Recap: Induced rep. (ρ, V) rep of H $\longrightarrow G$.

$G \times H$ on $\mathrm{Map}(G \to V)$

$$(g, h)\Phi(g_0) = \rho(h)\Phi(g^{-1} g_0 h)$$

H-equivariant Φ's. : fixed points of $\{1\} \times H$

$$(1, h)\Phi(g_0) = \rho(h)\Phi(g_0 h) = \Phi(g_0)$$

$$\Rightarrow \Phi(gh) = \rho(h^{-1}) \cdot \Phi(g)$$

$$\begin{tikzcd} G \arrow[r, "\Phi"] \arrow[d, "R(h)"'] & V \arrow[d, "\rho(h^{-1})"] \\ G \arrow[r, "\Phi"] & V \end{tikzcd}$$

$$\mathrm{Ind}_H^G V := \{\Phi : G \to V \mid \Phi(gh^{-1}) = \rho(h)\Phi(g) \quad \forall g \in G,\ h \in H\}$$

$$(g \cdot \Phi)(g_0) := \Phi(g^{-1} g_0)$$

$$\Phi(gh) = \rho(h^{-1}) \cdot \Phi(g)$$

$\mathrm{Supp}(\Phi)$ are left cosets

$$V_C = \{\Phi : G \to V \mid \mathrm{Supp}(\Phi) = C\} \subset \mathrm{Ind}\, V$$

① $V_c \cong V \qquad c \in \{c_i\}_i$

$ev_c : \Phi \longmapsto \Phi(g_c) \in V.$

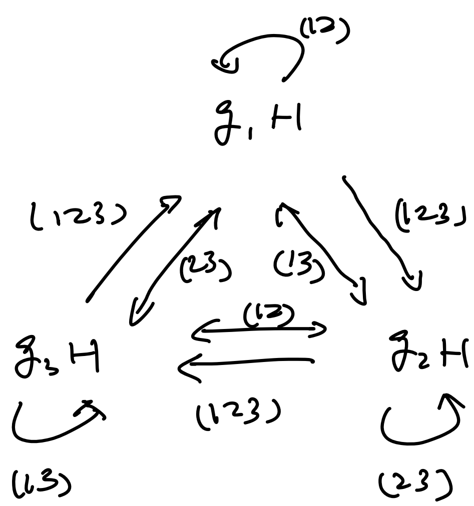
② $\mathrm{Ind}_H^G V \cong \bigoplus_c V_c \qquad \dim \mathrm{Ind}_H^G V = [G:H] \dim V$

③
$$\begin{tikzcd}
V_c \arrow[r, "(\rho_{\mathrm{Ind}V}(g))"] \arrow[d, "ev_c"', "\wr"] & V_{c'=g\cdot c} \arrow[d, "ev_{c'}", "\wr"'] \\
V \arrow[r, "(\rho_V(g_{c'}^{-1} g g_c))"'] & V
\end{tikzcd}$$

$$\chi_{\mathrm{Ind}V}(g) = \sum_{gc=c} \chi_V(g_c^{-1} g g_c)$$

$S_2 \xrightarrow{\mathrm{Ind}} S_3 :$

$\{e, (12)\}$



$\{(23), (123)\}$ $\qquad$ $\{(13), (123)\}$

$V = \mathbb{C}. \quad \longrightarrow \quad \dim \mathrm{Ind} V = 3$

$$f_1((12)) = f_2((13)) = f((23)) = \underline{\underline{1}}$$

$$f_1(e) = \underline{\rho_V((12))}\, f_1((12)) = \epsilon\, \cancel{f((12))}^{1}$$

$$f_2((123)) = \epsilon$$

$$f_3((132)) = \epsilon.$$

$$(g \cdot f_i) = ? \qquad \Rightarrow$$

$$\mathrm{Ind}_{S_2}^{S_3}(V(\epsilon)) = V(\epsilon) \oplus W_2$$

– Frobenius reciprocity

$$\mathrm{Rep}(G) \underset{\mathrm{Ind}_H^G}{\overset{\mathrm{Res}_H^G}{\rightleftarrows}} \mathrm{Rep}(H)$$

Theorem (F. R.) V rep H. W rep of G.

$$\mathrm{Hom}_G(W, \mathrm{Ind}_H^G V) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G W, V)$$

$$\begin{tikzcd}
W \arrow[r, "\varphi"] \arrow[d, "\rho_W"'] & \mathrm{Ind}_H^G V \arrow[d, "\rho_{\mathrm{Ind}}"] \\
W \arrow[r, "\varphi"] & \mathrm{Ind}_H^G V
\end{tikzcd}
\quad \longleftrightarrow \quad
\begin{tikzcd}
\mathrm{Res}_H^G W \arrow[r, "\phi"] \arrow[d, "\rho_W"'] & V \arrow[d, "\rho_V"] \\
\mathrm{Res}_H^G W \arrow[r, "\phi"] & V
\end{tikzcd}$$

Proof: ① given $\varphi: w \longmapsto \varphi(w) \in \mathrm{Ind}\, V.$

define $\phi: w \longmapsto (\varphi(w))(e) \in V$

show: $\underline{\rho_V(h)\cdot\phi(w)} = \underline{\phi(\rho_W(h)\cdot w)}$ check

② Inverse: given ϕ.

$W \longrightarrow \mathrm{Ind}\, V$.

define $\varphi: w \longmapsto (g \mapsto \phi(\rho_W(g^{-1})\, w))$

show: $\underline{[\rho_{\mathrm{Ind}}(g_1)\cdot\varphi(w)](g_2)}$

$= \underline{\varphi(\rho_W(g_1)\cdot w)(g_2)}$ check recall $\langle \chi_U \ \chi_V \rangle = \dim \mathrm{Hom}_G(U, V)$

↳ Corollary :

$$\langle \chi_{\mathrm{Ind}_H^G V}, \chi_W \rangle = \langle \chi_V, \chi_{\mathrm{Res}_H^G W} \rangle$$

$$\mathrm{Ind}_H^G V \cong \bigoplus_\mu W^\mu \otimes \mathrm{Hom}_G(W^\mu, \mathrm{Ind}_H^G V)$$

$$\overset{\text{F.R.}}{\cong} \bigoplus_\mu W^\mu \otimes \mathrm{Hom}_H(\mathrm{Res}_H^G W^\mu, V)$$

If V is an irrep of H

if $H \triangleleft G$, V irrep. $\xrightarrow{\text{some condition}}$ $\mathrm{Ind}_H^G V$. is an irrep of G

S_n irrep $\longrightarrow$ S_{n+1} irrep. $\longrightarrow$

Ref: ① Hamermesh. GT and application to physical problems (Dover)

② Fulton & Harris (GTM 129) Rep. theory Lec 4.

Example: $\mathrm{Ind}_{S_2}^{S_3} V$

$\rho_{\mathrm{Ind}}(e) = \mathbb{1}_3$ $\quad [(12)] = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & 0 & \epsilon \\ 0 & \epsilon & 0 \end{pmatrix}$ $\quad [123] = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

χ_{Ind} $\quad 3 \quad \epsilon \quad 0$

$W = W_3(1^+),\ W_3(1^-),\ W_3(2)$

$\mathrm{Res}^G_H W = V_2(1^+),\ V_2(1^-)$

$\mathrm{Ind}^G_H V = \bigoplus_\mu a_\mu W^\mu \qquad a_\mu = \langle \chi_V, \chi_{\mathrm{Res}^G_H W^\mu} \rangle$

$a_{W_3(1^\pm)} = 1$

$a_{W_3(2)} = \frac{1}{2}(2+0) = 1$

$\Rightarrow \underline{\mathrm{Ind}^G_H V(\epsilon) = W_3(\epsilon) \oplus W_3(2)}$

- Rep of SU(2) (induced)

$G = SU(2) \qquad H = D = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \cong U(1) \qquad \theta \in [0, 2\pi)$

rep of $U(1)$ $(\hat{U}(1) = \mathbb{Z})$

$\rho_k(\theta) = e^{ik\theta} \quad (k \in \mathbb{Z})$

$\mathrm{Ind}^G_H \rho_{-k} : \quad \Phi\Big[\underbrace{\begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix}}_{g} \underbrace{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}}_{h}\Big] = \underbrace{e^{ik\theta}}_{\rho_k(h^{-1})} \Phi\left(\begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix}\right)$

$\underline{\Phi(ue^{i\theta}, ve^{i\theta}) = e^{ik\theta}\Phi(u,v)}$

$\Phi(u,v) = \underline{u^k \cdot |v|^l}$

Ind ρ_{-k} is inf. dim.

Consider only the holomorphic sector.

$\hookrightarrow$ homogeneous polynomials $u^{k-i} v^{i}$

$$\mathcal{H}_k = \text{span}\{ u^k, u^{k-1}v, \dots uv^{k-1}, v^k \}$$

$$g \cdot \Psi(g_0) = \Psi(g^{-1} g_0) = \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} u & -\bar{u} \\ v & \bar{v} \end{pmatrix}$$

$\downarrow$

α, β

$$= \begin{pmatrix} \bar{\alpha} u + \bar{\beta} v & — \\ -\beta u + \alpha v & — \end{pmatrix}$$

$$(g \cdot \Psi)(u, v) = \Psi(\bar{\alpha} u + \bar{\beta} v, -\beta u + \alpha v)$$

$$\underset{\alpha}{|u|^2} + \underset{\beta}{|v|^2} = 1 \Longrightarrow | \quad |^2 + | \quad |^2 = 1$$

Connect with physics. $k = 2j$ j "spin"

$$V_j := \text{span}\{ \tilde{f}_{j,m}(u, v) := u^{j+m} v^{j-m}$$

$$m = -j, -j+1, \dots, j \}$$

$$\dim V_j = 2j+1 = k+1 = \dim \mathcal{H}_k$$

$$V_j \cong \mathcal{H}_{2j}$$

$$(g\cdot \tilde{f}_{j,m})(u,v) = \tilde{f}_{j,m}(\bar{\alpha}u+\bar{\beta}v, -\beta u+\alpha v)$$

$$= (\bar{\alpha}u+\bar{\beta}v)^{j+m}(-\beta u+\alpha v)^{j-m}$$

$$= \sum_{m'} \tilde{D}^{j}_{m'm}(g)\, \tilde{f}_{j,m'}(u,v)$$

Remarks: 1. consider $h \in H = D$ $h = \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}$

$$h\cdot \tilde{f}_{j,m} = \bar{\alpha}^{j+m}\alpha^{j-m}u^{j+m}v^{j-m}$$

$$= \alpha^{-2m}\tilde{f}_{j,m}$$

$$\tilde{D}^{j}_{m'm} = \delta_{m'm}\,\alpha^{-2m}$$

in physics: $\hat{J}_z$ diagonal in the $|j,m\rangle$

2.

$$\sum_{m'} \tilde{D}_{m'm}(g)\, \tilde{f}_{j,m'} = \sum_{s,t} \binom{j+m}{s}\binom{j-m}{t}\bar{\alpha}^{s}\bar{\beta}^{j+m-s}\cdot(-\beta)^{t}\alpha^{j-m-t}\, u^{s+t}v^{2j-s-t}$$

$$\Rightarrow \tilde{D}^{j}_{m'm}(g) = \sum_{s+t=j+m'} \binom{j+m}{s}\binom{j-m}{t}\bar{\alpha}^{s}\bar{\beta}^{j+m-s}(-\beta)^{t}\alpha^{j-m-t}$$

$$\tilde{D}^{j}_{m',-j}(g) = \binom{2j}{j+m'}\alpha^{j-m'}(-\beta)^{j+m'} \propto \tilde{f}_{j,-m'}(\alpha,\beta)$$

3. $\tilde{D}^j_{m'm}\left(f\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha}\end{pmatrix}\right) = \underline{(e^{i\alpha})^{-2m}}\, \tilde{D}^j_{m'm}(f)$

V_j : $j \geq |m|$ $\qquad$ H-equiv.: $\underline{-2m} = \underline{k}$

$\mathrm{Ind}^{SU(2)}_{U(1)}(\rho_{-k}) \cong V_{|k|/2} \oplus V_{|k|/2+1} \oplus \cdots$

V_j spanned by $\underline{\tilde{D}^j_{m',-j}}$ $\quad (m' = \underline{-j}, -j+1, \ldots \underline{j})$

$\dim = 2j+1$

4. $SO(3) \cong SU(2)/\mathbb{Z}_2$

($\pi : SU(2) \longrightarrow SO(3)$

$u \cdot \vec{x}\cdot\vec{\sigma}\, u^{-1} = (\pi(u)\,\vec{x})\cdot\vec{\sigma}$

$1 \longrightarrow \underline{\mathbb{Z}_2} \xrightarrow{\iota} SU(2) \xrightarrow{\pi} SO(3) \rightarrow 1$)

$\Rightarrow$ $g = \begin{pmatrix} -1 & 0 \\ 0 & -1\end{pmatrix}$ acts trivially

diagonal; acts as $\underbrace{(-1)^{-2m}}_{=1}\, 1\!\!1_{V_j}$

$\rightarrow$ m an integer

$\Rightarrow$ irreps of $SO(3)$ are given by

V_j with $j \in \mathbb{Z}$. $\qquad \dim_{\mathbb{C}} V_j = 2j+1$ odd.

5. Recall that a trivial rep of H.

induces function on $\underline{G/H}$.

$\underline{SU(2)/U(1)} \cong \underline{S^2 (\cong CP_1)}$

trivial rep of $U(1)$: $\rho_k = 1 \quad m=0$

$$\underline{\underline{\tilde{D}^j_{m0}}}(g) = \sum_{s+t=j+m} \binom{j}{s}\binom{j}{t} \bar{\alpha}^s \alpha^{j-t} \bar{\beta}^{j-s} (-1)^t$$

$$\left(\alpha = \cos\frac{\theta}{2} ,\ \beta = \sin\frac{\theta}{2} e^{i\phi} \right)$$

check. $\propto \underline{Y_j^m(\theta, \phi)}$

Characters of V_j

$$g \sim d(z) = \begin{pmatrix} z & 0 \\ 0 & z^{-1} = \bar{z} \end{pmatrix} \quad |z| = 1$$

$$\tilde{D}^j_{m'm}(g) = \delta_{m'm} z^{-2m}$$

$$\chi_j(g) = \sum_{m=-j}^{j} z^{-2m} = \underline{\frac{z^{2j+1} - z^{-2j-1}}{z - z^{-1}}} \quad \left(\text{even under } z \leftrightarrow z^{-1}\right)$$

Harr measure ?

$$[dg] = \frac{1}{16\pi^2} d\psi \, d\phi \, \sin\theta \, d\theta$$

$$\alpha = e^{i\frac{1}{2}(\phi+\psi)} \cos\theta/2$$

$$\beta = i e^{i\frac{1}{2}(\phi-\psi)} \sin\theta/2$$

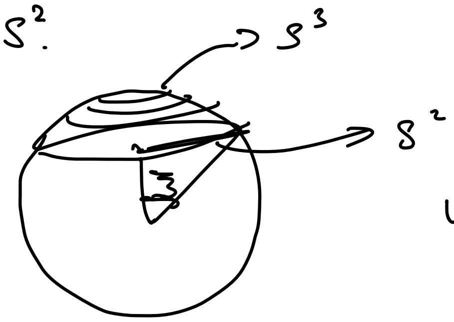
element specific

For a conjugacy class. $g = e^{i\xi \hat{n}\cdot\vec{\sigma}}$

$(= \cos\xi \, \mathbb{1} + i\sin\xi \, \hat{n}\cdot\vec{\sigma})$

$$g \sim d(\xi) = \begin{pmatrix} e^{i\xi} & 0 \\ 0 & e^{-i\xi} \end{pmatrix} \qquad \xi \in [0, \pi)$$

↪ think it as parametrization of S^3 by S^2.



$\mathrm{vol}(S^2) \sim \sin^2\xi$

class function F.

$$F\left(\begin{pmatrix} e^{i\xi} & 0 \\ 0 & e^{-i\xi} \end{pmatrix}\right) = f(\xi) \overset{z=e^{i\xi}}{=} g(z)$$

$$f(\xi) = f(-\xi)$$

$$\int_{SU(2)} F(u)[du] = \frac{2}{\pi}\int_0^{\pi} f(\xi)\sin^2\xi \, d\xi$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(\xi)\sin^2\xi \, d\xi$$

$$\overset{z=e^{i\xi}}{=} -\frac{1}{4\pi i}\oint f(z)\,(z-z^{-1})^2\,\frac{dz}{z}$$

$\Rightarrow \underline{\langle \chi_j , \chi_{j'} \rangle} = \underline{\delta_{jj'}}$ check.

$\Rightarrow$ V_j's are irreps.

$L^2(SU(2))^{\text{class}}$ spanned by χ_j's

$L^2(SU(2))$ spanned by $\tilde{D}$'s