BACHELORARBEIT

Growth, order and zeros of entire functions

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2.1 Bloch's Theorem

Lemma 2.1.1. *Let $G \subset \mathbb{C}$ be a bounded domain, $f : \overline{G} \to \mathbb{C}$ continuous and $f\big|_G : G \to \mathbb{C}$ open. If there exists $a \in G$ such that $s := \min_{z \in \partial G} |f(z) - f(a)| > 0$, then $B_s(f(a)) \subseteq f(G)$.*

Proof. The function $z \mapsto |z - f(a)|$ is continuous on the compact set $\partial f(G)$, hence it attains its minimum m at some $w_* \in \partial f(G)$. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in G, such that $\lim_{n\to\infty} f(z_n) = w_*$. Since $\overline{G}$ is compact, we can find a subsequence that converges to some $z_* \in \overline{G}$ and by continuity of f we have $f(z_*) = w_*$.

Assuming $z_* \in G$, since $f\big|_G$ is open, the image of any open set in G containing z_* under f must be an open set in $f(G)$ containing w_*, which is impossible since $w_* \in \partial G$.

Therefore $z_* \in \partial G$ and we have $m = |w_* - f(a)| = |f(z_*) - f(a)| \geq s$, from which it follows that $B_s(f(a)) \subseteq f(G)$. ■

Lemma 2.1.2. *Let $a \in \mathbb{C}, r > 0$ and $B := B_r(a)$. Suppose further that $f \in H(\overline{B})$ such that $\|f'\|_B \leq 2|f'(a)|$. Then $B_R(f(a)) \subseteq f(B)$, where $R := (3 - 2\sqrt{2})r|f'(a)|$.*

Proof. Without loss of generality we can assume $a = f(a) = 0$. Consider the function

$$\alpha_f : \begin{cases} B \to \mathbb{C}, \\ z \mapsto f(z) - f'(0)z, \end{cases}$$

which satifies, for the line γ connecting 0 to $z \in B$,

$$|\alpha_f(z)| = \left| \int_\gamma f'(\zeta) - f'(0) \, \mathrm{d}\zeta \right| \leq \int_0^1 |f'(tz) - f'(0)||z| \, \mathrm{d}t. \qquad (*)$$

Let $w \in B$, then Cauchy's integral formula gives

$$|f'(w) - f'(0)| = \frac{1}{2\pi} \left| \int_{\partial B} \frac{f'(\zeta)}{\zeta - w} - \frac{f'(\zeta)}{\zeta} \, \mathrm{d}\zeta \right| = \frac{1}{2\pi} \left| \int_{\partial B} \frac{w f'(\zeta)}{\zeta(\zeta - w)} \, \mathrm{d}\zeta \right| \leq$$
$$\leq \frac{1}{2\pi} \int_{\partial B} \frac{|w| \|f'\|_B}{r(r - |w|)} \, \mathrm{d}\zeta = \frac{|w|}{r - |w|} \|f'\|_B.$$

Combining the above with $(*)$ and our estimate on $\|f'\|_B$ yields

$$|\alpha_f(z)| \leq \int_0^1 \frac{|zt|\|f'\|_B}{r-|zt|}|z|\,\mathrm{d}t \leq \frac{|z|^2}{r-|z|}\|f'\|_B \int_0^1 t\,\mathrm{d}t \leq \frac{|z|^2}{r-|z|}|f'(0)|.$$

Let $0 < \rho < r$, then for $|z| = \rho$ we have

$$\begin{aligned} |f'(0)|\rho - |f(z)| \leq |\alpha_f(z)| &\leq \frac{\rho^2}{r-\rho}|f'(0)| \\ \iff \quad |f(z)| &\geq \left(\rho - \frac{\rho^2}{r-\rho}\right)|f'(0)|. \end{aligned}$$

Considering $\rho - \rho^2/(r-\rho)$ as a function of ρ, it attains its maximum value, $(3-2\sqrt{2})r$, at $\rho_* := (1-\sqrt{2}/2)r \in (0,r)$. Therefore,

$$|f(z)| \geq (3-2\sqrt{2})r|f'(0)| = R, \quad \text{for all} \quad |z| = \rho_*.$$

Invoking Lemma 2.1.1 with $G := B_{\rho_*}(0)$ thus yields $B_R(0) \subseteq f(G) \subseteq f(B)$. ∎

Theorem 2.1.3. *Let $f \in H(\overline{\mathbb{D}})$ be non-constant. Then there is a point $p \in \mathbb{D}$ and a constant $C_f > 0$ such that $B_R(f(p)) \subseteq f(\mathbb{D})$, where $R := (\frac{3}{2}-\sqrt{2})C_f \geq (\frac{3}{2}-\sqrt{2})|f'(0)|$.*

Proof. The function

$$\alpha_f : \begin{cases} \overline{\mathbb{D}} \to \mathbb{R} \\ z \mapsto |f'(z)|(1-|z|) \end{cases}$$

is continuous and attains its maximum $C_f > 0$ at some point $p \in \mathbb{D}$.

Set $t := \frac{1}{2}(1-|p|) > 0$, then we have $B_t(p) \subseteq \mathbb{D}$ and $1-|z| \geq t$ for $z \in B_t(p)$. Since $|f'(z)|(1-|z|) \leq C_f = 2t|f'(p)|$, this implies $|f'(z)| \leq 2|f'(p)|$. By Lemma 2.1.2, we obtain $B_R(f(p)) \subseteq f(\mathbb{D})$, where

$$R := (3-2\sqrt{2})t|f'(p)| = (\tfrac{3}{2}-\sqrt{2})|f'(p)|(1-|p|) > \tfrac{1}{12}|f'(0)|,$$

establishes the assertion. ∎

We immediately have:

Theorem 2.1.4 (Bloch)**.** *Let $f \in H(\overline{\mathbb{D}})$ and assume that $f'(0) = 1$. Then $f(\mathbb{D})$ contains a disk of radius $\frac{3}{2}-\sqrt{2}$.*

In the following we will denote by $\beta > 0$ any constant satisfying Bloch's Theorem, for example $\beta = \frac{1}{12} < \frac{3}{2}-\sqrt{2}$.

Corollary 2.1.5. *Let $G \subset \mathbb{C}$ be a domain and $f \in H(G)$ with $f'(c) \neq 0$ for some $c \in G$. Then $f(G)$ contains a disk of every radius $\beta s|f'(c)|$, where $0 < s < d(c, \partial G)$.*

Proof. Without loss of generality we may assume $c = 0$. Since f is analytic on $\overline{B_s(0)} \subseteq G$, we have $g(z) := f(sz)/sf'(0) \in H(\overline{\mathbb{D}})$. Since $g'(0) = 1$, Bloch's Theorem (2.1.4) yields a disk B of radius β such that $B \subseteq g(\mathbb{D})$. It follows that

$$s|f'(0)|B \subseteq s|f'(0)|g(\mathbb{D}) = f(B_s(0)) \subseteq f(G).$$

■

Corollary 2.1.6. *If $f \in H(\mathbb{C})$ is non-constant, then $f(\mathbb{C})$ contains a disk of every radius.*

2.2 Schottky's Theorem

For sets $G, E \subseteq \mathbb{C}$ we define by $H(G; E)$ the set of all $f \in H(G)$ such that $f(G) \subseteq E$.

Lemma 2.2.1. *It holds that:*

i. If $a, b \in \mathbb{R}$ with $\cos \pi a = \cos \pi b$, then $b = \pm a + 2n$ for some $n \in \mathbb{Z}$.

ii. For every $w \in \mathbb{C}$ there exists a $v \in \mathbb{C}$ such that $\cos \pi v = w$ and $|v| \leq 1 + |w|$.

Proof. For the first part, it suffices to notice that

$$0 = \cos \pi a - \cos \pi b = -2 \sin \tfrac{\pi}{2}(a + b) \sin \tfrac{\pi}{2}(a - b).$$

Since the complex cosine function is surjective and $\mathbb{R}$-periodic, we can choose $v = a + ib$ with $w = \cos \pi v$ and $|a| \leq 1$. Now we have

$$\begin{aligned}|w|^2 &= |\cos(\pi a + i\pi b)|^2 = |\cos \pi a \cos i\pi b + \sin \pi a \sin i\pi b|^2 = \\ &= |\cos \pi a \cosh \pi b - i \sin \pi a \sinh \pi b|^2 = \\ &= \cos^2 \pi a \cosh^2 \pi b + \sin^2 \pi a \sinh^2 \pi b = \\ &= \cos^2 \pi a + \cos^2 \pi a \sinh^2 \pi b + \sin^2 \pi a \sinh^2 \pi b = \\ &= \cos^2 \pi a + \sinh^2 \pi b \geq \sinh^2 \pi b \geq \pi^2 b^2,\end{aligned}$$

where the last inequality holds since $\sinh x \geq x$ for $x \geq 0$. We conclude

$$|v| = \sqrt{a^2 + b^2} \leq \sqrt{1 + |w|^2/\pi^2} \leq 1 + |w|$$

■

Lemma 2.2.2. *Let $G \subseteq \mathbb{C}$ be a simply connected domain and $f \in H(G; \mathbb{C} \setminus \{-1, 1\})$. Then there exists an $F \in H(G)$ such that $f = \cos F$.*

Proof. Since $1 - f^2$ vanishes nowhere in G it has a square root $g \in H(G)$. It follows that

$$1 = f^2 + g^2 = (f + ig)(f - ig).$$

Thus $f+ig$ vanishes nowhere and there exists an $F \in H(G)$ with $f+ig = e^{iF}$. Additionally we have $f - ig = e^{-iF}$ and therefore

$$f = \tfrac{1}{2}(e^{iF} + e^{-iF}) = \cos F.$$

■

Lemma 2.2.3. *Let $f \in H(\overline{\mathbb{D}}; \mathbb{C} \setminus \{0,1\})$. Then there exists a $g \in H(\overline{\mathbb{D}})$ such that:*

i. $f = \frac{1}{2}(1 + \cos\pi(\cos\pi g))$, *and* $|g(0)| \leq 3 + 2|f(0)|$.

ii. $g(\overline{\mathbb{D}})$ *contains no disk of radius* 1.

iii. $|g(z)| \leq |g(0)| + \theta/b(1-\theta)$, *for all* $|z| \leq \theta$ *where* $0 < \theta < 1$.

Proof. By Lemma 2.2.2 there exists a $\widetilde{F} \in H(\overline{\mathbb{D}})$ such that $2f-1 = \cos\pi\widetilde{F}$. By Lemma 2.2.1 there is a $b \in \mathbb{C}$ such that $\cos\pi b = 2f(0) - 1$ and $|b| \leq 1 + |2f(0) - 1| \leq 2 + 2|f(0)|$. Additionally we have that $b = \pm\widetilde{F}(0) + 2k$ for some $k \in \mathbb{Z}$. Then $F := \pm\widetilde{F} + 2k \in H(\overline{\mathbb{D}})$ satifies $F(0) = b$. Since F must omit all integers, there exists a $\widetilde{g} \in H(\overline{\mathbb{D}})$ such that $F = \cos\pi\widetilde{g}$. Again, there is an $a \in \mathbb{C}$ such that $\cos\pi a = b$ and $|a| \leq 1 + |b| \leq 3 + 2|f(0)|$. Like above, we have $a = \pm\widetilde{g}(0) + 2\ell$ for some $\ell \in \mathbb{Z}$, thus $g := \pm\widetilde{g} + 2\ell \in H(\overline{\mathbb{D}})$ satisfies $g(0) = a$. Ultimately, we obtain

$$f = \tfrac{1}{2}(1 + \cos\pi(\cos\pi g)), \quad \text{and} \quad |g(0)| = |a| \leq 3 + 2|f(0)|.$$

To show (ii) we consider the set

$$A := \{m \pm i\pi^{-1}\log(n + \sqrt{n^2 - 1}) : m \in \mathbb{Z}, n \in \mathbb{N} \setminus \{0\}\},$$

the points of which can be considered the vertices of a rectangular grid in $\mathbb{C}$. The width of said rectangles is 1, and since

$$\begin{aligned}\log((n+1) + \sqrt{(n+1)^2 - 1}) - \log(n + \sqrt{n^2 - 1}) &= \\ = \log\frac{1 + \frac{1}{n} + \sqrt{1 + \frac{2}{n}}}{1 + \sqrt{1 - \frac{1}{n^2}}} &\leq \log(2 + \sqrt{3}) < \pi\end{aligned}$$

their height is strictly bounded above by 1. Let $a = p + i\pi^{-1}\log(q + \sqrt{q^2 - 1}) \in A$, then

$$\begin{aligned}\cos\pi a = \tfrac{1}{2}(e^{i\pi a} + e^{-i\pi a}) &= \tfrac{1}{2}(-1)^p((q + \sqrt{q^2 - 1})^{-1} + (q + \sqrt{q^2 - 1})) = \\ &= (-1)^p \frac{1}{2}\frac{1 + q^2 + 2q\sqrt{q^2 - 1} + q^2 - 1}{q + \sqrt{q^2 - 1}} = (-1)^p q.\end{aligned}$$

Thus $\cos\pi(\cos\pi a) = \pm 1$. But $0, 1 \notin f(\overline{D})$, therefore $g(\overline{D}) \cap A = \emptyset$ and $g(\overline{D})$ cannot contain a disk of radius 1.

For (iii), let $0 < \theta < 1$, then for $|z| \leq \theta$ we have $1 - \theta \leq d(z, \partial\mathbb{D})$. By (ii) and Corollary 2.1.5

it follows that $\beta(1-\theta)|g'(z)| \leq 1$. Let γ denote the line connecting 0 and z, then

$$|g(z)| \leq |g(0)| + |g(z) - g(0)| \leq |g(0)| + \int_\gamma |g'(\zeta)|\,\mathrm{d}\zeta \leq |g(0)| + \frac{\theta}{\beta(1-\theta)}$$

yields our desired estimate. ■

Theorem 2.2.4 (Schottky)**.** *There exists a function $\psi(\theta, \omega) : (0,1) \times (0,\infty) \to (0,\infty)$ such that for any $f \in H(\overline{\mathbb{D}}; \mathbb{C} \setminus \{0,1\})$ with $|f(0)| \leq \omega$ it holds that*

$$|f(z)| \leq \psi(\theta, \omega), \quad |z| \leq \theta. \tag{2.1}$$

Proof. Set

$$\psi(\theta, \omega) := \exp\left(\pi \exp \pi \left(3 + 2\omega + \frac{\theta}{\beta(1-\theta)}\right)\right).$$

Note that for all $w \in \mathbb{C}$ we have $|\cos w| \leq e^{|w|}$ and $\frac{1}{2}|1 + \cos w| \leq e^{|w|}$. Hence, from Lemma 2.2.3, we conclude

$$\begin{aligned}|f(z)| = |\tfrac{1}{2}(1 + \cos \pi(\cos \pi g(z)))| &\leq \exp \pi |\cos \pi g(z)| \leq \\ &\leq \exp(\pi \exp \pi |g(z)|) \leq \exp(\pi \exp \pi(|g(0)| + \theta/\beta(1-\theta))) \leq \\ &\leq \exp(\pi \exp \pi(3 + 2\omega + \theta/\beta(1-\theta))) = \psi(\theta, \omega).\end{aligned}$$

■

2.3 Picard's Great Theorem

Definition 2.3.1. Let $G \subseteq \mathbb{C}$ be a domain and $\mathscr{F} \subseteq H(G)$. Then:

- $\mathscr{F}$ is called *locally bounded* if for every $w \in G$ there is a neighborhood U of w and a constant $C > 0$ such that $|f(z)| \leq C$ for all $f \in \mathscr{F}$ and $z \in U$.
- $\mathscr{F}$ is called *normal* in G if every sequence in $\mathscr{F}$ has a subsequence which converges compactly in G to some $f \in H(G)$ or to ∞.

The following well-known theorem establishes the equivalence of the two concepts:

Theorem 2.3.2 (Montel)**.** *Let $G \subseteq \mathbb{C}$ be a domain. Then a family $\mathscr{F} \subseteq H(G)$ is normal if and only if it is locally bounded.*

Theorem 2.3.3 (Hurwitz)**.** *Let $G \subseteq \mathbb{C}$ be a domain and $(f_n)_{n\in\mathbb{N}}$ a sequence in $H(G)$ that converges compactly to f. If for every $n \in \mathbb{N}$ the number of a-points of f_n is bounded by some $m \in \mathbb{N}_0$, then either the number of a-points of f are also bounded by m, or $f \equiv a$.*

The following lemma can be interpreted as a sharpened version of Montel's Theorem (2.3.2) and is sometimes referred to as the *fundamental normality test.*

Lemma 2.3.4. *For any domain $G \subseteq \mathbb{C}$ the family $\mathscr{F} := H(G, \mathbb{C} \setminus \{0,1\})$ is normal in G.*

Proof. We shall give the proof in three steps:

1. Let $w \in G$, $c > 0$ and $\mathscr{F}_*$ be a subfamily of $\mathscr{F}$ such that $|f(w)| \leq c$ for all $f \in \mathscr{F}_*$. We aim to show that there is a neighborhood of w in which $\mathscr{F}_*$ is bounded. Select $t > 0$ such that $\overline{B_t(w)} \subset G$. Without loss of generality we may assume $w = 0$ and $t = 1$. By the maximum principle and Schottky's Theorem (2.2.4) we obtain
$$\sup\{\|f\|_{B_{1/2}(0)} : f \in \mathscr{F}_*\} \leq \psi(1/2, c) < \infty.$$

2. Fix some $w_* \in G$ and set $\mathscr{F}_1 := \{f \in \mathscr{F} : |f(w_*)| \leq 1\}$. We aim to show that $\mathscr{F}_1$ is locally bounded (and by Montel's Theorem (2.3.2) therefore normal) in G. Consider the set
$$U := \{w \in G : \mathscr{F}_1 \text{ is bounded in a neighborhood of } w\},$$
which is open in G. By (1) we have that $w_* \in U$. For sake of contradiction, assume that $U \neq G$. Then there exists some $w \in \partial U \cap G$ and a sequence $f_n \in \mathscr{F}_1$ with $\lim_{n\to\infty} f_n(w) = \infty$. Set $g_n := 1/f_n \in \mathscr{F}$, then $\lim_{n\to\infty} g_n(w) = 0$ and by (1) the family $(g_n)_{n\in\mathbb{N}}$ must be bounded in a neighborhood of w. By Montel's Theorem (2.3.2) it is therefore normal and there exists a subsequence $(g_{n_k})_{k\in\mathbb{N}}$ which converges compactly to a $g \in H(B)$, for some disk B around w. The functions g_n have no zeros, but $g(w) = 0$; by Hurwitz's Theorem (2.3.3) we therefore have $g \equiv 0$. Then for any $z \in B \cap U$ we have
$$\lim_{k\to\infty} f_{n_k}(z) = \lim_{k\to\infty} 1/g_{n_k}(z) = \infty,$$
contradicting the assumption that $\mathscr{F}_1$ is bounded in a neighborhood of z.

3. We can now conclude the proof. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{F}$. If it has a subsequence in $\mathscr{F}_1$, then the claim follows from (2).

   On the other hand, if there are only finitely many f_n in $\mathscr{F}_1$, then $(1/f_n)_{n\in\mathbb{N}}$ has some subsequence, say g_n, in $\mathscr{F}_1$, which converges compactly in G to some $g \in H(G)$. If g has no zeros, then the subsequence $1/g_n$ of the sequence f_n converges compactly in G to $1/g$. If g has zeros, then by Hurwitz's Theorem (2.3.3) we have $g \equiv 0$, thus $1/g_n$ converges compactly in G to ∞.

■

Lemma 2.3.5. *Let $f \in H(\mathbb{D}^\times; \mathbb{C} \setminus \{0,1\})$. Then f or $1/f$ is bounded in a punctured neighborhood of zero.*

Proof. Set $f_n(z) := f(z/n) \in H(\mathbb{D}^\times; \mathbb{C} \setminus \{0,1\})$. By Lemma 2.3.4 there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that $(f_{n_k})_{k\in\mathbb{N}}$ or $(1/f_{n_k})_{k\in\mathbb{N}}$ is bounded on $\partial B_{1/2}(0)$.

In the first case, we have $|f(z/n_k)| \leq M$ for some $M > 0$ and all $|z| = \frac{1}{2}$, $k \in \mathbb{N}$. Thus f is bounded on every circle of radius $1/(2n_k)$. By the maximum principle, f must therefore be bounded on every annulus $1/(2n_{k+1}) \leq |z| \leq 1/(2n_k)$. This yields that f is bounded in a punctured neighborhood of zero.

In the second case, the same argument shows that $1/f$ is bounded in a punctured neighborhood of zero. ■

Theorem 2.3.6 (Picard's Great Theorem)**.** *Let $G \subseteq \mathbb{C}$ be open, $w \in G$ and suppose $f \in H(G \setminus \{w\})$ such that f has an essential singularity at w. Then f takes on all values in $\mathbb{C}$, with at most one exception, infinitely often in any punctured neighborhood of w.*

Proof. Aiming for contradiction, assume that f only takes on $z_0, z_1 \in \mathbb{C}$ finitely often in some punctured neighborhood W of w. Then W contains a punctured disk of radius $\varepsilon > 0$ around w, on which f does not assume z_0, z_1. The function

$$g(z) := \frac{f(\varepsilon z + w) - z_0}{z_1 - z_0} \in H(\mathbb{D}^\times; \mathbb{C} \setminus \{0, 1\})$$

has an essential singularity at zero. By Lemma 2.3.5, we have that either g or $1/g$ must be bounded in a neighborhood of zero. In the former case the singularity must therefore be removable, whereas in the latter case it must be a pole, yielding a contradiction. ■

Corollary 2.3.7. *Every entire transcendental function assumes every value in $\mathbb{C}$ infinitely often, with at most one exception.*

Proof. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $f(z^{-1}) = \sum_{n=-\infty}^{0} a_{-n} z^n$, thus f has an essential singularity at ∞ and Picard's Great Theorem establishes the assertion. ■

Corollary 2.3.8 (Picard's Little Theorem)**.** *Every nonconstant entire function omits at most one value.*

Proof. A non-constant entire function f is either a non-constant polynomial or transcendental. In the latter case, the claim follows from Corollary 2.3.7.

Otherwise let $w \in \mathbb{C}$, then $f(z) - w$ has a complex root by the Fundamental Theorem of Algebra and hence f assumes all values. ■

3 Growth and zeros distribution

3.1 Order and Type

Definition 3.1.1. Let f be an entire function. The *order* of f is defined by

$$\rho_f := \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}. \tag{3.1}$$

Constant functions, by convention, have order 0.

Note that, for any entire function f, we have $0 \le \rho_f \le \infty$. Furthermore, $\rho_f < \infty$ if and only if $M_f(r) = O(\exp(r^{\rho_f+\varepsilon}))$ for every $\varepsilon > 0$ and no $\varepsilon < 0$, as $r \to \infty$ [2].

An elementary fact regarding the order of the sum or product of two entire functions of finite order can immediately be obtained by naively estimating their respective maximum modulus:

Proposition 3.1.2. *Let f, g be entire functions of finite order. Then it holds that:*

i. $\rho_{f+g} \le \max\{\rho_f, \rho_g\}$

ii. $\rho_{fg} \le \max\{\rho_f, \rho_g\}$

Proposition 3.1.3. *Let f be an entire function of finite order, then $\rho_{f'} = \rho_f$.*

Proof. TODO. ∎

For functions of finite and positive order, we can obtain a natural refinement of the concept of order:

Definition 3.1.4. Let f be an entire function of finite and positive order. The *type* of f is defined by

$$\tau_f := \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}} \tag{3.2}$$

For any entire function f with $0 < \rho_f < \infty$, we have $0 < \tau_f \le \infty$. Additionally, we have $0 < \rho_f < \infty$ and $\tau_f < \infty$ if and only if $M_f(r) = O(\exp((\tau_f + \varepsilon)r^{\rho_f}))$ for every $\varepsilon > 0$ and no $\varepsilon < 0$, as $r \to \infty$ [2].

Imitating the proof of Proposition 3.1.3, we immediately get:

Proposition 3.1.5. *Let f be an entire function of finite type, then $\tau_{f'} = \tau_f$.*

Definition 3.1.6. Let f be an entire function. Then f is said to be of *growth* (a, b) if

- $\rho_f < a$, or
- $\rho_f = a$ and $\tau_f \leq b$.

Example 3.1.7. Maybe give some functions together with their order and type?

3.2 Hadamard's Theorem

Theorem 3.2.1 (Weierstrass [1, 3])**.** *Let $(z_j)_{j\in\mathbb{N}}$ be a sequence in $\mathbb{C}$ without accumulation points. Then there exists an entire function E (called the* Weierstrass canonical product *formed from said sequence) that has zeros precisely at $(z_j)_{j\in\mathbb{N}}$, with multiplicities equal to how often z_j occurs in the sequence.*

In particular, we have

$$E(z) = z^k \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{R_n(z/z_n)}, \tag{3.3}$$

where k is the order of the zero at $z = 0$ and R_n is a polynomial, namely a truncation of the power series for $-\log(1 - \frac{z}{z_n})$ chosen of smallest degree to ensure convergence of the product [2].

Weierstrass' Theorem (3.2.1) is also known as the *Weierstrass Factorization Theorem*, due to the following corollary:

Corollary 3.2.2. *Let f be an entire function with zeros $(z_j)_{j\in\mathbb{N}}$. Then there exists an entire function g, such that*

$$f(z) = e^{g(z)} E(z),$$

where E is a Weierstrass canonical product formed from the zeros of f.

Proof. Since f/E has removable singularities at all $(z_j)_{j\in\mathbb{N}}$, we have that f/E is entire and nowhere 0. Thus there exists an entire function g with $f/E = e^g$, which yields $f = e^g E$. ∎

Hadamard's Theorem will show that, for functions of finite order ρ, the function g in Corollary 3.2.2 can be taken to be a polynomial of degree less than ρ and the degree of the polynomials R_n in eq. (3.3) can be taken to be independant of n. To prove this we require the following lemma, which can be interpreted as a version of the maximum modulus theorem applied to the real part of an analytic function.

Lemma 3.2.3 (Borel-Carathéodory)**.** *Let f be analytic in $\overline{\mathbb{D}}$ and let*

$$A_f(r) = \max_{|z|=r} \Re f(z).$$

Then, for $0 < r < R$,

$$M_f(r) \le \frac{2r}{R-r} A_f(R) + \frac{R+r}{R-r} |f(0)|$$

and, if additionally $A_f(R) \ge 0$, *then for* $n \in \mathbb{N}$

$$\max_{|z|=r} |f^{(n)}(z)| \le \frac{2^{n+2} n! R}{(R-r)^{n+1}} (A_f(R) + |f(0)|).$$

Proof. TODO. ∎

Theorem 3.2.4 (Hadamard)**.** *Let* f *be an entire function of finite order with zeros* $(z_j)_{j\in\mathbb{N}}$. *Then there exists a polynomial* Q *with* $\deg Q \le \rho_f$, *such that*

$$f(z) = e^{Q(z)} E(z),$$

where E *is a Weierstrass canonical product formed from the zeros of* f.

Proof. TODO. ∎

3.3 Zeros

We recall a rather explicit connection between the moduli of the zeros of an analytic function and the modulus of the function itself:

Theorem 3.3.1 (Jensen [3])**.** *Let* f *be analytic on* $B_R(0)$ *with* $f(0) \ne 0$ *and let* $r_1, r_2, \ldots$ *denote the moduli of the zeros of* f *in* $B_R(0)$ *arranged in a non-decreasing sequence. Then, for* $r_n < r < r_{n+1}$, *we have*

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\vartheta})| \,\mathrm{d}\vartheta = \log |f(0)| + \log \frac{r^n}{r_1 \ldots r_n}.$$

Definition 3.3.2. Let f be analytic on $B_R(0)$. Then, for $0 < r < R$, we denote by $n_f(r)$ the number of zeros of f in $\overline{B_r(0)}$.

Corollary 3.3.3. *Let* f *be analytic on* $B_R(0)$ *with* $f(0) \ne 0$. *Then, for* $0 < r < R$, *we have*

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\vartheta})| \,\mathrm{d}\vartheta = \log |f(0)| + \int_0^r \frac{n_f(s)}{s} \,\mathrm{d}s$$

Proof. Let $r_1, r_2, \ldots$ denote the moduli of the zeros of f in $B_R(0)$ arranged in a non-decreasing sequence. Then, for any $r_n < r < r_{n+1}$, we obtain

$$\begin{aligned}\log \frac{r^n}{r_1 \dots r_n} &= \sum_{k=1}^{n} \log \frac{r}{r_k} = \sum_{k=1}^{n} \int_{r_k}^{r} \frac{1}{s} \, \mathrm{d}s = \\ &= \sum_{k=1}^{n} \int_{0}^{r} \mathbb{1}_{(r_k,\infty)}(s) \frac{1}{s} \, \mathrm{d}s = \int_{0}^{r} \left(\sum_{k=1}^{n} \mathbb{1}_{(r_k,\infty)}(s) \right) \frac{1}{s} \, \mathrm{d}s = \\ &= \int_{0}^{r} \frac{n_f(s)}{s} \, \mathrm{d}s\end{aligned}$$

and Theorem 3.3.1 concludes the claim. ∎

In particular, we observe that the more zeros a function $f(z)$ has, the faster its modulus must grow as $|z| \to \infty$. The converse is naturally false, as seen by iterated exponentials [2].

Definition 3.3.4. Let f be an entire function and let $(r_j)_{j\in\mathbb{N}}$ denote the non-zero moduli of its zeros (if any) arranged in non-decreasing order. Then

$$\lambda_f := \inf \left\{ \lambda > 0 : \sum_{n=1}^{\infty} \frac{1}{r_n^{\lambda}} < \infty \right\}$$

is called the *exponent of convergence of the zeros of f.* If f has finitely many zeros, then we set $\lambda_f = 0$ by convention.

Furthermore, the *exponent of convergence of the a-points of f* is defined as exponent of convergence of zeros of $f(z) - a$.

Theorem 3.3.5. *Let f be an entire function of finite order. Then $\lambda_f \leq \rho_f$.*

Proof. TODO. ∎

Example 3.3.6. Consider the entire function $f(z) := \sin(z)$, which has zeros at $(n\pi)_{n\in\mathbb{Z}}$, and let $\lambda > 0$. Since

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{|n\pi|^{\lambda}} = \frac{2}{\pi^{\lambda}} \sum_{n=1}^{\infty} \frac{1}{|n|^{\lambda}}$$

is finite if and only if $\lambda > 1$, we obtain $\lambda_f = 1$. Furthermore, we have $|\sin(z)| \leq e^{|\Im z|}$ and therefore

$$M_f(r) \leq \max_{|z|=r} |e^z|.$$

Since e^z is of order 1, this implies $\rho_f \leq 1$. Finally, Theorem 3.3.5 concludes $\rho_f = 1$.

Theorem 3.3.7. *Let E be a Weierstrass canonical product of finite order. Then $\lambda_E = \rho_E$.*

Proof. TODO. ∎

Theorem 3.3.8. *Let f be an entire function of finite, non-integer order. Then $\rho_f = \lambda_f$.*

Proof. By Theorem 3.3.5 we have $\lambda_f \leq \rho_f$. Invoking Hadamard's Theorem we can write $f = e^Q E$ for a polynomial Q with $\deg Q \leq \rho_f$. Since ρ_f is not an integer, this implies $\deg Q \leq \lfloor \rho_f \rfloor < \rho_f$. Now, again by Hadamard's Theorem, e^Q has order $\deg Q$ and by Theorem 3.3.7 E has order λ_f. Using Proposition 3.1.2 we obtain

$$\rho_f \leq \max\{\deg Q, \lambda_f\} = \lambda_f \leq \rho_f,$$

implying $\rho_f = \lambda_f$. ∎

Theorem 3.3.9. *Let f be an entire function of finite, non-integer order. Then f has infinitely many zeros.*

Proof. By Theorem 3.3.8 we have $\rho_f = \lambda_f$. Since ρ_f is not an integer, $\lambda_f > 0$, which implies that f has infinitely many zeros. ∎

Maybe introduce Borel exceptional values as a definition? But then again, I will never need them again. Maybe also add a remark on the relation to lacunary values (Picard).

Theorem 3.3.10 (Borel)**.** *Existence of Borel exceptional values.*

Proof. TODO. ∎

4 Composition of entire functions

4.1 Composition

As seen by Proposition 3.1.2, the order of the sum or product of two entire functions is reasonably bounded by the order of the functions involved. This is not the case when composition is involved. Indeed, consider e^{e^z}, which has infinite order, yet is the composition of two functions of order 1. Necessary conditions for the order of a composition to be finite will be illustrated by Pòlya's Theorem, the proof of which relies on the following result:

Lemma 4.1.1 (Bohr)**.** *Let $0 < R < 1$ and suppose f is analytic on $\overline{\mathbb{D}}$, such that $f(0) = 0$ and $M_f(R) = 1$. Let r_f denote the largest $r \geq 0$ such that $C_r \subseteq f(\overline{\mathbb{D}})$. Then we have $r_f > C > 0$, where C is a constant depending only on R.*

Proof. TODO. Note that the proof relies on the strong form of Schottky's Theorem. ■

Theorem 4.1.2 (Pólya)**.** *Let g, h be entire. For the order of $g \circ h$ to be finite, it must hold that either*

i. h is a polynomial and g of finite order, or

ii. h is of finite order, not a polynomial, and g is of order zero.

Proof. TODO. ■

Theorem 4.1.3 (Thron)**.** *Let g be an entire function of finite order, not a polynomial, which takes some value w only finitely often. Suppose further that there exists some function f such that $f \circ f = g$. Then f is not entire.*

Proof. Seeking contradiction, suppose f were entire. Since g is not a polynomial, Theorem 4.1.2 implies that f is of order 0 and not a polynomial. Let $(z_j)_{j\in J}$ denote the points where f equals w. For each $m \in J$ we additionally denote by $(z_{j,m})_{j\in J_m}$ the points where f equals z_m. Thus, for each $m \in J$ and $n \in J_m$ we have

$$g(z_{n,m}) = f(f(z_{n,m})) = f(z_m) = w.$$

Our assumption on g assures that there must only be finitely many distinct points among the $(z_{n,m})_{m\in J,n\in J_m}$. Thus, each point in $(z_j)_{j\in J}$ is only taken on by f finitely often.

By Corollary 2.3.7, f attains all values in the complex plane infinitely often, with at most one exception. This implies that that there is at most one z_0 in $(z_j)_{j\in J}$ that is only taken on finitely often by f.

If there is no such z_0, then $h(z) \coloneqq f(z) - w$ is entire, of order 0 and nowhere 0. Thus, by Hadamard's Theorem (3.2.4), h must be constant, and therefore f aswell, a contradiction.

If such a z_0 exists, then $h(z) \coloneqq f(z) - w$ has a zero of finite order $n \in \mathbb{N}$ at z_0. Therefore we can write $h(z) = (z - z_0)^n p(z)$, where p is entire, of order 0 and nowhere 0. Again, this implies that p is constant, and therefore f a polynomial, a contradiction. ■

Example 4.1.4. A natural application of Theorem 4.1.3 is taking g to be e^z, which never takes on 0 as a value. Indeed, this implies that there is no entire function f satisfying

$$f(f(z)) = e^z.$$

On the other hand, there does exist a real-analytic function satisfying the above, as demonstrated by H. Kneser. I probably still need a citation here.

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