Elastohydrodynamic force on a sphere that oscillates horizontally

Vincent Bertin

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1 Intro

In this note, I will derive the elastohydrodynamic force that act on a sphere that move horizontally near a soft wall. This model is a close version of the one developed in Urzay et al. 2007 Phys. Fluids : Link.

2 Hydrodynamic model

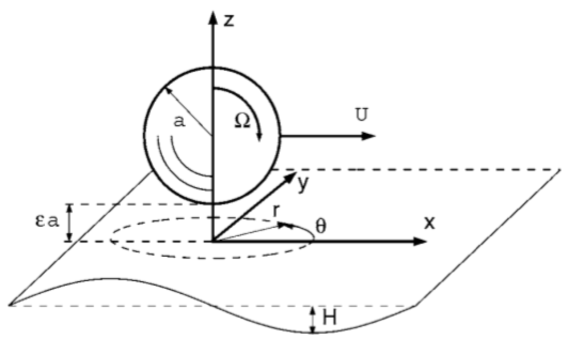

Figure 1: Schematic of the model (adapted from Urzay 2007).

2.1 General remarks

We take the same general formulation as in Urzay (see Figure 1). The major differences are :

- The sphere is free to move in the vertical direction (vertical motion neglected in the lift force calculation)
- The velocity U is time dependant (instationnary terms neglected too in the lift force calculation).
- The sphere does not rotate (i.e. $\Omega = 0$).

- We derive the hydrodynamic model in the reference frame of the lab (and not the one of the sphere). In addition, we consider that the sphere is fixed (horizontally) and that the substrate is moving (acutal experimental set-up).

I choose similar notation as in Salez 2015 JFM and notably :

- In the model, we rescale the radial length by the same lateral length $a\sqrt(2\epsilon)$ as in Salez 2015 JFM and Skotheim 2005 Phys. Fluids. and not $a\sqrt(\epsilon)$ (factor $\sqrt{2}$ of difference).
- The fluid viscosity, elastic parameters notation are the different than the one in Urzay.
- The sphere radius is denoted a and the vertical deflection of the subtrate δ. The gap thickness between the sphere and the substrate at rest is denote d.
- Note that we take into account the full displacement field of the substrate at the surface (denoted $\vec{u}$) and not only the vertical one.

2.2 Governing equations and scaling analysis.

We assume that the fluid is Newtonian and incompressible and that we can neglect gravity and inertia. The governing equations are the Stokes equations for the velocity field $\vec{v}$

$$\vec{\nabla} p = \eta \vec{\nabla}^2 \vec{v} \tag{1}$$

$$\vec{\nabla} . \vec{v} = 0 \tag{2}$$

A cylindrical coordinate system (r, θ, z) is choose such that the origine of the system is the basis of the sphere (see Fig. 1). The position of the fluid/sphere interface is

$$h_0(r,t) = d(t) + \frac{r^2}{2a} \tag{3}$$

The x-axis is the direction of the substrate motion. We introduce a small parameter $\epsilon = d/a \ll 1$ which is the ratio between the typical distance of the sphere with wall d and the sphere radius a. We will write dimensionless parameters with capital letters. The following lubrication rescaling are made

$$r = a(2\epsilon)^{1/2}\, R, \quad z = a\epsilon\, Z, \tag{4}$$

We introduce a typical velocity scale which is equal to $c = \sqrt{2} x_0 \omega$ where x_0 is the amplitude of the lateral oscillation, and ω its pulsation. The velocity[1] are then rescaled as

$$v_r = c\, V_R, \quad v_\theta = c\, V_\theta, \quad v_z = c(\epsilon/2)^{1/2} V_Z \tag{5}$$

The time is rescaled as

$$t = \frac{a(2\epsilon)^{1/2}}{c}\, T \tag{6}$$

And the pressure is rescaled to have $\partial_R P = \partial_Z^2 V_R$ at leading order which yields to

$$p = \frac{\eta c \sqrt{2}}{a} \epsilon^{-3/2}\, P \tag{7}$$

[1]Note that I correct a mistake made in the previous version of the note : $v_z = c\epsilon^{1/2} V_Z \rightarrow v_z = c(\epsilon/2)^{1/2} V_Z$

Finally, the interface position are also rescaled as $h_0 = a\,\epsilon H_0$, $d = a\,\epsilon D$, $\delta = a\,\epsilon\Delta$, such that the interface equation is

$$H_0(R,T) = D(T) + R^2 \tag{8}$$

Then at leading order in ϵ, the Stokes equation and incompressibility are

$$\partial_R P = \partial_Z^2 V_R \tag{9}$$

$$\frac{1}{R}\partial_\theta P = \partial_Z^2 V_\theta \tag{10}$$

$$\partial_Z P = 0 \tag{11}$$

$$\frac{1}{R}\partial_R\left(RV_R\right) + \frac{1}{R}\partial_\theta V_\theta + \partial_Z V_Z = 0 \tag{12}$$

We assume no-slip at both interface which give for the substrate/fluid interface. Therefore, the velocity of the fluid is equal to the velocity of the substrate:

$$\text{at } Z = \Delta, \quad V_{R,\text{fluide}} = V_{R,\,\text{substrate}} = \partial_T U_R + U(t)\cos(\theta), \qquad V_{\theta,\text{fluide}} = V_{\theta,\,\text{substrate}} = \partial_T U_\theta - U(t)\sin(\theta) \tag{13}$$

The velocity of the substrate can be decomposed as two terms. The first one, $\vec{U} = U(t)\vec{e}_x$ come from the overall displacement of the substrate driven experimentally by the piezzo. The second one comes from the substrate itself. When a external operator push on an elastic material, it deforms and expel some of the fluid. Therefore, we need to consider a the displacement field of the substrate at the interface called $\vec{u}$, or (u_r, u_θ). This displacements are rescaled by the contact length $a\sqrt{2\epsilon}$ such that where $(u_r, u_\theta) = a\sqrt{2\epsilon}(U_R, U_\theta)$. Note that in a thin compressible material, theoretician usually use the Winkler foundation which assume that these displacement vanish at leading order, valid when the material thickness (denoted h_{sub} in the following) is small compare with the contact length: $h_{\text{sub}}/\sqrt{2aD} \to 0$. These displacements also vanish in the limit of the thick material (which would be the main focus of this work). Nevertheless, for a thin incompressible material, this is no longer the case. For the specific case of a thin incompressible material the horizontal displacement are even large compare with vertical one because of the incompressibility condition: $\vec{\nabla}.\vec{u} = \frac{1}{r}\partial_r(ru_r) + \frac{1}{r}\partial_\theta u_\theta + \partial_z u_z$. If we evaluate this horizontal displacement order of magnitude from this equation, we find: $u_r/u_z \sim \sqrt{2aD}/h_{\text{sub}}$.

The sphere is assume underformable and fixed horizontally, and we assume a no-slip condition such that $\vec{V}_{\text{fluide}} = \vec{V}_{\text{sphere}} = 0$ which yields to the trivial boundary conditions

$$\text{at } Z = H_0, \quad V_{R,\text{fluide}} = V_{R,\,\text{sphere}} = 0, \qquad V_{\theta,\text{fluide}} = V_{\theta,\,\text{sphere}} = 0. \tag{14}$$

Given that P does not depend on the vertical position, we can integrate both Eqs. (9) and (10) and we get

$$V_R(Z) = \frac{1}{2}\partial_R P Z^2 + A\,Z + B \tag{15}$$

$$V_\theta(Z) = \frac{1}{2R}\partial_\theta P Z^2 + C\,Z + D \tag{16}$$

and the four integration constants can be found with the boundary conditions Eqs. (14) and (13). We find:

$$V_R(Z) = \frac{1}{2}\partial_R P\Big(Z^2 - (H_0+\Delta)Z + \Delta H_0\Big) + U(t)\cos(\theta)\frac{H_0 - Z}{H_0 - \Delta} + \partial_T U_R \frac{H_0 - Z}{H_0 - \Delta} \tag{17}$$

$$V_\theta(Z) = \frac{1}{2R}\partial_\theta P\Big(Z^2 - (H_0+\Delta)Z + \Delta H_0\Big) - U(t)\sin(\theta)\frac{H_0 - Z}{H_0 - \Delta} + \partial_T U_\theta \frac{H_0 - Z}{H_0 - \Delta} \tag{18}$$

We notice that the vertical velocity can be extracted from the incompressible condition if necessary.

We can integrate the Eq. (12) in with respect to the vertical coordinate from $Z = \Delta$ to $Z = H_0$ and we get

$$V_Z(Z = H_0) - V_Z(Z = \Delta) = -\frac{1}{R}\int_{\Delta}^{H_0} \partial_R R V_R(Z)\mathrm{d}Z - \frac{1}{R}\int_{\Delta}^{H_0} \partial_\theta V_\theta(Z)\mathrm{d}Z \tag{19}$$

Then we need to specify what is the vertical velocity at the two interfaces $V_Z(Z = H_0)$ and $V_Z(Z = \Delta)$. We invoke the kinematic boundary conditions

$$\begin{aligned} V_Z(Z = H_0) &= \partial_T H_0 + \vec{V}.\vec{\nabla} H_0 \\ &= \dot{D} + 0 = \dot{D} \end{aligned} \tag{20}$$

Given the fact that the sphere is fixed, the convective term is nul at this interface. The instationnary term can be considered in the calculation and would lead to the same kind of hydrodynamic force as in Elizabeth Charlaix's paper (Leroy & Charlaix 2011 JFM) for instance. This force is actually taken into account in the data process. The vertical piezzo impose a approaching velocity (of the order of the $\dot{d} \sim$ nm/s) which yields to a hydrodynamic vertical force of $F_z = \frac{6\pi\eta a^2 \dot{d}}{d}$. Then the kinematic boundary condition at the fluid/substrate interface is

$$\begin{aligned} V_Z(Z = \Delta) &= \partial_T \Delta + \vec{V}.\vec{\nabla}\Delta \\ &= \partial_T \Delta + U\cos(\theta)\partial_R \Delta - U\sin\theta \frac{1}{R}\partial_\theta \Delta + \partial_T U_R \partial_R \Delta + \frac{1}{R}\partial_T U_\theta \partial_\theta \Delta \end{aligned} \tag{21}$$

Note that the two last terms of the RHS of the last equation are of second order in the elastic perturbation and will be systematically neglected in all perturbative methods up to second order. Then we can write the full volume conservation as

$$\begin{aligned} \dot{D} - \partial_T \Delta =& \frac{1}{12\,R}\partial_R \left[R\partial_R P (H_0 - \Delta)^3 - 6R(H_0 - \Delta)\cos(\theta)U - 6R(H_0 - \Delta)\partial_T U_R \right] \\ &+ \frac{1}{12\,R}\partial_\theta \left[\frac{1}{R}\partial_\theta P (H_0 - \Delta)^3 + 6(H_0 - \Delta)\sin(\theta)U - 6(H_0 - \Delta)\partial_T U_\theta \right] \end{aligned} \tag{22}$$

This equation is equivalent to the volume conservation in lubrication. That makes total sence because we are integrating the (local) mass/volume conservation condition over an entire trench of fluid.

3 Elasticity

The equation (22) is not a close problem. We need to specify the rheology, i.e. elastic model (or viscoelastic), of the substrate. In other words, one need to specify the relationship between the substrate displacement at the surface (U_R, U_θ, Δ) and the pressure field.

3.1 Definition of the elastic problem

The elastic material is subjected to a pressure and shear, which means normal and tangential stresses that give rise to displacement. At the fluid/substrate interface, we do have the following stress balance[2]

$$\vec{\vec{\sigma}}_{\text{fluid}} . \vec{n}_{\text{interface}} = \vec{\vec{\sigma}}_{\text{substrate}} . \vec{n}_{\text{interface}} \tag{23}$$

where $\vec{\vec{\sigma}}$ denotes the stress field in the two materials and $\vec{n}$ is the normal vector. To stay consistent with the lubrication approach (small slopes), the normal vector is the vertical unit vector at leading order in ϵ, i.e. $\vec{n} = \vec{e}_z + \mathcal{O}(\epsilon)$. This leads to the stress continuity equations[3]

$$\sigma_{zz,\text{fluid}} = \sigma_{zz,\text{substrate}}, \qquad \sigma_{zr,\text{fluid}} = \sigma_{zr,\text{substrate}}, \qquad \sigma_{z\theta,\text{fluid}} = \sigma_{z\theta,\text{substrate}} \tag{24}$$

Then, let's express the stress field in the fluid (we recall $\vec{\vec{\sigma}}_{\text{fluid}} = -p\vec{\vec{I}} + \eta(\vec{\nabla}\vec{v} + \vec{\nabla}\vec{v}^T)$.

$$\sigma_{zz,\text{fluid}} = -p + 2\eta\partial_z v_z \tag{25}$$

$$\sigma_{zr,\text{fluid}} = \eta(\partial_z v_r + \partial_r v_z) \tag{26}$$

$$\sigma_{z\theta,\text{fluid}} = \eta(\partial_z v_\theta + \frac{1}{r}\partial_\theta v_z) \tag{27}$$

We now, apply the rescaling introduced in the previous section

$$\sigma_{zz,\text{fluid}} = \sqrt{2}\frac{\eta c}{a\epsilon^{3/2}}\left(- P + \epsilon\partial_Z V_Z\right) \tag{28}$$

$$\sigma_{zr,\text{fluid}} = \frac{\eta c}{a\epsilon^{3/2}}\left(\epsilon^{1/2}\partial_Z V_R + \epsilon^{3/2}\frac{1}{2}\partial_R V_Z\right) \tag{29}$$

$$\sigma_{z\theta,\text{fluid}} = \frac{\eta c}{a\epsilon^{3/2}}\left(\epsilon^{1/2}\partial_Z V_\theta + \epsilon^{3/2}\frac{1}{2R}\partial_\theta V_Z\right) \tag{30}$$

Then to the leading order in the lubrication parameter ϵ, the substrate is subjected to

$$\sigma_{zz,\text{fluid}} = -\frac{\eta c\sqrt{2}}{a\epsilon^{3/2}}P \tag{31}$$

$$\sigma_{zr,\text{fluid}} = 0 \tag{32}$$

$$\sigma_{z\theta,\text{fluid}} = 0 \tag{33}$$

at the interface $z = 0$.

[2] I wonder weather the stress boundary conditions should rather be simply $\vec{\vec{\sigma}}_{\text{fluid}} = \vec{\vec{\sigma}}_{\text{solid}}$ or the one I juste wrote. It does not change anything.

[3] Acutally, I realized afterward that doing this approximation, I neglect some terms that are of the same order of magnitude in the radial and orthoradial displacement. It doesnot change the conclusion of the paragraph, but I'm aware that this calculation should be done more carrefully.

3.2 Response of a thin compressible material

For a thin compressible material (Poisson coefficient strictly less than 1/2, i.e. $\nu < 1/2$) of Lamé coefficient (λ, μ), the stress/displacement relationship is

$$\begin{aligned}\vec{\vec{\sigma}}_{\text{substrate}} &= \mu(\vec{\nabla}\vec{u} + \vec{\nabla}\vec{u}^T) + \lambda(\vec{\nabla}.\vec{u})\vec{\vec{I}} \\ &= \mu(\vec{\nabla}\vec{u} + \vec{\nabla}\vec{u}^T) + \frac{\lambda}{2}\text{Trace}\Big((\vec{\nabla}\vec{u} + \vec{\nabla}\vec{u}^T)\Big)\vec{\vec{I}}\end{aligned} \tag{34}$$

We recall that the (bulk) stress balance is

$$\vec{\nabla}\vec{\vec{\sigma}}_{\text{substrate}} = \vec{0} \tag{35}$$

We assume that the elasticity material is subjected to the boundary conditions (see previous subsection) :

$$\text{at z} = 0, \quad \sigma_{zz} = -\frac{\eta c\sqrt{2}}{a\epsilon^{3/2}}P, \quad \sigma_{zr} = \sigma_{z\theta} = 0. \tag{36}$$

In addition, we assume that the substrate is rigidly attached to a wall at $z = -h_{\text{sub}}$:

$$\text{at z} = -h_{\text{sub}}, \quad u_r = u_\theta = u_z = 0 \tag{37}$$

I wont do the full calculation here. In the thin limit when the thickness of the substrate is much smaller than the contact length, i.e. $h_{\text{sub}} \ll \sqrt{2aD}$, one can show that the elastic material can be considered as a assembly of independant springs and that the elasticity relation is local:

$$U_Z(Z=0) = \Delta = -\kappa P = -\underbrace{\frac{\sqrt{2}h_{\text{sub}}\eta c}{a^2\epsilon^{5/2}(2\mu+\lambda)}}_{\kappa} P \tag{38}$$

and that the horizontal displacements are null to the leading order

$$U_R = U_\theta = 0. \tag{39}$$

Note that the calculus can be found in Skotheim and Mahadevan 2005 for a 2D problem, using a scale separation technic and in some other papers (Li & Chou (1997) and Nogi & Kato (1997) cited in Leroy and Charlaix 2011 JFM) that compute the Green function after introducing the Hankel transform (same method as the one developped in the appendix of this note but with the constitutive equation (34)).

3.3 Response of an incompressible material

For an incompressible material (Poisson coefficient strictly equal to 1/2, i.e. $\nu = 1/2$), the second Lamé coefficient $\lambda \to \infty$, and the stress/strain relationship becomes

$$\vec{\vec{\sigma}}_{\text{substrate}} = -\mathcal{P}\vec{\vec{I}} + \mu(\vec{\nabla}\vec{u} + \vec{\nabla}\vec{u}^T) \tag{40}$$

with $\mathcal{P}$ a Lagrange multiplier introduce to enforce the incompressibility condition (similar to the pressure in fluids).

$$\vec{\nabla}.\vec{u} = 0 \tag{41}$$

I notice that the equilibrium equation, $\vec{\nabla}\vec{\vec{\sigma}}_{\text{substrate}} = \vec{0}$, gives the same equation as the Stokes equation, for a incompressible Newtonian fluid. Here, we are using a Green function approach.

Since, the Green function $\vec{\mathcal{G}}$ is the displacement field at the interface of a elastic problem defined as

$$\vec{\vec{\sigma}}_{\text{substrate}} = -\mathcal{P}\vec{\vec{I}} + \mu(\vec{\nabla}\vec{u}_{\text{green}} + \vec{\nabla}\vec{u}_{\text{green}}^T), \quad \text{and} \quad \vec{\nabla}.\vec{u}_{\text{green}} = 0 \tag{42a}$$

$$\forall R \in \mathbb{R}^+, \quad \forall\theta \in [0, 2\pi[, \quad \forall Z \in [-H_{\text{sub}}, 0], \quad \vec{\nabla}.\vec{\vec{\sigma}}_{\text{substrate}} = \vec{0} \tag{42b}$$

$$\text{at } Z = -H_{\text{sub}}, \quad \vec{u}_{\text{green}} = 0 \tag{42c}$$

$$\text{at } Z = 0, \quad \sigma_{ZZ} = -\delta(\vec{r}), \quad \sigma_{ZR} = \sigma_{Z\theta} = 0 \tag{42d}$$

$$\vec{\mathcal{G}}(\vec{r}) = \vec{u}_{\text{green}}(\vec{r}, z = 0) \tag{42e}$$

Then the solution can be written as

$$\vec{u}(\vec{r}, z) = \int d^2\vec{r'}\vec{\mathcal{G}}(\vec{r} - \vec{r'}, z)p(\vec{r'}) \tag{43}$$

with $(\vec{r}, \vec{r'})$, 2D vectors on the interface, $d^2\vec{r'} = r'dr'd\theta'$. The Green function $\vec{\mathcal{G}}$ has been calculated explicitly in the Appendix. The orthoradial component of $\vec{\mathcal{G}}$ is null. We then have

$$\begin{aligned}\Delta(\vec{R}) = \Delta(R, \theta) &= \frac{u_z(z = 0)}{a\epsilon} \\ &= -\frac{\eta c}{2\pi a\mu\epsilon^2}\int d^2\vec{R}'P(\vec{R}')\int_0^\infty dK J_0(K|\vec{R} - \vec{R}'|)\frac{KH_{\text{sub}} - \sinh(KH_{\text{sub}})\cosh(KH_{\text{sub}})}{\cosh^2(KH_{\text{sub}}) + (KH_{\text{sub}})^2},\end{aligned} \tag{44}$$

$$U_R(\vec{R}) = \frac{u_r(R, \theta)}{a\sqrt{2\epsilon}} = -\frac{\eta c}{2\pi\sqrt{2}\mu\epsilon^{3/2}}\int d^2\vec{R}'P(\vec{R}')\vec{e}_r.\vec{e}_{r-r'}\int_0^\infty dK J_1(K|\vec{R} - \vec{R}'|)\frac{(KH_{\text{sub}})^2}{\cosh^2(KH_{\text{sub}}) + (KH_{\text{sub}})^2}, \tag{45}$$

$$U_R(\vec{R}) = -\frac{\eta c}{2\pi\sqrt{2}\mu\epsilon^{3/2}}\int d^2\vec{R}'P(\vec{R}')\frac{R - R'\cos(\theta - \theta')}{|\vec{R} - \vec{R}'|}\int_0^\infty dK J_1(K|\vec{R} - \vec{R}'|)\frac{(KH_{\text{sub}})^2}{\cosh^2(KH_{\text{sub}}) + (KH_{\text{sub}})^2}, \tag{46}$$

and for the orthoradial part

$$U_\theta(\vec{R}) = \frac{u_\theta(R, \theta)}{a\sqrt{2\epsilon}} = -\frac{\eta c}{2\pi\sqrt{2}\mu\epsilon^{3/2}}\int d^2\vec{R}'P(\vec{R}')\vec{e}_\theta.\vec{e}_{r-r'}\int_0^\infty dK J_1(K|\vec{R} - \vec{R}'|)\frac{(KH_{\text{sub}})^2}{\cosh^2(KH_{\text{sub}}) + (KH_{\text{sub}})^2}, \tag{47}$$

$$U_\theta(\vec{R}) = \frac{u_\theta(R, \theta)}{a\sqrt{2\epsilon}} = -\frac{\eta c}{2\pi\sqrt{2}\mu\epsilon^{3/2}}\int d^2\vec{R}'P(\vec{R}')\frac{R'\sin(\theta - \theta')}{|\vec{R} - \vec{R}'|}\int_0^\infty dK J_1(K|\vec{R} - \vec{R}'|)\frac{(KH_{\text{sub}})^2}{\cosh^2(KH_{\text{sub}}) + (KH_{\text{sub}})^2}, \tag{48}$$

with $H_{\text{sub}} = h_{\text{sub}}/\sqrt{2\epsilon}a$, a dimensionless substrate thickness.

3.4 Fourier transform expression in the incompressible case.

Following Appendix D, the deformation can be written as a function of its inverse Fourier transform (I keep dimensionnal variable here).

$$u_z(z=0) = \int_{\mathbb{R}^2} \mathrm{d}k_x \mathrm{d}k_y \, \hat{p}(k_x, k_y) \hat{\mathcal{G}}(k_x, k_y) \exp(-i(k_x x + k_y y) \tag{49}$$

with

$$\hat{p}(k_x, k_y) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathrm{d}x \mathrm{d}y \, p(x, y) \exp(i(k_x x + k_y y), \tag{50}$$

$$\hat{\mathcal{G}}(k_x, k_y) = \frac{k h_{\text{sub}} - \sinh(k h_{\text{sub}}) \cosh(k h_{\text{sub}})}{2\mu k[\cosh^2(k h_{\text{sub}}) + (k h_{\text{sub}})^2]}, \qquad k = \sqrt{k_x^2 + k_y^2} \tag{51}$$

This expression can be rewrite as a function of a convolution product as

$$u_z(z=0) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \mathrm{d}^2 \vec{r}' p(\vec{r}') \mathcal{G}(\vec{r} - \vec{r}') \tag{52}$$

with

$$\mathcal{G}(\vec{r}) = \int_0^\infty \mathrm{d}k \, \frac{k h_{\text{sub}} - \sinh(k h_{\text{sub}}) \cosh(k h_{\text{sub}})}{2\mu[\cosh^2(k h_{\text{sub}}) + (k h_{\text{sub}})^2]} J_0(|\vec{r}|). \tag{53}$$

Then for the specific case of a pressure field that can be write as

$$p(\vec{r}) = f(r) \cos(n\theta), \tag{54}$$

the deformation is then

$$u_z(r, \theta, z=0) = \underbrace{\int_{\mathbb{R}^+} k \mathrm{d}k \, \tilde{f}(k) \hat{\mathcal{G}}_z(k) J_n(kr)}_{F(r)} \cos(n\theta) = F(r) \, \cos(n\theta) \tag{55a}$$

$$\tilde{f}(k) = \int_{\mathbb{R}^+} r \mathrm{d}r \, f(r) J_n(kr) \tag{55b}$$

Given that in the perturbation analysis, the pressure field is of the form $P_0 = p(r)\cos(\theta)$ we will use the second expression of the deformation directly in the following and to define the elasticity parameter κ.

$$u_z(r, \theta, z=0) = \int_{\mathbb{R}^+} k \mathrm{d}k \, \frac{k h_{\text{sub}} - \sinh(k h_{\text{sub}}) \cosh(k h_{\text{sub}})}{2k[\cosh^2(k h_{\text{sub}}) + (k h_{\text{sub}})^2]} J_1(kr) \int_{\mathbb{R}^+} r' \mathrm{d}r' \, p(r') J_1(kr') \, \cos(\theta) \tag{56}$$

which gives in dimensionless form, *i.e.* $p(r) = \frac{\eta c \sqrt{2}}{a \epsilon^{3/2}} P(R)$, and $r = a\sqrt{2\epsilon} R$

$$\begin{aligned} \Delta(R, \theta) &= \frac{u_z(r, \theta, z=0)}{a\epsilon} \\ &= \underbrace{\frac{2\eta c}{\mu a \epsilon^2}}_{\kappa} \int_{\mathbb{R}^+} \mathrm{d}K \, \frac{K H_{\text{sub}} - \sinh(K H_{\text{sub}}) \cosh(K H_{\text{sub}})}{2[\cosh^2(K H_{\text{sub}}) + (K H_{\text{sub}})^2]} J_1(KR) \int_{\mathbb{R}^+} R' \mathrm{d}R' \, P(R') J_1(KR') \, \cos(\theta) \end{aligned} \tag{57}$$

For a semi-infinite material, the expression $\frac{KH_{\rm sub}-\sinh(KH_{\rm sub})\cosh(KH_{\rm sub})}{2[\cosh^2(KH_{\rm sub})+(KH_{\rm sub})^2]} \to -\frac{1}{2}$ and this expression is simplified as

$$\Delta(R,\theta) = -\underbrace{\frac{2\eta c}{\mu a \epsilon^2}}_{\kappa}\int_{\mathbb{R}^+} \mathrm{d}K\,\frac{1}{2}J_1(KR)\int_{\mathbb{R}^+} R'\mathrm{d}R'\,P(R')J_1(KR')\,\cos(\theta) \tag{58}$$

For a thin compressible material, the Green function $\hat{\mathcal{G}}(k)$ is equalivalent to $-\frac{h_{\rm sub}}{(\lambda+2\mu)}$ (e.g. *Leroy & Charlaix 2011* JFM for instance). Then, the expression becomes

$$u_z(r,\theta,z=0) = -\int_{\mathbb{R}^+} k\mathrm{d}k\,\frac{h_{\rm sub}}{2\mu+\lambda}J_1(kr)\int_{\mathbb{R}^+} r'\mathrm{d}r'\,p(r')J_1(kr')\,\cos(\theta) \tag{59}$$

we can use the orthogonality relation for J_1,

$$\int_0^\infty k\mathrm{d}k\,J_1(kr)J_1(kr') = \frac{\delta(r-r')}{r} \tag{60}$$

and we recover the Winkler foundation expression

$$u_z(r,\theta,z=0) = -\frac{h_{\rm sub}}{2\mu+\lambda}\int_{\mathbb{R}^+} r'\mathrm{d}r'\frac{\delta(r-r')}{r}p(r')\cos(\theta) = \frac{p(r')h_{\rm sub}}{2\mu+\lambda} \tag{61}$$

which gives in dimensionless form

$$\Delta(\vec{R}) = -\underbrace{\frac{\eta c\sqrt{2}h_{\rm sub}}{(2\mu+\lambda)a^2\epsilon^{5/2}}}_{\kappa}P(\vec{R}) \tag{62}$$

which is exactly the same expression as the one written above.

4 Perturbation analysis

4.1 Definition

The set of equations (22) and with an compressible substrate (Eqs.(38),(39)) or incompressible (Eqs. (44),(45),(48)) are highly complicated and non-linear. The idea is to solve this equation using perturbative method as in all the previous articles. The total fluid thickness $H = H_0 - \Delta$ that arise in the equation is written as the expended as

$$H = H_0 + \kappa H_1 \quad (= H_0 - \Delta) \tag{63}$$

with $\kappa \ll 1$ a small parameter and $H_1 = \mathcal{O}(1)$. The parameter κ will depend on the material elastic response. From the last section, we have :

$$\kappa = \frac{\sqrt{2} h_{\mathrm{sub}} \eta c}{a^2 \epsilon^{5/2} (2\mu + \lambda)} \quad \text{and} \quad H_1 = P_0 \qquad \text{for an compressible material.} \tag{64}$$

$$\kappa = \frac{2\eta c}{a \epsilon^2 \mu} \quad \text{and} \quad H_1 = \sum_{k \in \mathbb{N}} H_1^{(k)} \qquad \text{for a semi infinite material} \tag{65}$$

$$H_1^{(k)} = \cos(k\theta) \int_{\mathbb{R}^+} \mathrm{d}K \, \frac{J_k(KR)}{2} \int_{\mathbb{R}^+} R' \mathrm{d}R' \, P_0^{(k)}(R') J_k(KR') \tag{66}$$

For the semi infinite material, I used the mode decomposition. $P_0^{(k)}$ is kth the mode (see previous part), *i.e.* $P_0^{(k)}(R) = \frac{1}{\pi} \int_{-\pi}^{\pi} P_0(R,\theta) \cos(k\theta) \mathrm{d}\theta$.

The substrate deformation is the cause of the non antisymmetry in the pressure fields (i.e. $p(x) \neq -p(-x)$). Thus, we can anticipate that the leading order solution will be antisymmetric and will not exhibit a lift force. Such effects will arise at next leading order. The pressure field is also expanded as well as the radial deformation of the substrate.

$$P = P_0 + \kappa P_1 + \kappa^2 P_2 + ... \tag{67}$$

Note that the boundary condition of the pressure field is

$$P_0(R \to \infty) = 0 = P_1(R \to \infty) = 0 \tag{68}$$

4.2 Leading order pressure.

Then the leading order in $\kappa \to 0$ of the Eq. (22) is

$$12\dot{D} = \frac{1}{R} \partial_R \left[R \partial_R P_0 H_0^3 - 6 R H_0 \cos(\theta) U \right] + \frac{1}{R} \partial_\theta \left[\frac{1}{R} \partial_\theta P_0 H_0^3 + 6 H_0 \sin(\theta) U \right] \tag{69}$$

that can be simplified a little bit and rewriten as

$$R^2 \partial_R^2 P_0 + (R + \frac{6R^3}{H_0}) \partial_R P_0 + \partial_\theta^2 P_0 = \mathcal{L}.P_0 = R^2 \, \frac{12\dot{D} + 12 U \cos(\theta) R}{H_0^3} \tag{70}$$

with $\mathcal{L}$ a linear operator defined as

$$\mathcal{L}. = R^2 \partial_R^2 . + (R + \frac{R^3}{H_0(R)}) \partial_R . + \partial_\theta^2 . \tag{71}$$

We recall that $H_0(R) = D + R^2$. Inspired by the solution of all the previous papers, we are looking for a solution of the form

$$P_0(R,\theta) = \frac{\alpha + \beta\cos(\theta)R}{H_0^2} \tag{72}$$

With this assumption we can see that such solution exist with $\alpha = -3/2\dot{D}$ and $\beta = 6/5U(t)$. Then the leading order pressure is

$$P_0(R,\theta) = \frac{-15\dot{D} - 12U(t)\cos(\theta)R}{10(D+R^2)^2} \tag{73}$$

Justification of this solution: Usually, in linear equation, the superposition principle applies and we must write the solution as the sum of a "homogeneous" solution (i.e. solution of $\mathcal{L}.P_0 = 0$) and a particular solution that take into account the right hand side of the equation : $P_0 = P_0^{\text{homo}} + P_0^{\text{part}}$. We can use a seperation of variable technic to solve the equation.

$$P_0(R,\theta) = F(R)G(\theta) \tag{74}$$

and therefore

$$\mathcal{L}P_0 = 0 \quad \rightarrow \quad G''(\theta) + \frac{R^2F''(R) + (R + 6R^3/H_0)F'(R)}{F(R)}G = 0 \tag{75}$$

The angular dependance is trivial : solution of the type $\theta \rightarrow e^{im\theta}$. Given the periodicity, i.e.

$$P_0(R,\theta + 2\pi) = P_0(R,\theta), \tag{76}$$

we can write the solution as a sum of modes

$$P_0(R,\theta) = \sum_{k\in\mathbb{N}} \Big(f_k(R)\cos(k\theta) + g_k(R)\sin(k\theta) \Big). \tag{77}$$

We recall that the modes are orthogonal for the scalar product

$$\forall (k,k') \in \mathbb{N}^2, \quad < f_k, f_{k'} >= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta f_k \cos(k\theta) f_{k'}\cos(k'\theta) = \frac{\delta_{k,k'}}{2} f_k^2. \tag{78}$$

$$\forall (k,k') \in \mathbb{N}^2, \quad < g_k, g_{k'} >= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta g_k \sin(k\theta) g_{k'}\sin(k'\theta) = \frac{\delta_{k,k'}}{2} g_k^2. \tag{79}$$

$$\forall (k,k') \in \mathbb{N}^2, \quad < f_k, g_{k'} >= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta f_k \cos(k\theta) g_{k'}\sin(k'\theta) = 0 \tag{80}$$

Written as this, the amplitude of the mode follows the equations

$$\forall k \geqslant 2, \quad R^2 f_k''(R) + (R + \frac{6R^3}{D+R^2})f_k'(R) - k^2 f_k = 0 \tag{81}$$

$$\forall k \in \mathbb{N}^*, \quad R^2 g_k''(R) + (R + \frac{6R^3}{D+R^2})g_k'(R) - k^2 g_k = 0, \qquad \text{and} \qquad g_0 = 0 \tag{82}$$

$$R^2 f_1''(R) + (R + \frac{6R^3}{D+R^2})f_1'(R) - f_1 = R^2\frac{12UR}{(D+R^2)^3} \tag{83}$$

$$R^2 f_0''(R) + (R + \frac{6R^3}{D+R^2}) f_0'(R) = R^2 \frac{12\dot{D}}{(D+R^2)^2} \tag{84}$$

Each single equation can be solved. The homogeneous solution for each mode $k \geq 1$ is equal to the sum of a hypergeometric function and G-Meijer function (you can get them with formal calculus software such as Mathematica). Nevertheless the G-Meijer function has value in $\mathbb{C}$, and the Hypergeometric one diverges at $R \to \infty$. Then given the boundary condition on P_0

$$P_0(R \to \infty, \theta) = 0, \tag{85}$$

we can affirm that the homogeneous solution is null. Therefore, the solution is indeed the particular one exhibit in the section. We will write this solution as

$$P_0(R,\theta) = P_0^{(0)}(R) + P_0^{(1)}(R)\cos(\theta) = \frac{-3\dot{D}}{2(D+R^2)} - \frac{6UR}{5(D+R^2)^2}\cos(\theta) \tag{86}$$

where the superscript denote the corresponding mode.

Additionnal remark : Actually the homogeneous solution of the isotrope mode, *i.e.* $k = 0$, is not of the same form as the others.

The solution of the system

$$R^2 f''(R) + (R + \frac{6R^3}{1+R^2}) f'(R) = 0, \tag{87}$$

is

$$f : R \to K_1 + K_2 \left(\frac{3+2R^2}{4(1+R^2)^2} + \log(R) - \frac{1}{2}\log(1+R^2) \right) \tag{88}$$

with K_1, K_2 two constant. If we add the conditions $P(R \to \infty) = 0$ and $P(R = 0) < \infty$, these two constants are null and we recover that the homogeneous solution vanishes.

4.3 Next-leading order.

In what follows, I will consider the simpler case of a quasistatic approach of the substrate to the sphere such that $\dot{D}$ can be taken as zero. In fact, this assumption is quite tricky because we are measuring the position of the cantilever so in some way, we are measuring $D(t)$. However, the time dependance of D is not necessary to extract the lift force. This hydrodynamic force goes as $U(t)^2 \sim \cos^2(\omega t)$ and therefore as a zero frequency signature. We then take $D = 1$ or $H_0(R) = 1 + R^2$. The other major consequence is that only the mode $k = 1$ is taken into account in P_0 and not the $k = 0$ mode.

At next leading order, the Equation. (22) yields to

$$\begin{aligned}\partial_T H_1 =& \frac{1}{12\,R}\partial_R \Big[R\partial_R P_1 H_0^3 + 3R\partial_R P_0 H_0^2 H_1 - 6RH_1\cos(\theta)U - 6RH_0\partial_T U_R \Big] \\ &+ \frac{1}{12\,R}\partial_\theta \Big[\frac{1}{R}\partial_\theta P_1 H_0^3 + \frac{3}{R}\partial_\theta P_0 H_0^2 H_1 + 6H_1\sin(\theta)U - 6H_0\partial_T U_\theta \Big]\end{aligned} \tag{89}$$

That can be expand isolating P_1

$$\begin{aligned}R^2\partial_R^2 P_1 + (R+\frac{6R^3}{H_0})\partial_R P_1 + \partial_\theta^2 P_1 =& 12\frac{R^2}{H_0^3}\partial_T H_1 - H_1\Bigg(3\frac{R^2\partial_R^2 P_0 + (R+\frac{4R^3}{H_0})\partial_R P_0 + \partial_\theta^2 P_0}{H_0}\Bigg)\\ &+\partial_R H_1\Bigg(-3\frac{R^2\partial_R P_0}{H_0}+6\frac{UR^2\cos(\theta)}{H_0^3}\Bigg)\\ &+\partial_\theta H_1\Bigg(-3\frac{\partial_\theta P_0}{H_0}-6\frac{UR\sin(\theta)}{H_0^3}\Bigg)\end{aligned} \tag{90}$$

We notice that we recover the same operator, $\mathcal{L}$, as in the equation for the leading order pressure P_0 which is consistant. Using the solution P_0 written is the previous section, and after some heavy calculations, one find that the terms in the large brackets are equal to

$$\begin{aligned}\mathcal{L}.P_1 =& 12\frac{R^2}{H_0^3}\partial_T H_1 - H_1\frac{72R^3(3+R^2)U\cos(\theta)}{5H_0^5}\\ &+\partial_R H_1\frac{24R^2(2-R^2)U\cos(\theta)}{5H_0^4}\\ &-\partial_\theta H_1\frac{48RU\sin(\theta)}{5H_0^3}\end{aligned} \tag{91}$$

4.4 Compressible material

For a compressible material, I will solve analytically the pressure field P_1 following the same kind of method as in Urzay 2007. We can use directly the latter equation with $H_1 = P_0$ and $U_R = U_\theta = 0$. This yields to

$$\begin{aligned}\mathcal{L}.P_1 =& 12\frac{R^2}{H_0^3}\partial_T P_0 - P_0\frac{72R^3(3+R^2)U\cos(\theta)}{5H_0^5}\\ &+\partial_R P_0\frac{24R^2(2-R^2)U\cos(\theta)}{5H_0^4}\\ &-\partial_\theta P_0\frac{48RU\sin(\theta)}{5H_0^3}\end{aligned} \tag{92}$$

which gives

$$\mathcal{L}P_1 = \frac{144R^2(-2+6R^2-R^4)U^2}{25(1+R^2)^7} - \frac{72R^3\dot{U}}{5(1+R^2)^5}\cos(\theta) + \frac{144R^2(10+R^2)U^2}{25(1+R^2)^7}\cos(2\theta) \tag{93}$$

Note that we have used the following trigonometric relationships

$$\cos(\theta)^2 = \frac{1+\cos(2\theta)}{2}, \qquad \sin(\theta)^2 = \frac{1-\cos(2\theta)}{2} \tag{94}$$

in order to write the RHS of the equation as a sum of three orthogonal modes

$$\mathcal{L}P_1 = K_0(R) + K_1(R)\cos(\theta) + K_2(R)\cos(2\theta) \tag{95}$$

Using the same argument as for the leading order pressure, we can write the pressure P_1 as

$$P_1 = P_1^{(0)}(R) + P_1^{(1)}(R)\cos(\theta) + P_1^{(2)}(R)\cos(2\theta) \tag{96}$$

with $P_1^{(i)}(R)$ solution of the equation

$$R^2 \frac{\mathrm{d}^2}{dR^2} P_1^{(i)} + (R + \frac{6R^3}{D + R^2)} \frac{\mathrm{d}}{dR} P_1^{(i)} - i^2 P_1^{(i)} = K_i(R) \tag{97}$$

Each of this mode can be solved. The most important one is the zeroth order pressure that is the only one that contribute to the lift force

$$P_1^{(0)}(R) = U^2 \frac{9(7 - 5R^2)}{125(1 + R^2)^5} \tag{98}$$

which is obtained solving the mode equation with the boundary condition $P(R \to \infty) = 0$ and $P(R = 0) < \infty$.

Remark I don't know why but at this stage of the calculation, Urzay et al. wrote that the boundary condition to solve the mode equation (without clear justification) was $P_1^{(0)}(R \to \infty) = 0$ and $\frac{d}{dR} P_1^{(0)}(R = 0) = 0$. The solution that they found is the same as the one that I consider but the boundary condition written in their paper come out of nowhere to me.

Possible answer to this question: I assume that the reason is that if one evaluate both side of Eq. (22) of Urzay et al. at $R \to 0$, then we obtain $\gamma'(0) \to 0$. Anyway it does not seem very rigorous mathematically speaking and unclear in the paper.

4.5 Incompressible material

For the reader: all what is in between the two horizontale line can be skiped (it is just to keep a track on what I've done before finding a direct solution).

The relationship between the deformation (H_1, U_R, U_θ) and the leading order pressure P_0 are highly complicated. We will focus on the vertical displacement that is given by

$$\begin{aligned} H_1(\vec{R}) &= H_1(R, \theta) \\ &= \int d^2\vec{R}' P_0(\vec{R}') \int_0^\infty dK J_0(K|\vec{R} - \vec{R}'|) \frac{KH_{\text{sub}} - \sinh(KH_{\text{sub}})\cosh(KH_{\text{sub}})}{\cosh^2(KH_{\text{sub}}) + (KH_{\text{sub}})^2} \\ &= \int d^2\vec{R}' P_0(\vec{R}') \mathcal{G}_Z(|\vec{R} - \vec{R}'|) \end{aligned} \tag{99}$$

with $\mathcal{G}_Z$ the Green function, and H_{sub} the dimensionless substrate thickness. One can show that in the limite of a semi-infinite substrate, the Green function that the simpler form $\mathcal{G}_Z(R) = 1/R$. In this limit, the radial and Green function, $\mathcal{G}_R$ (see Appendix A) vanishes and thus the in plane displacements, (U_R, U_θ) can be neglected. Then, the deformation H_1 is

$$H_1(\vec{R}) = \int d^2\vec{R}' \frac{P_0(\vec{R}')}{|\vec{R} - \vec{R}'|} \tag{100}$$

In cylindrical coordinates, this gives

$$\begin{aligned} H_1(R, \theta) &= \int R' dR' d\theta' \frac{P_0(R', \theta')}{\sqrt{R^2 + R'^2 - 2RR' \cos(\theta - \theta')}} \\ &= -\frac{6U}{5} \int R' dR' d\theta' \frac{R' \cos(\theta'))}{(1 + R'^2)} \frac{1}{\sqrt{R^2 + R'^2 - 2RR' \cos(\theta - \theta')}} \end{aligned} \tag{101}$$

No matter what variable is integrate in the first place, *Mathematica* cannot find annalytical solution of this equations. I also tried to rewrite this equation in cartesian coordinate

$$H_1(x,y) = -\frac{6U}{5}\int dx'dy' \frac{x'}{(1+x'^2+y'^2)} \frac{1}{\sqrt{(x-x')^2+(y-y')^2}} \tag{102}$$

and it does not help at all with *Mathematica* which confirms that the deformation cannot be simply written as a polygonal fraction as in the Winkler foundation (or in the 2D case as in Skotheim & Mahadevan 2005 *Phys. Fluids.*). Then, I discretise space and write the integrale as the Riemman sum. I defined the shifted grids for the integration variables, (R', θ'), and the real variables (R, θ) in order to avoid divergence at $\vec{R} = \vec{R}'$ of the part "$\frac{1}{|\vec{R}-\vec{R}'|}$" of the integrale (see schematic in Figure 2(c)).

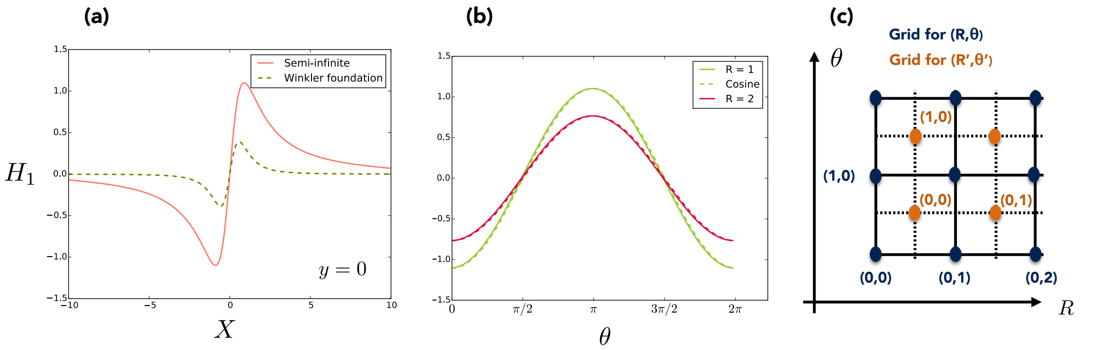

Figure 2: Deformation of a semi-infinite substrate subjected to the pressure field P_0. Panel (a) : the deformation is plotted along the axis of the motion of the sphere (solid line). In order to compare with the previous case, I also plotted the deformation for a thin compressible material in dashed line. Panel (b) : the same quantity is plotted in cylindrical coordinate at fixed radius versus the angle. A cosine function $H(\theta) = H(\theta = 0)\cos(\theta)$ is plotted is dashed line and seems to be a good approximation (no fit). I must recall that H_1 is defined as the opposite of the real deformation Δ with a proportionnality factor κ, *i.e.* $\Delta = -\kappa H_1$ which is why it is the opposite of what we would expect. Panel (c) : Schematic of the grid used in the integration scheme to avoid the divergence of the Green function.

In the Figure 2(a) left, I plotted the deformation of the substrate along the axis of the motion (at the center $y = 0$). We recover the same shape as in the cylindrical case. For your information, I also plotted the deformation of a material with local elasticity (Winkler fondation) is dashed lines, which caracterizes a thin incompressible material submitted to the same pressure field P_0. We can see that the lateral extension of the deformation is larger for the semi-infinite material which make sense. Note that $H_1 = -\Delta$ and that this graph corresponds to $U = 1$.

In Figure 2(b) right, I plotted the same quantity in cylindrical coordinates at fixed radius to observe the angular dependance of the deformation. I anticipate that the solution would have a symmetry around $\pi/2$ (y-axis), but it is quite remarkable that the profile $H_1(R,\theta)$ can be decomposed as

$$H_1(R,\theta) = F(R)\cos(\theta) \tag{103}$$

which was the same thing for the Winkler foundation. Therefore, the procedure in the Appendix B might actually works. I will try again in the future. The function F can be expanded as

$$F(R) = -\frac{6U}{5}\sum_{k=0}^{\infty}(\alpha_k(R) + \beta_k(R))\gamma_k \tag{104}$$

with the three auxilary series defined as

$$\alpha_k(R) = \int_0^R \mathrm{d}R' \frac{R'^2}{R(1+R'^2)^2}\left(\frac{R'}{R}\right)^{2k+1} \tag{105a}$$

$$\beta_k(R) = \int_R^\infty \mathrm{d}R' \frac{R'}{(1+R'^2)^2}\left(\frac{R}{R'}\right)^{2k+1} \tag{105b}$$

$$\gamma_k = \int_0^{2\pi} \mathrm{d}\tilde{\theta}\cos(\tilde{\theta})P_{2k+1}(\cos(\tilde{\theta})) \tag{105c}$$

We can plug the leading order pressure directly into the equation (65) and (66) and we find

$$H_1(R,\theta) = U\,F(R)\cos(\theta) \tag{106}$$

with

$$\begin{aligned}F(R) &= -\frac{6}{5}\int_{\mathbb{R}+}\mathrm{d}K\,\frac{J_1(KR)}{2}\int_{\mathbb{R}+}R'\mathrm{d}R'\,\frac{R'}{(1+R'^2)^2}J_1(KR') \\ &= -\frac{6}{5}\int_{\mathbb{R}+}\mathrm{d}K\,\frac{J_1(KR)}{2}\frac{K}{2}\,K_0(K) \\ &= -\frac{3}{10}\left(\frac{K(-R^2)}{R} - \frac{E(-R^2)}{R(1+R^2)}\right)\end{aligned} \tag{107}$$

(I did not make the proof of this solution but I found it with *Mathematica*). This solution is plotted in Figure 3.

Therefore, I can plug this into the Eq. (91). Again, this equation can be written as the sum of three modes

$$\mathcal{L}P_1 = K_0^{\mathrm{semi}}(R) + K_1^{\mathrm{semi}}(R)\cos(\theta) + K_2^{\mathrm{semi}}(R)\cos(2\theta) \tag{108}$$

and the solution written as

$$P_1^{\mathrm{semi}} = P_1^{(0)}(R) + P_1^{(1)}(R)\cos(\theta) + P_1^{(2)}(R)\cos(2\theta) \tag{109}$$

The only mode that will contribute to the lift force is the mode $i = 0$ for which the evolution is given by

$$\begin{aligned}R^2P_1^{(0)}(R)'' + \left(R + \frac{6R^3}{1+R^2}\right)P_1^{(0)}(R)' &= K_0^{\mathrm{semi}}(R) \\ &= UF(R)\left(\frac{12U}{5}\frac{R(2-5R^2-R^4)}{H_0^5}\right) + UF'(R)\left(\frac{12U}{5}\frac{R^2(2+R^2-R^4)}{H_0^5}\right) \\ &= U^2\,\frac{18R^2\Big((8+7R^2-R^4)K(-R^2) - 2(5-R^2)E(-R^2)\Big)}{25(1+R^2)^6}\end{aligned} \tag{110}$$

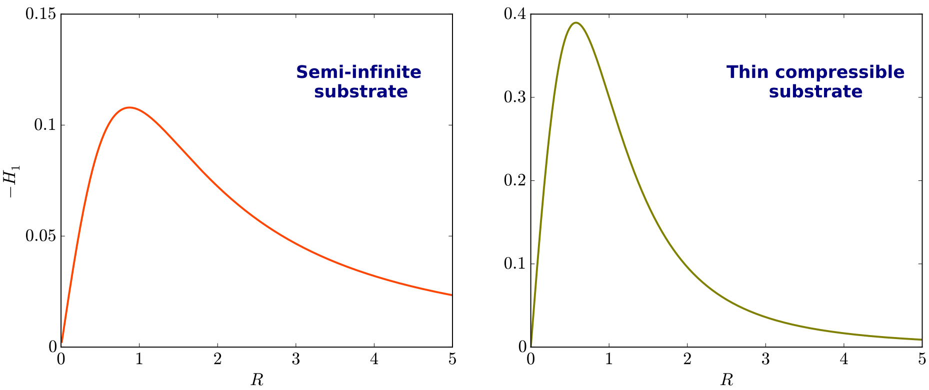

Figure 3: Deformation of a semi-infinite substrate subjected to the pressure field $P_0 = -\frac{6R}{5(1+R^2)^2}\cos(\theta)$. In the linear elasticity framework, the deformation is of the form $H_1 = F(R)\cos(\theta)$. The radial function F is the one plotted here **(a)** for a semi-infinite substrate and **(b)** : for a thin compressible substrate (Winkler foundation). Note that the y scale is not the same in both panel but it does not make sense to compare them because everything is dimensionless. However, one can observe that the radial extension of the deformation is larger in the semi infinite substrate which make sense.

Mathematica cannot solve this equation analytically so I solve this equation numerically using a 4th order Runge-Kutta scheme (with $U = 1$). I have to introduce boundary conditions for the numerical integration. Given the fact that the two homogeneous solution of the equation are not compatible with the boundary conditions, we can either evalutate both side of the equation at $R \gg 1$ or $R \ll 1$ and use this evalutation as a boundary. The behavior of F at large R is not trivial (it goes as $\log(R)/R^2$) so that I choose to use the other limit. At $R \ll 1$, the function F is equivalent to $F(R) \sim F'(0)\,R$, with $F'(0)$ a finite constante (see Appendix : $F'(0) = -\frac{3\pi}{40}$). Therefore, the RHS of the last equation, $K_0^{\text{semi}}(0)$ is equivalent to $\sim -\frac{18\pi}{25}R^2$ at small radius. We can assume that the next leading order pressure $P_1^{(0)}$ is of the form

$$P_1^{(0)}(R) = A + BR^2 \tag{111}$$

when $R \to 0$ with $A = P_1^{(0)}(R=0)$ and $B = \frac{\mathrm{d}^2}{\mathrm{d}R^2}P_1^{(0)}(R=0)/2$ two constants. With the limit $R \to 0$, we can determine B as

$$B = \frac{9}{50}\pi. \tag{112}$$

The equation that we solve is actually a 1st order differential equation for P'. Thus it require only one "initial condition" for P'. The other initial condition, *i.e.* $P(0)$, will just shift vertically the solution. This is some sort of shooting parameter that we will determine to satisfy the boundary condition $P_1^{(0)}(R \to \infty) = 0$.

The numerical solution is plotted in Figure 4. As a comparison, I also plotted the numerical solution of the other elastic model. Note that we have a analytical solution for the Winkler solution and that the numeric/analytic cannot be distinguish which validate the numerical scheme.

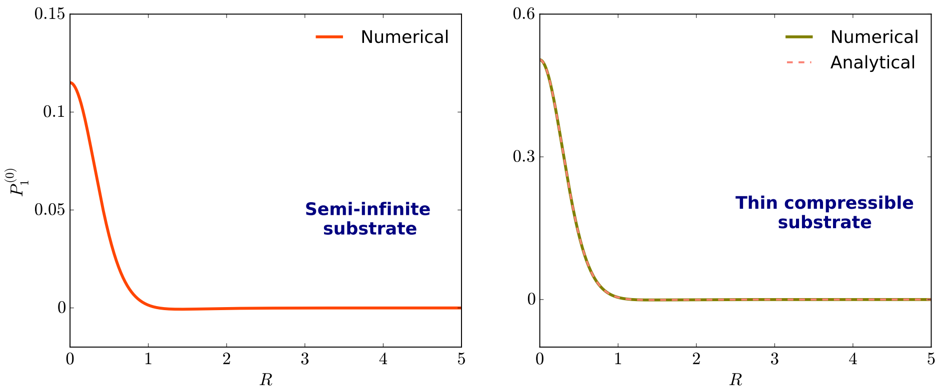

Figure 4: **Left :** Numerical solution of the equation (110). **Right :** Numerical & analytical solution for a thin compressible material (see previous section). I add this plot to confirm that the numerical scheme that I was using was correct. Note that the y scale is not the same in both panel but it does not make sense to compare them because everything is dimensionless.

5 Force expression in the asymptotic limit.

5.1 General expression

Once we have an expression of the pressure field, we can compute the leading order force, $\vec{F}$ that is exerted on the sphere by the flow. This latter is given by

$$\vec{F} = \int \vec{\vec{\sigma}}_{\text{fluid}} \vec{n} \mathrm{d}S \tag{113}$$

where $\vec{n} = \left(r/a,\, 0,\, -\sqrt{1-(r/a)^2} \right)_{\{r,\theta,z\}} = \left((2\epsilon)^{1/2} R, 0, 1 \right)_{\{r,\theta,z\}} + \mathcal{O}(\epsilon)$ is the unit normal vector of the surface, and $\mathrm{d}S = r\mathrm{d}r\mathrm{d}\theta = 2\epsilon R a^2 \mathrm{d}R\mathrm{d}\theta$ is the elementary surface in cylindrical coordinates. Finally, the fluid stress tensor $\vec{\vec{\sigma}}_{\text{fluid}}$ is given by the Newtonnian rheological law

$$\vec{\vec{\sigma}}_{\text{fluid}} = -p\vec{\vec{I}} + \eta(\vec{\nabla}\vec{v} + \vec{\nabla}\vec{v}^T) \tag{114}$$

Then, we can project Eq.(113) on to the vertical and horizontal coordinates and we find that

$$\begin{aligned} F_x &= \int \vec{e}_x \vec{\vec{\sigma}}_{\text{fluid}} \vec{n} \mathrm{d}S \\ &= \int \Big(\cos(\theta), \sin(\theta), 0 \Big)^T_{\{r,\theta,z\}} \Big(\sqrt{2\epsilon} R\sigma_{rr} - \sigma_{rz}, \sqrt{2\epsilon}\sigma_{r\theta} - \sigma_{\theta z}, [...] \Big)_{\{r,\theta,z\}} \mathrm{d}S \\ &= \int \Big[\cos(\theta)\Big(\sqrt{2\epsilon} R\sigma_{rr} - \sigma_{rz} \Big) + \sin(\theta)\Big(\sqrt{2\epsilon}\sigma_{r\theta} - \sigma_{\theta z} \Big) \Big] \mathrm{d}S \end{aligned} \tag{115}$$

$$\begin{aligned} F_z &= \int \vec{e}_z \vec{\vec{\sigma}}_{\text{fluid}} \vec{n} \mathrm{d}S \\ &= \int (\sqrt{2\epsilon} R\sigma_{rz} - \sigma_{zz}) \mathrm{d}S \end{aligned} . \tag{116}$$

We can then expand and take the leading order in the lubrication parameter, ϵ, and we get

$$F_x = -\eta c 2\sqrt{2}a \int \left(\sqrt{2}P + \partial_Z V_R\right)\cos(\theta) R\mathrm{d}R\mathrm{d}\theta + \mathcal{O}(\epsilon) \tag{117}$$

$$F_z = \eta c 2\sqrt{2}a\epsilon^{-1/2}\int P R\mathrm{d}R\mathrm{d}\theta + \mathcal{O}(\epsilon^{1/2}) \tag{118}$$

5.2 Leading order force

We write in this section the leading order force[4] (in the elastic parameter κ).

$$F_{x,0} = -\eta c 2\sqrt{2}a \int \left(\sqrt{2}P_0 + \partial_Z V_{R,0}\right)\cos(\theta) R\mathrm{d}R\mathrm{d}\theta \tag{119}$$

$$F_{z,0} = \eta c 2\sqrt{2}a\epsilon^{-1/2}\int P_0 R\mathrm{d}R\mathrm{d}\theta + \mathcal{O}(\epsilon^{1/2}) \tag{120}$$

We notice that it is not the same mode that contribute to the force horizontal and vertical expression.

$$\begin{aligned}
F_{x,0} &= -\eta c 2\sqrt{2}a \int \left(\sqrt{2}P_0^{(1)} + \partial_Z V_{R,0}^{(1)}\right)\cos(\theta)^2 R\mathrm{d}R\mathrm{d}\theta \\
&= -\eta c 2\sqrt{2}a\pi \int \left(\sqrt{2}P_0^{(1)} + \partial_R P_0^{(1)}\frac{H_0}{2} + \frac{U}{H_0}\right)R\mathrm{d}R, \qquad \left(\text{using} \quad \int_0^{2\pi}\cos^2(\theta) = \pi\right) \\
&= -\eta c 2\sqrt{2}a\pi U \int \left(\sqrt{2}\frac{18R}{5H_0^2} + \frac{9(D-3R^2)}{5H_0^2} + \frac{1}{H_0}\right)R\mathrm{d}R
\end{aligned} \tag{121}$$

Remark : The last term of the latter equation $\frac{R}{H_0(R)}$ is equivalent to $\frac{R}{H_0(R)} \underset{R\to\infty}{\sim} \frac{1}{R}$ at large R. Therefore, the integrale does not converge. There is not a physical problem because the interface profile cannot be approximate by H_0 at large R. As a result, I can't find the horizontal force with this approach. Nevertheless, it will not be such a big deal because this force is proportionnal to the velocity, U. And in the case of an oscillating velocity, i.e. $U(t) = x_0\cos(\omega t)$, this force is null on average.

Then the vertical force is

$$\begin{aligned}
F_{z,0} &= \eta c 2\sqrt{2}a\epsilon^{-1/2}\int P_0^0 R\mathrm{d}R\mathrm{d}\theta + \mathcal{O}(\epsilon^{1/2}) \\
&= \eta c 2\sqrt{2}a\epsilon^{-1/2}2\pi \int \frac{-3\dot{D}}{2H_0^2(R)}R\mathrm{d}R \\
&= \eta c 2\sqrt{2}a\epsilon^{-1/2}2\pi\frac{-3\dot{D}}{2}\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}R}\left(-\frac{1}{2}\frac{1}{D+R^2}\right)\mathrm{d}R \\
&= \eta c 2\sqrt{2}a\epsilon^{-1/2}2\pi\frac{-3\dot{D}}{4D} \\
&= -\frac{6\pi\eta a^2\dot{d}}{d}
\end{aligned} \tag{122}$$

[4] $V_{R,0}$ is the leading order velocity in the elastic parameter, i.e. $V_{R,0} = \frac{1}{2}\partial_R P_0\left(Z^2 - H_0 Z\right) + U(t)\cos(\theta)\frac{Z}{H_0} + \partial_T U_R\frac{H_0 - Z}{H_0}$, or the radial velocity Eq. (17) with $\Delta = 0$.

We recover the well known result from the 60's (papers of Cox, Brenner...). In the model, I computed the lift force in the case of a vanishing approaching speed $\dot{d}$. One could possibly apply the same method to get the contributing term in the next-leading order vertical force (would go as $\ddot{d}$ and $\dot{d}^2$).

5.3 Elastic contribution in a thin compressible material

The horizontal force is equal to

$$F_{x,1} = -\kappa\eta c 2\sqrt{2}a \int \left(\sqrt{2}P_1^{(1)} + \partial_Z V_{R,1}^{(1)}\right) \cos(\theta)^2 R\mathrm{d}R\mathrm{d}\theta \tag{123}$$

It does not worth compute it because this force is proportional to the mode $k = 1$ and therefore proportional to U (and not U^2). As a result, it would be zero on average.

The vertical one is

$$\begin{aligned} F_{z,1} &= \kappa\eta c 2\sqrt{2}a\epsilon^{-1/2} \int P_1^{(0)} R\mathrm{d}R\mathrm{d}\theta \\ &= \kappa\eta c 2\sqrt{2}a\epsilon^{-1/2} 2\pi \int_0^\infty P_1^{(0)} R\mathrm{d}R \\ &= 8\pi \frac{\eta^2 c^2 h_{\mathrm{sub}}}{(2\lambda+\mu)a} \left(\frac{a}{d}\right)^3 \int_0^\infty P_1^{(0)} R\mathrm{d}R \end{aligned} \tag{124}$$

With the pressure field found in the previous section for the Winkler foundation the numerical prefactor is

$$\int_0^\infty P_1^{(0)} R\mathrm{d}R = \int_0^\infty R\frac{9(7-5R^2)}{125(1+R^2)^5}\,\mathrm{d}R = \frac{6}{125} \tag{125}$$

which gives

$$F_{\mathrm{Lift}}^{\mathrm{Winkler}} = \frac{48\pi}{125}\,\frac{\eta^2 U^2 h_{\mathrm{sub}}}{(2\lambda+\mu)a}\left(\frac{a}{d}\right)^3 \sim 1.206\,\frac{\eta^2 U^2 h_{\mathrm{sub}}}{(2\lambda+\mu)a}\left(\frac{a}{d}\right)^3 \tag{126}$$

I recover the result in Urzay.

5.4 Elastic contribution in a incompressible material

The exact same equation can be used to compute the elastohydrodynamic force in a semi-incompressible material but with a different prefactor/scaling law since the elasticity parameters κ and the dimentionless pressure are different. I get

$$\begin{aligned} F_{\mathrm{Lift}}^{\mathrm{semi}} &= \kappa\eta c 2\sqrt{2}a\epsilon^{-1/2} \int P_1^{(0)} R\mathrm{d}R\mathrm{d}\theta \\ &= \frac{4\sqrt{2}\eta^2 c^2}{\mu}\left(\frac{a}{d}\right)^{5/2} 2\pi \int_0^\infty P_1^{(0)} R\mathrm{d}R \end{aligned} \tag{127}$$

The prefactor has been computed numerically and we have found

$$\int_0^\infty P_1^{(0)} R\mathrm{d}R = 0.01171.... \tag{128}$$

Then I get

$$F_{\mathrm{Lift}}^{\mathrm{Semi}} = 0.4162...\,\frac{\eta^2 v^2}{\mu}\left(\frac{a}{d}\right)^{5/2} \tag{129}$$

6 Non-linear solution

6.1 Mode decomposition

For the reader: the mode decomposition method in the following section did not give access to simpler numerical solution. Nevertheless, I keep it here in case I find an idea in the future.

In this section, I will write basic ideas to solve numerically the set of equations describe above for very soft substrate, when the elasticity parameter is large $\kappa = \mathcal{O}(1)$. Nevertheless, we are only consider linear elasticity model. I just make the following assumptions for simplicity

$$D = U = 1, \qquad U_R = U_\theta = 0 \tag{130}$$

which means that I consider a stationnary problem and with neglegible in-plane displacement at the substrate interface. Then the hydrodynamic equation becomes

$$\begin{aligned} -\cos\theta\partial_R\Delta + \sin\theta\frac{1}{R}\partial_\theta\Delta =& \frac{1}{12\,R}\partial_R\left[R\partial_R P(H_0-\Delta)^3 - 6R(H_0-\Delta)\cos(\theta)\right] + \\ & \frac{1}{12\,R}\partial_\theta\left[\frac{1}{R}\partial_\theta P(H_0-\Delta)^3 + 6(H_0-\Delta)\sin(\theta)\right] \end{aligned} \tag{131}$$

or rewritten as

$$\partial_R\left[R\partial_R P(H_0-\Delta)^3\right] + \partial_\theta\left[\frac{1}{R}\partial_\theta P(H_0-\Delta)^3\right] = 12R^2\cos(\theta) - 18R\cos\theta\partial_R\Delta + 18\sin\theta\partial_\theta\Delta \tag{132}$$

The displacement Δ is relate to the pressure field P via a convolution product to the Green function of the linear elasticity $\mathcal{G}$.

$$\Delta = \kappa\int d^2\vec{R}' P_0(\vec{R}')\mathcal{G}(|\vec{R}-\vec{R}'|) \tag{133}$$

We have shown in the previous section that a mode decomposition was relevant to get physical insight and to transform the partial differential equation into a set of ordinary differential equations. Then we try to make the following decomposition

$$P(R,\theta) = P^{(0)}(R) + P^{(1)}(R)\cos(\theta) \tag{134}$$

$$\Delta(R,\theta) = \Delta^{(0)}(R) + \Delta^{(1)}(R)\cos(\theta) \tag{135}$$

I choose to stop the expansion to the $\cos(\theta)$ and not going up to $\cos(2\theta)$. I'm aware that the perturbation analysis leads to a next-leading order pressure with a term in $\cos(2\theta)$, but let's keep the things as simple as possible. As seen in the Appendix, the deformation $\Delta^{(i)}$ can be written with the Hankel transform formalism

$$\Delta^{(0)} = \kappa\int_{\mathbb{R}_+}\hat{P}^{(0)}(k)\hat{\mathcal{G}}(k)J_0(kr)k\mathrm{d}k, \qquad \hat{P}^{(0)} = \int_{\mathbb{R}_+}P^{(0)}(r)J_0(kr)r\mathrm{d}r, \tag{136}$$

$$\Delta^{(1)} = \kappa\int_{\mathbb{R}_+}\hat{P}^{(1)}(k)\hat{\mathcal{G}}(k)J_1(kr)k\mathrm{d}k, \qquad \hat{P}^{(1)} = \int_{\mathbb{R}_+}P^{(1)}(r)J_1(kr)r\mathrm{d}r. \tag{137}$$

Then, I can introduce the mode decomposition into Eq. (132) and project onto each mode, i.e. $\int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi}...$ and $\int_0^{2\pi}\frac{\mathrm{d}\theta}{\pi}\cos\theta...$ and we get

$$\partial_R\left[R\partial_R P^{(0)}(H_0-\Delta)^3_{(0)}+\frac{1}{2}R\partial_R P^{(1)}(H_0-\Delta)^3_{(1)}\right]=-9R\partial_R\Delta^{(1)}-9\partial_\theta\Delta^{(1)} \tag{138}$$

$$\partial_R\left[R\partial_R P^{(0)}(H_0-\Delta)^3_{(1)}+R\partial_R P^{(1)}(H_0-\Delta)^3_{(0)}\right]-\frac{1}{R}P^{(1)}(H_0-\Delta)^3_{(0)}=12R^2-18R\partial_R\Delta^{(0)} \tag{139}$$

I can rewrite this two equations system in a Matrix form

$$\begin{pmatrix}\partial_R\left[(H_0-\Delta)^3_{(0)}R\partial_R...\right] & \frac{1}{2}\partial_R\left[(H_0-\Delta)^3_{(1)}R\partial_R...\right]\\ \partial_R\left[(H_0-\Delta)^3_{(1)}R\partial_R...\right] & \partial_R\left[(H_0-\Delta)^3_{(0)}R\partial_R...\right]-\frac{(H_0-\Delta)^3_{(0)}}{R}\end{pmatrix}\begin{pmatrix}P^{(0)}\\ P^{(1)}\end{pmatrix}=\begin{pmatrix}-9R\partial_R\Delta^{(1)}-9\partial_\theta\Delta^{(1)}\\ 12R^2-18R\partial_R\Delta^{(0)}\end{pmatrix} \tag{140}$$

where $(H_0-\Delta)^3_{(n)}$ is the projection of $(H_0-\Delta)^3$ on the mode n, *i.e.*

$$(H_0-\Delta)^3_{(0)}=\int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi}(H_0-\Delta)^3=(H_0-\Delta^{(0)})^3+\frac{3}{2}(H_0-\Delta^{(0)})(\Delta^{(1)})^2 \tag{141}$$

$$(H_0-\Delta)^3_{(1)}=\int_0^{2\pi}\frac{\mathrm{d}\theta}{\pi}(H_0-\Delta)^3\cos\theta=3(H_0-\Delta^{(0)})^2\Delta^{(1)}+\frac{3}{8}(\Delta^{(1)})^3. \tag{142}$$

If we diagonalize the governing Matrix, we would be able to decouple the two pressure modes equations and I would be able to solve the equation numerically. However, I have no idea how to threat that....

I wonder weather it exists other numerical method...

6.2 Numerical procedure in Skotheim/Mahadevan 2005

In a 2D geometry, the equation are much simpler. The Eq. (132) becomes (i.e. Eq(9) of Skotheim with my notation)

$$\partial_x\left((H_0-\Delta)^3\partial_x P\right)=-6\partial_x(H_0-\Delta) \tag{143}$$

and the convolution product Eq. (133) still holds for Δ. The numerical procedure described in the paper is the following

- (a) They first guess $\Delta_{\mathrm{old}}(x)$.
- (b) Then solve Eq. (143) numerically to get P.
- (c) After that, they compute $\Delta_{\mathrm{new}}(x)$ using the convolution product expression Eq. (133).
- $\Delta_{\mathrm{new}}\to\Delta_{\mathrm{old}}$ and Reiterate the three steps (a)-(c) until $||\Delta_{\mathrm{old}}-\Delta_{\mathrm{new}}||$ converges to zero.

The first idea would be to copy this scheme. The problem is that the step (b) is much more complicated for us. We don't have a ordinary differential equation (ODE) as in the 2D case but a partial differential one. The mode decomposition, which permits to get an ODE in the perturbation analysis does not help here because the Matrix system seems too complicated to me. I guess that a 2D numerical integration of (132) will be easier. I will have a look in the near future on that.

6.3 Finite Difference scheme

Finally, I change my mind and forget about the mode decomposition. I tried a more basic numerical solution which seems to work very well to make the step (b). First, to avoid any complication with the singularity $1/r$ that arises in cylindrical coordinates. Then, the lubrication equation in cartesian coordinate becomes

$$\partial_X\Big((H_0-\Delta)^3\partial_X P\Big) + \partial_Y\Big((H_0-\Delta)^3\partial_Y P\Big) = 6\partial_X(H_0-\Delta) \tag{144}$$

where I have neglected the instationnary terms (in $\partial_t\Delta$). The fluid thickness will be denoted H in what follow, *i.e.* $H_0-\Delta = H$. Both X and Y are discretized through $X \to X_i = dX\,(i-\frac{M+1}{2})$ with $i \in [0, M-1]$ (python convention). Therefore, the fields $P(X,Y), H(X,Y), \Delta(X,Y)$ becomes set of M^2 scalar $P_{i,j} = P(X_i, Y_j)$ and so one. I assume that $H_{i,j}$ and $\Delta_{i,j}$ are known fields and we try to get the pressure (*i.e.* step (b) described above) We introduce two discretization schemes.

$$\partial_X P_{i,j} = \frac{P_{i+1,j}-P_{i,j}}{dX}, \qquad \partial_Y P_{i,j} = \frac{P_{i,j+1}-P_{i,j}}{dX}, \tag{145}$$

$$\partial_{\bar X} P_{i,j} = \frac{P_{i,j}-P_{i-1,j}}{dX}, \qquad \partial_{\bar Y} P_{i,j} = \frac{P_{i,j}-P_{i,j-1}}{dX}, \tag{146}$$

The discretization of the LHS of the Eq. (144) is chosen as

$$\begin{aligned}\partial_{\bar X}\left[H^3_{i+\frac12,j}\partial_X P_{i,j}\right] &= \partial_{\bar X}\left[\left(\frac{H_{i+1,j}+H_{i,j}}{2}\right)^3\left(\frac{P_{i+1,j}-P_{i,j}}{dX}\right)\right] \\ &= \frac{H^3_{i+\frac12,j}P_{i+1,j} - \Big(H^3_{i+\frac12,j}+H^3_{i-\frac12,j}\Big)P_{i,j} + H^3_{i-\frac12,j}P_{i-1,j}}{dX^2}\end{aligned} \tag{147}$$

We evaluate the "mobility" H^3 as the mean between successif point on the grid to avoid grid decoupling. The exact same scheme is used for the Y component. The RHS of Eq. (144) is discretized as

$$12X - 6\partial_X\Delta \to 12X_i - 6\frac{\Delta_{i+1,j}-\Delta_{i,j}}{dX} \tag{148}$$

Then, the equation becomes in the bulk, *i.e.* $(i,j) \in [1, M-2]^2$

$$\begin{aligned}&\frac{H^3_{i+\frac12,j}P_{i+1,j} - \Big(H^3_{i+\frac12,j}+H^3_{i-\frac12,j}\Big)P_{i,j} + H^3_{i-\frac12,j}P_{i-1,j}}{dX^2} \\ &+ \frac{H^3_{i,j+\frac12}P_{i,j+1} - \Big(H^3_{i,j+\frac12}+H^3_{i,j-\frac12}\Big)P_{i,j} + H^3_{i,j-\frac12}P_{i,j-1}}{dX^2} \\ &= 12X_i - 6\frac{\Delta_{i+1,j}-\Delta_{i,j}}{dX}\end{aligned} \tag{149}$$

We choose Dirichlet boundary conditions, which means that I impose a zero pressure at the boundary,

$$P_{0,j} = P_{M-1,j} = 0, \qquad j \in [0, M-1] \tag{150}$$

$$P_{i,0} = P_{i,M-1} = 0, \qquad i \in [0, M-1] \tag{151}$$

We can then write these equations as in a tensor form

$$T_{ijkl}P_{kl} = \mathrm{RHS}_{ij} \tag{152}$$

with Einstein summation convention. The tensor needs to be inverted to get the numerical pressure field.

$$P_{ij} = T^{-1}_{ijkl}\mathrm{RHS}_{kl} \tag{153}$$

In practice, I have transformed the $(M \times M)$ matrix $P_{i,j}$ in a (M^2) vector P_I and the $(M \times M \times M \times M)$ tensor T_{ijkl} in a matrix $(M^2 \times M^2)$ matrix $T_{I,J}$ accordingly. It does not make any difference. Of course, I'm happy to share my code with as many details as you want if you are interested.

6.4 Results

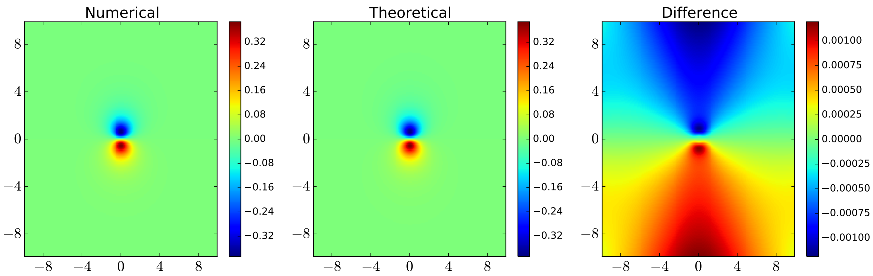

Figure 5: Numerical solution for a rigid wall ($\Delta = 0$).

In Fig. 5, I plotted the result of a the numerical solution for a grid step $dX = 0.1$ with $M = 100$ points (which means a domain $[-9.95, 9.95]^2$). The solution is compared with the theoretical one $P = -\frac{6}{5}\frac{X}{(1+X^2+Y^2)^2}$. We find a very good agreement. The difference between both solution is very small except near the boundary, which makes total sense because the chosen domain is not very large.

In Fig 6, I plotted the result of numerical integration with the Winkler foundation rheology and an elastic parameter $\kappa = 0.1$ as P_{num}. The procedure described in section 6.2 is applied mutiple times until it converges toward the "fixed point". The pressure field (top left) still look like the rigid wall solution (top right) but slitly larger. When I plot the difference (next leading order solution on the bottom left panel), I recover a dominant "monopolar" or axisymmetric (or mode 0) pressure as predicted by the asymptotic solution. The magnitude of this pressure is the same as the one given by the theoretical solution. When I remove the (theoretical) monopolar term $P_1^{(0)}$, I found the "quadripolar" mode ($k = 2$) predicted by the theory.

When I compute the dimensionless lift force $\int dXdY P(X,Y) = 2\pi \int dR\, R\, P(R)$, I found 0.03001 against 0.03016 predicted by the asymptotic theory which validate this numerical procedure.

In Fig. 7, I plotted the lift force extracted from the numerical integration versus the elastic parameter κ. We can see the lift force is very closed to the one predict by the asymptotic prediction up to $\kappa = 0.5$ and then this one decreases. This can be understood with a the hands. The lift force is actually proportional to the asymetry in the pressure field. For a quasi rigid material ($\kappa \ll 1$), the asymmetry is proportionnal to the displacement of the substrate. However, for a very soft material, $\kappa > 1$, the displacement becomes of the same order of magnitude of the thickness at rest no matter what is κ, and therefore the asymmetry and the lift force. This can explain the saturation.

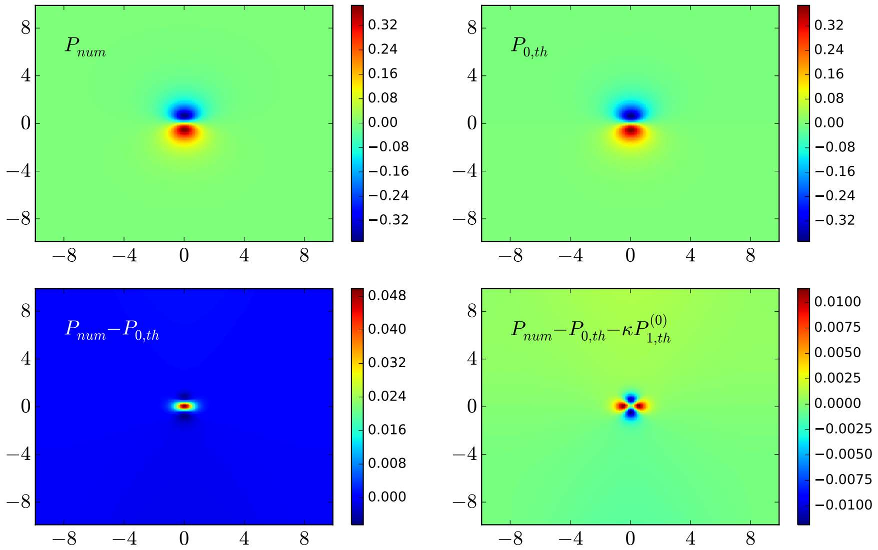

Figure 6: Numerical solution for a soft wall with the Winkler foundation rheology and $\kappa = 0.1$.

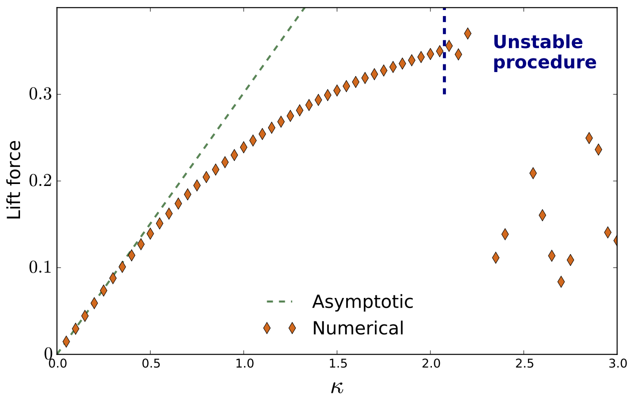

Figure 7: Numerical lift force in the Winkler rheology.

An important remark is that my numerical procedure (see section 6.2) does not converge for $\kappa > 2.1$. Thus the last points cannot be trusted. Compared with the previous note, I access a larger domain in κ I still can't access the highly non-linear regime. What I can do is rewrite the agorithm in cylindrical coordinates to get more point with the same computational time and I should therefore access a larger domain.

7 Viscoelasticity ?

7.1 Basic definition.

Following the work of Pandey/Kapitschka/Venner/Snoeijer in 2016 published in JFM (see Link), we investigate the effect of viscoelasticity of the substrate on the lift force expression. The two differences with their work in that

- We work with a sphere and note a cylinder
- We will take a oscillatory velocity $U = U_0 \cos(\omega_0 t)$ while they consider the case of a constant speed U.

We take the same constitutive law of viscoelasticity as them which is

$$\vec{\vec{\sigma}}_{\text{substrate}}(\vec{r},t) = -\mathcal{P}(\vec{r},t)\vec{\vec{I}} + \int_{-\infty}^{t} \psi(t-t')(\vec{\nabla}\vec{u} + \vec{\nabla}\vec{u}^T)(\vec{r},t')\mathrm{d}t'. \tag{154}$$

We still assume that the substrate is incompressible but the displacement fields resulting from the traction also depends on the "history" (or the past) through a shear relaxation function ψ. This relation is more convinent to use in (temporal) Fourier space.

$$\vec{\vec{\sigma}}_{\text{substrate}}(\vec{r},\omega) = -\mathcal{P}(\vec{r},\omega)\vec{\vec{I}} + \mu(\omega)(\vec{\nabla}\vec{u} + \vec{\nabla}\vec{u}^T)(\vec{r},\omega) \tag{155}$$

with $\mu(\omega)$ the complex shear modulus

$$\mu(\omega) = i\omega \int_0^{\infty} \psi(t)\exp(-i\omega t)\mathrm{d}t = G'(\omega) + iG''(\omega). \tag{156}$$

Then, in the Fourier domain, the constitutive law and boundary conditions are the same as in a pure elastic incompressible material. We can then use directly the result from the previous section which is that the vertical diplacement at the substrate can be written as

$$u_z(z=0,\omega) = \int_{\mathbb{R}^2} \mathrm{d}k_x \mathrm{d}k_y\, \hat{p}(k_x,k_y,\omega)\frac{\hat{\mathcal{G}}(k_x,k_y)}{\mu(\omega)}\exp(-i(k_x x + k_y y)) \tag{157}$$

with $\hat{\mathcal{G}}$ the exact same spatial green function. We will focus again on the specific case of a thick substrate in which this function take the simple form

$$\hat{\mathcal{G}(k)} \to \frac{1}{2\sqrt{k_x^2+k_y^2}}, \qquad (H_{\text{sub}} \to \infty) \tag{158}$$

7.2 Visco elastic asymptic model.

Let's put ourself in the framework of the asympotic model. We assume that the substrate displacement is small compare with the surface/sphere distance. We apply the same procedure as in the section 4, which means that we consider elastohydrodynamics as a perturbation of the lubrication problem of a sphere sliding near a boundary. Therefore, the leading order pressure is not affected by the viscoelasticity and we can write directly

$$P_0(R,\theta,T) = \frac{6\,U(T)R\cos(\theta)}{5(1+R^2)^2} \tag{159}$$

I will write explicitly the time dependance of the velocity since it affects the "history" and then the viscoelastic response of the substrate. First, I need to specify that the elastic parameter κ is defined as

$$\kappa = \frac{2\eta c}{a\epsilon^2\mu} \rightarrow \frac{2\eta c}{a\epsilon^2 G'(\omega_0)} \tag{160}$$

We can apply directly the law written above in Fourier (temporal space). The dimensionless deformation H_1 in Fourier space is related to leading order pressure field P_0 in this asymptotic model the then through the exact same expression.

$$H_1(R,\theta,\omega) = \frac{U(\omega)}{\mu(\omega)}\cos(\theta)\int_{\mathbb{R}+} \mathrm{d}K\,\frac{J_1(KR)}{2}\int_{\mathbb{R}+} R'\mathrm{d}R'\,\frac{6R}{5(1+R^2)^2}J_1(KR') \tag{161}$$

with

$$\begin{aligned} F(R) &= -\frac{6}{5}\int_{\mathbb{R}+} \mathrm{d}K\,\frac{J_1(KR)}{2}\int_{\mathbb{R}+} R'\mathrm{d}R'\,\frac{R'}{(1+R'^2)^2}J_1(KR') \\ &= -\frac{6}{5}\int_{\mathbb{R}+} \mathrm{d}K\,\frac{J_1(KR)}{2}\frac{K}{2}K_0(K) \\ &= -\frac{3}{10}\left(\frac{K(-R^2)}{R} - \frac{E(-R^2)}{R(1+R^2)}\right) \end{aligned} \tag{162}$$

With my definition of the elastic parameter, μ here is dimensionless and equal to

$$\mu(\omega) = \frac{G'(\omega)+G''(\omega)}{G'(\omega_0)} \tag{163}$$

If we take $U(T) = U_0\cos(\omega_0 T)$, then the Fourier transform of is $U(\omega)\frac{U_0}{2}\Big(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\Big)$ (where δ is the Dirac distribution here). Taking the Fourier transform of $H_1(R,\theta,\omega)$ we find that

$$H_1(R,\theta,T) = \frac{U_0}{\sqrt{1+\left(\frac{G''(\omega_0)}{G'(\omega_0)}\right)^2}}\cos(\omega_0 T-\phi)\,F(R)\cos(\theta) \tag{164}$$

with $\arctan(\phi) = \frac{G''(\omega_0)}{G'(\omega_0)}$. I guess that this expression is quite natural. If the imaginary part of the complex modulus $G''(\omega_0)$ vanish, we recover the former expression. If the substrate dissipates energy, *i.e.* $G''(\omega) > 0$, then the amplitude decrease and become out of phase with the motion. The spatial dependance of H_1 is not affected by this change and therefore the lift force will be exactly the same as before. We can directly write

$$F_{\text{Lift}}^{\text{Semi}} = 0.4162...\frac{\eta^2}{G'(\omega_0)}\left(\frac{a}{d}\right)^{5/2}\frac{U_0\cos(\omega_0 t-\phi)}{\sqrt{1+\left(\frac{G''(\omega_0)}{G'(\omega_0)}\right)^2}}U_0\cos(\omega_0 t) \tag{165}$$

If we take the mean force, it gives (using $<\cos(\omega_0 t-\phi)\cos(\omega_0 t)>=\frac{\cos(\phi)}{2}$)

$$< F_{\text{Lift}}^{\text{Semi}} >= 0.4162...\frac{\eta^2}{G'(\omega_0)}\left(\frac{a}{d}\right)^{5/2}\frac{U_0}{2}\frac{\cos(\phi)}{\sqrt{1+\left(\frac{G''(\omega_0)}{G'(\omega_0)}\right)^2}} \tag{166}$$

Finally, using $\cos(\phi) = \frac{G'(\omega_0)/G''(\omega_0)}{\sqrt{1+\left(\frac{G''(\omega_0)}{G'(\omega_0)}\right)^2}}$, we end up with

$$< F_{\text{Lift}}^{\text{Semi}} >= 0.4162... \frac{\eta^2 U_0^2}{2} \left(\frac{a}{d}\right)^{5/2} \frac{G'(\omega_0)}{G'(\omega_0)^2 + G''(\omega_0)^2} \tag{167}$$

We recover the scaling law derived in the last part of the article of Pandey et al. (see Eq.4.4). What is great is that, with a periodic forcing, the scaling law is exact :)

A Green function of a thin incompressible elastic subjected to a pressure.

We are looking for the solution of the elastic problem with an incompressible rheology and with a boundary condition

$$\text{at } z = 0, \quad \sigma_{zz} = -\mu\delta(\vec{r}), \quad \sigma_{zr} = \sigma_{z\theta} = 0, \tag{168}$$

and

$$\text{at } z = -h_{\text{sub}}, \quad u_r = u_\theta = u_z \tag{169}$$

The coefficient in front of the dirac function are choosen in order to have a dimensionless Green function. Since the boundary condition are axisymmetric, we assume that the solution is also axisymmetric. Then the equilibrium condtion yields to three equations.

$$\partial_r \mathcal{P} = \mu\left(\frac{1}{r}\partial_r(r\partial_r u_r) - \frac{1}{r^2}u_r + \partial_z^2 u_r\right), \tag{170a}$$

$$\partial_z \mathcal{P} = \mu\left(\frac{1}{r}\partial_r(r\partial_r u_z) + \partial_z^2 u_z\right), \tag{170b}$$

$$\frac{1}{r}\partial_r(r\, u_r) + \partial_z u_z = 0. \tag{170c}$$

We introduce the Hankel transform of index 1 for the radial displacement u_r and of index 0 for both the pressure $\mathcal{P}$, and the vertical velocity u_z.

$$\tilde{u}_r(k, z) = \int_0^\infty \mathrm{d}r\, r\, u_r(r, z)\, J_1(kr), \tag{171a}$$

$$\tilde{u}_z(k, z) = \int_0^\infty \mathrm{d}r\, r\, u_z(r, z)\, J_0(kr), \tag{171b}$$

$$\tilde{\mathcal{P}}(k, t) = \int_0^\infty \mathrm{d}r\, r\, \mathcal{P}(r, z)\, J_0(kr), \tag{171c}$$

with the corresponding inversion:

$$u_r(r, z) = \int_0^\infty \mathrm{d}k\, k\, \tilde{u}_r(k, z)\, J_1(kr), \tag{172a}$$

$$u_z(r, z) = \int_0^\infty \mathrm{d}k\, k\, \tilde{u}_z(k, z)\, J_0(kr), \tag{172b}$$

$$\mathcal{P}(r, t) = \int_0^\infty \mathrm{d}k\, k\, \tilde{\mathcal{P}}(k, z)\, J_0(kr), \tag{172c}$$

Balancing the derrivative of Eq.(170a) with respect to z, and the derrivative of Eq.(170b) with respect to r, we find that

$$\left(\frac{\partial}{\partial z}\right)^3 \tilde{u}_r(k, z) + k\left(\frac{\partial}{\partial z}\right)^2 \tilde{u}_z(k, z) - \left(\frac{\partial}{\partial z}\right)\tilde{u}_r(k, z) - k^3\tilde{u}_z(k, z) = 0. \tag{173}$$

Then taking the derrivative of the mass conservation with respect to r, we end up with a additionnal equation.

$$\left(\frac{\partial}{\partial z}\right)\tilde{u}_z(k, z) + k\tilde{u}_r(k, z) = 0 \tag{174}$$

We can then inject Eq.(174) into Eq.(173), and we find the same ODE equation for both $\tilde{u}_r$ and $\tilde{u}_z$. The system can be written as

$$\left(\frac{\partial}{\partial z}\right)^4 \tilde{u}_z(k,z) - 2k^2 \left(\frac{\partial}{\partial z}\right)^2 \tilde{u}_z(k,z) + k^4 \tilde{u}_z(k,z) = 0. \tag{175a}$$

$$k\tilde{u}_r(k,z) = -\left(\frac{\partial}{\partial z}\right) \tilde{u}_z(k,z) \tag{175b}$$

whose general solution is

$$\tilde{u}_r(k,z) = -\frac{1}{k}\Big(kA(k)+kzC(k)+D(k)\Big)\sinh(kz) - \frac{1}{k}\Big(kB(k)+kzD(k)+C(k)\Big)\cosh(kz) \tag{176a}$$

$$\tilde{u}_z(k,z) = \Big(A(k)+zC(k)\Big)\cosh(kz) + \Big(B(k)+zD(k)\Big)\sinh(kz) \tag{176b}$$

We then need to inject the boundary counditions. The rigidly attached boundary conditions lead to

$$\tilde{u}_r = \tilde{u}_z = 0, \qquad z = -h_{\rm sub} \tag{177}$$

and at the liquid-substrate interface, the no shear boundary condition yields to:

$$\partial_z u_r + \partial_r u_z = 0, \qquad z = 0, \tag{178}$$

and the normal stress balance[5]:

$$-\tilde{\mathcal{P}} + 2\mu\partial_z \tilde{u}_z = -\mu \tag{179}$$

which gives in terms of the (A,B,C,D) coefficients:

$$\Big(-kA(k)+kh_{\rm sub}C(k)-D(k)\Big)\sinh(kh_{\rm sub}) + \Big(kB(k)-kh_{\rm sub}D(k)+C(k)\Big)\cosh(kh_{\rm sub}) = 0 \tag{180a}$$

$$\Big(kA(k)-kh_{\rm sub}C(k)\Big)\cosh(kh_{\rm sub}) + \Big(-kB(k)+kh_{\rm sub}D(k)\Big)\sinh(kh_{\rm sub}) = 0 \tag{180b}$$

$$kA(k)+D(k) = 0 \tag{180c}$$

$$2kB(k) = -1 \tag{180d}$$

We have a 4x4 matrix system $\vec{\vec{M}}.(kA,kB,C,D) = (0,0,0,-1)$ that can be inversed and we get $((kA,kB,C,D) = \vec{\vec{M}}^{-1}.(0,0,0,-1)$

$$kA = \frac{kh_{\rm sub} - \sinh(kh_{\rm sub})\cosh(kh_{\rm sub})}{2(\cosh^2(kh_{\rm sub}) + (kh_{\rm sub})^2)} \tag{181a}$$

$$kB = -\frac{1}{2} \tag{181b}$$

$$C = \frac{\cosh^2(kh_{\rm sub})}{2(\cosh^2(kh_{\rm sub}) + (kh_{\rm sub})^2)} \tag{181c}$$

[5] I want to emphasis that if we are considering un point force $\delta(\vec{r}) = \delta(x)\delta(y)$, which is equal to $\delta(r)/r$, and thus the Hankel transform is equal to unity, i.e. $\tilde{\delta}(\vec{r}) = \widetilde{\frac{\delta(r)}{r}} = 1$

$$D = \frac{-kh_{\mathrm{sub}} + \sinh(kh_{\mathrm{sub}})\cosh(kh_{\mathrm{sub}})}{2(\cosh^2(kh_{\mathrm{sub}}) + (kh_{\mathrm{sub}})^2)} \tag{181d}$$

That finally gives

$$\tilde{u}_r(z=0) = \frac{-1}{k}(kB + C) = \frac{1}{k}\frac{(kh_{\mathrm{sub}})^2}{2(\cosh^2(kh_{\mathrm{sub}}) + (kh_{\mathrm{sub}})^2)} \tag{182}$$

$$\tilde{u}_z(z=0) = A = \frac{1}{k}\frac{kh_{\mathrm{sub}} - \sinh(kh_{\mathrm{sub}})\cosh(kh_{\mathrm{sub}})}{2(\cosh^2(kh_{\mathrm{sub}}) + (kh_{\mathrm{sub}})^2)} \tag{183}$$

or

$$u_r(z=0) = G_r = \frac{1}{2}\int_0^\infty \mathrm{d}k\, \frac{(kh_{\mathrm{sub}})^2}{\cosh^2(kh_{\mathrm{sub}}) + (kh_{\mathrm{sub}})^2} J_1(kr) \tag{184}$$

$$u_\theta(z=0) = G_\theta = 0 \tag{185}$$

$$u_z(z=0) = G_z = \frac{1}{2}\int_0^\infty \mathrm{d}k\, \frac{kh_{\mathrm{sub}} - \sinh(kh_{\mathrm{sub}})\cosh(kh_{\mathrm{sub}})}{\cosh^2(kh_{\mathrm{sub}}) + (kh_{\mathrm{sub}})^2} J_0(kr) \tag{186}$$

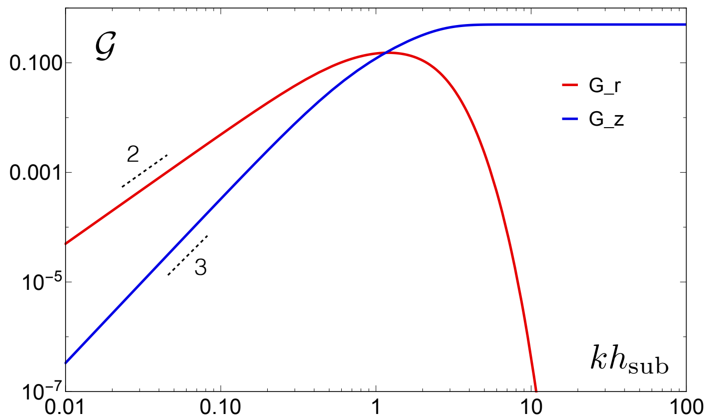

Figure 8: Green function of an finite size incompressible material. It is the same quantity as the Figure. 2 in Leroy & Charlaix 2011 JFM. The blue curve is the the value at $\nu = 1/2$ (i.e. incompressible). The radial Green function is added to this curve. **Notably**, the Bessel function J_1 is used for the radial Green function and J_0 for the vertical one.

Question : Can we perform Taylor expansion as in Leroy/Charlaix ? : if we assume that the substrate is thin, i.e. $H_{\mathrm{sub}} \ll 1$, one can expend the dimensionless part of the Green function as

$$\frac{(KH_{\mathrm{sub}})^2}{\cosh^2(KH_{\mathrm{sub}}) + (KH_{\mathrm{sub}})^2} \to \frac{2}{3}(KH_{\mathrm{sub}})^3, \qquad \text{if } H_{\mathrm{sub}} \ll 1 \tag{187}$$

$$\frac{(KH_{\mathrm{sub}})^2}{\cosh^2(KH_{\mathrm{sub}}) + (KH_{\mathrm{sub}})^2} \to (KH_{\mathrm{sub}})^2, \qquad \text{if } H_{\mathrm{sub}} \ll 1 \tag{188}$$

Nevertheless, this approach does not work because the following integrals are not defined.

$$\int_0^\infty dk J_0(kr) k^3 \tag{189}$$

$$\int_0^\infty dk J_1(kr)k^2 \tag{190}$$

Thus, the large K limit is necessary to ensure the convergence of the integral and one cannot find trivial functions of the Green function in this limit.

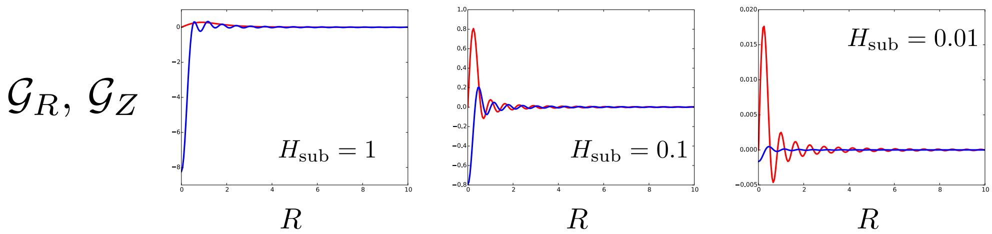

Figure 9: Green function of an finite size incompressible material. The same color code is used as in the Fig. 8 (Red = radial, Blue = Vertical). One can observe that the radial displacement become dominant as the substrate thickness decrease, i.e. $H_{\rm sub} \to 0$.

(For me : need to check that the integration scheme is correct.)

B Attempt to find a more elegant incompressible solution

Let's solve the same incompressible problem with a pressure at the elastic interface that verifies

$$P(r,\theta) = f(r)\cos(\theta) \tag{191}$$

as in the next-leading order calculation. We then have the following system

$$\vec{\vec{\sigma}}_{\text{substrate}} = -\mathcal{P}\vec{\vec{I}} + \mu(\vec{\nabla}\vec{u}_{\text{green}} + \vec{\nabla}\vec{u}_{\text{green}}^{T}), \quad \text{and} \quad \vec{\nabla}.\vec{u}_{\text{green}} = 0 \tag{192}$$

$$\forall r \in \mathbb{R}^{+}, \quad \forall \theta \in [0, 2\pi[, \quad \forall z \in [-h_{\text{sub}}, 0], \quad \vec{\nabla}.\vec{\vec{\sigma}}_{\text{substrate}} = \vec{0} \tag{193}$$

$$\text{at } z = -h_{\text{sub}}, \quad \vec{u}_{\text{green}} = 0 \tag{194}$$

$$\text{at } z = 0, \quad \sigma_{zz} = -f(r)\cos(\theta), \quad \sigma_{zr} = \sigma_{z\theta} = 0 \tag{195}$$

$$\vec{\mathcal{G}}(\vec{r}) = \vec{u}_{\text{green}}(\vec{r}, z = 0) \tag{196}$$

We project the stress balance and incompressibility condition as

$$\partial_r \mathcal{P} = \mu\left(\frac{1}{r}\partial_r(r\partial_r u_r) - \frac{u_r}{r^2} + \frac{2}{r^2}\partial_\theta u_\theta + \frac{1}{r^2}\partial_\theta^2 u_r + \partial_z^2 u_r\right) \tag{197}$$

$$\frac{1}{r}\partial_\theta \mathcal{P} = \mu\left(\frac{1}{r}\partial_r(r\partial_r u_\theta) - \frac{u_\theta}{r^2} - \frac{2}{r^2}\partial_\theta u_r + \frac{1}{r^2}\partial_\theta^2 u_\theta + \partial_z^2 u_\theta\right) \tag{198}$$

$$\partial_z \mathcal{P} = \mu\left(\frac{1}{r}\partial_r(r\partial_r u_z) + \frac{1}{r^2}\partial_\theta^2 u_z + \partial_z^2 u_z\right) \tag{199}$$

$$\frac{1}{r}\partial_r(ru_r) + \frac{1}{r}\partial_\theta u_\theta + \partial_z u_z = 0 \tag{200}$$

Inspired by the normal stress condition, it seems quite natural to assume that the pressure $\mathcal{P}$ can also be written as

$$\mathcal{P}(r,\theta,z) = \mathcal{P}^{*}(r,z)\cos(\theta) \tag{201}$$

and therefore, when we plug that into the last equations, we want to make the following assumptions

$$\begin{aligned} u_r(r,\theta,z) &= u_r^{*}(r,z)\cos(\theta), \\ u_\theta(r,\theta,z) &= u_\theta^{*}(r,z)\sin(\theta), \\ u_z(r,\theta,z) &= u_z^{*}(r,z)\cos(\theta). \end{aligned} \tag{202}$$

When we do that, we find that the "Stokes equations" become

$$\partial_r \mathcal{P}^{*} = \mu\left(\frac{1}{r}\partial_r(r\partial_r u_r^{*}) - \frac{2u_r^{*}}{r^2} + \frac{2u_\theta^{*}}{r^2} + \partial_z^2 u_r^{*}\right) \tag{203}$$

$$\frac{1}{r}\mathcal{P}^{*} = \mu\left(\frac{1}{r}\partial_r(r\partial_r u_\theta^{*}) - \frac{2u_\theta^{*}}{r^2} + \frac{2u_r^{*}}{r^2} + \partial_z^2 u_\theta^{*}\right) \tag{204}$$

$$\partial_z \mathcal{P}^{*} = \mu\left(\frac{1}{r}\partial_r(r\partial_r u_z^{*}) - \frac{u_z^{*}}{r^2} + \partial_z^2 u_z^{*}\right) \tag{205}$$

$$\frac{1}{r}\partial_r(ru_r^{*}) + \frac{u_\theta^{*}}{r} + \partial_z u_z^{*} = 0 \tag{206}$$

The two first equations are coupled, so we consider in the following the variables

$$u^+ = u_r + u_\theta, \qquad u^- = u_r - u_\theta \tag{207}$$

or respectively

$$u_r = \frac{u^+ + u^-}{2}, \qquad u_\theta = \frac{u^+ - u^-}{2} \tag{208}$$

Therefore, the two first equations gives

$$\partial_r \mathcal{P}^* + \frac{1}{r}\mathcal{P}^* = \mu\bigg(\frac{1}{r}\partial_r(r\partial_r u^+) + \partial_z^2 u^+\bigg) \tag{209}$$

$$\partial_r \mathcal{P}^* - \frac{1}{r}\mathcal{P}^* = \mu\bigg(\frac{1}{r}\partial_r(r\partial_r u^-) - \frac{4u^-}{r^2} + \partial_z^2 u^-\bigg) \tag{210}$$

$$\partial_z \mathcal{P}^* = \mu\bigg(\frac{1}{r}\partial_r(r\partial_r u_z^*) - \frac{u_z^*}{r^2} + \partial_z^2 u_z^*\bigg) \tag{211}$$

Written as this, the radial operator (of the RHS) of this last equation are

$$\mathcal{L}. = \partial_r^2 + \frac{1}{r}\partial_r - \frac{i^2}{r^2} \qquad \text{with} \quad i \in \{0, 1, 2\} \tag{212}$$

for which I know that the eigenvectors are the Bessel function of first and second kind of index i. We will consider only the Bessel function of first kind (finite in $r = 0$) and introduce the coresponding Hankel transform as

$$\tilde{u}^+(k,z) = \int_0^\infty \mathrm{d}r\, r\, u^+(r,z)\, J_0(kr), \tag{213a}$$

$$\tilde{u}^-(k,z) = \int_0^\infty \mathrm{d}r\, r\, u^-(r,z)\, J_2(kr), \tag{213b}$$

$$\tilde{u}_z(k,z) = \int_0^\infty \mathrm{d}r\, r\, u_z(r,z)\, J_1(kr), \tag{213c}$$

$$\tilde{\mathcal{P}}(k,t) = \int_0^\infty \mathrm{d}r\, r\, \tilde{\mathcal{P}}(r,z)\, J_1(kr), \tag{213d}$$

And the coresponding inversion

$$u^+(r,z) = \int_0^\infty \mathrm{d}k\, k\, \tilde{u}^+(k,z)\, J_0(kr), \tag{214a}$$

$$u^-(r,z) = \int_0^\infty \mathrm{d}k\, k\, \tilde{u}^-(k,z)\, J_2(kr), \tag{214b}$$

$$u_z(r,z) = \int_0^\infty \mathrm{d}k\, k\, \tilde{u}_z(k,z)\, J_1(kr), \tag{214c}$$

$$\mathcal{P}(r,t) = \int_0^\infty \mathrm{d}k\, k\, \tilde{\mathcal{P}}(k,z)\, J_1(kr), \tag{214d}$$

Then, the Stokes equation becomes

$$k\tilde{\mathcal{P}} = \Big(-k^2 + \partial_z^2\Big)\tilde{u}^+(k,z) \tag{215}$$

$$-k\tilde{\mathcal{P}} = \Big(-k^2 + \partial_z^2\Big)\tilde{u}^-(k,z) \tag{216}$$

$$\partial_z\tilde{\mathcal{P}} = \Big(-k^2 + \partial_z^2\Big)\tilde{u}_z(k,z) \tag{217}$$

This gives me three equations and four unknown. I should use the incompressibility condition to go further. I can first replace u_r and u_θ by their values in terms of u^+ and u^-

$$\frac{1}{r}\partial_r\left(r\frac{u^+ + u^-}{2}\right) + \frac{u^+ - u^-}{2r} + \partial_z u_z = 0, \tag{218}$$

or

$$\frac{1}{2}\partial_r(u^+ + u^-) + \frac{u^+}{r} + \partial_z u_z = 0. \tag{219}$$

Let's take the Hankel transform (of index 1) of the mass conservation equation

$$\int_0^\infty \mathrm{d}r\, rJ_1(kr)\Bigg[\underbrace{\frac{1}{2}\partial_r(u^+ + u^-)}_{(A)} + \underbrace{\frac{u^+}{r}}_{(B)} + \underbrace{\partial_z u_z}_{(C)} = 0\Bigg] \tag{220}$$

We called (A, B, C) the three term of the latter equation. The term (C) is trivial, i.e. $(C) = \partial_z\tilde{u}_z(k,z)$. For the two other terms we have to make some calculation.

$$(A) = \int_0^\infty \mathrm{d}r\, rJ_1(kr)\frac{1}{2}\partial_r(u^+ + u^-) \tag{221}$$

Let's integrate by part

$$(A) = -\frac{1}{2}\int_0^\infty \mathrm{d}r\, \partial_r\Big(rJ_1(kr)\Big)(u^+ + u^-) + \frac{1}{2}\underbrace{\Big[rJ_1(kr)(u^+ + u^-)\Big]_0^\infty}_{=0} \tag{222}$$

We then use the relationship $\partial_x(xJ_1(x) = xJ_0(x)$ such that

$$(A) = -\frac{1}{2}\int_0^\infty \mathrm{d}r\, krJ_0(kr)(u^+ + u^-) = -\frac{1}{2}k\tilde{u}^+(k) - \frac{1}{2}\int_0^\infty \mathrm{d}r\, krJ_0(kr)u^-(r) \tag{223}$$

The second term (B) is equal to

$$(B) = \int_0^\infty \mathrm{d}r\, rJ_1(kr)\frac{u^+}{r} = \int_0^\infty \mathrm{d}r\, J_1(kr)u^+(r) \tag{224}$$

Here, we will use the Bessel recurrence relationship $J_1(x) = x\frac{J_0(x)+J_2(x)}{2}$ such that

$$(B) = \frac{1}{2}\int_0^\infty \mathrm{d}r\, kr\Big(J_0(kr) + J_2(kr)\Big)u^+(r) = \frac{1}{2}k\tilde{u}^+(k) + \frac{1}{2}\int_0^\infty \mathrm{d}r\, krJ_2(kr)u^+(r). \tag{225}$$

Therefore, the two terms $\frac{1}{2}k\tilde{u}^+(k)$ vanish in the mass conservation and we can write

$$\frac{1}{2}\int_0^\infty \mathrm{d}r\, krJ_2(kr)u^+(r) - \frac{1}{2}\int_0^\infty \mathrm{d}r\, krJ_0(kr)u^-(r) + \partial_z\tilde{u}_z(k,z) = 0 \tag{226}$$

This equation makes sense because the vertical flow (term in J_1) is coupled to lateral one and the lateral flow is a combinaison of terms in J_0, J_2. I must notice that there is no way to write this equation in a simpler form because there is not "orthogonality" between J_2 and J_0. If I inject the Hankel transform of u^+, u^-, and try to invert it (I do it only for the first term)

$$\frac{1}{2}\int_0^\infty \mathrm{d}r\, krJ_2(kr)\int_0^\infty \mathrm{d}k'k'\tilde{u}^+(k')J_0(k'r) - \frac{1}{2}\int_0^\infty \mathrm{d}r\, krJ_0(kr)u^-(r) + \partial_z\tilde{u}_z(k,z) = 0 \quad (227)$$

and then inverting the integrales

$$\frac{1}{2}k\int_0^\infty \mathrm{d}k'\, k'\tilde{u}^+(k')\int_0^\infty \mathrm{d}rJ_2(kr)J_0(k'r) - \frac{1}{2}\int_0^\infty \mathrm{d}r\, krJ_0(kr)u^-(r) + \partial_z\tilde{u}_z(k,z) = 0 \quad (228)$$

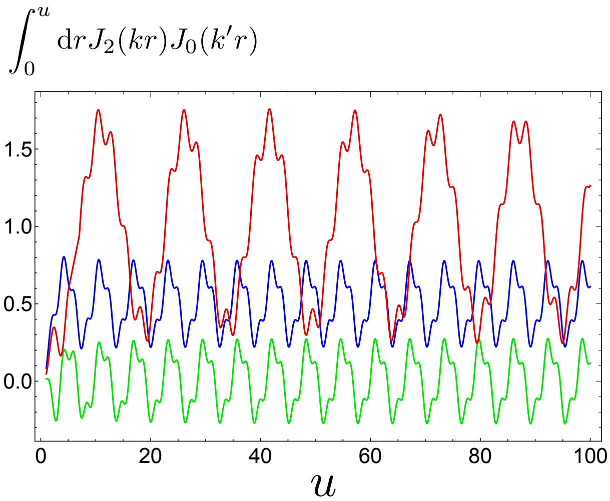

Figure 10: Value of the integral with a running integral limit. The different color are for different couple (k,k') : blue $= \{k=2, k'=1\}$, green $= \{k=1, k'=2\}$, red $= \{k=1, k'=\sqrt{2}\}$.

This integral $\int_0^\infty \mathrm{d}rJ_2(kr)J_0(k'r)$ is actually not defined. In Figure 10, I plotted $u \to \int_0^u \mathrm{d}rJ_2(kr)J_0(k'r)$ for different couples (k,k'). For each of this cases, we observe that this function is oscillatory at large u. Then it does not make any sense to define $\int_0^\infty \mathrm{d}rJ_2(kr)J_0(k'r)$ which is the limit $u \to \infty$ of the function plotted.

C Deformation for a semi-infinite substrate.

As we discussed above, the deformation of the interface for a semi infinite substrate subjected to a pressure $P(R,\theta) = \frac{R\cos(\theta)}{(1+R^2)^2}$ (I forget about the prefactor) is

$$\begin{aligned} H_1(R,\theta) &= \int \mathrm{d}^2\vec{R}' \frac{P(\vec{R}')}{|\vec{R}-\vec{R}'|} \\ &= \int_0^\infty R'\mathrm{d}R' \int_0^{2\pi} \mathrm{d}\theta' \frac{R'\cos(\theta')}{(1+R'^2)^2} \frac{1}{\sqrt{R^2+R'^2-2RR'\cos(\theta-\theta')}} \end{aligned} \tag{229}$$

Then we separate what only depends on the radial part and the angular part

$$H_1(R,\theta) = \int_0^\infty \mathrm{d}R' \frac{R'^2}{(1+R'^2)^2} \int_0^{2\pi} \mathrm{d}\theta' \frac{\cos(\theta')}{\sqrt{R^2+R'^2-2RR'\cos(\theta-\theta')}} \tag{230}$$

The magic trick is that the following ratio can be expand as a series and the coefficient are Legendre polynomial of order n, denoted P_n.

$$\forall |z|<1, \qquad \frac{1}{\sqrt{1+z^2-2zx}} = \sum_{k=0}^{\infty} P_n(x) z^n \tag{231}$$

This procedure is used in physical problems with spherical coordinate. An example is in electrostatic and when decomposing between the different multipoles (monopole, dipole, quadripole...). In Cartesian coordinate, it is not exactly the same but it is close from the mathematical point of view.

First, let split in two the integrale for the radial part and make a change of variable $\tilde{\theta} = \theta - \theta'$ (it is more convenient). Then it yields to

$$\begin{aligned} H_1(R,\theta) = &\int_0^R \mathrm{d}R' \frac{R'^2}{(1+R'^2)^2} \int_0^{2\pi} \mathrm{d}\tilde{\theta} \frac{\cos(\theta-\tilde{\theta})}{\sqrt{R^2+R'^2-2RR'\cos(\tilde{\theta})}} \\ &+ \int_R^\infty \mathrm{d}R' \frac{R'^2}{(1+R'^2)^2} \int_0^{2\pi} \mathrm{d}\tilde{\theta} \frac{\cos(\theta-\tilde{\theta})}{\sqrt{R^2+R'^2-2RR'\cos(\tilde{\theta})}} \end{aligned} \tag{232}$$

Then we factorize by R in the first integrale and R' in the second and we get

$$\begin{aligned} H_1(R,\theta) = &\int_0^R \mathrm{d}R' \frac{R'^2}{(1+R'^2)^2} \int_0^{2\pi} \mathrm{d}\tilde{\theta} \frac{\cos(\theta-\tilde{\theta})}{R\sqrt{1+(\frac{R'}{R})^2-2\frac{R'}{R}\cos(\tilde{\theta})}} \\ &+ \int_R^\infty \mathrm{d}R' \frac{R'^2}{(1+R'^2)^2} \int_0^{2\pi} \mathrm{d}\tilde{\theta} \frac{\cos(\theta-\tilde{\theta})}{R'\sqrt{1+(\frac{R}{R'})^2-2\frac{R}{R'}\cos(\tilde{\theta})}} \end{aligned} \tag{233}$$

and then apply the formula written above and we expand $\cos(\theta-\tilde{\theta})$ using basic trigonometric formula

$$\begin{aligned} H_1(R,\theta) = &\int_0^R \mathrm{d}R' \frac{R'^2}{R(1+R'^2)^2} \int_0^{2\pi} \mathrm{d}\tilde{\theta} \Big(\cos(\theta)\cos(\tilde{\theta}) + \sin(\theta)\sin(\tilde{\theta})\Big) \sum_{k=0}^{\infty} P_k(\cos(\tilde{\theta})) \left(\frac{R'}{R}\right)^k \\ &+ \int_R^\infty \mathrm{d}R' \frac{R'}{(1+R'^2)^2} \int_0^{2\pi} \mathrm{d}\tilde{\theta} \Big(\cos(\theta)\cos(\tilde{\theta}) + \sin(\theta)\sin(\tilde{\theta})\Big) \sum_{k=0}^{\infty} P_k(\cos(\tilde{\theta})) \left(\frac{R}{R'}\right)^k \end{aligned} \tag{234}$$

All the terms in $\int \mathrm{d}\tilde{\theta}\sin(\tilde{\theta})P_k(\cos(\tilde{\theta}))$ vanish by symmetry as well as the odd $\int \mathrm{d}\tilde{\theta}\cos(\tilde{\theta})P_{2k}(\cos(\tilde{\theta}))$. Then the integral can be written as

$$\begin{aligned} H_1(R,\theta) = \cos(\theta)\sum_{k=0}^{\infty}\int_0^R \mathrm{d}R' \frac{R'^2}{R(1+R'^2)^2}\left(\frac{R'}{R}\right)^{2k+1}\int_0^{2\pi}\mathrm{d}\tilde{\theta}\cos(\tilde{\theta})P_{2k+1}(\cos(\tilde{\theta})) \\ + \cos(\theta)\sum_{k=0}^{\infty}\int_R^{\infty}\mathrm{d}R'\frac{R'}{(1+R'^2)^2}\left(\frac{R}{R'}\right)^{2k+1}\int_0^{2\pi}\mathrm{d}\tilde{\theta}\cos(\tilde{\theta})P_{2k+1}(\cos(\tilde{\theta})) \end{aligned} \tag{235}$$

or in a more condensed form

$$H_1(R,\theta) = F(R)\cos(\theta) \tag{236}$$

with

$$F(R) = \sum_{k=0}^{\infty}(\alpha_k(R)+\beta_k(R))\gamma_k \tag{237}$$

with the three series defined as

$$\alpha_k(R) = \int_0^R \mathrm{d}R'\frac{R'^2}{R(1+R'^2)^2}\left(\frac{R'}{R}\right)^{2k+1} \tag{238a}$$

$$\beta_k(R) = \int_R^{\infty}\mathrm{d}R'\frac{R'}{(1+R'^2)^2}\left(\frac{R}{R'}\right)^{2k+1} \tag{238b}$$

$$\gamma_k = \int_0^{2\pi}\mathrm{d}\tilde{\theta}\cos(\tilde{\theta})P_{2k+1}(\cos(\tilde{\theta})) \tag{238c}$$

All these series can be computed explicitly and have "simple" analytical form. I will give the first three terms in the following to give you an idea of these functions.

$$\alpha_0(R) = \frac{1}{2}\frac{R^2}{1+R^2} - \frac{R^2-\log(1+R^2)}{2R^2} \tag{239a}$$

$$\alpha_1(R) = \frac{1}{2}\frac{R^2}{1+R^2} - \frac{-2R^2+R^4+2\log(1+R^2)}{2R^4} \tag{239b}$$

$$\alpha_2(R) = \frac{1}{2}\frac{R^2}{1+R^2} - \frac{6R^2-3R^4+2R^6-6\log(1+R^2)}{4R^6} \tag{239c}$$

$$\beta_0(R) = \frac{1}{2}\frac{1}{1+R^2} + \frac{1}{2}(1-R\arctan(\frac{1}{R})) \tag{240a}$$

$$\beta_1(R) = \frac{1}{2}\frac{1}{1+R^2} + \frac{1}{2}(-1+3R^2-3R^3\arctan(\frac{1}{R})) \tag{240b}$$

$$\beta_2(R) = \frac{1}{2}\frac{1}{1+R^2} + \frac{1}{2}(-1+\frac{5}{3}R^2-5R^4+5R^5\arctan(\frac{1}{R})) \tag{240c}$$

$$\gamma_0 = \pi, \quad \gamma_1 = \frac{3}{8}\pi, \quad \gamma_2 = \frac{15}{64}\pi, \ ... \tag{241}$$

In Figure 11, I plotted the first five term of the series expansion of F, defined as

$$F_k(R) = (\alpha_k(R)+\beta_k(R))\gamma_k \tag{242}$$

We can see that only the first terms are significant in the series. The term $k=0$ exhibits quite a particular behavior compare with the others. Notably, $F_0(R) \sim \frac{\pi^2}{4}R$ and all other terms $F_{k>0}(R) = \mathcal{O}(R^2)$ as $R \to 0$. Also at large radius, the zeroth term is also different from the other. $F_0(R) \sim \pi\frac{\log(R)}{R^2}$ while all the other terms are $F_{k>0} = \mathcal{O}(1/R^2)$.

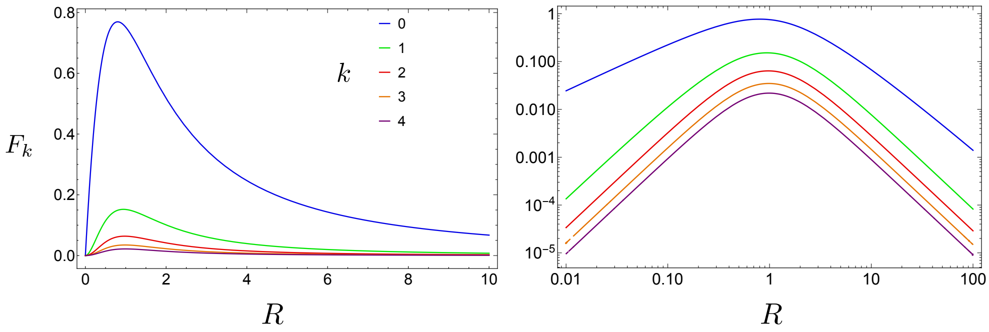

Figure 11: First five terms of the series expansion of the function F which is the radial dependance of the deformation of the interface for a semi-infinite substrate.

D Better formulation

Finally I found a way to make the integration. Let me remind you that I'm considering the problem

$$\vec{\vec{\sigma}}_{\text{substrate}} = -\mathcal{P}\vec{\vec{I}} + \mu(\vec{\nabla}\vec{u}_{\text{green}} + \vec{\nabla}\vec{u}_{\text{green}}^T), \quad \text{and} \quad \vec{\nabla}.\vec{u}_{\text{green}} = 0 \tag{243}$$

$$\forall r \in \mathbb{R}^+, \quad \forall \theta \in [0, 2\pi[, \quad \forall z \in [-h_{\text{sub}}, 0], \quad \vec{\nabla}.\vec{\vec{\sigma}}_{\text{substrate}} = \vec{0} \tag{244}$$

$$\text{at } z = -h_{\text{sub}}, \quad \vec{u}_{\text{green}} = 0 \tag{245}$$

$$\text{at } z = 0, \quad \sigma_{zz} = -p_{\text{ext}} = -f(r)\cos(\theta), \quad \sigma_{zr} = \sigma_{z\theta} = 0 \tag{246}$$

$$\vec{\mathcal{G}}(\vec{r}) = \vec{u}_{\text{green}}(\vec{r}, z = 0) \tag{247}$$

with $f(r) = \frac{r}{(1+r^2)^2}$ in the elastohydrodynamic problem. It is actually simpler to use the Fourier transform. I choose the exact same convention for the Fourier transform as in *Skotheim & Mahadevan 2005* since this calculation is very close to section VI of this paper.

$$u_z(x, y, z) = \int_{\mathbb{R}^2} \mathrm{d}k_x \mathrm{d}k_y \, \hat{u}_z(k_x, k_y, z) e^{-i(k_x x + k_y y)} \tag{248}$$

$$\hat{u}_z(k_x, k_y, z) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathrm{d}x \mathrm{d}y \, u_z(x, y, z) e^{i(k_x x + k_y y)} \tag{249}$$

We can directly uses the equation (63) of their paper which is the same as the one that I derived in the Appendix A but with the Fourier transform instead of the Hankel transform.

$$\hat{u}_z(k_x, k_y, z = 0) = \hat{p}_{\text{ext}}(k_x, k_y) \hat{\mathcal{G}}_z(k_x, k_y) \tag{250}$$

with $\hat{\mathcal{G}}_z$ (resp. $\hat{p}_{\text{ext}}$) the Fourier transform of the Green function (resp. the external pressure).

$$\hat{\mathcal{G}}_z(k_x, k_y) = \frac{2kh_{\text{sub}} - \sinh(2kh_{\text{sub}})}{2\mu k(1 + 2(kh_{\text{sub}})^2 + \cosh(2kh_{\text{sub}}))} = \frac{kh_{\text{sub}} - \sinh(kh_{\text{sub}})\cosh(kh_{\text{sub}})}{2\mu k(\cosh(kh_{\text{sub}})^2 + (kh_{\text{sub}})^2)} \tag{251}$$

with $k = \sqrt{k_x^2 + k_y^2}$ and that $k \to k\hat{\mathcal{G}}_z(k)$ is plotted in Figure 8, and

$$\hat{p}_{\text{ext}}(k, \theta) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathrm{d}x \mathrm{d}y \, p_{\text{ext}}(x, y) e^{i(k_x x + k_y y)} = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} r \mathrm{d}r \mathrm{d}\phi \, f(r) \cos(\phi) e^{ikr\cos(\theta - \phi)} \tag{252}$$

We integrate this latter equation with respect to the angular variable first

$$\begin{aligned}\int_0^{2\pi} \mathrm{d}\phi \cos(\phi) e^{ikr\ \cos(\theta-\phi)} &= \int_0^{2\pi} \mathrm{d}\phi' \cos(\theta-\phi') e^{ikr\ \cos(\phi')} \\ &= \int_0^{2\pi} \mathrm{d}\phi' \Big(\cos(\theta)\cos(\phi') + \sin(\theta)\sin(\phi') \Big) e^{ikr\ \cos(\phi')} \\ &= \int_{-\pi}^{\pi} \mathrm{d}\phi' \Big(\cos(\theta)\cos(\phi') + \underbrace{\sin(\theta)\sin(\phi')}_{=0} \Big) e^{ikr\ \cos(\phi')} \\ &= 2\cos(\theta) \int_0^{\pi} \mathrm{d}\phi' \cos(\phi') e^{ikr\ \cos(\phi')} = 2i\pi J_1(kr)\end{aligned} \tag{253}$$

The term in sin vanishes (odd function) and I have used the definition of the Bessel function of first kind and index 1[6]. Note that this calculation can eventually be generalized for other "mode"

$$\int_0^{2\pi} \mathrm{d}\phi \cos(n\phi) e^{ikr\ \cos(\theta-\phi)} = i^n 2\pi J_n(kr) \tag{254}$$

Then the Fourier transform of the pressure is

$$\begin{aligned}\hat{p}_{\mathrm{ext}}(k,\theta) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} r\mathrm{d}r\mathrm{d}\phi\, f(r)\cos(\phi) e^{ikr\ \cos(\theta-\phi)} \\ &= \frac{i}{(2\pi)} \cos(\theta) \underbrace{\int_{\mathbb{R}+} r\mathrm{d}r\, f(r) J_1(kr)}_{\tilde{f}(k)} = \frac{i}{(2\pi)} \tilde{f}(k)\ \cos(\theta)\end{aligned} \tag{255}$$

with $\tilde{f}$ the Hankel transform of index 1 of the function f. Then the Fourier transform of the displacement

$$\hat{u}_z(k,\theta,z=0) = \frac{i}{(2\pi)} \tilde{f}(k)\ \cos(\theta) \hat{\mathcal{G}}_z(k) \tag{256}$$

and therefore

$$\begin{aligned}u_z(r,\theta,z=0) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} k\mathrm{d}k\mathrm{d}\phi\, \hat{u}_z(k,\phi,z=0) e^{-ikr\cos(\theta-\phi)} \\ &= \frac{i}{2\pi} \int_{\mathbb{R}^2} k\mathrm{d}k\mathrm{d}\phi\, \tilde{f}(k)\ \cos(\phi) \hat{\mathcal{G}}_z(k) e^{-ikr\cos(\theta-\phi)} \\ &= \cos(\theta) \underbrace{\int_{\mathbb{R}+} k\mathrm{d}k\, \tilde{f}(k) \hat{\mathcal{G}}_z(k) J_1(kr)}_{F(r)} = F(r)\ \cos(\theta)\end{aligned} \tag{257}$$

We recover the same form as before. We know have an analytical form of F for any pressure field and any Green function (or elasticity model). For the case that we have considered in the main text, we can use the semi-infinite expression

$$\hat{\mathcal{G}}_z(k) = \frac{1}{2k} \tag{258}$$

and the leading order pressure (without prefactor)

$$f(r) = \frac{r}{(1+r^2)^2} \tag{259}$$

[6] $J_1(x) = \frac{1}{i\pi}\int_0^{\pi} \mathrm{d}\phi \cos(\phi) e^{ix\cos(\phi)}$

such that

$$\tilde{f}(k) = \int_0^\infty k\mathrm{d}k \, \frac{r}{(1+r^2)^2} J_1(kr) = \frac{k}{2} K_0(k) \tag{260}$$

with K_0 the modified Bessel function of index zero[7]. Then for this specific case, *Mathematica* gave me a analytical solution.

$$\begin{aligned} F(r) &= \int_{\mathbb{R}+} k\mathrm{d}k \, \tilde{f}(k)\hat{\mathcal{G}}_z(k)J_1(kr) \\ &= \frac{1}{2}\int_{\mathbb{R}+} \mathrm{d}k \, \tilde{f}(k)J_1(kr) \\ &= \frac{1}{4}\int_{\mathbb{R}+} \mathrm{d}k \, K_0(k)kJ_1(kr) \\ &= \frac{1}{4}\left(\frac{K(-r^2)}{r} - \frac{E(-r^2)}{r(1+r^2)}\right) \end{aligned} \tag{261}$$

Where K and E are complete Elliptic function

$$K(x) = \int_0^{\pi/2} \mathrm{d}\theta \frac{1}{\sqrt{1-k^2\sin^2(\theta)}} \tag{262}$$

$$E(x) = \int_0^{\pi/2} \mathrm{d}\theta \sqrt{1-k^2\sin^2(\theta)} \tag{263}$$

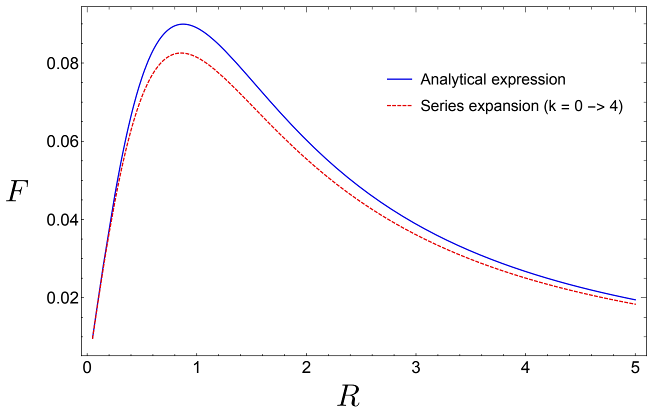

Figure 12: Radial function $F(r)$ defined in the main text with $U = 1$ and the other prefactor set to unity as well. The sum of the 5 first terms of the series expansion is plotted for comparison.

I plotted this analytical function in Figure 12. We recover the same shape as in the series expansion method (or numerics). This analytical follows the same asymptotic behavior as before $F(r) \sim r$ for $r \to 0$ and $F(r) \sim \log(r)/r^2$ for $r \to \infty$. I just have a factor 4π of difference in the amplitude of F. That makes total sense because I forgot a factor $1/2\pi$ in the convolution product, and a factor $1/2$ in the Green function.

[7] K_0 solution of $x^2u''(x) + xu'(x) - x^2u(x) = 0$ that diverges in 0.

For latter : As I suspected, the Hankel transform appears in the mode decomposition solution found with the Fourier transform. In a near future, I will look again into the solution found with the Fourier transform framework to understand why the approach in the Appendix B does not work.