HW:

P7. $SU(2)$ $\quad T: \mathbb{C}^2 \to \mathbb{C}^2$

$$T(\vec{z}) = \alpha \vec{z}, \quad \alpha \in \mathbb{C}.$$

linear map : $f: V \to V$

$$f(v+w) = f(v) + f(w)$$

$$f(\alpha v) = \alpha f(v)$$

$$\cancel{f(\vec{v}) = c\vec{v}}$$

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow TU = UT \Rightarrow T \propto \mathbb{1}_2$$

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \Rightarrow$$

$$U_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad U_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \Rightarrow T = \alpha \mathbb{1}.$$

P8 ——

P9. (1) <u>cycle decomposition</u> : disjoint cycles

<u>$(1\,n)(2\;n-1)$ ——</u>

(2) $\lceil \frac{n-1}{2} \rceil$ even $\Rightarrow$ $n = 4k, 4k+1$

odd $\Rightarrow$ $4k+2 \quad 4k+3$

(3) $\underline{\sigma_i \sigma_{i+1} \cdots \sigma_j} \neq \underline{(i\ j+1)}$

$= (i, i+1, \cdots j+1)$

$\underline{(i\ j)} = (i\ \ i+1)(i+1\ \ j)(i, i+1)$

$= \sigma_i (i+1, j) \sigma_i$

$\uparrow$

$\phi = (n-1, n)(n-2, n-1, n) \cdots \underline{\quad} \ \underline{(1 2 3 \cdots n)}$

Recap : Group actions.

effective : $\forall g \neq 1 \ \exists x . \ gx \neq x$

transitive : $\forall x, y \in X . \ \exists g \ \ y = gx$

free : $\forall g \neq 1 \ \ \forall x . \ gx \neq x$

Stabilizer group. / (isotropy grp.)

$\mathrm{Stab}_G(x) := \{ g \in G . \ g \cdot x = x \} \subset G.$

(G^x)

free $\Leftrightarrow$ $G^x = \{1\} \ \ \forall x \in X$

$\mathrm{Fix}_X(g) := \{ x \in X : g \cdot x = x \} \subset X.$

(X^g)

free $\Leftrightarrow$ $X^g = \phi$

Stabilizer — orbit theorem.

$$\varphi : \underline{O_G(x)} \longrightarrow \underline{G/G^x}$$
$$gx \longmapsto g \cdot G^x$$

$$|O_G(x)| = [G : G^x] \overset{\text{finite}}{=} |G| / |G^x|$$

G acts on G:

① conj. $\longrightarrow$ centralizer (of h)

$$\mathrm{Stab}_G(h) := \{ g \in G .\ ghg^{-1} = h \} =: C_G(h)$$

$$\downarrow$$

$$C_G(H) = \{ \ — \ —— \ .\ \forall h \in H \}$$

② $X = \{ \text{subgroups } H \text{ of } G \}$

$$N_G(H) = \{ g \in G : gHg^{-1} = H \}$$

$$C_G \subset N_G$$

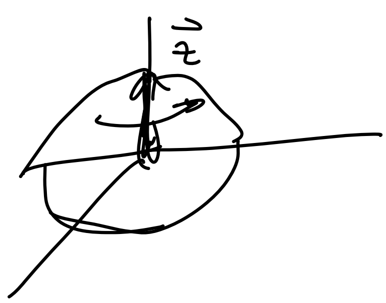
$$H \lhd N_G(H)$$

Examples.

① $\mathbb{Z}_p$ acts on any X.

$$|O_G(x)| = |G/G^x| = \begin{matrix} 1 \\ p \end{matrix} \qquad |G^x| = \begin{matrix} p \\ 1 \end{matrix}$$

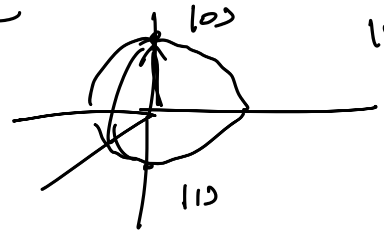
② $SO(3)$ on S^2



$$\mathrm{Stab}_{SO(3)}(\vec{z}) \cong SO(2)_{\vec{z}}$$

$$S^2 \cong SO(3)/SO(2)_{\vec{n}}$$

③ $SU(2)$ on $\mathbb{C}^2$



$|\varphi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$

$$e^{-\frac{i}{2}\sigma_z\alpha}|0\rangle = e^{-\frac{i}{2}\alpha}|0\rangle$$

$$\xrightarrow{\text{mat}} \begin{pmatrix} z & 0 \\ 0 & z^* \end{pmatrix} \qquad |z|^2 = 1$$

$$\mathrm{Stab}_{SU(2)}(|0\rangle) = \{ e^{-\frac{i}{2}\sigma_z\alpha},\ \alpha \in [0, 2\pi) \}$$

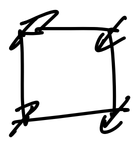
$$\cong U(1)$$

$$S^2 \cong SU(2)/U(1)$$

Burnside's theorem / orbit counting

$$|\{\text{orbit}\}| = \frac{1}{|G|}\sum_g |X^g|$$

$$\Downarrow$$



$$\begin{pmatrix} + & - \\ + & - \end{pmatrix} \begin{pmatrix} + & + \\ - & - \end{pmatrix}$$

physical config. $\Longleftrightarrow$ $\{\text{orbit}\}$

$G = C_4 \qquad |C_4| = 4$

$X = 2^4 = 16$

$$\# \text{ orbits} = \frac{1}{|G|} \sum_{g} |X^g|$$

$$= \frac{1}{4} \left(X^0 + X^{\frac{\pi}{2}} + X^{\pi} + X^{\frac{3}{2}\pi} \right)$$

$$= 6$$

8. Centralizer subgroups

$$C_G(h) := \{ g \in G.\ \ g h g^{-1} = h \}$$

$H \quad F \quad , \forall h \in H$

$$|\underline{C(h)}| = [G : C_G(h)]$$

$$\varphi : G/C_G(g) \longrightarrow C(g)$$

$$g_i C_G(g) \longmapsto g_i g g_i^{-1} \in C(g)$$

For finite G. $(|G| < \infty)$

$$|C(g)| = \frac{|G|}{|C_G(g)|}$$

$$|G| = \sum_{\substack{\text{distinct} \\ \text{conj. classes } \{C(g)\}}} |C(g)|$$

$$\Rightarrow |G| = \sum_{\{C(g)\}} \frac{|G|}{|C_G(g)|} \qquad \text{"class equation"}$$

$$C_G(G) = Z(G)$$

$$\forall g \in Z(G) : \ C(g) = \{ h g h^{-1},\ h \in G \} = \{ g \}$$

$$|G| = \sum_{g \in Z(G)} |C(g)| + \sum_{\text{others}} |C(g)|$$

$$= |Z(G)| + \sum_{\substack{\{C(g),\\ g \notin Z(G)\}}} \frac{|G|}{|C_G(g)|} \qquad \text{"class eq."}$$

Applications:

① previously: $1 \to \mathbb{Z}_p \to \underbrace{\begin{array}{c} \mathbb{Z}_{p^2} \\ \mathbb{Z}_p \times \mathbb{Z}_p \end{array}}_{G} \to \mathbb{Z}_p \to 1$

$$|G| = p^2$$

Theorem: If $|G| = p^n$. (p prime) then the center $Z(G)$ is nontrivial

i.e. $Z(G) = \{1\}$

Proof:

① if $|C_G(g)| = G$.

② if not.

$$|C_G(g)| = p^{n-n_i} \qquad (n_i < n)$$

$$\Rightarrow p \,\Big|\, \sum \frac{|G|}{|C_G(g)|}$$

$$\underline{|G|} = \underline{|Z(G)|} + \boxed{\sum_{\substack{\{C(g),\\ g \notin Z(G)\}}} \frac{|G|}{|C_G(g)|}} \quad \Rightarrow p \,\big|\, |Z(G)|$$

$$\Rightarrow Z(G) \neq \{1\}$$

Lemma : $\boxed{G \text{ abelian}}$.

$p \mid |G|$, p prime $\Rightarrow \exists g \in G$. of order p

Proof: $|G| = pm$

$$\left(\begin{array}{l} m=1 \\ |G| = p \Rightarrow G = \langle g \rangle \\ \qquad\qquad\quad g^p = 1 \end{array} \right)$$

$h \in G$. of order t. $h^t = 1$

① $p \mid t$ $\quad (h^{t/p})^p = 1$ $\quad$ of order p.

② $p \nmid t$. $\langle h \rangle$ is a normal subgroup.

$$|G/\langle h \rangle| = pm/t$$

m/t integer $< m$

$G/\langle h \rangle$ has an element of order p

$$\varphi : G \to G/\langle h \rangle$$
$$g \mapsto g\langle h \rangle$$

if $g_0\langle h \rangle$ has order p.

$$\varphi(g_0^p) = g_0^p\langle h \rangle = 1\langle h \rangle$$

$g_0^p = h^x$ $\quad$ $h^x = 1$ $\quad g_0$ order p

$h^x \neq 1 \Rightarrow (h^x)^y = 1$

$$\Rightarrow (g_0^p)^y = 1 \Rightarrow (g_0^y)^p = 1.$$

Theorem (Cauchy). G any finite group

$p \mid |G|$, p prime $\Rightarrow \exists g \in G$ of order p

Proof. $|G| = pm$.

$$|C(g)| = [G : C_G(g)] = |G| / |C_G(g)|$$

pick $g \notin Z(G)$. $|C(g)| > 1$

$$\Rightarrow \quad |C_G(g)| < |G|$$

① $p \mid |C_G(g)| \Rightarrow C_G(g)$ has an element of order p.

② $\forall g \in G$. $p \nmid |C_G(g)|$.

$$|G| = [G : C_G(g)] \cdot |C_G(g)|$$

$$\Rightarrow p \mid [G : C_G(g)]$$

$$|G| = |Z(G)| + \underbrace{\sum \frac{|G|}{|C_G(g)|}}$$

$$\Rightarrow \quad p \mid |Z(G)| \xrightarrow{\text{Lemma}} Z(G) \text{ has an element of order } p$$

9. Representation theory

Review of basic definitions.

① V: vector space over field k $\mathbb{R}$, $\mathbb{C}$

$GL(V)$, $Aut(V)$: invertible linear transformations

$$V \longrightarrow V$$

② a representation of G is a group homomorphism

$$T: G \to GL(V)$$
$$g \mapsto T(g)$$

$$T(g_1) \cdot T(g_2) = T(g_1 g_2)$$

V: carrier space / representation space.

$(\underline{V}, T)$ denotes the rep.

V finite dim with basis $\{e_1, \cdots e_n\}$

$GL(V) \cong GL(n, k)$ n: dim / degree of rep.

$$\underline{T(g) \hat{e}_i} = \sum_j \underline{M(g)}_{ji} \hat{e}_j$$

mat. rep.

$$T(g_1)[T(g_2)\hat{e}_i] = T(g_1)\sum_j M(g_2)_{ji}\hat{e}_j$$

$$= \sum_j M(g_2)_{ji}[T(g_1)\hat{e}_j]$$

$$= \sum_j M(g_2)_{ji}\sum_k M(g_1)_{kj}\hat{e}_k$$

$$= \sum_{jk} M(g_1)_{kj}M(g_2)_{ji}\hat{e}_k$$

$$= \sum_k [M(g_1)M(g_2)]_{ki}\hat{e}_k$$

$$T(g_1)T(g_2) = T(g_1g_2) \iff M(g_1)M(g_2) = M(g_1g_2)$$

In terms of group actions. rep. of G.
is a G-action on V that respects the linearity:

$$g\cdot(\alpha_1v_1+\alpha_2v_2) = \alpha_1(g\cdot v_1)+\alpha_2(g\cdot v_2)$$

$$\left(\begin{array}{c} v_i\in V \\ \alpha_i\in K\end{array}\right)$$

Examples:

1. rep. of degree / dim 1:

$$T: G\to \mathbb{C}^*$$

$\exists g$. of order n. $g^n=1$

$T(g)$ are roots of 1.

$\mathbb{Z}_3 \cong \mu_3 \cong A_3 = \langle g\rangle \qquad T(g) = \omega = e^{i\frac{2\pi}{3}}$

"trivial representation" $T(g) = 1$ (unit)

2. <u>regular representation</u> of a finite group

Let $\dim V = |G| = n$, with an ordered basis $\{\hat{e}_g\}$ $(g \in G)$

$$T(g_1)\cdot e_{g_2} = e_{g_1 g_2} \qquad (X = G)$$

3. More generally. G acts on set X.

$$x \longmapsto g\cdot x$$

V vector space with basis $\{e_x\}$ $(x \in X)$

$$\underline{T(g)}\cdot \underline{e_x} = \underline{e_{gx}}$$

"permutation rep." (associated with X)

4 group: "V" $= \langle a, b \mid a^2 = b^2 = (ab)^2 = e \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$T: \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow GL(4)$

$V = \{\hat{e}_e, \hat{e}_a, \hat{e}_b, \hat{e}_c\}$

$T(e)\cdot\hat{e}_g = \hat{e}_g$ $\qquad T(e) = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} = \mathbb{1}_4$

$$T(a)\cdot\hat{e}_e = \hat{e}_a$$

$$\begin{cases} T(a)\,\hat{e}_a = \hat{e}_e \\ T(a)\,\hat{e}_b = \hat{e}_c \\ T(a)\,\hat{e}_c = \hat{e}_b \end{cases} \qquad T(a) = \begin{pmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{pmatrix}$$

$$\mathrm{Tr}[T(g\neq 1)] = 0$$

4. $G = \mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$. $\quad T : G \to GL(\mathbb{C})$

$$\underline{n \mapsto a^n} \quad (a \in \mathbb{C}^*)$$

$$m+n \mapsto a^{m+n}$$

5. $G = \mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$; $\quad T : G \to GL(2,K)$

$$n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

$$m+n \mapsto \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & m+n \\ 0 & 1 \end{pmatrix}$$

6. 1+1 dim Lorentz.

$$x'^0 = \cosh\theta\, x^0 + \sinh\theta\, x^1$$

$$x'^1 = \sinh\theta\, x^0 + \cosh\theta\, x^1$$

$$\begin{pmatrix} x'^0 \\ x'^1 \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} =: B(\theta)\begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

$$B(\theta) \in O(1,1) = \{ A \mid A^T \eta A = \eta \}$$

$$\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

<u>Definition</u>: Let (V_1, T_1) and (V_2, T_2) be two reps of a group G. An <u>intertwiner</u> between these reps is a linear transformation $A : V_1 \to V_2$ s.t. $\forall g \in G$. the diagram

$$\begin{tikzcd} V_1 \arrow[r, "A"] \arrow[d, "T_1(g)"'] & V_2 \arrow[d, "T_2(g)"] \\ V_1 \arrow[r, "A"] & V_2 \end{tikzcd}$$

commutes.

(i.e. A is an equivariant map of G spaces $V_1 \to V_2$)

$$T_1(g) \cdot A = A \cdot T_2(g)$$

Definition : Two reps (V_1, T_1) and (V_2, T_2) are equivalent : $(V_1, T_1) \cong (V_2, T_2)$ if there is an intertwiner

$$A : V_1 \to V_2$$

which is an <u>isomorphism</u>.

$$T_2(g) = A T_1(g) A^{-1} \quad (\forall g \in G)$$

(isomorphism $\Rightarrow$ invertible)

- Unitary representations

Let V be a complex vector space. over K $\begin{pmatrix} \mathbb{R} \\ \mathbb{C} \end{pmatrix}$

Define the inner product on V as a sesquilinear map. $\langle \cdot , \cdot \rangle : V \times V \to K$. s.t

(1) $\langle v, \cdot \rangle$ is linear for all fixed v.

(2) $\langle w, v \rangle = \overline{\langle v, w \rangle}$

(3) $\langle v, v \rangle \geq 0$ equal iff $v = 0$

$$\left(\begin{array}{ll} \text{linear} & \langle v, \alpha_1 w_1 + \alpha_2 w_2 \rangle = \alpha_1 \langle v, w_1 \rangle + \alpha_2 \langle v, w_2 \rangle \\ \text{anti linear :} & \langle \alpha_1 v_1 + \alpha_2 v_2, w \rangle = \alpha_1^* \langle v_1, w \rangle + \alpha_2^* \langle v_2, w \rangle \end{array} \right)$$

Definition. Let V be an inner product space. A unitary representation is a rep (U, T) s.t. $\forall g \in G$. $U(g)$ is a unitary operator on V. i.e.

$$\underline{\langle U(g)v, U(g)w \rangle = \langle v, w \rangle} \quad \begin{pmatrix} \forall v, w \in V \\ \forall g \in G \end{pmatrix}$$

In QM. symmetry operators have to preserve the probability $\underline{\underline{|\langle \varphi | \phi \rangle|^2}}$

① In Hilbert space $\mathcal{H}$.

symmetry group $\Leftrightarrow$ unitary operators $U(g)$

or antiunitary operators (unitary . antilinear)

$\left(\langle U(g) v, U(g) w \rangle = \overline{\langle v, w \rangle} = \langle w, v \rangle \right)$

e.g. time-reversal symmetry.

② $[U, H] = 0$ i.e. $U H U^{-1} = H$

if H has certain symmety represented by U.

$\Rightarrow$ structure U. $\Leftrightarrow$ structure of $\mathcal{H}$.

$\Downarrow$

selection rules, degeneracies