Untitled

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1 Marginal Structural Linear Odds Model (MSLOM)

In addition to linear odds model, the marginal structural linear odds models add the weight w^i to each observation, defined as

$$w^i = \frac{1}{p(G = g_i|C = c_i)} \times \frac{1}{p(E = e_i|G = g_i, C = c_i)}$$

where the two denominators are fitted using the control data, $Y_i = 0$, only. The model is exactly the same as the one without covariates, which is

$$O_i = \text{odds}_i = \exp(\beta_0)(1 + \beta_1 G_i + \beta_2 E_i + \beta_3 G_i E_i)$$

where

$$O_i = \frac{p_i}{1 - p_i} \Leftrightarrow p_i = \frac{O_i}{1 + O_i}, \quad p_i = p(Y_i = 1 \mid G_i = g, E_i = e)$$

The only difference is the MSLOM adds the weight to each observation, the likelihood function is now:

$$\mathcal{L}(\boldsymbol{\beta}) = \prod_{i=1}^{n} p_i^{w_i y_i} (1 - p_i)^{w_i(1-y_i)}$$

and the log-likelihood:

$$\ell(\boldsymbol{\beta}) = \sum_{i=1}^{n} w_i \left\{ y_i \log(p_i) + (1 - y_i) \log(1 - p_i) \right\}$$

It can be also written in terms of beta explicitly:

$$\ell(\boldsymbol{\beta}) = \sum_{i=1}^{n} w_i \left\{ y_i \log\left(\frac{\exp(\beta_0) z_i}{1 + \exp(\beta_0) z_i}\right) + (1 - y_i) \log\left(\frac{1}{1 + \exp(\beta_0) z_i}\right) \right\}$$

where

$$p_i = \frac{\exp(\beta_0) z_i}{1 + \exp(\beta_0) z_i}, \quad z_i = 1 + \beta_1 G_i + \beta_2 E_i + \beta_3 G_i E_i$$

Score function:

Define the score vector:

$$S(\boldsymbol{\beta})_j = \frac{\partial \ell(\boldsymbol{\beta})}{\partial \beta_j}$$

It can be further expanded as

$$\begin{aligned}
\frac{\partial \ell}{\partial \beta_j} &= \sum_{i=1}^{n} w_i \left(\frac{y_i}{p_i} - \frac{1 - y_i}{1 - p_i} \right) \frac{\partial p_i}{\partial \beta_j} \\
&= \sum_{i=1}^{n} w_i \frac{y_i - p_i}{p_i(1 - p_i)} \frac{\partial p_i}{\partial O_i} \frac{\partial O_i}{\partial \beta_j} \\
&= \sum_{i=1}^{n} w_i \frac{r_i}{O_i} \frac{\partial O_i}{\partial \beta_j}
\end{aligned}$$

where

$$r_i = y_i - p_i, \quad \frac{dp_i}{dO_i} = \frac{1}{(1+O_i)^2}$$

For each β_j, we have:

$$\frac{\partial O_i}{\partial \beta_j} = \begin{cases} O_i & j = 0 \\ \exp(\beta_0) G_i & j = 1 \\ \exp(\beta_0) E_i & j = 2 \\ \exp(\beta_0) G_i E_i & j = 3 \end{cases}$$

Consequently

$$S(\boldsymbol{\beta}) = \left(\sum_{i=1}^{n} w_i r_i, \quad \sum_{i=1}^{n} w_i \frac{r_i G_i}{z_i}, \quad \sum_{i=1}^{n} w_i \frac{r_i E_i}{z_i}, \quad \sum_{i=1}^{n} w_i \frac{r_i G_i E_i}{z_i} \right)^\top$$

Observed Information Matrix

The information is defined as

$$\mathcal{I}(\boldsymbol{\beta})_{ij} = -\frac{\partial}{\partial \beta_j} S(\boldsymbol{\beta})_i$$

Denote w_i as

$$\omega_i = \frac{w_i O_i}{(1+O_i)^2}$$

By derivations, the final $\mathcal{I}(\boldsymbol{\beta})$ is written as

$$I(\boldsymbol{\beta}) = \begin{pmatrix} \sum \omega_i & \sum \omega_i \frac{G_i}{z_i} & \sum \omega_i \frac{E_i}{z_i} & \sum \omega_i \frac{G_i E_i}{z_i} \\ \sum \omega_i \frac{G_i}{z_i} & \sum (\omega_i + w_i r_i) \frac{G_i^2}{z_i^2} & \sum (\omega_i + w_i r_i) \frac{G_i E_i}{z_i^2} & \sum (\omega_i + w_i r_i) \frac{G_i^2 E_i}{z_i^2} \\ \sum \omega_i \frac{E_i}{z_i} & \sum (\omega_i + w_i r_i) \frac{G_i E_i}{z_i^2} & \sum (\omega_i + w_i r_i) \frac{E_i^2}{z_i^2} & \sum (\omega_i + w_i r_i) \frac{G_i E_i^2}{z_i^2} \\ \sum \omega_i \frac{G_i E_i}{z_i} & \sum (\omega_i + w_i r_i) \frac{G_i^2 E_i}{z_i^2} & \sum (\omega_i + w_i r_i) \frac{G_i E_i^2}{z_i^2} & \sum (\omega_i + w_i r_i) \frac{(G_i E_i)^2}{z_i^2} \end{pmatrix}$$

The Newton-Raphson Algorithm can be therefore written as

$$\hat{\boldsymbol{\beta}}^{(r+1)} = \hat{\boldsymbol{\beta}}^{(r)} + I\left(\hat{\boldsymbol{\beta}}^{(r)}\right)^{-1} S\left(\hat{\boldsymbol{\beta}}^{(r)}\right)$$

where I and S are defined above.