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Classification of Spherical Fusion Categories of Frobenius–Schur Exponent 2

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Abstract. In this paper, we propose a new approach towards the classification of spherical fusion categories by their Frobenius–Schur exponents. We classify spherical fusion categories of Frobenius–Schur exponent 2 up to monoidal equivalence. We also classify modular categories of Frobenius–Schur exponent 2 up to braided monoidal equivalence. It turns out that the Gauss sum is a complete invariant for modular categories of Frobenius–Schur exponent 2. This result can be viewed as a categorical analog of Arf's theorem on the classification of non-degenerate quadratic forms over fields of characteristic 2.

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1 Introduction

Let $\mathcal{C}$ be a spherical fusion category over $\mathbb{C}$. The higher Frobenius–Schur indicators $\nu_n(V)$ of $V \in \mathrm{Ob}(\mathcal{C})$ and $n \in \mathbb{Z}$ are generalizations of the classical Frobenius–Schur indicator for irreducible finite group representations (see [16] and the references therein). The Frobenius–Schur indicators are important invariants of a spherical fusion category, especially when the category is in addition non-degenerately braided (in other words, modular). For example, the congruence subgroup conjecture on the $\mathrm{SL}(2,\mathbb{Z})$ representations arising from modular categories can be resolved using generalized Frobenius–Schur indicators [18].

The Frobenius–Schur exponent of a spherical fusion category $\mathcal{C}$, denoted by $\mathrm{FSexp}(\mathcal{C})$, is the smallest positive integer n such that $\nu_n(V) = \dim_{\mathcal{C}}(V)$ for any object $V \in \mathrm{Ob}(\mathcal{C})$, where $\dim_{\mathcal{C}}(V)$ is the categorical dimension of V in $\mathcal{C}$. It is shown in [16] that $\mathrm{FSexp}(\mathcal{C})$ is equal to the order of the T-matrix of $Z(\mathcal{C})$, the Drinfeld center of $\mathcal{C}$. Moreover, the Cauchy theorem for spherical fusion categories asserts that the prime ideals dividing $\mathrm{FSexp}(\mathcal{C})$ and those dividing the global dimension $\dim(\mathcal{C})$ are the same in the ring of algebraic integers [3]. It is then reasonable to pursue a classification of spherical fusion categories by their Frobenius–Schur exponents, as opposed to the usual method of classification by rank [4, 19, 20].

In this paper we give a full classification of the spherical fusion categories of Frobenius–Schur exponent 2. We show that such a spherical fusion category $\mathcal{C}$ is equivalent, as a fusion category, to $\mathrm{Rep}(\mathbb{Z}_2^n)$ for some positive integer n. In particular, the associativity constraints of $\mathcal{C}$ are all identities. We also show that if $\mathcal{C}$ is in addition modular, then $\mathcal{C}$ can be decomposed into a Deligne tensor product of two types of modular categories called $\mathcal{C}(\mathbb{Z}_2^2, q_1)$ and $\mathcal{C}(\mathbb{Z}_2^2, q_2)$. It is worth mentioning that in [5, Theorem 3.2] the authors showed that any modular category of Frobenius–Schur exponent 2 is a braided fusion subcategory of $\mathrm{Rep}(D^{\omega}(\mathbb{Z}_2^{2n}))$ for some positive integer n. Here we completely classify these modular categories by a categorical analog of Arf's theorem on the classification of non-degenerate quadratic forms over fields of characteristic 2. It turns out, in this case, that the positive Gauss sum is a complete invariant.

The paper is structured as follows. In Section 2, we give a quick review of basic concepts and set up notations for future use. We also discuss the braided monoidal structure on the category of G-graded vector spaces for a finite abelian group G. In Section 3, we classify spherical fusion categories of Frobenius–Schur exponent 2. Finally, in Section 4, we classify modular categories of Frobenius–Schur exponent 2.

2 Preliminaries

• **Basic concepts and notations.** Let $\mathcal{C}$ be a *fusion category* over $\mathbb{C}$ [8]. In particular, $\mathcal{C}$ is rigid monoidal, $\mathbb{C}$-linear, and semisimple with finitely many isomorphism classes of simple objects such that the tensor unit $\mathbb{1} \in \mathrm{Ob}(\mathcal{C})$ is simple. We fix a choice of representatives from the isomorphism class of simple objects and denote the set of all such representatives by $\Pi_{\mathcal{C}}$. The *Frobenius–Perron dimension* of $V \in \mathrm{Ob}(\mathcal{C})$, denoted by $\mathrm{FPdim}_{\mathcal{C}}(V)$, is the largest non-negative eigenvalue of the fusion matrix of V. In addition, we define the Frobenius–Perron dimension of $\mathcal{C}$ by $\mathrm{FPdim}(\mathcal{C}) = \sum_{V \in \Pi_{\mathcal{C}}} \mathrm{FPdim}_{\mathcal{C}}(V)^2$.

A fusion category $\mathcal{C}$ is called *spherical* if it has a pivotal structure such that the left and right pivotal traces coincide on all endomorphisms. In this case, the left (or right) pivotal trace of id_V, the identity of $V \in \mathrm{Ob}(\mathcal{C})$, is called the *categorical dimension* of V. We denote the categorical dimension of $V \in \mathrm{Ob}(\mathcal{C})$ by $\dim_{\mathcal{C}}(V)$, and we define the *global dimension* $\mathcal{C}$ by $\dim(\mathcal{C}) = \sum_{V \in \Pi_{\mathcal{C}}} \dim_{\mathcal{C}}(V)^2$.

A spherical fusion category admitting a braiding is called a braided spherical fusion category (or premodular category). A braided spherical fusion category is called *modular* if the braiding is non-degenerate, or equivalently, if its S-matrix is non-degenerate [15]. For example, $Z(\mathcal{C})$, the Drinfeld center of a spherical fusion category $\mathcal{C}$, is modular [15]. Objects of $Z(\mathcal{C})$ are pairs (X, σ_X), where $X \in \mathrm{Ob}(\mathcal{C})$ and $\sigma_X \colon X \otimes - \xrightarrow{\sim} - \otimes X$ is a half braiding. Since the pivotal structure of $Z(\mathcal{C})$ is inherited from $\mathcal{C}$, we have

$$\dim_{Z(\mathcal{C})}(V, \sigma_V) = \dim_{\mathcal{C}}(V) \tag{2.1}$$

for any $V \in \mathrm{Ob}(\mathcal{C})$.

Let $\mathcal{C}$ be a spherical fusion category. For any $n \in \mathbb{Z}$ and for any $V \in \mathrm{Ob}(\mathcal{C})$, the *$n$th Frobenius–Schur indicator* ν_n of V is defined to be the operator trace of a linear operator $E_V^{(n)} \colon \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, V^{\otimes n}) \to \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, V^{\otimes n})$ satisfying $(E_V^{(n)})^n = \mathrm{id}$. Here, $V^{\otimes n}$ is understood as inductively defined by $V^{\otimes(m+1)} = V \otimes V^{\otimes m}$ for $1 \leq m < n$, and associativity constraints are included in the definition of $E_V^{(n)}$ (see [17]). In particular, if V is simple, then

$$\nu_1(V) = \delta_{\mathbb{1},V}. \tag{2.2}$$

We also have

$$\nu_2(V) = 0 \ \text{ if } \ V \ncong V^*, \quad \nu_2(V) = 1 \text{ or } -1 \ \text{ if } \ V \cong V^* \tag{2.3}$$

for all $V \in \Pi_{\mathcal{C}}$.

The *Frobenius–Schur exponent of an object* V in a spherical fusion category $\mathcal{C}$, denoted by $\mathrm{FSexp}(V)$, is defined to be the smallest positive integer n such that $\nu_n(V) = \dim_{\mathcal{C}}(V)$. The *Frobenius–Schur exponent of* $\mathcal{C}$, denoted by $\mathrm{FSexp}(\mathcal{C})$, is defined to be the smallest positive integer n such that $\nu_n(V) = \dim_{\mathcal{C}}(V)$ for all $V \in \mathcal{C}$ [16]. When $\mathcal{C}$ is the category of finite-dimensional H-modules for a semisimple Hopf algebra H over $\mathbb{C}$, $\mathrm{FSexp}(V)$ is equal to the exponent of V as a finite-dimensional H-module. In other words, $\mathrm{FSexp}(V)$ is equal to the exponent of the image of G in $\mathrm{GL}(V, \mathbb{C})$ [11].

It is immediate from the definition and equation (2.2) that if $\mathcal{C}$ is a spherical fusion category such that $\mathrm{FSexp}(\mathcal{C}) = 1$, then $\dim_{\mathcal{C}}(V) = \delta_{\mathbb{1},V}$ for any $V \in \Pi_{\mathcal{C}}$. According to [9, Theorem 2.3], $\dim_{\mathcal{C}}(V) \neq 0$ for all $V \in \Pi_{\mathcal{C}}$, and hence $\mathcal{C}$ has the tensor unit $\mathbb{1}$ as its only simple object. Therefore, $\mathcal{C}$ is monoidally equivalent to $\mathrm{Vec}_{\mathbb{C}}$, the category of finite-dimensional vector spaces over $\mathbb{C}$.

It is worth mentioning that by [16], for any $V \in \mathrm{Ob}(\mathcal{C})$, $\mathrm{FSexp}(V)$ does not depend on the choice of pivotal structures. In addition, $\mathrm{FSexp}(\mathcal{C})$ of a spherical fusion category $\mathcal{C}$ depends only on the equivalence class of the modular category $Z(\mathcal{C})$.

• **Braided monoidal structure on G-graded vector spaces.** Let G be a finite multiplicative abelian group. Recall that the category Vec_G^{ω} of finite-dimensional G-graded vector spaces has simple objects $\{V_g \mid g \in G\}$, where $(V_g)_h = \delta_{g,h}\mathbb{C}$ for all $h \in G$. The tensor product is given by $V_g \otimes V_h = V_{gh}$ and the tensor unit is V_1, where 1 is the identity of G. The associator is given by a normalized 3-cocycle $\omega \in Z^3(G, \mathbb{C}^\times)$:

$$\omega(x, y, z) \colon V_x \otimes (V_y \otimes V_z) \longrightarrow (V_x \otimes V_y) \otimes V_z.$$

Now we equip Vec_G^{ω} with a braiding given by a normalized 2-cochain c in $C^2(G, \mathbb{C}^\times)$:

$$c(x, y) \colon V_x \otimes V_y \longrightarrow V_y \otimes V_x$$

satisfying the hexagon axioms

$$\frac{c(xy,z)}{c(x,z)c(y,z)} \cdot \frac{\omega(x,y,z)\omega(z,x,y)}{\omega(x,z,y)} = 1 = \frac{c(x,yz)}{c(x,y)c(x,z)} \cdot \frac{\omega(y,x,z)}{\omega(x,y,z)\omega(y,z,x)} \tag{2.4}$$

for all $x, y, z \in G$. In other words, the pair (ω, c) is an *Eilenberg–MacLane* 3*-cocycle* of G. Finally, we equip Vec_G^{ω} with the canonical (spherical) pivotal structure, which is simply given by identities on objects, so that the categorical dimensions are all positive. We denote this braided spherical fusion category by $\mathrm{Vec}_G^{(\omega,c)}$.

An Eilenberg–MacLane 3-cocycle (ω, c) is called a *coboundary* if there exists a 2-cochain $h \in C^2(G, \mathbb{C}^\times)$ such that $\omega = \delta h$ and $c(x,y) = h(x,y)/h(y,x)$. The *Eilenberg–MacLane cohomology group* $H^3_{ab}(G, \mathbb{C}^\times)$ is then defined by

$$H^3_{ab}(G, \mathbb{C}^\times) = \frac{Z^3_{ab}(G, \mathbb{C}^\times)}{B^3_{ab}(G, \mathbb{C}^\times)},$$

where $Z^3_{ab}(G, \mathbb{C}^\times)$ and $B^3_{ab}(G, \mathbb{C}^\times)$ are respectively the abelian groups of Eilenberg–MacLane 3-cocycles and 3-coboundaries. To $(\omega, c) \in Z^3_{ab}(G, \mathbb{C}^\times)$ one can assign the function $q(x) := c(x,x)$, called its *trace*. It is easy to show that $q(x)$ is a *quadratic form* (or a quadratic function). In other words, we have the following:

(1) $q(x^a) = q(x)^{a^2}$ for any $a \in \mathbb{Z}$, and

(2) $b_q(x,y) := \frac{q(xy)}{q(x)q(y)}$ defines a bicharacter of G.

We will use the pair (G, q) to denote a quadratic form q on the finite abelian group G. When the group G is clear from the context, we will sometimes simply write q. Note that given two quadratic forms (G, q) and (G', q'), we can define a quadratic form on $G \oplus G'$, denoted by $q \oplus q'$, via the formula

$$(q \oplus q')(x, x') = q(x)q'(x')$$

for all $(x, x') \in G \oplus G'$. The quadratic form $(G \oplus G',\ q \oplus q')$ is called the *direct sum* of (G, q) and (G', q').

We recall a theorem of Eilenberg–MacLane (see [6, 7]).

Theorem 2.1. (Eilenberg–MacLane) *The map assigning (ω, c) to its trace induces an isomorphism of groups*

$$H^3_{ab}(G, \mathbb{C}^\times) \xrightarrow{\cong} Q(G, \mathbb{C}^\times),$$

where $Q(G, \mathbb{C}^\times)$ is the abelian group of quadratic forms from G to $\mathbb{C}^\times$.

Now we introduce the following notations before proceeding. Given a group homomorphism $f\colon G \to G'$ and a positive integer n, we use the standard notation for the n-fold product of f:

$$f^n\colon G^n \longrightarrow (G')^n, \quad f^n(g_1, \ldots, g_n) := (f(g_1), \ldots, f(g_n)).$$

For any n-cochain $\mu \in C^n(G', \mathbb{C}^\times)$, we define $f^*(\mu) = \mu \circ f^n$.

Two quadratic forms $q\colon G \to \mathbb{C}^\times$ and $q'\colon G' \to \mathbb{C}^\times$ are *equivalent* if there exists a group isomorphism $f\colon G \to G'$ such that $q = f^*(q')$.

Lemma 2.2. *$\mathrm{Vec}_G^{(\omega,c)}$ and $\mathrm{Vec}_{G'}^{(\omega',c')}$ are equivalent braided monoidal categories if and only if the traces of (ω,c) and (ω',c') are equivalent quadratic forms.*

Proof. If $F\colon \mathrm{Vec}_G^{(\omega,c)} \to \mathrm{Vec}_{G'}^{(\omega',c')}$ is a braided monoidal equivalence with the natural isomorphism $\mu(x,y)\colon F(V_x)\otimes F(V_y) \to F(V_x \otimes V_y)$, then F induces a group isomorphism $f\colon G \to G'$ on simple objects. Moreover, the following diagrams commute:

$$\begin{tikzcd}
(F(V_x)\otimes F(V_y))\otimes F(V_z) \arrow[r, "\mu(x,y)\otimes \mathrm{id}"] & F(V_x\otimes V_y)\otimes F(V_z) \arrow[r, "\mu(xy,z)"] & F((V_x\otimes V_y)\otimes V_z) \\
F(V_x)\otimes (F(V_y)\otimes F(V_z)) \arrow[u, "\omega'(f(x),f(y),f(z))"] \arrow[r, "\mathrm{id}\otimes\mu(y,z)"] & F(V_x)\otimes F(V_y\otimes V_z) \arrow[r, "\mu(x,yz)"] & F(V_x\otimes (V_y\otimes V_z)) \arrow[u, "F(\omega(x,y,z))"']
\end{tikzcd}$$

$$\begin{tikzcd}
F(V_x)\otimes F(V_y) \arrow[r, "c'(f(x),f(y))"] & F(V_y)\otimes F(V_x) \\
F(V_x\otimes V_y) \arrow[u, "\mu(x,y)"'] \arrow[r, "F(c(x,y))"] & F(V_y\otimes V_x) \arrow[u, "\mu(y,x)V"]
\end{tikzcd}$$

Hence, $f^*(\omega') = \omega\cdot\delta\mu$ and $f^*(c')(x,y) = c(x,y)\mu(x,y)/\mu(y,x)$. Therefore, (ω,c) and $(f^*(\omega'), f^*(c'))$ differ by an Eilenberg–MacLane 3-coboundary. By the theorem of Eilenberg–MacLane, $q = f^*(q')$.

Conversely, assume that there exists a group isomorphism $f\colon G\to G'$ such that $q = f^*(q')$. By the theorem of Eilenberg–MacLane, (ω,c) and $(f^*(\omega'), f^*(c'))$ differ by an Eilenberg–MacLane 3-coboundary. In other words, there exists a 2-cochain μ of G such that $f^*(\omega') = \omega\cdot\delta\mu$ and $f^*(c')(x,y) = c(x,y)\mu(x,y)/\mu(y,x)$. Define $F(V_x) = V_{f(x)}$ and $\mu(x,y)\colon F(V_x)\otimes F(V_y)\to F(V_x\otimes V_y)$; then F together with μ extends to a braided monoidal equivalence between $\mathrm{Vec}_G^{(\omega,c)}$ and $\mathrm{Vec}_{G'}^{(\omega',c')}$. □

Remark 2.3. In the light of the Eilenberg–MacLane theorem, we will denote any representative in the braided monoidal equivalence class of some $\mathrm{Vec}_G^{(\omega,c)}$ by $\mathcal{C}(G,q)$, where q is the trace of (ω,c). Therefore, Lemma 2.2 can be rewritten as follows: $\mathcal{C}(G,q)\cong\mathcal{C}(G',q')$ as braided monoidal categories if and only if q and q' are equivalent quadratic forms.

3 Classification of Spherical Fusion Categories of Frobenius–Schur Exponent 2

In this section, we classify spherical fusion categories of Frobenius–Schur exponent 2 up to monoidal equivalence. Let $\mathcal{C}$ be such a category. Then the Frobenius–Schur exponent of $Z(\mathcal{C})$ is also 2 by [16, Corollary 7.8]. Consequently, for any $V\in\mathrm{Ob}(\mathcal{C})$, $\nu_2(V) = \dim_{\mathcal{C}}(V)$. In addition, if V is simple, then $\nu_2(V) = 0, \pm 1$ (cf. equation

(2.3)). By [9, Theorem 2.3], $\dim_{\mathcal{C}}(V) \neq 0$. Hence, we have

$$\dim_{\mathcal{C}}(V) = \nu_2(V) = \pm 1 \tag{3.1}$$

for any $V \in \Pi_{\mathcal{C}}$. By [9, Proposition 8.22],

$$(\mathrm{FPdim}(\mathcal{C}))^2 = \frac{(\dim(\mathcal{C}))^2}{\dim_{Z(\mathcal{C})}((V, \sigma_V))^2}$$

for some $(V, \sigma_V) \in \Pi_{Z(\mathcal{C})}$. Since $(V, \sigma_V) \in \Pi_{Z(\mathcal{C})}$ implies $V \in \Pi_{\mathcal{C}}$ [15], by equations (2.1) and (3.1) we have $(\mathrm{FPdim}(\mathcal{C}))^2 = (\dim(\mathcal{C}))^2$. As both $\mathrm{FPdim}(\mathcal{C})$ and $\dim(\mathcal{C})$ are positive [9, Theorem 2.3], we have $\mathrm{FPdim}(\mathcal{C}) = \dim(\mathcal{C})$. Hence, $\mathcal{C}$ is pseudo-unitary [9]. By [9, Proposition 8.23] there exists a unique spherical pivotal structure on $\mathcal{C}$ such that $\dim_{\mathcal{C}}(V) = \mathrm{FPdim}_{\mathcal{C}}(V) > 0$ for all $V \in \Pi_{\mathcal{C}}$. Since our classification is up to monoidal equivalence, we can assume without loss of generality that $\mathcal{C}$ is equipped with its unique spherical pivotal structure described above.

According to equation (3.1), for any $V \in \Pi_{\mathcal{C}}$, V is self-dual. As a result, we have

$$\dim_{\mathcal{C}}(V \otimes V^*) = \dim_{\mathcal{C}}(V \otimes V) = \dim_{\mathcal{C}}(V)^2 = 1.$$

By rigidity, pseudo-unitarity and the fact that categorical dimension is a character of the fusion ring, we have $V \otimes V \cong \mathbb{1}$. Therefore, $\Pi_{\mathcal{C}}$ is a group of exponent 2, or $\Pi_{\mathcal{C}} = \mathbb{Z}_2^n$ for some positive integer n. As a result, $\mathcal{C} = \mathrm{Vec}_{\mathbb{Z}_2^n}^{\omega}$ for some $\omega \in H^3(\mathbb{Z}_2^n, \mathbb{C}^\times)$. By [16, Theorem 9.2], for any finite group G we have

$$\mathrm{FSexp}(\mathrm{Vec}_G^{\omega}) = \mathrm{lcm}_{g \in G}\, \mathrm{ord}(\omega|_{\langle g \rangle})\, \mathrm{ord}(g),$$

where $\omega|_{\langle g \rangle}$ denotes the restriction of ω on the subgroup generated by g. Since $\mathrm{FSexp}(\mathcal{C}) = 2$, we see that $\omega|_{\langle x \rangle}$ is trivial for all $x \in \mathbb{Z}_2^n$.

For any $n \in \mathbb{Z}$, consider the map

$$b\colon H^3(\mathbb{Z}_2^n, \mathbb{C}^\times) \longrightarrow \{\pm 1\}^{2^n - 1}, \quad \lambda \longmapsto (\dots, \lambda_C, \dots),$$

where C ranges over the subgroups of $\mathbb{Z}_2^n$ of order 2, and

$$\lambda_C = \begin{cases} 1 & \text{if the restriction of } \lambda \text{ on } C \text{ is trivial,} \\ -1 & \text{otherwise.} \end{cases}$$

By [13, Proposition 2.2], b is injective. Therefore, $\omega|_{\langle x \rangle}$ being trivial for all $x \in \mathbb{Z}_2^n$ implies that ω itself is cohomologous to the trivial 3-cocycle. Let $[\omega]$ be the cohomology class of ω in $H^3(G, \mathbb{C}^\times)$; we see that $[\omega] = 1$, and $\mathrm{Vec}_{\mathbb{Z}_2^n}^{\omega}$ is monoidally equivalent to $\mathrm{Vec}_{\mathbb{Z}_2^n}^{1}$ by a standard argument. Note that the more familiar category of finite-dimensional representations of $\mathbb{Z}_2^n$, denoted by $\mathrm{Rep}(\mathbb{Z}_2^n)$, is nothing but an incarnation of $\mathrm{Vec}_{\mathbb{Z}_2^n}^{1}$ as a fusion category.

We summarize the above discussion in the following theorem.

Theorem 3.1. *If $\mathcal{C}$ is a spherical fusion category of Frobenius–Schur exponent 2, then $\mathcal{C}$ is pseudo-unitary. Moreover, $\mathcal{C}$ is monoidally equivalent to $\mathrm{Rep}(\mathbb{Z}_2^n)$ for some positive integer n.*

Remark 3.2. We can also obtain this result by the explicit formula of the normalized 3-cocycle (see [10]):

$$\omega(x,y,z) = \prod_{r=1}^{n} (-1)^{a_r i_r [\frac{j_r+k_r}{2}]} \prod_{1\le r<s\le n} (-1)^{a_{rs} k_r [\frac{i_s+j_s}{2}]} \prod_{1\le r<s<t\le n} (-1)^{a_{rst} k_r j_s i_t},$$

where $x = (i_1,\dots,i_n)$, $y = (j_1,\dots,j_n)$, $z = (k_1,\dots,k_n)$, $i_r, j_r, k_r, a_r, a_{rs}, a_{rst} \in \{0,1\}$;

$$\omega(x,x,x) = \prod_{r=1}^{n} (-1)^{a_r i_r^2} \prod_{1\le r<s\le n} (-1)^{a_{rs} i_r i_s} \prod_{1\le r<s<t\le n} (-1)^{a_{rst} i_r i_s i_t} = 1.$$

Take $x = (0,\dots,0,1,0,\dots,0)$, where 1 is at the rth position; we get $a_r = 0$ for $1 \le r \le n$. Take $x = (0,\dots,0,1,0,\dots,0,1,0,\dots,0)$, where the first 1 is at the rth position and the second 1 is at the sth position; we get $a_{rs} = 0$ for $1 \le r < s \le n$. Take $x = (0,\dots,0,1,0,\dots,0,1,0,\dots,0,1,0,\dots,0)$, where the first 1 is at the rth position, the second 1 is at the sth position, and the third 1 is at the tth position; we get $a_{rst} = 0$ for $1 \le r < s < t \le n$. Hence, $[\omega] = 1$.

4 Classification of Modular Categories of Frobenius–Schur Exponent 2

In this section, we use the result in the previous section to classify modular categories of Frobenius–Schur exponent 2 up to braided monoidal equivalence. Let $\mathcal{C}$ be such a modular category. By the same argument as in the previous section, $\mathcal{C}$ is pseudo-unitary, and we will equip $\mathcal{C}$ with its canonical spherical pivotal structure such that $\dim_{\mathcal{C}}(V) = \mathrm{FPdim}_{\mathcal{C}}(V) > 0$ for all $V \in \Pi_{\mathcal{C}}$. According to Theorem 3.1, $\mathcal{C}$ is equivalent to $\mathrm{Rep}(\mathbb{Z}_2^n)$ as a fusion category for some n. Consequently, as a braided fusion category, $\mathcal{C} \cong \mathrm{Vec}_{\mathbb{Z}_2^n}^{(\omega,c)}$ for some Eilenberg–MacLane 3-cocycle (ω, c). By the same argument as in the previous section, $[\omega] = 1$.

Therefore, $\mathcal{C} \cong \mathrm{Vec}_{\mathbb{Z}_2^n}^{(1,c)}$ with $(1,c)$ an Eilenberg–MacLane 3-cocycle. By equation (2.4), we have $c(1,x) = c(x,1) = 1$, and $q(x)^2 = c(x,x)^2 = 1$ for all $x \in \mathbb{Z}_2^n$. In particular, q takes value in $\{\pm 1\}$. Therefore, by definition (cf. Section 2), the bilinear form associated to q is given by

$$b_q \colon \mathbb{Z}_2^n \oplus \mathbb{Z}_2^n \longrightarrow \{\pm 1\}, \quad b_q(x,y) = \frac{q(xy)}{q(x)q(y)} = c(x,y)c(y,x)$$

for any $(x,y) \in \mathbb{Z}_2^n \oplus \mathbb{Z}_2^n$. Moreover, since $b_q(x,y)$ is the entry of the S-matrix of $\mathcal{C}$ [8], the modularity of $\mathcal{C}$ implies that q is a non-degenerate quadratic form. Hence, b_q is a non-degenerate alternating form (in particular, $b_q(x,x) = 1$ for any $x \in \mathbb{Z}_2^n$). Therefore, $n = 2m$ is even. Moreover, there exists a symplectic basis $\{e_1,\dots,e_m,f_1,\dots,f_m\}$ of $\mathbb{Z}_2^{2m}$, with respect to which $b_q(e_j,e_k) = b_q(f_j,f_k) = 1$, and $b_q(e_j,f_k) = (-1)^{\delta_{j,k}}$ for any $j,k = 1,\dots,m$.

For any non-degenerate quadratic form $q \colon \mathbb{Z}_2^{2m} \to \{\pm 1\}$, we define its additive version $Q \colon \mathbb{Z}_2^{2m} \to \mathbb{Z}_2$ such that $(-1)^{Q(x)} = q(x)$ for any $x \in \mathbb{Z}_2^{2m}$. Then the *Arf invariant* of q, denoted by $\mathrm{Arf}(q)$, is given by the classical Arf invariant of Q. More precisely, we have

$$\mathrm{Arf}(q) := \mathrm{Arf}(Q) = \sum_{j=1}^{m} Q(e_j)Q(f_j),$$

where $\{e_1, \ldots, e_m, f_1, \ldots, f_m\}$ is the symplectic basis given above. Note that the Arf invariant takes value in $\mathbb{Z}_2$, where we use the standard notation $\mathbb{Z}_2 = \{0, 1\}$. We also view $\mathbb{Z}_2$ as a field here.

Arf showed in [1] that the Arf invariant is independent of the choice of basis, and is additive with respect to the direct sum of quadratic forms. More importantly, Arf showed that the dimension $2m$ (of $\mathbb{Z}_2^{2m}$ as a vector space over $\mathbb{Z}_2$) and the Arf invariant $\mathrm{Arf}(q)$ completely determine the equivalence class of a non-degenerate quadratic form $(\mathbb{Z}_2^{2m}, q)$ over $\mathbb{Z}_2$. The readers, especially those who are not fluent in German, are highly recommended to consult [14, Appendix 1] for a beautiful exposition of Arf invariant.

As a consequence of Arf's theorem, for any positive integer m, there are only two equivalence classes of non-degenerate quadratic forms on $\mathbb{Z}_2^{2m}$, and they can be obtained as direct sums from two inequivalent quadratic forms on $\mathbb{Z}_2^2$. We give explicit representatives for the two equivalence classes of non-degenerate quadratic forms on $\mathbb{Z}_2^2$ as follows:

$$q_1 \colon \mathbb{Z}_2^2 \longrightarrow \{\pm 1\}, \quad q_1(x, y) = (-1)^{xy} \tag{4.1}$$

and

$$q_2 \colon \mathbb{Z}_2^2 \longrightarrow \{\pm 1\}, \quad q_2(x, y) = (-1)^{x^2+xy+y^2} \tag{4.2}$$

for any $x, y \in \mathbb{Z}_2$. In other words, we have $Q_1(x, y) = xy$, $Q_2(x, y) = x^2 + xy + y^2$. Therefore, any quadratic form $(\mathbb{Z}_2^{2m}, q)$ is equivalent to $q_1^a \oplus q_2^{m-a}$ for some $a \geq 0$. The presentation of q may not be unique, but they are all equivalent to the representatives given as follows.

By direct computation, we obtain $\mathrm{Arf}(q_1) = 0$ and $\mathrm{Arf}(q_2) = 1$. Therefore, $\mathrm{Arf}(q_1 \oplus q_1) = \mathrm{Arf}(q_2 \oplus q_2) = 0$. Since both $q_1 \oplus q_1$ and $q_2 \oplus q_2$ are quadratic forms on $\mathbb{Z}_2^4$, by Arf's theorem $q_1 \oplus q_1$ is equivalent to $q_2 \oplus q_2$. As a result, if a non-degenerate quadratic form $(\mathbb{Z}_2^{2m}, q)$ is equivalent to $q_1^a \oplus q_2^{m-a}$ for some $a \geq 0$, then its Arf invariant is given by

$$\mathrm{Arf}(q) = \begin{cases} 0 & \text{if } m - a \text{ is even,} \\ 1 & \text{otherwise} \end{cases}$$

by the additivity of the Arf invariant. Now that we can change any summand of the form $q_2 \oplus q_2$ into $q_1 \oplus q_1$ without changing the equivalence class of q, we know that q is equivalent to q_1^m if $\mathrm{Arf}(q) = 0$, and q is equivalent to $q_1^{m-1} \oplus q_2$ if $\mathrm{Arf}(q) = 1$. We will assume in the rest of this article that any non-degenerate quadratic form $(\mathbb{Z}_2^{2m}, q)$ is represented in this way.

Next, we analyze the categorical interpretation of the direct sum of quadratic forms (cf. Section 2). Let (G, q) and (G', q') be two non-degenerate quadratic forms. We consider the Deligne tensor product of the modular categories $\mathcal{C}(G, q)$

and $\mathcal{C}(G', q')$, denoted by $\mathcal{D} = \mathcal{C}(G, q) \boxtimes \mathcal{C}(G', q')$ [8]. By definition, $\mathcal{D}$ is also a modular category, and its fusion rule is given by the multiplication of the abelian group $G \oplus G'$. Therefore $\Pi_{\mathcal{D}} = G \oplus G'$, and hence $\mathcal{D} \cong \mathrm{Vec}_{G\oplus G'}^{(\omega,c)}$ for some Eilenberg–MacLane 3-cocycle (ω, c) of $G \oplus G'$. Let $p(x) = c(x, x)$ be the corresponding trace. In other words, $\mathcal{D} \cong \mathcal{C}(G \oplus G', p)$.

Let $(\omega_1, c_1), (\omega_2, c_2)$ be representatives of the Eilenberg–MacLane 3-cohomology classes corresponding to q and q', respectively. By the definition of Deligne tensor product, the associativity constraints in $\mathcal{D}$ is the tensor product of those in $\mathcal{C}(G, q)$ and $\mathcal{C}(G', q')$. In other words, for any $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in G \oplus G'$, we have

$$\omega((x_1, x_2), (y_1, y_2), (z_1, z_2)) = \omega_1(x_1, y_1, z_1)\omega_2(x_2, y_2, z_2).$$

Similarly, we have the following equality from the definition of the braiding on $\mathcal{D}$:

$$c((x_1, x_2), (y_1, y_2)) = c_1(x_1, y_1)c_2(x_2, y_2).$$

In particular, for any $(x_1, x_2) \in G \oplus G'$, we have

$$p(x_1, x_2) = q(x_1)q'(x_2) = (q \oplus q')(x_1, x_2).$$

Therefore, by Lemma 2.1 we have $\mathcal{D} \cong \mathcal{C}(G \oplus G', p) \cong \mathcal{C}(G \oplus G', q \oplus q')$.

We summarize the above discussion in the following lemma.

Lemma 4.1. *$\mathcal{C}(G \oplus G', q \oplus q') \cong \mathcal{C}(G, q) \boxtimes \mathcal{C}(G', q')$ as modular categories.*

Combining the discussions in this section gives rise to the following classification result.

Theorem 4.2. *If $\mathcal{C}$ is a modular category of Frobenius–Schur exponent 2, then $\mathcal{C}$ is pseudo-unitary, and $\mathcal{C}$ is braided monoidally equivalent to $\mathcal{C}(\mathbb{Z}_2^{2m}, q)$ for a positive integer m and a non-degenerate quadratic form q. Moreover, we have the following Deligne tensor product decomposition:*

$$\mathcal{C} \cong \begin{cases} \mathcal{C}(\mathbb{Z}_2^2, q_1)^{\boxtimes m} & \text{if } \mathrm{Arf}(q) = 0, \\ \mathcal{C}(\mathbb{Z}_2^2, q_1)^{\boxtimes (m-1)} \boxtimes \mathcal{C}(\mathbb{Z}_2^2, q_2) & \text{if } \mathrm{Arf}(q) = 1, \end{cases}$$

where q_1 and q_2 are given in equations (4.1) *and* (4.2).

Remark 4.3. A braiding of $\mathcal{C}(\mathbb{Z}_2^2, q_1)$ can be given by $c_1((x, y), (a, b)) = (-1)^{xb}$, and a braiding of $\mathcal{C}(\mathbb{Z}_2^2, q_2)$ can be given by $c_2((x, y), (a, b)) = (-1)^{xa+yb+ay}$.

We would like to interpret the Arf invariant in the modular category setting. Firstly, note that for any non-degenerate quadratic form $(\mathbb{Z}_2^{2m}, q)$, by direct computation we have

$$(-1)^{\mathrm{Arf}(q)} = \frac{1}{\sqrt{|\mathbb{Z}_2^{2m}|}} \sum_{x \in \mathbb{Z}_2^{2m}} q(x) = \frac{1}{2^m} \sum_{x \in \mathbb{Z}_2^{2m}} q(x)$$

(by Arf's theorem, we only have to check this equality for $(\mathbb{Z}_2^2, q_1)$ and $(\mathbb{Z}_2^2, q_2)$, which is immediate). In the literature, the above quantity is also referred to as the *Gaussian sum* for the quadratic form q on the finite abelian group $\mathbb{Z}_2^{2m}$ (for example, see [21]).

On the category-theoretical side, recall (for example, [8]) that the *positive Gauss sum* of a modular category $\mathcal{C}$ is defined by

$$\tau_+ = \sum_{X \in \Pi_{\mathcal{C}}} \theta_X \dim_{\mathcal{C}}(X)^2,$$

where θ_X is the twist of the simple object X. It is standard (see [2]) that in a modular category $\mathcal{C}$, the global dimension is the square of the complex absolute value of τ_+. In other words, $\dim(\mathcal{C}) = |\tau_+|^2$. The *multiplicative central charge* of $\mathcal{C}$ is defined by

$$\xi = \frac{\tau_+(\mathcal{C})}{\sqrt{\dim(\mathcal{C})}} = \frac{\tau_+}{|\tau_+|}.$$

Note that $\xi(\mathcal{C})$ is well-defined as $\dim(\mathcal{C})$ is a totally positive algebraic integer [8].

In particular, when $\mathcal{C} = \mathcal{C}(\mathbb{Z}_2^{2m}, q)$ for a non-degenerate quadratic form $(\mathbb{Z}_2^{2m}, q)$, we can compute the dimension m and the Arf invariant $\mathrm{Arf}(q)$ of $(\mathbb{Z}_2^{2m}, q)$ by the positive Gauss sum τ_+ as follows. We have $\Pi_{\mathcal{C}} = \mathbb{Z}_2^{2m}$. We also have $\dim_{\mathcal{C}}(x) = 1$ for any $x \in \mathbb{Z}_2^{2m}$, and hence

$$|\tau_+|^2 = \dim(\mathcal{C}) = \sum_{x \in \mathbb{Z}_2^{2m}} \dim_{\mathcal{C}}(x)^2 = |\mathbb{Z}_2^{2m}| = 2^{2m}; \tag{4.3}$$

in particular, $|\tau_+| = 2^m$, or $m = \log_2(|\tau_+|)$. Moreover, since for any $x \in \mathbb{Z}_2^{2m}$, $\theta_x = q(x)$ [8], we have

$$\frac{\tau_+}{2^m} = \frac{\tau_+}{|\tau_+|} = \xi(\mathcal{C}(\mathbb{Z}^{2m}, q)) = \frac{1}{\sqrt{|\mathbb{Z}_2^{2m}|}} \sum_{x \in \mathbb{Z}_2^{2m}} q(x) = (-1)^{\mathrm{Arf}(q)}. \tag{4.4}$$

Hence, $\mathrm{Arf}(q)$ is 0 or 1 depending on whether τ_+ is positive or negative, respectively.

Conversely, by equations (4.3) and (4.4), we have $\tau_+ = (-1)^{\mathrm{Arf}(q)} 2^m$.

The argument above shows that both the dimension and the Arf invariant of the quadratic form $(\mathbb{Z}_2^{2m}, q)$ are completely determined by the positive Gauss sum τ_+ of the modular category $\mathcal{C}(\mathbb{Z}_2^{2m}, q)$ and vice versa.

Recall that by Arf, a non-degenerate quadratic form is completely determined (up to equivalence) by its dimension and its Arf invariant. In the same vein, we restate Theorem 4.2 as a categorical analog of Arf's theorem.

Theorem 4.4. *If $\mathcal{C}$ is a modular category of Frobenius–Schur exponent 2, then $\mathcal{C}$ is pseudo-unitary, and $\mathcal{C}$ is completely determined, up to braided monoidal equivalence, by its positive Gauss sum τ_+. More precisely, we have*

$$\mathcal{C} \cong \begin{cases} \mathcal{C}(\mathbb{Z}_2^2, q_1)^{\boxtimes \log_2(|\tau_+|)} & \text{if } \tau_+ > 0, \\ \mathcal{C}(\mathbb{Z}_2^2, q_1)^{\boxtimes (\log_2(|\tau_+|)-1)} \boxtimes \mathcal{C}(\mathbb{Z}_2^2, q_2) & \text{if } \tau_+ < 0. \end{cases}$$

Finally, we make a remark on the prime factorization of modular categories.

A modular category is *non-trivial* if its rank is larger than 1. A non-trivial modular category is called a *prime modular category* if it is not braided monoidally equivalent to a Deligne tensor product of two non-trivial modular categories.

A direct consequence of Theorem 4.2 is that there are only two (pseudo-unitary) prime modular categories of Frobenius–Schur exponent 2. In view of [3, Lemma 2.4], there are finitely many prime modular categories of Frobenius–Schur exponent 2.

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