Group theory and its application (in Physics)

0. Logistics:

1. 教学线下: 勇 A-202 (三) 14:00 - ~~16:00~~ 15:40

   A-201 (五)

2. ① 教学云: TFTJH2N2

   ② QQ: 729571175

   ③ yilu.me/teaching

3. 成绩: 40 + 60% ~10次

4. 教材参考:

   ① Greg Moore (Rutgers)

   赵导的

http://www.physics.rutgers.edu/~gmoore/618Spring2022/GroupTheory-Spring2022.html

   ② Zee, GT in nutshell, PUP

   Dresselhaus, GT, springer

   Ramond, GT, cambridge UP.

   ③ 中文教材:

   马中骐、陶瑞宝

   陈金全《群表示论的新途径》

1. Introduction

19th Galois. "groupe" group

roots of polynomial $P(x)$

order <5 radical solution

>5 not necessarily

杨振宁 二十世纪 量子化. 对称性. 相因子.

20th group $\Leftrightarrow$ symmetry

$\Updownarrow$

conservation

① General relativity Diff(M)

special relativity. $(t, x, y, z) \to t^2 - x^2 - y^2 - z^2$

$O(1, d)$ Lorentz group

② Lie algebra, Jacobi identity

$$[T^A, T^B] = i\epsilon^{ABC} T^C$$

$$[T^A, [T^B, T^C]] + [T^B, [T^C, T^A]] + [T^C, [T^A, T^B]] = 0$$

$$\Downarrow$$

$SU(2)$, $SO(3)$, $Sp(2n)$

$\uparrow$ spin, $\uparrow$ H atom, $\uparrow$ $\{q_i, p_j\} = \delta_{ij}$

③ Standard model $\underbrace{U(1) \times SU(2)}_{\text{electroweak}} \times \underbrace{SU(3)}_{\text{strong}}$

4-dim, 8-dim, 8-gluons

photon, $2\times W$, Z

④ Condensed matter.

$\begin{cases} \text{A. translation sym.} & (T^n = 1) \longleftrightarrow \text{momentum} \\ \text{B. rotation} & . \quad 32 \text{ point group} \end{cases}$

A + B → space group 17 wallpaper group 2D

230 in 3D

atomic spectra, lattice vibration, bandstructure

SC

2. Groups: Basic definitions & Examples

Definition: A group is a quartet $(G, \underline{m}, \underline{I}, e)$ where

1. G is a set.
2. $\underline{m}: \underline{G} \times \underline{G} \to \underline{G}$. multiplication map
3. $\underline{I}: G \to G$ inverse map
4. $e \in G$. identity element.

They satisfy conditions:

1 (associativity)

$$\underline{m}(\underline{m}(g_1, g_2), g_3) = \underline{m}(g_1, \underline{m}(g_2, g_3))$$

$$(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$$

not associative : Octonian

$$e_i e_j = \begin{cases} e_i & i=0 \\ e_j & j=0 \\ -\delta_{ij} e_0 + \varepsilon_{ijk} e_k & \text{otherwise} \end{cases}$$

$$(e_i e_j) e_k = - e_i (e_j e_k)$$

2. (existence of e) $\exists\, e$. s.t. $\forall g$.

$$e \cdot g = g \cdot e = g$$

3. (existence of $\underline{I}$) $\forall g \in G$. $\exists\, \underline{I}(g) =: g^{-1}$. s.t.

$$g \cdot g^{-1} = g^{-1} \cdot g = e$$

Remarks .

1. e, $\underset{\mathrm{O}}{\underset{\mathrm{U}}{1_G}}$, g_0 .

2. related defs:

$$\underbrace{\underbrace{\underbrace{\text{associtivity}}_{\text{semigroup / monoid.}} , \quad \exists\, e ,}_{\text{unital monoid}} \quad \exists\, g^{-1}}_{\text{group}}$$

3. topological group. if G is a topological space

$\underline{m}$, $\underline{I}$ continuous

Lie group. G is a manifold (locally homeomorphic to $\mathbb{R}^n$)

$\underline{m}, \underline{I}$ real analytic in local coord.

4. $G \Longleftrightarrow (G, \underline{m}, \underline{I}, e)$

Exercise:

a. e unique?

$e_1 = e_1 e_2 = e_2$

b. a^{-1} unique?

$ab = ba = ac = ca = e$

$b = b(ac) = (ba)c = c$

Examples:

1. $G = \mathbb{Z}, \mathbb{R}$, or $\mathbb{C}$.

$\underline{m}(a,b) := a+b$

e? 0

$\underline{I}$? $a \to -a$

closed? ✓

$\underline{m}(a,b) := ab$

2. $G = \mathbb{R}^* := \mathbb{R} - \{0\}$ ✓

$\mathbb{C}^* := \mathbb{C} - \{0\}$

$\mathbb{Z}^* := \mathbb{Z} - \{0\}$ $\quad 2 \to \frac{1}{2} \notin \mathbb{Z}$

Definition (subgroup)

(G, m, I, e) a group. set $H \subset G$

m, I preserve H. i.e.

$$m: H \times H \to H$$

$$I: H \to H$$

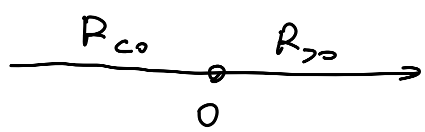
(H, m, I, e) is a subgroup of $(G \cdots)$

proper subgroup $H \neq G$ $H \subsetneq G$

Exercise:

1. $\mathbb{Z} \subset \mathbb{R} \subset \mathbb{C}$. subgroups m: "+"

2. $\mathbb{R}^*$.



$\mathbb{R}_{<0}$ subgroup of $\mathbb{R}^*$? ×

$\mathbb{R}_{>0}$ ✓

3. $\mathbb{R}_{<0}$: $\underline{m}(a, b) = -ab$

$a \cdot (-1) = a$ ✓

$\underline{a} \cdot \frac{1}{\underline{a}} = -1$ ✓

4. $H_1 \subset G$, $H_2 \subset G$.

(a) $H_1 \cap H_2$ subgroup ?

① $e \in H_1$ $e \in H_2$ $e \in H_1 \cap H_2$ ✓

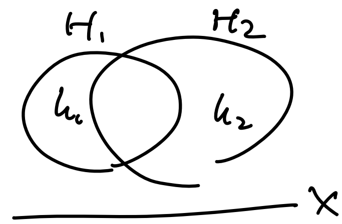
② $h \in H_1 \to h^{-1} \in H_1$ $\to$ $h \in H_1 \cap H_2$

$h \in H_2 \to h^{-1} \in H_2$ $\Downarrow$

$h^{-1} \in H_1 \cap H_2$

③ $\forall h_1, h_2 \in H_1 \cap H_2$ $\quad h_1 h_2 \in H_1$, $h_2 h_1 \in H_2$ $\rightarrow h_1 h_2 \in H_1 \cap H_2$

(b) $H_1 \cup H_2$ subgroup ?



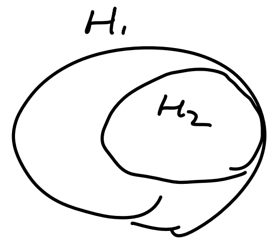
if $\exists h_1 \in H_1 \quad \notin H_2$

$\exists h_2 \in H_2, \notin H_1$

$h_3 = h_1 \cdot h_2 \in H_1 \cup H_2 \quad$ WLOG $. \in H_1$

Contra.

$h_2 = h_1^{-1} \cdot h_3 \in H_1$



$\Rightarrow H_1 \subset H_2$ or $H_2 \subset H_1$

<u>Definition</u> (order of a group) $|G|$ cardinality of G.

finite group $\quad |G| < \infty$

infinite group $\quad$ otherwise

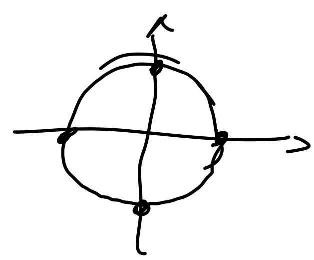
<u>Example</u>: N^{th} roots of unity

$\mu_N = \{1, \omega, \cdots \omega^{N-1}\} \equiv \{z \in \mathbb{C} \mid z^N = 1\}$

$\omega = e^{2\pi i/N}$ for $N \in \mathbb{N}$

$|\mu_N| = N \quad \xrightarrow{N=\infty} \quad U(1) = e^{i\phi}$

$0 \leq \phi < 2\pi$



$\omega^i \omega^j = e^{i\frac{2\pi}{N}(i+j \bmod N)} \equiv \omega^k \qquad k = (i+j) \bmod N$

"equivalence" relation $\sim \qquad \underline{k \sim k + lN}$

Example: residue classes modulo N.

$1 \leq j < N-1 \quad \bar{j} \equiv [j] = \{ n \in \mathbb{Z} \mid j = n \bmod N \}$

$\underline{m}(\bar{r}_1, \bar{r}_2) := \overline{r_1 + r_2}$

$\mathbb{Z}_N$, or $\mathbb{Z}/N\mathbb{Z}$

$P: x \to -x$

$P = \pm 1 \qquad \mathbb{Z}_2$

Definition: (direct product of groups) $G_1 \times G_2$, or $G_1 \otimes G_2$

$$\underline{m}_{G_1 \times G_2}((g_1, g_2), (g_1', g_2')) = (m_{G_1}(g_1, g_1'), m_{G_2}(g_2, g_2'))$$

$$\left(\begin{array}{l} g_1, g_1' \in G_1 \\ g_2, g_2' \in G_2 \end{array} \right)$$

Example: Klein's Viergruppe (4-group) $\mathbb{Z}_2 \times \mathbb{Z}_2$

$I = (1, 1)$, $a_1 = (-1, 1)$, $a_2 = (1, -1)$, $a_3 = (-1, -1)$

so far, $ab = ba$.

Definitions (Abelian & non-Abelian groups)

$\forall a, b \in G$. $a \cdot b = b \cdot a$ Abelian

$\exists a, b \in G$. s.t. $a \cdot b \neq b \cdot a$ non-Abelian

Abelian: $m(a, b) := a \underline{+} b$

$e = 0$

Example . (The general linear group)

$M_n(K)$: all $n\times n$ matrices defined on field K $(\mathbb{R}, \mathbb{C})$

$$GL(n,K) := \{A \in M_n(K) \mid A \text{ non-singular} = \text{invertible} = \det A \neq 0\}$$

in general $AB \neq BA$ $(n \geq 2)$

Definition (center of a group) $Z(G)$

$$Z(G) := \{z \in G \mid zg = gz, \ \forall g \in G\}$$

↳ $Z(G)$ is an Abelian subgroup of G.

$Z(GL(n,K)) = \{\lambda \mathbb{1}_n, \ \lambda \in K^*\}$ (Moore notes p. 13)

Examples : standard matrix groups

1 Special linear group

$$SL(n,K) = \{A \in GL(n,K) \mid \det A = 1\}$$

2. (special) orthogonal groups

$$O(n,K) := \{A \in GL(n,K) \mid AA^T = \mathbb{1}\}$$

$$SO(n,K) := \{A \in O(n,K) \mid \det A = 1\}$$

$AA^T = 1 \Leftrightarrow A^T A = 1$

$\hookrightarrow A^T = A^{-1} \to A^T A = 1$

$\det(AA^T) = (\det A)^2 = 1 \quad \det A = \pm 1$

3. (special) unitary group $\quad A^+ = (A^T)^*$

$U(n) = \{A \in GL(n, \underline{\mathbb{C}}) \mid \underline{AA^+ = 1}\}$

$SU(n) = \{A \in U(n) \mid \det A = 1\}$

$\det(AA^+) = (\det A)(\det A^*) = 1$

$|\det A| = 1$

def. by bilinear form: $\quad A\hat{\ } A^T = 1$ generalized.

4 indefinite orthogonal group.

$O(p,q) := \{A \in GL(p+q, \underline{\mathbb{R}}) \mid A^T J_{p,q} A = J_{p,q}\}$

$$J_{p,q} = \left(\begin{array}{c|c} -1_p & 0 \\ \hline 0 & 1_q \end{array}\right)$$

Lorentz group $O(1,d)$ $\quad d+1$ space-time

5 Symplectic group.

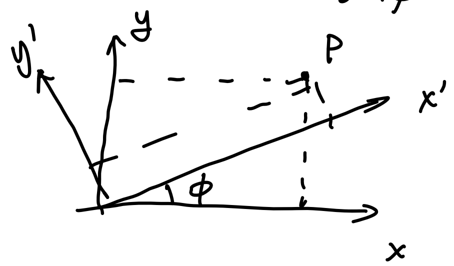
$Sp(2n, K) := \{A \in GL(2n, K) \mid A^T J A = J\}$

$$J = \left(\begin{array}{c|c} 0 & 1_n \\ \hline -1_n & 0 \end{array}\right) \qquad \underline{J = J^* = -J^T = -J^{-1}}$$

Remarks :

1. $SO(2,\mathbb{R}) \ni g = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ $a^2+b^2=1$

$R(\phi_1)R(\phi_2)=R(\phi_1+\phi_2) \leftarrow R(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \equiv e^{\phi J}$ $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

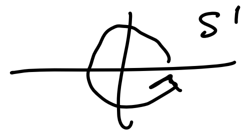


$x' = \cos\phi\, x + \sin\phi\, y$

$y' = -\sin\phi\, x + \cos\phi\, y$

2. $U(1)$: $z(\phi) = e^{i\phi}$ $z(\phi_1)z(\phi_2) = z(\phi_1+\phi_2)$

$SO(2)$ " $\simeq$ " $U(1)$



μ_N " $\simeq$ " $\mathbb{Z}_N$

3. $SU(2)$ $g = \begin{pmatrix} z & -w^* \\ w & z^* \end{pmatrix}$ $|z|^2+|w^2|=1$

$z = x_0 + i x_1$
$w = x_2 + i x_3$ $\sum_{i=0}^{3} x_i^2 = 1$ $\rightarrow S^3$

4. $SU(3)$ 8-dim

" S^3 - bundle over S^5 "

5. $Sp(2n,K)$. $A^T J A = J$ $(\det A)^2 = 1 \Rightarrow \det A = \pm 1$

"Pfaffian": $Pf(A^T J A) = (\det A)\cdot Pf(J) \Rightarrow \det A = 1$
$\quad\quad\quad\;\; \| $
$\quad\quad\; Pf(J)$