Recap $GL(n, k)$

$Sp(2n, k) \Rightarrow \{q^i, p_i\}$

$\Rightarrow \{Q^i, P_i\}$

$$\begin{pmatrix} Q^1 \\ Q^2 \\ \vdots \\ Q^n \\ P_1 \\ \vdots \\ P_n \end{pmatrix} = A \begin{pmatrix} q^1 \\ \vdots \\ q^n \\ P_1 \\ \vdots \\ P_n \end{pmatrix}$$

$A \subset Sp(2n, \mathbb{R})$

$X \subset G \qquad \langle X \rangle = G \qquad |X| < \infty$ "finitely generated"

$G = \langle g_1 \cdots g_n \mid R_1 \cdots R_r \rangle$

$\mathbb{Z}_N, \mu_N \quad \langle A \mid A^N = 1 \rangle$

$\mathbb{Z}_2 \otimes \mathbb{Z}_2 \quad \langle A\, B \mid A^2 = B^2 = (AB)^2 = 1 \rangle$

$D_n \quad \langle A\, B \mid A^n = B^2 = (AB)^2 = 1 \rangle$

$\Big\}\ D_2 \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$

Quaternion group. $\quad i^2 = j^2 = k^2 = -1$

$ij = -ji = k \quad \cdots$

$Q = \{\pm 1, \pm i, \pm j, \pm k\} = \langle x, y \mid x^4 = 1,\ x^2 = y^2,$
$y^{-1} x y = x^{-1} \rangle$
$= \langle i\, j \rangle$

Pauli matrices. $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\underline{i} = -i\sigma^1, \quad \underline{j} = -i\sigma^2, \quad \underline{k} = -i\sigma^3$$

$$Q = \{\pm 1, \pm i\sigma^1, \pm i\sigma^2, \pm i\sigma^3\} \equiv \langle -i\sigma^1, -i\sigma^2 \rangle$$

Pauli group

$$P_1 = \{\pm 1, \pm i, \quad \cdots \quad \} \equiv \langle \sigma^1, \sigma^2, \sigma^3 \rangle$$

$$|P_1| = 16$$

$X = \sigma^1, \quad Z = \sigma^3.$

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$X|0\rangle = |0\rangle$

$X|1\rangle = |0\rangle$ bit flip

$Z|0\rangle = |0\rangle$

$Z|1\rangle = -|1\rangle$ phase flip

$P_1 \rightarrow$ n-qubit $\quad P_n = (P_1)^n$

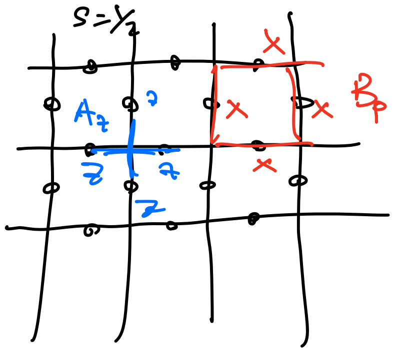
$$V_S = \{|\psi\rangle : s|\psi\rangle = |\psi\rangle, \ \forall s \in S\}$$

3-qubit bit-flip code

$$S = \langle Z_1 Z_2, Z_2 Z_3 \rangle$$

$$V_S = \{|000\rangle, |111\rangle\}$$

Z_1Z_2 ,	Z_2Z_3	$\rightarrow$
$+1$	$+1$	✓
$+1$	-1	3 - flip
-1	$+1$	1
-1	-1	2

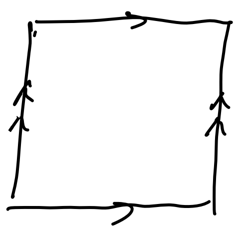


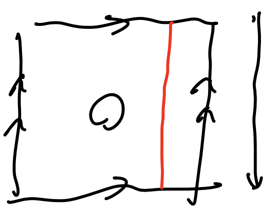
Toric code

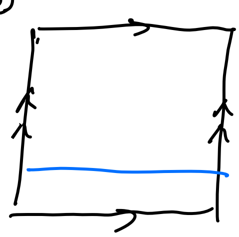
$$\left|\begin{array}{l} A_v = \prod Z_j \\ \\ B_j = \prod X_j \end{array}\right|$$

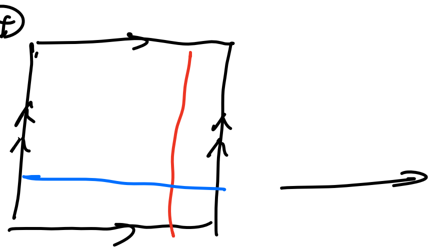
$$H = -\sum_v A_v - \sum_p B_p \qquad |GS\rangle = \sum |\text{closed loops}\rangle$$

$\mathbb{Z}_2 \times \mathbb{Z}_2$

① 

② 

③ 

④ 

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3. Homomorphism & Isomorphism.

<u>Definition</u>. Let $(G, \underline{m}, \underline{I}, e)$ and $(G', \underline{m}', \underline{I}', e')$ be two groups.

Homomorphism. $\varphi : G \to G'$. s.t. $\forall g_1, g_2 \in G$

$$\varphi(\underline{m}(g_1, g_2)) \longrightarrow \underline{m}'(\varphi(g_1), \varphi(g_2))$$

a mapping preserving the group law.

$$\varphi(g_1 \cdot g_2) = \varphi(g_1) \cdot \varphi(g_2)$$

$$\begin{tikzcd} G \times G \arrow[r, "m"] \arrow[d, "\varphi \times \varphi"'] & G \arrow[d, "\varphi"] \\ G' \times G' \arrow[r, "m'"] & G' \end{tikzcd}$$

"commutative diagram"

$$\varphi(e) = \varphi(e \cdot e) = \varphi(e)\varphi(e) \Rightarrow \varphi(e) = e'$$

$$e' = \varphi(e) = \varphi(g \cdot g^{-1}) = \varphi(g) \cdot \varphi(g^{-1}) \Rightarrow \varphi(g^{-1}) = \varphi(g)^{-1}$$

$$\begin{tikzcd} G \arrow[r, "I"] \arrow[d, "\varphi"'] & G \arrow[d, "\varphi"] \\ G' \arrow[r, "I'"] & G' \end{tikzcd}$$

(1) $\varphi(g) = e'$ iff $g = e$ $\quad\varphi$ injective

or

$\varphi(g_1) = \varphi(g_2) \Rightarrow g_1 = g_2$

$\exists\, g_1, g_2 \in G \quad g_1 \neq g_2$

$e' = \varphi(g_1)\cdot\varphi(g_2)^{-1} = \varphi(g_1 \cdot g_2^{-1}) = \varphi(g_3 \neq e)$

(2) $\forall g' \in G'$. $\exists g \in G$. s.t $\varphi(g) = g'$ $\quad\varphi$ is surjective

(3) (Def) φ is an isomorphism if (1) & (2)

(bijective)

$G \underset{\varphi^{-1}}{\overset{\varphi}{\rightleftarrows}} G'$ $\quad\varphi$ isomorphism

φ^{-1} isomorphism

$\forall g_1', g_2' \in G' \quad \varphi^{-1}(g_1', g_2') = \varphi^{-1}(g_1')\,\varphi^{-1}(g_2')$?

$\exists\, g_1, g_2 \in G$ s.t. $\varphi(g_1) = g_1'$, $\varphi(g_2) = g_2'$.

$\varphi(g_1)\varphi(g_2) = \varphi(g_1 g_2)$

$\varphi^{-1}(\varphi(g_1)\,\varphi(g_2)) = g_1 g_2 = \varphi^{-1}(g_1')\,\varphi^{-1}(g_2')$

Isomorphic groups are the same.

4. (Def) φ an isomorphism $G \xrightarrow{\varphi} G$

φ : automorphism

Definition (kernel & Image)

Let φ be an homomorphism

$$\varphi : \underline{G} \to \underline{H}$$

(a) kernel of φ

$$K = \ker\varphi := \{ g \in G : \varphi(g) = \underline{1_H} \}$$

(b) image of φ:

$$\operatorname{im}\varphi := \{ h \in H : h = \varphi(g) \text{ for some } g \in G \}$$
$$= \varphi(G)$$

Remarks

(a) $\underline{\varphi(G)} \subset H$ is a subgroup.

① $\varphi(\underline{1_G}) = \underline{1_H}$ ✓

② $\forall\, h_1 = \varphi(g_1),\ h_2 = \varphi(g_2)$

$$h_1 \cdot h_2 = \varphi(g_1)\varphi(g_2) = \varphi(g_1 g_2) \in \varphi(G)$$ ✓

③ $h_1 = \varphi(g_1)$ ✓

$$1 = \varphi(g_1)\varphi(g_1^{-1}) = h_1 h_2 \qquad h_2 = \varphi(g_1^{-1}) \in \varphi(G)$$

(b) $K \subset G$ is a subgroup

① $\varphi(1_G) = 1_H$ $\quad 1_G \in K$ ✓

② $\forall g_1, g_2 \in K$ $\quad \varphi(g_1)\varphi(g_2) = \varphi(g_1 g_2) = 1_H$

$g_1 g_2 \in K$ ✓

③ $\underbrace{\varphi(g_1)}_{=1} \cdot \underbrace{\varphi(g_1^{-1})}_{=1} = \varphi(1_G) = 1_H$ $\quad (g_1 \in K)$

$g_1^{-1} \in K$ ✓

(c) φ isomorphism. $\quad G \xrightarrow{\varphi} H.$

$K = \{1\}$

$\mathrm{im}\,\varphi = H$

<u>Example</u> : μ_N & $\mathbb{Z}_N$

$\mu_N = \{1, \omega, \omega^2 \dots \omega^{N-1}\}$ $\quad \omega = e^{i\frac{2\pi}{N}}$

$\mathbb{Z}_N = \{\bar{0}, \bar{1}, \dots \overline{N-1}\}$ $\quad \bar{i} = i + \ell \cdot N$ $\quad (0 \le i < N)$

$\varphi : \mathbb{Z}_N \to \mu_N$

$$\boxed{\varphi(\bar{r} = r + N\mathbb{Z}) := e^{i\frac{2\pi}{N}r}}$$

① $\varphi(\bar{r}_1 + \bar{r}_2) = \varphi(\bar{r}_1) \cdot \varphi(\bar{r}_2)$

$$e^{\frac{2\pi i}{N}(r_1 + r_2)} = e^{\frac{2\pi i}{N}r_1} \cdot e^{\frac{2\pi i}{N}r_2}$$ homomor.

② $\varphi(\bar{r}) = 1 \Rightarrow \bar{r} = \bar{0}$ $\quad$ injective

③ $\forall\, \omega^j \in \mu_n$, $\exists\, \varphi(\bar{r_j}) = \omega^j$ — surjective

①+②+③ : φ is an $\boxed{\text{isomorphism}}$

<u>Example</u>

$\mu_n \to \mu_n$ | $\mathbb{Z}_n \to \mathbb{Z}_n$

$P_k : \mu_k \to \mu_k$ | $m_k : \mathbb{Z}_n \to \mathbb{Z}_n$

$P_k(z) = z^k$ | $m_k(\bar{r}) = \overline{mr}$

$(z_1 z_2)^k = z_1^k z_2^k$

$P_k(z_1 z_2) = P_k(z_1) \cdot P_k(z_2)$ ✓

is P_k or m_k an isomorphism? $\gcd(k, n) = 1$

① $\mu_3 \cong \mathbb{Z}_3$ $k=2$ $\gcd(2,3)=1$

$\bar{0} \to \bar{0}$

$\bar{1} \to \bar{2}$

$\bar{2} \to \bar{1}$

② P_2 & μ_4, m_2 & $\mathbb{Z}_4$ $\gcd(2,4) = 2 \neq 1$

$\bar{0} \to \bar{0}$

$\bar{1} \to \bar{2}$

$\bar{2} \to \bar{0}$

$\bar{3} \to \bar{2}$

$\ker(m_2) = \{\bar{0}, \bar{2}\}$

$\mathrm{im}(m_2) = \{\bar{0}, \bar{2}\}$

$\cong \mathbb{Z}_2$

$$\varphi(\bar r) = e^{i\frac{2\pi}{N}\cdot r}$$

$$\begin{tikzcd}
\mathbb{Z}_N \arrow[r, "m_k", "m_{k_1}"'] \arrow[d, "\varphi"'] & \mathbb{Z}_N \arrow[d, "\varphi"] \\
\mu_N \arrow[r, "p_k"', "p_{k_2}"] & \mu_N
\end{tikzcd}$$

$$\varphi \circ m_k = p_k \circ \varphi$$

$$p_k = \varphi \circ m_k \circ \varphi^{-1}$$

$$\begin{tikzcd}
\mathbb{Z}_\varphi \arrow[r, "m_2"] \arrow[d, "\varphi"'] & \mathbb{Z}_\varphi \arrow[d, "\{1,-1\}"] \\
\mu_N \arrow[r, "m_\varphi", "\{1\}"'] & \mu_N
\end{tikzcd}$$

$$\neq$$

diagram commutes iff $k_1 = k_2 \pmod N$

Example $\varphi : U(1) \to SU(2)$

$$\varphi(z) = \begin{pmatrix} z^N & 0 \\ 0 & z^{-N} \end{pmatrix} \in SU(2) \qquad \det \varphi(z) = 1$$

$$\ker(\varphi) = \langle z \mid z^N = 1 \rangle \cong \mu_N$$

Example $\pi : SU(2) \to SO(3)$

$$\vec{x} \in \mathbb{R}^3$$

$$\vec{x}\cdot\vec{\sigma} = \sum x_i \sigma^i$$

$\forall u \in SU(2)$, $\exists\ \pi(u) \in SO(3)$ s.t.

$$u\, \vec{x}\cdot\vec{\sigma}\, u^\dagger = (\underbrace{\pi(u)\cdot\vec{x}}_{\vec{y}})\cdot\vec{\sigma} \qquad \det(\vec{y}\cdot\vec{\sigma}) = -\vec{y}^2$$

$$= \sum_{ij} (\pi(u)_{ij}\, x_j)\, \sigma_i \qquad \vec{x}^2 = \vec{y}^2$$

$$\Leftrightarrow\ u\,\sigma_i\,u^\dagger = \sum_j \pi(u)_{ji}\,\sigma_j$$

$$(u_1u_2)\,\sigma_i\,(u_1u_2)^\dagger = u_1(u_2\sigma_i u_2^\dagger)u_1^\dagger$$

$$= u_1\Big(\sum_j \sigma_j\, \pi(u_2)_{ji}\Big)u_1^\dagger$$

$$= \sum_j \pi_{ji}(u_2)\,(u_1\sigma_j u_1^\dagger)$$

$$= \sum_{j,k} \pi_{ji}(u_2)\,\pi_{kj}(u_1)\,\sigma_k$$

$$= \sum_k \pi_{ki}(u_1u_2)\,\sigma_k$$

$$\pi(u_1)\cdot\pi(u_2) = \pi(u_1u_2) \quad ✓$$

$SU(2)$, $SO(3)$ double cover

$$\pi(u) = \pi(-u)$$

$\pi(u) \in SO(3)$?

① $\pi(u) \in O(3)$

LHS $\det(u \cdot \vec{x} \cdot \vec{\sigma} \cdot u^{+}) = \det(\vec{x} \cdot \vec{\sigma}) = \det\begin{pmatrix} x_3 & x_1 - i x_2 \\ x_1 + i x_2 & -x_3 \end{pmatrix}$

RHS $(\pi(u) \cdot \vec{x}) \cdot \vec{\sigma} = \vec{y} \cdot \vec{\sigma}$ $\quad \det(\ ) = \underbrace{-\vec{y}^2 \quad = -\vec{x}^2}_{\vec{x}^2 = \vec{y}^2}$ norm preserving

$\pi(u) \in O(3)$ ✓

② $u = \mathbb{1}$. $\pi(u) = \mathbb{1}$ $\quad \pi(u) \in SO(3)$.

$$
\begin{aligned}
2i &= \mathrm{tr}(\sigma_1 \sigma_2 \sigma_3) \qquad u^{+}u \\
&= \mathrm{tr}(u\sigma_1 u^{+} u \sigma_2 u^{+} u \sigma_3 u^{+}) \\
&= \sum \pi(u)_{i1} \pi(u)_{j2} \pi(u)_{k3} \, \mathrm{tr}(\sigma_i \sigma_j \sigma_k) \\
&= \underbrace{2i \, \epsilon^{ijk} \pi(u)_{i1} \pi(u)_{j2} \pi(u)_{k3}}_{\det(\pi(u)) = 1}
\end{aligned}
$$

Definition: A matrix representation of a group G is a homomorphism

$$
\begin{aligned}
T : \ & G \longrightarrow GL(n, K) \qquad (K = \mathbb{R}, \mathbb{C}) \\
& g \longmapsto T(g)
\end{aligned}
$$

The set of matrices in $GL(n, K)$. can be thought as invertible linear transformations on a n-dim. vector space V with

a given basis set $b = \{ \hat{e}_1, \hat{e}_2, \cdots \hat{e}_n \}$

$(\underline{GL(V)} \cong \underline{GL(n, k)})$

$$T(g_1) \cdot T(g_2) = T(g_1 g_2) \quad ?$$

$$T(g)\hat{e}_i = \sum_j \hat{e}_j T(g)_{ji}$$

$$(T(g_1) \cdot T(g_2))\hat{e}_i = T^1 \Big(\sum_j \hat{e}_j T^2_{ji} \Big) \qquad (T^i := T(g_i))$$

$$= \sum_j T^2_{ji} (T^1 \hat{e}_j)$$

$$= \sum_{jk} T^2_{ji} T^1_{kj} \hat{e}_k$$

$$= \sum_k \hat{e}_k \underline{\sum_j (T^1_{kj} T^2_{ji})}$$

$$= \sum_k \hat{e}_k (\underline{T^1 T^2})_{ki}$$

$$\equiv \sum_k \hat{e}_k [\underline{T(g_1 g_2)}]_{ki}$$

$$\boxed{T(g_1) \cdot T(g_2) = T(g_1 g_2)} \quad \checkmark$$

basis dependent, $\quad \hat{e}_i = \sum_{j=1}^{n} \underline{S_{ji}} \hat{e}'_j$

$$T'(g) = \underline{S}\, \underline{T(g)} S^{-1}$$

<u>Definition</u> (equivalent representation). T, T' are n-dim matrix. rep. of G.

$T \cong T'$ (equivalent). if $\exists S \in GL(n, k)$, s.t

$$T'(g) = S\,T(g)\,S^{-1}$$

Example. $\mathbb{Z}, \mathbb{R}, \mathbb{C} \ni a$

$$T(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

$$T(a)\,T(b) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a+b & 1 \end{pmatrix}$$

Example μ_3 $\{1, \omega, \omega^2\}$ $\omega = e^{i\frac{2\pi}{3}}$

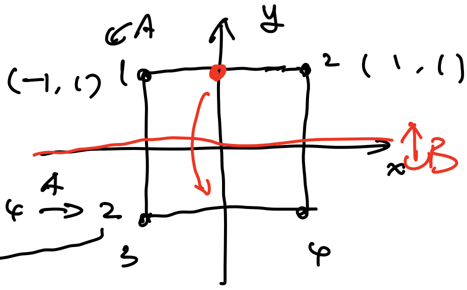
$$T(1) = \mathbb{1}_3$$

$$T(\omega) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$T(\omega^2) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Example D_4

$$D_4 = \langle A, B \mid A^4 = B^2 = (AB)^2 = 1 \rangle$$

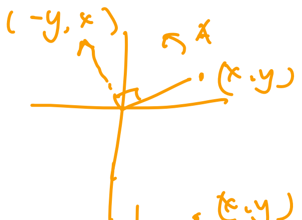


A : $1 \xrightarrow{A} 3 \xrightarrow{A} 4 \xrightarrow{A} 2 \xrightarrow{A} 1$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

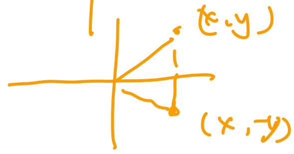
$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

A: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$



$\frac{\pi}{2}$ rotation

B: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$



reflection

"faithful representation": injective

$A = B = \mathbb{1}_n$ "not faithful"