1. Introduction

19th Galois. "groupe" group

roots of polynomial $P(x)$

order <5 radical solution

>5 not necessarily

杨振宁 二十世纪 量子化、对称性、相因子

20th group $\Leftrightarrow$ symmetry

$\Updownarrow$

conservation

① General relativity $\mathrm{Diff}(M)$

special relativity, $(t, x, y, z) \to t^2 - x^2 - y^2 - z^2$

$O(1, d)$ Lorentz group

② Lie algebra, Jacobi identity

$$[T^A, T^B] = i\epsilon^{ABC} T^C$$

$$[T^A, [T^B, T^C]] + [T^B, [T^C, T^A]] + [T^C, [T^A, T^B]] = 0$$

$\Downarrow$

$SU(2)$, $SO(3)$, $Sp(2n)$

$\uparrow$ spin, $\uparrow$ H atom, $\uparrow$ $\{q_i, p_j\} = \delta_{ij}$

③ Standard model $\underbrace{U(1) \times SU(2)}_{\text{electroweak}} \times \underbrace{SU(3)}_{\text{strong}}$

4-dim, 8-dim, 8-gluons

photon, $2\times W$, Z

④ Condensed matter.

$$\begin{cases} \text{A. translation sym.} \quad (T^n = 1) \longleftrightarrow \text{momentum} \\ \text{B. rotation} \quad . \quad 32 \text{ point group} \end{cases}$$

A + B → space group 17 wallpaper group 2D

230 in 3D

atomic spectra, lattice vibration, bandstructure

SC

2. Groups: Basic definitions & Examples

<u>Definition</u>: A group is a quartet $(G, \underline{m}, \underline{I}, e)$ where

1. G is a set.
2. $\underline{m}: \underline{G} \times \underline{G} \to \underline{G}$. multiplication map
3. $\underline{I}: G \to G$ inverse map
4. $e \in G$. identity element.

They satisfy conditions:

1 (associativity)

$$\underline{m}(\underline{m}(g_1, g_2), g_3) = \underline{m}(g_1, \underline{m}(g_2, g_3))$$

$$(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$$

not associative : Octonian

$$e_i e_j = \begin{cases} e_i & i=0 \\ e_j & j=0 \\ -\delta_{ij} e_0 + \varepsilon_{ijk} e_k & \text{otherwise} \end{cases}$$

$$(e_i e_j) e_k = - e_i (e_j e_k)$$

2. (existence of e) $\exists e$. s.t. $\forall g$.

$$e \cdot g = g \cdot e = g$$

3. (existence of $\underline{I}$) $\forall g \in G$. $\exists \underline{I}(g) =: g^{-1}$. s.t.

$$g \cdot g^{-1} = g^{-1} \cdot g = e$$

<u>Remarks</u> .

1. e, 1_G , g_0 .

$\quad\;\; \begin{matrix} u \\ 0 \end{matrix}$

2. related defs :

$$\underbrace{\underbrace{\underbrace{\text{associtivity}}_{\text{semigroup / monoid.}} , \quad \exists e,}_{\text{unital monoid}} \quad \exists g^{-1}}_{\text{group}}$$

3. topological group. if G is a <u>topological space</u>

$\underline{m}$, $\underline{I}$ continuous

Lie group. G is a manifold locally homeomorphic to $\mathbb{R}^n$

$\underline{m}$, $\underline{I}$ real analytic in local coord.

4. $G \Longleftrightarrow (G, \underline{m}, \underline{I}, e)$

Exercise:

a. e unique?

$e_1 = e_1 e_2 = e_2$

b. a^{-1} unique?

$ab = ba = ac = ca = e$

$b = b(ac) = (ba)c = c$

Examples:

1. $G = \mathbb{Z}, \mathbb{R}$, or $\mathbb{C}$.

$\underline{m}(a,b) := a+b$

e? 0

$\underline{I}$? $a \to -a$

closed? ✓

$\underline{m}(a,b) := ab$

2. $G = \mathbb{R}^* := \mathbb{R} - \{0\}$ ✓

$\mathbb{C}^* := \mathbb{C} - \{0\}$

$\mathbb{Z}^* := \mathbb{Z} - \{0\}$ $\quad 2 \to \frac{1}{2} \notin \mathbb{Z}$

Definition (subgroup)

(G, m, I, e) a group. set $H \subset G$

m, I preserve H. i.e.

$$m : H \times H \to H$$

$$I : H \to H$$

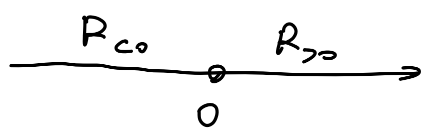
(H, m, I, e) is a subgroup of $(G - -)$

proper subgroup $H \neq G$ $H \subsetneq G$

Exercise.

1. $\mathbb{Z} \subset \mathbb{R} \subset \mathbb{C}$. subgroups m: "+"

2. $\mathbb{R}^*$.



$\mathbb{R}_{<0}$ subgroup of $\mathbb{R}^*$? ×

$\mathbb{R}_{>0}$ ✓

3. $\mathbb{R}_{<0}$: $m(a, b) = -ab$

$a \cdot (-1) = a$ ✓

$a \cdot \frac{1}{a} = -1$ ✓

4. $H_1 \subset G$, $H_2 \subset G$.

(a) $H_1 \cap H_2$ subgroup ?

① $e \in H_1$ $e \in H_2$ $e \in H_1 \cap H_2$ ✓

② $h \in H_1 \to h^{-1} \in H_1$

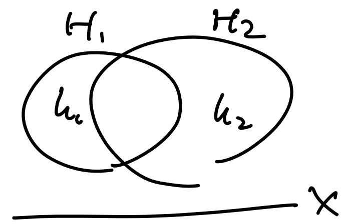
$h \in H_2 \to h^{-1} \in H_2$ $\to$ $h \in H_1 \cap H_2$

$\Downarrow$

$h^{-1} \in H_1 \cap H_2$

③ $\forall h_1, h_2 \in H_1 \cap H_2$ $\quad h_1 h_2 \in H_1$, $h_2 h_1 \in H_2$ $\rightarrow h_1 h_2 \in H_1 \cap H_2$

(b) $H_1 \cup H_2$ subgroup ?



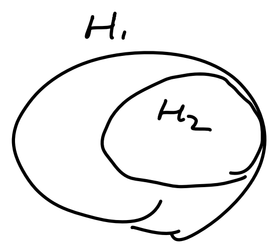
if $\exists h_1 \in H_1$, $\notin H_2$

$\exists h_2 \in H_2$, $\notin H_1$

$h_3 = h_1 \cdot h_2 \in H_1 \cup H_2$ $\quad$ WLOG. $\in H_1$

Contra. $\quad h_2 = \underline{h_1^{-1}} \cdot \underline{h_3} \in H_1$

$\Rightarrow H_1 \subset H_2$. or $H_2 \subset H_1$



<u>Definition</u> (order of a group) $|G|$ cardinality of G.

finite group $\quad |G| < \infty$

infinite group $\quad$ otherwise

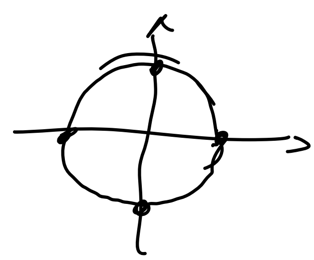
<u>Example</u>: Nth roots of unity

$$\mu_N = \{1, \omega, \dots \omega^{N-1}\} \equiv \{z \in \mathbb{C} \mid z^N = 1\}$$

$$\omega = e^{2\pi i/N} \quad \text{for } N \in \mathbb{N}$$

$|\mu_N| = N \quad \xrightarrow{N=\infty} \quad U(1) = e^{i\phi}$

$0 \le \phi < 2\pi$



$$\omega^i \omega^j = e^{i\frac{2\pi}{N}(i+j \bmod N)} \equiv \omega^k \qquad k = (i+j) \bmod N$$

"equivalence" relation $\sim$ $\quad \underline{k \sim k + lN}$

Example: residue classes modulo N.

$1 \leq j < N-1 \quad \bar{j} \equiv [j] = \{ n \in \mathbb{Z} \mid j = n \bmod N \}$

$\underline{m}(\bar{r}_1, \bar{r}_2) := \overline{r_1 + r_2}$

$\mathbb{Z}_N$, or $\mathbb{Z}/N\mathbb{Z}$

$P : x \to -x$

$P = \pm 1 \qquad \mathbb{Z}_2$

Definition: (direct product of groups) $G_1 \times G_2$, or $G_1 \otimes G_2$

$$\underline{m}_{G_1 \times G_2}((g_1, g_2), (g_1', g_2')) = (m_{G_1}(g_1, g_1'), m_{G_2}(g_2, g_2'))$$

$$\left(\begin{matrix} g_1, g_1' \in G_1 \\ g_2, g_2' \in G_2 \end{matrix} \right)$$

Example: Klein's Vierergruppe (4-group) $\mathbb{Z}_2 \times \mathbb{Z}_2$

$I = (1, 1)$, $a_1 = (-1, 1)$, $a_2 = (1, -1)$, $a_3 = (-1, -1)$

so far, $ab = ba$.

Definitions (Abelian & non-Abelian groups)

$\forall a, b \in G. \quad a \cdot b = b \cdot a \qquad$ Abelian

$\exists a, b \in G. \quad$ s.t. $a \cdot b \neq b \cdot a \qquad$ non-Abelian

Abelian: $m(a, b) := \underline{a+b}$

$e = 0$

Example . (The general linear group)

$M_n(K)$: all $n \times n$ matrices defined on field K $(\mathbb{R}, \mathbb{C})$

$GL(n, K) := \{ A \in M_n(K) \mid A$ non-singular $\}$
= invertible =
$\det A \neq 0$

in general $AB \neq BA$ $(n \geq 2)$

Definition (center of a group) $Z(G)$

$Z(G) := \{ z \in G \mid zg = gz \ , \forall g \in G \}$

↳ $Z(G)$ is an Abelian subgroup of G.

$Z(GL(n, K)) = \{ \lambda \mathbb{1}_n \ . \ \lambda \in K^* \}$ (Moore notes p. 13)

Examples : standard matrix groups

1 Special linear group

$SL(n, K) = \{ A \in GL(n, K) \mid \underline{\det A = 1} \}$

2. (special) orthogonal groups

$O(n, K) := \{ A \in GL(n, K) \mid \underline{AA^T = \mathbb{1}} \}$

$SO(n, K) := \{ A \in O(n, K) \mid \det A = 1 \}$

$AA^T = 1 \Rightarrow A^T A = 1$

$\hookrightarrow A^T = A^{-1} \to A^T A = 1$

$\det(AA^T) = (\det A)^2 = 1 \qquad \det A = \pm 1$

3. (special) unitary group $\qquad A^+ = (A^T)^*$

$$U(n) = \{ A \in GL(n, \underline{\mathbb{C}}) \mid \underline{AA^+ = 1} \}$$

$$SU(n) = \{ A \in U(n) \mid \det A = 1 \}$$

$$\det(AA^+) = (\det A)(\det A^*) = 1$$

$$|\det A| = 1$$

def. by bilinear form: $\quad \hat{A} A^T = 1 \quad$ generalized.

4 indefinite orthogonal group.

$$O(p, q) := \{ A \in GL(p+q, \underline{\mathbb{R}}) \mid A^T J_{p,q} A = J_{p,q} \}$$

$$J_{p,q} = \left(\begin{array}{c|c} -1_p & 0 \\ \hline 0 & 1_q \end{array} \right)$$

Lorentz group $\quad O(1, d) \quad d+1$ space-time

5 Symplectic group.

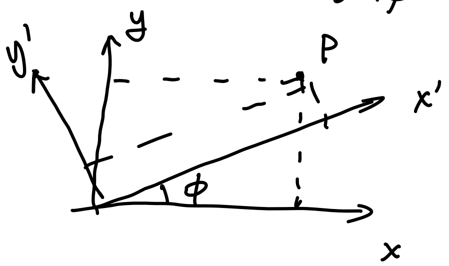
$$Sp(2n, K) := \{ A \in GL(2n, K) \mid A^T J A = J \}$$

$$J = \left(\begin{array}{c|c} 0 & 1_n \\ \hline -1_n & 0 \end{array} \right) \qquad \underline{J = J^* = -J^T = -J^{-1}}$$

Remarks :

1. $SO(2,\mathbb{R}) \ni g = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad a^2+b^2=1$

$R(\phi_1)R(\phi_2) = R(\phi_1+\phi_2) \leftarrow R(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \equiv e^{\phi J} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

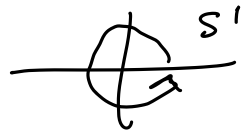


$x' = \cos\phi\, x + \sin\phi\, y$

$y' = -\sin\phi\, x + \cos\phi\, y$

2. $U(1)$: $z(\phi) = e^{i\phi} \qquad z(\phi_1)\, z(\phi_2) = z(\phi_1+\phi_2)$

$SO(2)$ "$\cong$" $U(1)$



μ_N "$\cong$" $\mathbb{Z}_N$

3. $SU(2) \qquad g = \begin{pmatrix} z & -w^* \\ w & z^* \end{pmatrix} \qquad |z|^2 + |w|^2 = 1$

$z = x_0 + i x_1$
$w = x_2 + i x_3$
$\sum_{i=0}^{3} x_i^2 = 1 \quad \rightarrow S^3$

4. $SU(3)$ 8-dim

"S^3 – bundle over S^5"

5. $Sp(2n, K)$. $A^T J A = J \quad (\det A)^2 = 1 \Rightarrow \det A = \pm 1$

"Pfaffian": $Pf(A^T J A) = (\det A) \cdot Pf(J) \Rightarrow \det A = 1$

$\quad\quad\quad \| $
$\quad\quad Pf(J)$