Problem 1:

1. Since each color will be associated with a weight $W_a \in R$ for each $a \in \{1, \cdots K\}$, when $W_a = a$;

the joint probability $p(x) = \frac{1}{Z} \prod_{i \in V} e^{x_i} \prod_{(i,j)\in E} 1_{x_i \neq x_j}$

with $Z = \sum_{x_V \in \{1,\cdots K\}^{|V|}} \prod_{i \in V} e^{x_i} \prod_{(i,j)\in E} 1_{x_i \neq x_j}$;

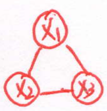
since max assignment $x^* = \arg\max_x p(x)$, for the cycle G with even number of vertices ($|V| = 2t$, $t \in N$), in order to maximize $p(x)$, it's conceivable to assign the color $(K-1)$ and (K) repetitively to the adjacent vertices. More specifically, the MAP assignment will be for x_i ($i \in \{1, 2, \cdots 2t\}$),

$$\begin{cases} \text{when } i = 2\ell - 1 \ (\ell \in \{1,2,\cdots t\}),\ x_i = K-1; \\ \text{and when } i = 2\ell \ (\ell \in \{1,2,\cdots t\}),\ x_i = K; \end{cases}$$

or

$$\begin{cases} \text{when } i = 2\ell - 1 \ (\ell \in \{1,2,\cdots t\}),\ x_i = K; \\ \text{and when } i = 2\ell \ (\ell \in \{1,2,\cdots t\}),\ x_i = K-1; \end{cases}$$

As a specific example, when $|V| = 6$, we can find x^* as $= (K-1, K, K-1, K, K-1, K)$ or $(K, K-1, K, K-1, K, K-1)$.

2. 

for MAP, we need to

$$\max_{q \in Q} \sum_{i \in V} \sum_{x_i} q_i(x_i) \log \phi_i(x_i) + \sum_{(i,j)\in E} \sum_{x_i, x_j} q_{ij}(x_i, x_j) \log \psi_{ij}(x_i, x_j)$$

in our example, $\phi_i(x_i) = e^{x_i}$, $\psi_{ij}(x_i, x_j) = 1_{x_i \neq x_j}$;

$$= \max_{q \in Q} \sum_{i \in V} \sum_{x_i} q_i(x_i) \cdot x_i + \sum_{(i,j)\in E} \sum_{x_i, x_j} q_{ij}(x_i, x_j) \log \psi_{ij}(x_i, x_j)$$

for both IP and LP, the common constraints include:

Set 1: when $x_i = x_j$, $1_{x_i \neq x_j} = 0$, and its log will be negative infinity, thus in those cases,

$q_{ij}(x_i, x_j) = 0$;

$q_{12}(x_1 = 1, x_2 = 1) = 0$, $q_{12}(x_1 = 2, x_2 = 2) = 0$, $q_{12}(x_1 = 3, x_2 = 3) = 0$

$q_{13}(x_1 = 1, x_3 = 1) = 0$, $q_{13}(x_1 = 2, x_3 = 2) = 0$, $q_{13}(x_1 = 3, x_3 = 3) = 0$

$q_{23}(x_2 = 1, x_3 = 1) = 0$, $q_{23}(x_2 = 2, x_3 = 2) = 0$, $q_{23}(x_2 = 3, x_3 = 3) = 0$

Set 2: $q_1(x_1 = 1) + q_1(x_1 = 2) + q_1(x_1 = 3) = 1$

$q_2(x_2 = 1) + q_2(x_2 = 2) + q_2(x_2 = 3) = 1$

$q_3(x_3 = 1) + q_3(x_3 = 2) + q_3(x_3 = 3) = 1$

Set 3: $\sum_{x_j} q_{ij}(x_i, x_j) = q_i(x_i)$ for all $(i,j) \in E$, x_i;

such as: $q_{12}(x_1 = 1, x_2 = 1) + q_{12}(x_1 = 1, x_2 = 2) + q_{12}(x_1 = 1, x_2 = 3) = q_1(x_1 = 1)$

$q_{12}(x_1 = 2, x_2 = 1) + q_{12}(x_1 = 2, x_2 = 2) + q_{12}(x_1 = 2, x_2 = 3) = q_1(x_1 = 2)$

Then in addition, for IP:

set 4: all $q_i(X_i) \in \{0,1\}$, and $q_{ij}(X_i, X_j) \in \{0,1\}$

for LP:

set 4: all $q_i(X_i) \in [0,1]$, and $q_{ij}(X_i, X_j) \in [0,1]$

Thus: $\max\limits_{q \in Q} \sum\limits_{i \in V} \sum\limits_{X_i} q_i(X_i) \cdot X_i + \sum\limits_{(i,j) \in E} \sum\limits_{X_i, X_j} q_{ij}(X_i, X_j) \cdot \log 1_{X_i \neq X_j}$

$= \max\limits_{q \in Q} [\, 2 \cdot q_1(X_1=2) + 1 \cdot q_1(X_1=1) + 3 \cdot q_1(X_1=3) + 1 \cdot q_2(X_2=1) + 2 \cdot q_2(X_2=2) + 3 \cdot q_2(X_2=3)$

$+ 1 \cdot q_3(X_3=1) + 2 \cdot q_3(X_3=2) + 3 \cdot q_3(X_3=3 \,]$

For IP: since X_1, X_2, X_3 are symmetric, if we assign $q_1(X_1=3)=1$, thus giving it's largest possible sum of $1 \cdot q_1(X_1=1) + 2 \cdot q_1(X_1=2) + 3 \cdot q_1(X_1=3)$.

Then for the constraints, we can derive that either $q_2(X_2=2)=1$, and $q_3(X_3=1)=1$,

or $q_2(X_2=1)=1$, and $q_3(X_3=2)=1$,

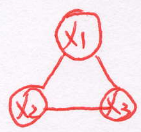
Thus in all cases, the maximal value will be $3 \cdot 1 + 2 \cdot 1 + 1 \cdot 1 = 6$;

For LP: if we subject the $\max\limits_{q \in Q}$ function to all the constraints, and use linear programming solver (such as www.comnuan.com/cmnn03/cmnn03004),

The solutions are $q_1(X_1=2) = q_1(X_1=3) = q_2(X_2=2) = q_2(X_2=3) = q_3(X_3=2) = q_3(X_3=3) = 1/2$;

Thus the maximal value will be $(2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2}) \cdot 3 = 7.5$;

Problem 2. Part 1: Here I use a simple example to illustrate the procedures of the MATLAB code:



$$\phi_i(x_i) = e^{x_i}, \quad \psi_{ij}(x_i, x_j) = 1_{x_i \neq x_j}; \quad x_i \in \{1,2,3\}$$

Step 1: calculate the messages:

first assign $m^0_{1\to 2}(x_2) = m^0_{1\to 3}(x_3) = m^0_{2\to 3}(x_3) = m^0_{2\to 1}(x_1) = m^0_{3\to 1}(x_1) = m^0_{3\to 2}(x_2) =$

Then $m^t_{i\to j}(x_j) = \sum_{x_i} \left[\phi_i(x_i)\,\psi_{ij}(x_i, x_j) \prod_{k\in N(i)\setminus j} m^{t-1}_{k\to i}(x_i) \right]$

for example. $m^1_{1\to 2}(x_2=1) = \sum_{x_1} \left[\phi_1(x_1)\psi_{12}(x_1, x_2=1)\cdot m^0_{3\to 1}(x_1) \right]$

$= \phi_1(x_1=1)\psi_{12}(x_1=1, x_2=1)\cdot m^0_{3\to 1}(x_1=1) + \phi_1(x_1=2)\cdot\psi_{12}(x_1=2, x_2=1)\cdot m^0_{3\to 1}(x_1=2)$
$+ \phi_1(x_1=3)\,\psi_{12}(x_1=3, x_2=1)\cdot m^0_{3\to 1}(x_1=3)$

$= e^2\cdot 1 + e^3\cdot 1$

similarly, $m^1_{1\to 2}(x_2=2) = e^1\cdot 1 + e^3\cdot 1$

$m^1_{1\to 2}(x_2=3) = e^1\cdot 1 + e^2\cdot 1$

In order to avoid overflow, we sum $\sum_{x_j} m^t_{i\to j}(x_j)$, and use $1/\left(\sum_{x_j} m^t_{i\to j}(x_j)\right)$ as the $\eta^t_{i\to j}$, in above case, after normalization:

$m^1_{1\to 2}(x_2=1) = (e^2+e^3)/(2e+2e^2+2e^3)$

$m^1_{1\to 2}(x_2=2) = (e+e^3)/(2e+2e^2+2e^3)$

$m^1_{1\to 2}(x_2=3) = (e+e^2)/(2e+2e^2+2e^3)$

We will repeat this for all other five messages, normalize each, and finish this iteration;

We will repeat this whole iteration its times, and hopefully it will converge;

For this specific example, after converging (11 iterations). the values are like:

$m_{1\to 2}(x_2=1) = 0.4293$, $m_{1\to 2}(x_2=2) = 0.3453$, $m_{1\to 2}(x_2=3) = 0.2254$
$m_{1\to 3}(x_3=1) = 0.4291$, $m_{1\to 3}(x_3=2) = 0.3445$, $m_{1\to 3}(x_3=3) = 0.2265$
$m_{2\to 1}(x_1=1) = 0.4291$, $m_{2\to 1}(x_1=2) = 0.3445$, $m_{2\to 1}(x_1=3) = 0.2265$
$m_{2\to 3}(x_3=1) = 0.4293$, $m_{2\to 3}(x_3=2) = 0.3453$, $m_{2\to 3}(x_3=3) = 0.2255$
$m_{3\to 1}(x_1=1) = 0.4293$, $m_{3\to 1}(x_1=2) = 0.3453$, $m_{3\to 1}(x_1=3) = 0.2254$
$m_{3\to 2}(x_2=1) = 0.4294$, $m_{3\to 2}(x_2=2) = 0.3459$, $m_{3\to 2}(x_2=3) = 0.2247$

Step 2: calculate the beliefs:

for $b_i(x_i) = \phi_i(x_i) \prod_{k\in N(i)} m_{k\to i}(x_i)$

for example, $b_1(x_1=1) = \phi_1(x_1=1)\cdot m_{2\to 1}(x_1=1)\cdot m_{3\to 1}(x_1=1)$

$= e^1 \cdot 0.4291 \cdot 0.4293$

$b_1(x_1=2) = \phi_1(x_1=2)\cdot m_{2\to 1}(x_1=2)\cdot m_{3\to 1}(x_1=2)$

$= e^2 \cdot 0.3445 \cdot 0.3453$

$b_1(x_1=3) = \phi_1(x_1=3)\cdot m_{2\to 1}(x_1=3)\cdot m_{3\to 1}(x_1=3)$

$= e^3 \cdot 0.2265 \cdot 0.2254$

since $\sum_{x_i} b_i(x_i) = 1$, we normalize by $\gamma_1 = 1/(b_1(x_1=1) + b_1(x_1=2) + b_1(x_1=3))$

and we get $b_1(x_1=1) = 0.21$, $b_1(x_1=2) = 0.36$, $b_1(x_1=3) = 0.43$;

we repeat this for $b_2(x_2)$ and $b_3(x_3)$;

and $b_2(x_2=1) = 0.21$, $b_2(x_2=2) = 0.36$, $b_2(x_2=3) = 0.43$;

$b_3(x_3=1) = 0.21$, $b_3(x_3=2) = 0.36$, $b_3(x_3=3) = 0.43$;

for $b_{ij}(x_i, x_j) = \phi_i(x_i)\phi_j(x_j)\psi_{ij}(x_i, x_j) \cdot \prod_{k\in N(i)\setminus j} m_{k\to i}(x_i) \cdot \prod_{k\in N(j)\setminus i} m_{k\to j}(x_j)$

for example, $b_{12}(x_1=1, x_2=1) = \phi_1(x_1=1)\cdot\phi_2(x_2=1)\,\psi_{12}(x_1=1, x_2=1)\cdot m_{3\to 1}(x_1=1)\cdot m_{3\to 2}(x_2=1) = 0$

$b_{12}(x_1=1, x_2=2) = e^1\cdot e^2\cdot m_{3\to 1}(x_1=1)\cdot m_{3\to 2}(x_2=2)$

$b_{12}(x_1=1, x_2=3) = e^1\cdot e^3\cdot m_{3\to 1}(x_1=1)\cdot m_{3\to 2}(x_2=3)$

$b_{12}(x_1=2, x_2=1) = e^2\cdot e\cdot m_{3\to 1}(x_1=2)\cdot m_{3\to 2}(x_2=1)$

$b_{12}(x_1=2, x_2=2) = 0$

$b_{12}(x_1=2, x_2=3) = e^2\cdot e^3\cdot m_{3\to 1}(x_1=2)\cdot m_{3\to 2}(x_2=3)$

$b_{12}(x_1=3, x_2=1) = e^3\cdot e\cdot m_{3\to 1}(x_1=3)\cdot m_{3\to 2}(x_2=1)$

$b_{12}(x_1=3, x_2=2) = e^3\cdot e^2\cdot m_{3\to 1}(x_1=3)\cdot m_{3\to 2}(x_2=2)$

$b_{12}(x_1=3, x_2=3) = 0$

since $\sum_{x_i x_j} b_{ij}(x_i, x_j) = 1$, we normalize $b_{12} = 1/\sum_{x_1 x_2} b_{12}(x_1, x_2)$;

and we get: $b_{12}(x_1=1, x_2=1) = 0$, $b_{12}(x_1=1, x_2=2) = 0.0773$, $b_{12}(x_1=1, x_2=3) = 0.1330$

$b_{12}(x_1=2, x_2=1) = 0.0772$, $b_{12}(x_1=2, x_2=2) = 0$, $b_{12}(x_1=2, x_2=3) = 0.2908$

$b_{12}(x_1=3, x_2=1) = 0.1335$, $b_{12}(x_1=3, x_2=2) = 0.2923$, $b_{12}(x_1=3, x_2=3) = 0$

We repeat this for $b_{13}(x_1, x_3)$ and $b_{23}(x_2, x_3)$,

and we get: $b_{13}(x_1=1, x_3=1)=0$, $b_{13}(x_1=1, x_3=2)=0.0750$, $b_{13}(x_1=1, x_3=3)=0.1331$

$b_{13}(x_1=2, x_3=1)=0.0749$, $b_{13}(x_1=2, x_3=2)=0$, $b_{13}(x_1=2, x_3=3)=0.2906$;

$b_{13}(x_1=3, x_3=1)=0.1338$, $b_{13}(x_1=3, x_3=2)=0.2926$, $b_{13}(x_1=3, x_3=3)=0$;

$b_{23}(x_2=1, x_3=1)=0$, $b_{23}(x_2=1, x_3=2)=0.0749$, $b_{23}(x_2=1, x_3=3)=0.1338$

$b_{23}(x_2=2, x_3=1)=0.0750$, $b_{23}(x_2=2, x_3=2)=0$, $b_{23}(x_2=2, x_3=3)=0.2926$;

$b_{23}(x_2=3, x_3=1)=0.1331$, $b_{23}(x_2=3, x_3=2)=0.2905$, $b_{23}(x_2=3, x_3=3)=0$;

All match the results produced by the MATLAB code.

Step 3: calculate the partition function Z;

since $p(x) = \frac{1}{Z} \prod_{i \in V} \phi_i(x_i) \cdot \prod_{(i,j) \in E} 1_{x_i \neq x_j}$

$$D(q \| p) = -H(q) + \log Z - \sum_x q(x) \left[\sum_{i \in V} \log \phi_i(x_i) + \sum_{(i,j) \in E} \log 1_{x_i \neq x_j} \right]$$

$$= -H(q) + \log Z - \sum_{i \in V} \sum_{x_i} q_i(x_i) \log \phi_i(x_i) - \sum_{(i,j) \in E} \sum_{x_i x_j} q_{ij}(x_i, x_j) \cdot \log 1_{x_i \neq x_j}$$

Thus when local maximization is achieved;

$$\log Z = H(q) + \sum_{i \in V} \sum_{x_i} b_i(x_i) \log \phi_i(x_i) + \sum_{(i,j) \in E} \sum_{x_i x_j} b_{ij}(x_i, x_j) \cdot \log 1_{x_i \neq x_j}$$

$$H(q) = -\sum_{i \in V} \sum_{x_i} b_i(x_i) \cdot \log b_i(x_i) - \sum_{(i,j) \in E} \sum_{x_i x_j} b_{ij}(x_i, x_j) \cdot \log \frac{b_{ij}(x_i, x_j)}{b_i(x_i) \cdot b_j(x_j)}$$

In above case, the Z is calculated as 4493.2;

Notes for the matlab code:

For a graph $G:(V,E)$, it has N nodes, each node can take k value;

Step 1: create the messages. The final matrix is named as C, and it's $N^2 \times k$ dimension as

$$\begin{array}{ll|llll} & & k=1 & k=2 & \cdots & k=K \\ m_{1\to 1} \rightsquigarrow & 11 & & & & \\ & 12 & & & & \\ & \vdots & & & & \\ & 1N & & & & \\ & \vdots & & & & \\ & N1 & & & & \\ & \vdots & & & & \\ & NN & & & & \end{array}$$

;

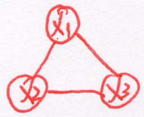
Step 2: create the singleton, named as B, and it's $N \times k$ dimension as

$$\begin{array}{ll|llll} & & k=1 & k=2 & \cdots & k=K \\ b_1(x_1) \rightsquigarrow & 1 & & & & \\ & 2 & & & & \\ & \vdots & & & & \\ & N & & & & \end{array}$$

and pairwise, named as E, and it's $N(N-1)/2 \times k^2$ dimension as

$$\begin{array}{ll|llll} & & k,k=1,1 & k,k=1,2 & \cdots\cdots & k,k=K,K \\ b_{12}(x_1,x_2) \rightsquigarrow & 12 & & & & \\ & 13 & & & & \\ & \vdots & & & & \\ & 1N & & & & \\ & 23 & & & & \\ & \vdots & & & & \\ & 2N & & & & \\ & \vdots & & & & \\ & (N-1)N & & & & \end{array}$$

Problem 2. Part 2: using the same example.



$\phi_i(x_i) = e^{x_i}$, $\psi_{ij}(x_i, x_j) = 1_{x_i \neq x_j}$; $x_i \in \{1,2,3\}$

Step 1: Calculate the messages:

first assign $m^0_{1\to2}(x_2) = m^0_{1\to3}(x_3) = m^0_{2\to3}(x_3) = m^0_{2\to1}(x_1) = m^0_{3\to1}(x_1) = m^0_{3\to2}(x_2) = 1$;

Then $m^t_{i\to j}(x_j) = \max\limits_{x_i} [\phi_i(x_i)\psi_{ij}(x_i, x_j) \prod\limits_{k \in N(i)\setminus j} m^{t-1}_{k\to i}(x_i)]$

for example, $m^1_{1\to2}(x_2=1) = \max\limits_{x_1} [\phi_1(x_1)\psi_{12}(x_1, x_2=1) \cdot m^0_{3\to1}(x_1)]$

$= \max\limits_{x_1} [\phi_1(x_1=1)\cdot\psi_{12}(x_1=1, x_2=1)\cdot m^0_{3\to1}(x_1=1),\ \phi_1(x_1=2)\cdot\psi_{12}(x_1=2, x_2=1)\cdot m^0_{3\to1}(x_1=2),$
$\phi_1(x_1=3)\psi_{12}(x_1=3, x_2=1)\cdot m^0_{3\to1}(x_1=3)]$

$= \max\limits_{x_1} (0, e^2\cdot1, e^3\cdot1) = e^3$, record $x_1=3$;

similarly, $m^1_{1\to2}(x_2=2) = \max\limits_{x_1}(e\cdot1, 0, e^3\cdot1) = e^3$, record $x_1 = 3$;

$m^1_{1\to2}(x_2=3) = \max\limits_{x_1}(e\cdot1, e^2\cdot1, 0) = e^2$, record $x_1 = 2$;

In order to avoid overflow, we sum $\sum\limits_{x_j} m^t_{i\to j}(x_j)$, and use $1/(\sum\limits_{x_j} m^t_{i\to j}(x_j))$ as the $\eta^t_{i\to j}$, in above case, after normalization:

$m^1_{1\to2}(x_2=1) = e^3/(2e^3+e^2)$; when $x_1=3$
$m^1_{1\to2}(x_2=2) = e^3/(2e^3+e^2)$; when $x_1=3$
$m^1_{1\to2}(x_2=3) = e^2/(2e^3+e^2)$; when $x_1=2$.

We will use this to calculate all other 5 messages, normalize each, and finish this iteration (note: also need to record the maximum assignment, such as above $x_1=3$ for $m^1_{1\to2}(x_2=1)$).

We will then repeat this whole iteration its times, and each time update the assignment matrix as well.

For this specific example, after 11 iterations, the values are like:

$\underset{x_1=3}{m_{1\to2}}(x_2=1)=0.4223$, $\underset{x_1=3}{m_{1\to2}}(x_2=2)=0.4223$, $\underset{x_1=2}{m_{1\to2}}(x_2=3)=0.1554$;

$\underset{x_1=2}{m_{1\to3}}(x_3=1)=0.3333$, $\underset{x_1=3}{m_{1\to3}}(x_3=2)=0.3333$, $\underset{x_1=2}{m_{1\to3}}(x_3=3)=0.3333$;

$\underset{x_2=2}{m_{2\to1}}(x_1=1)=0.3333$, $\underset{x_2=3}{m_{2\to1}}(x_1=2)=0.3333$, $\underset{x_2=2}{m_{2\to1}}(x_1=3)=0.3333$;

$\underset{x_2=2}{m_{2\to3}}(x_3=1)=0.3333$, $\underset{x_2=3}{m_{2\to3}}(x_3=2)=0.3333$, $\underset{x_2=2}{m_{2\to3}}(x_3=3)=0.3333$;

$\underset{x_3=3}{m_{3\to1}}(x_1=1)=0.4223$, $\underset{x_3=3}{m_{3\to1}}(x_1=2)=0.4223$, $\underset{x_3=2}{m_{3\to1}}(x_1=3)=0.1554$;

$\underset{x_3=3}{m_{3\to2}}(x_2=1)=0.4223$, $\underset{x_3=3}{m_{3\to2}}(x_2=2)=0.4223$, $\underset{x_3=2}{m_{3\to2}}(x_2=3)=0.1554$;

Note for Matlab code maxprod.m;

Step 1: using Max-product to calculate the message matrix, which is named as C, and it's $N^2 \times K$ dimension, as:

$$
\begin{array}{ll|cccc}
 & & k=1 & k=2 & \cdots & k=K \\
m_{i\to j} \sim & 11 & & & & \\
 & 12 & & & & \\
 & \vdots & & & & \\
 & 1N & & & & \\
 & 21 & & & & \\
 & \vdots & & & & \\
 & N1 & & & & \\
 & \vdots & & & & \\
 & NN & & & &
\end{array} ;
$$