(Five Ways in which) The Fourier Inversion Formula Implies the Fundamental Theorem of Algebra

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Abstract

This paper presents five particular ways to prove the fundamental theorem of algebra. What these five ways have in common is the usage of the Fourier inversion formula, hence the title. To provide a self-contained while easily intelligible account of these proofs, we first introduce the two theorems, supply a suitable amount of background knowledge behind them, prove the Fourier inversion formula given the background knowledge, and proceed to presenting four direct proofs. For the fifth proof, we show that the Fourier inversion formula also implies Liouville's theorem, of which the fundamental theorem of algebra is famously a consequence.

Introduction

This paper builds upon Lazer and Leckband's 2010 article "The Fundamental Theorem of Algebra via the Fourier Inversion Formula" [1] to present in detail five particular ways to prove the FTA–the fundamental theorem of algebra. What these four ways have in common is the usage of the FIF–the Fourier inversion formula–hence the title.

To accomplish our goal, we break the process into sections. The first section is where we introduce the two major theorems that are in the title. The second section provides a suitable amount of background knowledge behind the FIF, which leads to a self-contained proof of it. Next, in the third section, we present a number of Corollaries and Lemmas that will aid our arguments, and in the fourth section we develop the four direct arguments. In the fifth and final section, we show that the Fourier inversion formula also implies Liouville's theorem, of which the fundamental theorem of algebra is a famous consequence.

1 The Two Theorems

First, the FTA. We write it as such:

Theorem 1.1 (The Fundamental Theorem of Algebra)**.** *For any nonconstant polynomial P with complex coefficients, $P(z) = 0$ has a solution.*

Readers who are familiar with the FTA may also be familiar with a different version:

Theorem 1.2 (The Fundamental Theorem of Algebra, a different version)**.** *For any nonconstant polynomial P of degree n with complex coefficients, $P(z) = 0$ has exactly n solutions.*

We may wonder if these two statements are equivalent; and with some brief staring, we see immediately that the second version clearly implies the first. But how about the other direction? It turns out that with the help of a particular proposition, we can show that the first version also implies the second quite easily. The proposition is as follows:

Proposition 1.3. *If f is a nonconstant polynomial with complex coefficients of degree n, then given any $w \in \mathbb{C}$, $f(w) = 0$ if and only if there exists nonconstant polynomial g with complex coefficients of degree $n-1$, such that for all $z \in \mathbb{C}$,*

$$f(z) = (z - w)g(z).$$

A proof for this statement, for both directions, involves only simple algebraic manipulations; and as one may probably guess, with Proposition 1.3, the proof for Theorem 1.1 implies Theorem 1.2 can be constructed using mathematical induction. So the two versions are indeed equivalent, and it is good enough to prove either one. It is just so the case that our proofs will be directed at the first version, so Theorem 1.1 is what we will be referring to when we say "the FTA," or "the Fundamental Theorem of Algebra," from now on.

Then, we have the Fourier Inversion Formula:

Theorem 1.4 (The Fourier Inversion Formula)**.** *Let $f(x) : \mathbb{R} \to \mathbb{C}$ be continuous. If $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$, then its Fourier transform*

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x \xi} f(x) dx$$

is continuous. If $\int_{-\infty}^{\infty} |\hat{f}(\xi)| d\xi < \infty$, then the inversion formula

$$f(x) = \int_{-\infty}^{\infty} e^{2\pi i x t} \hat{f}(\xi) d\xi$$

holds everywhere.

Readers familiar with Fourier Analysis and the Fourier transform are probably also familiar with this theorem. Conversely, readers who are unfamiliar with these topics should find the statement somewhat arbitrary: the "inversion" element is readily apparent, but the significance of defining $\hat{f}$, the "Fourier transform" of f, in the particular way it is defined is not at all obvious.

I am going to admit here that if we proceed now—albeit blindly, perhaps—with the FIF as stated above, we will be able to prove the FTA without encountering trouble in understanding these proofs. It may, however, be helpful to see the FIF as being more than a black box. If you are unfamiliar with the Fourier transform or the FIF, and wish to learn about this crucial tool we are using somewhat beyond what can only be suitably described as the very surface, the next section is for you. Otherwise, feel free to go directly to section 3.

2 Story of the FIF and Its Proof

We begin from the very basic: by introducing a context in which it even makes sense to take an integral from negative infinity to infinity. The final goal of this section—the organization of which generally follows that of the first part of Stein and Shakarchi's *Fourier Analysis* [2], Chapter 5—is to prove the Fourier inversion formula, and to get there we will necessarily acquire much background knowledge surrounding the Fourier transform and the Fourier inversion formula.

Definition 2.1 (Schwartz Space)**.** The **Schwartz space** on $\mathbb{R}$, denoted $\mathcal{S}(\mathbb{R})$, is the set of all infinitely differentiable f on $\mathbb{R}$ such that f and all its derivatives f', f'', ..., $f^{(l)}$, ..., satisfies

$$\sup_{x\in\mathbb{R}} |x|^k |f^{(l)}(x)| < \infty \text{ for every } k, l \geq 0.$$

Stein and Shakarchi call functions f and their derivatives that satisfy the final expression above to be of "rapid decrease," (Chapter 5, Section 1.3) [2] (as opposed to being of "moderate decrease," which we will not explicitly discuss in this paper) which is an apt way of thinking about these functions.

An example of a function in the Schwartz space is the Gaussian $f(x) = e^{-ax^2}$, for any $a > 0$. We can quickly verify this fact by considering that derivatives of f are all of the form $P(x)e^{-ax^2}$, where P is a polynomial, and the exponential term decreases more rapidly than the polynomial term. The Gaussian is an important mathematical object in a study of the Fourier transform, and we will make good use of these functions by taking $a = \pi$ later in the section.

As promised, the following proposition discusses the validity of taking an infinite integral of f given $f \in \mathcal{S}(\mathbb{R})$.

Proposition 2.2. *For $f \in \mathcal{S}(\mathbb{R})$,*

$$\lim_{N\to\infty} \int_{-N}^{N} f(x)dx = \int_{-\infty}^{\infty} f(x)dx$$

exists.

Proof. Let $f \in \mathcal{S}(\mathbb{R})$. Since f is continuous, we know that $I_N(x) = \int_{-N}^{N} f(x)dx$ is well defined for all $N > 0$. So all we need is to show that $\{I_N\}$ is Cauchy. Suppose $M > N$. Since $f \in \mathcal{S}(\mathbb{R})$, there exists $L > 0$ such that $|x|^N|f(x)| < L$. Thus we have

$$\begin{aligned}
|I_M - I_N| &= \left| \int_{-M}^{M} f(x)dx - \int_{-N}^{N} f(x)dx \right| \\
&\leq \int_{N\leq|x|\leq M} |f(x)|dx \\
&< \int_{N\leq|x|\leq M} \frac{L}{|x|^N} dx \\
&= L \left(\left[\frac{x^{1-N}}{1-N} \right]_N^M + \left[\frac{x(-x)^{-N}}{1-N} \right]_{-M}^{-N} \right) \\
&= L \left(\frac{2M^{1-N}}{1-N} - \frac{2N}{N^N - N^{N+1}} \right)
\end{aligned}$$

which goes to 0 as $N \to \infty$ (the first term clearly goes to zero, and one can verify that the second term goes to zero by applying l'Hôpital's rule). □

We now know that infinite integrals make sense when applied to $f \in \mathcal{S}(\mathbb{R})$. We can thus comfortably work with them, and define the Fourier transform on $\mathcal{S}(\mathbb{R})$.

Definition 2.3 (Fourier transform on $\mathcal{S}(\mathbb{R})$)**.** For a function $f \in \mathcal{S}(\mathbb{R})$, we call

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ix\xi}dx,$$

where $\xi \in \mathbb{R}$, the Fourier transform of f.

For simplicity, we can write the Fourier transform as $f \to \hat{f}$.

If you are familiar with Fourier series, you would notice that this expression looks terribly like a continuous version of the equation for calculating Fourier coefficients of a periodic function. Stein and Shakarchi motivate the idea of the Fourier transform precisely by comparing $f \in \mathcal{S}(\mathbb{R})$ and $\hat{f}$ to a periodic function f and its Fourier coefficients, and claiming that the relationships are analogous [2]. Here we take an approach that does not assume understanding of Fourier series.

To do so, we observe what the integral is describing. The integrand is the product of some rapidly decreasing function f and a complex number in polar form. Examining the $e^{-2\pi ix\xi}$ term first, we notice first that its modulus is always one, and by continuously adjusting the values of x and ξ, we travel along the unit circle centered around the origin on the complex plane. Multiplying this complex number by $f(x)$ makes $f(x)$ in charge of how far we are from the origin at each moment ($f(x)$ becomes the modulus if we view the integrand as one single complex number). So by following the integrand on the complex plane, we are in a sense drawing a winded up graph of f. If we interpret x as "time," it turns out that ξ determines the frequency of our winded up graph, i.e., the number of cycles our winded up graph undergoes per unit time, or per unit x.

It might seem odd to interpret x as "time," however. This interpretation makes more sense in an applied setting, of which there are many for the Fourier transform, but we will leave this aside for the moment. Now we look at the integral. Integration is of course continuous summation, so in essence we are computing the sum of all the points on our winded graph. So naturally, if the points in our graph lie predominantly in one direction relative to the origin, the sum is going to be located in that direction.

The significance of finding this "sum" of all points on this winded graph of f may be best illustrated by considering it in, as mentioned, an applied setting. Say our f is a periodic function, e.g. $f(x) = \cos(\pi x)$. Then this function has a "frequency." (number of cycles per unit time or x) In the case of our example, the frequency is 0.5. If we plot the Fourier transform of f, and adjust the winding frequency ξ (so there are two frequencies we are dealing with here), we will see that the value of the sum wobbles around near the origin or around zero most of the time, except when ξ matches the frequency of f, i.e. becomes 0.5, then there would be quite a sudden spike in the value of the sum/integral. Now, if we add a wave of a different frequency to our f, say $\cos(2\pi x)$ which has a frequency of one, so that $f(x) = \cos(\pi x) + \cos(2\pi x)$, and adjust the winding frequency ξ, we see that now the value of the integral spikes at both $\xi = 0.5$ and $\xi = 1$. So if we are given a heavily superpositioned frequency, the Fourier transform can easily tell us which frequency is in there. I have created a visual demonstration of this on Desmos, feel free to play around with it at https://www.desmos.com/calculator/ou6sf68p8l.

We now introduce a number of properties of the mathematical objects we have introduced thus far in this section that will help us prove the FIF.

Proposition 2.4. *(Scaling under dilation of $f \in \mathcal{S}(\mathbb{R})$) Let $f \in \mathcal{S}(\mathbb{R})$. Then for $\delta > 0$,*

$$\delta \int_{-\infty}^{\infty} f(\delta x)dx = \int_{-\infty}^{\infty} f(x)dx.$$

Proof. Let $\frac{dF}{dx} = f$. Then we have

$$\delta \int_{-N}^{N} f(\delta x)dx = \delta \left(\frac{F(\delta N)}{\delta} - \frac{F(-\delta N)}{\delta} \right) = F(\delta N) - F(-\delta N) = \int_{-\delta N}^{\delta N} f(x)dx$$

Let $N \to \infty$, we have the proposition. □

This proposition helps us prove some of the identities below.

Proposition 2.5 (Identities of the Fourier transform)**.** *Let $f \in \mathcal{S}(\mathbb{R})$, then*

(i) $f(\delta x) \to \delta^{-1}\hat{f}(\delta^{-1}\xi)$,

(ii) $f'(x) \to 2\pi i\xi\hat{f}(\xi)$,

(iii) $-2\pi i x f(x) \to \frac{d}{d\xi}\hat{f}(\xi)$,

(iv) $f(x+h) \to \hat{f}(\xi)e^{2\pi i h\xi}$.

Proof. (i) Let $g(x) = f(\delta x)$. By the previous proposition and the definition of the Fourier transform,

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} f(\delta x)e^{-2\pi i x\xi}dx = \delta^{-1}\int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi\delta^{-1}}dx = \delta^{-1}\hat{f}(\delta^{-1}\xi).$$

□

Proof. (ii) Following Stein and Shakarchi's argument of this proposition (Chapter 5, Proposition 1.2 (iv)) [2], we integrate by parts, and find

$$\int_{-N}^{N} f'(x)e^{-2\pi i x\xi}dx = \left[f(x)e^{-2\pi i x\xi}\right]_{-N}^{N} + 2\pi i\xi\int_{-N}^{N} f(x)e^{-2\pi i x\xi}dx.$$

Letting $N \to \infty$, the first term goes to zero since $f \in \mathcal{S}(\mathbb{R})$, and we get (ii). □

Proof. (iii) This proof again follows Stein and Shakarchi's argument of this proposition (Chapter 5, Proposition 1.2 (v)) [2], but on top of it provides more details. Let $\epsilon > 0$. We show that $\hat{f}$ is differentiable and its derivative is as we claim. In particular, we show that there exists h such that:

$$\left| \frac{\hat{f}(\xi+h) - \hat{f}(\xi)}{h} - (\widehat{-2\pi i x f})(\xi) \right| < \epsilon.$$

First, we simplify the expression on the left hand side to find

$$\frac{\hat{f}(\xi+h) - \hat{f}(\xi)}{h} - (\widehat{-2\pi i x f})(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\left(\frac{e^{-2\pi i x h} - 1}{h} + 2\pi i x \right) dx.$$

Since $f \in \mathcal{S}(\mathbb{R})$, there exists $N \in \mathbb{N}$ such that

$$\int_{|x|\geq N} |f(x)| \left| \frac{e^{-2\pi i x h} - 1}{h} \right| dx < \epsilon/4 \text{ and } \int_{|x|\geq N} |f(x)||2\pi i x| dx < \epsilon/4.$$

This is true since we have proven that $\{I_N\}$, as defined in the proof of Proposition 2.2, is a Cauchy sequence. Now, let $M = \int_{|x|\leq N} |f(x)|dx$. Then for $|x| \leq N$, there exists h_0 such that whenever $|h| < h_0$,

$$\left| \frac{e^{-2\pi i x h} - 1}{h} + 2\pi i x \right| < \frac{\epsilon}{2M}.$$

This can be verified by applying l'Hôpital's rule on the fraction on the left hand side as $h < h_0 \to 0$. Now we have

$$\begin{aligned} \left| \frac{\hat{f}(\xi + h) - \hat{f}(\xi)}{h} - (\widehat{-2\pi i x f})(\xi) \right| &\leq \int_{-\infty}^{\infty} \left| f(x) e^{-2\pi i x \xi} \left(\frac{e^{-2\pi i x h} - 1}{h} + 2\pi i x \right) \right| dx \\ &\leq \int_{|x|\geq N} |f(x)| \left| \frac{e^{-2\pi i x h} - 1}{h} \right| dx \\ &\quad + \int_{|x|\geq N} |f(x)||2\pi i x| dx \\ &\quad + \int_{|x|\leq N} |f(x)| \left| \frac{e^{-2\pi i x h} - 1}{h} + 2\pi i x \right| dx \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2M} M = \epsilon. \end{aligned}$$

□

Proof. (iv) Let $g(x) = f(x + h)$. Then by translation invariance of the integral, we have

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} f(x + h) e^{-2\pi i x \xi} dx = \int_{-\infty}^{\infty} f(x) e^{-2\pi i (x-h)\xi} dx = \hat{f}(\xi) e^{2\pi i h \xi} dx.$$

□

The following theorem will help us construct a special family of function, $K_\delta(x)$ (we will see this in Corollary 2.7), which has particular qualities that will tremendously aid our argument for the FIF.

Theorem 2.6 (The Gaussian $e^{-\pi x^2}$ is its own Fourier transform)**.** *If* $f(x) = e^{-\pi x^2}$, *then* $\hat{f}(\xi) = f(\xi)$.

Proof. We still follow Stein and Shakarchi (Chapter 5, Theorem 1.4) [2] for this one. Let $f(x) = e^{-\pi x^2}$. First, notice that since all derivatives of f have the form $P(x)e^{-\pi x^2}$, where $P(x)$ is a polynomial, and $e^{\pi x^2}$ grows quicker than $P(x)$ as $x \to \infty$, $f(x) \in \mathcal{S}(\mathbb{R})$. We claim that $\int_{-\infty}^{\infty} f(x)dx = 1$. We can verify this by computing this integral squared, applying

Fubini's theorem (see appendix), and switching to polar coordinates:

$$\begin{aligned}\left(\int_{-\infty}^{\infty} e^{-\pi x^2} dx\right)^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)} dxdy \\ &= \int_0^{2\pi}\int_0^{\infty} re^{-\pi r^2} drd\theta \\ &= 2\pi\int_0^{\infty} re^{\pi r^2} dr \\ &= \left[-e^{-\pi r^2}\right]_0^{\infty} \\ &= 1,\end{aligned}$$

which implies our claim.
Now, define

$$F(\xi) = \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ix\xi}dx.$$

From our claim above, we see that $F(0) = 1$. We will use this fact shortly after.
Consider Proposition 2.5(iii), which claims that the Fourier transform of $-2\pi ixf(x)$ is $\hat{f}'(\xi)$. We can thus write

$$\hat{F}'(\xi) = \int_{-\infty}^{\infty}(-2\pi ix)f(x)e^{-2\pi ix\xi}dx = i\int_{-\infty}^{\infty} f'(x)e^{-2\pi ix\xi}dx.$$

Consider Proposition 2.5(ii), which states that the Fourier transform of $f'(x)$ is $2\pi it\hat{f}(\xi)$. We have

$$\hat{F}'(\xi) = i(2\pi i\xi)\hat{f}(\xi) = -2\pi\xi F(\xi).$$

Define $G(\xi) = F(\xi)e^{\pi\xi^2}$. Then $G'(\xi) = -2\pi\xi F(\xi)e^{pi\xi^2} + 2\pi\xi F(\xi)e^{\pi\xi^2} = 0$ by the chain rule. Since $F(0) = 1$, $G(\xi) = G(0) = F(0) = 1$, and so $F(\xi) = \hat{f}(\xi) = e^{-\pi\xi^2} = f(\xi)$ as desired. ☐

Now, using the first identity (Proposition 2.5(i))—replacing δ with $\delta^{-1/2}$—we get the following corollary:

Corollary 2.7. *Let $\delta > 0$. If $K_\delta(x) = \delta^{-1/2}e^{-\pi x^2/\delta}$, then $\widehat{K_\delta}(\xi) = e^{-\pi\delta\xi^2}$; and if $G_\delta(x) = e^{-\pi\delta x^2}$, $\widehat{G_\delta}(\xi) = K_\delta(\xi)$.*

Apart having clear and structured Fourier transforms, as presented in the corollary above, the family of functions $\{K_\delta\}_{\delta>0}$ is in fact a family of **good kernels**, which has a catalogue of neat properties. To avoid a dedicated digression on good kernels, we will constrict ourselves in only discussing a few properties particular to K_δ, and will be directly useful for us.

Proposition 2.8 (First property of $K_\delta(x)$)**.** *If $K_\delta(x) = \delta^{-1/2}e^{-\pi x^2/\delta}$, then*

$$\int_{-\infty}^{\infty} K_\delta(x)dx = 1.$$

Proof. Observe that

$$\int_{-\infty}^{\infty} K_\delta(x)dx = \widehat{K_\delta}(0) = 1.$$

☐

Proposition 2.9 (Second property of $K_\delta(x)$)**.** *If $K_\delta(x) = \delta^{-1/2}e^{-\pi\xi^2}$, then for every $N > 0$,*

$$\int_{|x|>N} |K_\delta(x)|\, dx \to 0$$

as $\delta \to 0$.

Proof. We follow Stein and Shakarchi's argument (Chapter 5, Corollary 1.5 (iii)) [2] and let $y = x/\delta^{1/2}$. Then

$$\int_{|x|>N} |K_\delta(x)| dx = \int_{|x|>N} \left|\delta^{-1/2}e^{-\pi x^2/\delta}\right| dx = \int_{|y|>N/\delta^{1/2}} \left|e^{-\pi y^2}\right| dy = \int_{|y|>N/\delta^{1/2}} e^{-\pi y^2} dy,$$

which converges to 0 as $\delta \to 0$ because $f(y) = e^{-\pi y^2} \in \mathcal{S}(\mathbb{R})$, as discussed before. $\square$

The culmination of this discussion of the family of functions $K_\delta(x)$ is the following lemma.

Lemma 2.10. *If $f \in \mathcal{S}(\mathbb{R})$, then*

$$\int_{-\infty}^{\infty} f(x+t)K_\delta(t)dt \to f(x)$$

uniformly as $\delta \to 0$.

Proof. Although the statement above differs from the one constructed by Stein and Shakarchi in notation and presentation, its argument here again follows Stein and Shakarchi's argument of the corresponding proposition (Chapter 5, Corollary 1.7) [2]. Let $\epsilon > 0$. First we claim that there f is uniformly continuous. Since f is of rapid decrease, there exists $R > 0$ such that $|f(x)| < \epsilon/4$ whenever $|x| \geq R$. We also know that f is continuous, so it is uniformly continuous on the bounded interval $[-R, R]$. Combining these two facts, we know that there exists $\eta > 0$ such that $|x - y| < \eta$ implies $|f(x) - f(y)| < \epsilon$. By Proposition 2.8, we can carry out the following simplification:

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty} f(x+t)K_\delta(t)dt - f(x)\right| \\
=&\left|\int_{-\infty}^{\infty} f(x+t)K_\delta(t)dt - \int_{-\infty}^{\infty} f(x)K_\delta(t)dt\right| \\
\leq&\int_{-\infty}^{\infty} |K_\delta(t)||f(x+t) - f(x)|dt \\
=&\int_{|t|\geq\eta} |K_\delta(t)||f(x+t) - f(x)|dt + \int_{|t|<\eta} |K_\delta(t)||f(x+t) - f(x)|dt.
\end{aligned}$$

Let $\delta \to 0$. By Proposition 2.9 and since f is bounded, the first integral goes to zero. By the result we have obtained earlier, we know that $|f(x+t) - f(x)|$ is small for $|t| < \eta$. So by Proposition 2.8, the second integral is also small, which is exactly what we want. $\square$

This gives us one crucial piece that will directly assist our argument for the Fourier inversion formula. Now we develop the second crucial piece.

Proposition 2.11. *If there exists $A > 0$ such that $|F(x,y)| \leq \frac{A}{(1+x^2)(1+y^2)}$ for all $x, y \in \mathbb{R}$, then for fixed x,*

$$\lim_{N\to\infty} \int_{-N}^{N} F(x,y)dy = \int_{-\infty}^{\infty} F(x,y)dy;$$

for fixed y,

$$\lim_{N\to\infty} \int_{-N}^{N} F(x,y)dx = \int_{-\infty}^{\infty} F(x,y)dx.$$

Proof. Define

$$I_N(x) = \int_{-N}^{N} F(x,y)dy.$$

Let $M > N$. Then

$$\begin{aligned}
|I_M(x) - I_N(x)| &= \left| \int_{-M}^{M} F(x,y)dy - \int_{-N}^{N} F(x,y)dy \right| \\
&\leq \int_{|y|\in[N,M]} |F(x,y)|\, dy \\
&\leq \int_{|y|\in[N,M]} \frac{A}{(1+x^2)(1+y^2)} dy \\
&= \frac{A}{1+x^2} \int_{|y|\in[N,M]} \frac{1}{1+y^2} dy \\
&= \frac{A}{1+x^2} \left[(\arctan M - \arctan N) + (\arctan(-N) - \arctan(-M)) \right]
\end{aligned}$$

which converges to 0 as $N \to \infty$, since $\lim_{x\to\infty} \arctan(x) = \pi/2$ and $\lim_{x\to\infty} \arctan(-x) = -\pi/2$. The argument for fixed y goes similarly. $\square$

Proposition 2.12. *If $F_1(x) = \int_{-\infty}^{\infty} F(x,y)dy$ and $F_2(y) = \int_{-\infty}^{\infty} F(x,y)dx$, $\lim_{N\to\infty} \int_{-N}^{N} F_1(x)dx = \int_{-\infty}^{\infty} F_1(x)dx$ exists, so does $\lim_{N\to\infty} \int_{-N}^{N} F_2(y)dy = \int_{-\infty}^{\infty} F_2(y)dy$, and*

$$\int_{-\infty}^{\infty} F_1(x)dx = \int_{-\infty}^{\infty} F_2(y)dy.$$

Proof. Familiarly, define

$$I_N = \int_{-N}^{N} F_1(x)dx.$$

We assert that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \pi.$$

See 7.1 in the appendix for a detailed, self-contained proof of this fact. Using this, we can carry out the following simplification:

$$
\begin{aligned}
|I_M - I_N| &= \left| \int_{-M}^{M} F_1(x)dx - \int_{-N}^{N} F_1(x)dx \right| \\
&\le \int_{|x|\in[N,M]} \int_{-\infty}^{\infty} |F(x,y)| dydx \\
&\le \int_{|x|\in[N,M]} \frac{A}{1+x^2} \int_{-\infty}^{\infty} \frac{1}{1+y^2} dydx \\
&= A\pi \int_{|x|\in[N,M]} \frac{1}{1+x^2} dx \\
&= A\pi \left[(\arctan M - \arctan N) + (\arctan(-N) - \arctan(-M)) \right],
\end{aligned}
$$

which converges to 0 as $N \to \infty$, as argued in the proof for Proposition 2.11. The argument for the limit existing when integrating $F_2(y)$ from negative infinity to infinity goes similarly. Also, we can similarly prove that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x,y)| dydx < \infty,$$

which allows us to apply Fubini's theorem, and get

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x,y) dydx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x,y) dxdy \Longrightarrow \int_{-\infty}^{\infty} F_1(x)dx = \int_{-\infty}^{\infty} F_2(y)dy.$$

□

Following Stein and Shakarchi, we can cleverly set $F(x,y) = f(x)g(y)e^{-2\pi ixy}$. (It can be easily verified that this expression satisfies the condition of Proposition 2.11 because all three components are of rapid decrease.) Notice that this makes $F_1(x) = f(x)\hat{g}(x)$ and $F_2(y) = \hat{f}(y)g(y)$. Applying the proposition above, we have:

Lemma 2.13 (The Multiplication Formula)**.** *If $f, g \in \mathcal{S}(\mathbb{R})$,*

$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)dx = \int_{-\infty}^{\infty} g(y)\hat{f}(y)dy.$$

This is the second and final crucial piece of our proof of the FIF.

Now comes what we have been waiting for.

Theorem 2.14 (Fourier Inversion Formula for $f \in \mathcal{S}(\mathbb{R})$)**.** *If $f \in \mathcal{S}(\mathbb{R})$, then*

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi ix\xi} d\xi.$$

Proof. We have followed Stein and Shakarchi's set-up (mostly) up until this point in this section precisely for the construction of this proof, thus we are going to borrow Stein and Shakarchi's argument here once again, for the last time in this paper. Now, let $G_\delta(x) = e^{-\pi\delta x^2}$, then $\widehat{G_\delta}(\xi) = K_\delta(\xi)$ by Corollary 2.7. Applying the multiplication formula (Lemma 2.13), we get

$$\int_{-\infty}^{\infty} f(x)K_\delta(x)dx = \int_{-\infty}^{\infty} \hat{f}(\xi)G_\delta(\xi)d\xi.$$

If we let $\delta \to 0$, by Lemma 2.10, the left hand side integral goes to $f(0)$. The right hand side converges to $\int_{-\infty}^{\infty} \hat{f}(\xi)d\xi$. This gives us

$$f(0) = \int_{-\infty}^{\infty} \hat{f}(\xi)d\xi.$$

Now, let $F(y) = f(y+x)$. Then by Proposition 2.5(iv), we have

$$f(x) = F(0) = \int_{-\infty}^{\infty} \hat{F}(\xi)d\xi = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x\xi}d\xi.$$

□

Recall that the conditions of Theorem 2.14 differs form that of Theorem 1.4. Also recall that there is also a statement on the continuity of $\hat{f}$ in Theorem 1.4. The continuity part, as one might anticipate, is not difficult to show. The actual "inversion" part, now with different conditions, however, is slightly more nuanced because an approach different than that used in proving Theorem 2.14 needs to be taken due to the changed condition. We will be using a number of familiar results that are adjusted slightly to fit the occasion; however, we will not present fully developed proofs for these slightly altered versions of old results, because they all involve only slight and apparent changes to the arguments previously supplied.

Proof. (Theorem 1.4) We first show that if $\int_{-\infty}^{\infty} |f(x)|dx < \infty$ (i.e. $f \in L^1$), the Fourier transform $\hat{f}(\xi)$ is continuous. So suppose $f \in L^1$. We begin by claiming that the function $y(\xi) = e^{-2\pi i x\xi}$ is continuous. This is simply because

$$y(\xi) = e^{-2\pi i\xi} = \cos(-2\pi x t\xi) + i\sin(-2\pi x\xi),$$

and both $u(t) = \cos(-2\pi x\xi)$ and $v(t) = \sin(-2\pi x\xi)$ are continuous.
Let $\epsilon > 0$. Since $f \in L^1$, there exists $L > 0$ such that $\int_{-\infty}^{\infty} |f(x)|dx < L$. Since $e^{-2\pi i x\xi}$ is continuous, there exists $\delta > 0$ such that $|e^{-2\pi i x\xi} - e^{-2\pi i x\xi_0}| < \epsilon/L$ whenever $|\xi - \xi_0| < \delta$. Let $|\xi - \xi_0| < \delta$, then we have

$$\begin{aligned}
|\hat{f}(\xi) - \hat{f}(\xi_0)| &= \left| \int_{-\infty}^{\infty} e^{-2\pi i x\xi} f(x)dx - \int_{-\infty}^{\infty} e^{-2\pi i x\xi_0} f(x)dx \right| \\
&\leq \int_{-\infty}^{\infty} |f(x)| \left| e^{-2\pi i x\xi} - e^{-2\pi i x\xi_0} \right| dx \\
&< \int_{-\infty}^{\infty} |f(x)| \frac{\epsilon}{L} dx < \epsilon.
\end{aligned}$$

So $\hat{f}(\xi)$ is continuous.
Now, suppose also that $\hat{f} \in L^1$. Define

$$I_\delta(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)e^{2\pi i(y-x)t} G_\delta(\xi)dxdt.$$

If we solve the inner integral first, we get

$$I_\delta(x) = \int_{-\infty}^{\infty} G_\delta(\xi)e^{2\pi i y\xi} \hat{f}(\xi)d\xi.$$

Notice that since $G_\delta(\xi) = e^{-\pi\delta\xi^2}$,

$$|G_\delta(\xi)e^{2\pi i y\xi}\hat{f}(\xi)| \leq |\hat{f}(\xi)|.$$

And since $\hat{f} \in L^1$, the infinite integral over the dominating expression $|\hat{f}(\xi)|$ converges, which means that we can apply Lebesgue dominated convergence theorem (see appendix) to take the following limit:

$$\lim_{\delta\to 0} I_\delta(x) = \lim_{\delta\to 0}\int_{-\infty}^{\infty} G_\delta(\xi)e^{2\pi i y\xi}\hat{f}(\xi)d\xi = \int_{-\infty}^{\infty} e^{2\pi i y\xi}\hat{f}(\xi)d\xi.$$

Notice further that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|f(x)e^{2\pi i(y-x)t}G_\delta(\xi)\right| dxd\xi < \infty.$$

This is easily verifiable using techniques that we have used before. Thus we can apply Fubini's theorem and Corollary 2.7, and integrate over t first in the double integral $I_\delta(x)$, as such:

$$\begin{aligned}
I_\delta(x) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)e^{2\pi i(y-x)\xi}G_\delta(\xi)dxd\xi \\
&= \int_{-\infty}^{\infty} f(x)\int_{-\infty}^{\infty} e^{2\pi i(y-x)\xi}G_\delta(\xi)d\xi dx \\
&= \int_{-\infty}^{\infty} f(x)K_\delta(y-x)dx.
\end{aligned}$$

Let $\delta \to 0$, apply a version of Lemma 2.10 where $f \in L^1$ (again, easily demonstrable with familiar techniques), and we have exactly what we want. □

The choice to not organize this section around this proof for $f \in L^1$ but instead around that for $f \in \mathcal{S}(\mathbb{R})$ is mainly because a completely self-contained application of the Lebesgue dominated convergence theorem requires a long winded digression around measurability and integrability (these issues are bypassed in our narrative by introducing the Schwartz space), which are less relevant to the Fourier transform than, say, the multiplication formula. The key is to gain knowledge of the tool we are going to use, and see how it can be proven—and both of these goals should have been adequately fulfilled as of now.

3 Corollaries and Lemmas: Our Preliminary Toolbox

Here is something that will be quite useful for our proofs of the FTA:

Corollary 3.1. *If $f \in L^1$, then $\hat{f}(\xi) = 0$ implies $f(x) = 0$.*

This corollary follows as automatically from the FIF as it gets (because if $\hat{f}(\xi) = 0$, $\hat{f} \in L^1$, and by the inversion formula $f(x)$ equals the integral of zero, which is indeed zero), and seems almost trivial, but it is in fact rather powerful for our purpose. Every single proof of the FTA that we show in the next section, as it turns out, directly uses this corollary in this way: assume, towards a contradiction, that the FTA is false; define f with an impossible polynomial P where f can never be zero; show that the Fourier transform of f, $\hat{f}$, equals zero; use the corollary to conclude that f equals zero, and arrive at a contradiction.

Now we catalogue the statements of two particular theorems that will be useful.

Theorem 3.2 (Cauchy's theorem for entire functions)**.** *If f entire, and γ is a closed curve on the complex plane, then*

$$\int_\gamma f(z)dz = 0.$$

Theorem 3.3 (Cauchy-Riemann equations)**.** *Write $f(x+iy) = u(x,y) + iv(x,y)$. If f is holomorphic on Ω, then u, v satisfy*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Here are two statements that smooth the small hiccups we might experience.

Proposition 3.4. *If $g(x)$ is entire and never zero, then $f(x) = 1/g(x)$ is entire.*

Proof. Let $z_0 \in \mathbb{C}$. Since f is continuous

$$\lim_{h\to 0} \frac{1}{g(z_0+h)} = \frac{1}{g(z_0)}.$$

And since g is entire,

$$\lim_{h\to 0} \frac{g(z_0+h) - g(z_0)}{h} = g'(z_0).$$

Using these facts we get

$$\begin{aligned}
\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} &= \lim_{h\to 0} \frac{\frac{1}{g(z_0+h)} - \frac{1}{g(z_0)}}{h} \\
&= -\frac{1}{g(z_0)} \lim_{h\to 0} \left(\frac{1 - \frac{g(z_0)}{g(z_0+h)}}{h} \right) \\
&= -\frac{1}{g(z_0)} \lim_{h\to 0} \frac{1}{g(z_0+h)} \frac{g(z_0+h) - g(z_0)}{h} \\
&= -\frac{g'(z_0)}{g(z_0)^2}.
\end{aligned}$$

□

Proposition 3.5. *Let P be a nonconstant polynomial with complex coefficients such that deg(P) ≥ 2. Then the restriction of $f(z) = 1/P(z)$ to $\mathbb{R}$, $f(x)$, is in L^1.*

Proof. Since $\deg(P) \geq 2$, there exists $A, x_0 > 0$ such that $|f(x)| \leq A/x^2$ whenever $x > x_0$. Define $I_N(x) = \int_{-N}^{N} f(x)dx$. We need to show that I_N is Cauchy, so we can set $N > x_0$. Without loss of generality, suppose $M > N$. Then

$$|I_M - I_N| \leq \int_{N\leq|x|\leq M} |f(x)|dx \leq \int_{N\leq|x|\leq M} \frac{A}{x^2}dx \leq \frac{2A}{N},$$

which goes to 0 as $N \to \infty$. □

4 4 Proofs: The FIF implies the FTA

4.1 First Proof

Towards a contradiction, suppose the FTA is false. Then there exists noncontant polynomial P with $\deg(P) \geq 2$ that is zero free, since polynomial P with degree one can be written as $P(z) = c_0 + c_1 z$ and thus clearly has a zero at $z = -c_0/c_1$. Define $f(z) = 1/P(z)$. By Proposition 3.4, f is entire. Let $\hat{f}$ be the Fourier transform of the restriction of f to $\mathbb{R}$:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ix\xi}dx.$$

Consider first the case in which $\xi \leq 0$. Define the upper half disc $D^+ = \{x + iy| - L \leq x \leq L, 0 \leq y \leq \sqrt{L^2 - x^2}$. Let C^+ denote the semicircle portion of the half disc (going counterclockwise), then by Cauchy's theorem, we have

$$\int_{\partial D^+} f(z)e^{-2\pi iz\xi}dz = \int_{-L}^{L} f(x)e^{-2\pi ix\xi}dx + \int_{C^+} f(z)e^{-2\pi iz\xi}dz = 0.$$

We claim that the second integral is $O(1/L)$. To verify this, notice first that $|f(z)e^{-2\pi iz\xi}| = |f(z)e^{-2\pi i(x+iy)\xi}| = e^{2\pi y\xi}|f(z)| \leq |f(z)|$, and since $\deg(P) \geq 2$, $|f(z)|$ is $O(1/|z|^2)$. Since the radius of C^+ is L, $|z| = L$; and since the length of the semicircle is $2\pi L$, we get that the integral is $O(1/L)$, as claimed. Let $L \to \infty$, we get

$$\int_{\infty}^{\infty} f(x)e^{-2\pi ix\xi}dx = \hat{f}(\xi) = 0.$$

This result can be reached identically in the case for $\xi > 0$ by defining and integrating along the lower half disc. By Corollary 3.1, this implies that $f(x) = 0$, which it cannot be, and we get the contradiction as desired. $\square$

4.2 Second, Third, and Fourth Proof

Again towards a contradiction, suppose the FTA is false. Then similar as before we can get nonconstant polynomial P with complex coefficients with $\deg(P) \geq 2$ that is zero free. Define $f(x + iy) = 1/P(x + iy)$. By Proposition 3.5, $f \in L^1$ for fixed y. So for fixed y

$$\hat{f}(\xi, y) = \int_{-\infty}^{\infty} e^{-2\pi ix\xi} f(x + iy)dx$$

is continuous. We claim that

$$\hat{f}(\xi, y) = e^{-2\pi y\xi}\hat{f}(\xi, 0). \tag{1}$$

This identity is motivated by the technique of u-substitution integration: by substituting $u = x + iy$, you can get something very similar to equation (1). It cannot be proven in this way, however, as we will see later. Now, assume that (1) is true. Since $\deg(P) \geq 2$, $|1/P(x + iy)| \leq A/(x^2 + 1)$, so

$$\hat{f}(\xi, y) \leq \int_{-\infty}^{\infty} |f(x + iy)|dx \leq A\int_{-\infty}^{\infty} \frac{1}{x^2 + 1} = A\pi,$$

and is thus bounded. Consider first $\xi < 0$. If $y > 0$, the right hand side of (1) $e^{-2\pi y\xi}\hat{f}(\xi, 0)$ is unbounded unless $\hat{f}(\xi, 0) = 0$. For $\xi > 0$, consideration of $y < 0$ again forces $\hat{f}(\xi, 0) = 0$. Since $\hat{f}$ is continuous, $\hat{f}(\xi, 0) = 0$ when $\xi = 0$. So $\hat{f}(\xi, 0) = 0$ always, and we can apply Corollary 3.1 to conclude that $f(x) = 0$, but it cannot be, and we get the desired contradiction that proves the FTA. □

One may wonder why this constitutes as three proofs—that is because we are going to supply three proof of equation (1).

First Proof of Equation (1). By the Lebesgue dominated convergence theorem, we can find the following partial derivative by moving $\partial/\partial y$ inside the integral:

$$\begin{aligned}
\hat{f}_y(\xi, y) &= \int_{-\infty}^{\infty} e^{2\pi i x\xi} \frac{\partial}{\partial y} f(x+iy)dx \\
&= i\int_{-\infty}^{\infty} e^{2\pi i x\xi} \frac{\partial}{\partial x} f(x+iy)dx \\
&= i\left(\left[e^{2\pi i x\xi} f(x+iy)\right]_{-\infty}^{\infty} + 2\pi i\xi \int_{-\infty}^{\infty} e^{2\pi i x\xi} f(x+iy)dx\right) \\
&= i\left(\left[\frac{e^{2\pi i x\xi}}{P(x+iy)}\right]_{-\infty}^{\infty} + 2\pi i\xi \hat{f}(\xi, y)\right) \\
&= -2\pi\xi\hat{f}(\xi, y).
\end{aligned}$$

The second equality uses the Cauchy-Riemann equations, and the subsequent equality uses integration by parts. Using this equality, we can calculate:

$$\int_0^{y'} e^{2\pi y\xi}\hat{f}_y(\xi, y)dy + \int_0^{y'} 2\pi\xi e^{2\pi y\xi}\hat{f}(\xi, y)dy = 0,$$

again using integration by parts on both integrals. This is a lengthy derivation, but it gives us $\hat{f}(\xi, y') = e^{-2\pi y'\xi}\hat{f}(\xi, 0)$, which is exactly what we wished. □

Second Proof of Equation (1). Consider first the case $y > 0$. Define rectangle $R = \{x + iv | -L \le x \le L, 0 \le v \le y\}$. We are going to use Cauchy's theorem and integrate $f(z)e^{-2\pi izt}$ along ∂R. First, observe that on the vertical line (we use the one on the right as an example), we have

$$\int_0^y e^{-2\pi i(L+iv)\xi} f(L+iv)idv \le \int_0^y e^{2\pi v\xi}|f(L+iv)|dv.$$

Since y is fixed, this integral is $O(1/L)$ by definition of f. The exact same goes for the one on the left. So, by Cauchy's theorem, we get

$$\int_{-L}^{L} e^{-2\pi ix\xi} f(x)dx + \int_{L}^{-L} e^{-2\pi i(x+iy)\xi} f(x+iy)dx + O(1/L) = 0.$$

The $y = 0$ case obviously holds. In the $y < 0$ case, we can simply define $R = \{x + iv | -L \le x \le L, y \le v \le 0\}$, make a similar argument, and arrive at the same result. Let $L \to \infty$, we get $\hat{f}(\xi, 0) - \hat{f}(\xi, y)e^{2\pi y\xi} = 0$, as desired. □

Third Proof of Equation (1). By translation invariance of the integral, for real s, we can write
$$\int_{-\infty}^{\infty} e^{-2\pi i x\xi} f(x+s)dx = e^{2\pi i s t}\int_{-\infty}^{\infty} e^{-2\pi i x\xi} f(x)dx.$$
By the identity theorem (see appendix), the relation above holds for $s \in \mathbb{C}$, and we can set $s = iy$ to arrive at equation (1). $\square$

5 The FIF Implies Liouville's Theorem, which Implies the FTA

Theorem 5.1 (Liouville's theorem)**.** *If f is entire and bounded, then f is constant.*

Proof. (Liouville's theorem via the FIF) Let $g : \mathbb{C} \to \mathbb{C}$ be entire and bounded. Define
$$f(z) = \begin{cases} \left(\frac{g(z)-g(0)}{z}\right)^2 & \text{if } z \neq 0, \\ (g'(0))^2 & \text{if } z = 0. \end{cases}$$
Since g is entire, f is continuous everywhere and entire. Again define
$$\hat{f}(\xi, y) = \int_{-\infty}^{\infty} e^{-2\pi i x\xi} f(x+iy)dx.$$
We claim that equation (1) holds for this $\hat{f}(\xi, y)$. Since f is $O(1/L)$ by definition, we can follow the argument the Second Proof of Equation (1), i.e., by defining and integrating along a rectangle to reach our claim. Following now the same argument as laid out for the part of the proof shared by the second, third, and fourth proof of the FTA after the establishment of equation (1), we get $f(x, 0) = 0$, which implies that $g(z)$ is constant. $\square$

The significance of obtaining Liouville's theorem from the FIF, besides the simple fact that Liouville's theorem implies the FTA, also lies in how "naturally"—many believe and I agree—Liouville's theorem implies the FTA, as shown below.

Proof. (FTA via Liouville's theorem) Suppose the FTA is false. Then we once again has a zero free nonconstant P with complex coefficients. For $z \neq 0$, write
$$\frac{P(z)}{z^n} - c_n = \frac{c_{n-1}}{+} \cdots + \frac{a_0}{z^n},$$
we see that the right-hand-side goes to zero as $|z| \to \infty$. Let $A = |c_n|/2$, then there exists $R > 0$ such that whenever $|z| > R$,
$$|P(z)| \geq A|z|^n.$$
This gives us that $1/P(z)$ is bounded when $|z| > R$. Furthermore, since P has no zeros when $|z| < R$, $1/P(z)$ is bounded when $|z| < R$ as well. And since $1/P(z)$ is entire by Proposition 3.4, $1/P(z)$ is constant by Liouville's theorem. But P is nonconstant, and we have a contradiction. $\square$

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7 Appendix

7.1 Integrating $1/(1+x^2)$ on $(-\infty, \infty)$ Gives π

This identity is used multiple times in the text, thus deserves a somewhat complete treatment. The following proof uses Cauchy's theorem and Cauchy's integral formula (see next section of Appendix).

Proof. Let γ_R be the outline of the upper half disc with radius R. Call the curved part of γ_R C_R. Then

$$\int_{\gamma_R} \frac{1}{1+z^2} dz = \int_{-R}^{R} \frac{1}{1+x^2} dx + \int_{C_R} \frac{1}{1+z^2} dz.$$

We first need to show is that the second term on the right hand side goes to zero as $R \to \infty$. Let $z = Re^{it}$. Then $z : [0, \pi] \mapsto \mathbb{C}$ parametrizes C_R. We have

$$\begin{aligned} \left| \int_{C_R} \frac{1}{1+z^2} dz \right| &\leq \int_0^\pi \frac{|-iRe^{-it}|}{|1+R^2 e^{-i2t}|} dt \\ &\leq \int_0^\pi \frac{R}{|R^2 e^{-i2t}| - 1} dt \\ &\leq \int_0^\pi \frac{R}{R^2 - 1} dt \\ &= \frac{R\pi}{R^2 - 1}. \end{aligned}$$

Applying l'Hopital's rule, we know that this expression goes to 0 as $R \to \infty$.
We then construct a keyhole around the point i with width of corridor δ and radius of the hole ϵ. Let $\delta \to 0$. Since $1/(1+z^2)$ is continuous on that corridor, we can apply Cauchy's theorem on our keyhole and get

$$\int_{-R}^{R} \frac{1}{1+x^2} dx + \int_{C_\epsilon} \frac{1}{1+z^2} dz + \int_{C_R} \frac{1}{1+z^2} dz = 0.$$

Letting $R \to \infty$ and using our findings from part (a), we have

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \int_{C'_\epsilon} \frac{1}{1+z^2} dz.$$

Where C'_ϵ has the opposite orientation as C_ϵ. Since f is holomorphic on C'_ϵ and its interior, we can apply Cauchy's integral formula:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \int_{C'_\epsilon} \frac{f(z)}{z-i} dz = 2\pi i f(i) = \frac{2\pi i}{2i} = \pi,$$

which is precisely what we desire. □

7.2 Statements of Theorems Used

Theorem 7.1 (Fubini's theorem)**.** *If either*

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)| dy \right) dx < \infty$$

or

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)| dx \right) dy < \infty,$$

then

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)| dy \right) dx = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)| dx \right) dy = \int_{\mathbb{R}\times\mathbb{R}} f(x,y) d(x,y).$$

As we see here and have seen in the text, Fubini's theorem is used to exchange the order of integration in a double integral under appropriate circumstances. Lebesgue dominated convergence theorem, on the other hand, is used to exchange the limit and the integral.

Theorem 7.2 (Lebesgue Dominated Convergence Theorem)**.** *Let $f_n(x) : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable functions such that there exists f such that $\lim_{n\to\infty} f_n(x) \to f(x)$ pointwise. If there exists an integrable $g : \mathbb{R} \to \mathbb{R}^+$ such that $|f_n(x)| \leq g(x)$ for all $x \in \mathbb{R}$, then f and f_n are integrable for all n, and*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x) dx = \int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(x) dx = \int_{-\infty}^{\infty} f(x) dx.$$

Theorem 7.3 (Cauchy's Integral Formula)**.** *Suppose f is holomorphic in an open set that contains the closure of a disc D. If C denotes the boundary circle of this disc with the positive orientation, then*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z} dw$$

for any point $z \in D$.

Theorem 7.4 (Identity Theorem)**.** *Suppose f and g are holomorphic in Ω and $f(z) = g(z)$ for all z in some non-empty open subset of Ω. Then $f(z) = g(z)$ in Ω.*

Proofs of these results can be found in Stein and Shakarchi's *Complex Analysis* and Nelson's *A User-Friendly Introduction to Lebesgue Measure and Integration.*