HW 06

P16. $M_{\tilde{T}} = S M_T(g) S^{-1}$

$\exists$ some S s.t. M is totally real

$\sum \chi(g^2) = |G|$ "totally real rep"

$\nexists$ pseudo real

$\sum \chi(g^3) = 0$

P17. $\mathrm{Hom}(V, W) = \underline{V^* \otimes W}$

$(\tilde{T}(g) \cdot \phi)(v) = T_W(g) \cdot \phi(T_V(g^{-1}) v)$

(1) ———

(2) $V^* := \mathrm{Hom}(V, k) \cong \underline{V^* \otimes k}$ $\quad$ $\underline{T_W}$ acts trivially on k

$\underline{(T^*(g)\, \underline{v_i^*})(v_j) = v_i^*(\underline{T(g)^{-1} \cdot v_j})}$

(3) $e_{ai}(v_j) = w_a \delta_{ij}$ $\quad$ $\underline{e_{ai} = w_a \otimes v_i^*}$

$[\underline{\tilde{T}(g)\, e_{ai}}](v_j) = T_W(g) \{ e_{ai} (\sum_k \underline{[M(g)^{-1}]_{kj}} v_k) \}$

$= T_W(g) (\sum_k [M(g)^{-1}]_{kj} \underline{e_{ai}(v_k)})$

$= \underline{T_W(g)} (\sum_k [M(g)^{-1}]_{kj} \underline{w_a \delta_{ik}})$

$= [M(g)^{-1}]_{ij} \sum_b M(g)_{ba} w_b$

$= \underline{\sum_b [M(g)]_{ba} [M(g)^{tr,-1}]_{ji}\, e_{bj}(v_j)}$

P 18 $\langle T(g)v, T(g)w\rangle = \langle v, w\rangle$

$$\int_G dg \langle T(g)v, T(g)w\rangle \qquad \left(\int_G dg = 1\right)$$

$$\frac{1}{|G|}\sum_g \langle T(g)v, T(g)w\rangle$$

$\langle v, w\rangle = v^{\dagger}\cdot w$ $\qquad$ $\langle T(g)v, T(g)w\rangle = v^{\dagger}T^{\dagger}Tw$

$= v^{\dagger}w$

$\langle v, w\rangle = \overline{\langle w, v\rangle}$

Recap. Schur's lemma $\quad V_1, V_2$ irreps

$$\begin{array}{ccc} V_1 & \xrightarrow{A} & V_2 \\ {\scriptstyle T_1(g)}\downarrow & & \downarrow{\scriptstyle T_2(g)} \\ V_1 & \xrightarrow{A} & V_2 \end{array} \qquad T_2(g)A = A T_1(g)$$

A either 0 or isomorphism.

$\ker A$. $\operatorname{im} A$

$V_1 = V_2 \qquad A \propto$ identity

$$M_1(g) A = A M(g_2) \quad \to \quad A \propto \lambda \mathbb{1}_V$$

$$\mathcal{H} \cong \oplus_\mu \mathcal{H}^{(\mu)} \qquad \mathcal{H}^{(\mu)} := D_\mu \otimes V^{(\mu)}$$

$[H, T(G)] = 0 \qquad H$ is an intertwiner

hamiltonian

$$H = \oplus_\mu H^{(\mu)} \otimes \mathbb{1}_{V^{(\mu)}}$$

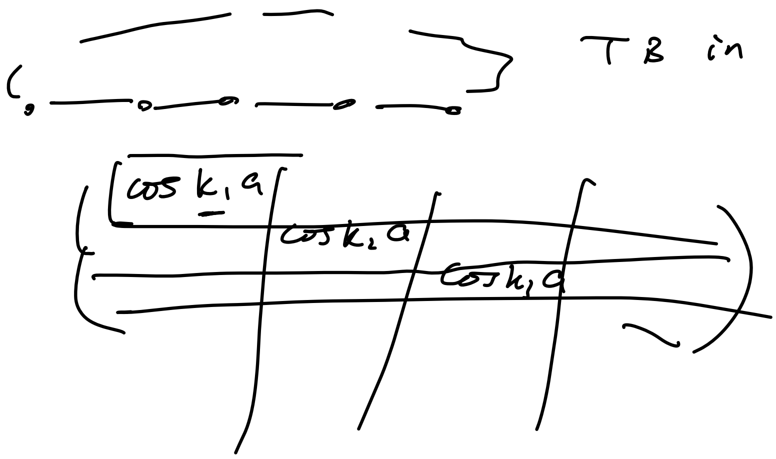
$$S H S^{-1} = \begin{pmatrix} H_{11} & 0 & 0 & 0 \\ & H_{22} & 0 & 0 \\ & & H_{33} & \\ & & & \ddots \end{pmatrix}$$

$[O, T(G)] = 0$

$$O = \oplus O^{(\mu)} \otimes \mathbb{1}_{V^\mu}$$

$$\langle \psi_1, O \psi_2 \rangle = 0$$

$\psi_1 \in V^\mu \quad \psi_2 \in V^\nu \quad (\mu \neq \nu)$



TB in 1D

- Pontryagin duality

Abelian group $S \longrightarrow U(1)$

① $\hat{S} := \mathrm{Hom}(S, \underline{U(1)})$ character group

$\chi \in \hat{S}$

$$\chi : S \longrightarrow U(1)$$
$$s \longmapsto \chi(s)$$

② $ev_s \in \mathrm{Hom}(\hat{S}, U(1)) = \mathrm{Hom}(\mathrm{Hom}(S, U(1)), U(1)) =: \hat{\hat{S}}$

$$ev_s : \hat{S} \longrightarrow U(1)$$
$$\chi \longmapsto \chi(s)$$

③ $$S \longrightarrow \hat{\hat{S}}$$
$$s \longmapsto ev_s$$

PvK : $\underline{S \in LCA}$ $\quad \hat{\hat{S}} = S$

Examples: 1. $\hat{\mathbb{Z}}_n = \mathbb{Z}_n$

PUK ✓

$\hat{\mathbb{Q}} = \mathbb{R}$

$= \mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$

2. $\underline{\hat{\mathbb{R}}} = \underline{\mathbb{R}}$

3. $S = (\mathbb{Z}, +)$

$\chi \in \hat{\mathbb{Z}} = \mathrm{Hom}(\mathbb{Z}, U(1))$

$\mathbb{Z} = \langle 1 \rangle$

$\chi(1) = \xi \qquad \xi \in U(1)$

$\chi(n) = \xi^n$

$(\underline{\chi_{\xi_1}} \underline{\chi_{\xi_2}})(n) = \chi_{\xi_1}(n) \chi_{\xi_2}(n) = (\xi_1 \xi_2)^n$

$= \underline{\chi_{\xi_1 \xi_2}}(n)$

$\underline{\hat{\mathbb{Z}}} \cong \underline{U(1)}$

discrete $\xrightarrow{\text{dual}}$ compact

compact $\longrightarrow$ discrete

4. $\widehat{U(1)} = \hat{\mathbb{Z}} = \mathbb{Z}$?

$S = U(1) = \{ e^{i\phi}, \phi \in [0, 2\pi) \}$

$\chi(\phi + 2\pi\mathbb{Z}) = \exp[ik(\phi + 2\pi\mathbb{Z})] \in U(1)$

$1 = \chi(2\pi\mathbb{Z}) = \exp[\underline{ik}\, 2\pi \underline{\mathbb{Z}}]$

$\Rightarrow k \in \mathbb{Z}$

$\chi(\phi + 2\pi\mathbb{Z}) = \exp(i 2\pi n \phi) \quad n \in \mathbb{Z}.$

$\underline{\chi_{n_1}} \underline{\chi_{n_2}}(\phi) = \chi_{n_1}(\phi) \chi_{n_2}(\phi) = \underline{\chi_{n_1 + n_2}}(\phi)$

$$\begin{cases} \widehat{U(1)} \cong \mathbb{Z}. & \hat{\hat{U}}(1) = U(1) \\ \hat{\mathbb{Z}} \cong U(1) & \hat{\hat{\mathbb{Z}}} = \mathbb{Z} \end{cases}$$

PvK ✓

S	$\hat{S}$
$\underline{\mathbb{R}}$	$\underline{\mathbb{R}}$
$U(1)$	$\mathbb{Z}$.
$\mathbb{Z}$	$U(1)$
$\mathbb{Z}_n$	$\mathbb{Z}_n$

FT

$x \leftrightarrow k$

periodic func. $\leftrightarrow$ summation of sin/cos

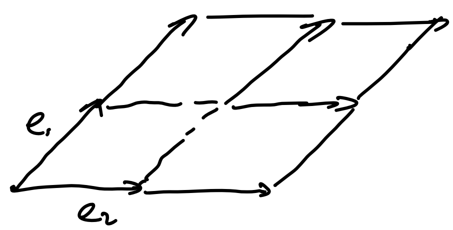
$U(1) \to$ discrete-time FT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-i\omega n}$$

$\mathbb{Z}_n \to$ discrete FT

5. Torus $G = \mathbb{Z}^d$

Γ lattice



dual lattice.

$$\Gamma^\vee = \{ g \in \mathbb{R}^d \mid \underline{g \cdot \gamma \in \mathbb{Z}} , \forall \gamma \in \Gamma \}$$

$$\underline{\cong \mathbb{Z}^d}$$

reciprocal lattice in solid state phys. (2π)

$\mathbb{R}/\mathbb{Z} \cong U(1)$

$$T^\vee \cong \mathbb{R}^d / \Gamma^\vee \cong U(1)^d.$$

Brillouin zone / torus

$$\hat{\Gamma} : \gamma \longrightarrow U(1)$$

$$\chi_{\bar{k}}(\gamma) = \exp[2\pi i \cdot k \cdot \gamma]$$

$$\bar{k} = k + \underline{\Gamma^\vee} \qquad \bar{k} \in T^\vee \cong U(1)^d.$$

$$\left(\chi_{k_1} = \chi_{k_2} \quad k_1 - k_2 \in \Gamma^\vee \right)$$

$$\chi_{\underline{\bar{k}_1}} \chi_{\underline{\bar{k}_2}}(\gamma) = \chi_{\bar{k}_1}(\gamma)\chi_{\bar{k}_2}(\gamma) = \chi_{\underline{\overline{k_1+k_2}}}$$

$$\hat{\mathbb{Z}}^d \cong U(1)^d$$

$\bar{k} \in$ Brillouin zone / torus

labels different irreps of the translational group.

- Application : Bloch's theorem

$$H = -\frac{\hbar^2}{2m}\nabla^2 + U(x)$$

$$U(x) = U(x+a) \qquad a \text{ lattice unit vector}$$

$$T_{\underline{n}} T_m = T_{n+m}$$

$$\underline{T_n \varphi(x) = \varphi(x + na)}$$

$$\chi_{\bar{k}}(\gamma) = \exp[2\pi i k\cdot\gamma] \qquad \bar{k} = k + \Gamma^\vee \in \mathbb{R}^d/\Gamma^\vee$$

$$\underline{L_\gamma\varphi(x)} = \underline{\varphi(x+\gamma)} = \underline{\chi_{\bar{k}}(\gamma)\varphi(x)}$$

if we write $\varphi(x) = e^{2\pi i k\cdot x} u_k(x)$

$$\underline{e^{2\pi i k(x+\gamma)}} u_k(x+\gamma) = \underline{e^{2\pi i k\cdot\gamma} e^{2\pi i kx}} u_k(x)$$

$$\Rightarrow \boxed{u_k(x+\gamma) = u_k(x)} \quad (\gamma \in \Gamma)$$

Bloch theorem.

$$\mathcal{H} = L^2(\mathbb{R}^d)$$

$$\underline{\mathcal{H} \cong \oint_{T^\vee} d\bar{k}\, \mathcal{H}_{\bar{k}}}$$

$$\underline{\mathcal{H}_{\bar{k}}} = \text{span}\{\varphi(x) : \varphi(x+\underline{\gamma}) = \underline{\chi_{\bar{k}}(\gamma)}\varphi(x)\}$$

Solving the eigenproblem

$$H\varphi(x) = E\varphi(x)$$

choose $k\in\bar{k}$

$$H e^{2\pi i kx} u_k(x) = E_k e^{2\pi i kx} u_k(x)$$

$$\Rightarrow H_k \underline{u_k(x)} = E_k u_k(x)$$

$$H_k = e^{-2\pi i kx} H e^{2\pi i kx}$$

$$\left(\begin{array}{l} H\varphi = E\varphi \\ (uHu^{-1})u\varphi = uE\varphi = E(u\varphi) \end{array} \right)$$

Choose $k' = k + g \quad g \in \Gamma^\vee$

$$H_k = u H_{k'} u^{-1} \qquad u = e^{2\pi i g x}$$

$H_k, H_{k'}$ have the same spectrum.

$$H_k = -\frac{\hbar^2}{2m}\nabla^2 - 4\pi \frac{\hbar^2}{2m} k \cdot (i\nabla) + \left(u + \frac{\hbar^2}{2m} 4\pi^2 k^2\right)$$

- Orthogonality relations of matrix elements of representations;

Peter-Weyl theorem

① $L^2(G) = \{ f : G \to \mathbb{C} \mid \int_G |f(g)|^2 dg < \infty \}$

unitary rep. of $G \times G$.

② $\mathrm{End}(V) := \mathrm{Hom}(V, V)$ unitary rep of $G \times G$

$S \in \mathrm{End}(V)$: $(g_1, g_2) \cdot S = T(g_1) \cdot S \cdot T(g_2)^{-1}$

Peter - Weyl theorem : G compact group.

Then there is an isomorphism of $G \times G$ representations.

$$L^2(G) \cong \oplus_\mu \mathrm{End}(V^{(\mu)})$$

sum over all isomorphism class of each irrep once.

To prove Peter - Weyl. first prove:

I. Let (T, V) be a unitary irrep of a compact group G. on a complex vector space V.

Then V is finite dimensional.

II. G compact. The Hermitian inner product on $L^2(G)$

$$\langle \varphi_1, \varphi_2 \rangle := \int_G \varphi_1^*(g)\, \varphi_2(g)\, dg$$

$$\left(\int_G dg = 1 \right)$$

Let $\{V^\mu\}$ be a set of representatives of distinct isomorphism classes of unitary irreps.

Each equiped with basis $w_i^{(\mu)}$, $(i=1,\cdots \underline{n_\mu})$

$$n_\mu = \dim_{\mathbb{C}} V^{(\mu)} \qquad (\because \text{ I})$$

$$T^{(\mu)}(g)\, w_i^{(\mu)} = \sum_{j=1}^{n_\mu} M^{\mu}_{ji}(g)\, w_j^{(\mu)}$$

<u>Statement:</u> M^{μ}_{ij} form a complete orthogonal set of functions on $L^2(G)$

$$\langle M^{\mu_1}_{i_1 j_1}, M^{\mu_2}_{i_2 j_2} \rangle = \frac{1}{n_\mu} \delta^{\mu_1 \mu_2} \delta_{i_1 i_2} \delta_{j_1 j_2}$$

$$\phi^{\mu}_{ij} := \sqrt{n_\mu}\, M^{\mu}_{ij}$$

$$\langle \phi^{\mu_1}_{i_1 j_1}, \phi^{\mu_2}_{i_2 j_2} \rangle = \delta^{\mu_1 \mu_2} \delta_{i_1 i_2} \delta_{j_1 j_2} \qquad \underline{\text{OC}}$$

<u>Proof</u>:

I. unitary irreps of compact G is finite dim.

choose $v \in V$. $\quad A: V \to V$. s.t.

$\forall w \in V$.

$$A(w) = \int_G dg \langle T(g) v, w \rangle T(g) \cdot v$$

$$A(\underline{T(g_0)} w) = \int_G dg \langle T(g) v, T(g_0) w \rangle T(g) v$$

$$= \int_G dg \langle T(g_0^{-1} g) v, w \rangle T(g) v$$

left inv. $$= \int_G dg \langle T(g) v, w \rangle T(g_0 g) v$$

$$= T(g_0) \underbrace{\int_G dg \langle T(g) v, w \rangle T(g) v}_{A(w)}$$

$$\Rightarrow \forall g_0 \in G. \quad A\, T(g_0) = T(g_0) A$$

A is an intertwiner

$$\begin{tikzcd} V \arrow[r, "A"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ V \arrow[r, "A"] & V \end{tikzcd}$$

Schur's lemma $\Rightarrow A = \lambda \mathbb{1}_V$

$$\forall v. \quad \langle v, A(v) \rangle = \int_G |\langle T(g) v, v \rangle|^2 dg \equiv \lambda \|v\|^2 \quad (A)$$

$$Tr(A) = \sum_i \langle v_i, A(v_i) \rangle$$

$$A \propto \lambda \mathbb{1}_V \quad = \sum_i \int_G |\langle v_i, T(g) v \rangle|^2 dg$$

$$= \int_G \sum_i |\langle v_i, T(g) v \rangle|^2 dg$$

$$= \int_G \|T(g) v\|^2 dg = \|v\|^2 \cdot Vol(G) (= 1) \quad (B)$$

unit vector v. & $\int_G dg = 1$

$$Tr(A) = \lambda \cdot \dim V = 1.$$

A + B :

$$\dim V = \frac{1}{\lambda} = \frac{1}{\int_G |\langle v, T(g) v \rangle|^2 g}$$

II. $\langle M^{\mu_1}_{i_1 j_1}, M^{\mu_2}_{i_2 j_2} \rangle = \frac{1}{n_\mu} \delta^{\mu_1 \mu_2} \delta_{i_1 i_2} \delta_{j_1 j_2}$

A. ortogonal.

B. complete

<u>Proof</u>:

$$\forall A : V^\mu \longrightarrow V^\nu$$

$$\tilde{A} := \int_G T^\nu(g) A \, T^\mu(g^{-1}) dg$$

$$\begin{aligned}
T^\nu(h) \tilde{A} &= \int_G T^\nu(hg) \cdot A \cdot T^\mu(g^{-1}) dg \\
&= \int_G T^\nu(g) A \, T^\mu((h^{-1}g)^{-1}) dg \\
&= \underline{\underline{\int_G T^\nu(g) A \, T^\mu(g^{-1}) dg}} \cdot T^\mu(h) \\
&= \tilde{A} \, T^\mu(h)
\end{aligned}$$

$$\begin{tikzcd}
V^\mu \arrow[r, "\tilde{A}"] \arrow[d, "T^\mu"] & V^\nu \arrow[d, "T^\nu"] \\
\underline{V^\mu} \arrow[r, "\tilde{A}"] & \underline{V^\nu}
\end{tikzcd}$$

Schur's lemma :

$$\tilde{A} = C_A \cdot \mathbb{1}_V \, \delta_{\mu\nu}$$

$\hookrightarrow$ introduce a basis $\quad \forall A \in Mat_{n_\mu \times n_\nu}(\mathbb{C})$

$$[\tilde{A}]_{ia} = \int_G dg \, M^\nu_{ii'}(g) A_{i'a'} M^\mu_{a'a}(g^{-1}) = \delta_{\mu\nu} \cdot (C_A \cdot \delta_{ia}) \quad (*)$$

set $\mu = \nu$. $i = a$.

$$C_A = \int_G dg\, \underbrace{M^{\mu}_{ii'}(g) A_{i'a'} M^{\mu}_{a'i}(g^{-1})}_{\Rightarrow}$$

<u>sum over i</u>

$$n_\mu C_A = \int_G dg\, \mathrm{Tr}\, M^\mu(g) A M^\mu(g^{-1})$$

$$= \int_G dg\, \mathrm{Tr}\, A = \mathrm{Tr}\, A .$$

$$\Rightarrow C_A = \frac{1}{n_\mu} \mathrm{Tr}\, A .$$

$$A = e_{jk} \qquad [e_{jk}]_{i'a'} = \delta_{ji'} \delta_{ka'}$$ insert into (*)

$$\int_G dg\, M^\nu_{ii'} (\delta_{ji'} \delta_{ka'}) M^\mu_{a'a}(g^{-1}) = \frac{\mathrm{Tr}\, e_{jk}}{n_\mu} \delta_{\mu\nu} \delta_{ia}$$

$$\Rightarrow \int_G dg\, M^\nu_{ij}(g) \underset{\substack{\| \\ M^{\mu *}_{ak}(g)}}{M^\mu_{ka}(g^{-1})} = \frac{1}{n} \delta_{\mu\nu} \delta_{ia} \delta_{jk}$$

$$\Rightarrow \langle M^\mu_{ij}, M^\nu_{i'j'} \rangle = \frac{1}{n_\mu} \delta_{\mu\nu} \delta_{ii'} \delta_{jj'}$$

$(\phi^\mu = \sqrt{n_\mu} M^\mu)$ <u>normalized.</u>