Average treatment effect on the treated, under lack of positivity

Yi Liu

Department of Statistics

NC STATE UNIVERSITY

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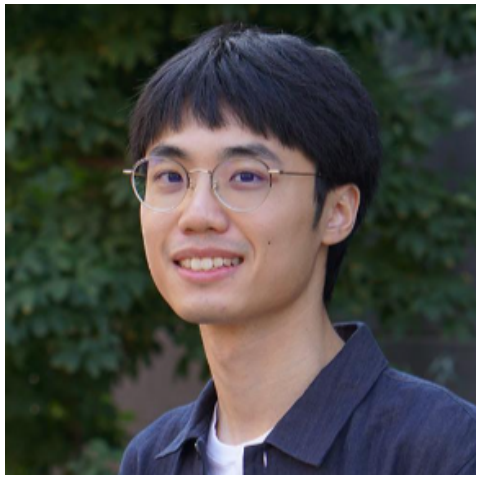
Acknowledgement





- Roland Matsouaka, PhD (Duke Biostat & Bioinfo, DCRI)
- Huiyue Li (Baim Institute for Clinical Research, Boston)
- Yunji Zhou (University of Washington Biostat)

Publication

- **Liu Y**, Li H, Zhou Y, and Matsouaka RA (2024). Average treatment effect on the treated, under lack of positivity. *Statistical Methods in Medical Research*. Forthcoming.

Overview

- Sufficient **positivity** plays a very key role in propensity score methods for causal inference.
- Each participant should have certain (strictly non-zero) probabilities to receive either treatment or control, given their baseline covariates.
- Analogy in missing data analysis: the missing data mechanism for each participant gives a non-zero probability for their data is missing.
- In this talk, we focus on the positivity violation when the interest is identifying average treatment effect on the treated (ATT), i.e.,

$$\mathbb{E}\{\text{treatment effect} \mid \text{treated subjects}\}.$$

Overview

The literature defines two types of violation of positivity.[1]

- Random violation (i.e., by chance): due to small sample sizes, model misspecifications, etc.
- Structural violation: expected due to the inherent characteristics of the target population.
  - ATT is technically not identifiable in this case!

[1] Petersen, M. L. *et al.* Diagnosing and responding to violations in the positivity assumption. *Statistical methods in medical research* **21,** 31–54 (2012).

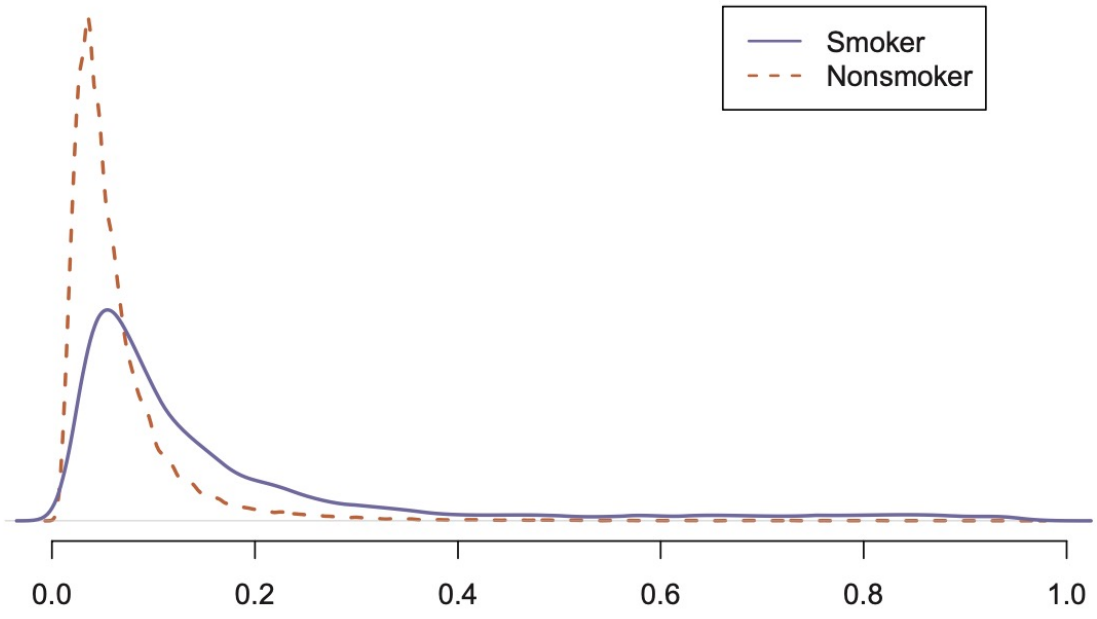
Set-up

- We assume the complete data is from a super-population model $f(x, z, y(0), y(1))$.
- We have a covariate vector X, a binary treatment $Z \in \{0, 1\}$, and two potential outcomes $Y(0)$ and $Y(1)$.
- We can observe the outcome associated with the treatment value of each subject in the data, i.e., $Y = Y(Z)$.
- Assume also $Y(z) \perp\!\!\!\perp Z \mid X$ for both $z = 0, 1$.
- **Positivity**: the propensity score $e(X) = P(Z = 1 \mid X)$ (a subject-specific score) must satisfy $0 < e(X) < 1$ w.p.1.
- Or **strict positivity**[2]: $c_1 \leq e(X) \leq c_2$ for some $0 < c_1 < c_2 < 1$ w.p.1.

[2] Hirano, K. *et al.* Efficient estimation of average treatment effects using the estimated propensity score. *Econometrica* **71,** 1161–1189 (2003), D'Amour, A. *et al.* Overlap in observational studies with high-dimensional covariates. *Journal of Econometrics* **221,** 644–654 (2021).

Positivity violation example

Extreme propensity scores: NC birth weights data[3]



	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
Smoker	0.0044	0.06	0.10	0.17	0.18	0.98
Nonsmoker	0.0014	0.03	0.05	0.07	0.08	0.97

[3] Zhou, Y. *et al.* Propensity score weighting under limited overlap and model misspecification. *Statistical Methods in Medical Research* **29,** 3721–3756 (2020).

Trimming or truncating extreme weights

Two common practices for excluding/capping extreme weights:

- Trimming: exclude participants with estimated $e(X)$ outside a range $[c_1, c_2]$, where $0 < c_1 < c_2 < 1$.
- Truncation: a weight capping, i.e., assign c_1 as the new propensity score to those $e(X) < c_1$ and c_2 to those $e(X) > c_2$.

*Moving the **goalposts**:*

- There is a (even asymptotically) non-negligible bias when using trimming or truncation.
- In fact, they moved the target (goalposts)[4] of inference. For example, the trimming targets $\mathcal{O}(X) = \{X : c_1 < e(X) < c_2\}$.

[4] Crump, R. *et al. Moving the goalposts: Addressing limited overlap in the estimation of average treatment effects by changing the estimand.* 2006.

Average treatment effect on the treated (ATT)

- This talk focuses on the "ATT-type inference".
- First, the ATT is defined by
  $\tau_{att} = \mathbb{E}\{Y(1) - Y(0) \mid Z = 1\} = \mathbb{E}\{Y \mid Z = 1\} - \mathbb{E}\{Y(0) \mid Z = 1\}$.
- $Y(0)$ is missing (unobserved) for $Z = 1$ group.
- However, we can re-write ATT using the propensity score:

$$\tau_{att} = \frac{\mathbb{E}(ZY)}{\mathbb{E}(Z)} - \frac{\mathbb{E}\{w_0(X)(1-Z)Y\}}{\mathbb{E}\{w_0(X)(1-Z)\}},$$

  where $w_0(X) = \dfrac{e(X)}{1-e(X)}$.

- A weighting estimator for ATT:

$$\widehat{\tau}_{att} = \frac{\sum_{i=1}^N Z_i Y_i}{\sum_{i=1}^N Z_i} - \frac{\sum_{i=1}^N (1-Z_i)\widehat{w}_0(X_i) Y_i}{\sum_{i=1}^N (1-Z_i)\widehat{w}_0(X_i)}.$$

- Extreme weights occur when $e(X) \approx 1$ in control participants.

Positivity issue in ATT identification

- Positivity assumptions for identifying ATT using propensity score weighting:
  - (a) $P(Z = 1) > 0$. We need a fraction of the population to receive treatment.
  - (b) $e(X) < 1$ with probability 1 on control participants.
- More insights can be found from Abadie et al. and Heckman et al. for these assumptions.[5]

[5] Abadie, A. & Imbens, G. W. *Matching on the estimated propensity score.* Tech. rep. (National Bureau of Economic Research, 2009), Heckman, J. J. *et al.* Matching as an econometric evaluation estimator. *The review of economic studies* **65,** 261–294 (1998).

Moving the goalpost: weighted ATT (WATT)

The WATT is defined by:

$$\tau^h_{watt} = \frac{\mathbb{E}(ZY)}{\mathbb{E}(Z)} - \frac{\mathbb{E}\{\omega_{0h}(X)(1-Z)Y\}}{\mathbb{E}\{\omega_{0h}(X)(1-Z)\}}, \quad \text{with } \omega_{0h}(x) = w_0(x)h(x) = \frac{e(x)h(x)}{1-e(x)}.$$

- $h(x)$ is a tilting function. It generalizes the weights on control and thus generalizes the estimand.
- A weighting estimator for ATT:

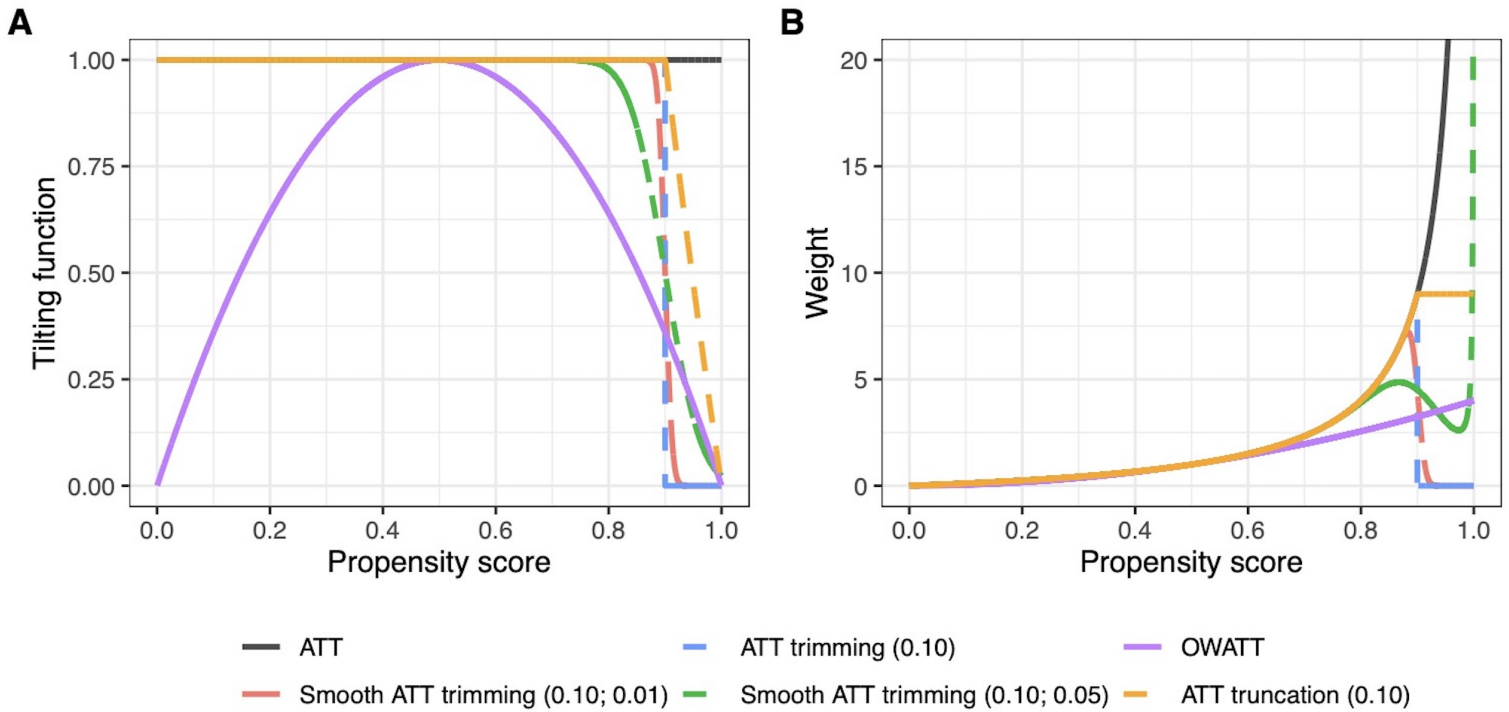
$$\widehat{\tau}^h_{watt} = \frac{\sum_{i=1}^N Z_i Y_i}{\sum_{i=1}^N Z_i} - \frac{\sum_{i=1}^N (1-Z_i)\widehat{\omega}_{0h}(X_i) Y_i}{\sum_{i=1}^N (1-Z_i)\widehat{\omega}_{0h}(X_i)}.$$

- The idea of defining this WATT mimics the idea of weighted average treatment effect (WATE).[6]

[6] Hirano, K. *et al.* Efficient estimation of average treatment effects using the estimated propensity score. *Econometrica* **71,** 1161–1189 (2003), Li, F. *et al.* Balancing covariates via propensity score weighting. *Journal of the American Statistical Association* **113,** 390–400 (2018).

Overlap weighted ATT (OWATT)

A: $h(x)$ vs. $e(x)$, and **B**: weights $\omega_{0h}(x)$ on the controls.



The purple curves correspond to $h(x) = e(x)\{1 - e(x)\}$ (**overlap** function). We call the WATT when choosing the overlap function as the tilting functions by **"overlap weighted ATT (OWATT)"**.

Inference

Assuming we use a GLM for the propensity score $e(x) = e(x'\beta)$, the estimator $\widehat{\tau}^h_{watt}$ is regular and asymptotic linear (RAL), with

$$\sqrt{N}(\widehat{\tau}^h_{watt} - \tau^h_{watt}) \to_d \mathcal{N}(0, \sigma^2 + b_1' \mathcal{I}(\beta^*)^{-1} b_1 - b_2' \mathcal{I}(\beta^*)^{-1} b_2),$$

where $\sigma^2 = \sum_{z=0}^{1} \mathbb{E}\left\{\eta_z(X)\{\mu\{z, e(X)\}^2 + \sigma^2\{z, e(X)\} + \sigma^2(z, X)\}\right\}$ with

$$\eta_1(X) = \frac{e(X)}{\mathbb{E}\{e(X)\}^2}, \quad \eta_0(X) = \frac{\omega_{0h}(X)^2\{1 - e(X)\}}{\mathbb{E}\{e(X)h(X)\}^2},$$

$$\mu\{z, e(X)\} = \mathbb{E}\{Y \mid e(X), Z = z\},$$

$$\sigma^2\{z, e(X)\} = \text{var}\{Y \mid e(X), Z = z\},$$

$$\sigma^2(z, X) = \text{var}\{Y \mid X, Z = z\}, \quad \text{for } z = 0, 1,$$

where $\mathcal{I}(\beta^*)$ is the Fisher's information matrix of β, with β^* the truth of β, and

$$b_1' = \mathbb{E}\left\{\frac{\partial}{\partial \beta'}\left[\frac{e(X'\beta^*)}{\mathbb{E}\{e(X'\beta^*)\}}\right]\mu(1, X) - \frac{\partial}{\partial \beta'}\left[\frac{e(X'\beta^*)h(X'\beta^*)}{\mathbb{E}\{e(X'\beta^*)h(X'\beta^*)\}}\right]\mu(0, X)\right\},$$

$$b_2' = \mathbb{E}\left\{\left[\frac{\mathbb{E}\{X\mu(1, X) \mid e(X)\}}{\mathbb{E}\{e(X)\}} + \frac{\omega_{0h}(X)\mathbb{E}\{X\mu(0, X) \mid e(X)\}}{\mathbb{E}\{e(X)h(X)\}}\right] f(X)\right\}.$$

Inference

Remarks:

- The asymptotic linearity allows the use of bootstrap for variance estimation.
- In the asymptotic variance term, $\eta_0(X) = \frac{\omega_{0h}(X)^2\{1-e(X)\}}{\mathbb{E}\{e(X)h(X)\}^2}$. Thus,
  - when $h(x) \propto 1$ (ATT), $\eta_0(X) \propto e(x)^2/\{1-e(x)\}$, which can still be extreme.
  - when $h(x) \propto e(x)\{1-e(x)\}$ (OWATT), $\eta_0(x) \propto e(x)^4\{1-e(x)\}$, which is always bounded.

Inference

We demonstrated that, when the propensity score is possibly misspecified and converges to a limit $\widetilde{e}(x)$, the asymptotic biases of estimating ATT and OWATT are, respectively,

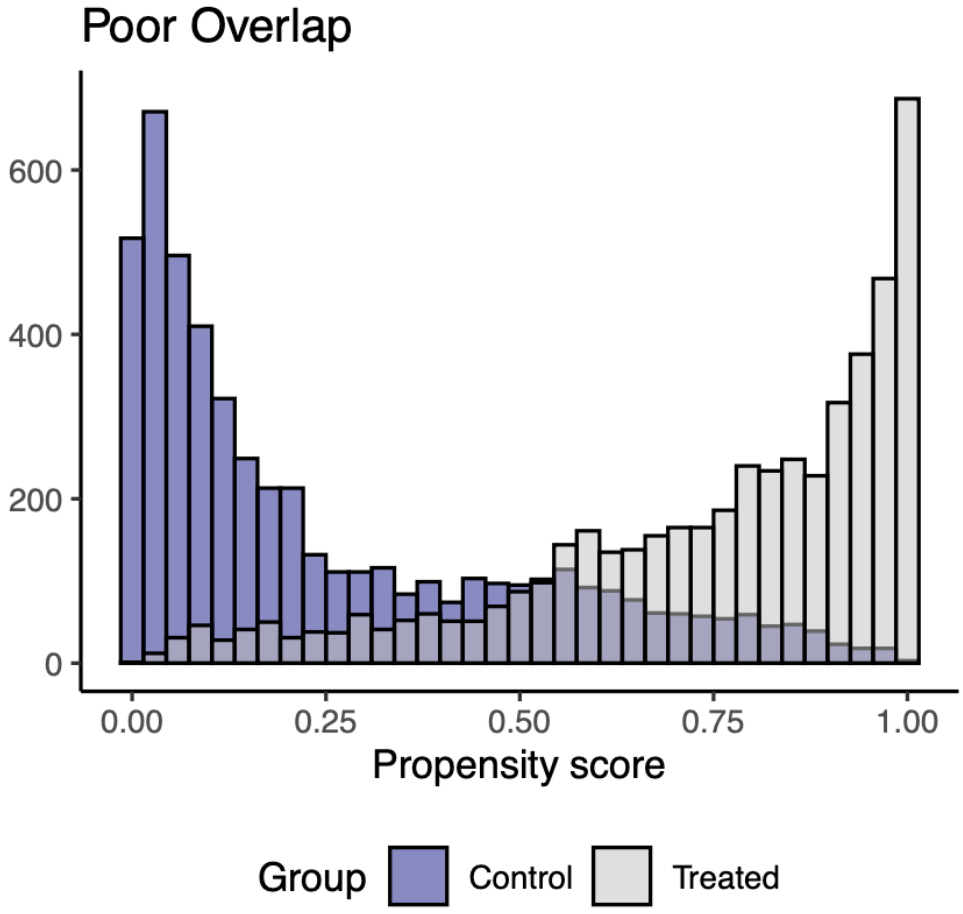
$$\text{ABias}(\widehat{\tau}_{att}) = \frac{\mathbb{E}\{e(X)m_0(X)\}}{\mathbb{E}\{e(X)\}} - \frac{\mathbb{E}\left\{\dfrac{\widetilde{e}(X)}{1-\widetilde{e}(X)}\{1-e(X)\}m_0(X)\right\}}{\mathbb{E}\left\{\dfrac{\widetilde{e}(X)}{1-\widetilde{e}(X)}\{1-e(X)\}\right\}},$$

$$\text{ABias}(\widehat{\tau}_{owatt}) = \frac{\mathbb{E}\{e(X)^2\{1-e(X)\}m_0(X)\}}{\mathbb{E}\{e(X)^2\{1-e(X)\}\}} - \frac{\mathbb{E}\left\{\widetilde{e}(X)^2\{1-e(X)\}m_0(X)\right\}}{\mathbb{E}\left\{\widetilde{e}(X)^2\{1-e(X)\}\right\}}.$$

The teal parts can incur extreme values when $\widetilde{e}(x) \to 1$.

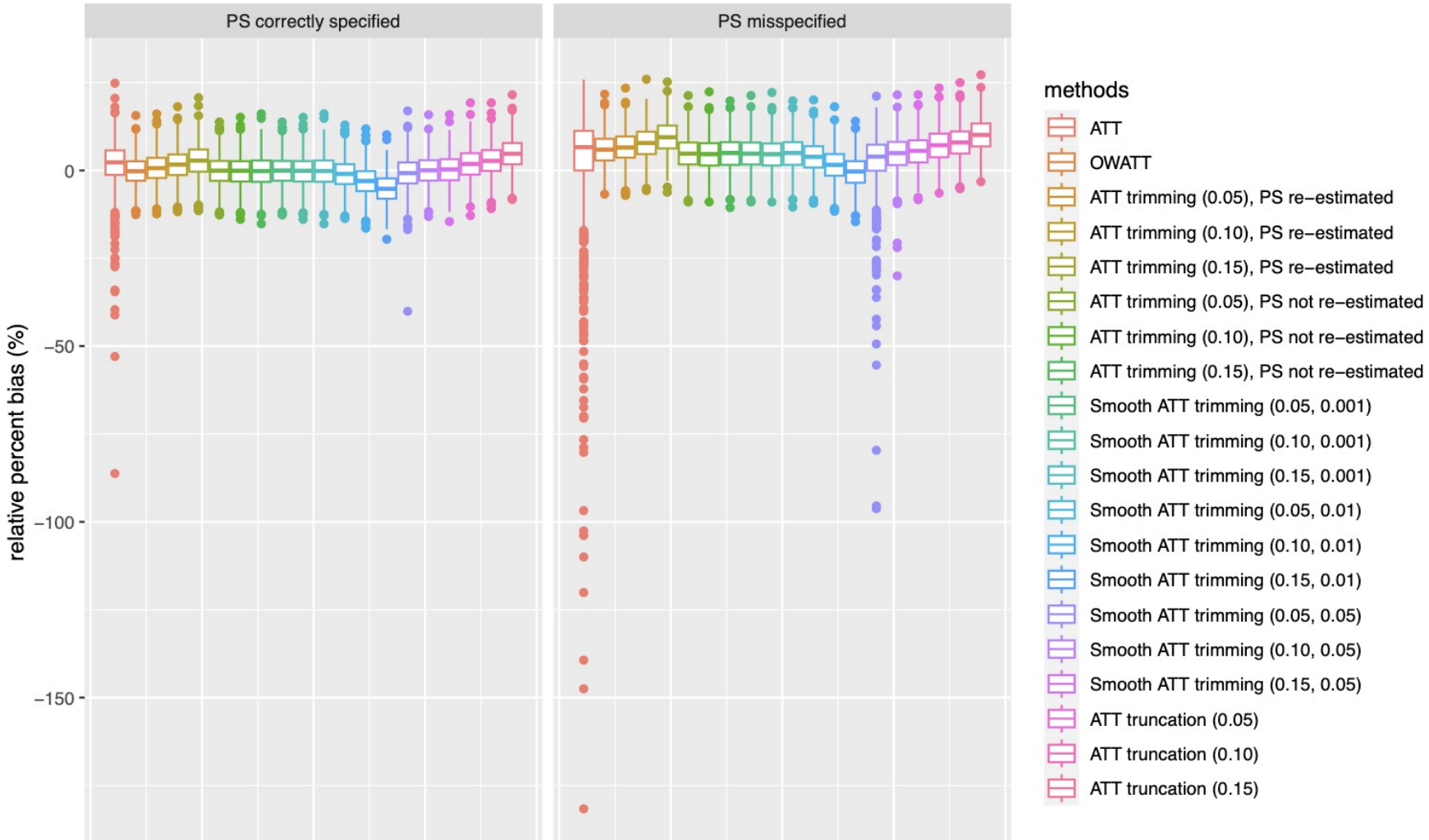
Simulation study

We conducted a simulation study with propensity score model such that the overlap is as follows. There are certain extreme weights as well by this model.



Simulation study

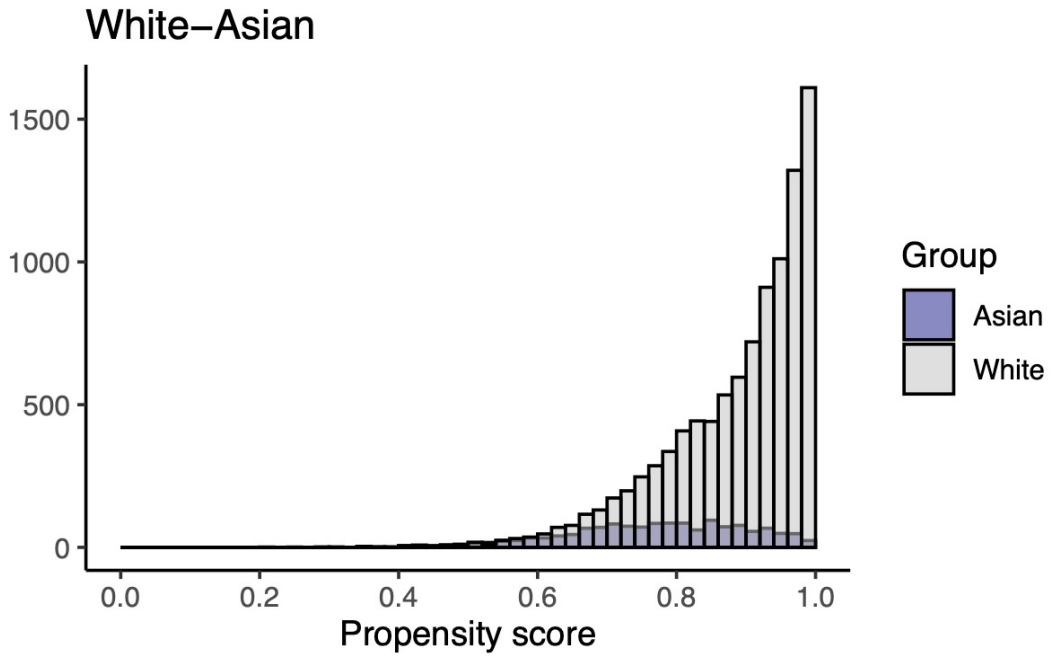
Boxplots of relative biases:



For smooth ATT trimming methods, the parameter in the bracket is (α, ε), i.e., trimming threshold and standard error of the normal cdf in the tilting function, respectively.

Racial disparities in health care expenditure

- Data from the Medical Expenditure Panel Survey (MEPS): `https://www.meps.ahrq.gov/mepsweb/`
- We include 11276 individuals, with 9830 (87.18%) non-Hispanic White as treated and 1446 (12.82%) Asian as control. We included 31 covariates, and considered the health care expenditure as the outcome of interest.



Racial disparities in health care expenditure

Method	Point estimate	Standard error	p-value
ATT	2399.32	787.37	0.002
OWATT	**2511.91**	**255.20**	< 0.001
ATT trimming ($\alpha = 0.05$), PS re-estimated	2363.09	403.42	< 0.001
ATT trimming ($\alpha = 0.10$), PS re-estimated	2666.13	356.62	< 0.001
ATT trimming ($\alpha = 0.15$), PS re-estimated	3054.09	352.98	< 0.001
ATT trimming ($\alpha = 0.05$), PS not re-estimated	2487.25	352.16	< 0.001
ATT trimming ($\alpha = 0.10$), PS not re-estimated	2928.39	286.52	< 0.001
ATT trimming ($\alpha = 0.15$), PS not re-estimated	3286.90	270.04	< 0.001
Smooth ATT trimming ($\alpha = 0.05, \varepsilon = 0.001$)	2488.98	348.88	< 0.001
Smooth ATT trimming ($\alpha = 0.10, \varepsilon = 0.001$)	2926.52	285.92	< 0.001
Smooth ATT trimming ($\alpha = 0.15, \varepsilon = 0.001$)	3291.05	268.68	< 0.001
Smooth ATT trimming ($\alpha = 0.05, \varepsilon = 0.01$)	2419.59	327.68	< 0.001
Smooth ATT trimming ($\alpha = 0.10, \varepsilon = 0.01$)	2881.88	277.57	< 0.001
Smooth ATT trimming ($\alpha = 0.15, \varepsilon = 0.01$)	3229.41	259.47	< 0.001
Smooth ATT trimming ($\alpha = 0.05, \varepsilon = 0.05$)	2337.55	373.65	< 0.001
Smooth ATT trimming ($\alpha = 0.10, \varepsilon = 0.05$)	2638.19	250.78	< 0.001
Smooth ATT trimming ($\alpha = 0.15, \varepsilon = 0.05$)	3014.23	232.06	< 0.001
ATT truncation ($\alpha = 0.05$)	1945.35	385.00	< 0.001
ATT truncation ($\alpha = 0.10$)	2211.56	307.63	< 0.001
ATT truncation ($\alpha = 0.15$)	2419.23	271.39	< 0.001

Discussion

Summary

- We proposed overlap weighted ATT (OWATT) under lack of positivity.
- OWATT has some practical advantages:
  - No selection on any threshold parameters.
  - Statistically sound and efficient under lack of positivity.
- The methodology can easily be extended to the average treatment effect on the controls (ATC).

Future research

- Semiparametric efficiency estimation via augmentation, other robust estimator, empirical sandwich variance estimation, etc.
- Extensions to multi-valued treatment data, survival data, etc.

Discussion

Other related work

- Data-driven based trimming/truncation for targeting ATT under lack of positivity[7]
  - Practical limitation: too technical and no available software developed.
  - Only reliable when the violation of positivity is random.

Open question

- Can we develop methods to distinguish random and structural violations of positivity, e.g., similar to sensitivity analysis?

[7] Ma, X. & Wang, J. Robust inference using inverse probability weighting. *Journal of the American Statistical Association* **115,** 1851–1860 (2020), Chaudhuri, S. & Hill, J. B. *Heavy tail robust estimation and inference for average treatment effects.* Tech. rep. (Working paper, 2014), Sasaki, Y. & Ura, T. Estimation and inference for moments of ratios with robustness against large trimming bias. *Econometric Theory* **38,** 66–112 (2022).

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1. Petersen, M. L., Porter, K. E., Gruber, S., Wang, Y. & van der Laan, M. J. Diagnosing and responding to violations in the positivity assumption. *Statistical methods in medical research* **21,** 31–54 (2012).
2. Hirano, K., Imbens, G. W. & Ridder, G. Efficient estimation of average treatment effects using the estimated propensity score. *Econometrica* **71,** 1161–1189 (2003).
3. D'Amour, A., Ding, P., Feller, A., Lei, L. & Sekhon, J. Overlap in observational studies with high-dimensional covariates. *Journal of Econometrics* **221,** 644–654 (2021).
4. Zhou, Y., Matsouaka, R. A. & Thomas, L. Propensity score weighting under limited overlap and model misspecification. *Statistical Methods in Medical Research* **29,** 3721–3756 (2020).
5. Crump, R., Hotz, V. J., Imbens, G. & Mitnik, O. *Moving the goalposts: Addressing limited overlap in the estimation of average treatment effects by changing the estimand.* 2006.
6. Abadie, A. & Imbens, G. W. *Matching on the estimated propensity score.* Tech. rep. (National Bureau of Economic Research, 2009).
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8. Li, F., Morgan, K. L. & Zaslavsky, A. M. Balancing covariates via propensity score weighting. *Journal of the American Statistical Association* **113,** 390–400 (2018).

References II

9. Ma, X. & Wang, J. Robust inference using inverse probability weighting. *Journal of the American Statistical Association* **115,** 1851–1860 (2020).

10. Chaudhuri, S. & Hill, J. B. *Heavy tail robust estimation and inference for average treatment effects.* Tech. rep. (Working paper, 2014).

11. Sasaki, Y. & Ura, T. Estimation and inference for moments of ratios with robustness against large trimming bias. *Econometric Theory* **38,** 66–112 (2022).

Thank you!

Email: yliu297@ncsu.edu

https://yiliu1998.github.io/