An overview of common functors in algebraic geometry

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Todo list

1 Introduction

In algebraic geometry we often associate certain categories to an object. Most of the times these are categories of sheaves on a scheme, or more generally a ringed space or even topological space. A list of candidates we consider is given in section 2.

Given these associated categories we would like to study their relations under a functor induced by a morphism $f : X \to Y$. The main questions are:

1. How can we create meaningful functors between these categories? This is addressed in section 3.

2. What categorical properties do these functors have? By categorical properties we understand adjointness, preservation of (co)limits, etc. This is addressed in section 4.

3. What "geometric properties" do these functors have? By geometric properties we understand mostly properties that are dependent on geometric conditions, and imply certain preservations. This is addressed in section 5.

Whenever something is *not* true in an interesting way (for our interpretation of interesting), we will provide a counterexample in section 6.

Throughout this text we will refer to [Stacks] or any of the relevant EGA's whenever possible.

Add reference: no references are given at this moment, look up the definitions

2 Categories of sheaves

We will consider the following categories:

1. The category of sheaves of abelian groups on a topological space X, denoted Sh_X.
2. The category of $\mathcal{O}_X$-modules on a ringed space X, denoted $\mathcal{O}_X$-Mod.
3.

Expand: derived categories

In section 5 we will discuss preservation of specific subcategories. These will be:

1. The category of coherent sheaves on a ringed space X, denoted Coh_X.
2. The category of quasicoherent sheaves on a ringed space X, denoted Qcoh_X.
3.

Check this: or just on schemes?

Check this: or just on schemes?

Expand: derived versions

3 Functors

The first two functors are the well-known direct and inverse image, defined in a topological situation.

Definition 1. Let $f : X \to Y$ be a morphism of topological spaces. The *direct image functor* $f_*\colon \mathrm{Sh}_X \to \mathrm{Sh}_Y$ is defined by sending $\mathscr{F} \in \mathrm{Ob}(\mathrm{Sh}_X)$ to the presheaf on Y defined by

$$f_*\mathscr{F} : U \mapsto \mathscr{F}(f^{-1}(U)) \tag{1}$$

for $U \subseteq Y$ open. This presheaf is already a sheaf.

Definition 2. Let $f : X \to Y$ be a morphism of topological spaces. The *inverse image functor* $f^{-1}\colon \mathrm{Sh}_Y \to \mathrm{Sh}_X$ is defined by sending $\mathscr{F} \in \mathrm{Ob}(\mathrm{Sh}_Y)$ to the sheafification of the presheaf defined by

$$f^{-1} : U \mapsto \varinjlim_{V \supseteq f(U)} \mathscr{F}(V) \tag{2}$$

for $U \subseteq X$ open.

The following functors are "exceptional" functors, although the first one is only given to introduce the notation that is used in table 1.

Definition 3. Let $i\colon Z \hookrightarrow X$ be the inclusion of a closed subspace Z in a topological space X. The *direct image functor* $i_*\colon \mathrm{Sh}_Z \to \mathrm{Sh}_X$ is defined by sending $\mathscr{F} \in \mathrm{Ob}(\mathrm{Sh}_Z)$ to the presheaf on X defined by

$$i_*\mathscr{F} : U \mapsto \mathscr{F}(i^{-1}(U)) = \mathscr{F}(Z \cap U) \tag{3}$$

for $U \subseteq X$ open. This presheaf is already a sheaf.

Expand: describe right adjoint of this functor (hence in our terminology it is *left adjoint to* and we get two adjoints in the table)

Definition 4. Let $f : X \to Y$ be a morphism of ringed spaces. The *direct image functor* $f_*\colon \mathscr{O}_X\text{-Mod} \to \mathscr{O}_Y\text{-Mod}$ is defined by sending $\mathscr{F} \in \mathrm{Ob}(\mathscr{O}_X\text{-Mod})$ to the presheaf defined on Y by

$$f_*\mathscr{F} : U \mapsto \mathscr{F}(f^{-1}(U)) \tag{4}$$

for $U \subseteq Y$ open. This presheaf is already a sheaf.

Definition 5. Let $f : X \to Y$ be a morphism of ringed spaces. The *inverse image functor* $f^*\colon \mathscr{O}_Y\text{-Mod} \to \mathscr{O}_X\text{-Mod}$ is defined by sending $\mathscr{F} \in \mathrm{Ob}(\mathscr{O}_Y\text{-Mod})$ to the sheaf defined as

$$f^*\mathscr{F} := f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X. \tag{5}$$

4 Adjointness and other categorical properties

5 Geometric properties

6 Counterexamples

References

[EGA I] Jean Dieudonné and Alexander Grothendieck. *Éléments de géométrie algébrique: I. Le langage des schémas*. French. 4. Publications Mathématiques de l'IHÉS, 1960, pages 5–228.

[EGA II] Jean Dieudonné and Alexander Grothendieck. *Éléments de géométrie algébrique: II. Étude globale élémantaire de quelques classes de morphismes*. French. 8. Publications Mathématiques de l'IHÉS, 1961, pages 5–222.

[EGA III_1] Jean Dieudonné and Alexander Grothendieck. *Éléments de géométrie algébrique: III. Étude cohomologique des faisceaux cohérents, Première partie*. French. 11. Publications Mathématiques de l'IHÉS, 1961, pages 5–167.

[EGA III_2] Jean Dieudonné and Alexander Grothendieck. *Éléments de géométrie algébrique: III. Étude cohomologique des faisceaux cohérents, Deuxième partie*. French. 17. Publications Mathématiques de l'IHÉS, 1963, pages 5–91.

[EGA IV_1] Jean Dieudonné and Alexander Grothendieck. *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Première partie*. French. 20. Publications Mathématiques de l'IHÉS, 1964, pages 5–259.

[EGA IV_2] Jean Dieudonné and Alexander Grothendieck. *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Deuxième partie*. French. 24. Publications Mathématiques de l'IHÉS, 1965, pages 5–231.

[EGA IV_3] Jean Dieudonné and Alexander Grothendieck. *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Troisième partie*. French. 28. Publications Mathématiques de l'IHÉS, 1966, pages 5–255.

	left exact	right exact	left adjoint to	right adjoint to	remarks
$f_*\colon \mathrm{Sh}_X \to \mathrm{Sh}_Y$	yes			f^{-1}	
$f^{-1}\colon \mathrm{Sh}_Y \to \mathrm{Sh}_X$	yes [01AJ]	yes	f_*		
$i_*\colon \mathrm{Sh}_Z \to \mathrm{Sh}_X$	yes [01AX]	yes [01AX]	$i^!$	i^{-1}	i^{-1} is nothing but f^{-1} for $i = f$
$i^!\colon \mathrm{Sh}_X \to \mathrm{Sh}_Z$				i_*	
$f_*\colon \mathscr{O}_X\text{-Mod} \to \mathscr{O}_Y\text{-Mod}$	yes [01AJ]			f^* [0096]	
$f^*\colon \mathscr{O}_Y\text{-Mod} \to \mathscr{O}_X\text{-Mod}$		yes [01AJ]	f_* [0096]		

Table 1: Categorical properties of functors

[EGA IV_4] Jean Dieudonné and Alexander Grothendieck. *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Quatrième partie*. French. 32. Publications Mathématiques de l'IHÉS, 1967, pages 5–361.

[Stacks] Aise Johan de Jong. *The Stacks Project*. 2013. URL: `http://stacks.math.columbia.edu`.