12月21日 (周三) 复习

考试：暂定 1月5号 2:00 p.m.

Recap:

Character tables

C_{3v} ($\cong S_3$)

C_{3v}	E	[(123)] $2C_3(z)$	[(12)] $3\sigma_v$	linear	quadratic ...
A_1	+1	+1	+1	z	
A_2	+1	+1	−1	R_z	
E	+2	−1	0	(x,y) (R_x, R_y)	

Mulliken symbols

A / B	1D	sym. / antisym $\chi(C_n) = \pm 1$
E	2D	
T	3D	
⋮		

1/2 : vertical mirror plane

g/u : inversion

′/″ : horizontal plane

V_1, V_2 reps of G.

$$V_1 = \bigoplus a_\mu V^\mu \qquad V_2 = \bigoplus b_\nu V^\nu.$$

$$V_1 \otimes V_2 = \bigoplus_{\mu\nu} a_\mu b_\nu \, V^\mu \otimes V^\nu$$

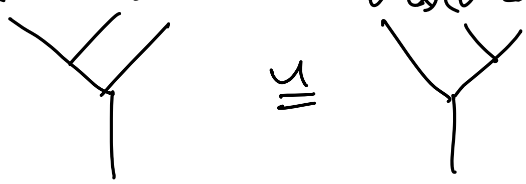
$$V^\mu \otimes V^\nu = \bigoplus_\lambda N_{\mu\nu}^{\lambda} V^\lambda, \quad N_{\mu\nu}^{\lambda} = \dim_k \mathrm{Hom}_G(V^\lambda, V^\mu \otimes V^\nu)$$

$$\chi_\mu \cdot \chi_\nu = \sum_\lambda N_{\mu\nu}^{\lambda} \chi_\lambda$$

$$N_{\mu\nu}^{\lambda} = \langle \chi_\lambda, \chi_\mu \chi_\nu \rangle$$

$\boxed{\text{Category theory}}$ $\quad \left\{ \begin{array}{l} Ob(\mathcal{C}) \\ \\ Mor(\mathcal{C}) \end{array} \right.$

$(V^\mu \otimes V^\nu) \otimes V^\lambda \qquad V^\mu \otimes (V^\nu \otimes V^\lambda)$



— Group algebra (of finite groups)

Let G be a finite group of order n.

Define n-dim vector space R_G with basis $\{g,\ g\in G\}$

$$x = \sum_{g\in G} x(g)\cdot g \qquad x(g) \in \mathbb{C}.$$

$$x = y. \quad \text{iff} \quad \forall g\in G \quad x(g) = y(g)$$

$$\begin{cases} \underline{x+y} = \sum x(g)\cdot g + \sum y(g)\cdot g = \sum (x(g)+y(g))\cdot g \\ \alpha x = \sum \alpha x(g)\cdot g \\ \underline{0} = \sum 0\cdot g \end{cases}$$

$$x\cdot y = \sum_{g\in G} x(g)\cdot g \cdot \sum_{h\in G} y(h)\cdot h = \sum_{g,h\in G} [x(g)\cdot y(h)]\cdot(\underbrace{gh}_{k})$$

$$= \sum_{k} \underbrace{[\sum_{g} x(g)\, y(g^{-1}k)]}_{\text{convolution}} \cdot k$$

$\Rightarrow$ R_G is a group ring / group algebra

$\mathbb{C}[G]$

Recall regular rep. $G \times G$ action on G.

$$(g_1, g_2) \longmapsto L(g_1) R(g_2^{-1})$$

$$(g_1, g_2)\cdot g = g_1 g g_2^{-1} \qquad g_i, g \in G$$

$L(g)$ & $R(g)$: $G \longrightarrow GL(R_G)$

LRR: $L(g)\cdot x = g x \qquad x \in R_G$

RRR: $R(g)\cdot x = x g^{-1} \qquad g \in G.$

$$L(h)\cdot x = L(h)\cdot \sum_g x(g)\cdot g = \sum_g x(g)(hg) = \sum_g x(h^{-1}g)\, g$$

$$[L(h)\cdot x](g) = x(h^{-1}g)$$

$$[R(h)\cdot x](g) = x(g\cdot h)$$

define: $\langle x, y\rangle = \int_G \overline{x(g)}\, y(g)\, dg$ $\qquad L(h)$ & $R(h)$ unitary

$$\langle L(h)x, L(h)y\rangle = \int_G \overline{x(h^{-1}g)}\, y(h^{-1}g)\, dg = \langle x, y\rangle$$

previously, $\delta_h(g) = \begin{cases} 1 & h = g \\ 0 & \text{otherwise} \end{cases}$ $\qquad h\cdot\delta_g = \delta_{hg}$

$$h = \sum_g h(g)\cdot g = 1\cdot h \implies \underset{\overset{|||}{\delta_h(g)}}{h(g)} = \begin{cases} 1 & h = g \\ 0 & \text{otherwise} \end{cases}$$

$$\underline{\delta_h \delta_g} = \sum_k \Big(\sum_l \underbrace{\delta_h(l)}_{l=h} \underbrace{\delta_g(l^{-1}\cdot k)}_{g=h^{-1}\cdot k} \Big)\cdot k = 1\cdot(hg) = \underline{\delta_{hg}}$$

in group algebra. the group elements can be thought both as operators & vectors.

The basis for $L^2(G)^{\text{class}}$:

$$\delta_{C_i}(g) = \begin{cases} 1 & g\in C_i \\ 0 & \text{otherwise} \end{cases}$$

$$\underline{\delta_{C_i}} = \sum_{g\in G} \underline{\delta_{C_i}(g)}\cdot g = \sum_{g\in C_i} g \equiv \underline{C_i}$$

$\forall h\in G$: $\quad \underline{h C_i h^{-1}} = \sum_{g\in C_i} h \underline{g h^{-1}} = \underline{C_i}$

center. $Z[\mathbb{C}(G)] = \text{span}\{C_i\}$

<u>projectors</u>:

completely reducible rep $V = \oplus_i W^i$

P a projector onto W. $\quad V = W \oplus W^\perp$

$\forall x \in R_G \qquad x = w + w^\perp \qquad w\in W,\ w^\perp \in W^\perp$

$$g\cdot(Px) = \underline{g\cdot w} = \underline{Pgw} = Pgx$$

$\boxed{P \text{ commutes with } \forall g \in G.}$

$(Pe \equiv e')$

$$Px = \sum_g x(g) \cdot P(g \cdot e) = \underbrace{\sum x(g) \cdot g}_{x}(Pe) = x \cdot e'$$

$\underline{W} = \{ xe' : x \in R_G \} =: R_G e'$

$(e'^2 = e'$ idempotent $)$

W is irreducible $\Leftrightarrow$ e' (primitive idempotent)

cannot be written as

$e' = e_1 + e_2$, $(e_i \neq 0)$

s.t. $e_i^2 = e_i$

$e_1 e_2 = e_2 e_1 = 0$

Is there relation between c_i's & P's ?

Motivations from physics :

Hamiltonian H. symmetry group G.

$$[H, T(G)] = 0$$

$\{\varphi_\mu\}$ eigenvectors of H span an invariant subspace W of $L^2(G)$

$$H \varphi_\mu = E_\mu \varphi_\mu$$

$$H\,T(g)\varphi_\mu = T(g)\,H\varphi_\mu = E_\mu T(g)\varphi_\mu \qquad \forall g\in G.$$

$$V \cong \oplus\, W^\mu$$

W^μ is still reducible (there is degeneracy)

$\Rightarrow$ find another operator that commutes with $T(g)$

until one finds all irreps.

— <u>Construction of character table</u>

With a <u>complete set of commuting operators</u> (CSCO), we can achieve a complete reduction of representations / find all irreps.

This is an idea explored systematically by 陈金全 (南大)

① 《群表示论的新途径》 1984

② [Eng] Group representation theory for physicists (2nd ed.) (World scientific. 2002)

(③ RMP 57, 211 (1985)

Let G be a finite group with r

conjugacy classes $[C_i]$ $(i = 1, \cdots r)$

$|[C_i]| = m_i$.

Correspondingly, there are r irreps V^μ

and characters χ_μ.

Consider the <u>class operators</u>

$$c_i = \sum_{g \in C_i} g \qquad \mathrm{span}\{c_i\} = Z(\mathbb{C}[G])$$

<u>Some properties.</u>

① $\forall h \in G$. $[c_i, h] = 0$

② $\forall i, j$ $[c_i, c_j] = 0$

③ closed : $\boxed{c_i c_j = \sum_{k=1}^{r} c_{ij}^k c_k}$ $(*)$

For a given G, $\{c_i\}$, c_{ij}^k are "easily" computed.

$$\hat{c}_i \delta_{c_j} = \sum_{k=1}^{r} [c^i]_{jk} \delta_{c_k} \qquad (*)$$

(*) defines an eigen problem. defined on orthogonal basis;

$$\left(\langle \delta_{C_j}, \delta_{C_k} \rangle = \frac{1}{|G|} \sum_g \delta_{C_j}(g)\, \delta_{C_k}(g) = \frac{m_i}{|G|} \delta_{jk} \right)$$

Suppose for $\hat{C}_i$ we find its eigenvectors $\{\phi^\mu\}$

$$\hat{C}_i \phi^\mu = \lambda_i^\mu \phi^\mu$$

$$\hat{C}_i (\phi^\mu \phi^\nu) = \lambda_i^\mu (\phi^\mu \phi^\nu)$$

$$\hookrightarrow = \hat{C}_i (\phi^\nu \phi^\mu) = \lambda_i^\nu (\phi^\nu \phi^\mu)$$

Assuming there is no degeneracy.

$$\phi^\mu \phi^\nu = \alpha_\mu \delta_{\mu\nu} \phi^\mu \qquad (\alpha_\mu \in \mathbb{C})$$

Define $p^\mu = \alpha_\mu^{-1} \phi^\mu \Rightarrow \underline{p^\mu p^\nu = \delta_{\mu\nu} p^\nu}$

$\{p^\mu e\}$ are primitive idempotents of R_G.

And $\underline{C_i = \sum_{\mu=1}^{r} \lambda_i^\mu p^\mu}$ is linear combination of projection operators onto irreps.

Observations:

C_i restricted to an irrep V^μ

$$\left\{ \begin{aligned} & \underline{C_i^\mu} = \lambda_i^\mu \cdot \mathbb{1}_{V^\mu} \\ & \chi_\mu(C_i^\mu) = \sum_{g \in C_i} \chi_\mu(g) = m_i \chi([C_i]) \end{aligned} \right.$$

$$\underline{C_i^\mu = \frac{m_i}{n_\mu} \chi_\mu([C_i]) \cdot \mathbb{1}_{V^\mu}} \qquad (n_\mu = \dim V^\mu)$$

$$\Rightarrow \underline{\lambda_i^\mu = \frac{m_i^2}{n_\mu} \chi_\mu([C_i])}$$

$$\frac{1}{|G|} \sum_{\{C_i\}} m_i \chi_\mu(C_i) \overline{\chi_\nu(C_i)} = \delta_{\mu\nu}$$

$$\Rightarrow \underbrace{\frac{1}{|G|} \sum_{\{C_i\}}}_{} m_i \lambda_i^\mu \overline{\lambda_i^\nu} = \delta_{\mu\nu} \left(\frac{m_i}{n_\mu}\right)^2$$

$$=: \langle \lambda_i^\nu , \lambda_i^\mu \rangle$$

$$\Rightarrow n_\mu = \frac{m_i}{\sqrt{\langle \lambda^\mu, \lambda^\mu \rangle}}$$

$$\chi_\mu = \frac{\lambda_i^\mu}{\sqrt{\langle \lambda_i^\mu, \lambda_i^\mu \rangle}}$$

Now try to diagonalize all C_i's:

$$(*) \Rightarrow \frac{m_i}{n_\mu} \chi_\mu([C_i]) \frac{m_j}{n_\mu} \chi_\mu([C_j]) = \sum_{k=1}^{r} C_{ij}^k \frac{m_k}{n_\mu} \chi_\mu([C_k])$$

To keep track of different c_i's, introduce auxilary variables y^i $(i=1 \cdots r)$

LHS: $\sum_{i=1}^{r} m_i m_j \chi_\mu([C_i])\chi_\mu([C_j]) y^i = \sum_{i=1}^{r} (\varphi_i y^i)\varphi_j$

$(\varphi_i = m_i \chi_\mu([C_i]))$

RHS: $\sum_{i=1}^{r} n_\mu \sum_{k=1}^{r} c_{ij}^{k} m_k \chi_\mu([C_k]) y^i = n_\mu \sum_{k=1}^{r} L_j^k \varphi_k$

$(L_j^k = \sum_i c_{ij}^k y^i)$

$$\Rightarrow \sum_{k=1}^{r} L_j^k \varphi_k = \lambda \varphi_j \qquad \left(\lambda = \frac{1}{n_\mu} \sum_{i=1}^{r} \varphi_i y^i\right)$$

Solviy $(L - \lambda \mathbb{1})\varphi = 0$

$$\lambda_\mu = \frac{1}{n_\mu} \sum_{i=1}^{r} m_i \chi_\mu([C_i]) y^i$$

via the orthoponal realation of χ_μ's:

$$n_\mu = \left[\frac{|G|}{\sum_{i=1}^{r} m_i \left| \frac{\chi_\mu([C_i])}{n_\mu} \right|^2} \right]^{\frac{1}{2}}$$

Example : $G = S_3$

S_3: E ; $(12), (13), (23)$; $(123), (132)$

① class operator:

$$C_1 = E$$

$$C_2 = (12) + (13) + (23)$$

$$C_3 = (123) + (132)$$

② class multiplication table

	C_1	C_2	C_3
C_1	C_1	C_2	C_3
C_2	C_2	$3C_1 + 3C_3$	$2C_2$
C_3	C_3	$2C_2$	$2C_1 + C_3$

$C_{22}^{1} = 3$ $\quad C_{22}^{3} = 3$

$C_{23}^{2} = 2$

$$L = \begin{pmatrix} y^1 & y^2 & y^3 \\ 3y^2 & y^1 + 2y^3 & 3y^2 \\ 2y^3 & 2y^2 & y^1 + y^3 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \lambda_a = y^1 + 3y^2 + 2y^3 \\ \lambda_b = y^1 - 3y^2 + 2y^3 \\ \lambda_c = y^1 + 0y^2 - y^3 \end{cases}$$

$$\lambda_\mu = \frac{1}{n_\mu} \sum_{i=1}^{r} m_i \chi_\mu([c_i]) y^i$$

$\chi_a = n_a (1, 1, 1)$ $\longrightarrow$ $n_a = 1$

$\chi_b = n_b (1, -1, 1)$ $\quad n_b = 1$

$\chi_c = n_c (1, 0, -\frac{1}{2})$ $\quad n_c = 2$

	$[1]$	$3[(12)]$	$2[(123)]$
1^+	1	1	1
1^-	1	-1	1
2	2	0	-1

C_2 is already a CSCO by itself.