BACHELORARBEIT

Picard's Theorem
AND
growth, zeros and composition of entire functions

ausgeführt am

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Some words why the subject is of interest.

A reference to the primary literature used.

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$M_f(r) \coloneqq \max_{|z|=r} |f(z)|$

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Introduction, mention E. Picard, essential singularities, Casorati-Weierstrass.

2.1 Bloch's Theorem

Bloch's Theorem will show that for any $f \in H(\overline{\mathbb{D}})$ with $f'(0) = 1$ the image of $\overline{\mathbb{D}}$ under f contains a disk of a fixed radius. The point at which said disk is centered at is given in the proof.

If $G \subseteq \mathbb{C}$ is a domain and $f \in H(G)$ is non-constant, then $f(G)$ is a domain as well. Since such an f is an open mapping on G, we first observe a general criterion for the size of disks in the image domain of such functions:

Lemma 2.1.1. *Let $G \subset \mathbb{C}$ be a bounded domain, $f : \overline{G} \to \mathbb{C}$ continuous and $f\big|_G : G \to \mathbb{C}$ open. If there exists $a \in G$ such that $s := \min_{z \in \partial G} |f(z) - f(a)| > 0$, then $B_s(f(a)) \subseteq f(G)$.*

Proof. Since G is bounded, $\overline{G}$ is compact and by continuity of f, so is $\overline{f(G)}$. The function $z \mapsto |z - f(a)|$ is continuous on the compact set $\partial f(G)$, hence it assumes its minimum m at some $w_* \in \partial f(G)$. Choose a sequence $(z_n)_{n \in \mathbb{N}}$ in G with $\lim_{n\to\infty} f(z_n) = w_*$ then, since $\overline{G}$ is compact, we can find a subsequence that converges to some $z_* \in \overline{G}$. By continuity of f we have $f(z_*) = w_*$.

If $z_* \in G$, since $f\big|_G$ is open, the image of any open set in G containing z_* under f is an open set in $f(G)$ containing w_*, which is impossible since $w_* \in \partial f(G)$.

Therefore $z_* \in \partial G$ and we have

$$d(f(a), \partial f(G)) = m = |w_* - f(a)| = |f(z_*) - f(a)| \geq s,$$

which implies $B_s(f(a)) \subseteq f(G)$. ∎

Lemma 2.1.2. *Fix $a \in \mathbb{C}, r > 0$ and let $B := B_r(a)$. Suppose further that $f \in H(\overline{B})$ such that $\|f'\|_B \leq 2|f'(a)|$. Then $B_R(f(a)) \subseteq f(B)$, where $R := (3 - 2\sqrt{2})r|f'(a)|$.*

Proof. We can assume $a = f(a) = 0$, otherwise we consider $f_1(z) := f(z + a) - f(a)$. The function

$$\alpha_f : \begin{cases} B \to \mathbb{C}, \\ z \mapsto f(z) - f'(0)z, \end{cases}$$

satifies, for all $z \in B$,

$$|\alpha_f(z)| = \left| \int_{[0,z]} f'(\zeta) - f'(0) \,\mathrm{d}\zeta \right| \leq \int_0^1 |f'(tz) - f'(0)||z| \,\mathrm{d}t. \qquad (*)$$

We wish to further estimate the integrand. Let $w \in B$, then Cauchy's integral formula gives

$$|f'(w) - f'(0)| = \frac{1}{2\pi} \left| \int_{\partial B} \frac{f'(\zeta)}{\zeta - w} - \frac{f'(\zeta)}{\zeta} \,\mathrm{d}\zeta \right| = \frac{1}{2\pi} \left| \int_{\partial B} \frac{w f'(\zeta)}{\zeta(\zeta - w)} \,\mathrm{d}\zeta \right| \leq$$
$$\leq \frac{1}{2\pi} \int_{\partial B} \frac{|w| \|f'\|_B}{r(r - |w|)} \,\mathrm{d}\zeta = \frac{|w|}{r - |w|} \|f'\|_B.$$

Combining the above with $(*)$ and our estimate on $\|f'\|_B$ yields

$$|\alpha_f(z)| \leq \int_0^1 \frac{|zt| \|f'\|_B}{r - |zt|} |z| \,\mathrm{d}t \leq \frac{|z|^2}{r - |z|} \|f'\|_B \int_0^1 t \,\mathrm{d}t \leq \frac{|z|^2}{r - |z|} |f'(0)|.$$

Let $0 < \rho < r$, then for $|z| = \rho$ we have

$$|f'(0)|\rho - |f(z)| \leq |\alpha_f(z)| \leq \frac{\rho^2}{r - \rho} |f'(0)|$$
$$\iff \quad |f(z)| \geq \left(\rho - \frac{\rho^2}{r - \rho} \right) |f'(0)|.$$

The function $\rho \mapsto \rho - \rho^2/(r - \rho)$ assumes its maximum value at $\rho_* := (1 - \sqrt{2}/2)r \in (0, r)$, namely $(3 - 2\sqrt{2})r$. Therefore,

$$|f(z)| \geq (3 - 2\sqrt{2})r|f'(0)| = R, \quad \text{for all} \quad |z| = \rho_*.$$

In particular, $\min_{z \in \partial B_{\rho_*}} |f(z)| \geq R > 0$, thus invoking Lemma 2.1.1 with the domain $B_{\rho_*}(0)$ yields $B_R(0) \subseteq f(B_{\rho_*}(0)) \subseteq f(B)$. ∎

Theorem 2.1.3. *Let $f \in H(\overline{\mathbb{D}})$ be non-constant. Then there is a point $p \in \mathbb{D}$ and a constant $C_f > 0$ such that $B_R(f(p)) \subseteq f(\mathbb{D})$, where $R := (\frac{3}{2} - \sqrt{2})C_f \geq (\frac{3}{2} - \sqrt{2})|f'(0)|$.*

Proof. The function

$$\alpha_f : \begin{cases} \overline{\mathbb{D}} \to \mathbb{R} \\ z \mapsto |f'(z)|(1 - |z|) \end{cases}$$

is continuous and assumes its maximum $C_f > 0$ at some point $p \in \overline{\mathbb{D}}$. Note that $C_f \geq |f'(0)|$ and since f is non-constant and $\alpha_f\big|_{\partial\mathbb{D}} = 0$ we even have $p \in \mathbb{D}$.

Set $t := \frac{1}{2}(1 - |p|) > 0$, then we have $B_t(p) \subseteq \mathbb{D}$. Furthermore, for $z \in B_t(p)$, we have

$$1 - |z| \geq 1 - |z - p| - |p| \geq 1 - t - |p| = t$$

and since $|f'(z)|(1-|z|) \leq C_f = 2t|f'(p)|$, this implies $|f'(z)| \leq 2|f'(p)|$ for all $z \in B_t(p)$. By Lemma 2.1.2, we get $B_R(f(p)) \subseteq f(\mathbb{D})$, where $R := (3-2\sqrt{2})t|f'(p)| = (\frac{3}{2}-\sqrt{2})C_f$, which establishes the assertion. ∎

We now immediately obtain:

Theorem 2.1.4 (Bloch). *Let $f \in H(\overline{\mathbb{D}})$ and assume $f'(0) = 1$. Then $f(\mathbb{D})$ contains a disk of radius $\frac{3}{2} - \sqrt{2}$.*

In the following we will denote by $\beta > 0$ any constant less than or equal to the radius in Bloch's Theorem, for example $\beta = \frac{1}{12} < \frac{3}{2} - \sqrt{2}$.

Corollary 2.1.5. *Let $G \subseteq \mathbb{C}$ be a domain and $f \in H(G)$ with $f'(c) \neq 0$ for some $c \in G$. Then $f(G)$ contains a disk of every radius $\beta s|f'(c)|$, where $0 < s < d(c, \partial G)$.*

Proof. We may assume $c = 0$, otherwise we consider $f_1(z) := f(z+c)$. Let $0 < s < d(c, \partial G)$, then f is analytic on $\overline{B_s(0)} \subseteq G$, thus we have $g(z) := f(sz)/sf'(0) \in H(\overline{\mathbb{D}})$. Since $g'(0) = 1$, Bloch's Theorem yields a disk B of radius β with $B \subseteq g(\mathbb{D})$. Then $D := s|f'(0)|B$ is a disk of radius $\beta s|f'(0)|$ and we have

$$D = s|f'(0)|B \subseteq s|f'(0)|g(\mathbb{D}) = f(B_s(0)) \subseteq f(G).$$

∎

Corollary 2.1.6. *If $f \in H(\mathbb{C})$ is non-constant, then $f(\mathbb{C})$ contains a disk of every radius.*

2.2 Schottky's Theorem

Holomorphic functions which omit the values 0 and 1 have a universal estimate on the growth of their modulus, which will be given by Schottky's Theorem.

For a domain $G \subseteq \mathbb{C}$ and a set $E \subseteq \mathbb{C}$ we define $H(G; E)$ as the set of all $f \in H(G)$ such that $f(G) \subseteq E$.

Lemma 2.2.1. *It holds that:*

i. If $a, b \in \mathbb{R}$ with $\cos \pi a = \cos \pi b$, then $b = \pm a + 2n$ for some $n \in \mathbb{Z}$.

ii. For every $w \in \mathbb{C}$ there exists a $v \in \mathbb{C}$ such that $\cos \pi v = w$ and $|v| \leq 1 + |w|$.

Proof. For the first part, it suffices to notice that

$$0 = \cos \pi a - \cos \pi b = -2 \sin \tfrac{\pi}{2}(a+b) \sin \tfrac{\pi}{2}(a-b).$$

Since the complex cosine function is surjective and $\mathbb{R}$-periodic, we can choose $v = a + ib$ with $w = \cos \pi v$ and $|a| \leq 1$. Now we have

$$\begin{aligned} |w|^2 &= |\cos(\pi a + i\pi b)|^2 = |\cos \pi a \cos i\pi b + \sin \pi a \sin i\pi b|^2 = \\ &= |\cos \pi a \cosh \pi b - i \sin \pi a \sinh \pi b|^2 = \\ &= \cos^2 \pi a \cosh^2 \pi b + \sin^2 \pi a \sinh^2 \pi b = \\ &= \cos^2 \pi a + \cos^2 \pi a \sinh^2 \pi b + \sin^2 \pi a \sinh^2 \pi b = \\ &= \cos^2 \pi a + \sinh^2 \pi b \geq \sinh^2 \pi b \geq \pi^2 b^2, \end{aligned}$$

where the last inequality holds since $\sinh x \geq x$ for $x \geq 0$. We conclude

$$|v| = \sqrt{a^2 + b^2} \leq \sqrt{1 + |w|^2/\pi^2} \leq 1 + |w|$$

■

We recall the following result: Let $G \subseteq \mathbb{C}$ be a simply connected domain and $f \in H(G)$ such that f vanishes nowhere on G. Then there is a $g \in H(G)$ such that $f = e^g$. This can also be used to obtain multiplicative n-th roots of such functions, by defining $\sqrt[n]{f} := e^{g/n}$.

Lemma 2.2.2. *Let $G \subseteq \mathbb{C}$ be a simply connected domain and $f \in H(G; \mathbb{C} \setminus \{-1, 1\})$. Then there exists an $F \in H(G)$ such that $f = \cos F$.*

Proof. Since $1 - f^2$ vanishes nowhere in G it has a square root $g \in H(G)$. It follows that

$$1 = f^2 + g^2 = (f + ig)(f - ig).$$

Thus $f + ig$ vanishes nowhere and there exists an $F \in H(G)$ with $f + ig = e^{iF}$. By the above we also have $f - ig = e^{-iF}$ and therefore

$$f = \tfrac{1}{2}(e^{iF} + e^{-iF}) = \cos F.$$

■

Lemma 2.2.3. *Let $G \subseteq \mathbb{C}$ be a simply connected domain and $f \in H(G; \mathbb{C} \setminus \{0, 1\})$. Then there exists a $g \in H(G)$ such that:*

i. $f = \frac{1}{2}(1 + \cos \pi(\cos \pi g))$.

ii. $|g(0)| \leq 3 + 2|f(0)|$.

iii. *$g(G)$ contains no disk of radius* 1.

iv. *If $\mathbb{D} \subseteq G$ then $|g(z)| \leq |g(0)| + \theta/b(1 - \theta)$, for all $|z| \leq \theta$ where* $0 < \theta < 1$.

Proof. By Lemma 2.2.2 there exists a $\widetilde{F} \in H(G)$ such that $2f - 1 = \cos \pi \widetilde{F}$ and by Lemma 2.2.1 there is a $b \in \mathbb{C}$ such that $\cos \pi b = 2f(0) - 1$ and $|b| \leq 1 + |2f(0) - 1| \leq$

$2+2|f(0)|$. Furthermore, since $\cos\pi b = \cos\pi\widetilde{F}(0)$, we have $b = \pm\widetilde{F}(0)+2k$ for some $k \in \mathbb{Z}$. Then $F := \pm\widetilde{F} + 2k \in H(G)$ satifies $F(0) = b$ and $2f - 1 = \cos\pi F$.

Since F must omit all integers, there exists a $\widetilde{g} \in H(G)$ such that $F = \cos\pi\widetilde{g}$. Similarly, there is an $a \in \mathbb{C}$ such that $\cos\pi a = b$ and $|a| \le 1 + |b| \le 3 + 2|f(0)|$. Like above, since $\cos\pi a = \cos\pi\widetilde{g}(0)$, we have $a = \pm\widetilde{g}(0) + 2\ell$ for some $\ell \in \mathbb{Z}$, thus $g := \pm\widetilde{g} + 2\ell \in H(G)$ satisfies $g(0) = a$ and $F = \cos\pi g$. Ultimately, we obtain

$$f = \tfrac{1}{2}(1 + \cos\pi(\cos\pi g)), \quad \text{and} \quad |g(0)| = |a| \le 3 + 2|f(0)|$$

and have thus shown (i) and (ii).

To show (iii) we consider the set

$$A := \{m \pm i\pi^{-1}\log(n + \sqrt{n^2-1}) : m \in \mathbb{Z}, n \in \mathbb{N} \setminus \{0\}\},$$

the points of which can be considered the vertices of a rectangular grid in $\mathbb{C}$. The width of such a rectangular cell is 1, and since

$$\begin{aligned}\log((n+1) + \sqrt{(n+1)^2-1}) - \log(n + \sqrt{n^2-1}) &= \\ = \log\frac{1 + \frac{1}{n} + \sqrt{1 + \frac{2}{n}}}{1 + \sqrt{1 - \frac{1}{n^2}}} &\le \log(2+\sqrt{3}) < \pi\end{aligned}$$

their height is bounded above by some $C < 1$. Therefore, for all $z \in \mathbb{C}$ there is a $w_z \in A$ such that $|\operatorname{Re} z - \operatorname{Re} w_z| \le \frac{1}{2}$ and $|\operatorname{Im} z - \operatorname{Im} w_z| \le \frac{C}{2}$. Thus we have

$$|z - w_z| \le |\operatorname{Re} z - \operatorname{Re} w_z| + |\operatorname{Im} z - \operatorname{Im} w_z| \le \frac{1}{2} + \frac{C}{2} < 1.$$

If we can show that $g(G) \cap A = \emptyset$, then $g(G)$ therefore cannot contain a disk of radius 1. Let $a = p + i\pi^{-1}\log(q + \sqrt{q^2-1}) \in A$, then

$$\begin{aligned}\cos\pi a = \tfrac{1}{2}(e^{i\pi a} + e^{-i\pi a}) &= \tfrac{1}{2}(-1)^p((q + \sqrt{q^2-1})^{-1} + (q + \sqrt{q^2-1})) = \\ &= (-1)^p\frac{1}{2}\frac{1 + q^2 + 2q\sqrt{q^2-1} + q^2 - 1}{q + \sqrt{q^2-1}} = (-1)^p q\end{aligned}$$

and thus $\cos\pi(\cos\pi a) = \pm 1$. But $0, 1 \notin f(G)$, therefore $a \notin g(G)$ and $g(G) \cap A = \emptyset$, proving (iii).

For (iv), if $\mathbb{D} \subseteq G$, then $g\big|_{\mathbb{D}} \in H(\mathbb{D})$. Fix $0 < \theta < 1$, then for $|z| \le \theta$ we have

$$d(z, \partial\mathbb{D}) = \inf_{w \in \partial\mathbb{D}} |z - w| \ge \inf_{w \in \partial\mathbb{D}} (|w| - |z|) \ge 1 - \theta.$$

From (iii) it follows that $g\big|_{\mathbb{D}}(\mathbb{D})$ does not contain a disk of radius 1. Let $0 < s < 1 - \theta$, then applying Corollary 2.1.5 to $g\big|_{\mathbb{D}}$ implies that $\beta s|g'(z)| < 1$. Taking the supremum over

s and rearranging yields $|g'(z)| \leq 1/(\beta(1-\theta))$. Thus our desired estimate is shown by

$$|g(z)| \leq |g(0)| + |g(z) - g(0)| \leq |g(0)| + \int_{[0,z]} |g'(\zeta)|\, \mathrm{d}\zeta \leq |g(0)| + \frac{\theta}{\beta(1-\theta)}.$$

■

The result we just proved is quite powerful, as it contains both Schottky's Theorem, as well as Picard's Little Theorem:

Theorem 2.2.4 (Picard's Little Theorem)**.** *Let $f \in H(\mathbb{C}; \mathbb{C} \setminus \{a, b\})$ for distinct points $a, b \in \mathbb{C}$. Then f is constant.*

Proof. Consider $f_1(z) := \frac{f(z)-a}{b-a} \in H(\mathbb{C}; \mathbb{C}\setminus\{0,1\})$. By Lemma 2.2.3 there is some $g \in H(\mathbb{C})$ such that $f_1 = \frac{1}{2}(1 + \cos \pi(\cos \pi g))$ and $g(\mathbb{C})$ does not contain a disk of radius 1. By Corollary 2.1.6 we thereby have that g must be constant and therefore so are f_1 and f. ■

Theorem 2.2.5 (Schottky)**.** *There exists a function $\psi(\theta, \omega) : (0, 1) \times (0, \infty) \to (0, \infty)$ such that for any $f \in H(\mathbb{D}; \mathbb{C} \setminus \{0, 1\})$ with $|f(0)| \leq \omega$ it holds that*

$$|f(z)| \leq \psi(\theta, \omega), \quad |z| \leq \theta. \tag{2.1}$$

Proof. Note that for all $w \in \mathbb{C}$ we have $|\cos w| \leq e^{|w|}$ and $\frac{1}{2}|1 + \cos w| \leq e^{|w|}$. Hence, from Lemma 2.2.3, we get

$$\begin{aligned} |f(z)| = |\tfrac{1}{2}(1 + \cos \pi(\cos \pi g(z)))| &\leq \exp(\pi \exp \pi |g(z)|) \leq \\ &\leq \exp(\pi \exp \pi(|g(0)| + \theta/\beta(1-\theta))) \leq \\ &\leq \exp(\pi \exp \pi(3 + 2\omega + \theta/\beta(1-\theta))), \end{aligned}$$

and defining $\psi(\theta, \omega)$ as the final term establishes the assertion. ■

2.3 Normal families

We first recall a generalized form of uniform convergence:

Definition 2.3.1. Let $G \subseteq \mathbb{C}$ be a domain, $f \in H(G)$ and $(f_n)_{n\in\mathbb{N}}$ a sequence in $H(G)$. We say that f_n *converges compactly in G to f*, or f_n *converges compactly in G to ∞* as $n \to \infty$, if for every compact set $K \subset G$

$$\lim_{n\to\infty} \sup_{z\in K} |f_n(z) - f(z)| = 0, \quad \text{or} \quad \lim_{n\to\infty} \inf_{z\in K} |f_n(z)| = \infty.$$

Maybe a note on how compact convergence can also been seen as uniform convergence on the compactified complex plane.

A well-known theorem on compact convervence is:

Theorem 2.3.2 (Hurwitz)**.** *Let $G \subseteq \mathbb{C}$ be a domain and $(f_n)_{n\in\mathbb{N}}$ a sequence in $H(G)$ that converges compactly to $f \in H(G)$. If for every $n \in \mathbb{N}$ the number of a-points of f_n is bounded by some $m \in \mathbb{N}_0$, then either the number of a-points of f are also bounded by m, or $f \equiv a$.*

Compact convergence is, in some ways, compatible with reciprocals:

Lemma 2.3.3. *Let $G \subseteq \mathbb{C}$ be a domain and $(f_n)_{n\in\mathbb{N}}$ a sequence in $H(G;\mathbb{C}\setminus\{0\})$. If it converges compactly to some $f \in H(G)$. Then it holds that either:*

- *$0 \notin f(G)$ and $1/f_n \to 1/f$ compactly in G, or*
- *$f \equiv 0$ and $1/f_n \to \infty$ compactly in G.*

If it converges compactly to ∞, then $1/f_n \to 0$ compactly in G.

Proof. For the equivalence "$f_n \to 0$ if and only if $1/f_n \to \infty$" it suffices to notice that for any compact $K \subset G$ we have

$$\frac{1}{\sup_{z\in K}|f_n(z)|} = \inf_{z\in K}\left|\frac{1}{f_n(z)}\right|.$$

Since the f_n vanish nowhere, by Hurwitz' Theorem we either have $0 \notin f(G)$ or $f \equiv 0$. In the latter case we have just shown that $1/f_n \to \infty$ compactly.

In the former case we have, again for any compact $K \subset G$, that $m := \min_{z\in K}|f(z)| > 0$ and $\sup_{z\in K}|f_n(z) - f(z)| < \frac{m}{2}$ for sufficiently large $n \in \mathbb{N}$. Thus for all $z \in K$

$$\frac{m}{2} \geq |f(z) - f_n(z)| \geq |f(z)| - |f_n(z)| \geq m - |f_n(z)|$$

and $|f_n(z)| \geq \frac{m}{2}$. We obtain, for large $n \in \mathbb{N}$,

$$\sup_{z\in K}\left|\frac{1}{f_n(z)} - \frac{1}{f(z)}\right| = \sup_{z\in K}\left|\frac{f(z) - f_n(z)}{f(z)f_n(z)}\right| \leq \sup_{z\in K}|f_n(z) - f(z)| \cdot \frac{1}{m}\cdot\frac{2}{m}$$

and therefore, after letting $n \to \infty$, that $1/f_n \to 1/f$ compactly in G. ■

Definition 2.3.4. Let $G \subseteq \mathbb{C}$ be a domain and $\mathscr{F} \subseteq H(G)$. Then $\mathscr{F}$ is called:

- *locally bounded* if for every $w \in G$ there is a neighborhood U of w and a constant $C > 0$ such that $|f(z)| \leq C$ for all $f \in \mathscr{F}$ and $z \in U$.
- *normal* in G if every sequence in $\mathscr{F}$ has a subsequence which converges compactly in G to some $f \in H(G)$. If the limit ∞ is also permitted it is instead called $*$-*normal.*

The former two concepts are equivalent by the following well-known theorem:

Theorem 2.3.5 (Montel)**.** *Let $G \subseteq \mathbb{C}$ be a domain. Then a family $\mathscr{F} \subseteq H(G)$ is normal if and only if it is locally bounded.*

The following theorem can be interpreted as a sharpened version of Montel's Theorem and is sometimes referred to as the *fundamental normality test.*

Theorem 2.3.6. *Let $G \subseteq \mathbb{C}$ be a domain. Then any family $\mathscr{F} \subseteq H(G, \mathbb{C} \setminus \{0,1\})$ is $*$-normal in G.*

Proof. We shall give the proof in three steps:

1. Let $w \in G$, $c > 0$ and $\mathscr{F}_* \subseteq \mathscr{F}$ such that $|f(w)| \le c$ for all $f \in \mathscr{F}_*$. We aim to show that there is an open disk at w in which $\mathscr{F}_*$ is bounded. Select $t > 0$ such that $B_t(w) \subseteq G$. Let $f \in \mathscr{F}_*$, then $g(z) := f(tz + w) \in H(\mathbb{D})$. By the maximum modulus principle and Schottky's Theorem we obtain
$$\sup_{z \in B_{t/2}(w)} |f(z)| \le \sup_{z \in B_{1/2}(0)} |g(z)| \le \sup_{|z|=1/2} |g(z)| \le \psi(1/2, c)$$
and f is bounded on the disk $B_{t/2}(w)$. Since f was arbitrary, $\mathscr{F}_*$ is bounded aswell.
2. Fix some $w_* \in G$ and set $\mathscr{F}_1 := \{f \in \mathscr{F} : |f(w_*)| \le 1\}$. We aim to show that $\mathscr{F}_1$ is locally bounded in G. Consider the set
$$U := \{w \in G : \mathscr{F}_1 \text{ is bounded in a neighborhood of } w\},$$
by (1) we have that $w_* \in U$. Note that U is open in G, since if $\mathscr{F}_1$ is bounded in a disk $B_r(w)$, then for any $w' \in B_r(w)$ there is a disk $B_{r'}(w') \subseteq B_r(w)$, on which $\mathscr{F}_1$ is bounded as well.

   For sake of contradiction, suppose that $U \ne G$. Then there exists some $w \in \partial U \cap G$ such $\mathscr{F}_1$ is unbounded in every neighborhood of w.

   If there were some $c > 0$ such that $|f(w)| \le c$ for all $f \in \mathscr{F}_1$, then by (1) there would exist an open disk centered at w on which $\mathscr{F}_1$ would be bounded – contradicting our assumption on w. Thus for every $n \in \mathbb{N}$ we can find some $f_n \in \mathscr{F}_1$ such that $|f_n(w)| \ge n$ and we obtain that $\lim_{n\to\infty} |f_n(w)| = \infty$.

   Set $g_n := 1/f_n \in \mathscr{F}$, then $\lim_{n\to\infty} |g_n(w)| = 0$. In particular, the family $(g_n)_{n\in\mathbb{N}}$ is bounded at w by some constant, thus by (1) the family is bounded in some disk B around w. By Montel's Theorem it is therefore normal in B, and there exists a subsequence $(g_{n_k})_{k\in\mathbb{N}}$ which converges compactly to a $g \in H(B)$. The functions g_{n_k} have no zeros, but $g(w) = 0$; by Hurwitz's Theorem we therefore have $g \equiv 0$. Then for any $z \in B \cap U$ we have
$$\lim_{k\to\infty} f_{n_k}(z) = \lim_{k\to\infty} 1/g_{n_k}(z) = \infty,$$
contradicting the assumption that $\mathscr{F}_1$ is bounded in a neighborhood of such z. We thus have $U = G$, therefore $\mathscr{F}_1$ is locally bounded and by Montel's Theorem therefore

normal.

3. We can now conclude the proof. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{F}$, we claim that it has some subsequence which converges compactly to some function in $H(G)$ or to ∞.

   If infinitely many f_n lie in $\mathscr{F}_1$, then there is a subsequence $(f_{n_m})_{m\in\mathbb{N}}$ in $\mathscr{F}_1$, which by (2) has a subsequence $(f_{n_{m_k}})_{k\in\mathbb{N}}$ in $\mathscr{F}_1$ which convergences compactly in G to some $f \in H(G)$. This sequence is also a subsequence of $(f_n)_{n\in\mathbb{N}}$, concluding the claim in this case.

   On the other hand, if there are only finitely many f_n in $\mathscr{F}_1$, then infinitely many $1/f_n$ lie in $\mathscr{F}_1$. As above, we thus obtain some subsequence in $\mathscr{F}_1$, say $(g_n)_{n\in\mathbb{N}}$, converging compactly in G to some $g \in H(G)$. The sequence $(1/g_n)_{n\in\mathbb{N}}$ is a subsequence of $(f_n)_{n\in\mathbb{N}}$, which – by Lemma 2.3.3 – converges compactly to $1/g$ if $0 \notin g(G)$, and to ∞ otherwise.

■

2.4 Picard's Great Theorem

The following lemma is integral in showing that functions which omit two values on the punctured unit disk cannot have an essential singularity at 0.

Lemma 2.4.1. *Let $f \in H(\mathbb{D}^\times; \mathbb{C} \setminus \{0,1\})$. Then f or $1/f$ is bounded in a punctured neighborhood of zero.*

Proof. For $n \in \mathbb{N}$ set $f_n(z) \coloneqq f(z/n) \in H(\mathbb{D}^\times; \mathbb{C} \setminus \{0,1\})$. By Theorem 2.3.6 the sequence $(f_n)_{n\in\mathbb{N}}$ has a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ that converges compactly to a $f \in H(\mathbb{D}^\times)$ or to ∞.

Assume the former case. Then there is some $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have $\|f_{n_k} - f\|_{\partial B_{1/2}(0)} < 1$ and thus

$$\|f\|_{\partial B_{1/2n_k}(0)} = \|f_{n_k}\|_{\partial B_{1/2}(0)} \leq \|f_{n_k} - f\|_{\partial B_{1/2}(0)} + \|f\|_{\partial B_{1/2}(0)} \leq 1 + \|f\|_{\partial B_{1/2}(0)} =: C.$$

By the maximum modulus principle, f must therefore be bounded on every annulus $1/(2n_{k+1}) \leq |z| \leq 1/(2n_k)$, for $k \geq k_0$. Thus f is bounded on

$$V \coloneqq \bigcup_{k\geq k_0} \left\{ z \in \mathbb{C} : \frac{1}{2n_{k+1}} \leq |z| \leq \frac{1}{2n_k} \right\},$$

which is a punctured neighborhood of zero.

In the latter case, $(1/f_{n_k})_{k\in\mathbb{N}}$ converges compactly to 0 by Lemma 2.3.3. Replacing f_{n_k} with $1/f_{n_k}$ and f with 0 in the above we likewise obtain that $1/f$ is bounded in a punctured neighborhood of zero. ■

Theorem 2.4.2 (Picard's Great Theorem)**.** *Let $G \subseteq \mathbb{C}$ be open, $w \in G$ and suppose $f \in H(G \setminus \{w\})$ such that f has an essential singularity at w. Then f assumes all values in $\mathbb{C}$, with at most one exception, infinitely often in any punctured neighborhood of w.*

Proof. Aiming for contradiction, assume that f only takes on $z_0, z_1 \in \mathbb{C}$ finitely often in some punctured neighborhood W of w. Then W contains a punctured disk of radius $t > 0$ around w, on which f does not assume z_0, z_1, and

$$g(z) := \frac{f(tz + w) - z_0}{z_1 - z_0} \in H(\mathbb{D}^\times; \mathbb{C} \setminus \{0, 1\})$$

has an essential singularity at zero. By Lemma 2.4.1, we have that either g or $1/g$ must be bounded in a punctured neighborhood of zero. By the classification of isolated singularities, in the former case the singularity must therefore be removable, whereas in the latter case it must be a pole, yielding a contradiction. ■

The following corollary is sometimes also referred to as Picard's Great Theorem:

Corollary 2.4.3. *Every entire transcendental function assumes every value in $\mathbb{C}$ infinitely often, with at most one exception.*

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be transcendental and entire, and consider $g(z) := f(1/z) \in H(\mathbb{C}^\times)$. Then the Laurent series expansion of g at 0 has infinite principal part, therefore g has an essential singularity at zero. By Picard's Great Theorem g assumes all values in $\mathbb{C}$ on $B_1(0) \setminus \{0\}$ infinitely often, except at most one, and so f does the same on $\mathbb{C} \setminus B_1(0)$. ■

Remark 2.4.4. Unsurprisingly, Picard's Little Theorem is contained in Picard's Great Theorem; a non-constant $f \in H(\mathbb{C})$ is either a non-constant polynomial or transcendental. In the latter case, f omits at most one value by Corollary 2.4.3.

In the former case let $w \in \mathbb{C}$, then $f(z) - w$ has a zero in $\mathbb{C}$ by the Fundamental Theorem of Algebra and hence f assumes all values.

3 Growth and zeros distribution

3.1 Order and Type

Definition 3.1.1. Let f be an entire function. The *order* of f is defined by

$$\rho_f := \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}. \tag{3.1}$$

Constant functions, by convention, have order 0.

Note that, for any entire function f, we have $0 \le \rho_f \le \infty$. Equivalently, ρ_f can also be defined as the infimum over all $\rho > 0$ such that $M_f(r) = O(\exp(r^\rho))$ as $r \to \infty$.

An elementary fact regarding the order of the sum or product of two entire functions of finite order can immediately be obtained by naively estimating their respective maximum modulus:

Proposition 3.1.2. *Let f, g be entire functions of finite order. Then it holds that:*

i. $\rho_{f+g} \le \max\{\rho_f, \rho_g\}$

ii. $\rho_{fg} \le \max\{\rho_f, \rho_g\}$

Proposition 3.1.3. *Let f be an entire function of finite order, then $\rho_{f'} = \rho_f$.*

Proof. TODO. ∎

For functions of finite and positive order, we can obtain a natural refinement of the concept of order:

Definition 3.1.4. Let f be an entire function of finite and positive order. The *type* of f is defined by

$$\tau_f := \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}} \tag{3.2}$$

For any entire function f with $0 < \rho_f < \infty$, we have $0 \le \tau_f \le \infty$. Equivalently, τ_f can also be defined as the infimum over all $\tau > 0$ such that $M_f(r) = O(\exp(\tau r^{\rho_f}))$ as $r \to \infty$.

Imitating the proof of Proposition 3.1.3, we immediately get:

Proposition 3.1.5. *Let f be an entire function of finite type, then $\tau_{f'} = \tau_f$.*

Definition 3.1.6. Let f be an entire function. Then f is said to be of *growth* (a, b) if

- $\rho_f < a$, or
- $\rho_f = a$ and $\tau_f \leq b$.

Example 3.1.7. Maybe give some functions together with their order and type?

3.2 Hadamard's Theorem

Theorem 3.2.1 (Weierstrass [1, 4])**.** *Let $(z_j)_{j\in\mathbb{N}}$ be a sequence in $\mathbb{C}$ without accumulation points. Then there exists an entire function E (called the* Weierstrass canonical product *formed from said sequence) that has zeros precisely at $(z_j)_{j\in\mathbb{N}}$, with multiplicities equal to how often z_j occurs in the sequence.*

In particular, we have

$$E(z) = z^k \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{R_n(z/z_n)}, \tag{3.3}$$

where k is the order of the zero at $z = 0$ and R_n is a polynomial, namely a truncation of the power series for $-\log(1 - \frac{z}{z_n})$ chosen of smallest degree to ensure convergence of the product [3].

Weierstrass' Theorem is also known as the *Weierstrass Factorization Theorem*, due to the following corollary:

Corollary 3.2.2. *Let f be an entire function with zeros $(z_j)_{j\in\mathbb{N}}$. Then there exists an entire function g, such that*

$$f(z) = e^{g(z)} E(z),$$

where E is a Weierstrass canonical product formed from the zeros of f.

Proof. Since f/E has removable singularities at all $(z_j)_{j\in\mathbb{N}}$, we have that f/E is entire and nowhere zero. Thus there exists an entire function g with $f/E = e^g$, which yields $f = e^g E$. ∎

Hadamard's Theorem will show that, for functions of finite order ρ, the function g in Corollary 3.2.2 can be taken to be a polynomial of degree less than ρ and the degree of the polynomials R_n in (3.3) can be taken to be independant of n. To prove this we require the following lemma, which can be interpreted as a version of the maximum modulus principle applied to the real part of an analytic function.

Lemma 3.2.3 (Borel-Carathéodory)**.** *Let f be analytic in $\overline{\mathbb{D}}$ and let*

$$A_f(r) = \max_{|z|=r} \operatorname{Re} f(z).$$

Then, for $0 < r < R$,

$$M_f(r) \le \frac{2r}{R-r} A_f(R) + \frac{R+r}{R-r}|f(0)|$$

and, if additionally $A_f(R) \ge 0$*, then for* $n \in \mathbb{N}$

$$\max_{|z|=r} |f^{(n)}(z)| \le \frac{2^{n+2} n! R}{(R-r)^{n+1}} (A_f(R) + |f(0)|).$$

Proof. TODO. ∎

Theorem 3.2.4 (Hadamard)**.** *Let* f *be an entire function of finite order with zeros* $(z_j)_{j\in\mathbb{N}}$*. Then there exists a polynomial* Q *with* $\deg Q \le \rho_f$*, such that*

$$f(z) = e^{Q(z)} E(z),$$

where E *is a Weierstrass canonical product formed from the zeros of* f*.*

Proof. TODO. ∎

3.3 Zeros distribution

We recall a rather explicit connection between the moduli of the zeros of an analytic function and the modulus of the function itself:

Theorem 3.3.1 (Jensen [4])**.** *Let* f *be analytic on* $B_R(0)$ *with* $f(0) \neq 0$ *and let* $r_1, r_2, \dots$ *denote the moduli of the zeros of* f *in* $B_R(0)$ *arranged in a non-decreasing sequence. Then, for* $r_n < r < r_{n+1}$*, we have*

$$\frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})| \,\mathrm{d}\theta = \log|f(0)| + \log \frac{r^n}{r_1 \dots r_n}.$$

Definition 3.3.2. Let f be analytic on $B_R(0)$. Then, for $0 < r < R$, we denote by $n_f(r)$ the number of zeros of f in $\overline{B_r(0)}$.

Corollary 3.3.3. *Let* f *be analytic on* $B_R(0)$ *with* $f(0) \neq 0$*. Then, for* $0 < r < R$*, we have*

$$\frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})| \,\mathrm{d}\theta = \log|f(0)| + \int_0^r \frac{n_f(s)}{s} \,\mathrm{d}s$$

Proof. Let $r_1, r_2, \dots$ denote the moduli of the zeros of f in $B_R(0)$ arranged in a non-decreasing sequence. Then, for any $r_n < r < r_{n+1}$, we obtain

$$
\begin{aligned}
\log \frac{r^n}{r_1 \dots r_n} &= \sum_{k=1}^{n} \log \frac{r}{r_k} = \sum_{k=1}^{n} \int_{r_k}^{r} \frac{1}{s} \,\mathrm{d}s = \\
&= \sum_{k=1}^{n} \int_0^r \mathbb{1}_{(r_k,\infty)}(s) \frac{1}{s} \,\mathrm{d}s = \int_0^r \left(\sum_{k=1}^{n} \mathbb{1}_{(r_k,\infty)}(s) \right) \frac{1}{s} \,\mathrm{d}s = \\
&= \int_0^r \frac{n_f(s)}{s} \,\mathrm{d}s
\end{aligned}
$$

and Theorem 3.3.1 concludes the claim. ■

In particular, we observe that the more zeros a function $f(z)$ has, the faster its modulus must grow as $|z| \to \infty$. The converse is naturally false, as seen by iterated exponentials [3].

Definition 3.3.4. Let f be an entire function and let $(r_j)_{j\in\mathbb{N}}$ denote the non-zero moduli of its zeros (if any) arranged in non-decreasing order. Then

$$
\lambda_f := \inf \left\{ \lambda > 0 : \sum_{n=1}^{\infty} \frac{1}{r_n^\lambda} < \infty \right\}
$$

is called the *exponent of convergence of the zeros of f.* If f has finitely many zeros, then we set $\lambda_f = 0$ by convention.

Furthermore, the *exponent of convergence of the a-points of f* is defined as exponent of convergence of zeros of $f(z) - a$.

Theorem 3.3.5. *Let f be an entire function of finite order. Then $\lambda_f \le \rho_f$.*

Proof. TODO. ■

Example 3.3.6. Consider the entire function $f(z) := \sin(z)$, we want to calculate λ_f and ρ_f. First, let $\lambda > 0$ and consider that f has zeros at $(n\pi)_{n\in\mathbb{Z}}$. Since

$$
\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{|n\pi|^\lambda} = \frac{2}{\pi^\lambda} \sum_{n=1}^{\infty} \frac{1}{|n|^\lambda}
$$

is finite if and only if $\lambda > 1$, we obtain $\lambda_f = 1$. Furthermore, we have $|\sin(z)| \le e^{|\operatorname{Im} z|}$ and therefore

$$
M_f(r) \le \max_{|z|=r} |e^z|.
$$

Since e^z is of order 1, this implies $\rho_f \le 1$. Finally, Theorem 3.3.5 concludes $\rho_f = 1$.

Theorem 3.3.7. *Let E be a Weierstrass canonical product of finite order. Then $\lambda_E = \rho_E$.*

Proof. TODO. ∎

Theorem 3.3.8. *Let f be an entire function of finite, non-integer order. Then $\rho_f = \lambda_f$.*

Proof. By Theorem 3.3.5 we have $\lambda_f \leq \rho_f$. Invoking Hadamard's Theorem we can write $f = e^Q E$ for a polynomial Q with $\deg Q \leq \rho_f$. Since ρ_f is not an integer, this implies $\deg Q \leq \lfloor \rho_f \rfloor < \rho_f$. Now, again by Hadamard's Theorem, e^Q has order $\deg Q$ and by Theorem 3.3.7 E has order λ_f. Using Proposition 3.1.2 we obtain

$$\rho_f \leq \max\{\deg Q, \lambda_f\} = \lambda_f \leq \rho_f,$$

implying $\rho_f = \lambda_f$. ∎

Theorem 3.3.9. *Let f be an entire function of finite, non-integer order. Then f has infinitely many zeros.*

Proof. By Theorem 3.3.8 we have $\rho_f = \lambda_f$. Since ρ_f is not an integer, $\lambda_f > 0$, which implies that f has infinitely many zeros. ∎

Maybe introduce Borel exceptional values as a definition? But then again, I will never need them again. Maybe also add a remark on the relation to lacunary values (Picard).

Theorem 3.3.10 (Borel)**.** *Existence of Borel exceptional values.*

Proof. TODO. ∎

4 Composition of entire functions

As seen by Proposition 3.1.2, the order of the sum or product of two entire functions is reasonably bounded by the order of the functions involved. This is not the case when composition is involved. Indeed, consider $\exp\exp z$, which has infinite order, yet is the composition of two functions of order 1.

4.1 Pólya's Theorem

Necessary conditions for the order of a composition to be finite will be given by Pòlya's Theorem, the proof of which relies on the following result:

Theorem 4.1.1 (Bohr)**.** *Let $0 < \theta < 1$ and $f \in H(\overline{\mathbb{D}})$, such that $f(0) = 0$ and $M_f(\theta) = 1$. Let r_f denote the largest $r > 0$ such that $\partial B_r(0) \subseteq f(\overline{\mathbb{D}})$. Then we have $r_f \geq C_\theta$, where $C_\theta > 0$ is a constant depending only on θ.*

Proof. Suppose g satisfies the hypothesis of the theorem and let $C > 0$ such that for all $r \geq C$ there exists some point $w_r \in \partial B_r(0)$ with $w_r \notin f(\overline{\mathbb{D}})$. Choose such points w_C, w_{2C} and define

$$h(z) := \frac{g(z) - w_C}{w_{2C} - w_C} \in H(\overline{\mathbb{D}}, \mathbb{C} \setminus \{0, 1\}).$$

Since

$$|h(0)| = \left| \frac{g(0) - w_C}{w_{2C} - w_C} \right| \leq \frac{C}{2C - C} = 1,$$

by Schottky's Theorem we have $|h(z)| \leq \psi(\theta, 1)$ for all $|z| \leq \theta$. Therefore

$$|g(z)| - C \leq |g(z) - w_C| \leq |w_{2C} - w_C|\psi(\theta, 1) \leq 3C\psi(\theta, 1)$$

and thus $|g(z)| \leq C + 3C\psi(\theta, 1)$ for all $|z| \leq \theta$. Using the hypothesis that $M_g(\theta) = 1$ and the maximum modulus principle we obtain $C \geq \frac{1}{1+3\psi(\theta,1)}$ and choosing C_θ to be less than this constant yields the theorem. ■

Theorem 4.1.2 (Pólya)**.** *Let $g, h \in H(\mathbb{C})$. For the order of $g \circ h$ to be finite, it must hold that either*

i. h is a polynomial and $\rho_g < \infty$, or

ii. h is not a polynomial, $\rho_h < \infty$ and $\rho_g = 0$.

Proof. Without loss of generality we can assume $h(0) = 0$; otherwise we just consider $h_0(z) \coloneqq h(z) - h(0)$ and $g_0(w) \coloneqq g(w + h(0))$. Set $f \coloneqq g \circ h$ and define

$$k_r(z) \coloneqq \frac{h(rz)}{M_h(r/2)} \in H(\overline{\mathbb{D}}), \quad \text{for } r > 0.$$

Note that by definition we have $M_{k_r}(1/2) = 1$ and $k_r(0) = 0$, thus by Theorem 4.1.1 there is some constant $C > 0$ and an $R/M_h(r/2) > C$ such that $\partial B_{R/M_h(r/2)}(0) \subseteq k_r(\overline{\mathbb{D}})$ and thus $\partial B_R(0) \subseteq h(\overline{B_r(0)})$. By the maximum modulus principle, $|g|$ assumes its maximum over $\overline{B_R(0)}$ at some $w_0 \in \partial B_R(0)$. By the above there is a $z_0 \in \overline{B_r(0)}$ with $h(z_0) = w_0$. Thus we get

$$M_g(CM_h(r/2)) < M_g(R) = |g(w_0)| = |g(h(z_0))| = |f(z_0)| \leq M_f(r).$$

Assuming $\rho_f < \infty$, we have $M_f(r) < K \exp(r^\alpha)$ for some $\alpha > \rho_f$. Consider the power series expansion $h(z) = \sum_{n=0}^\infty a_n z^n$ and let a_m denote any non-zero coefficient; note that since $h(0) = 0$ we have $m \geq 1$. By Cauchy's integral formula we have, for all $s > 0$,

$$|a_m| = \left| \frac{h^{(m)}(0)}{m!} \right| = \frac{1}{2\pi} \left| \int_{\partial B_s(0)} \frac{h(\zeta)}{\zeta^{n+1}} \,\mathrm{d}\zeta \right| \leq \frac{M_h(s)}{s^m} \tag{$*$}$$

and thus

$$M_g(C|a_m|(r/2)^m) \leq M_g(CM_h(r/2)) < M_f(r) < K \exp(r^\alpha), \quad \text{for all } r > 0.$$

Replacing $(r/2)^m$ with r we obtain $\rho_g \leq \alpha/m$. If h is not a polynomial we may let $m \to \infty$, thus $\rho_g = 0$.

Now consider $g(z) = \sum_{n=0}^\infty b_n z^n$. Replacing h with g in $(*)$ we obtain $|b_n| s^n \leq M_g(s)$ for all $s > 0$ and $n \geq 1$ and thus

$$|b_n|(CM_h(r/2))^n \leq M_g(CM_h(r/2)) < M_f(r) < K \exp(r^\alpha),$$

which implies $\rho_h \leq \alpha < \infty$. ∎

Theorem 4.1.3 (Thron)**.** *Let $g \in H(\mathbb{C})$ be transcendental, of finite order, such that g assumes some value $w \in \mathbb{C}$ only finitely often. Then there does not exist any $f \in H(\mathbb{C})$ such that $f \circ f = g$.*

Proof. Seeking contradiction, suppose there were such a $f \in H(\mathbb{C})$. Since g is not a polynomial, Pólya's Theorem implies that f is of order 0 and not a polynomial. Let $(z_j)_{j \in J}$ denote the points where f equals w. For each $m \in J$ we additionally denote by $(z_{j,m})_{j \in J_m}$ the points where f equals z_m. Thus, for each $m \in J$ and $n \in J_m$ we have

$$g(z_{n,m}) = f(f(z_{n,m})) = f(z_m) = w.$$

Our assumption on g assures that there must only be finitely many distinct points among the $(z_{n,m})_{m \in J, n \in J_m}$. Thus, each point in $(z_j)_{j \in J}$ is only taken on by f finitely often.

By Corollary 2.4.3, f assumes all values in the complex plane infinitely often, with at most one exception. This implies that that there is at most one z_0 in $(z_j)_{j \in J}$ that is only taken on finitely often by f.

If there is no such z_0, then $h(z) \coloneqq f(z) - w$ is entire, of order 0 and nowhere 0. Thus, by Hadamard's Theorem, h must be constant, and therefore f aswell, a contradiction.

If such a z_0 exists, then $h(z) \coloneqq f(z) - w$ has a zero of finite order $n \in \mathbb{N}$ at z_0. Therefore we can write $h(z) = (z - z_0)^n p(z)$, where p is entire, of order 0 and nowhere 0. Again, this implies that p is constant, and therefore f a polynomial, a contradiction. ■

Example 4.1.4. A natural application of Thron's Theorem is taking g to be e^z, which never assumes zero as a value. Indeed, this implies that there is no entire function f satisfying

$$f(f(z)) = e^z.$$

On the other hand, there does exist a real-analytic function satisfying the above, as demonstrated by H. Kneser. I probably still need a citation here.

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