DETERMINING NOISE FROM DETERMINSTIC FORCES

Consider the following deterministic equations

(1) $$dX_\alpha = V_\alpha dt$$

and

(2) $$dV_\alpha = -U_\alpha dt - \nabla\phi(\mathbf{X})dt.$$

We assume that U_α are generated by hydrodynamic interactions which do not however affect the equilibrium Gibbs Boltzmann distribution which is

(3) $$P_{eq}(\mathbf{X},\mathbf{V}) = \frac{1}{\bar{Z}}\exp(-\frac{\beta\mathbf{V}^2}{2} - \beta\phi(\mathbf{X}))$$

The Fokker Planck equation at finite temperature which introduces white noise and possibly temperature dependent drifts is $\phi(\mathbf{X})$ is

(4) $$\frac{\partial P}{\partial t} = \frac{\partial}{\partial V_\alpha}\left[T\gamma_{\alpha\beta}\frac{\partial P}{\partial V_\beta} + U_\alpha P + \frac{\partial\phi}{\partial X_\alpha}P\right] - \frac{\partial}{\partial X_\alpha}V_\alpha P$$

We obtain the GB distribution for the steady state if

(5) $$U_\alpha = \gamma_{\alpha\beta}V_\beta$$

We have for small velocities that

(6) $$U_\alpha = \lambda_{\alpha\beta}(\mathbf{X})V_\beta + \Lambda_{\alpha\beta\gamma}V_\beta V_\gamma,$$

and so we find

(7) $$\gamma_{\alpha\beta}V_\beta = \lambda_{\alpha\beta}(\mathbf{X})V_\beta + \Lambda_{\alpha\beta\gamma}(\mathbf{X})V_\beta V_\gamma$$

Written this way the term $\lambda_{\alpha\beta}(\mathbf{X})$ is just the friction tensor in the absence of any elastic effects. We can thus write

(8) $$\gamma_{\alpha\beta} = \lambda_{\alpha\beta} + \gamma_{2\alpha\beta}$$

and we write

(9) $$\gamma_{2\alpha\beta} = \Gamma_{\alpha\beta\gamma}V_\gamma$$

and

(10) $$\Gamma_{\alpha\beta\gamma}V_\beta V_\gamma = \Lambda_{\alpha\beta\gamma}V_\beta V_\gamma,$$

where we without loss of generality take $\Lambda_{\alpha\beta\gamma} = \Lambda_{\alpha\gamma\beta}$ This then gives

(11) $$\Gamma_{\alpha\beta\gamma} + \Gamma_{\alpha\gamma\beta} = 2\Lambda_{\alpha\beta\gamma}.$$

We have to solve this system with the constraint that $\Gamma_{\alpha\beta\gamma}V_\gamma = \Gamma_{\beta\alpha\gamma}V_\gamma$. In Thomas' problem we have

$$U_z = \xi\frac{V_z}{Z^{\frac{3}{2}}} + \frac{21\kappa\xi}{4}\frac{V_z^2}{Z^{\frac{9}{2}}} - \frac{\kappa\xi}{4}\frac{V_x^2}{Z^{\frac{7}{2}}} \tag{12}$$

and

$$U_x = 2\xi\epsilon\frac{V_x}{3Z^{\frac{1}{2}}} + \frac{19\kappa\xi\epsilon}{24}\frac{V_zV_x}{Z^{\frac{7}{2}}} \tag{13}$$

Form this we find that

$$\begin{aligned}\sum_{\alpha\beta}\Lambda_{z\alpha\beta}V_\alpha V_\beta &= \frac{21\kappa\xi}{4}\frac{V_z^2}{Z^{\frac{9}{2}}} - \frac{\kappa\xi}{4}\frac{V_x^2}{Z^{\frac{7}{2}}}\\ \sum_{\alpha\beta}\Lambda_{x\alpha\beta}V_\alpha V_\beta &= \frac{19\kappa\xi\epsilon}{24}\frac{V_zV_x}{Z^{\frac{7}{2}}} \end{aligned}\tag{14}$$

This gives the set of equations

$$\Gamma_{zzz} = \frac{21\kappa\xi}{4Z^{\frac{9}{2}}} \tag{15}$$

$$\Gamma_{zxx} = -\frac{\kappa\xi}{4Z^{\frac{7}{2}}} \tag{16}$$

$$\Gamma_{zxz} + \Gamma_{zzx} = 0 \tag{17}$$

$$\Gamma_{xzz} = 0 \tag{18}$$

$$\Gamma_{xxx} = 0 \tag{19}$$

$$\Gamma_{xxz} + \Gamma_{xzx} = \frac{19\kappa\xi\epsilon}{24Z^{\frac{7}{2}}}. \tag{20}$$

The symmetry $\Gamma_{\alpha\beta\gamma} = \Gamma_{\beta\alpha\gamma}$ now gives

$$\Gamma_{xxz} = \frac{19\kappa\xi\epsilon}{24Z^{\frac{7}{2}}} - \Gamma_{xzx} = \frac{19\kappa\xi\epsilon}{24Z^{\frac{7}{2}}} - \Gamma_{zxx} = \frac{\kappa\xi}{Z^{\frac{7}{2}}}(\frac{19\epsilon}{24} + \frac{1}{4}), \tag{21}$$

as well as

$$\Gamma_{zxz} = \Gamma_{zzx} = 0 \tag{22}$$

The Langevin equation corresponding to this is, using the Ito convention,

$$\frac{dV_\alpha}{dt} = -U_\alpha - \frac{\partial\phi(\mathbf{X})}{\partial X_\alpha} + T\frac{\partial\gamma_{\alpha\beta}}{\partial V_\beta} + \eta_\alpha(t) \tag{23}$$

which can be written as

$$\frac{dV_\alpha}{dt} = -U_\alpha - \frac{\partial\phi(\mathbf{X})}{\partial X_\alpha} + T\Gamma_{\alpha\beta\beta} + \eta_\alpha(t), \tag{24}$$

where we use the Einstein summation convention and the noise correlator is given by

$$\langle\eta_\alpha(t)\eta_\beta(t')\rangle = 2T\gamma_{\alpha\beta}\delta(t-t') = 2T[\lambda_{\alpha\beta}(\mathbf{X}) + \Gamma_{\alpha\beta\gamma}(\mathbf{X})V_\gamma)]\delta(t-t'). \tag{25}$$

Putting this together we find

$$\begin{aligned}
\frac{dV_z}{dt} &= -V'(Z) - \xi \frac{V_z}{Z^{\frac{3}{2}}} - \frac{21\kappa\xi}{4}\frac{V_z^2}{Z^{\frac{9}{2}}} + \frac{\kappa\xi}{4}\frac{V_x^2}{Z^{\frac{7}{2}}} + T[\frac{21\kappa\xi}{4Z^{\frac{9}{2}}} - \frac{\kappa\xi}{4Z^{\frac{7}{2}}}] + \eta_z(t) \\
\frac{dV_x}{dt} &= -2\xi\epsilon\frac{V_x}{3Z^{\frac{1}{2}}} - \frac{19\kappa\xi\epsilon}{24}\frac{V_zV_x}{Z^{\frac{7}{2}}} + \eta_x(t)
\end{aligned} \tag{26}$$