Errata on Theorem 2 in "False Data Inject Attacks in Control Systems"

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Let X be a strictly positive definite matrix of proper dimension. Define the X-norm of a vector v to be

$$\|v\|_X = \sqrt{v^T X v}.$$

The following lemma is needed to prove Theorem 2.

Lemma 1. *Given any matrix A, there exists a positive semidefinite X matrix (depending on A), such that if $\|Ap\|_X \geq \|p\|_X$, then $\lim_{k\to\infty} A^k p \neq 0$.*

Proof. Let us decompose A as

$$V \begin{bmatrix} A_u & 0 \\ 0 & A_s \end{bmatrix} V^{-1},$$

where A_s is strictly stable and A_u contains all the unstable and critically stable eigenvalues. Define

$$\tilde{A} \triangleq V \begin{bmatrix} 0 & 0 \\ 0 & A_s \end{bmatrix} V^{-1}.$$

Since $\tilde{A}$ is stable, there exists an positive semidefinite X,

$$X > \tilde{A}^T X \tilde{A}.$$

Now suppose that

$$V^{-1} p = \begin{bmatrix} p_u \\ p_s \end{bmatrix}.$$

If $p_u = 0$, then

$$Ap = V \begin{bmatrix} A_u p_u \\ A_s p_s \end{bmatrix} = V \begin{bmatrix} 0 \\ A_s p_s \end{bmatrix} = \tilde{A} p.$$

As a result,

$$\|Ap\|_X^2 = \|\tilde{A}p\|_X^2 = p^T \tilde{A}^T X \tilde{A} p < p^T X p = \|p\|_X^T.$$

Thus, if $\|Ap\|_X \geq \|p\|_X$, we can conclude that p_u cannot be zero. For any k,

$$A^k p = V \begin{bmatrix} A_u^k p_u \\ A_l^k p_s \end{bmatrix}.$$

Notice that

$$\|A_u^k p_u\| \geq \frac{\|p_u\|}{\|A_u^{-k}\|},$$

by the fact that $\|A_u^{-k}\| \to 0$, we can prove that

$$\lim A^k p \neq 0.$$

□

The proof of Theorem 2 will remain mostly the same, except the norm need to be changed into X-norm. Notice that by the equivalence of norms, $\limsup_{k\to\infty} \|\Delta e(k)\| = \infty$ implies that $\limsup_{k\to\infty} \|\Delta e(k)\|_X = \infty$.