4 Random Walk + Soft substance

4.1 Fundamentals

Gibbs-Boltzmann distribution The Boltzmann distribution is a probability distribution that gives the probability of a certain state as a function of that state's energy and temperature of the system to which the distribution is applied. It is given as

$$p_i = \frac{\exp(-\beta\varepsilon_i)}{\sum_{j=1}^{M}\exp(-\beta\varepsilon_j)}$$

Langevin equation The original Langevin equation describes Brownian motion, the apparently random movement of a particle in a fluid due to collisions with molecules of the fliud,

$$m\frac{dv}{dt} = -\lambda v + \eta(t)$$

where v is the velocity of the particle, and m is the mass. The force acting on the particle is written as a sum of a viscous force proportional to the particles's velocity, and a noise term $\eta(t)$ representing the effect of the collisions with the molecules of the fluid. The force $\eta(t)$ has a Gaussian probability distribution with correlation function $\left\langle \eta_i(t)\eta_j(t')\right\rangle = 2\lambda k_B T\delta_{ij}\delta(t-t')$

There are two common choices of discretization: the Itô and the Stratonovich conventions. Discretization of the Langevin equation:

$$\frac{x_{t+\Delta}-x_t}{\Delta} = -V'(x_t) + \xi_t$$

with an associated discretization of the correlations:

$$\langle f\,[x(t)]\rangle \to \langle f(x_t)\rangle \qquad \langle f\,[x(t)]\,\xi(t)\rangle \to \langle f(x_t)\xi_t\rangle \qquad \langle f\,[x(t)]\,\dot{x}(t)\rangle \to \left\langle f(x_t)\frac{x_{t+\Delta}-x_t}{\Delta}\right\rangle$$

which leads to **Itô's chain rule**:

$$\frac{d}{dt}\langle f\,[x(t)]\rangle = \left\langle f'\,[x(t)]\,\frac{dx}{dt}\right\rangle + T\,\langle f''\,[x(t)]\rangle$$

Fokker-Planck equation In one spatial dimension x, for an Itô process driven by the standard Wiener process W_t and described by the stochastic differential equation (SDE)

$$dX_t = \mu(X_t,t)dt + \sigma(X_t,t)dW_t$$

with drift $\mu(X_t,t)$ and diffusion coefficient $D(X_t,t) = \sigma^2(X_t,t)/2$, the Fokker-Planck equation for the probability density $p(x,t)$ of the random variable X_t is

$$\frac{\partial}{\partial t}p(x,t) = -\frac{\partial}{\partial x}\left[\mu(x,t)p(x,t)\right] + \frac{\partial^2}{\partial x^2}\left[D(x,t)p(x,t)\right]$$

Derivation from the over-damped Langevin equation

Let $\mathbb{P}(x,t)$ be the probability density density function to find a particle in $[x, x+dx]$ at time t, and let x satisfy:

$$\dot{x}(t) = -V'(x) + \xi(t)$$

if f is a function, we have:

$$\frac{d}{dt}\langle f[x(t)]\rangle = \frac{d}{dt}\int \mathbb{P}(x,t)f(x)dx = \int \frac{\partial \mathbb{P}(x,t)}{\partial t} f(x)dx$$

but using Itô's chain rule:

$$\frac{d}{dt}\langle f[x(t)]\rangle = \left\langle f'[x(t)]\frac{dx}{dt}\right\rangle + T\langle f''[x(t)]\rangle$$

with Langevin's equation

$$\frac{d}{dt}\langle f[x(t)]\rangle = \langle f'[x(t)]\{-V'[x(t)] + \xi(t)\}\rangle + T\langle f''[x(t)]\rangle$$

since $\langle f'[x(t)]\xi(t)\rangle = 0$, we have

$$\frac{d}{dt}\langle f[x(t)]\rangle = \int \left[\frac{df(x)}{dx}\left(-\frac{dV(x)}{dx}\right) + T\frac{d^2 f(x)}{dx^2}\right]\mathbb{P}(x,t)dx$$

performing an integration by parts, and using that $\mathbb{P}(x,t)$ is a probability density vanishing at $x \longrightarrow \infty$:

$$\int \frac{\partial \mathbb{P}(x,t)}{\partial t} f(x)dx = \int \frac{\partial}{\partial x}\left[\frac{dV(x)}{dx} + T\frac{\partial}{\partial x}\right]\mathbb{P}(x,t)f(x)dx$$

this is true for any function f, thus

$$\boxed{\frac{\partial \mathbb{P}(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[\frac{dV(x)}{dx} + T\frac{\partial}{\partial x}\right]\mathbb{P}(x,t)}$$

It could be written as $\partial_t \mathbb{P}(x,t) = -H_{FP}\mathbb{P}(x,t)$ with H_{FP} the Fokker-Planck operator shown above.

4.2 Salez2015: Elastohydrodynamics of a sliding, spinning and sedimenting cylinder near a soft wall

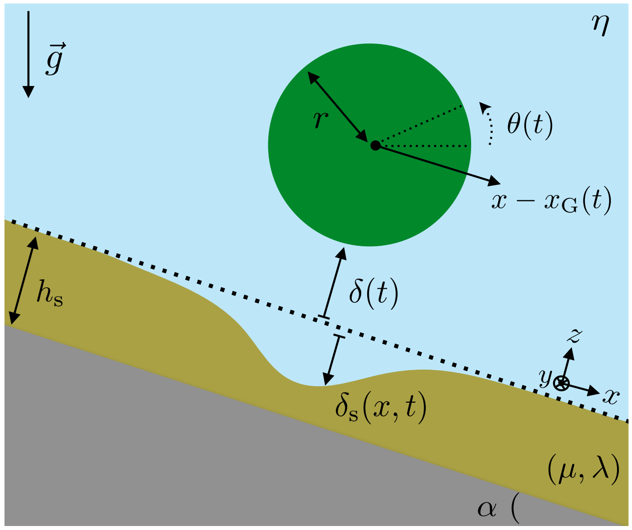

FIG. 1: *Schematic of the system. A negatively buoyant cylinder (green) falls down under the acceleration of gravity $\vec{g}$, inside a viscous fluid (blue), in the vicinity of a thin soft wall (brown). The ensemble lies atop a tilted, infinitely rigid support (grey).*

$$\delta_s(x,t) = -\frac{h_s p(x,t)}{2\mu + \lambda} \tag{Salez2015.2}$$

... we non-dimensionalize the problem using the following choices: $z = Zr\varepsilon$, $h = Hr\varepsilon$, $\delta = \Delta r\varepsilon$, $x = Xr\sqrt{2\varepsilon}$, $x_G = X_G r\sqrt{2\varepsilon}$, $\theta = \Theta\sqrt{2\varepsilon}$, $t = Tr\sqrt{2\varepsilon}/c$, $u = Uc$, and $p = P\eta c\sqrt{2}/(r\varepsilon^{3/2})$, where we have introduced a free fall velocity scale $c = \sqrt{2gr\rho^*/\rho}$ and the dimensionless parameter:

$$\xi = \frac{3\sqrt{2}\eta}{r^{3/2}\varepsilon\sqrt{\rho\rho^* g}}\kappa = \frac{2h_s\eta\sqrt{g\rho^*}}{r^{3/2}\varepsilon^{5/2}(2\mu+\lambda)\sqrt{\rho}}$$

With perturbation theory in first-order correction, the soft compressible wall gives

$$\ddot{X}_G + \frac{2\varepsilon\xi}{3}\frac{\dot{X}_G}{\sqrt{\Delta}} + \frac{\kappa\varepsilon\xi}{6}\left[\frac{19}{4}\frac{\dot{\Delta}\dot{X}_G}{\Delta^{7/2}} - \frac{\dot{\Delta}\dot{\Theta}}{\Delta^{7/2}} + \frac{1}{2}\frac{\ddot{\Theta} - \ddot{X}_G}{\Delta^{5/2}}\right] - \sqrt{\frac{\varepsilon}{2}}\sin\alpha = 0 \tag{Salez2015.50}$$

$$\ddot{\Delta} + \xi\frac{\dot{\Delta}}{\Delta^{3/2}} + \frac{\kappa\xi}{4}\left[21\frac{\dot{\Delta}^2}{\Delta^{9/2}} - \frac{(\dot{\Theta} - \dot{X}_G)^2}{\Delta^{7/2}} - \frac{15}{2}\frac{\ddot{\Delta}}{\Delta^{7/2}}\right] + \cos\alpha = 0 \tag{Salez2015.51}$$

$$\ddot{\Theta} + \frac{4\varepsilon\xi}{3}\frac{\dot{\Theta}}{\sqrt{\Delta}} + \frac{\kappa\varepsilon\xi}{3}\left[\frac{19}{4}\frac{\dot{\Delta}\dot{\Theta}}{\Delta^{7/2}} - \frac{\dot{\Delta}\dot{X}_G}{\Delta^{7/2}} + \frac{1}{2}\frac{\ddot{X}_G - \ddot{\Theta}}{\Delta^{5/2}}\right] = 0 \tag{Salez2015.52}$$

where Δ refers to z and X_G refers to x after the scaling.

For the plan case, we set $\alpha = 0$. Also, there would be no rotation, thus $\theta(t) = 0$.

$$\ddot{X}_G + \frac{2\varepsilon\xi}{3}\frac{\dot{X}_G}{\sqrt{\Delta}} + \frac{\kappa\varepsilon\xi}{6}\left[\frac{19}{4}\frac{\dot{\Delta}\dot{X}_G}{\Delta^{7/2}} + \frac{1}{2}\cdot\frac{-\ddot{X}_G}{\Delta^{5/2}}\right] = 0$$

$$\ddot{\Delta} + \xi\frac{\dot{\Delta}}{\Delta^{3/2}} + \frac{\kappa\xi}{4}\left[21\frac{\dot{\Delta}^2}{\Delta^{9/2}} - \frac{(-\dot{X}_G)^2}{\Delta^{7/2}} - \frac{15}{2}\frac{\ddot{\Delta}}{\Delta^{7/2}}\right] + \cos\alpha = 0$$

4.3 David's note: Determining noise from deterministic forces

Consider the following deterministic equations

$$dX_\alpha = V_\alpha dt \tag{David.1}$$

and

$$dV_\alpha = -U_\alpha dt - \nabla\phi(\mathbf{X})dt \tag{David.2}$$

We assume that U_α are generated by hydrohynamic interactions which do not however affect the equilibrium Gibbs-Boltzmann distribution which is

$$P_{eq}(\mathbf{X},\mathbf{V}) = \frac{1}{\bar{Z}}\exp\left(-\frac{\beta\mathbf{V}^2}{2} - \beta\phi(\mathbf{X})\right) \tag{David.3}$$

Exploit the Fokker-Planck operator

$$\frac{\partial P}{\partial t} = -H_{FP}P = \frac{\partial}{\partial x}\left[\frac{dV}{dx}P + T\frac{\partial}{\partial x}P\right] = \frac{\partial}{\partial V_\alpha}\left[(U_\alpha + \nabla_\alpha\phi)P + T\gamma_{\alpha\beta}\frac{\partial P}{\partial V_\beta}\right] + \frac{\partial}{\partial X_\alpha}[\cdots]$$

Note $\frac{\partial P}{\partial X_\alpha} = P\left(-\beta\frac{\partial\phi}{\partial X_\alpha}\right)$ and $\frac{\partial P}{\partial V_\alpha} = P(-\beta V_\alpha)$. Consider the gravity $\phi(\mathbf{X}) = -mg\Delta$, and then we could derive the eq. David.4

$$\begin{aligned}\frac{\partial}{\partial X_\alpha}\left[\frac{dV}{dx}P + T\frac{\partial}{\partial x}P\right] &= \frac{\partial}{\partial X_\alpha}\left[\frac{dV}{dX_\alpha}P + T\frac{\partial}{\partial X_\alpha}P + T\frac{\partial}{\partial V_\alpha}P\right] \\ &= \frac{\partial}{\partial X_\alpha}\left[(\nabla_\alpha\phi)P + \cancel{T}\cdot P\left(-\cancel{\beta}\frac{\partial\phi}{\partial X_\alpha}\right) + T\frac{\partial}{\partial V_\alpha}P\right] = \frac{\partial}{\partial X_\alpha}\left[T\frac{\partial}{\partial V_\alpha}P\right] \\ &= \frac{\partial}{\partial X_\alpha}\left[\cancel{T}\cdot P(-\cancel{\beta}V_\alpha)\right] = -\frac{\partial}{\partial X_\alpha}V_\alpha P\end{aligned}$$

The Fokker Planck equation at finite temperature which introduces white noise and possibly temperature dependent drifts is $\phi(\mathbf{X})$ is

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial V_\alpha}\left[T\gamma_{\alpha\beta}\frac{\partial P}{\partial V_\beta} + U_\alpha P + \frac{\partial\phi}{\partial X_\alpha}P\right] - \frac{\partial}{\partial X_\alpha}V_\alpha P \tag{David.4}$$

The last two terms would vanish since

$$\frac{\partial}{\partial V_\alpha}\left(\frac{\partial \phi}{\partial X_\alpha}P\right) = \left(\cancel{\frac{\partial}{\partial V_\alpha}\frac{\partial \phi}{\partial X_\alpha}}\right)\cdot P + \frac{\partial \phi}{\partial X_\alpha}\cdot\frac{\partial P}{\partial V_\alpha} = \frac{\partial \phi}{\partial X_\alpha}\cdot P(-\beta V_\alpha)$$

$$\frac{\partial}{\partial X_\alpha}V_\alpha P = \left(\cancel{\frac{\partial V_\alpha}{\partial X_\alpha}}\right)P + V_\alpha\left(\frac{\partial P}{\partial X_\alpha}\right) = V_\alpha\cdot P\cdot\left(-\beta\frac{\partial \phi}{\partial X_\alpha}\right)$$

Therefore, at equilibrium $\frac{\partial P}{\partial t} = 0$

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial V_\alpha}\left[T\gamma_{\alpha\beta}\frac{\partial P}{\partial V_\beta} + U_\alpha P\right] = \frac{\partial}{\partial V_\alpha}\left[\cancel{T}\gamma_{\alpha\beta}P\cdot(-\cancel{\beta}V_\beta) + U_\alpha P\right] = \frac{\partial}{\partial V_\alpha}\left[(U_\alpha - \gamma_{\alpha\beta}V_\beta)\cdot P\right] = 0$$

We obtain the GB distribution for the steady state if

$$U_\alpha = \gamma_{\alpha\beta}V_\beta \qquad \text{(David.5)}$$

We have for small velocities that

$$U_\alpha = \lambda_{\alpha\beta}(\mathbf{X})V_\beta + \Lambda_{\alpha\beta\gamma}(\mathbf{X})V_\beta V_\gamma \qquad \text{(David.6)}$$

and so we find

$$\gamma_{\alpha\beta}V_\beta = \lambda_{\alpha\beta}(\mathbf{X})V_\beta + \Lambda_{\alpha\beta\gamma}(\mathbf{X})V_\beta V_\gamma \qquad \text{(David.7)}$$

Written this way the term $\lambda_{\alpha\beta}(\mathbf{X})$ is just the friction tensor in the absence of any elastic effects. We can thus write

$$\gamma_{\alpha\beta} = \lambda_{\alpha\beta} + \gamma_{2\alpha\beta} \qquad \text{(David.8)}$$

and we write

$$\gamma_{2\alpha\beta} = \Gamma_{\alpha\beta\gamma}V_\gamma \qquad \text{(David.9)}$$

and

$$\Gamma_{\alpha\beta\gamma}(\mathbf{X})V_\beta V_\gamma = \Lambda_{\alpha\beta\gamma}(\mathbf{X})V_\beta V_\gamma \qquad \text{(David.10)}$$

where we without loss of generality take $\Lambda_{\alpha\beta\gamma} = \Lambda_{\alpha\gamma\beta}$, which then gives

$$\Gamma_{\alpha\beta\gamma} + \Gamma_{\alpha\gamma\beta} = 2\Lambda_{\alpha\beta\gamma} \qquad \text{(David.11)}$$

We have to solve this system with the constraint that $\Gamma_{\alpha\beta\gamma}V_\gamma = \Gamma_{\beta\alpha\gamma}V_\gamma$. In Thomas' problem [1412.0162, Journal of Fluid Mechanics, 779 181 (2015)], we have

$$\ddot{\Delta} + \xi\frac{\dot{\Delta}}{\Delta^{3/2}} + \frac{\kappa\xi}{4}\left[21\frac{\dot{\Delta}^2}{\Delta^{9/2}} - \frac{(\dot{\Theta} - \dot{X}_G)^2}{\Delta^{7/2}} - \frac{15}{2}\frac{\ddot{\Delta}}{\Delta^{7/2}}\right] + \cos\alpha = 0$$

$$\ddot{X}_G + \frac{2\varepsilon\xi}{3}\frac{\dot{X}_G}{\sqrt{\Delta}} + \frac{\kappa\varepsilon\xi}{6}\left[\frac{19}{4}\frac{\dot{\Delta}\dot{X}_G}{\Delta^{7/2}} - \frac{\dot{\Delta}\dot{\Theta}}{\Delta^{7/2}} + \frac{1}{2}\frac{\ddot{\Theta} - \ddot{X}_G}{\Delta^{5/2}}\right] - \sqrt{\frac{\varepsilon}{2}}\sin\alpha = 0$$

where Δ refers to z and X_G refers to x. Note $\dot{\Delta} = -U_z$ and $\dot{X}_G = -U_x$, we write

$$U_z = \xi\frac{V_z}{Z^{3/2}} + \frac{21\kappa\xi}{4}\frac{V_z^2}{Z^{9/2}} - \frac{\kappa\xi}{4}\frac{V_x^2}{Z^{7/2}} \quad \text{(David.12)}$$

$$U_x = 2\xi\varepsilon\frac{V_x}{3Z^{1/2}} + \frac{19\kappa\xi\varepsilon}{24}\frac{V_zV_x}{Z^{7/2}} \quad \text{(David.13)}$$

Form this we find that

$$\begin{aligned}\sum_{\alpha\beta}\Lambda_{z\alpha\beta}V_\alpha V_\beta &= \frac{21\kappa\xi}{4}\frac{V_z^2}{Z^{9/2}} - \frac{\kappa\xi}{4}\frac{V_x^2}{Z^{7/2}}\\ \sum_{\alpha\beta}\Lambda_{x\alpha\beta}V_\alpha V_\beta &= \frac{19\kappa\xi\varepsilon}{24}\frac{V_zV_x}{Z^{7/2}}\end{aligned} \quad \text{(David.14)}$$

This gives the set of equations

$$\Gamma_{zzz} = \frac{21\kappa\xi}{4Z^{9/2}} \quad \text{(David.15)}$$

$$\Gamma_{zxx} = -\frac{\kappa\xi}{4Z^{7/2}} \quad \text{(David.16)}$$

$$\Gamma_{zxz} + \Gamma_{zzx} = 0 \quad \text{(David.17)}$$

$$\Gamma_{xzz} = 0 \quad \text{(David.18)}$$

$$\Gamma_{xxx} = 0 \quad \text{(David.19)}$$

$$\Gamma_{xxz} + \Gamma_{xzx} = \frac{19\kappa\xi\varepsilon}{24Z^{7/2}} \quad \text{(David.20)}$$

The symmetry $\Gamma_{\alpha\beta\gamma} = \Gamma_{\beta\alpha\gamma}$ now gives

$$\Gamma_{xxz} = \frac{19\kappa\xi\varepsilon}{24Z^{7/2}} - \Gamma_{xzx} = \frac{19\kappa\xi\varepsilon}{24Z^{7/2}} - \Gamma_{zxx} = \frac{\kappa\xi}{Z^{7/2}}\left(\frac{19\varepsilon}{24} + \frac{1}{4}\right) \quad \text{(David.21)}$$

as well as

$$\Gamma_{zxz} = \Gamma_{zzx} = 0 \quad \text{(David.22)}$$

The Langevin equation corresponding to this is, using the Ito convention,

$$\frac{dV_\alpha}{dt} = -U_\alpha - \frac{\partial\phi(\mathbf{X})}{\partial X_\alpha} + T\frac{\partial\gamma_{\alpha\beta}}{\partial V_\beta} + \eta_\alpha(t) \quad \text{(David.23)}$$

which can be written as

$$\frac{dV_\alpha}{dt} = -U_\alpha - \frac{\partial\phi(\mathbf{X})}{\partial X_\alpha} + T\Gamma_{\alpha\beta\beta} + \eta_\alpha(t) \quad \text{(David.24)}$$

where we use the Einstein summation convention and the noise correlator is given by

$$\left\langle \eta_\alpha(t)\eta_\beta(t')\right\rangle = 2T\gamma_{\alpha\beta}\delta(t-t') = 2T\left[\lambda_{\alpha\beta}(\mathbf{X}) + \Gamma_{\alpha\beta\gamma}(\mathbf{X})V_\gamma\right]\delta(t-t') \tag{David.25}$$

Putting this together we find (from eq. David.24) with all $\Gamma_{\alpha\beta\beta}$ with y index vanishing.

$$\begin{aligned}\frac{dV_z}{dt} &= -V'(Z) - \xi\frac{V_z}{Z^{3/2}} - \frac{21\kappa\xi}{4}\frac{V_z^2}{Z^{9/2}} + \frac{\kappa\xi V_x^2}{4Z^{7/2}} + T\left[\frac{21\kappa\xi}{4Z^{9/2}} - \frac{\kappa\xi}{4Z^{7/2}}\right] + \eta_z(t)\\ \frac{dV_x}{dt} &= -2\xi\varepsilon\frac{V_x}{3Z^{1/2}} - \frac{19\kappa\xi V_zV_x}{24Z^{7/2}} + \eta_x(t)\end{aligned} \tag{David.26}$$

4.4 Modification in 3D

We re-write the differential equations as

$$\begin{aligned}-U_z &= \dot{v}_\Delta = \ddot{\Delta} = F_\Delta(\Delta,\dot{\Delta},\dot{X},\dot{\Theta},\ddot{\Theta}) + \eta_\Delta\\ -U_X &= \dot{v}_X = \ddot{X} = F_X(\Delta,\dot{\Delta},\dot{X},\dot{\Theta},\ddot{X}_G,\ddot{\Theta}) + \eta_X\\ -U_\theta &= \dot{v}_\Theta = \ddot{\Theta} = F_\Theta(\Delta,\dot{\Delta},\dot{X},\dot{\Theta},\ddot{X}_G,\ddot{\Theta}) + \eta_\Theta\end{aligned}$$

In 3D system, we have independent coordinates $\Delta,\dot{\Delta},\dot{X},\dot{\Theta}$. Consider the second derivative in the eq. Salez2015.51, we could obtain

$\mathrm{U}_z = \frac{8\Delta^{9/2}+2\xi\left(-\Delta\kappa v_X^2+4\Delta^3 v_z+21\kappa v_z^2+2\Delta\kappa v_X v_\theta-\Delta\kappa v_\theta^2\right)}{8\Delta^{9/2}-15\Delta\kappa\xi}$

Then combine eqs Salez2015.50 and Salez2015.52, we could solve

$\mathrm{U}_X = \frac{\epsilon\xi\left(\kappa\left(16\Delta^3\epsilon\xi+\left(-24\Delta^{5/2}+23\epsilon\kappa\xi\right)v_z\right)v_\theta+v_X\left(-4\epsilon\kappa^2\xi v_z+\left(6\Delta^{5/2}-\epsilon\kappa\xi\right)\left(16\Delta^3+19\kappa v_\theta\right)\right)\right)}{36\left(4\Delta^6-\Delta^{7/2}\epsilon\kappa\xi\right)}$

$\mathrm{U}_\theta = \frac{\epsilon\xi\left(\left(16\Delta^3\left(12\Delta^{5/2}-\epsilon\kappa\xi\right)+\kappa\left(228\Delta^{5/2}-23\epsilon\kappa\xi\right)v_z\right)v_\theta+\kappa v_X\left(\left(-48\Delta^{5/2}+4\epsilon\kappa\xi\right)v_z+\epsilon\xi\left(16\Delta^3+19\kappa v_\theta\right)\right)\right)}{36\left(4\Delta^6-\Delta^{7/2}\epsilon\kappa\xi\right)}$

After some calculations with the help of *Mathematica*, we list all $\lambda_{\alpha\beta}$ and $\Gamma_{\alpha\beta\gamma}$

$$\begin{aligned}\lambda_{\mathrm{zz}} &= \frac{8\Delta^2\xi}{8\Delta^{7/2}-15\kappa\xi}\\ \lambda_{\mathrm{xx}} &= -\frac{4\epsilon\xi\left(-6\Delta^{5/2}+\epsilon\kappa\xi\right)}{36\Delta^3-9\sqrt{\Delta}\epsilon\kappa\xi}\\ \lambda_{\theta\theta} &= -\frac{4\epsilon\xi\left(-12\Delta^{5/2}+\epsilon\kappa\xi\right)}{36\Delta^3-9\sqrt{\Delta}\epsilon\kappa\xi}\\ \Gamma_{\mathrm{zzz}} &= \frac{42\kappa\xi}{8\Delta^{9/2}-15\Delta\kappa\xi}\\ \Gamma_{\mathrm{xzx}} = \Gamma_{\mathrm{zxx}} &= \frac{2\kappa\xi}{-8\Delta^{7/2}+15\kappa\xi}\\ \Gamma_{\theta z\theta} = \Gamma_{\mathrm{z}\theta\theta} &= \frac{2\kappa\xi}{-8\Delta^{7/2}+15\kappa\xi}\end{aligned}$$

$$\Gamma_{\mathrm{xxz}} = \frac{1}{9}\kappa\xi\left(\frac{18}{8\Delta^{7/2} - 15\kappa\xi} + \frac{\epsilon^2\kappa\xi}{-4\Delta^6 + \Delta^{7/2}\epsilon\kappa\xi}\right)$$

$$\Gamma_{\mathrm{xx}\theta} = \frac{19\epsilon\kappa\xi\left(-6\Delta^{5/2} + \epsilon\kappa\xi\right)}{-144\Delta^6 + 36\Delta^{7/2}\epsilon\kappa\xi}$$

$$\Gamma_{\theta\theta\mathrm{x}} = \frac{19\epsilon^2\kappa^2\xi^2}{36\left(4\Delta^6 - \Delta^{7/2}\epsilon\kappa\xi\right)}$$

$$\Gamma_{\theta\theta\mathrm{z}} = \frac{\epsilon\kappa\xi\left(-228\Delta^{5/2} + 23\epsilon\kappa\xi\right)}{-144\Delta^6 + 36\Delta^{7/2}\epsilon\kappa\xi}$$

$$\Gamma_{\mathrm{zx}\theta} = \Gamma_{\mathrm{xz}\theta} = -\frac{25}{18\Delta} - \frac{19\epsilon\kappa\xi}{72\Delta^{7/2}} + \frac{2\kappa\xi}{8\Delta^{7/2} - 15\kappa\xi} + \frac{50\Delta^{3/2}}{36\Delta^{5/2} - 9\epsilon\kappa\xi}$$

$$\Gamma_{\mathrm{z}\theta\mathrm{x}} = \Gamma_{\theta\mathrm{zx}} = \frac{25}{18\Delta} + \frac{19\epsilon\kappa\xi}{72\Delta^{7/2}} + \frac{2\kappa\xi}{8\Delta^{7/2} - 15\kappa\xi} + \frac{50\Delta^{3/2}}{9\left(-4\Delta^{5/2} + \epsilon\kappa\xi\right)}$$

$$\Gamma_{\mathrm{x}\theta\mathrm{z}} = \Gamma_{\theta\mathrm{xz}} = \frac{2}{15} - \frac{1}{2\Delta} - \frac{3\epsilon\kappa\xi}{8\Delta^{7/2}} + \frac{16}{15\left(-8 + \frac{15\kappa\xi}{\Delta^{7/2}}\right)} + \frac{1}{2\Delta - \frac{\epsilon\kappa\xi}{2\Delta^{3/2}}}$$