Recap.

(T, V)

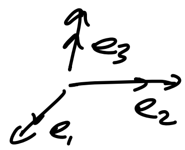
reducible: $\exists\, W \subset V$ proper, nontrivial invariable subspace

$W \Rightarrow V \backslash W$

$V \cong \bigoplus_i W_i$ completely reducible

$$M(g) = \left(\begin{array}{c|c|c} M_{11}(g) & 0 & 0 \\ \hline 0 & M_{22}(g) & 0 \\ \hline 0 & 0 & M_{33}(g) \end{array}\right) \quad \forall g \in G.$$

Examples. S_3 on $\mathbb{R}^3$

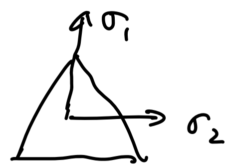


$u_0 = e_1 + e_2 + e_3 \qquad T(\sigma) u_0 = u_0$

$W = \{u_0\}$

$W^{\perp} = \text{span}\{u_1, u_2\}$ a. $u_1 = e_1 - e_2$, $u_2 = e_2 - e_3$

b.



$\Downarrow$

S_n on $\mathbb{R}^n$

$L = \sum_{i=1}^{n} e_i$ trivial rep.

$L^{\perp} = \{ \sum x_i e_i \mid \sum x_i = 0, x_i \in \mathbb{R} \}$

$\mathbb{R}^n \cong L \oplus L^\perp$ $L, L^\perp$ both irreducible.

finite. dim rep of non compact groups

$H \rtimes_\alpha G$: $(\underline{h_1, g_1}) \cdot (\underline{h_2, g_2}) = (\underline{h_1 \alpha_{g_1}(h_2)}, \underline{g_1 g_2})$

$\begin{cases} H \times \underline{\{1\}} \\ \underline{\{1\}} \times G \end{cases}$

α : trivial $\longrightarrow$ direct product

nontrivial group action.

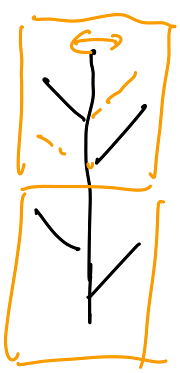
$\{\alpha | \vec{\tau}\} \in$ Euclidean group.

symmorphic space group.

$(\vec{\tau}_1, \alpha_1) \cdot (\vec{\tau}_2, \alpha_2) = (\vec{\tau}_1 + \underline{R(\alpha_1)}\vec{\tau}_2, \alpha_1 \alpha_2)$

$T \rtimes_\alpha PG$ $PG \subset SG$.

$\{\alpha | \vec{\tau}\} = \{\alpha | 0\}$ $\vec{\tau} = \vec{R}_n$ symmorphic



$\underline{\vec{\tau} = \frac{1}{2}}$

– Isotypic decomposition / components

G. set of irreps (up to isomorphism) of G is countable.

$$V \cong \oplus_{\mu} \underbrace{\oplus_{i=1}^{a_\mu} V^{(\mu)}}_{} \rightarrow \text{isotypic components}$$

$(T^{(\mu)}, V^{(\mu)})$ is a representative of its isomorphism class.

a_μ number of times $V^{(\mu)}$ appears.

Example:

$$T : U(1) \to U(N)$$

$$T(z) = \mathrm{diag}(\underbrace{z^j, z^j}_{n_j} \cdots \quad)$$

$$(T, V) \cong \oplus_{j \in \mathbb{Z}} n_j e_j \qquad e_j := z^j .$$

$$\underbrace{V^{(\mu)} \oplus V^{(\mu)} \oplus \cdots \oplus V^{(\mu)}}_{a_\mu} \cong K^{a_\mu} \otimes V^{(\mu)} =: a_\mu V^{(\mu)}$$

$$T(g) = \mathbb{1}_{a_\mu} \otimes T^{(\mu)}(g)$$

Example, rep. of $\mathbb{Z}_2$ on a vector space V $\Leftrightarrow$ linear operator

$$T : V \to V .$$

$$T^2 = \mathbb{1}$$

$$P_\pm = \frac{1}{2}(1 \pm T)$$

$$T^2 = 1 \quad \Rightarrow \quad T^2 v = v \quad \begin{cases} Tv = v \\ Tv = -v \end{cases}$$

$$\ker(P_+) : \{ v , \ P_+ v = 0 \}$$

$$\frac{1}{2}(1+T)v = 0 \quad \Rightarrow \quad Tv = -v$$

eigen space of $T = -1$

$\mathrm{im}(P_+) = V \setminus \ker(P_+)$: eigen space of $T = 1$

$$V \cong \ker(P_+) \otimes \underbrace{T_{1-}}_{Tv = -v} \oplus \ \mathrm{im}(P_+) \otimes \underbrace{T_{1+}}_{Tv := v}$$

- ## Schur's lemma

recall a intertwiner A is a morphism of G spaces

$$\begin{tikzcd} V_1 \arrow[r, "A"] \arrow[d, "T_1(g)"'] & V_2 \arrow[d, "T_2(g)"] \\ V_1 \arrow[r, "A"] & V_2 \end{tikzcd}$$

$$T_2(g) A = A T_1(g)$$

Schur's lemma :

Let G be <u>any</u> group. V_1, V_2 be vector spaces over <u>any</u> field k, they are carrier spaces of $\boxed{\text{irreps}}$ of G.

$A : V_1 \to V_2$ intertwiner between these two irreps.

A is either <u>zero</u> or an <u>isomorphism</u> of representations

<u>Proof</u> :

$$\ker A := \{ v_1 \in V_1 \mid A(v_1) = 0 \}$$

$$\operatorname{im} A := \{ v_2 \in V_2 \mid \exists v_1 \in V_1 \text{ s.t. } v_2 = A(v_1) \}$$

① $\underline{v_1} \in \ker A$ $\quad A(\underline{T_1(g) \cdot v_1}) = T_2(g)(\underline{A v_1})$

$$= 0 \qquad (\forall g \in G)$$

$$\Rightarrow T_1(g) v \in \ker A$$

$\ker A$ invariant subspace

② $v_2 \in \operatorname{im} A \subset V_2$

$$T_2(g) v_2 \overset{\exists v_1}{=} T_2(g)(A v_1) = A(\underbrace{T_1(g) v_1}_{\in V_1})$$

$$\in \operatorname{im} A$$

$\operatorname{im} A$ invariant

V_1 is an irrep $\Rightarrow$ $\ker A$ either 0 or V_1

a. $\ker A = V_1 \Rightarrow A = 0$

b. $\ker A = 0 \Rightarrow$ $\underline{A \text{ is injective.}}$ (b.1)

$(Av_1 = Av_2 \Rightarrow A(v_1 - v_2) \Rightarrow$
$(v_1 \neq v_2) \Rightarrow v_1 - v_2 \in \ker A)$

$\Rightarrow \mathrm{im}\, A$. ~~non-empty~~ nonzero

$\mathrm{im}\, A$ invariant subspace of V_2 $\begin{cases} \cancel{0} \\ \underline{V_2} \end{cases}$

$\Rightarrow \underline{\mathrm{im}\, A = V_2}$ surjective (b.2)

b.1 + b.2 $\Rightarrow$ $\underline{\underline{A \text{ is an isomorphism.}}}$

Now we set $V_1 = V_2 = V$.

$A : V \to V \in \mathrm{End}(V) := \mathrm{Hom}_G(V, V)$

all A's form an endomorphism $\underline{\underline{\text{ring}}}$ $(+, \times)$

$$\begin{cases} (A_1 \cdot A_2) v = A_1 \cdot (A_2 \cdot v) \\ (A_1 + A_2) v = A_1 v + A_2 v \end{cases}$$

Schur's lemma $\Rightarrow$ A isomorphism / invertible

$(A \cdot A^{-1} = 1)$

$\Rightarrow$ division ring / algebra $\begin{cases} \text{commutative: field} \\ \text{non-com. skew field.} \end{cases}$

Example: R. G. $H \trianglelefteq \mathrm{span}\{1, \sigma^k\}$

Theorem: Suppose (T, V) is an irrep of G.

V a complex vector space

$A: V \to V$ intertwiner

$\Rightarrow A$ is proportional to the identity transformation.

$$A(v) = \lambda v \qquad (\lambda \in \mathbb{C})$$

Proof. $\exists v$. s.t. $Av = \lambda v$. i.e. it is an eigen vector of A.

($p(x) = \det(x\mathbb{1} - A)$ always has a root in $\mathbb{C}$)

Eigen space $C = \{ w : Aw = \lambda w \}$ nonzero

$A\,\underline{T(g)w} = T(g)Aw = \lambda\,\underline{T(g)w} \qquad \forall g \in G$

$\Rightarrow C$ invariant subspace.

$\Rightarrow C = V$.

Remarks :

1. block diagonalization of Hamiltonians.

$\mathcal{H}$ a Hilbert space, is a representation of some symmetry group G. & completely reducible.

$$\mathcal{H} \cong \oplus_\mu H^{(\mu)} \qquad \underbrace{H^{(\mu)} := D_\mu \otimes V^{(\mu)}}_{\text{isotypic component.}}$$

H an Hamiltonian. $H : \mathcal{H} \to \mathcal{H}$. is an intertwiner $([H, G] = 0)$

$$\underline{H\,T(g) = T(g)\,H \quad (g \in G)}$$

Schur's lemma

$$H \cong \oplus_\mu \underbrace{H^{(\mu)}}_{\text{Hermitian operator on } D_\mu} \otimes \underline{\mathbb{1}_{V^{(\mu)}}}$$

With a proper choice of basis

$$SHS^{-1} = \begin{pmatrix} H_{11} & & \\ & H_{22} & \\ & & H_{33} \\ & & & \ddots \end{pmatrix}$$

Block diagonal with blocks labeled by irreps / "quantum number".

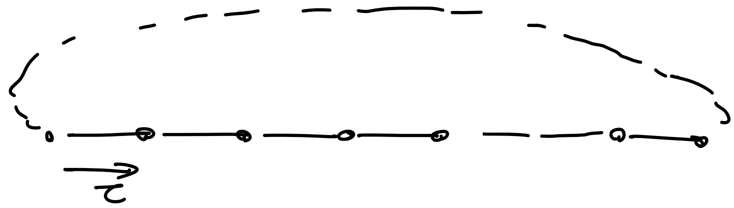
Any operator, $[O, G] = 0$

$$O = \oplus_\mu O^{(\mu)} \otimes \mathbb{1}_{V^{(\mu)}}$$

$$\langle \varphi_1, O\varphi_2 \rangle = 0 \qquad \varphi_1 \in \mathcal{H}^{(\mu)},\ \varphi_2 \in \mathcal{H}^{(\nu)}$$
$$(\mu \neq \nu)$$

<u>Example</u> 1D - tight-binding model

$$\hat{H} = -t \sum_{\langle ij \rangle} a_i^\dagger a_j + h.c$$



PBC

$$G = C_N = \langle \tau \mid \tau^N = 1 \rangle$$

$$H = \begin{pmatrix} 0 & -t & & & -t \\ -t & 0 & -t & & \\ & -t & \ddots & & \\ & & & \ddots & -t \\ -t & & & -t & 0 \end{pmatrix}$$

basis transformation,

$$a_k^\dagger = \frac{1}{\sqrt{N}} \sum_i^N e^{ikr_i} a_i^\dagger$$

$$\underline{\sum_i a_i^\dagger a_{i\pm1}} = 2\cos ka\ \underline{a_k^\dagger a_k}$$

$$-\frac{H}{2t} = \begin{pmatrix} \cos k_1 a & & & \\ & \cos k_2 a & & \\ & & \ddots & \\ & & & \end{pmatrix} \qquad k_i = \frac{2\pi}{N} i$$

$$\mathcal{H} \cong \oplus_k \mathcal{H}^k \qquad \mathcal{H}^k = \frac{1}{\sqrt{N}} \sum_i e^{ikr_i} |i\rangle$$

different irreps labeled by k.

2. $\mathbb{C}$.

$$\mathrm{Hom}_G(V^{(\mu)}, V^{(\mu)}) = \mathrm{End}(V^{(\mu)})$$

division ring/algebra

finite dimensional (associative) division algebra over $k=\mathbb{R}$ isomorphic to either $\mathbb{R}$. $\mathbb{C}$. $\mathbb{H}$

(Frobenius theorem)

- Pontryagin duality (Pontrjagin)

Abelian group $S \Leftrightarrow U(1)$

Definition : Let S be an Abelian group

The Pontryagin dual group $\hat{S}$

is the group of homomorphisms

$\mathrm{Hom}(S, U(1))$. For $\chi_1, \chi_2 \in \mathrm{Hom}(S, U(1))$

define

$$(\chi_1 \cdot \chi_2)(s) := \chi_1(s) \cdot \chi_2(s)$$

$\hat{S} = \mathrm{Hom}(S, U(1))$ is also an Abelian group.

Remarks.

1. $\hat{S}$ is the group of all complex one-dimensional unitary representations of S.

2 elements of $\hat{S}$ called characters.

$\hat{S}$ is the character group.

For a fixed $s \in S$. define homomorphism

$\hat{\hat{S}} = \mathrm{Hom}(\hat{S}, U(1))$ $\hat{S} = \mathrm{Hom}(S, U(1))$

$$\hat{S} \longrightarrow U(1)$$

$$\hat{s} : \chi \longmapsto \chi(s) \qquad \hat{s}(\chi) = \chi(s)$$

$$\begin{aligned}
(\hat{s}_1 \hat{s}_2)(\chi) &= \hat{s}_1(\chi) \cdot \hat{s}_2(\chi) \\
&= \chi(s_1) \cdot \chi(s_2) \\
&= \chi(s_1 s_2) \\
&= \widehat{s_1 s_2}(\chi)
\end{aligned}$$

Theorem (Pontryagin – van Kampen duality)

If G is a locally compact Abelian group then the canonical homomorphism $S \to \hat{\hat{S}}$ is an isomorphism: $S \cong \hat{\hat{S}}$

(locally compact . $\forall x \in X$. x has a compact neighborhood. $x \in$ open set $U \subset$ compact set K)

$\mathbb{R}, \mathbb{C}, \mathbb{Z}$ locally compact. ; $\mathbb{Q}$ rationals not locally compact.)

<u>Summary</u> : S Abelian.

① $\chi \in \mathrm{Hom}(S, U(1)) =: \hat{S}$

$$\chi : S \to U(1)$$
$$s \mapsto \chi(s)$$

② $ev_s \equiv \hat{\hat{s}} \in \mathrm{Hom}(\hat{S}, U(1)) = \mathrm{Hom}(\overbrace{\mathrm{Hom}(S, U(1))}, \underline{U(1)})$

$$ev_s : \hat{S} \longrightarrow U(1)$$
$$\chi \mapsto \chi(s)$$

③ $S \longrightarrow \hat{\hat{S}}$

$$s \longmapsto ev_s$$

$$(ev_{s_1} \cdot ev_{s_2})(\chi) = ev_{s_1 s_2}(\chi)$$

PvK theorem: $S \in LCA$. $S \cong \hat{\hat{S}}$

Examples:

1. $S = \mathbb{Z}/n\mathbb{Z}$

$\chi : S \to U(1)$

$\chi(\bar{1}) = \omega \in U(1)$

$\chi(\bar{0}) = \chi(\bar{n}) = \omega^n = 1 \qquad \omega_k = e^{i\frac{2\pi}{n}k}$

$\hat{\mathbb{Z}}_n = \{ \underline{\chi_{\omega_k}} \mid \omega_k = e^{i\frac{2\pi}{n}k},\ k = 1, \dots, n \}$

$\chi_{\omega_k}(\bar{\ell}) := (\omega_k)^\ell$

$$
\begin{aligned}
(\underline{\chi_{\omega_{k_1}}} \cdot \underline{\chi_{\omega_{k_2}}})(\bar{\ell}) &= \chi_{\omega_{k_1}}(\ell) \cdot \chi_{\omega_{k_2}}(\ell) \\
&= (\omega_{k_1})^\ell \cdot (\omega_{k_2})^\ell \\
&= e^{i\frac{2\pi}{n}(k_1+k_2)\cdot\ell} \\
&= \underline{\chi_{\omega_{k_1}\omega_{k_2}}}(\bar{\ell})
\end{aligned}
$$

$\hat{\mathbb{Z}}_n \cong \mu_n \cong \mathbb{Z}_n \qquad$ Pontryagin self-dual

$\hat{\hat{\mathbb{Z}}}_n \cong \hat{\mathbb{Z}}_n \cong \mathbb{Z}_n \qquad$ <u>PvK</u> ✓

2. $S = (\mathbb{R}, +)$

$\chi(x \cdot_S y) = \chi(x+y) = \chi(x) \cdot \chi(y)$

$\Rightarrow \chi(x) = e^{ax} \in U(1) \Rightarrow \chi(x) = e^{ikx} \quad \underline{(k \in \mathbb{R})}$

$(\underline{\chi_k} \cdot \underline{\chi_\ell})(x) = e^{i(k+\ell)x} = \underline{\chi_{k+\ell}}$

$\hat{\mathbb{R}} \cong \mathbb{R}$ $\quad (\hat{\mathbb{R}}^n \cong \mathbb{R}^n)$ self-dual

$\hat{\hat{\mathbb{R}}} \cong \hat{\mathbb{R}} \cong \mathbb{R}$

$S = (\mathbb{Q}, +)$ (induced top) $\quad \hat{\mathbb{Q}} = \mathbb{R}$

$\hat{\hat{\mathbb{Q}}} \neq \mathbb{Q}$