \in \mathbb{C}$ only finitely often, then clearly $\lambda_f^{(a)} = 0$.

Theorem 3.2.6. *If $f \in H(\mathbb{C})$ is of finite order, then $\lambda_f \leq \rho_f$.*

Proof. We may assume $f(0) \neq 0$. If $\lambda_f = 0$ there is nothing to show. Let $\rho > \rho_f$ then Proposition 3.2.4 yields a constant $K_1 > 0$ such that for sufficiently large r we have $n_f(r) \leq K_1 r^\rho$.

If $(r_j)_{j\in\mathbb{N}}$ denote the non-zero moduli of the zeros of f, arranged in non-decreasing order, then for any $m \in \mathbb{N}$ we have $m = n_f(r_m) \leq K_1 r_m^\rho$. Let $0 < \lambda < \lambda_f$ then this implies

$$\left(\frac{1}{r_m}\right)^\lambda \leq K_2 \left(\frac{1}{m}\right)^{\lambda/\rho}.$$

for some constant $K_2 > 0$. Therefore

$$\sum_{m=1}^{\infty} \left(\frac{1}{r_m}\right)^\lambda \leq K_2 \sum_{m=1}^{\infty} \left(\frac{1}{m}\right)^{\lambda/\rho}.$$

The left-hand side diverges by Definition 3.2.5, thus so does the right-hand side. But the latter diverges if and only if $\frac{\lambda}{\rho} \leq 1$, therefore letting $\lambda \nearrow \lambda_f$ and then $\rho \searrow \rho_f$ yields $\lambda_f \leq \rho_f$. ■

Remark 3.2.7. The function $f(z) := \exp z$ is of order one and has no zeros – thus we observe that we may have $\lambda_f < \rho_f$ in some cases. //

As Proposition 3.2.4 might indicate, a more precise analysis of the density of the zeros of an entire function f is contained in the growth of the zero-counting function n_f.

Definition 3.2.8. Let $f \in H(\mathbb{C})$. As the *order* of the zero-counting function n_f we define

$$\nu_f := \limsup_{r\to\infty} \frac{\log n_f(r)}{\log r}. \tag{3.8}$$

If ν_f is finite we additionally define the *upper density* of the zeros of f as

$$\Delta_f := \limsup_{r\to\infty} \frac{n_f(r)}{r^{\nu_f}}. \tag{3.9}$$

If the limit exists then Δ_f is simply called the *density*. //

Proposition 3.2.9. *Let $f \in H(\mathbb{C})$, then λ_f is finite if and only if ν_f is finite and in either case we have $\lambda_f = \nu_f$.*

Proof. TODO. ∎

3.2.2 Canonical Products

By the Fundamental Theorem of Algebra every complex polynomial can be written as a product of linear factors involving only its roots and some scaling factor. Conversely, for a finite sequence of points $z_1, \ldots, z_n$ the polynomial $p(z) := \prod_{k=1}^{n}(z - z_k)$ vanishes only at said points.

The question arises if one can construct an entire function that vanishes precisely at the points of some infinite sequence $(z_k)_{k\in\mathbb{N}}$ and nowhere else. Clearly the product $\prod_{k=1}^{\infty}(z - z_k)$ need not be convergent in the entire complex plane. This issue is solved by introducing additional exponential factors, which improve convergence without introducing new zeros. We first study sufficient conditions for convergence of infinite products of numbers and holomorphic functions.

Lemma 3.2.10. *Suppose $G \subseteq \mathbb{C}$ is open and let $(f_k)_{k\in\mathbb{N}}$ be a sequence of non-vanishing functions in $H(G)$. If $\sum_{k=1}^{\infty}(1 - f_k(z))$ converges absolutely compactly in G, then $\prod_{k=1}^{\infty} f_k(z)$ converges compactly in G to a non-vanishing $f \in H(G)$.*

Proof. Let $K \subset G$ be compact, then the absolute, compact convergence of the sum implies that there is a k_0 such that $|1 - f_k(z)| \le 1/2$ for $z \in K, k > k_0$. For $|w| \le 1/2$ we have

$$|\log(1 - w)| \le \sum_{k=1}^{\infty} \left|\frac{1}{k} w^k\right| \le |w| \sum_{k=0}^{\infty} |w|^k = \frac{|w|}{1 - |w|} \le 2|w|.$$

Therefore

$$\sum_{k=1}^{\infty} |\log f_k(z)| = \sum_{k=1}^{\infty} |\log(1 - (1 - f_k(z)))| \le \sum_{k=1}^{k_0-1} |\log f_k(z)| + \sum_{k=k_0}^{\infty} |1 - f_k(z)|$$

and since the rightmost sum converges compactly it follows that $\sum_{k=1}^{\infty} \log f_k(z)$ converges absolutely compactly in G. Since the exponential function is uniformly continuous, the sequence

$$\exp \sum_{k=k_0}^{n} \log f_k(z) = \prod_{k=k_0}^{n} f_k(z), \quad n \in \mathbb{N}$$

converges absolutely compactly to some $f \in H(G)$. Again by continuity of the exponential function we have

$$f(z) = \prod_{k=1}^{\infty} f_k(z) = \exp \sum_{k=1}^{\infty} \log f_k(z) \neq 0$$

and thus f is non-vanishing. ■

Definition 3.2.11. The *canonical factors* are defined at $z \in \mathbb{C}$ by

$$W(z;0) := 1 - z, \quad \text{and} \quad W(z;n) := (1-z)\exp\left(\sum_{k=1}^{n} \frac{z^k}{k}\right), \quad n \in \mathbb{N}.$$

The number n is called the *degree* of the canonical factor. //

Lemma 3.2.12. *Let $n \in \mathbb{N}_0$ and $|z| \le 1/2$, then $|1 - W(z;n)| \le 2e|z|^{n+1}$.*

Proof. For $n = 0$ there is nothing to show. Note that for such given z we can expand $\log(1-z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}$, therefore

$$W(z;n) = \exp\left(\log(1-z) + \sum_{k=1}^{n} \frac{z^k}{k}\right) = \exp\zeta,$$

where $\zeta := -\sum_{k=n+1}^{\infty} z^k/k$. Furthermore, we have that

$$|\zeta| \le |z|^{n+1} \sum_{k=n+1}^{\infty} |z|^{k-n-1}/k \le |z|^{n+1} \sum_{j=0}^{\infty} 2^{-j} \le 2|z|^{n+1}$$

and in particular $|\zeta| \le 1$. We conclude

$$|1 - W(z;n)| = |\exp\zeta - 1| \le \sum_{k=1}^{\infty} \frac{|\zeta|^k}{k!} \le |\zeta| \sum_{k=0}^{\infty} \frac{1}{k!} \le 2e|z|^{n+1}.$$

■

The next theorem – which is also known as the *Weierstrass Product Theorem* – now constructively asserts the existence of entire function with prescribed zeros.

Theorem 3.2.13. *Let $M \subseteq \mathbb{N}$ and $\mathbf{z} = (z_m)_{m\in M}$ be a family in $\mathbb{C}^\times$ without accumulation points. Then there is a family $(p_m)_{m\in M}$ in $\mathbb{N}_0$ such that*

$$\Pi(z) := \prod_{m\in M} W\left(\frac{z}{z_m}; p_m\right) \tag{3.10}$$

is an entire function that vanishes at all $z_m, m \in M$ and nowhere else. If

$$\mu := \min\left\{p \in \mathbb{N}_0 : \sum_{m\in M} \frac{1}{|z_m|^{p+1}} < \infty\right\} \tag{3.11}$$

is finite then one can take $p_m = \mu, m \in M$.

Proof. If M is finite there is nothing to show. We may thus assume that M is infinite and without loss of generality $M = \mathbb{N}$. Thus $\mathbf{z}$ is a sequence, which we can assume to be arranged in non-decreasing order according to the moduli its elements.

We claim that there exists a sequence $(p_n)_{n\in\mathbb{N}}$ in $\mathbb{N}_0$ such that

$$\sum_{n=1}^{\infty} \left| \frac{z}{z_n} \right|^{p_n+1} \tag{3.12}$$

converges compactly in $\mathbb{C}$. Let $K \subset B_R(0) \subset \mathbb{C}$ be compact. Note that since $\mathbf{z}$ has no accumulation points we necessarily have $\lim_{n\to\infty} |z_n| = \infty$, thus there is some n_0 such that for $n > n_0$ we have $|z_n| \geq R + 1$. Therefore, for any $z \in K$,

$$\sum_{n=1}^{\infty} \left| \frac{z}{z_n} \right|^{p_n+1} \leq \sum_{n=1}^{n_0-1} \left| \frac{z}{z_n} \right|^{p_n+1} + \sum_{n=n_0}^{\infty} \left(\frac{R}{R+1} \right)^{p_n+1},$$

and the rightmost series is finite if, for example, $p_n = n, n \in \mathbb{N}$. Therefore the series is uniformly bounded and thus compactly convergent, concluding our claim. Note that if μ is finite then we can take $p_n = \mu, n \in \mathbb{N}$, since for $z \in K$

$$\sum_{n=1}^{\infty} \left| \frac{z}{z_n} \right|^{\mu+1} \leq |z|^{\mu} \sum_{n=1}^{\infty} \frac{1}{|z_n|^{\mu+1}} \leq R^{\mu} \sum_{n=1}^{\infty} \frac{1}{|z_n|^{\mu+1}},$$

where the rightmost sum is a finite constant by the definition of μ.

It remains to show that is Π is an entire function that satisfies the desired properties.Let $R > 0$, then it suffices to show that Π has is holomorphic in $B_R(0)$ and vanishes only at points of $\mathbf{z}$ that are also in $B_R(0)$. We write

$$\Pi(z) = \left(\prod_{\substack{n=1 \\ |z_n|<2R}}^{\infty} W\left(\frac{z}{z_n}; p_n \right) \right) \left(\prod_{\substack{n=1 \\ |z_n|\geq 2R}}^{\infty} W\left(\frac{z}{z_n}; p_n \right) \right) =: \Pi_1(z)\Pi_2(z).$$

Clearly Π_1 is a finite product which vanishes at points of $\mathbf{z}$ in $B_R(0)$ and nowhere else. Note that for $|z_m| \geq 2R$ we have $|z/z_m| \leq R/|z_m| \leq 1/2$, hence by Lemma 3.2.12 we have

$$\sum_{\substack{n=1 \\ |z_n|\geq 2R}}^{\infty} \left| 1 - W\left(\frac{z}{z_n}; p_n \right) \right| \leq 2e \sum_{\substack{n=1 \\ |z_n|\geq 2R}}^{\infty} \left| \frac{z}{z_n} \right|^{p_n+1}$$

and the compact convergence of the right sum, which stems from our choice of the $p_n, n \in \mathbb{N}$, implies compact convergence of the left sum. By Lemma 3.2.10 we thereby have that $\Pi_2 \in H(B_R(0))$ and vanishes nowhere. Therefore Π has the desired properties and, since R was arbitrary, this concludes the proof. ∎

Definition 3.2.14. With the notation of the Theorem 3.2.13, Π is called a *Weierstrass* *product*[1] formed from the family $\mathbf{z}$. If $\mu < \infty$ then

$$\Pi(z) := \prod_{m \in M} W\left(\frac{z}{z_m}; \mu\right) \tag{3.13}$$

is called a *(Weierstrass) canonical product* and μ is called the *genus* of the canonical product, which will also be denoted as μ_Π. Otherwise the genus of Π is said to be infinite. //

Remark 3.2.15. By comparing Definition 3.2.5 and Definition 3.2.14 we immediately observe that if $\lambda_\Pi < \infty$ then $\mu_\Pi \leq \lambda_\Pi \leq \mu_\Pi + 1$. If therefore λ_f is an integer, and Π has infinitely many zeros, the series

$$\sum_{m \in M} \frac{1}{|z_m|^\lambda},$$

will converge if $\lambda_\Pi = \mu_\Pi + 1$ and diverge if $\lambda_\Pi = \mu_\Pi$. //

Just as any polynomial decomposes into linear factors involving their roots by to the Fundamental Theorem of Algebra, Weierstrass products allow us to obtain a similar factorization result for arbitrary entire functions, known as the *Weierstrass Factorization Theorem.*

Theorem 3.2.16 (Weierstrass)**.** *Let $f \in H(\mathbb{C}), f \neq 0$ and let $m \in \mathbb{N}_0$ be the order of the zero of f at the origin. Then there exists a $g \in H(\mathbb{C})$ such that*

$$f(z) = z^m e^{g(z)} \Pi(z), \tag{3.14}$$

where Π denotes a Weierstrass product formed from the zeros of f.

Proof. Since $f(z)$ and $z^m \Pi(z)$ have the same zeros with identical multiplicities the function

$$\varphi(z) := \frac{f(z)}{z^m \Pi(z)}$$

is entire and vanishes nowhere. Therefore we can write

$$\varphi(z) = e^{g(z)}$$

for some $g \in H(\mathbb{C})$ and rearranging terms yields the desired representation. ■

Definition 3.2.17. In the context of Weierstrass' Theorem, if Π is of finite genus and g is a polynomial, then the *genus of the entire function* f is defined as

$$\mu_f := \max\{\mu_\Pi, \deg g\}. \tag{3.15}$$

If g is not a polynomial or if Π is of infinite genus then f is said to be of infinite genus. //

[1] Note that this product is not uniquely determined, since the sequence $(p_m)_{m \in M}$ is not unique.

We wish to further estimate canonical products. To do this we first focus on the canonical factors.

Lemma 3.2.18. *For $p \in \mathbb{N}$ and all $z \in \mathbb{C}$ we have*

$$\log|W(z;p)| < A_p \frac{|z|^{p+1}}{1+|z|}, \quad \textit{where} \quad A_p := 3e(2+\ln p). \tag{3.16}$$

For $p = 0$ we have

$$\log|W(z;0)| \le \log(1+|z|). \tag{3.17}$$

Proof. The second assertion is clear since

$$\log|W(z;0)| = \log|1-z| \le \log(1+|z|).$$

TODO. ■

Lemma 3.2.19. *Let $\Pi \in H(\mathbb{C})$ be a canonical product of genus $p \in \mathbb{N}_0$. Then for any $z \in \mathbb{C}$ and $r = |z|$ we have*

$$\log|\Pi(z)| < k_p r^p \left(\int_0^r \frac{n_f(t)}{t^{p+1}} \,\mathrm{d}t + r \int_r^\infty \frac{n_f(t)}{t^{p+2}} \,\mathrm{d}t \right), \tag{3.18}$$

where

$$k_0 := 1, \quad \textit{and} \quad k_p := 3e(p+1)(2+\log p), \quad p \in \mathbb{N}.$$

Proof. TODO. ■

Theorem 3.2.20. *Let $\Pi \in H(\mathbb{C})$ be a canonical product of finite order, then $\lambda_\Pi = \rho_\Pi$.*

Proof. TODO. ■

Theorem 3.2.21. *Let $\Pi \in H(\mathbb{C})$ be a canonical product of finite order. If λ_Π is not an integer, then Π is of maximal, minimal or normal type according to whether Δ_Π is equal to infinity, zero, or a positive, real number.*

Proof. TODO. ■

3.2.3 Hadamard's Theorem

Our main goal in this section will be to refine the factorization given by the Weierstrass Factorization Theorem for entire functions of finite order. Such a refinement is given by Hadamard's Theorem, the proof of which relies on the following lemma, which can be considered as a version of the maximum modulus principle applied to the real part of a holomorphic function.

Lemma 3.2.22 (Borel-Carathéodory)**.** *Let $G \subseteq \mathbb{C}$ be a domain, $R > 0$ and suppose $\overline{B_R(0)} \subseteq G$ and $f \in H(G)$. Define $A_f(r) \coloneqq \max_{|z|=r} \operatorname{Re} f(z)$, then, for $0 < r < R$,*

$$M_f(r) \leq \frac{2r}{R-r} A_f(R) + \frac{R+r}{R-r} |f(0)| \tag{3.19}$$

and, if additionally $A_f(R) \geq 0$, then for any $n \in \mathbb{N}$

$$M_{f^{(n)}}(r) \leq \frac{2^{n+2} n! R}{(R-r)^{n+1}} (A_f(R) + |f(0)|). \tag{3.20}$$

Proof. If f is constant there is nothing to show.

We assume f being non-constant. First, for $r > 0$ we have

$$A_f(r) = \max_{|z|=r} \operatorname{Re} f(z) = \log \max_{|z|=r} \exp \operatorname{Re} f(z) = \log \max_{|z|=r} |\exp f(z)| = \log M_{\exp f}(r)$$

and since $M_{\exp f}$ is strictly increasing and continuous therefore so is A_f.

We will show (3.19) by considering two cases. First assume f is non-constant and $f(0) = 0$. Then by the above we have $A_f(R) > A_f(0) = 0$. Define

$$\phi(z) \coloneqq \frac{f(z)}{2A_f(R) - f(z)} \in H(B_R(0)).$$

Note that we have $\phi(0) = 0$ and

$$\begin{aligned} |\phi(z)|^2 = \phi(z)\overline{\phi(z)} &= \frac{|f(z)|^2}{(2A_f(R) - f(z))(2A_f(R) - \overline{f(z)})} = \\ &= \frac{|f(z)|^2}{(2A_f(R) - \operatorname{Re} f(z))^2 + |f(z)|^2 - (\operatorname{Re} f(z))^2} \leq 1, \end{aligned}$$

since clearly $2A_f(R) - \operatorname{Re} f(z) \geq \operatorname{Re} f(z)$. Now, since $\phi(zR) \in H(\mathbb{D})$ and $|\phi(zR)| \leq 1$ for $z \in \mathbb{D}$, Schwartz' Lemma implies $|\phi(zR)| \leq |z|$ for $z \in \mathbb{D}$. Therefore, for $z \in B_R(0)$ we have $|\phi(z)| \leq \frac{|z|}{R}$, and for $z \in B_r(0)$ we have $|\phi(z)| < \frac{r}{R} < 1$. Since

$$f(z) = 2A_f(R) \frac{\phi(z)}{1 + \phi(z)}$$

we obtain

$$|f(z)| = 2A_f(R) \left| \sum_{n=1}^{\infty} (-1)^n \phi(z)^n \right| \leq 2A_f(R) \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n = \frac{2r}{R-r} A_f(R).$$

Now assume that f is non-constant and $f(0) \neq 0$. Set $g(z) \coloneqq f(z) - f(0)$, then by the above we have, for $w \in B_r(0)$,

$$|f(w)| - |f(0)| \le |g(w)| \le M_g(r) \le \frac{2r}{R-r} A_g(R) \le \frac{2r}{R-r} A_f(R) - \frac{2r}{R-r} \operatorname{Re} f(0).$$

Since $-\operatorname{Re} f(0) \le |f(0)|$, this implies

$$|f(w)| \le \frac{2r}{R-r} A_f(R) + \frac{2r}{R-r}|f(0)| + |f(0)| \le \frac{2r}{R-r} A_f(R) + \frac{R+r}{R-r}|f(0)|,$$

thus proving (3.19).

To show (3.20), let $z \in \partial B_r(0)$ and set $s := \frac{R-r}{2}$, then by Cauchy's integral formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{\partial B_s(z)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, \mathrm{d}\zeta.$$

Replacing r with $\frac{R+r}{2} < R$ in (3.19) we get

$$\begin{aligned} |f^{(n)}(z)| &\le \frac{n!}{2\pi} 2\pi \frac{R-r}{2} \left(\frac{2}{R-r} \right)^{n+1} \left(\frac{2(R+r)/2}{R-(R+r)/2} A_f(R) + \frac{R+(R+r)/2}{R-(R+r)/2} |f(0)| \right) \le \\ &\le \frac{2^{n+1} n!}{(R-r)^{n+1}} \frac{R-r}{2} \left(2\frac{R+r}{R-r} A_f(R) + \frac{3R+r}{R-r}|f(0)| \right) \le \\ &\le \frac{2^{n+1} n!}{(R-r)^{n+1}} \left((R+r) A_f(R) + \frac{3R+r}{2}|f(0)| \right) \le \frac{2^{n+2} n! R}{(R-r)^{n+1}} (A_f(R) + |f(0)|). \end{aligned}$$

■

Theorem 3.2.23 (Hadamard)**.** *Let $f \in H(\mathbb{C}), f \neq 0$ be of finite order and let $m \in \mathbb{N}_0$ be the order of the zero of f at the origin. Then there exists a polynomial Q with* $\deg Q \le \rho_f$ *such that*

$$f(z) = z^m e^{Q(z)} \Pi(z), \tag{3.21}$$

where Π denotes the canonical product formed from the zeros of f.

Hadamard's Theorem can also be formulated more concisely: Let $f \in H(\mathbb{C})$ be of finite order, then $\mu_f \le \rho_f$, that is the genus of f does not exceed its order.

Proof. We may assume $f(0) \neq 0$. By the Weierstrass Factorization Theorem we have $f(z) = e^{Q(z)} \Pi(z)$ for some $Q \in H(\mathbb{C})$ and a Weierstrass product Π. Since

$$\mu_\Pi \le \lambda_\Pi = \lambda_f \le \rho_f$$

we may take Π to be a canonical product. Set $\nu := \lfloor \rho_f \rfloor$, then taking logarithms in the above factorization and differentiating $\nu+1$ times we get

$$\begin{aligned}
\frac{\mathrm{d}^\nu}{\mathrm{d}z^\nu}\left(\frac{f'(z)}{f(z)}\right) &= Q^{(\nu+1)}(z) + \frac{\mathrm{d}^{\nu+1}}{\mathrm{d}z^{\nu+1}} \log \prod_{m\in M} W\left(\frac{z}{z_m};\mu_\Pi\right) = \\
&= Q^{(\nu+1)}(z) + \frac{\mathrm{d}^{\nu+1}}{\mathrm{d}z^{\nu+1}} \sum_{m\in M}\left(\log\left(1-\frac{z}{z_m}\right)+\sum_{k=1}^{\mu_\Pi}\frac{z^k}{k}\right) = \\
&= Q^{(\nu+1)}(z) - \sum_{m\in M}\frac{1}{z_m}\frac{\mathrm{d}^\nu}{\mathrm{d}z^\nu}\left(1-\frac{z}{z_m}\right)^{-1} = \\
&= Q^{(\nu+1)}(z) - \nu! \sum_{m\in M}\frac{1}{(z_m-z)^{\nu+1}}.
\end{aligned}$$

We now aim to show that $Q^{(\nu+1)}$ is identically zero. For $R>0$ we set

$$g_R(z) \coloneqq \frac{f(z)}{f(0)} \prod_{\substack{m\in M\\ |z_m|\le R}} \left(1-\frac{z}{z_m}\right)^{-1} \in H(\mathbb{C}).$$

Note that $g_R(0)=1$. For $|z|=2R$ and $|z_m|\le R$ we have $|1-z/z_m| \ge |z|/|z_m| - 1 \ge 2R/R-1=1$, therefore

$$M_{g_R}(2R) \le \max_{|z|=2R}\left|\frac{f(z)}{f(0)}\right| = \mathcal{O}(\exp(2R)^\alpha), \quad \text{as } R\to\infty$$

for all $\alpha>\rho_f$. By the maximum modulus theorem this also holds for $|z|\le 2R$, and in particular we have

$$\log M_{g_R}(R) = \mathcal{O}(R^\alpha).$$

Note that g_R has no zeros in $\overline{B_R(0)}$, therefore we can define $h_R(z) \coloneqq \log g_R(z)$, choosing the branch of the logarithm for which $h_R(0)=0$, then h_R is holomorphic on some domain containing $\overline{B_R(0)}$. We have

$$A_{h_R}(R) = \max_{|z|=R} \operatorname{Re} h_R(z) = \max_{|z|=R} \log|g_R(z)| = \log M_{g_R}(R) = \mathcal{O}(R^\alpha).$$

Since by the maximum modulus theorem $|g_R(z)| \ge |g_R(0)| = 1$ we have $\operatorname{Re} h_R(z)\ge 0$. Invoking the Borel-Carathéodory lemma we therefore get, for $0<r<R$ and sufficiently large R,

$$M_{h_R^{(\nu+1)}}(R) \le \frac{2^{\nu+3}(\nu+1)!R}{(R-r)^{\nu+2}} A_{h_R}(R) \le \frac{K_0 2^{\nu+3}(\nu+1)!R^{\alpha+1}}{(R-r)^{\nu+2}},$$

where $K_0>0$ is some constant depending only on α. With $r\coloneqq R/2$ it follows that

$$M_{h_R^{(\nu+1)}}(R) \le K_1 R^{\alpha-(\nu+1)},$$

where $K_1 > 0$ is some constant depending only on α and ν. Note that we have

$$h_R^{(\nu+1)}(z) = \frac{\mathrm{d}^\nu}{\mathrm{d}z^\nu}\left(\frac{f'(z)}{f(z)}\right) + \nu! \sum_{\substack{m\in M\\ |z_m|\le R}} \frac{1}{(z_m - z)^{\nu+1}} = Q^{(\nu+1)}(z) - \nu! \sum_{\substack{m\in M\\ |z_m|>R}} \frac{1}{(z_m - z)^{\nu+1}}.$$

For $|z| = R/2$ and $z_m > R$ we have $|z_m - z| \ge |z_m| - R/2 \ge |z_m|/2$, therefore

$$|Q^{(\nu+1)}(z)| \le \nu! \sum_{\substack{m\in M\\ |z_m|>R}} \frac{1}{|z_m - z|^{\nu+1}} + |h_R^{(\nu+1)}(z)| \le K_2 \sum_{\substack{m\in M\\ |z_m|>R}} \frac{1}{|z_m|^{\nu+1}} + K_1 R^{\alpha-(\nu+1)},$$

where K_2 is some constant dependin only on ν. By the maximum modulus principle this holds for all $|z| \le R/2$. Since $\nu + 1 > \rho_f \ge \lambda_f$ both terms on the right side tend to zero as $R \to \infty$, if $\rho_f < \alpha < \nu + 1$. Therefore $Q^{(\nu+1)}$ vanishes identically, thus Q is a polynomial of degree at most $\nu \le \rho_f$. ■

Remark 3.2.24. An immediate consequence of Hadamard's Theorem is that entire functions of order $\rho < 1$ are of genus 0, and are therefore – like polynomials – solely determined by their roots and some scaling factor. Indeed, in this case the polynomial Q in the factorization is constant and the canonical product Π is of genus 0 as well. In particular, any non-vanishing function of order zero is constant.

As shown in Remark 3.1.6 the entire function

$$f(z) := \sum_{k=0}^{\infty} \frac{z^k}{(k^2)!}$$

is of order zero. Since f is clearly not a polynomial, f therefore must have infinitely many zeros. //

Hadamard's Theorem has a few consequences for entire functions of finite order, which we will explore further. Combining it with Theorem 3.2.20 allows us to prove easily prove two results regarding functions of finite, non-integer order.

Theorem 3.2.25. *Let $f \in H(\mathbb{C})$ be of finite, non-integer order. Then $\rho_f = \lambda_f$.*

Proof. We may assume $f(0) \ne 0$. By Theorem 3.2.6 we have $\lambda_f \le \rho_f$. Invoking Hadamard's Theorem we can write

$$f(z) = e^{Q(z)}\Pi(z)$$

for some polynomial Q with $\deg Q \le \rho_f$ and a canonical product Π. Since ρ_f is not an integer we have $\deg Q \le \lfloor \rho_f \rfloor < \rho_f$. By Proposition 3.1.4 e^Q has order $\deg Q$ and by Theorem 3.2.20 Π has order λ_f, therefore using Proposition 3.1.7 we obtain

$$\rho_f \le \max\{\deg Q, \lambda_f\} = \lambda_f \le \rho_f,$$

since $\rho_f \le \max\{\deg Q, \lambda_f\} = \deg Q < \rho_f$ would be a contradiction, and we get $\rho_f = \lambda_f$. ■

Theorem 3.2.26. *Let $f \in H(\mathbb{C})$ be of finite, non-integer order. Then f has infinitely many zeros.*

Proof. By Theorem 3.2.25 we have $\rho_f = \lambda_f$. Since ρ_f is not an integer it follows that $\lambda_f > 0$, which implies that f has infinitely many zeros. ■

Theorem 3.2.27 (Borel)**.** *Let $f \in H(\mathbb{C})$ be of finite, integer order. Then for any $a \in \mathbb{C}$ we have $\lambda_f^{(a)} = \rho_f$, except possibly for one value of a.*

Proof. TODO. ■

Relate to Picard exceptional values, "substantial deepening" of Picard's Little Theorem ...

4 Composition of Entire Functions

As seen by Proposition 3.1.7, the order of the sum or product of two entire functions of finite order is bounded above by the higher of the respective two orders. This is no longer the case when composition is involved; indeed, consider the function $f(z) \coloneqq \exp\exp z \in H(\mathbb{C})$, which is of infinite order, yet is the composition of two functions of order 1.

4.1 Pólya's Theorem

Necessary conditions for the order of a composition to be finite will be given by Pólya's Theorem, the proof of which relies on a result that was first proven by Harald Bohr. While Bloch's Theorem dealt with disks contained in the image of holomorphic functions, we now focus on circles contained in such images.

Proposition 4.1.1. *Let $G \subset \mathbb{C}$ be a bounded domain with $0 \in G$. Then the set*

$$S \coloneqq \{r \geq 0 : \partial B_r(0) \subseteq \overline{G}\} \tag{4.1}$$

has a positive maximum; that is, $\overline{G}$ contains a circle of positive, maximal radius[1]*.*

Proof. Since G is open and contains 0 there is some $t > 0$ such that $B_t(0) \subseteq G$. Since $\partial B_t(0) \subset \overline{B_t(0)} \subseteq \overline{G}$ we thus have $t \in S$ and S is non-empty. It now suffices to show that S is compact, since then it would contain its maximum $m \geq t$. Clearly S is bounded from below by 0 and since $\overline{G}$ is bounded it must also be bounded from above. Thus it remains to show that S is closed.

Let $(r_n)_{n\in\mathbb{N}}$ be a convergent sequence in S with limit $r \in [0, \infty)$. If $r = 0$, then $r \in S$. Otherwise we claim that $\partial B_r(0) \subseteq \overline{G}$. Let $z \in \partial B_r(0)$, and set $z_n \coloneqq r_n z/r$ for $n \in \mathbb{N}$. Now $|z_n| = r_n$, thus $z_n \in \partial B_{r_n}(0) \subseteq \overline{G}$ and therefore $(z_n)_{n\in\mathbb{N}}$ is a sequence in $\overline{G}$. Clearly $z_n \to z$ and since $\overline{G}$ is closed we therefore have $z \in \overline{G}$. Since z was arbitrary, this concludes the claim, we have $r \in S$ and thus S is closed. ■

If $G \subset \mathbb{C}$ is a bounded domain and $f \in H(G) \cap C(\overline{G}; \mathbb{C})$ is non-constant with $f(0) = 0$, then $f(G)$ is a bounded domain with $0 \in f(G)$. This, together with Proposition 4.1.1, justifies the following definition:

Definition 4.1.2. Let $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}}; \mathbb{C})$ be non-constant with $f(0) = 0$, then we define r_f as the positive maximum of the set S in (4.1), where $G = f(\mathbb{D})$. //

[1] The circle of radius 0 is understood as the singleton $\{0\}$.

Bohr's Theorem now asserts that r_f does (almost) not depend on f itself:

Theorem 4.1.3 (Bohr)**.** *There exists a function $\phi : (0,1) \to (0,\infty)$ such that for any $\theta \in (0,1)$ and $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}};\mathbb{C})$ with $f(0) = 0$ and $M_f(\theta) = 1$ it holds that $r_f \geq \phi(\theta)$.*

Proof. Suppose f satisfies the hypothesis of the theorem, let $\varepsilon > 0$ and set $R_\varepsilon := r_f + \varepsilon$. By definition of r_f, for all $r \geq R_\varepsilon$ there exists some point $w_r \in \partial B_r(0)$ with $w_r \notin f(\overline{\mathbb{D}})$. Choose such points $w_{R_\varepsilon}, w_{2R_\varepsilon}$ and define

$$h(z) := \frac{f(z) - w_{R_\varepsilon}}{w_{2R_\varepsilon} - w_{R_\varepsilon}} \in H(\mathbb{D}; \mathbb{C} \setminus \{0,1\}).$$

Since

$$|h(0)| = \left| \frac{f(0) - w_{R_\varepsilon}}{w_{2R_\varepsilon} - w_{R_\varepsilon}} \right| \leq \frac{|w_{R_\varepsilon}|}{||w_{2R_\varepsilon}| - |w_{R_\varepsilon}||} = \frac{R_\varepsilon}{2R_\varepsilon - R_\varepsilon} = 1,$$

by Schottky's Theorem we have $|h(z)| \leq \psi(\theta, 1)$ for all $|z| \leq \theta$. Therefore

$$|f(z)| - R_\varepsilon \leq |g(z) - w_{R_\varepsilon}| \leq |w_{2R_\varepsilon} - w_{R_\varepsilon}|\psi(\theta,1) \leq 3R_\varepsilon \psi(\theta,1)$$

and thus $|f(z)| \leq R_\varepsilon + 3R_\varepsilon\psi(\theta,1)$ for all $|z| \leq \theta$. Using the hypothesis that $M_f(\theta) = 1$ and the maximum modulus principle we obtain $1 \leq R_\varepsilon + 3R_\varepsilon\psi(\theta,1)$, and letting $\varepsilon \to 0$ we have $1 \leq r_f + 3r_f\psi(\theta,1)$. Thus

$$r_f \geq \frac{1}{1 + 3\psi(\theta,1)}$$

and defining $\phi(\theta)$ as the right-hand side establishes the assertion. ∎

Theorem 4.1.4 (Pólya)**.** *Let $g, h \in H(\mathbb{C})$ be non-constant. For the order of $g \circ h$ to be finite, it must hold that either*

i. h is a polynomial and $\rho_g < \infty$, or

ii. h is not a polynomial, $\rho_h < \infty$ and $\rho_g = 0$.

Proof. Without loss of generality we can assume $h(0) = 0$; otherwise we just consider $h_0(z) := h(z) - h(0)$ and $g_0(w) := g(w + h(0))$. Set $f := g \circ h$ and define

$$k_r(z) := \frac{h(rz)}{M_h(r/2)} \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}};\mathbb{C}), \quad \text{for } r > 0.$$

Note that by definition we have $M_{k_r}(1/2) = 1$ and $k_r(0) = 0$, thus by Bohr's Theorem there is some constant $C > 0$ and an $R > CM_h(r/2)$ such that $\partial B_{R/M_h(r/2)}(0) \subseteq k_r(\overline{\mathbb{D}})$ and thus $\partial B_R(0) \subseteq h(\overline{B_r(0)})$. By the maximum modulus principle, g assumes its maximum modulus over $\overline{B_R(0)}$ at some $w_0 \in \partial B_R(0)$. By the above there is a $z_0 \in \overline{B_r(0)}$ with $h(z_0) = w_0$. Thus we get

$$M_g(CM_h(r/2)) < M_g(R) = |g(w_0)| = |g(h(z_0))| = |f(z_0)| \leq M_f(r).$$

Assuming $\rho_f < \infty$, we have $M_f(r) < K\exp(r^\alpha)$ for every $\alpha > \rho_f$, with suitable $K > 0$. Consider the power series expansion $h(z) = \sum_{n=0}^\infty a_n z^n$. Let a_m denote any non-zero coefficient; note that since $h(0) = 0$ we have $m \geq 1$. By Cauchy's integral formula we have, for all $s > 0$,

$$|a_m| = \left|\frac{h^{(m)}(0)}{m!}\right| = \frac{1}{2\pi}\left|\oint_{\partial B_s(0)} \frac{h(\zeta)}{\zeta^{n+1}}\,\mathrm{d}\zeta\right| \leq \frac{M_h(s)}{s^m} \tag{$*$}$$

and thus

$$M_g(C|a_m|(r/2)^m) \leq M_g(CM_h(r/2)) < M_f(r) < K\exp(r^\alpha), \quad \text{for all } r > 0.$$

Replacing $(r/2)^m$ with r we obtain $\rho_g \leq \alpha/m$. If h is not a polynomial we may let $m \to \infty$, thus $\rho_g = 0$.

Now consider $g(z) = \sum_{n=0}^\infty b_n z^n$. Replacing h with g in $(*)$ we obtain $|b_n|s^n \leq M_g(s)$ for all $s > 0$ and $n \geq 1$ and thus

$$|b_n|(CM_h(r/2))^n \leq M_g(CM_h(r/2)) < M_f(r) < K\exp(r^\alpha),$$

which implies $\rho_h \leq \alpha < \infty$. ■

Theorem 4.1.5 (Thron)**.** *Let $g \in H(\mathbb{C})$ be transcendental, $\rho_g < \infty$ and suppose that g assumes some value $w \in \mathbb{C}$ only finitely often. Then there exists no $f \in H(\mathbb{C})$ with $f \circ f = g$.*

Proof. Assume towards a contradiction that $f \in H(\mathbb{C})$ with $f \circ f = g$ exists. Since g is not a polynomial, f is not a polynomial either. Thus Pólya's Theorem implies $\rho_f = 0$.

Consider the sets

$$Z := f^{-1}(\{w\}), \quad Z' := \bigcup_{z\in Z} f^{-1}(\{z\}).$$

By definition, for each $z' \in Z'$ there is some $z \in Z$ with $z' \in f^{-1}(\{z\})$, thus

$$g(z') = f(f(z')) = f(z) = w.$$

Our hypothesis on g implies that Z' must be finite. Since pre-images of singletons are disjoint, Z' is a disjoint union, therefore

$$\sum_{z\in Z} |f^{-1}(\{z\})| = \left|\bigcup_{z\in Z} f^{-1}(\{z\})\right| = |Z'| < \infty$$

and thus all points in Z are only assumed finitely often by f. But by Corollary 2.4.3, f assumes at most one value only finitely often; therefore $|Z| \leq 1$.

If $Z = \emptyset$, then $h(z) := f(z) - w$ is entire, of order 0 and nowhere 0. As discussed in Remark 3.2.24, as a consequence of Hadamard's Theorem we have that h is constant and consequently so is f, a contradiction.

If $Z = \{z_0\}$, then $h(z) := f(z) - w$ has a single zero of finite order $n \in \mathbb{N}$ at z_0. Therefore we can write $h(z) = (z - z_0)^n p(z)$, where p is entire, of order 0 and nowhere 0. Again, this implies that p is constant, and therefore f a polynomial, a contradiction. ■

Example 4.1.6. A natural application of Thron's Theorem is to set $g(z) := \exp z$, which never assumes zero as a value. Indeed, this implies that there is $f \in H(\mathbb{C})$ satisfying

$$f(f(z)) = \exp z.$$

On the other hand, there does exist a real-analytic function satisfying the above. The construction of such a function is difficult, but was demonstrated by H. Kneser [2]. //

Bibliography

[1] M. Blümlinger. Komplexe Analysis, 2021.

[2] H. Kneser. Reelle analytische Lösungen der Gleichung $\varphi(\varphi(x)) = e^x$ und verwandter Funktional-gleichungen. *J. Reine Angew. Math.*, 187:56–67, 1949.

[3] R. Remmert. *Classical topics in complex function theory*, volume 172 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998. Translated from the German by Leslie Kay.

[4] S. L. Segal. *Nine introductions in complex analysis*, volume 208 of *North-Holland Mathematics Studies*. Elsevier Science B.V., Amsterdam, revised edition, 2008.

[5] E. M. Stein and R. Shakarchi. *Complex analysis*, volume 2 of *Princeton Lectures in Analysis*. Princeton University Press, Princeton, NJ, 2003.