HW: P4: $\underline{A^T J A = J}$ $J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$

$\underline{A J A^T = J}$ A^{-1} $(A^T)^{-1}$ $SO(n)$: $\underline{AA^T = 1}$

$\Leftrightarrow$ $J = A^{-1} J (A^T)^{-1}$ $\underline{A^T A = 1}$

$\Leftrightarrow$ $J^{-1} = A^T J^{-1} A$ $J^2 = -1$ $A^{-1} = A^T$

$\Leftrightarrow$ $\underline{J = A^T J A}$

P1: $\varphi : Q \longrightarrow V$

$\{\pm 1, \pm i, \pm j, \pm k\}$ $V = \{1, a, b, ab\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

$\varphi(i) = a$ $\varphi(i) = b$

$\varphi(j) = b$ $\varphi(j) = a$

$\varphi(k) = \varphi(i)\varphi(j) = ab$

i

$\ker \varphi = \{\pm 1\}$ $\operatorname{im} \varphi = V$

$\phi(Q) \to 1_V$ "trivial homs."

Recap: Conj. classes in S_n

r-cycles conjugate

$(i_1 i_2 \cdots i_r) \sim (j_1 j_2 \cdots j_r)$

τ ____ $\tau^{-1} =$ ____

$\tau(i_k) = j_k$

$\phi \to$ cycle decomposition

$\Rightarrow \phi_1 \sim \phi_2$ same cycle decomp.

n: $\quad \underline{n} = \sum_{j=1}^{n} \underline{j} \cdot \underline{\alpha_j}$

plus: "partition function" of n

Young diagram

$4 = 1 + 1 + 1 + 1$

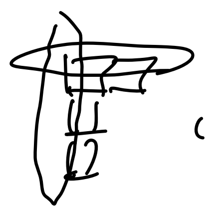
$(1)^4$

λ_i len. i-th

$\lambda_i \geq \lambda_{i+1}$

tableau

$2 + 1 + 1$



$(1)^2 (2)$

$\vdots$

Normal subgroup

$N \subset G$.

$gNg^{-1} = N \quad \forall g \in G$

$\underline{N \triangleleft G}$

$Z(G) = \{ z \in G : zg = gz . \forall g \in G \}$

$gzg^{-1} = z$

$Z(G)$ is a normal subgroup

Examples.

1. Abelian group

2. $\ker(\phi)$

$\phi : G \to G'$

$N \triangleleft G$

$G/N = \{ gN, g \in G \}$ natural group structure.

$\underline{(g_1 N) \cdot (g_2 \cdot N) = (g_1 g_2) N.}$

$H = \{ e, (1,2) \} \subset S_3$

$N \trianglelefteq G$

$$\phi : G \to G/N$$

$$g \mapsto gN$$

$\underline{\ker\phi = N}$.

Examples:

1. $\phi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$

$$\ker\phi = \underline{n}\mathbb{Z}$$

2. $A_3 \trianglelefteq S_3$ $\quad |A_3| = 3 = \frac{1}{2}|S_3|$

$$[G : H] = 2 \Rightarrow H \trianglelefteq G$$

3. D_4

Examples of normal subgroups (cont.)

4. det (M):

$$GL(n, k) \xrightarrow{\det} k$$

$$A \longmapsto \det A$$

$(\det(AB) = \det A \det B)$

ker(det)? $\underline{SL(n, k)}$ $\det = 1$

$\underline{SL(n, k) \triangleleft GL(n, k)}$

$(\det(gAg^{-1}) = \det(A))$

① $GL(n, k)/SL(n, k) \cong \underline{k^*}$

$\forall g \in GL$ $g = \underline{z}\,\underline{A}$

$A \in SL$

② $U(n)/SU(n) \cong U(1)$

③ $O(n)/SO(n) = \{SO(n), RSO(n)\} \cong \mathbb{Z}_2$

5. homomorphism

$$\pi : SU(2) \longrightarrow SO(3)$$

$$u\,\vec{x}\cdot\vec{\sigma}\cdot u^{+} := (\pi(u)\cdot\vec{x})\cdot\vec{\sigma}$$

$$u \in \ker\pi : \quad \underline{u}\,\vec{x}\cdot\vec{\sigma}\cdot\underline{u}^{+} = \vec{x}\cdot\vec{\sigma}$$

$$u = \lambda\mathbb{1}$$

$$[u, \vec{x}\cdot\vec{\sigma}] = 0$$

$$\underline{\lambda = \pm 1}$$

$$\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$$

$$\lambda^2 = 1$$

$$\ker\pi = \mathbb{Z}_2$$

$$\underline{SU(2)/\mathbb{Z}_2 \cong SO(3)}$$

6. G = space group $\quad \vec{r} \in \mathbb{R}^3$

$$g = \underline{\{R_\alpha | \tau\}} \qquad g.\vec{r} = R_\alpha\cdot\vec{r} + \vec{\tau}$$

$$\{\underline{R_\alpha} | \tau\}\{R_\beta | \underline{\tau'}\} = \{\underline{R_\alpha R_\beta} \,/\, \underline{R_\alpha\tau' + \tau}\}$$

$$\Rightarrow \underline{g^{-1} = \{R_\alpha^{-1} \,|\, -R_\alpha^{-1}\vec{\tau}\}} \qquad (g g^{-1} = \mathbb{1})$$

$$\{R_\alpha | \vec{\tau}\}\underline{\{e | \vec{t}\}}\{R_\alpha^{-1} | -R_\alpha^{-1}\vec{\tau}\} = \{R_\alpha | \vec{\tau}\}\{R_\alpha^{-1} | -R_\alpha^{-1}\vec{\tau} + \vec{t}\}$$

$$= \{e \,|\, R_\alpha(-R_\alpha^{-1}\vec{\tau} + \vec{t}) + \vec{\tau}\}$$

$$= \underline{\{e \,|\, R_\alpha\vec{t}\}} \in T$$

$T := \langle \vec{t}_1, \vec{t}_2, \vec{t}_3 \rangle$ $\vec{t}_i$ primitive lattice vectors

$\Rightarrow T \triangleleft SG$

7. $\{1\} \triangleleft G$. $G \triangleleft G$. trivial normal subgroups

Definition A group with no nontrivial normal subgroup is called a simple group.

① $\mu_p \cong \mathbb{Z}_p$ is prime

is a simple group

② $A_3 \cong \mathbb{Z}_3$ simple

A_4 : $V \triangleleft A_4$

A_n $(n \geqslant 5)$: simple.

— Quotient groups & short exact sequences

Let $K \subset G$ be the kernel of homo.

$$\mu : G \to G'$$

$$K \triangleleft G.$$

$$\forall g_1 g_2 \in G \qquad (g_1 K)(g_2 K) = (g_1 g_2) K$$

<u>Theorem</u> (1st isomorphism theorem)

$$\mu : G \to G' \qquad \text{homomorphism}$$

$$\underline{G/K \cong \operatorname{im} \mu}$$

<u>Proof</u>:

$$\varphi : G/K \to \operatorname{im} \mu$$

$$\underline{gK} \mapsto \underline{\mu(g)}$$

① φ well defined.

$$\underline{g_1 K} = \underline{g_2 K} \Rightarrow \exists k \in K. \quad g_1 = g_2 \cdot k$$

$$\Rightarrow g_2^{-1} g_1 = k \in K$$

$$\Rightarrow \mu(g_2^{-1} g_1) = \mu(g_2)^{-1} \mu(g_1) = 1_{G'}$$

$$\Rightarrow \underline{\mu(g_1)} = \underline{\mu(g_2)}$$

(2) $\varphi(g_1K\, g_2K) = \varphi(g_1g_2K)$

$$= \mu(g_1g_2)$$

$$= \mu(g_1)\mu(g_2)$$

$$= \varphi(g_1K)\varphi(g_2K)$$

$\varphi(g_1K) = \varphi(g_2K) \Leftrightarrow \mu(g_1) = \mu(g_2)$

$$\Rightarrow g_1g_2^{-1} \in K \quad (g_1 = g_2 \cdot k)$$

$$\Rightarrow g_1K = g_2K$$

bijective $\Rightarrow$ isomorphism.

Now we introduce a sequence of homomorphisms.

$$\cdots\ G_{i-1} \xrightarrow{f_{i-1}} G_i \xrightarrow{f_i} G_{i+1} \xrightarrow{f_{i+1}} \cdots$$

The sequence is exact at G_i if

$$\text{im } f_{i-1} = \ker f_i$$

A short exact sequence (SES) is of the form

$$1 \to G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \to 1$$

Abelian: "0" $\qquad 0$

1. "1". trivial group : $\{1\}$

2. $1 \xrightarrow{f} G_1$ & $G_3 \to 1$

$1 \mapsto 1_{G_1}$ $\qquad$ $g \mapsto 1 \quad \forall g \in G_3$

3. exact at :

$$1 \longrightarrow G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \xrightarrow{f_3} 1$$

a. G_1: $\ker f_1 = \operatorname{im}(1 \to G_1)$

$= \{1\}$ $\qquad$ <u>f_1 injective</u>

b. G_2: $\ker f_2 = \operatorname{im} f_1$

c. G_3: $\ker f_3 = G_3 = \operatorname{im} f_2$

<u>f_2 surjective</u>

Consider $\mu : G \to G'$ $\qquad$ $K = \ker \mu$.

$$1 \to K \xhookrightarrow{i} G \xrightarrow{\mu} \operatorname{im}\mu \to 1$$

is a SES.

Check exactness:

1. K: $\ker i = \{1_G\}$

2. G: $\underline{\ker \mu} = \operatorname{im}(i) = \underline{K}$

③ $\operatorname{im}\mu$: $\ker(\operatorname{im}\mu \to 1) = \operatorname{im}\mu$

$$1 \to K \to G \to G/K \to \underline{1}$$

Remarks

1. If we have a SES

$$1 \to N \to G \to Q \to 1$$

$N \cong H \trianglelefteq G$

kernel of $G \to Q$ homomorphism

We sometimes write Q as G/N.

$G/f(N)$

" G is an extension of Q by N "

Examples

1. $\varphi : \mu_6 \to \mu_2$ $\qquad \mathbb{Z}_6 \to \mathbb{Z}_2$

$\omega \mapsto \omega^2$ $\qquad \bar{i} \mapsto \overline{2i}$

$\ker\varphi = \{\pm 1\} \cong \mathbb{Z}_2$

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{Z}_2 & \to & \mathbb{Z}_6 & \to & \mathbb{Z}_2 & \to & 1 \\ & & \cong \ker\varphi & & G & & Q & & \end{array}$$

$$1 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 1$$

HW: $1 \to \mathbb{Z}_p \to \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p \to 1$

2. $\varphi : O(n) \to \mathbb{Z}_2$

$$M \mapsto \det(M)$$

$$\ker \varphi = SO(n)$$

$$1 \to SO(n) \to O(n) \to \mathbb{Z}_2 \to 1$$

&

$$O(n)/SO(n) \cong \mathbb{Z}_2$$

3. $1 \to SU(n) \to U(n) \to U(1) \to 1$

4. $\pi : SU(2) \to SO(3)$

① $\ker \pi = \{\pm 1_2\} \cong \mathbb{Z}_2$

$$1 \to \underset{\cong}{\overset{N}{\mathbb{Z}_2}} \to \underline{\underline{\overset{G}{SU(2)}}} \xrightarrow{\pi} \underline{\overset{Q}{SO(3)}} \to 1$$

" $SU(2)$ is an extension of $SO(3)$

by $\mathbb{Z}_2$ "

" $1 \to N \to G^{\text{quantum}} \to G^{\text{classical}} \to 1$ "

Definition central extension

$$1 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1$$

1. A is abelian.
2. $A \subset Z(E)$: $i(a)b = b\,i(a)$ $\begin{pmatrix} a \in A \\ b \in E \end{pmatrix}$

in QM: $\varphi_1 \sim \varphi_2$ $\quad \varphi_1 = \lambda \varphi_2 \quad \lambda \in \mathbb{C}^*$

$$P(H) := (H \setminus \{0\}) / \sim$$

U: representation of symmetries

$$1 \to \underline{U(1)} \to \underline{\tilde{U}(H)} \to \underline{U(P)} \to 1$$

$\forall f, g \in H \; \boxed{\langle f, g \rangle = \langle Uf, Ug \rangle}$

"projective representation"

$U(g_1) U(g_2) = \underline{c(g_1, g_2)}\, U(g_1 g_2)$

(unitary) representation

$\tilde{U}(g_1)\, \tilde{U}(g_2) = \tilde{U}(g_1 g_2)$ $\qquad c \in U(1)$

[GM] Sec. 14

$\underline{H^2(G, A)}$

7. Group actions (cont.)

Recall. $\phi : G \times X \to X$

① $\phi(g_1, \phi(g_2, x)) = \phi(g_1 g_2, x)$ (left)

$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$

② $\phi(1_G, x) = x$

A G-action is

① effective. $\forall g \neq 1 \ \exists x$. s.t $gx \neq x$

② transitive: $\forall x, y \in X$. $\exists g \in G$. s.t. $y = g \cdot x$

(there is only one orbit)

③ free $\forall g \neq 1$, $gx \neq x$ $\forall x \in X$

Definitions

1 Stabilizer group / isotropy group

$\mathrm{Stab}_G(x) := \{ g \in G : \underline{g \cdot x = x} \} < G$

(G^x)

free $\iff G^x = \{1\}$ $\forall x \in X$.

⑥

2. $\exists\ g \neq 1$. st $gx = x$ $\quad x$ is a fixed point

$\mathrm{Fix}_X(g) := \{ x \in X : g \cdot x = x \} \subset X$

(X^g)

free $\Leftrightarrow X^g = \phi$

3. $\mathcal{O}_G(x) = \{ gx, \ g \in G \}$