Mixing Langevin + ElastoHydroDynamic of cylinder

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Update: February 23, 2022

1 Theoretical Framework

1.1 Fundamentals

Gibbs-Boltzmann distribution The Boltzmann distribution is a probability distribution that gives the probability of a certain state as a function of that state's energy and temperature of the system to which the distribution is applied. It is given as

$$p_i = \frac{\exp(-\beta\varepsilon_i)}{\sum_{j=1}^{M}\exp(-\beta\varepsilon_j)}$$

Langevin equation The original Langevin equation describes Brownian motion, the apparently random movement of a particle in a fluid due to collisions with molecules of the fliud,

$$m\frac{dv}{dt} = -\lambda v + \eta(t)$$

where v is the velocity of the particle, and m is the mass. The force acting on the particle is written as a sum of a viscous force proportional to the particles's velocity, and a noise term $\eta(t)$ representing the effect of the collisions with the molecules of the fluid. The force $\eta(t)$ has a Gaussian probability distribution with correlation function $\langle\eta_i(t)\eta_j(t')\rangle = 2\lambda k_B T\delta_{ij}\delta(t-t')$

There are two common choices of discretization: the Itô and the Stratonovich conventions. Discretization of the Langevin equation:

$$\frac{x_{t+\Delta} - x_t}{\Delta} = -V'(x_t) + \xi_t$$

with an associated discretization of the correlations:

$$\langle f\left[x(t)\right]\rangle \to \langle f(x_t)\rangle \qquad \langle f\left[x(t)\right]\xi(t)\rangle \to \langle f(x_t)\xi_t\rangle \qquad \langle f\left[x(t)\right]\dot{x}(t)\rangle \to \left\langle f(x_t)\frac{x_{t+\Delta}-x_t}{\Delta}\right\rangle$$

which leads to **Itô's chain rule**:

$$\frac{d}{dt}\langle f\left[x(t)\right]\rangle = \left\langle f'\left[x(t)\right]\frac{dx}{dt}\right\rangle + T\left\langle f''\left[x(t)\right]\right\rangle$$

Fokker-Planck equation In one spatial dimension x, for an Itô process driven by the standard Wiener process W_t and described by the stochastic differential equation (SDE)

$$dX_t = \mu(X_t,t)dt + \sigma(X_t,t)dW_t$$

with drift $\mu(X_t,t)$ and diffusion coefficient $D(X_t,t) = \sigma^2(X_t,t)/2$, the Fokker-Planck equation for the probability density $p(x,t)$ of the random variable X_t is

$$\frac{\partial}{\partial t}p(x,t) = -\frac{\partial}{\partial x}\left[\mu(x,t)p(x,t)\right] + \frac{\partial^2}{\partial x^2}\left[D(x,t)p(x,t)\right]$$

Derivation from the over-damped Langevin equation

Let $\mathbb{P}(x,t)$ be the probability density density function to find a particle in $[x, x+dx]$ at time t, and let x satisfy:

$$\dot{x}(t) = -V'(x) + \xi(t)$$

if f is a function, we have:

$$\frac{d}{dt}\langle f\left[x(t)\right]\rangle = \frac{d}{dt}\int \mathbb{P}(x,t)f(x)dx = \int \frac{\partial \mathbb{P}(x,t)}{\partial t}f(x)dx$$

but using Itô's chain rule:

$$\frac{d}{dt}\langle f\left[x(t)\right]\rangle = \left\langle f'\left[x(t)\right]\frac{dx}{dt}\right\rangle + T\left\langle f''\left[x(t)\right]\right\rangle$$

with Langevin's equation

$$\frac{d}{dt}\langle f\left[x(t)\right]\rangle = \langle f'\left[x(t)\right]\{-V'\left[x(t)\right]+\xi(t)\}\rangle + T\left\langle f''\left[x(t)\right]\right\rangle$$

since $\langle f'\left[x(t)\right]\xi(t)\rangle = 0$, we have

$$\frac{d}{dt}\langle f\left[x(t)\right]\rangle = \int\left[\frac{df(x)}{dx}\left(-\frac{dV(x)}{dx}\right) + T\frac{d^2f(x)}{dx^2}\right]\mathbb{P}(x,t)dx$$

performing an integration by parts, and using that $\mathbb{P}(x,t)$ is a probability density vanishing at $x \to \infty$:

$$\int \frac{\partial \mathbb{P}(x,t)}{\partial t} f(x) dx = \int \frac{\partial}{\partial x} \left[\frac{dV(x)}{dx} + T \frac{\partial}{\partial x} \right] \mathbb{P}(x,t) f(x) dx$$

this is true for any function f, thus

$$\boxed{\frac{\partial \mathbb{P}(x,t)}{\partial t} = \frac{\partial}{\partial x} \left[\frac{dV(x)}{dx} + T \frac{\partial}{\partial x} \right] \mathbb{P}(x,t)}$$

It could be written as $\partial_t \mathbb{P}(x,t) = -H_{FP} \mathbb{P}(x,t)$ with H_{FP} the Fokker-Planck operator shown above.

1.2 Salez2015: Elastohydrodynamics of a sliding, spinning and sedimenting cylinder near a soft wall

arxiv: 1412.0162; Journal of Fluid Mechanics, 779 181 (2015)

This article describes the sedimentation, sliding, and spinning motions of a cylinder near a thin compressible elastic wall by thin-film lubrication dynamics. Below is the illustration.

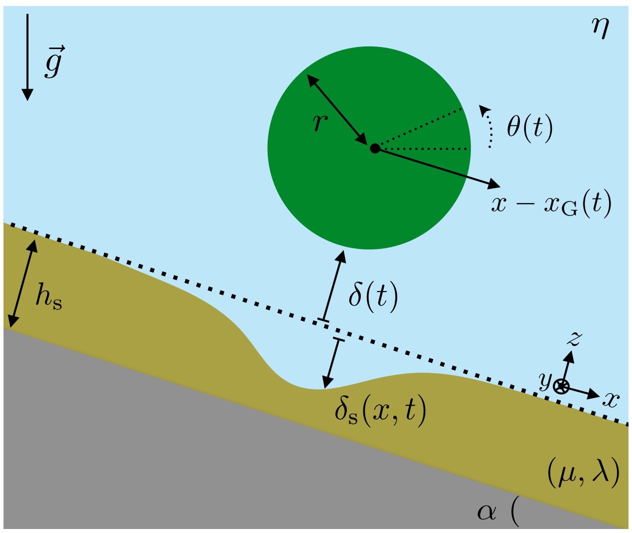

FIG. 1: *Schematic of the system. A negatively buoyant cylinder (green) falls down under the acceleration of gravity $\vec{g}$, inside a viscous fluid (blue), in the vicinity of a thin soft wall (brown). The ensemble lies atop a tilted, infinitely rigid support (grey).*

The deformation of the soft wall reads:

$$\delta_s(x,t) = -\frac{h_s p(x,t)}{2\mu + \lambda} \qquad \text{(Salez2015.2)}$$

as well as the dimensionless parameters: $z = Zr\varepsilon$, $h = Hr\varepsilon$, $\delta = \Delta r\varepsilon$, $x = Xr\sqrt{2\varepsilon}$, $x_G = X_G r\sqrt{2\varepsilon}$, $\theta = \Theta\sqrt{2\varepsilon}$, $t = Tr\sqrt{2\varepsilon}/c$, $u = Uc$, and $p = P\eta c\sqrt{2}/(r\varepsilon^{3/2})$; a free fall velocity scale $c = \sqrt{2gr\rho^*/\rho}$

and one dimensionless parameter ξ measures the ratio of the free fall time $\sqrt{\rho r\epsilon/(\rho^* g)}$ and the typical lubrication damping time $m\epsilon^{3/2}/\eta$ over which the inertia of the cylinder vanishes.

$$\xi = \frac{3\sqrt{2}\eta}{r^{3/2}\varepsilon\sqrt{\rho\rho^* g}}\kappa = \frac{2h_s\eta\sqrt{g\rho^*}}{r^{3/2}\varepsilon^{5/2}(2\mu+\lambda)\sqrt{\rho}}$$

With perturbation theory in first-order correction of κ, the soft compressible wall gives

$$\ddot{X}_G + \frac{2\varepsilon\xi}{3}\frac{\dot{X}_G}{\sqrt{\Delta}} + \frac{\kappa\varepsilon\xi}{6}\left[\frac{19}{4}\frac{\dot{\Delta}\dot{X}_G}{\Delta^{7/2}} - \frac{\dot{\Delta}\dot{\Theta}}{\Delta^{7/2}} + \frac{1}{2}\frac{\ddot{\Theta}-\ddot{X}_G}{\Delta^{5/2}}\right] - \sqrt{\frac{\varepsilon}{2}}\sin\alpha = 0 \tag{Salez2015.50}$$

$$\ddot{\Delta} + \xi\frac{\dot{\Delta}}{\Delta^{3/2}} + \frac{\kappa\xi}{4}\left[21\frac{\dot{\Delta}^2}{\Delta^{9/2}} - \frac{(\dot{\Theta}-\dot{X}_G)^2}{\Delta^{7/2}} - \frac{15}{2}\frac{\ddot{\Delta}}{\Delta^{7/2}}\right] + \cos\alpha = 0 \tag{Salez2015.51}$$

$$\ddot{\Theta} + \frac{4\varepsilon\xi}{3}\frac{\dot{\Theta}}{\sqrt{\Delta}} + \frac{\kappa\varepsilon\xi}{3}\left[\frac{19}{4}\frac{\dot{\Delta}\dot{\Theta}}{\Delta^{7/2}} - \frac{\dot{\Delta}\dot{X}_G}{\Delta^{7/2}} + \frac{1}{2}\frac{\ddot{X}_G-\ddot{\Theta}}{\Delta^{5/2}}\right] = 0 \tag{Salez2015.52}$$

where Δ refers to z and X_G refers to x after the scaling. For the plan case, we set $\alpha = 0$.

1.3 David's note: Determining noise from deterministic forces

Here is the note of David Dean, considering the Brownian motion but only in two dimension (Δ, X). The rotation had been neglected ($\dot{\Theta} = 0$), and the second derivatives in the first-order correction of κ as well. To be clear, Fokker-Planck equation would be carefully discussed. Other personal comments are also written in Italic.

Consider the following deterministic equations *(α refers to Δ, X these two directions)*

$$dX_\alpha = V_\alpha dt \tag{David.1}$$

and *($\mathbf{X}, \mathbf{V}$ refer to the position and the velocity, respectively)*

$$dV_\alpha = -U_\alpha dt - \nabla\phi(\mathbf{X})dt \tag{David.2}$$

We assume that U_α are generated by hydrodynamic interactions which do not however affect the equilibrium Gibbs-Boltzmann distribution which is

$$P_{eq}(\mathbf{X},\mathbf{V}) = \frac{1}{\bar{Z}}\exp\left(-\frac{\beta\mathbf{V}^2}{2} - \beta\phi(\mathbf{X})\right) \tag{David.3}$$

Exploit the Fokker-Planck operator ($\cdots$ refers the similar terms about X_α)

$$\frac{\partial P}{\partial t} = -H_{FP}P = \frac{\partial}{\partial x}\left[\frac{dV}{dx}P + T\frac{\partial}{\partial x}P\right] = \frac{\partial}{\partial V_\alpha}\left[(U_\alpha + \nabla_\alpha\phi)P + T\gamma_{\alpha\beta}\frac{\partial P}{\partial V_\beta}\right] + \frac{\partial}{\partial X_\alpha}[\cdots]$$

Note $\frac{\partial P}{\partial X_\alpha} = P\left(-\beta\frac{\partial\phi}{\partial X_\alpha}\right)$ *and* $\frac{\partial P}{\partial V_\alpha} = P\left(-\beta V_\alpha\right)$. *Consider the gravity* $\phi(\mathbf{X}) = -mg\Delta$, *and then we could derive the eq. David.4, regarding* k_B *as 1*

$$\begin{aligned}
\frac{\partial}{\partial X_\alpha}\left[\frac{dV}{dx}P + T\frac{\partial}{\partial x}P\right] &= \frac{\partial}{\partial X_\alpha}\left[\frac{dV}{dX_\alpha}P + T\frac{\partial}{\partial X_\alpha}P + T\frac{\partial}{\partial V_\alpha}P\right] \\
&= \frac{\partial}{\partial X_\alpha}\left[(\nabla_\alpha\phi)P + \cancel{T}\cdot P\left(-\cancel{\beta}\frac{\partial\phi}{\partial X_\alpha}\right) + T\frac{\partial}{\partial V_\alpha}P\right] = \frac{\partial}{\partial X_\alpha}\left[T\frac{\partial}{\partial V_\alpha}P\right] \\
&= \frac{\partial}{\partial X_\alpha}\left[\cancel{T}\cdot P\left(-\cancel{\beta}V_\alpha\right)\right] = -\frac{\partial}{\partial X_\alpha}V_\alpha P
\end{aligned}$$

The Fokker Planck equation at finite temperature which introduces white noise and possibly temperature dependent drifts is $\phi(\mathbf{X})$ is

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial V_\alpha}\left[T\gamma_{\alpha\beta}\frac{\partial P}{\partial V_\beta} + U_\alpha P + \frac{\partial\phi}{\partial X_\alpha}P\right] - \frac{\partial}{\partial X_\alpha}V_\alpha P \tag{David.4}$$

The last two terms would vanish since

$$\frac{\partial}{\partial V_\alpha}\left(\frac{\partial\phi}{\partial X_\alpha}P\right) = \left(\cancel{\frac{\partial}{\partial V_\alpha}\frac{\partial\phi}{\partial X_\alpha}}\right)\cdot P + \frac{\partial\phi}{\partial X_\alpha}\cdot\frac{\partial P}{\partial V_\alpha} = \frac{\partial\phi}{\partial X_\alpha}\cdot P(-\beta V_\alpha)$$

$$\frac{\partial}{\partial X_\alpha}V_\alpha P = \left(\cancel{\frac{\partial V_\alpha}{\partial X_\alpha}}\right)P + V_\alpha\left(\frac{\partial P}{\partial X_\alpha}\right) = V_\alpha\cdot P\cdot\left(-\beta\frac{\partial\phi}{\partial X_\alpha}\right)$$

Therefore, at equilibrium $\frac{\partial P}{\partial t} = 0$

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial V_\alpha}\left[T\gamma_{\alpha\beta}\frac{\partial P}{\partial V_\beta} + U_\alpha P\right] = \frac{\partial}{\partial V_\alpha}\left[\cancel{T}\gamma_{\alpha\beta}P\cdot(-\cancel{\beta}V_\beta) + U_\alpha P\right] = \frac{\partial}{\partial V_\alpha}\left[(U_\alpha - \gamma_{\alpha\beta}V_\beta)\cdot P\right] = 0$$

We obtain the GB distribution for the steady state if

$$U_\alpha = \gamma_{\alpha\beta}V_\beta \tag{David.5}$$

We have for small velocities that

$$U_\alpha = \lambda_{\alpha\beta}(\mathbf{X})V_\beta + \Lambda_{\alpha\beta\gamma}(\mathbf{X})V_\beta V_\gamma \tag{David.6}$$

and so we find

$$\gamma_{\alpha\beta}V_\beta = \lambda_{\alpha\beta}(\mathbf{X})V_\beta + \Lambda_{\alpha\beta\gamma}(\mathbf{X})V_\beta V_\gamma \tag{David.7}$$

Written this way the term $\lambda_{\alpha\beta}(\mathbf{X})$ is just the friction tensor in the absence of any elastic effects. We can thus write

$$\gamma_{\alpha\beta} = \lambda_{\alpha\beta} + \gamma_{2\alpha\beta} \tag{David.8}$$

and we write

$$\gamma_{2\alpha\beta} = \Gamma_{\alpha\beta\gamma}V_\gamma \tag{David.9}$$

and

$$\Gamma_{\alpha\beta\gamma}(\mathbf{X})V_\beta V_\gamma = \Lambda_{\alpha\beta\gamma}(\mathbf{X})V_\beta V_\gamma \tag{David.10}$$

where we without loss of generality take $\Lambda_{\alpha\beta\gamma} = \Lambda_{\alpha\gamma\beta}$, which then gives

$$\Gamma_{\alpha\beta\gamma} + \Gamma_{\alpha\gamma\beta} = 2\Lambda_{\alpha\beta\gamma} \tag{David.11}$$

We have to solve this system with the constraint that $\Gamma_{\alpha\beta\gamma}V_\gamma = \Gamma_{\beta\alpha\gamma}V_\gamma$. In Thomas' problem *(see subsection 1.2)* we have

$$\ddot{\Delta} + \xi\frac{\dot{\Delta}}{\Delta^{3/2}} + \frac{\kappa\xi}{4}\left[21\frac{\dot{\Delta}^2}{\Delta^{9/2}} - \frac{(\dot{\Theta} - \dot{X}_G)^2}{\Delta^{7/2}} - \frac{15}{2}\frac{\ddot{\Delta}}{\Delta^{7/2}}\right] + \cos\alpha = 0$$

$$\ddot{X}_G + \frac{2\varepsilon\xi}{3}\frac{\dot{X}_G}{\sqrt{\Delta}} + \frac{\kappa\varepsilon\xi}{6}\left[\frac{19}{4}\frac{\dot{\Delta}\dot{X}_G}{\Delta^{7/2}} - \frac{\dot{\Delta}\dot{\Theta}}{\Delta^{7/2}} + \frac{1}{2}\frac{\ddot{\Theta} - \ddot{X}_G}{\Delta^{5/2}}\right] - \sqrt{\frac{\varepsilon}{2}}\sin\alpha = 0$$

where Δ refers to z and X_G refers to x. Note $\dot{\Delta} = -U_z$ and $\dot{X}_G = -U_x$, we write

$$U_z = \xi\frac{V_z}{Z^{3/2}} + \frac{21\kappa\xi}{4}\frac{V_z^2}{Z^{9/2}} - \frac{\kappa\xi}{4}\frac{V_x^2}{Z^{7/2}} \tag{David.12}$$

$$U_x = 2\xi\varepsilon\frac{V_x}{3Z^{1/2}} + \frac{19\kappa\xi\varepsilon}{24}\frac{V_zV_x}{Z^{7/2}} \tag{David.13}$$

Note $\dot{\Delta} = V_z$ and $\dot{X}_G = V_x$, we could extract the relevant coefficients. But Attention! Here $\dot{\Theta}$ was assumed 0, and the second derivatives in first-order correlation had been ignored.

Form this we find that

$$\begin{aligned}\sum_{\alpha\beta}\Lambda_{z\alpha\beta}V_\alpha V_\beta &= \frac{21\kappa\xi}{4}\frac{V_z^2}{Z^{9/2}} - \frac{\kappa\xi}{4}\frac{V_x^2}{Z^{7/2}}\\ \sum_{\alpha\beta}\Lambda_{x\alpha\beta}V_\alpha V_\beta &= \frac{19\kappa\xi\varepsilon}{24}\frac{V_zV_x}{Z^{7/2}}\end{aligned} \tag{David.14}$$

This gives the set of equations

$$\Gamma_{zzz} = \frac{21\kappa\xi}{4Z^{9/2}} \tag{David.15}$$

$$\Gamma_{zxx} = -\frac{\kappa\xi}{4Z^{7/2}} \tag{David.16}$$

$$\Gamma_{zxz} + \Gamma_{zzx} = 0 \tag{David.17}$$

$$\Gamma_{xzz} = 0 \tag{David.18}$$

$$\Gamma_{xxx} = 0 \tag{David.19}$$

$$\Gamma_{xxz} + \Gamma_{xzx} = \frac{19\kappa\xi\varepsilon}{24Z^{7/2}} \tag{David.20}$$

The symmetry $\Gamma_{\alpha\beta\gamma} = \Gamma_{\beta\alpha\gamma}$ now gives

$$\Gamma_{xxz} = \frac{19\kappa\xi\varepsilon}{24Z^{7/2}} - \Gamma_{xzx} = \frac{19\kappa\xi\varepsilon}{24Z^{7/2}} - \Gamma_{zxx} = \frac{\kappa\xi}{Z^{7/2}}\left(\frac{19\varepsilon}{24} + \frac{1}{4}\right) \tag{David.21}$$

as well as

$$\Gamma_{zxz} = \Gamma_{zzx} = 0 \tag{David.22}$$

The Langevin equation corresponding to this is, using the Itô convention,

$$\frac{dV_\alpha}{dt} = -U_\alpha - \frac{\partial\phi(\mathbf{X})}{\partial X_\alpha} + T\frac{\partial\gamma_{\alpha\beta}}{\partial V_\beta} + \eta_\alpha(t) \tag{David.23}$$

which can be written as

$$\frac{dV_\alpha}{dt} = -U_\alpha - \frac{\partial\phi(\mathbf{X})}{\partial X_\alpha} + T\Gamma_{\alpha\beta\beta} + \eta_\alpha(t) \tag{David.24}$$

where we use the Einstein summation convention and the noise correlator is given by

$$\left\langle \eta_\alpha(t)\eta_\beta(t')\right\rangle = 2T\gamma_{\alpha\beta}\delta(t-t') = 2T\left[\lambda_{\alpha\beta}(\mathbf{X}) + \Gamma_{\alpha\beta\gamma}(\mathbf{X})V_\gamma\right]\delta(t-t') \tag{David.25}$$

Putting this together we find *(from eq. David.24) with all $\Gamma_{\alpha\beta\beta}$ only depending on Δ, X.*

$$\begin{aligned}
\frac{dV_z}{dt} &= -V'(Z) - \xi\frac{V_z}{Z^{3/2}} - \frac{21\kappa\xi}{4}\frac{V_z^2}{Z^{9/2}} + \frac{\kappa\xi V_x^2}{4Z^{7/2}} + T\left[\frac{21\kappa\xi}{4Z^{9/2}} - \frac{\kappa\xi}{4Z^{7/2}}\right] + \eta_z(t) \\
\frac{dV_x}{dt} &= -2\xi\varepsilon\frac{V_x}{3Z^{1/2}} - \frac{19\kappa\xi V_z V_x}{24Z^{7/2}} + \eta_x(t)
\end{aligned} \tag{David.26}$$

1.4 New Coefficients: $\lambda_{\alpha\beta}, \Gamma_{\alpha\beta\gamma}$ in 3D (Δ, X, Θ)

In this part, we would renew coefficients for the motion in 3D (Δ, X, Θ). Based on the previous subsection, we could repeat the calculation by Fokker-Planck operator, finding the similar results with additional terms about Θ.

For the sake of convenience, we re-write Thomas' differential equations (see subsection 1.2) with $\dot{v}_i$, (and X refers to X_G)

$$\begin{aligned}
-U_Z &= \dot{v}_\Delta = \ddot{\Delta} = F_\Delta(\Delta, v_\Delta, v_X, v_\Theta, \dot{v}_\Delta) \\
-U_X &= \dot{v}_X = \ddot{X} = F_X(\Delta, v_\Delta, v_X, v_\Theta, \dot{v}_X, \dot{v}_\Theta) \\
-U_\Theta &= \dot{v}_\Theta = \ddot{\Theta} = F_\Theta(\Delta, v_\Delta, v_X, v_\Theta, \dot{v}_X, \dot{v}_\Theta)
\end{aligned} \tag{1}$$

However, we'd like to derive equations for each $\dot{v}$ only depending on Δ and v, without $\dot{v}$. Therefore, we have to find the proper expression for each $\dot{v}_i$.

Consider the second derivative in the eq. Salez2015.51,

$$\ddot{\Delta} + a_1 \frac{\dot{\Delta}}{\Delta^{3/2}} + a_2 \frac{\dot{\Delta}^2}{\Delta^{9/2}} + a_3 \frac{\dot{\Theta}^2}{\Delta^{7/2}} + a_3 \frac{\dot{X}^2}{\Delta^{7/2}} + a_4 \frac{\dot{\Theta}\dot{X}}{\Delta^{7/2}} + a_5 \frac{\ddot{\Delta}}{\Delta^{7/2}} + a_6 = 0 \tag{2}$$

$$\ddot{\Delta} = (a_1 \frac{\dot{\Delta}}{\Delta^{3/2}} + a_2 \frac{\dot{\Delta}^2}{\Delta^{9/2}} + a_3 \frac{\dot{\Theta}^2}{\Delta^{7/2}} + a_3 \frac{\dot{X}^2}{\Delta^{7/2}} + a_4 \frac{\dot{\Theta}\dot{X}}{\Delta^{7/2}} + a_6) \times \frac{-1}{1 + a_5/\Delta^{7/2}} \tag{3}$$

We know that $a_1 = \xi$, $a_2 = \frac{21\kappa\xi}{4}$, $a_3 = -\frac{\kappa\xi}{4}$, $a_4 = \frac{\kappa\xi}{2}$, $a_5 = -\frac{15\kappa\xi}{8}$, $a_6 = \cos(\alpha = 0) = 1$. After simple calculation, we could obtain $\dot{v}_\Delta$ ($\dot{v}_z$) namely $\ddot{\Delta}$

$$-\dot{v}_\Delta = U_z = \frac{8\Delta^{9/2} + 2\xi\left(-\Delta\kappa v_X^2 + 4\Delta^3 v_z + 21\kappa v_z^2 + 2\Delta\kappa v_X v_\theta - \Delta\kappa v_\theta^2\right)}{8\Delta^{9/2} - 15\Delta\kappa\xi} \tag{4}$$

Similarly, we write the eqs Salez2015.50 and Salez2015.52 as

$$\ddot{X} + b_1 \frac{\dot{X}}{\sqrt{\Delta}} + b_2 \frac{\dot{\Delta}\dot{X}}{\Delta^{7/2}} + b_3 \frac{\dot{\Delta}\dot{\Theta}}{\Delta^{7/2}} + b_4 \frac{\ddot{\Theta}}{\Delta^{5/2}} + b_5 \frac{\ddot{X}}{\Delta^{5/2}} + b_6 = 0 \tag{5}$$

$$\ddot{\Theta} + c_1 \frac{\dot{\Theta}}{\sqrt{\Delta}} + c_2 \frac{\dot{\Delta}\dot{\Theta}}{\Delta^{7/2}} + c_3 \frac{\dot{\Delta}\dot{X}}{\Delta^{7/2}} + c_4 \frac{\ddot{X}}{\Delta^{5/2}} + c_5 \frac{\ddot{\Theta}}{\Delta^{5/2}} + c_6 = 0 \tag{6}$$

with all coefficients we need: $b_1 = \frac{2\varepsilon\xi}{3}$, $b_2 = \frac{19\kappa\xi\varepsilon}{24}$, $b_3 = -\frac{\kappa\xi\varepsilon}{6}$, $b_4 = \frac{\kappa\xi\varepsilon}{12}$, $b_5 = -\frac{\kappa\xi\varepsilon}{12}$, $b_6 = \sin(\alpha = 0) = 0$; and $c_1 = \frac{4\varepsilon\xi}{3}$, $c_2 = \frac{19\kappa\xi\varepsilon}{12}$, $c_3 = -\frac{\kappa\xi\varepsilon}{3}$, $c_4 = \frac{\kappa\xi\varepsilon}{6}$, $c_5 = -\frac{\kappa\xi\varepsilon}{6}$, $c_6 = 0$. For this system of linear equations, the coefficient matrix has full rank.

$$\begin{pmatrix} 1 + (b_5) & (b_4) \\ (c_4) & 1 + (c_5) \end{pmatrix} \begin{pmatrix} \ddot{X} \\ \ddot{\Theta} \end{pmatrix} = \begin{pmatrix} (b_1 + b_2 + b_3 + b_6) \\ (c_1 + c_2 + c_3 + c_6) \end{pmatrix}$$

Then we could solve $\ddot{X} = \dot{v}_X$ and $\ddot{\Theta} = \dot{v}_\Theta$ directly

$$-\dot{v}_X = U_X = \frac{\epsilon\xi\left(\kappa\left(16\Delta^3\epsilon\xi + \left(-24\Delta^{5/2} + 23\epsilon\kappa\xi\right)v_z\right)v_\theta + v_X\left(-4\epsilon\kappa^2\xi v_z + \left(6\Delta^{5/2} - \epsilon\kappa\xi\right)\left(16\Delta^3 + 19\kappa v_\theta\right)\right)\right)}{36\left(4\Delta^6 - \Delta^{7/2}\epsilon\kappa\xi\right)} \tag{7}$$

$$-\dot{v}_\Theta = U_\Theta = \frac{\epsilon\xi\left(\left(16\Delta^3\left(12\Delta^{5/2} - \epsilon\kappa\xi\right) + \kappa\left(228\Delta^{5/2} - 23\epsilon\kappa\xi\right)v_z\right)v_\theta + \kappa v_X\left(\left(-48\Delta^{5/2} + 4\epsilon\kappa\xi\right)v_z + \epsilon\xi\left(16\Delta^3 + 19\kappa v_\theta\right)\right)\right)}{36\left(4\Delta^6 - \Delta^{7/2}\epsilon\kappa\xi\right)} \tag{8}$$

Compare with the eq. 1, we finally remove the second derivatives inside each expression

$$\begin{aligned} \dot{v}_\Delta &= F_\Delta(\Delta, v_\Delta, v_X, v_\Theta) \\ \dot{v}_X &= F_X(\Delta, v_\Delta, v_X, v_\Theta) \\ \dot{v}_\Theta &= F_\Theta(\Delta, v_\Delta, v_X, v_\Theta) \end{aligned} \tag{9}$$

See eqs David.5 ~ David.9, we could extract these $\lambda_{\alpha\beta}$ by

$$\lambda_{\alpha\beta} = \mathrm{Coefficient}[\mathrm{U}_\alpha, \mathrm{v}_\beta] - \mathrm{Coefficient}[\mathrm{U}_\alpha, \mathrm{v}_\beta \mathrm{v}_\gamma] \times \mathrm{v}_\gamma \tag{10}$$

and $\Gamma_{\alpha\beta\beta}$ by

$$\Gamma_{\alpha\beta\beta} = \mathrm{Coefficient}[\mathrm{U}_\alpha, \mathrm{v}_\beta \mathrm{v}_\beta] \tag{11}$$

As for $\Gamma_{\alpha\beta\gamma}$, we should resolve them by

$$2\Lambda_{\alpha\beta\gamma} = \mathrm{Coefficient}[\mathrm{U}_\alpha, \mathrm{v}_\beta \mathrm{v}_\gamma] = \Gamma_{\alpha\beta\gamma} + \Gamma_{\alpha\gamma\beta} \tag{12}$$

as well as the constraint $\Gamma_{\alpha\beta\gamma} = \Gamma_{\beta\alpha\gamma}$.

After some calculations with the help of *Mathematica*, we list all $\lambda_{\alpha\beta}$

$$\begin{aligned}
\lambda_{\mathrm{zz}} &= \frac{8\Delta^2\xi}{8\Delta^{7/2} - 15\kappa\xi} \\
\lambda_{\mathrm{xx}} &= -\frac{4\epsilon\xi\left(-6\Delta^{5/2} + \epsilon\kappa\xi\right)}{36\Delta^3 - 9\sqrt{\Delta}\epsilon\kappa\xi} \\
\lambda_{\theta\theta} &= -\frac{4\epsilon\xi\left(-12\Delta^{5/2} + \epsilon\kappa\xi\right)}{36\Delta^3 - 9\sqrt{\Delta}\epsilon\kappa\xi}
\end{aligned} \tag{13}$$

$$\lambda_{x\theta} = \lambda_{\theta x} = \frac{4\epsilon^2\kappa\xi^2}{36\Delta^3 - 9\sqrt{\Delta}\epsilon\kappa\xi} \tag{14}$$

$$\lambda_{zx} = \lambda_{xz} = \lambda_{z\theta} = \lambda_{\theta z} = 0 \tag{15}$$

and then $\Gamma_{\alpha\beta\gamma}$

$$\begin{aligned}
\Gamma_{\mathrm{zzz}} &= \frac{42\kappa\xi}{8\Delta^{9/2} - 15\Delta\kappa\xi} \\
\Gamma_{\mathrm{xzx}} = \Gamma_{\mathrm{zxx}} &= \frac{2\kappa\xi}{-8\Delta^{7/2} + 15\kappa\xi} \\
\Gamma_{\theta z\theta} = \Gamma_{\mathrm{z}\theta\theta} &= \frac{2\kappa\xi}{-8\Delta^{7/2} + 15\kappa\xi}
\end{aligned} \tag{16}$$

$$\Gamma_{zxz} = \Gamma_{zzx} = \Gamma_{zz\theta} = \Gamma_{z\theta z} = 0 \tag{17}$$

$$\Gamma_{xzz} = \Gamma_{xxx} = \Gamma_{\theta zz} = \Gamma_{\theta\theta\theta} = 0 \tag{18}$$

$$\Gamma_{\theta xx} = \Gamma_{x\theta x} = \Gamma_{x\theta\theta} = \Gamma_{\theta x\theta} = 0 \tag{19}$$

$$
\begin{aligned}
\Gamma_{\mathrm{xxz}} &= \frac{1}{9}\kappa\xi\left(\frac{18}{8\Delta^{7/2}-15\kappa\xi}+\frac{\epsilon^2\kappa\xi}{-4\Delta^6+\Delta^{7/2}\epsilon\kappa\xi}\right)\\
\Gamma_{\mathrm{xx}\theta} &= \frac{19\epsilon\kappa\xi\left(-6\Delta^{5/2}+\epsilon\kappa\xi\right)}{-144\Delta^6+36\Delta^{7/2}\epsilon\kappa\xi}\\
\Gamma_{\theta\theta\mathrm{x}} &= \frac{19\epsilon^2\kappa^2\xi^2}{36\left(4\Delta^6-\Delta^{7/2}\epsilon\kappa\xi\right)}\\
\Gamma_{\theta\theta\mathrm{z}} &= \frac{\epsilon\kappa\xi\left(-228\Delta^{5/2}+23\epsilon\kappa\xi\right)}{-144\Delta^6+36\Delta^{7/2}\epsilon\kappa\xi}
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
\Gamma_{\mathrm{zx}\theta} = \Gamma_{\mathrm{xz}\theta} &= -\frac{25}{18\Delta}-\frac{19\epsilon\kappa\xi}{72\Delta^{7/2}}+\frac{2\kappa\xi}{8\Delta^{7/2}-15\kappa\xi}+\frac{50\Delta^{3/2}}{36\Delta^{5/2}-9\epsilon\kappa\xi}\\
\Gamma_{\mathrm{z}\theta\mathrm{x}} = \Gamma_{\theta\mathrm{zx}} &= \frac{25}{18\Delta}+\frac{19\epsilon\kappa\xi}{72\Delta^{7/2}}+\frac{2\kappa\xi}{8\Delta^{7/2}-15\kappa\xi}+\frac{50\Delta^{3/2}}{9\left(-4\Delta^{5/2}+\epsilon\kappa\xi\right)}\\
\Gamma_{\mathrm{x}\theta\mathrm{z}} = \Gamma_{\theta\mathrm{xz}} &= \frac{2}{15}-\frac{1}{2\Delta}-\frac{3\epsilon\kappa\xi}{8\Delta^{7/2}}+\frac{16}{15\left(-8+\frac{15\kappa\xi}{\Delta^{7/2}}\right)}+\frac{1}{2\Delta-\frac{\epsilon\kappa\xi}{2\Delta^{3/2}}}
\end{aligned} \tag{21}
$$

1.5 $\gamma_{\alpha\beta}$ and linear approximation of κ

Since $\gamma_{\alpha\beta} = \lambda_{\alpha\beta} + \Gamma_{\alpha\beta\gamma}V_\gamma$, we have

$$
\begin{aligned}
\gamma_{zz} &= \frac{8\Delta^2\xi}{8\Delta^{7/2}-15\kappa\xi}+\frac{42\kappa\xi v_z}{8\Delta^{9/2}-15\Delta\kappa\xi}\\
&= \frac{\xi}{\Delta^{3/2}}+\left(\frac{15\xi^2}{8\Delta^5}+\frac{21\xi v_z}{4\Delta^{9/2}}\right)\kappa+\left(\frac{225\xi^3}{64\Delta^{17/2}}+\frac{315\xi^2 v_z}{32\Delta^8}\right)\kappa^2+O[\kappa]^3
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
\gamma_{zx} = \gamma_{xz} &= \frac{2\kappa\xi v_X}{-8\Delta^{7/2}+15\kappa\xi}+\left(-\frac{25}{18\Delta}-\frac{19\epsilon\kappa\xi}{72\Delta^{7/2}}+\frac{2\kappa\xi}{8\Delta^{7/2}-15\kappa\xi}+\frac{50\Delta^{3/2}}{36\Delta^{5/2}-9\epsilon\kappa\xi}\right)v_\theta\\
&= -\frac{\left(\xi\left(3v_X-3v_\theta-\epsilon v_\theta\right)\right)\kappa}{12\Delta^{7/2}}+\frac{5\xi^2\left(-27v_X+27v_\theta+5\Delta\epsilon^2 v_\theta\right)\kappa^2}{288\Delta^7}+O[\kappa]^3
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
\gamma_{z\theta} = \gamma_{\theta z} &= \left(\frac{25}{18\Delta}+\frac{19\epsilon\kappa\xi}{72\Delta^{7/2}}+\frac{2\kappa\xi}{8\Delta^{7/2}-15\kappa\xi}+\frac{50\Delta^{3/2}}{9\left(-4\Delta^{5/2}+\epsilon\kappa\xi\right)}\right)v_X+\frac{2\kappa\xi v_\theta}{-8\Delta^{7/2}+15\kappa\xi}\\
&= -\frac{\left(\xi\left(-3v_X+\epsilon v_X+3v_\theta\right)\right)\kappa}{12\Delta^{7/2}}-\frac{5\left(\xi^2\left(-27v_X+5\Delta\epsilon^2 v_X+27v_\theta\right)\right)\kappa^2}{288\Delta^7}+O[\kappa]^3
\end{aligned} \tag{24}
$$

$$\gamma_{xx} = -\frac{4\epsilon\xi\left(-6\Delta^{5/2}+\epsilon\kappa\xi\right)}{36\Delta^3-9\sqrt{\Delta}\epsilon\kappa\xi} + \frac{1}{9}\kappa\xi\left(\frac{18}{8\Delta^{7/2}-15\kappa\xi} + \frac{\epsilon^2\kappa\xi}{-4\Delta^6+\Delta^{7/2}\epsilon\kappa\xi}\right)v_z + \frac{19\epsilon\kappa\xi\left(-6\Delta^{5/2}+\epsilon\kappa\xi\right)v_\theta}{-144\Delta^6+36\Delta^{7/2}\epsilon\kappa\xi}$$
$$= \frac{2\epsilon\xi}{3\sqrt{\Delta}} + \frac{\left(4\sqrt{\Delta}\epsilon^2\xi^2+18\xi v_z+57\epsilon\xi v_\theta\right)\kappa}{72\Delta^{7/2}} + \left(\frac{\epsilon^3\xi^3}{72\Delta^{11/2}} - \frac{\left(-135+8\Delta\epsilon^2\right)\xi^2 v_z}{288\Delta^7} + \frac{19\epsilon^2\xi^2 v_\theta}{288\Delta^6}\right)\kappa^2 + O[\kappa]^3 \tag{25}$$

$$\gamma_{\theta\theta} = -\frac{4\epsilon\xi\left(-12\Delta^{5/2}+\epsilon\kappa\xi\right)}{36\Delta^3-9\sqrt{\Delta}\epsilon\kappa\xi} + \frac{19\epsilon^2\kappa^2\xi^2 v_X}{36\left(4\Delta^6-\Delta^{7/2}\epsilon\kappa\xi\right)} + \kappa\xi\left(\frac{23\epsilon}{36\Delta^{7/2}} + \frac{2}{8\Delta^{7/2}-15\kappa\xi} + \frac{34\epsilon}{36\Delta^{7/2}-9\Delta\epsilon\kappa\xi}\right)v_z$$
$$= \frac{4\epsilon\xi}{3\sqrt{\Delta}} + \left(\frac{2\epsilon^2\xi^2}{9\Delta^3} + \frac{(3+19\epsilon)\xi v_z}{12\Delta^{7/2}}\right)\kappa + \left(\frac{\epsilon^3\xi^3}{18\Delta^{11/2}} + \frac{19\epsilon^2\xi^2 v_X}{144\Delta^6} + \frac{\left(135+68\Delta\epsilon^2\right)\xi^2 v_z}{288\Delta^7}\right)\kappa^2 + O[\kappa]^3 \tag{26}$$

$$\begin{aligned}\gamma_{x\theta} = \gamma_{\theta x} &= \frac{4\epsilon^2\kappa\xi^2}{36\Delta^3-9\sqrt{\Delta}\epsilon\kappa\xi} + \left(\frac{2}{15} - \frac{1}{2\Delta} - \frac{3\epsilon\kappa\xi}{8\Delta^{7/2}} + \frac{16}{15\left(-8+\frac{15\kappa\xi}{\Delta^{7/2}}\right)} + \frac{1}{2\Delta-\frac{\epsilon\kappa\xi}{2\Delta^{3/2}}}\right)v_z \\ &= \left(\frac{\epsilon^2\xi^2}{9\Delta^3} - \frac{(\xi+\epsilon\xi)v_z}{4\Delta^{7/2}}\right)\kappa + \left(\frac{\epsilon^3\xi^3}{36\Delta^{11/2}} + \frac{\left(-15+\Delta\epsilon^2\right)\xi^2 v_z}{32\Delta^7}\right)\kappa^2 + O[\kappa]^3\end{aligned} \tag{27}$$

Finally, we have obtained the all components of $\gamma_{\alpha\beta}$.

$$\boldsymbol{\gamma} = \begin{pmatrix} \boldsymbol{\gamma_{zz}} & \boldsymbol{\gamma_{zx}} & \boldsymbol{\gamma_{z\theta}} \\ \boldsymbol{\gamma_{xz}} & \boldsymbol{\gamma_{xx}} & \boldsymbol{\gamma_{x\theta}} \\ \boldsymbol{\gamma_{\theta z}} & \boldsymbol{\gamma_{\theta x}} & \boldsymbol{\gamma_{\theta\theta}} \end{pmatrix}$$

As we could see, this is a symmetric matrix. In addition, only three diagonal elements, namely $\gamma_{zz}, \gamma_{xx}, \gamma_{\theta\theta}$ have zero-order term of κ, which describes the compliance. In genenral, this parameter is about $10^{-4} \sim 10^{-3}$. Hence we just consider the first-order correction in the following calculations.

Up to now, we have elucidated the drift force. Indeed, there exist the external potential (like gravity) and white noises at the same time.

$$\dot{v}_i = F_i + TF_i^{drift} + \eta_i \tag{28}$$

Since three components are coupled, the equation turns to

$$\begin{pmatrix} \dot{v}_\Delta \\ \dot{v}_X \\ \dot{v}_\Theta \end{pmatrix} = -\begin{pmatrix} \gamma_{\mathrm{zz}} & \gamma_{\mathrm{zx}} & \gamma_{\mathrm{z}\theta} \\ \gamma_{\mathrm{xz}} & \gamma_{\mathrm{xx}} & \gamma_{\mathrm{x}\theta} \\ \gamma_{\theta\mathrm{z}} & \gamma_{\theta\mathrm{x}} & \gamma_{\theta\theta} \end{pmatrix}\begin{pmatrix} v_\Delta \\ v_X \\ v_\Theta \end{pmatrix} + \begin{pmatrix} \eta_\Delta \\ \eta_X \\ \eta_\Theta \end{pmatrix} \tag{29}$$

where the noise correlator has been shown in the eq. David.25:

$$\left\langle \eta_\alpha(t)\eta_\beta(t')\right\rangle = 2T\gamma_{\alpha\beta}\delta(t-t') = 2T\left[\lambda_{\alpha\beta}(\mathbf{X}) + \Gamma_{\alpha\beta\gamma}(\mathbf{X})V_\gamma\right]\delta(t-t')$$

So we'd like to find the expression of $\gamma^{1/2}$.

Suppose that

$$\gamma = \Psi + \kappa\Phi + O[\kappa]^2 \tag{30}$$

where Ψ is zero-order matrix of κ, and Φ the first-order one. So $\gamma^{1/2}$ would show a form such as $\gamma^{1/2} \approx \psi + \kappa\chi$,

$$\gamma = \gamma^{1/2}\gamma^{1/2} = (\psi + \kappa\chi)(\psi + \kappa\chi) = \psi\psi + \kappa(\psi\chi + \chi\psi) + O[\kappa]^2$$

we have $\Phi = \chi\psi + \psi\chi$, and $\psi = \sqrt{\Psi}$. Note Ψ is a symmetric matrix and all non-diagonal elements are equal to 0.

$$\Psi = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} \frac{\xi}{\Delta^{3/2}} & 0 & 0 \\ 0 & \frac{2\epsilon\xi}{3\sqrt{\Delta}} & 0 \\ 0 & 0 & \frac{4\epsilon\xi}{3\sqrt{\Delta}} \end{pmatrix} \tag{31}$$

Then we could solve all components of χ

$$\begin{aligned} \chi_{11} &= \frac{3\xi\left(5\xi + 14\sqrt{\Delta}v_z\right)}{16\Delta^5\sqrt{\frac{\xi}{\Delta^{3/2}}}} \\ \chi_{22} &= \frac{\xi\left(18v_z + \epsilon\left(4\sqrt{\Delta}\epsilon\xi + 57v_\theta\right)\right)}{48\sqrt{6}\Delta^{7/2}\sqrt{\frac{\epsilon\xi}{\sqrt{\Delta}}}} \\ \chi_{33} &= \frac{\xi\left(8\sqrt{\Delta}\epsilon^2\xi + (9+57\epsilon)v_z\right)}{48\sqrt{3}\Delta^{7/2}\sqrt{\frac{\epsilon\xi}{\sqrt{\Delta}}}} \end{aligned} \tag{32}$$

$$\begin{aligned} \chi_{12} = \chi_{21} &= \frac{\xi\left(-3v_X + (3+\epsilon)v_\theta\right)}{4\Delta^{7/2}\left(3\sqrt{\frac{\xi}{\Delta^{3/2}}} + \sqrt{6}\sqrt{\frac{\epsilon\xi}{\sqrt{\Delta}}}\right)} \\ \chi_{13} = \chi_{31} &= -\frac{\xi\left(3v_\theta + (\epsilon-3)v_X\right)}{4\Delta^{7/2}\left(3\sqrt{\frac{\xi}{\Delta^{3/2}}} + 2\sqrt{3}\sqrt{\frac{\xi\epsilon}{\sqrt{\Delta}}}\right)} \\ \chi_{23} = \chi_{32} &= \frac{\xi\left(4\sqrt{\Delta}\epsilon^2\xi - 9(1+\epsilon)v_z\right)}{12\sqrt{3}\left(2+\sqrt{2}\right)\Delta^{7/2}\sqrt{\frac{\epsilon\xi}{\sqrt{\Delta}}}} \end{aligned} \tag{33}$$