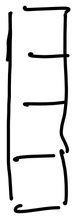
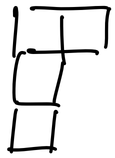
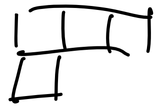
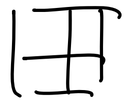
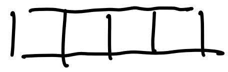
Recap. rep. of S_n

S_4:

 $(1)^4$  $(1)^2(2)$  $(1)(3)$

 $(2)^2$  (4)

Standard tableaux:

1	2	3
4		

✓

2	1	3
4		

✗

$R(T) = \{$ row permutations $\}$

$C(T) = \{$ column perm. $\}$

$$P = \sum_{p \in R(T)} p \qquad Q = \sum_{q \in C(T)} \operatorname{sgn}(q) \cdot q$$

$$c = PQ = \sum \operatorname{sgn}(q)\, p q$$

$c^2 = \lambda c \qquad \lambda \in \mathbb{R}^+$

$cc' = 0 \qquad T$ & T' different.

- Schur–Weyl duality : irreps of $GL(d, K)$ ①

Symmetry class of tensors

Simple example from before:

Recall representation of S_2 on $V \otimes V$.

$(V = K^d, K = \mathbb{R}, \mathbb{C})$

$$S_2 = \{e, \underset{(12)}{\sigma}\} \cong \mathbb{Z}_2$$

$$\sigma : v_1 \otimes v_2 \longmapsto v_2 \otimes v_1$$

$$\chi(e) = d^2 \qquad \chi(\sigma) = d \qquad \underline{v_i \otimes v_i}$$

$$V \otimes V \cong \bigoplus a_\mu V^\mu \qquad a_\mu = \langle \chi_\mu, \chi_{V\otimes V} \rangle$$

S_2	$()$	(12)
1^+	1	1
1^-	1	-1

$$a_{1^+} = \frac{d^2+d}{2} = \dim_K D^{1^+}$$

$$a_{1^-} = \frac{d^2-d}{2} = \dim_K D^{1^-}$$

$$V \otimes V \cong \underline{D^{1^+}} \otimes \underline{V^{1^+}} \oplus \underline{D^{1^-}} \otimes \underline{V^{1^-}}$$

$$D^{1^+} \otimes 1^+ \cong \operatorname{span}\{\underline{v_i \otimes v_j + v_j \otimes v_i}\} =: \operatorname{span}\{v_i \cdot v_j,\ i \leq j\}$$
$$=: \operatorname{Sym}^2 V$$

$$D^{1^-} \otimes 1^- \cong \operatorname{span}\{v_i \otimes v_j - v_j \otimes v_i\} =: \operatorname{span}\{v_i \wedge v_j,\ i < j\}$$
$$=: \Lambda^2 V$$

$\mathrm{Sym}^n U$ / $\Lambda^n V$ embedded into $V^{\otimes n}$

$$c(v_{i_1} \cdots v_{i_n}) = \sum_\sigma v_{\sigma(i_1)} \otimes v_{\sigma(i_2)} \cdots$$

$$c(v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_n}) = \sum_\sigma \mathrm{sgn}(\sigma)\, v_{\sigma(i_1)} \otimes v_{\sigma(i_2)} \cdots$$

$\mathrm{Sym}^n U$ is the quotient space $V^{\otimes n}/W$.

$$W = \mathrm{span}\{\underline{v_1 \otimes v_2 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}}\}$$

$\Lambda^n U$ $\qquad\qquad V^{\otimes n}/W'$

$W' = \mathrm{span}\{ v_1 \otimes v_2 \otimes \cdots \otimes v_n$ with any two v_i equal $\}$

$(v_i \wedge v_j = -v_j \wedge v_i \Rightarrow v_i \wedge v_i = 0)$

$\boxed{1|2}$ $\qquad c = e + (12)$

$$c \cdot V^{\otimes 2} = \mathrm{span}\{ v_i \otimes v_j + v_j \otimes v_i \} = \mathrm{Sym}^2 U$$

$\boxed{\begin{array}{c}1\\ \hline 2\end{array}}$ $\qquad c' = e - (12)$

$$c' \cdot V^{\otimes 2} = \mathrm{span}\{ v_i \otimes v_j - v_j \otimes v_i \} = \Lambda^2 V.$$

$V \otimes V$ is a rep of S_2

Consider any element $\in V^{\otimes 2}$, given by a rank-2 tensor: $\quad t = \sum_{ij} a_{ij}\, v_i \otimes v_j$

The action of S_2:

$$\underline{\sigma}\cdot t = \sum_{ij} a_{ij}\, v_{\sigma(i)}\otimes v_{\sigma(j)} = \sum_{ij} a_{\sigma^{-1}(i),\sigma^{-1}(j)}\, v_i\otimes v_j$$

$$(\sigma\cdot a)_{ij} = a_{\sigma^{-1}(i)\,\sigma^{-1}(j)} \qquad (a\in K^{d^2})$$

Let (T, V) be a rep of group $\underline{G}$.

$$T(g)^{\otimes 2}(v_1\otimes v_2) := \underline{T(g)\cdot v_1}\otimes \underline{T(g)\, v_2}$$

$$T(g)\cdot t = \sum_{ij} a_{ij}\left[T(g)\, v_i\otimes T(g)\, v_j\right]$$

$$= \sum_{\substack{ij\\kl}} a_{ij}\, M(g)_{ki}\, M(g)_{lj}\, v_k\otimes v_l$$

$$(g\cdot a)_{kl} = \sum_{ij} M(g)_{ki}\, M(g)_{lj}\, a_{ij}$$

easy to show that actions of G & S_2 commute:

$$[\sigma\cdot(g\cdot a)]_{ij} = (ga)_{\overset{j}{\overset{\shortparallel}{\underline{\sigma^{-1}(i)}}}\,\overset{i}{\overset{\shortparallel}{\underline{\sigma^{-1}(j)}}}}$$

$$= \sum_{kl} M(g)_{il}\, M(g)_{jk}\, a_{kl}$$

$$[g(\sigma a)]_{ij} \quad /\!/$$

check this

project $\underline{a}$ into different symmetry sectors:

$$(a_s)_{ij} = a_{ij} + a_{ji} \qquad \dim a_s = \frac{d(d+1)}{2}$$

$$(a_\wedge)_{ij} = a_{ij} - a_{ji} \qquad \dim a_\wedge = \frac{d(d-1)}{2}$$

$$\boxed{\text{The degeneracy space of different irreps of } S_2 \text{ becomes the representation space of } G.}$$

We can generalize to S_n:

$$V^{\otimes n} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{n}$$

$$T(g)^{\otimes n} (w_1 \otimes w_2 \otimes \cdots \otimes w_n) = \bigotimes_i T(g)\, w_i$$

$$\sigma \in S_n: \quad \sigma \cdot (w_1 \otimes w_2 \otimes \cdots \otimes w_n) = w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes \cdots \otimes w_{\sigma(n)}$$

$$T(g)^{\otimes n} \cdot \sigma = \sigma \cdot T(g)^{\otimes n}$$ Check: actions commute

$$V^{\otimes n} \cong \bigoplus_\lambda \mathcal{D}_\lambda \otimes V^\lambda \qquad \mathcal{D}_\lambda = \mathrm{Hom}_{S_n}(V^\lambda, V^{\otimes n})$$

λ denotes different partitions of n.
(Young diagrams)

$\mathcal{D}_\lambda$'s are reps of G.

Now consider $G = GL(d, K)$ $(K = \mathbb{R}, \mathbb{C})$

(and some subgroups $H \subset G$. e.g. $U(d)$)

$V = K^d$ is the defining rep. irreducible.

Schur – Weyl duality :

The representations D_λ are irreducible representations of $GL(d, K)$.

All irreps can be obtained by changing n.

↳ and all tensors can be projected into different symmetry sectors determined by Young diagrams using the Young symmetrizers.

Example $V^{\otimes 3}$

$\chi([1]) = d^3$

$\chi([12]) = d^2$ $\quad \overbrace{v_i \otimes v_i}^{d} \otimes \overbrace{v_k}^{d}$

$\chi([123]) = d$

	$[1]$	$3[(12)]$	$2[(123)]$
1^+	1	1	1
1^-	1	-1	1
2	2	0	-1

$$a_{1^+} = \langle \chi_{1^+}, \chi \rangle = \frac{1}{6}(d^3 + 3 \cdot d^2 + 2 \cdot d) = \frac{1}{6} d(d+1)(d+2)$$

$$a_1 = \frac{1}{6} d(d-1)(d-2)$$

$$a_2 = \frac{1}{3} d(d+1)(d-1)$$

① $\boxed{1|2|3}$ $\quad C = \sum_{\sigma \in S_3} \sigma$

$$C \cdot V^{\otimes 3} = \mathrm{span}\{ \sum_\sigma v_{\sigma(i)} \otimes v_{\sigma(j)} \otimes v_{\sigma(k)} \}$$

$$= \mathrm{Sym}^3 V$$

$$(a_S)_{ijk} = \sum_\sigma a_{\sigma^{-1}(i)\sigma^{-1}(j)\sigma^{-1}(k)}$$

$$\left(= \sum_\sigma a_{\sigma(i)\sigma(j)\sigma(k)} \right)$$

② $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$ $\quad C = \sum_{\sigma \in S_3} \mathrm{sgn}(\sigma)\sigma$

$$(a_\Lambda)_{ijk} = \sum_\sigma \mathrm{sgn}(\sigma)\, a_{\sigma^{-1}(i)\sigma^{-1}(j)\sigma^{-1}(k)}$$

$$(a_\Lambda)_{ijk} = -(a_\Lambda)_{jik}$$

Note: if $d = 2 (<3)$ this irrep is 0

($\dim_K V = d \geq$ # rows of Young diagram to be nonzero)

3 $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$

$$c_{(2,1)} = (e+(12))(e-(13))$$
$$= e + (12) - (13) - (132)$$

$$c_{(2,1)} V^{\otimes 3} = \operatorname{span}\{ v_i \otimes v_j \otimes v_k + v_j \otimes v_i \otimes v_k - v_k \otimes v_j \otimes v_i - v_k \otimes v_i \otimes v_j \}$$

$$(a_{(2,1)})_{ijk} = a_{ijk} + a_{jik} - a_{kji} - a_{jki}$$

$$\left(\sigma \cdot a_{ijk} \to a_{\sigma^{-1}(i)\sigma^{-1}(j)\sigma^{-1}(k)} \quad \sigma \cdot v_i \otimes v_j \otimes v_k \to v_{\sigma(i)} \otimes v_{\sigma(j)} \otimes v_{\sigma(k)} \right)$$

We can verify that:

- $(a_{(2,1)})_{ijk} + (\quad)_{jki} + (\quad)_{kij} = 0$ — A

and

$$(a_{(2,1)})_{ijk} = -(a_{(2,1)})_{kji} \quad — \quad B$$

$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ Check. $(a)_{ijk} = -a_{jik}$

A: i, j, k not all equal   all equal

cuts $\frac{1}{3}(d^3-d) + d = \frac{1}{3}(d^3+2d)$

remaining $d^3 - \frac{1}{3}(d^3+2d) = \frac{2}{3}(d^3-d)$

B cuts by $\frac{1}{2}$.

remaining $\underline{\underline{\frac{1}{3}(d^3-d)}}$ same as $\dim_{\mathbb{C}} D_2$

Consider the basis:

$$v_i \otimes v_j \otimes v_k + v_j \otimes v_i \otimes v_k - v_k \otimes v_j \otimes v_i - v_k \otimes v_i \otimes v_j$$

If we define the embedding

$$\Lambda^2 V \otimes V \longrightarrow V^{\otimes 3}$$

$$(v_1 \wedge v_3) \otimes v_2 \longmapsto v_1 \otimes v_2 \otimes v_3 - v_3 \otimes v_2 \otimes v_1$$

the preimage of $C_{(2,1)} V^{\otimes d} = \operatorname{span}\{(v_i \wedge v_k) \otimes v_j + (v_j \wedge v_k) \otimes v_i\}$

$$\subset \Lambda^2 V \otimes V$$

Consider the canonical map

$$\Lambda^m V \otimes \Lambda^n V \xrightarrow{\wedge} \Lambda^{m+n} V$$

$$(v_1 \wedge \cdots \wedge v_m) \otimes (v_{m+1} \wedge \cdots \wedge v_{m+n})$$

$$\longmapsto v_1 \wedge \cdots \wedge v_m \wedge v_{m+1} \cdots \wedge v_{m+n}$$

then $\underline{C_{(2,1)} V^{\otimes d} = \ker(\Lambda^2 V \otimes V \longrightarrow \Lambda^3 V)}$

Physical examples.

1 $C_{(n)} \cdot V^{\otimes n} = \mathrm{Sym}^n V$ projects into the totally symmetric tensor. $\Rightarrow$ Bosons

($V = \mathbb{k}^d = \mathcal{H}$ single-particle Hilbert space)

$$\dim \mathrm{Sym}^n V = \binom{n+d-1}{n}$$

$v_{i_1} \cdot v_{i_2} \cdots v_{i_n}$ $i_1 \leq i_2 \cdots i_n \leq d$

$\Downarrow$

$i_1 < i_2 < \cdots i_n \leq d+n-1$

$$H = \sum_j^d a_j^+ a_j \qquad (\hbar\omega = 1)$$

$$Z = \Big(\sum_{n \geq 0}^{\infty} q^n \Big)^d = \frac{1}{(1-q)^d} \qquad (q = e^{-\beta})$$

$$= (1 + q + q^2 + \cdots)^d$$

$$= \sum_{n \geq 0}^{\infty} q^n \dim(\mathrm{Sym}^n V)$$

$\dim(\mathrm{Sym}^n V)$ is the degeneracy of eigenstates with total energy n.

2. $C_{(1)^n}$ projects into totally antisym.

Fermions

$$\dim(\Lambda^n V) = \binom{d}{n} \qquad (= 0 \quad d < n)$$

$v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_n}$ $i_1 < i_2 < \cdots i_n \leq d$

$$H = \sum_j a_j^\dagger a_j \qquad (\text{Fermionic})$$

$$Z = (1+q)^d = \sum_{n=0}^{d} q^n \underline{\dim(\Lambda^n V)}$$

$U(2)$ and $SU(2) \subset GL(2, \mathbb{C})$

$U(2)$ action on $\Lambda^2 V$: $\qquad (V = \mathbb{C}^2)$

$$u \in U(2): \quad \underline{u} \cdot (v_1 \wedge v_2) = \sum u_{i1} u_{j2} \, v_i \wedge v_j$$

$$= u_{11} u_{22} \underline{v_1 \wedge v_2} + u_{21} u_{12} \underline{v_2 \wedge v_1}$$

$$= \underline{(\det u)} (v_1 \wedge v_2)$$

$V^{\otimes n}$:

$$T = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & - & \cdot\ \cdot & - & 2k-1 & & & \\ \hline 2 & - & \cdot\ \cdot & - & 2k \\ \cline{1-5} \end{array} \qquad T'$$

$$\underbrace{\qquad\qquad}_{k} \quad \underbrace{\qquad\quad}_{\ell}$$

$$n = 2k + \ell$$

Young symmetrizer: $\quad c = PQ \qquad P = \sum_{p \in R} p$

$$Q = \sum_{q \in C} \epsilon(q)\, q$$

$$c \cdot v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n} \qquad (v_{i_j} \in \{v_1, v_2\})$$

$$= P \, \underline{(v_{i_1} \wedge v_{i_2}) \otimes (v_{i_3} \wedge v_{i_4}) \otimes \cdots \otimes (v_{i_{2k-1}} \wedge v_{i_{2k}})}$$

$$\otimes \, v_{i_{2k+1}} \otimes \cdots \otimes v_{i_{2k+\ell}}$$

$v_{i_{2j-1}} \wedge v_{i_{2j}} \neq 0$ iff $i_{2j-1} = 1$, $i_{2j} = 2$

non-zero images of c is then

$$c \bigotimes_{j=1}^{n} v_{i_j} = \underline{p}\Big[\underline{\bigotimes_{i}^{k} (v_1 \wedge v_2)}\Big] \otimes v_{i_{2k+1}} \otimes \cdots \otimes v_{i_{2k+l}}$$

$$\propto \Big[\bigotimes \underline{(v_1 \wedge v_2)}\Big] \otimes P_{T'}\big(v_{i_{2k+1}} \otimes \cdots \otimes v_{i_{2k+l}}\big)$$

T': $\overbrace{\boxed{\;|\;|\;|\;|\;|\;}}^{l}$

$$\underline{u^{\otimes n} \cdot \Big(c \bigotimes_{j}^{n} v_{i_j}\Big)} = \cancel{(\det u)^k} \bigotimes_{i}^{k} (v_1 \wedge v_2) \otimes \underline{u^{\otimes l} P_{T'}(v \cdots)}$$

1 for $su(2)$

Irreps of $su(2)$ are in 1-1 correspondence to Young diagrams in the form of <u>single row of l boxes</u>

Dimension of irreps? $i_j \le i_{j'}$ ($j < j'$)

$$\underbrace{\boxed{1\,|\,1\,|\,1\,|\,1\,|\,1\,|\,1\,|\,1}}_{l} \quad \boxed{1\,|\,1\,|\,1\,|\,1\,|\,1\,|\,1\,|\,2} \quad \cdots \quad \boxed{2\,|\,2\,|\,2\,|\,2\,|\,2\,|\,\cdots\,|\,2}$$

$\dim = l + 1$ ($l = 2j$, j angular momentum)