Journal of Knot Theory and Its Ramifications
Vol. 28, No. 5 (2019) 1950037 (20 pages)

DOI: 10.1142/S0218216519500378

World Scientific
www.worldscientific.com

Modular group representations associated to $SO(p)_2$-TQFTS

Yilong Wang
Department of Mathematics, Louisiana State University,
Baton Rouge, LA 70803, USA
yilongwang@lsu.edu

Received 10 July 2017
Accepted 22 December 2017
Published 29 March 2019

ABSTRACT

In this paper, we prove that for any odd prime p greater than 3, the modular group representation associated to the $SO(p)_2$-topological quantum field theory can be defined over the ring of integers of a cyclotomic field. We will provide explicit integral bases. In the last section, we will relate these representations to the Weil representations over finite fields.

Keywords: RT-TQFT; modular group representation; integrality.

Mathematics Subject Classification 2010: 57R56, 18D10, 11R04

1. Introduction

According to [23], to each modular tensor category, we can associate a Reshetikhin–Turaev topological quantum field theory (TQFT). This TQFT not only gives rise to quantum invariants of 3-manifolds, but also to a series of projective representations of the mapping class groups $\mathrm{MCG}(\Sigma_g)$ of closed-oriented surfaces Σ_g of genus g. In particular, in genus one, we get a projective representation of the modular group $\mathrm{MCG}(\Sigma_1) = \mathrm{SL}(2, \mathbb{Z})$.

A systematic way to construct modular categories is to consider the representation theory of quantum groups at roots of unity. The TQFT representations of mapping class groups arising from such modular categories are finite-dimensional and can be defined over a cyclotomic field $\mathbb{Q}(\zeta)$ where ζ is a root of unity. As a result, the corresponding quantum invariants are elements of $\mathbb{Q}(\zeta)$.

The first integrality result was obtained by Murakami [15, 16], who showed that the SU(2)- and SO(3)-invariants are algebraic integers when the order of ζ is prime. More precisely, those invariants are elements of the ring of integers of $\mathbb{Q}(\zeta)$, namely, $\mathbb{Z}[\zeta]$. The result was reproved in [13], generalized to all classical Lie types in [14, 24], then to all Lie types by Le [12]. These results helped us relate the

quantum invariants to other invariants such as the Casson invariant [15, 16] and the Ohtsuki series [12, 20, 21].

A natural question to ask then is whether one can define the whole TQFT over $\mathbb{Z}[\zeta]$, or at least can one define the representations of the mapping class groups over $\mathbb{Z}[\zeta]$. If that is the case, we can get more information about these representations. For example, in [7], the authors studied the Frohman–Kania-Bartoszynska ideal [4] using the integral SO(3)-TQFT, and it is natural to apply this method to other integral TQFTs. Moreover, given integral TQFTs, we can reduce them by the natural reduction map $\mathbb{Z}[\zeta] \to \mathbb{Z}/p\mathbb{Z}$ to get the so-called p-modular TQFTs. The p-modular TQFTs have rich connections to the topological information of 3-manifolds, such as the Casson-Lescop invariant and the Milnor torsion [10]. We may also answer questions such as the finiteness of the images of these representations by the integrality results.

For the SO(3)-TQFT, Gilmer, Masbaum and van Wamelen first constructed integral bases for the genus one and two [8] cases. Then Gilmer and Masbaum generalized their result to arbitrary genus in [7], hence completed the construction of an integral SO(3)-TQFT.

In this paper, we will focus on the integrality properties of the $\mathrm{SO}(p)_2$-TQFTs for $p \geq 5$, which comes from the representation theories of quantum groups associated to the Lie algebra $\mathfrak{so}(p)$ at certain roots of unity. These TQFTs emerge as important objects in the context of topological quantum computing [9], and as interesting examples of classical Lie type quantum groups themselves. We will establish the integrality of them in genus one by proving

Theorem 1.1. *Suppose $p \geq 5$ is an odd prime. Then the genus one mapping class group representation given by the $\mathrm{SO}(p)_2$-TQFT can be defined over $\mathcal{O}$, where*

$$\mathcal{O} = \begin{cases} \mathbb{Z}[\zeta_p, i] = \mathbb{Z}[\zeta_{4p}], & \text{if } p \equiv 3 \pmod 4, \\ \mathbb{Z}[\zeta_p], & \text{if } p \equiv 1 \pmod 4. \end{cases} \tag{1.1}$$

Note that the i in $\mathcal{O}$ essentially comes from the twists of the simple objects of $\mathrm{SO}(p)_2$ (cf. (2.3)) and the quadratic Gauss sum formula (cf. Proposition 3.3).

In the proof, we give an explicit-integral basis as the authors of [8] did for the SO(3)-TQFT. As a byproduct, we show that a (irreducible) summand of the genus one mapping class group representation of $\mathrm{SL}(2,\mathbb{Z})$ factors through the even part of the Weil representation of $\mathrm{SL}(2,\mathbb{Z}/p\mathbb{Z})$. We then conclude as a corollary that the image of the $\mathrm{SO}(p)_2$-TQFT representation in genus one is finite. This confirms the theorem by [19] saying that the $\mathrm{SL}(2,\mathbb{Z})$ TQFT representation given by any modular category is finite.

The paper is organized as follows. In Sec. 2, we give a quick review of preliminaries on the TQFT representation and the $\mathrm{SO}(p)_2$-TQFT. In particular, we give explicit matrix presentations of the $\mathrm{SL}(2,\mathbb{Z})$ representation under a fixed basis. In Sec. 3, we use number-theoretic tools to construct a new basis for the representation

space and show that it is indeed integral. In Sec. 4, we briefly recall the definition of the Weil representation of $\mathrm{SL}(2,\mathbb{Z}/p\mathbb{Z})$. We prove in Theorem 4.1 that a summand of the genus one TQFT representation factors through the even part of the Weil representation. As a result, we prove that the image of the $\mathrm{SL}(2,\mathbb{Z})$ representation is finite.

Notations and conventions. In the discussion below, we will assume that $p \geq 5$ is an odd prime. Let $r = \frac{p-1}{2}$. Let $\zeta_n = e^{\frac{2\pi i}{n}}$ be an nth root of unity. We will let $\mathcal{O}$ be as in Theorem 1.1, and use $\mathcal{O}^\times$ to denote the group of units of $\mathcal{O}$. It is a well-known fact that $\sqrt{p} \in \mathcal{O}$, hence $\frac{1}{\sqrt{p}} \in \mathbb{Q}(\zeta_p)$ or $\mathbb{Q}(\zeta_{4p})$ depending on p mod 4. Let $*^t$ denote the transpose of $*$, and let Id_n stand for the $n \times n$ identity matrix. We call a representation *integral* if the matrix coefficients of the representation are in $\mathcal{O}$ with respect to a choice of basis. We will also call a matrix with entries in $\mathcal{O}$ integral.

2. Preliminaries

In this section, we briefly recall the definition of the modular category $\mathrm{SO}(p)_2$ and the Reshetikhin–Turaev TQFT associated to it. For more details, the readers are referred to [9, 23].

The $\mathrm{SO}(p)_2$ modular category is the unitary modular tensor category obtained from the representation theory of the quantum group $U_q(\mathfrak{so}(p))$, where $q = e^{\frac{\pi i}{2p}}$. It has $(r+4)$ simple objects labeled by

$$\mathrm{Irr}(\mathrm{SO}(p)_2) = \{\mathbf{1}, \mathbf{Z}, \mathbf{Y}_1, \ldots, \mathbf{Y}_r, \mathbf{X}, \mathbf{X}'\}.$$

Here, $\mathbf{1}$ is the tensor unit. The fusion rules can be completely determined by the following listed ones:

$$\mathbf{Z} \otimes \mathbf{Z} \cong \mathbf{1},$$

$$\mathbf{Z} \otimes \mathbf{X} \cong \mathbf{X}',$$

$$\mathbf{Z} \otimes \mathbf{Y}_j \cong \mathbf{Y}_j, \quad \forall\, j = 1, \ldots, r,$$

$$\mathbf{X} \otimes \mathbf{X} \cong \mathbf{1} \oplus \bigoplus_{j=1}^{r} \mathbf{Y}_j,$$

$$\mathbf{X} \otimes \mathbf{X}' \cong \mathbf{Z} \oplus \bigoplus_{j=1}^{r} \mathbf{Y}_j,$$

$$\mathbf{X} \otimes \mathbf{Y}_j \cong \mathbf{X} \oplus \mathbf{X}', \quad \forall\, j = 1, \ldots, r,$$

$$\mathbf{Y}_j \otimes \mathbf{Y}_j \cong \mathbf{1} \oplus \mathbf{Z} \oplus \mathbf{Y}_{\min\{2j, p-2j\}}, \quad \forall\, j = 1, \ldots, r,$$

$$\mathbf{Y}_j \otimes \mathbf{Y}_k \cong \mathbf{Y}_{|j-k|} \oplus \mathbf{Y}_{\min\{j+k, p-j-k\}}, \quad \forall\, 1 \leq j, \quad k \leq r, \;\; j \neq k.$$

Let

$$H := \mathrm{span}_{\mathbb{C}}\{\mathbf{1}, \mathbf{Z}, \mathbf{Y}_1, \ldots, \mathbf{Y}_r, \mathbf{X}, \mathbf{X}'\}.$$

We will view $\mathrm{Irr}(\mathrm{SO}(p)_2)$ as the fixed basis of H.

To any modular tensor category, we can construct a Reshetikhin–Turaev TQFT. The TQFT is, roughly speaking, a tensor functor from a suitably defined cobordism category to the category of finite-dimensional vector spaces. The cobordism category has closed-oriented surfaces as objects and 3-manifolds bounding two such surfaces as morphisms. The tensor structure on the cobordism category is the disjoint union, and the tensor structure on the category of vector spaces is the tensor product over the ground field.

In particular, for any orientation preserving diffeomorphism Ψ of a surface Σ_g of genus g, the image of the mapping cylinder of Ψ under the TQFT functor is a linear automorphism of the vector space associated to Σ_g. It is unique up to scalar multiples and is invariant under isotopy. As a result, the TQFT gives rise to a projective representation of the mapping class groups of the surface of genus g for each natural number g.

In this paper, we consider the representation of the genus one mapping class group $\mathrm{MCG}(\Sigma_1) = \mathrm{SL}(2,\mathbb{Z})$ associated to the $\mathrm{SO}(p)_2$-TQFT. We give the representation explicitly in terms of the generators of $\mathrm{SL}(2,\mathbb{Z})$ as follows:

Let $\sigma = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and $\tau = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ be the generators of $\mathrm{SL}(2,\mathbb{Z})$. Let A be the $(r \times r)$-matrix with entries given by

$$A_{jk} = \frac{2}{\sqrt{p}} \cos\left(\frac{2\pi jk}{p}\right) = \frac{1}{\sqrt{p}}(\zeta_p^{jk} + \zeta_p^{-jk}), \tag{2.1}$$

for all $1 \leq j, k \leq r$.

Let

$$a = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \tag{2.2}$$

be an $(r \times 1)$-dimensional vector, and let

$$\psi = \zeta_8^r = e^{2\pi i \cdot \frac{r}{8}}. \tag{2.3}$$

Note that $\psi \notin \mathcal{O}$, but $\psi^2 \in \mathcal{O}$.

From now on, we will suppress the subscript of ζ_p by simply writing ζ, while letting

$$\theta_j = \zeta^{rj^2} = e^{\frac{2\pi i}{p} rj^2} = e^{\frac{2\pi i}{p} \cdot \frac{j(p-j)}{2}}, \quad \forall\, j \in \mathbb{Z}. \tag{2.4}$$

The projective representation of $\mathrm{SL}(2,\mathbb{Z})$ derived from the $\mathrm{SO}(p)_2$-TQFT is given by (see, for example, [17])

$$\rho_1 : \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{PGL}(H),$$

$$\rho_1(\sigma) = \begin{bmatrix} \frac{1}{2\sqrt{p}} & \frac{1}{2\sqrt{p}} & \frac{1}{\sqrt{p}} \cdot a^t & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2\sqrt{p}} & \frac{1}{2\sqrt{p}} & \frac{1}{\sqrt{p}} \cdot a^t & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{p}} \cdot a & \frac{1}{\sqrt{p}} \cdot a & A & 0_{r\times 1} & 0_{r\times 1} \\ \frac{1}{2} & -\frac{1}{2} & 0_{1\times r} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0_{1\times r} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}, \tag{2.5}$$

and

$$\rho_1(\tau) = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \theta_1 & & & & \\ & & & \ddots & & & \\ & & & & \theta_r & & \\ & & & & & \psi & \\ & & & & & & -\psi \end{bmatrix}, \tag{2.6}$$

Here $0_{\mu\times\nu}$ in a matrix M is understood as a zero block of M of size $\mu \times \nu$.

Now we are ready to proceed to the proof of Theorem 1.1.

3. Proof of Theorem 1.1

To prove the theorem, we will give an explicit change of basis matrix $W \in \mathrm{GL}(H)$ so that $W^{-1}\rho_1(\sigma)W$ and $W^{-1}\rho_1(\tau)W$ have entries in $\mathcal{O}$. We will find W in several steps. First, we decompose H into a direct sum of two invariant subspaces in Lemma 3.1, and reduce the problem to Claim 3.2. We then investigate properties of the column vectors of the representation after the change of basis proposed in Claim 3.2. We will prove integrality of one of the column vectors in Proposition 3.6. Finally, we will prove the integrality of the rest of the column vectors by proving Claim 3.8.

Let

$$U = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & & \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & 0 & & & 1 \\ \hline 0 & 1 & \psi & 0 & 0 & \cdots & 0 \\ 0 & 1 & -\psi & 0 & 0 & \cdots & 0 \end{array}\right]. \tag{3.1}$$

Lemma 3.1. *After the change of basis by U, H splits into a direct sum of two invariant subspaces $H \cong H_1 \oplus H_2$, with $\rho_1|_{H_1}$ integral under the new basis.*

Proof. From the form of U, it is easy to see that the new basis corresponding to U is $\mathcal{B}_U = \{\mathbf{1}-\mathbf{Z}, \mathbf{X}+\mathbf{X}', \psi(\mathbf{X}-\mathbf{X}'), \mathbf{1}+\mathbf{Z}, \mathbf{Y}_1, \ldots, \mathbf{Y}_r\}$. To determine the matrix coefficients of $\rho_1(\sigma)$ and $\rho_1(\tau)$ after the change of basis, we simply have to calculate how the two linear operations act on the new basis vectors. Then we will write the resulting vectors as linear combinations of vectors in $\mathcal{B}_U$.

By (2.5), we have

$$\rho_1(\sigma)(\mathbf{1}) = \frac{1}{2\sqrt{p}}\mathbf{1} + \frac{1}{2\sqrt{p}}\mathbf{Z} + \frac{1}{\sqrt{p}}\sum_{k=1}^{r}\mathbf{Y}_k + \frac{1}{2}\mathbf{X} + \frac{1}{2}\mathbf{X}',$$

$$\rho_1(\sigma)(\mathbf{Z}) = \frac{1}{2\sqrt{p}}\mathbf{1} + \frac{1}{2\sqrt{p}}\mathbf{Z} + \frac{1}{\sqrt{p}}\sum_{k=1}^{r}\mathbf{Y}_k - \frac{1}{2}\mathbf{X} - \frac{1}{2}\mathbf{X}',$$

$$\rho_1(\sigma)(\mathbf{Y}_j) = \frac{1}{\sqrt{p}}\mathbf{1} + \frac{1}{\sqrt{p}}\mathbf{Z} + \sum_{k=1}^{r}A_{kj}\mathbf{Y}_k, \quad \forall\, j = 1, \ldots, r,$$

$$\rho_1(\sigma)(\mathbf{X}) = \frac{1}{2}\mathbf{1} - \frac{1}{2}\mathbf{Z} + \frac{1}{2}\mathbf{X} - \frac{1}{2}\mathbf{X}',$$

$$\rho_1(\sigma)(\mathbf{X}') = \frac{1}{2}\mathbf{1} - \frac{1}{2}\mathbf{Z} - \frac{1}{2}\mathbf{X} + \frac{1}{2}\mathbf{X}'.$$

So the action of $\rho_1(\sigma)$ on the new basis vectors (written as linear combinations of them) is given by

$$\rho_1(\sigma)(\mathbf{1}-\mathbf{Z}) = \mathbf{X}+\mathbf{X}',$$

$$\rho_1(\sigma)(\mathbf{X}+\mathbf{X}') = \mathbf{1}-\mathbf{Z},$$

$$\rho_1(\sigma)(\psi(\mathbf{X}-\mathbf{X}')) = \psi(\mathbf{X}-\mathbf{X}'),$$

$$\rho_1(\sigma)(\mathbf{1}+\mathbf{Z}) = \frac{1}{\sqrt{p}}(\mathbf{1}+\mathbf{Z}) + \frac{2}{\sqrt{p}}\sum_{k=1}^{r}\mathbf{Y}_k,$$

$$\rho_1(\sigma)(\mathbf{Y}_j) = \frac{1}{\sqrt{p}}(\mathbf{1}+\mathbf{Z}) + \sum_{k=1}^{r}A_{kj}\mathbf{Y}_k.$$

Therefore, the linear map $\rho_1(\sigma)$ has the following matrix presentation in the new basis $\mathcal{B}_U$:

$$U^{-1}\rho_1(\sigma)U = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & & \\ 1 & 0 & 0 & \multicolumn{2}{c}{0_{3\times(r+1)}} \\ 0 & 0 & 1 & & \\ \hline & & & \dfrac{1}{\sqrt{p}} & \dfrac{1}{\sqrt{p}}\cdot a^t \\ \multicolumn{3}{c|}{0_{(r+1)\times 3}} & & \\ & & & \dfrac{2}{\sqrt{p}}\cdot a & A \end{array}\right]. \tag{3.2}$$

By a similar argument, we have

$$\rho_1(\tau)(\mathbf{1}-\mathbf{Z}) = \mathbf{1}-\mathbf{Z},$$

$$\rho_1(\tau)(\mathbf{X}+\mathbf{X}') = \psi(\mathbf{X}-\mathbf{X}'),$$

$$\rho_1(\tau)(\psi(\mathbf{X}-\mathbf{X}')) = \psi^2(\mathbf{X}+\mathbf{X}'),$$

$$\rho_1(\tau)(\mathbf{1}+\mathbf{Z}) = \mathbf{1}+\mathbf{Z},$$

$$\rho_1(\tau)(\mathbf{Y}_j) = \theta_j\mathbf{Y}_j.$$

Therefore, under $\mathcal{B}_U$, $\rho_1(\tau)$ has matrix presentation

$$U^{-1}\rho_1(\tau)U = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ 0 & 0 & \psi^2 & & & \\ 0 & 1 & 0 & & & \\ \hline & & & 1 & & \\ & & & & \theta_1 & \\ & & & & & \ddots \\ & & & & & \theta_r \end{array}\right]. \tag{3.3}$$

The empty slots in the matrix are considered as 0-matrices of suitable size.

It is easy to see, either from the actions of $\rho_1(\sigma)$ and $\rho_1(\tau)$ or from the block form of their matrix presentations in the basis $\mathcal{B}_U$, that they preserve the subspaces $H_1 =$

$\mathrm{span}_{\mathbb{C}}\{\mathbf{1}-\mathbf{Z}, \mathbf{X}+\mathbf{X}', \psi(\mathbf{X}-\mathbf{X}')\}$ and the subspace $H_2 = \mathrm{span}_{\mathbb{C}}\{\mathbf{1}+\mathbf{Z}, \mathbf{Y}_1, \ldots, \mathbf{Y}_r\}$ of H. So we have

$$H \cong H_1 \oplus H_2.$$

In addition, the matrix coefficients of $U^{-1}\rho_1(\sigma)U$ and $U^{-1}\rho_1(\tau)U$ restricted to H_1 are in $\mathcal{O}$. □

Given Lemma 3.1, we just have to find a change of basis for the $(r+1)$-dimensional subspace H_2 so that ρ_1 restricted to H_2 is integral. For convenience, we introduce the following notations:

$$S' = \rho_1|_{H_2}(\sigma) = \begin{bmatrix} \frac{1}{\sqrt{p}} & \frac{1}{\sqrt{p}} \cdot a^t \\ \frac{2}{\sqrt{p}} \cdot a & A \end{bmatrix} \tag{3.4}$$

and

$$T' = \rho_1|_{H_2}(\tau) = \begin{bmatrix} 1 & & & \\ & \theta_1 & & \\ & & \ddots & \\ & & & \theta_r \end{bmatrix}. \tag{3.5}$$

Instead of S' and T', we would prefer to work with their transposes. We define

$$S := (S')^t = D^{-1}S'D = \begin{bmatrix} \frac{1}{\sqrt{p}} & \frac{2}{\sqrt{p}} \cdot a^t \\ \frac{1}{\sqrt{p}} \cdot a & A \end{bmatrix} \tag{3.6}$$

and

$$T := (T')^t = D^{-1}T'D = T' = \begin{bmatrix} 1 & & & \\ & \theta_1 & & \\ & & \ddots & \\ & & & \theta_r \end{bmatrix}. \tag{3.7}$$

Here,

$$D = \begin{bmatrix} \frac{1}{2} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}. \tag{3.8}$$

Theorem 1.1 now follows from the claim below:

Claim 3.2. Let V be the following Vandermonde matrix:

$$V = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \theta_1 & \theta_1^2 & \cdots & \theta_1^r \\ 1 & \theta_2 & \theta_2^2 & \cdots & \theta_2^r \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \theta_r & \theta_r^2 & \cdots & \theta_r^r \end{bmatrix}. \tag{3.9}$$

Then $V^{-1}SV$ and $V^{-1}TV$ are both integral.

Proof (Proof of Theorem 1). The change of basis matrix $(Id_3 \oplus DV)$ makes the block matrices $U^{-1}\rho_1(\sigma)U$ and $U^{-1}\rho_1(\tau)U$ integral. Hence $W = U(Id_3 \oplus DV)$ is the desired change of basis matrix. □

To prove the claim, we need the following property of SV. In the following discussions, we will index the matrix entries from 0, and recall that by definition $\theta_0 = 1$. The fractions $1/b$ for $b \in \mathbb{Z}$ in the powers of ζ, θ_j, or in the (mod p)-equations are understood as the multiplicative inverse of b mod p. In other words, $1/b$ in the above cases represents an integer $1 \leq w \leq p-1$ such that $wb \equiv 1 \mod p$.

Proposition 3.3. *The* (j,k)*th matrix coefficient of* SV *is given by*

$$(SV)_{jk} = \begin{cases} \sqrt{p}, & \text{if } j = k = 0, \\ 0, & \text{if } 1 \leq j \leq r, \quad \text{and} \quad k = 0, \\ \left(\dfrac{rk}{p}\right)_J \cdot \iota(p) \cdot \theta_j^{-\frac{1}{k}}, & \text{if } k \geq 1. \end{cases} \tag{3.10}$$

Here $(\frac{*}{*})_J$ *stands for the Jacobi symbol and*

$$\iota(p) = \begin{cases} 1, & \text{if } p \equiv 1 \mod 4, \\ i, & \text{if } p \equiv 3 \mod 4. \end{cases} \tag{3.11}$$

Proof. Case 1. When $j = k = 0$, a direct computation shows that

$$(SV)_{00} = \frac{1}{\sqrt{p}} + \frac{2}{\sqrt{p}} \times r = \frac{1}{\sqrt{p}} \times p = \sqrt{p}.$$

Case 2. When $1 \leq j \leq r$, and $k = 0$, we have,

$$\begin{aligned} (SV)_{j0} &= \frac{1}{\sqrt{p}} + \sum_{l=1}^{r} A_{jl} \times 1 \\ &= \frac{1}{\sqrt{p}} + \sum_{l=1}^{r} \frac{1}{\sqrt{p}} (\zeta^{jl} + \zeta^{-jl}) \end{aligned}$$

$$= \frac{1}{\sqrt{p}} \sum_{l=0}^{2r} \zeta^{jl}$$
$$= 0.$$

The third equality results from the fact that $-l \equiv p - l \mod p$. The last equality stands on the fact that for any $1 \leq j \leq r$, ζ^j is a pth root of unity, hence is a solution to the minimal polynomial $\Phi_p(x) = 1 + x + \cdots + x^{2r}$ (recall that by assumption p is an odd prime).

Case 3. When $1 \leq k \leq r$, recall (2.4), we have

$$(SV)_{jk} = \frac{1}{\sqrt{p}} + \sum_{l=1}^{r} \frac{1}{\sqrt{p}} (\zeta^{jl} + \zeta^{-jl}) \times \theta_l^k$$
$$= \frac{1}{\sqrt{p}} + \frac{1}{\sqrt{p}} \sum_{l=1}^{r} (\zeta^{jl} + \zeta^{-jl}) \times \zeta^{rkl^2}$$
$$= \frac{1}{\sqrt{p}} \sum_{l=0}^{2r} \zeta^{jl+rkl^2}$$
$$= \frac{1}{\sqrt{p}} \sum_{l=0}^{2r} \zeta^{rk(l^2+\frac{j}{rk}l)}.$$

Note that by assumption, $1 \leq k \leq r$, hence $\frac{j}{rk}$ is well defined in the finite field $\mathbb{Z}/p\mathbb{Z}$. Letting $\gamma = \frac{j}{2rk} \in \mathbb{Z}/p\mathbb{Z}$, we can continue our calculation as follows:

$$\frac{1}{\sqrt{p}} \sum_{l=0}^{2r} \zeta^{rk(l^2+2\gamma l)} = \frac{1}{\sqrt{p}} \sum_{l=0}^{2r} \zeta^{rk(l+\gamma)^2 - rk\gamma^2}$$
$$= \frac{1}{\sqrt{p}} \zeta^{-rk\gamma^2} \sum_{l=0}^{2r} \zeta^{rk(l+\gamma)^2}.$$

Hence by the quadratic Gauss sum formula, we have

$$(SV)_{jk} = \frac{1}{\sqrt{p}} \times \zeta^{-rk\gamma^2} \times \left(\frac{rk}{p}\right)_J \times \iota(p) \times \sqrt{p}$$
$$= \left(\frac{rk}{p}\right)_J \times \iota(p) \times \zeta^{-rk\gamma^2}.$$

Note that

$$4r^2 - 1 = (2r+1)(2r-1) = p(2r-1) \equiv 0 \mod p.$$

Therefore,

$$k\gamma^2 = \frac{j^2}{4r^2 k} \equiv \frac{j^2}{k} \mod p,$$

and consequently,

$$(SV)_{jk} = \left(\frac{rk}{p}\right)_J \times \iota(p) \times \zeta^{(rj^2)\times(-\frac{1}{k})} = \left(\frac{rk}{p}\right)_J \times \iota(p) \times \theta_j^{-\frac{1}{k}},$$

as desired. □

To proceed further, let's recall some basic facts in number theory.

Lemma 3.4. *Let* $\epsilon = (-1)^r \times \zeta^{-\frac{r(r+1)}{2}} \in \mathcal{O}^\times$. *Then*

$$p = 2r+1 = \epsilon \prod_{k=1}^{r} (1-\zeta^k)^2. \tag{3.12}$$

Proof. Recall that

$$\Phi_p(x) = 1 + x + \cdots + x^{2r} = \prod_{l=1}^{2r} (x - \zeta^l).$$

Putting $x = 1$, we have

$$\begin{aligned} p = 2r+1 &= \prod_{l=1}^{2r} (1-\zeta^l) \\ &= \prod_{k=1}^{r} (1-\zeta^k) \times (1-\zeta^{-k}) \\ &= \prod_{k=1}^{r} (1-\zeta^k) \times \zeta^{-k} \times (\zeta^k - 1) \\ &= \prod_{k=1}^{r} (-\zeta^{-k}) \times \prod_{k=1}^{r} (1-\zeta^k)^2 \\ &= (-1)^r \times \zeta^{-\frac{r(r+1)}{2}} \times \prod_{k=1}^{r} (1-\zeta^k)^2 \\ &= \epsilon \prod_{k=1}^{r} (1-\zeta^k)^2. \end{aligned}$$

□

Consider the numbers $\omega_\pm := \pm i^r \times \zeta^{\frac{-r(r+1)}{4}}$, where the fraction $\frac{-r(r+1)}{4}$ is viewed as an element of $\mathbb{Z}/p\mathbb{Z}$ as before. As a product of roots of unity, $\omega_\pm \in \mathcal{O}^\times$. Moreover, we have

$$(\omega_+)^2 = (\omega_-)^2 = i^{2r} \times \zeta^{\frac{-2r(r+1)}{4}} = (-1)^r \times \zeta^{-\frac{r(r+1)}{2}} = \epsilon.$$

In other words, $\omega_\pm$ are the two square roots of ϵ.

Note that for any integers α, β such that $g.c.d(\alpha, p) = g.c.d.(\beta, p) = 1$, we have

$$\frac{1-\zeta^\alpha}{1-\zeta^\beta} \in \mathcal{O}^\times. \tag{3.13}$$

This is because in $\mathbb{Z}/p\mathbb{Z}$, we can write α as a multiple of β, so the quotient in (3.13) becomes a sum of elements in $\mathcal{O}$, so it is in $\mathcal{O}$. On the other hand, we can write β as a multiple of α, then the inverse of the quotient in (3.13) is also a sum of elements in $\mathcal{O}$, hence in $\mathcal{O}$.

Combining Lemma 3.4 and the above observations, we have the following lemma.

Corollary 3.5.

$$\sqrt{p} = u \prod_{k=1}^{r} (1 - \theta_k), \tag{3.14}$$

and $u \in \mathcal{O}^\times$.

Proof. By Lemma 3.4, there exists a choice of square root of ϵ (either ω_+ or ω_- depending on p), denoted by ω_p, such that

$$\sqrt{p} = \omega_p \prod_{k=1}^{r} (1 - \zeta^k).$$

Recall that $\omega_p \in \mathcal{O}^\times$ as shown above. Also by the observation above, we have, for any $1 \leq k \leq 2r$,

$$\eta_k = \frac{1 - \zeta^k}{1 - \theta_k} = \frac{1 - \zeta^k}{1 - \zeta^{rk^2}} \in \mathcal{O}^\times.$$

Let $u := \omega_p \prod_{k=1}^{r} \eta_k$, we have

$$\sqrt{p} = \omega_p \prod_{k=1}^{r} (1 - \zeta^k) = \left(\omega_p \prod_{k=1}^{r} \eta_k \right) \times \prod_{k=1}^{r} (1 - \theta_k) = u \prod_{k=1}^{r} (1 - \theta_k).$$

Note that u, as product of elements in $\mathcal{O}^\times$, is in $\mathcal{O}^\times$. □

Proposition 3.6. *The 0th column of* $V^{-1}SV$ *is a vector in* $\mathcal{O}^{r+1}$.

Proof. By Proposition 3.3, for any j, we have

$$(V^{-1}SV)_{j0} = \sum_{l=0}^{r} (V^{-1})_{jl} (SV)_{l0} = (V^{-1})_{j0} \times \sqrt{p}.$$

To prove the proposition, we simply have to show that $(V^{-1})_{j0} \times \sqrt{p} \in \mathcal{O}$.

By definition, we have

$$V \cdot (V^{-1}) = Id_{r+1}.$$

In other words, for any $0 \leq k \leq r$,

$$\sum_{j=0}^{r} \theta_k^j \times (V^{-1})_{j0} = \delta_{k,0}, \tag{3.15}$$

where $\delta_{*,*}$ is the Kronecker delta function. Consider the polynomial

$$P_0(x) = \sum_{j=0}^{r} (V^{-1})_{j0} \times x^j.$$

By (3.15), we have

$$P_0(\theta_0) = 1, \quad P_0(\theta_k) = 0, \quad k = 1, \ldots, r.$$

Therefore, by the Lagrangian interpolation formula, we have

$$P_0(x) = \sum_{j=0}^{r} (V^{-1})_{j0} \times x^j = \prod_{n=1}^{r} \frac{x - \theta_n}{1 - \theta_n}.$$

By comparing coefficients, we can write down explicit formulas for $(V^{-1})_{j0}$. But what is more important here is that

$$(V^{-1})_{j0} \times \prod_{n=1}^{r} (1 - \theta_n) \in \mathcal{O},$$

since it is a coefficient of the integral polynomial $\prod_{n=1}^{r}(x - \theta_n)$. On the other hand, by Corollary 3.5, we have $\sqrt{p} = \prod_{n=1}^{r}(1 - \theta_n) \times u$ for some unit $u \in \mathcal{O}^\times$, hence

$$(V^{-1})_{j0} \times \sqrt{p} = (V^{-1})_{j0} \times \prod_{n=1}^{r} (1 - \theta_n) \times u \in \mathcal{O}.$$

□

By Proposition 3.6, we are left to show that the lth column vector of SV for $1 \leq l \leq r$ and all the column vectors of TV have the property that after multiplying them V^{-1} from left we get vectors in $\mathcal{O}^{r+1}$.

In light of Proposition 3.3, we have the following observation:

Lemma 3.7. *The lth column vector of SV for* $1 \leq l \leq r$ *and all the column vectors of TV are, up to a scalar multiplication by* $\pm i$ *or* ± 1*, of the form* $[1, {\theta_1}^j, \theta_2^j, \ldots, \theta_r^j]^t$ *for some* $0 \leq j \leq 2r$*.*

Proof. This is a direct result of Proposition 3.3 and the definition of T. □

Hence we reduce our problem to the problem of showing

Claim 3.8. The vectors in Lemma 3.7, after being multiplied by V^{-1} from left, become vectors in $\mathcal{O}^{r+1}$.

Recall the following facts from linear algebra. Let $f(x) = (x - x_0)(x - x_1)(x - x_2)\cdots(x - x_r) = x^{r+1} + a_1 x^r + \cdots + a_r x + a_{r+1}$. Then the companion

matrix

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_{r+1} & -a_r & -a_{r-1} & -a_{r-2} & \cdots & -a_1 \end{bmatrix}$$

has $[1, x_i, x_i^2, \ldots, x_i^r]^t$ as eigenvectors corresponding to eigenvalues x_i for any $0 \leq i \leq r$. We immediately have the following proposition.

Proposition 3.9. *$V^{-1}TV$ has entries in $\mathcal{O}$. Consequently, $V^{-1}T^jV$ has entries in $\mathcal{O}$ for every $0 \leq j \leq 2r$. In particular, their first columns are vectors in $\mathcal{O}^{r+1}$.*

Proof. Let $x_k = \theta_k$ in the discussions above. Then we have the corresponding polynomial $h(x) = (x-1)(x-\theta_1)\cdots(x-\theta_r) = x^{r+1} + b_1x^r + \cdots + b_{r+1}$ with its companion matrix

$$C_h = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -b_{r+1} & -b_r & -b_{r-1} & -b_{r-2} & \cdots & -b_1 \end{bmatrix}.$$

Since V^t diagonalizes C_h,

$$(V^t)^{-1}C_h(V^t) = T.$$

Taking the transpose of both sides, we have

$$V(C_h)^t(V^{-1}) = T.$$

Note that T is diagonal, then

$$V^{-1}TV = (C_h)^t.$$

As the $\mathcal{O}$ is a ring, $b_k \in \mathcal{O}$ for all k. Hence all entries in C_h are in $\mathcal{O}$, so is its transpose therefore $V^{-1}TV$. The rest of the corollary follows immediately. □

Note that the first columns of T^jV correspond exactly to the vectors in Claim 3.8, we can conclude that Claim 3.8 is true. As a result, Claim 3.2, therefore Theorem 1.1, is true.

4. Weil Representation Over Finite Fields

In Sec. 3 of [11], the genus one representation of $\mathrm{SL}(2,\mathbb{Z})$ associated to the SO(3)-TQFT for a fixed odd prime $p \geq 5$ (in the sense of [1]) was considered, where the authors identified the representation with the odd part of the Weil representation of $\mathrm{SL}(2,\mathbb{Z}/p\mathbb{Z})$ (see also [3]). Here we will prove a result in some sense "dual" to that in [11]. Namely, for the fixed prime p, a factor of ρ_1 factors through the even part of the Weil representation of $\mathrm{SL}(2,\mathbb{Z}/p\mathbb{Z})$.

To clarify the above paragraph, let us briefly recall the definition of the Weil representation over finite fields. The basic idea is to realize elements in $\mathrm{SL}(2,\mathbb{Z}/p\mathbb{Z})$ as intertwining operators of the Heisenberg representation of the Heisenberg group, which will be defined below. There is a vast amount of research on the Weil representations, and we will only extract some essential ingredients of the representation of $\mathrm{SL}(2,\mathbb{Z}/p\mathbb{Z})$ here. The interested readers are referred to [5].

Fix an odd prime $p \geq 5$. We start by looking at a group called the Heisenberg group $\mathcal{H}_p$, defined by

$$\mathcal{H}_p = \left\{ \begin{bmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix},\ x,\ y,\ z \in \mathbb{Z}/p\mathbb{Z} \right\}. \tag{4.1}$$

Here the group multiplication is the matrix multiplication. Considering the embedding

$$\mathbb{Z}/p\mathbb{Z} \to \mathcal{H}_p, \quad z \mapsto \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we can view the group $\mathcal{H}_p$ as a central extension of $(\mathbb{Z}/p\mathbb{Z})^2$ by $\mathbb{Z}/p\mathbb{Z}$. More precisely, we have a short exact sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathcal{H}_p \to (\mathbb{Z}/p\mathbb{Z})^2 \to 0.$$

The quotient map is given by

$$\mathcal{H}_p \to (\mathbb{Z}/p\mathbb{Z})^2, \quad \begin{bmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix}.$$

With a suitable choice of section to the quotient map above, it is not difficult to show that the defining action of $\mathrm{SL}(2,\mathbb{Z}/p\mathbb{Z})$ on $(\mathbb{Z}/p\mathbb{Z})^2$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}, \quad \text{where } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z}/p\mathbb{Z}), \begin{bmatrix} x \\ y \end{bmatrix} \in (\mathbb{Z}/p\mathbb{Z})^2,$$

can be lifted to $\mathcal{H}_p$, and the lifted action is trivial on the center $Z(\mathcal{H}_p)$ of $\mathcal{H}_p$.

Let $\mathcal{L}^2(\mathbb{Z}/p\mathbb{Z})$ denote the space of complex-valued functions on $\mathbb{Z}/p\mathbb{Z}$. It is easily seen that $\dim(\mathcal{L}^2(\mathbb{Z}/p\mathbb{Z})) = p$. Given any irreducible central character $\varphi : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}^\times$, we can define a representation $\pi_\varphi : \mathcal{H}_p \to \mathrm{GL}(\mathcal{L}^2(\mathbb{Z}/p\mathbb{Z}))$ by

$$\left(\pi_\varphi\left(\begin{bmatrix}1 & y & z\\ 0 & 1 & x\\ 0 & 0 & 1\end{bmatrix}\right)(f)\right)(a) = \varphi(-xa+z)f(a-y), \tag{4.2}$$

for any $\begin{bmatrix}1 & y & z\\ 0 & 1 & x\\ 0 & 0 & 1\end{bmatrix} \in \mathcal{H}_p$ and $f \in \mathcal{L}^2(\mathbb{Z}/p\mathbb{Z})$.

Since π_φ is p-dimensional, by the representation theory of finite groups, it is either a direct sum of p 1-dimensional representations or irreducible. However, in the first case, $\pi_\varphi|_{Z(\mathcal{H}_p)}$ should be trivial, which contradicts to our assumption on φ.

By Theorem 3.1 of [22], if two irreducible representations of $\mathcal{H}_p$ coincide on the center $Z(\mathcal{H}_p)$, then they are equivalent. Now let φ be any nontrivial irreducible central character. For any $\alpha \in \mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$, consider the representation $\pi_\varphi \circ \alpha$, a p-dimensional representation of $\mathcal{H}_p$ with the property

$$(\pi_\varphi \circ \alpha)|_{Z(\mathcal{H}_p)} = \varphi = \pi_\varphi|_{Z(\mathcal{H}_p)}.$$

By a similar argument as above, we know that $\pi_\varphi \circ \alpha$ is also irreducible. Hence $\pi_\varphi \circ \alpha$ is equivalent to π_φ, i.e. there is an intertwining operator (unique up to scalar by Schur's lemma), denoted by $W_\varphi(\alpha) \in \mathrm{GL}(\mathcal{L}^2(\mathbb{Z}/p\mathbb{Z}))$ such that the diagram

$$\begin{array}{ccc}
\mathcal{L}^2(\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\pi_\varphi(h)} & \mathcal{L}^2(\mathbb{Z}/p\mathbb{Z}) \\
\Big\downarrow{\scriptstyle W_\varphi(\alpha)} & & \Big\downarrow{\scriptstyle W_\varphi(\alpha)} \\
\mathcal{L}^2(\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\pi_\varphi(\alpha(h))} & \mathcal{L}^2(\mathbb{Z}/p\mathbb{Z}),
\end{array}$$

commutes for all $h \in \mathcal{H}_p$.

If we consider the class of $W_\varphi(\alpha)$ in the group $\mathrm{PGL}(\mathcal{L}^2(\mathbb{Z}/p\mathbb{Z}))$ instead of $W_\varphi(\alpha) \in \mathrm{GL}(\mathcal{L}^2(\mathbb{Z}/p\mathbb{Z}))$, we can eliminate the scaling ambiguity and get a well-defined projective representation of $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$ (by abuse of notation this map is also denoted by W_φ):

$$W_\varphi : \mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{PGL}(\mathcal{L}^2(\mathbb{Z}/p\mathbb{Z})). \tag{4.3}$$

We call this projective representation the Weil representation of $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$ (with respect to φ).

Remark. We may omit the word "projective" when it does not cause confusions, and we will present an element in $\mathrm{PGL}(\mathcal{L}^2(\mathbb{Z}/p\mathbb{Z}))$ by one of its representatives in $\mathrm{GL}(\mathcal{L}^2(\mathbb{Z}/p\mathbb{Z}))$.

Again, let $\sigma = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and $\tau = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ be the generators of $\mathrm{SL}(2,\mathbb{Z})$. Their reductions mod p generate $\mathrm{SL}(2,\mathbb{Z}/p\mathbb{Z})$. By an abuse of notation, we will not distinguish σ and τ from their reductions. For $j \in \mathbb{Z}/p\mathbb{Z}$, let $f_j : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ be the jth Kronecker delta function defined by

$$f_j(x) = \delta_{j,x}, \quad \forall\, x \in \mathbb{Z}/p\mathbb{Z}. \tag{4.4}$$

The set $\{f_j \mid j \in \mathbb{Z}/p\mathbb{Z}\}$ is a basis of $\mathcal{L}^2(\mathbb{Z}/p\mathbb{Z})$, which we fix in the following.

Now, to describe the Weil representation with respect to a nontrivial character φ, it suffices to give the matrices of $W_\varphi(\sigma)$ and $W_\varphi(\tau)$ under the fixed basis defined above. According to [2, Sec. 3.2] (see also [18]),

$$W_\varphi(\sigma) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \varphi(1) & \varphi(2) & \cdots & \varphi(p-1) \\ 1 & \varphi(2) & \varphi(4) & \cdots & \varphi(2(p-1)) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \varphi(p-1) & \varphi(2(p-1)) & \cdots & \varphi((p-1)^2) \end{bmatrix} \tag{4.5}$$

and that

$$W_\varphi(\tau) = \begin{bmatrix} 1 & & & & & \\ & \varphi\left(-\frac{1^2}{2}\right) & & & & \\ & & \varphi\left(-\frac{2^2}{2}\right) & & & \\ & & & \varphi\left(-\frac{3^2}{2}\right) & & \\ & & & & \ddots & \\ & & & & & \varphi\left(-\frac{(p-1)^2}{2}\right) \end{bmatrix}. \tag{4.6}$$

As before, $\frac{1}{2}$ is understood as the multiplicative reciprocal of 2 in $\mathbb{Z}/p\mathbb{Z}$.

Note that this representation is reducible. Indeed, it is easy to see that the $\mathbb{C}$-span of $\{f_k + f_{p-k} \mid k = 0, 1, \ldots, r\}$ and $\{f_k - f_{p-k} \mid k = 0, 1, \ldots, r\}$ are two invariant subspaces. If we denote the former vector space by E^{even} and the latter by E^{odd}, then we have a decomposition of representation spaces $\mathcal{L}^2(\mathbb{Z}/p\mathbb{Z}) \cong E^{\mathrm{even}} \oplus E^{\mathrm{odd}}$.

We are mainly interested in the restriction of the Weil representation on the even subspace E^{even}. By (4.5) and (4.6), we have

$$W_\varphi^{\mathrm{even}}(\sigma) = W_\varphi|_{E^{\mathrm{even}}}(\sigma) = \begin{bmatrix} 1 & a^t \\ 2 \cdot a & B \end{bmatrix}. \tag{4.7}$$

Here B is an $r \times r$-matrix with entries given by

$$B_{jk} = \varphi(jk) + \varphi(-jk), \quad \forall\, j,k = 1,\ldots,r. \tag{4.8}$$

In addition, we have

$$W_\varphi^{\text{even}}(\tau) = W_\varphi|_{E^{\text{even}}}(\tau) = \begin{bmatrix} 1 & & & & \\ & \varphi\left(-\frac{1^2}{2}\right) & & & \\ & & \varphi\left(-\frac{2^2}{2}\right) & & \\ & & & \ddots & \\ & & & & \varphi\left(-\frac{r^2}{2}\right) \end{bmatrix}. \tag{4.9}$$

If we choose the special character $\varphi : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}^\times$ defined by

$$\varphi(j) = \zeta^j, \tag{4.10}$$

we will have $\sqrt{p}A = B$ and

$$2r \equiv -1 \mod p \Rightarrow r \equiv -\frac{1}{2} \mod p \Rightarrow \varphi\left(-\frac{j^2}{2}\right) = \zeta^{rj^2} = \theta_j.$$

Therefore,

$$W_\varphi^{\text{even}}(\sigma) = \sqrt{p}\begin{bmatrix} \frac{1}{\sqrt{p}} & \frac{1}{\sqrt{p}} \cdot a^t \\ \frac{2}{\sqrt{p}} \cdot a & A \end{bmatrix}, \tag{4.11}$$

and

$$W_\varphi^{\text{even}}(\tau) = \begin{bmatrix} 1 & & & \\ & \theta_1 & & \\ & & \ddots & \\ & & & \theta_r \end{bmatrix}. \tag{4.12}$$

Recall from the previous sections that $H \cong H_1 \oplus H_2$ and that H_2 is an $(r+1)$-dimensional vector space. We can then identify H_2 and E^{even} via

$$\mathbf{1} + \mathbf{Z} \leftrightarrow 2f_0, \quad \mathbf{Y}_j \leftrightarrow f_j + f_{-j}, \quad \forall\, j = 1,\ldots,r. \tag{4.13}$$

With all the ingredients ready, we now state the theorem of this section:

Theorem 4.1. *Let φ be choosen as in* (4.10)*, then the restriction of the* $\mathrm{SO}(p)_2$*-TQFT representation of* $\mathrm{SL}(2,\mathbb{Z})$ *to* H_2*,* $\rho_1|_{H_2}$*, factors through* W_φ^{even}*, the even part of the Weil representation of* $\mathrm{SL}(2,\mathbb{Z}/p\mathbb{Z})$ *associated to* φ*. In other words, we have*

the following commutative diagram:

$$\begin{array}{ccc} \mathrm{SL}(2,\mathbb{Z}) & \xrightarrow{\rho_1|_{H_2}} & \mathrm{PGL}(H_2) \\ \Big\downarrow{\scriptstyle \text{mod } p} & & \Big\downarrow{\scriptstyle \cong} \\ \mathrm{SL}(2,\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{W_\varphi^{\text{even}}} & \mathrm{PGL}(E^{\text{even}}) \end{array} .$$

Proof. By (3.4), (3.5), (4.11) and (4.12), we know that $\rho_1|_{H_2}$ and W_φ^{even} are only different by a scalar multiple, hence as projective representations, they are the same. □

Remark. Combining the above theorem with [6, Proposition 5.4], we are able to obtain the integrality of $\rho_1|_{H_2}$, hence the integrality of ρ_1 (cf. Lemma 3.1). However, an explicit-integral basis (the one given in Sec. 3) cannot be produced by the mentioned results.

We immediately have the following corollaries.

Corollary 4.2. *The image of* $\rho_1|_{H_2}$ *is finite.*

By (3.3) and (3.2), it is easy to see that the image of $\rho_1|_{H_1}$ can be viewed as a subgroup of the permutation group of the finite set $\{\pm 1, \pm i\} \times \{\mathbf{1} - \mathbf{Z}, \mathbf{X} \pm \mathbf{X}'\}$, so $\rho_1|_{H_1}(\mathrm{SL}(2,\mathbb{Z}))$ is also finite. Hence, together with the above corollary, we have:

Corollary 4.3. *The image of* ρ_1 *is finite.*

Remark. The above corollary is a special case of the famous finiteness result in [19].

Acknowledgments

The author would like to thank his advisor Professor Thomas Kerler for his guidance and many stimulating discussions. The author would like to thank Professor James W. Cogdell for helping the author understand the Weil representation. The author is also grateful to Professor Patrick Gilmer, Professor Eric Rowell and Professor Zhenghan Wang for discussions and their advice. Finally, the author would like to thank the referees for many helpful suggestions.

References

[1] C. Blanchet, N. Habegger, G. Masbaum and P. Vogel, Topological quantum field theories derived from the Kauffman bracket, *Topol.* **34**(4) (1995) 883–927.
[2] C. Chan, The Weil representation, *Senior Honors Thesis, Stanford University* (2012).
[3] M. Freedman and V. Krushkal, On the asymptotics of quantum SU(2) representations of mapping class groups, *Forum Math.* **18**(2) (2006) 293–304.
[4] C. Frohman and J. Kania-Bartoszyńska, A quantum obstruction to embedding, *Math. Proc. Cambridge Philos. Soc.* **131**(2) (2001) 279–293.

[5] P. Gérardin, Weil representations associated to finite fields, *J. Algebr.* **46**(1) (1977) 54–101.
[6] P. M. Gilmer, Congruence and quantum invariants of 3-manifolds, *Algebr. Geom. Topol.* **7** (2007) 1767–1790.
[7] P. M. Gilmer and G. Masbaum, Integral lattices in TQFT, *Ann. Sci. École Norm. Sup.* (*4*) **40**(5) (2007) 815–844.
[8] P. M. Gilmer, G. Masbaum and P. van Wamelen, Integral bases for TQFT modules and unimodular representations of mapping class groups, *Comment. Math. Helv.* **79**(2) (2004) 260–284.
[9] M. B. Hastings, C. Nayak and Z. Wang, On metaplectic modular categories and their applications, *Comm. Math. Phys.* **330**(1) (2014) 45–68.
[10] T. Kerler, p-modular TQFT's, Milnor torsion and the Casson-Lescop invariant, in *Invariants of Knots and 3-Manifolds* (*Kyoto, 2001*), Vol. 4 of *Geom. Topol. Monogr.*, (Geometry and Topology Publications, Coventry, 2002), p. 119–141.
[11] M. Larsen and Z. Wang, Density of the SO(3) TQFT representation of mapping class groups, *Comm. Math. Phys.* **260**(3) (2005) 641–658.
[12] T. T. Q. Le, Quantum invariants of 3-manifolds: Integrality, splitting, and perturbative expansion, in *Proc. Pacific Institute for the Mathematical Sciences Workshop "Invariants of Three-Manifolds"* (*Calgary, AB, 1999*), Vol. 127 (2003), p. 125–152.
[13] G. Masbaum and J. D. Roberts, A simple proof of integrality of quantum invariants at prime roots of unity, *Math. Proc. Cambridge Philos. Soc.* **121**(3) (1997) 443–454.
[14] G. Masbaum and H. Wenzl, Integral modular categories and integrality of quantum invariants at roots of unity of prime order, *J. Reine Angew. Math.* **505** (1998) 209–235.
[15] H. Murakami, Quantum SU(2)-invariants dominate Casson's SU(2)-invariant, *Math. Proc. Cambridge Philos. Soc.* **115**(2) (1994) 253–281.
[16] H. Murakami, Quantum SO(3)-invariants dominate the SU(2)-invariant of Casson and Walker, *Math. Proc. Cambridge Philos. Soc.* **117**(2) (1995) 237–249.
[17] D. Naidu and E. C. Rowell, A finiteness property for braided fusion categories, *Algebr. Represent. Theory* **14**(5) (2011) 837–855.
[18] M. Neuhauser, An explicit construction of the metaplectic representation over a finite field, *J. Lie Theory* **12**(1) (2002) 15–30.
[19] S.-H. Ng and P. Schauenburg, Congruence subgroups and generalized Frobenius–Schur indicators, *Comm. Math. Phys.* **300**(1) (2010) 1–46.
[20] T. Ohtsuki, A polynomial invariant of integral homology 3-spheres, *Math. Proc. Cambridge Philos. Soc.* **117**(1) (1995) 83–112.
[21] T. Ohtsuki, A polynomial invariant of rational homology 3-spheres, *Invent. Math.* **123**(2) (1996) 241–257.
[22] A. Prasad, On character values and decomposition of the Weil representation associated to a finite abelian group, *J. Anal.* **17** (2009) 73–85.
[23] N. Reshetikhin and V. G. Turaev, Invariants of 3-manifolds via link polynomials and quantum groups, *Invent. Math.* **103**(3) (1991) 547–597.
[24] T. Takata and Y. Yokota, The PSU(N) invariants of 3-manifolds are algebraic integers, *J. Knot Theory Ramifications* **8**(4) (1999) 521–532.