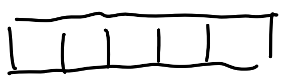
Recap.

$V = k^d$ a rep of $GL(d, k)$

$$V^{\otimes n} \cong \bigoplus_\lambda \underline{D_\lambda} \otimes \underline{V_\lambda} \qquad D_\lambda = \mathrm{Hom}_{S_n}(V_\lambda, V^{\otimes n})$$

irreps of $GL(d,k)$ ↗ ↖ irreps of S_n

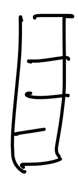
tensors projected into different symmetry sectors.

T: 

$c = \sum_{\sigma \in R(T)} \sigma \to \mathrm{Sym}^n V$

$= \mathrm{Span}(\sum v_{\sigma(i)} \otimes v_{\sigma(j)} \otimes \cdots)$

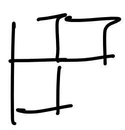
$\tilde{c}_{ijk} = \sum_\sigma c_{\sigma^{-1}(i)\, \sigma^{-1}(j)\, \sigma^{-1}(k)}$



$c = \sum \mathrm{sgn}(\sigma)\sigma \qquad \Lambda^n V$

$= \mathrm{Span}(v_i \wedge v_j \wedge k \cdots)$

$i < j < k < \cdots$



mixed. symmetry

Induced representations

Let $\underline{V}$ be a rep of G.

$H \subset G$ a subgroup

V restricts to be a rep of H: $\mathrm{Res}_H^G V$

What if we want to find rep. of

G using reps of $H \subset G$.

$$D \subset SU(2) \qquad d(z) = \begin{pmatrix} z & 0 \\ 0 & z^{+} \end{pmatrix} \in D \cong U(1)$$

$$\underline{\rho_k(d) = z^k}$$

Known data:

- G.

- $H \subset G$ subgroup.

- a rep H: (ρ, V)

$$\rho : H \to GL(V)$$

Consider H-equivariant maps $Map(G, V) \ni \Psi$

$$
\begin{tikzcd}
G \arrow[r, "\Psi"] \arrow[d, "R(h)"'] & V \arrow[d, "\rho(h^{-1})"] \\
G \arrow[r, "\Psi"] & V
\end{tikzcd}
$$

equivalently:

$$
\begin{tikzcd}
G \arrow[r, "\Psi"] \arrow[d, "L(h)"'] & V \arrow[d, "\rho(h)"] \\
G \arrow[r] & V
\end{tikzcd}
$$

$\Psi(hg) = \rho(h) \cdot \Psi(g)$

$R(h): g \mapsto gh$

$$\underline{\Psi(gh) = \rho(h^{-1}) \Psi(g)} \qquad (\forall g \in G,\ h \in H)$$

Consider group actions of $G \times H$ on $Map(G, V)$

$$(g, h) \Psi(g_0) = \rho(h) \cdot \Psi(g^{-1} g_0 h)$$

then the Ψ's are fixed points of $\underline{\{1\} \times H}$

$$h \cdot \Psi(g_0) = \rho(h) \cdot \Psi(g_0 h) = \rho(h) \cdot \rho(h^{-1}) \Psi(g_0)$$
$$= \Psi(g_0)$$

$$\mathrm{Ind}_H^G V := \{ \Psi : G \to V \mid \Psi(gh^{-1}) = \rho(h) \Psi(g), \ \forall g \in G,\ \forall h \in H \}$$

Define the group action of g

$$(\underline{g \cdot \Psi})(g_0) = \Psi(g^{-1} g_0)$$

$\Rightarrow \quad \mathrm{Ind}_H^G(\oplus_i V_i) \overset{u}{=} \oplus_i \mathrm{Ind}_H^G V_i$

<u>Example</u> $\quad V = \mathbb{C} \quad$ trivial rep of any $\underline{H}$.

$$\rho(h) = 1 \quad (\forall h \in H)$$

$$\mathrm{Ind}_H^G V = \{ \Psi : G \to \mathbb{C} \mid \Psi(\underline{gh^{-1}}) = \overset{1}{\cancel{\rho(h)}} \underline{\Psi(g)} \quad \forall h \in H \}$$

$\Psi(G)$ desends to functions on $\underline{G/H}$.

If $H = \{e\}$. $\quad \underline{\mathrm{Ind}_{\{e\}}^G (V)} = L^2(G)$

$$g \cdot \underline{\Psi}(g_0) := \Psi(g^{-1} g_0)$$

before: $\quad g \underline{\delta}_h(g_0) = \underline{\delta_h(g^{-1} g_0)} = \underline{\delta_{gh}(g_0)}$

$$\underline{g\delta_h = \delta_{gh}}$$

Let G be a <u>finite group</u>.

First notice : $\quad g \in \mathrm{Supp}(\Psi) \quad (\Psi(g) \neq 0)$

$$\Psi(gh) = \underline{\rho(h^{-1})} \cdot \Psi(g) \neq 0$$

$gh \in \mathrm{Supp}(\Psi)$

The support of Ψ is a union of left cosets

If we fix coset representatives g_i s.t.

$$G = \coprod_i C_i \qquad (C_i = g_i H \in G/H.)$$

$$\Phi(g) = \rho(h)^{-1}\, \Phi(g_i) \qquad (g = g_i h,\ \exists h \in H)$$

Any $\Phi(g)$ is fixed by $\{\Phi(g_i)\}_i$

Define $V_C := \{\Phi : G \to V \mid \mathrm{supp}(\Phi) = C\} \subset \mathrm{Ind}\, V$

Lemma 1: As vector spaces $V_C \cong V$ $\quad C \in \{C_i\}_i$

isomorphism defined as $ev_C : \Phi \mapsto \Phi(g_C)$

Proof: $\Phi \in V_C$ uniquely determined by $\Phi(g_C)$

inverse of the map:

$$V \longrightarrow V_C$$
$$v \longmapsto \Phi(g_C h^{-1}) = \rho_v(h) \cdot v$$

Lemma 2. There is a natural isomorphism

$$\bigoplus_C V_C \longrightarrow \mathrm{Ind}_H^G V.$$

$$(\Phi_{C_1}, \Phi_{C_2}, \cdots \Phi_{C_r}) \longmapsto \Phi := \sum_{\{C_i\}} \Phi_{C_i}$$

$$\dim \mathrm{Ind}_H^G V = [G:H] \dim V_c$$
$$= [G:H] \dim V.$$

Proof. define inverse : $\forall \Psi \in \mathrm{Ind}_H^G V$. define

$$\Psi_c(g) = \begin{cases} \Psi(g) & g \in g_c H \\ 0 & \text{otherwise} \end{cases}$$

Lemma 3. $\forall g \in G$.

$\tilde{\rho}(g) := \rho_{\mathrm{Ind}_H^G}(g) : \mathrm{Ind}_H^G V \longrightarrow \mathrm{Ind}_H^G V$

restricts to isomorphism :

① $\tilde{\rho}(g)|_{V_c} : V_c \longrightarrow V_{gC}$

In addition:

$$\begin{tikzcd} V_c \arrow[r, "\tilde{\rho}"] \arrow[d, "ev_c"'] & V_{gC} \arrow[d, "ev_{gC}"] \\ V \arrow[r, "\rho_V(g_{c'}^{-1} g g_c)"'] & V \end{tikzcd} \quad (C' = gC)$$

Proof. ① Let $\Psi \in V_c$

$$(\tilde{\rho}(g)\Psi)(g') = \Psi(g^{-1}g')$$

is 0 unless $g^{-1} \cdot g' \in g_c H$

$g' \in g \cdot C \ (=: C')$

$\Rightarrow \mathrm{Supp}(\tilde{\rho}(g)\Psi) = C'$

Check: define an inverse map to show it's an isomorphism.

$$\Psi \in V_{gC} \Rightarrow \tilde{\rho}(g^{-1})\Psi \in V_C$$

show:

$$\begin{tikzcd}
V_C \arrow[r, "\tilde{\rho}"] \arrow[d, "ev_c"'] & V_{gC} \arrow[d, "ev_{gc}"] \\
V \arrow[r, "\rho_V(g_{c'}^{-1} g g_c)"'] & V
\end{tikzcd}$$

$$g_{c'} \in g\cdot C \Rightarrow g_{c'} \overset{(\exists h)}{=} g\cdot g_c \cdot h$$

$$\Rightarrow g_{c'}^{-1}\cdot g \cdot g_c = h^{-1} \in H$$

$\rightharpoondown$: $ev_{c'} \circ \tilde{\rho}(g)\Psi = (\tilde{\rho}(g)\Psi)(g_{c'})$

$$= \Psi(g^{-1} g_{c'})$$

$\hookrightarrow$: $\rho_V(g_{c'}^{-1} g g_c) \circ ev_c(\Psi)$

$$= \rho_V(g_{c'}^{-1} g g_c)\, \Psi(g_c)$$

$$= \Psi(g_c \cdot (g_{c'}^{-1} g g_c)^{-1})$$

$$= \Psi(g^{-1} g_{c'})$$

Above lemmas tell us:

① dim of $\mathrm{Ind}_H^G V$.

② How $\tilde{\rho}(g)$ is defined.

<u>Characters</u> : $\mathrm{Ind}_H^G V \cong \oplus V_c$

$$\begin{tikzcd} V_c \arrow[r, "\tilde{\rho}"] \arrow[d, "ev_c"'] & V_c \arrow[d, "ev_c"] \\ V \arrow[r, "\rho_V(g_c^{-1} g g_c)"'] & V \end{tikzcd} \qquad (c' = c)$$

$$\tilde{\chi}(g)\big|_{V_c} = \chi_V(g_c^{-1} g g_c)$$

$$\tilde{\chi}(g) = \sum_{c : gC = C} \chi_V(g_c^{-1} g g_c)$$

<u>Example</u> :

1. Let (U, ρ_U) be reg. rep. of H with basis

$$\delta_h(h') = \begin{cases} 1 & h = h' \\ 0 & \text{otherwise} \end{cases}$$

Define $\Psi_g \in \mathrm{Ind}_H^G : G \to V$

- $$\Psi_g(g') = \begin{cases} \delta_h & \text{if } g' = g h^{-1} \text{ for } h \in H \\ 0 & \text{otherwise} \end{cases}$$

- $$\Psi_{g_1}(g_2 h_2) \overset{?}{=} \rho(h_2^{-1}) \cdot \Psi_{g_1}(g_2)$$

$g_2 = g_1 \cdot h_1^{-1} \cdot h_2^{-1}$
$= g_1 \cdot (h_2 h_1)^{-1}$

Let $g_2 \cdot h_2 = g_1 \cdot h_1^{-1}$ $\qquad LHS = \delta_{h_1}$

$$RHS = \delta_{h_2^{-1}} \cdot \delta_{h_2 h_1} = \delta_{h_1} = LHS$$

g action: $[g_1 \cdot \Phi_{g_2}](g_0) = \Phi_{g_2}(g_1^{-1} g_0) = \Phi_{g_1 g_2}(g_0)$

$\exists h$ s.t. $g_1^{-1} g_0 = g_2 \cdot h^{-1} \Rightarrow g_0 h = g_1 \cdot g_2$

$$\Rightarrow \underline{g_1 \cdot \Phi_{g_2} = \Phi_{g_1 g_2}} \qquad (g_1 \cdot \delta_{g_2} = \delta_{g_1 g_2})$$

This is the reg. rep. of G.

2. Induced rep of S_3 from S_2

$G = S_3 = \{e, (12), (13), (23), (123), (132)\}$

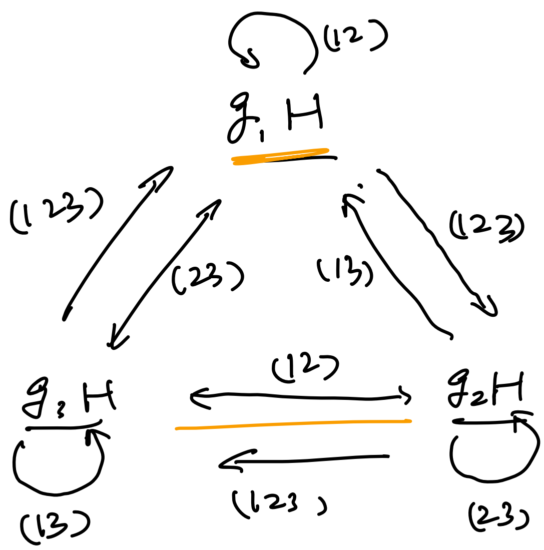
$H = S_2 = \{e, (12)\}$

H: $\rho((12)) = \pm 1$ $\qquad \underline{V(\epsilon) \quad (\epsilon = \pm 1)}$ $\qquad V = \mathbb{C}$

The cosets:

$G = \{e, (12)\} \cup \{(13), (123)\} \cup \{(23), (132)\}$

$g_1 = (12) \qquad g_2 = (13) \qquad g_3 = (23)$



$\mathrm{Ind}_H^G V(\epsilon)$

$= V_1 \oplus V_2 \oplus V_3$

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$\Phi(gh^{-1}) = \rho_V(h)\Phi(g)$ therefore

on V_1 : $\Phi_1(e) = \rho_{\mathbb{C}}((12))\Phi_1((12)) = \epsilon\,\Phi((12))$

V_2 : $\Phi_2((123)) = \Phi((13)(12)) = \epsilon\,\Phi_2((13))$

V_3 : $\Phi_3((132)) = \Phi((12)(13)) = \epsilon\,\Phi_3((23))$

We choose $\underline{\Phi_1((12)) = \Phi_2((13)) = \Phi_3((23)) = 1}$

Now try to find the rep.

① (123) :

a. $[(123)\Phi_1](g) = \Phi_1[(123)g]$

($\mathrm{Supp}((123)\Phi_1) = g_2 H = \{(13), (123)\}$

$[(123)\Phi_1][(13)] = \Phi_1[(12)] = 1 = \Phi_2[(13)]$

$[(123)]$ $= \epsilon = \Phi_2[(123)]$

$\Rightarrow (123)\Phi_1 = \Phi_2$

b. $(123)\Phi_2 = \Phi_3$

c. $(123)\Phi_3 = \Phi_1$

Check

$$\tilde{\rho}[(123)] = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\tilde{\chi}(g) = 0 = \underline{\sum_{c:\, c = gc}} \chi_V(g_c^{-1} g g_c)$$

② (12)

$$[(12)\Psi_1] = \epsilon \Psi_1$$

$$[(12)\Psi_2] = \epsilon \Psi_3$$

$$[(12)\Psi_3] = \epsilon \Psi_2$$

$$\tilde{\rho}[(12)] = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & 0 & \epsilon \\ 0 & \epsilon & 0 \end{pmatrix}$$

$$\tilde{\chi}[(12)] = \epsilon \overset{?}{=} \sum_{gc=c} \chi_V(g_c^{-1} g g_c) = \chi_V[(12)] = \epsilon$$

$$\tilde{\rho}(e) = \mathbb{1}_3 \qquad \tilde{\rho}[(12)] = \begin{pmatrix} \epsilon & & \\ & 0 & \epsilon \\ & \epsilon & 0 \end{pmatrix} \qquad \tilde{\rho}[(123)] = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\tilde{\chi} \qquad 3 \qquad \epsilon \qquad 0$$

$$\langle \tilde{\chi}, \tilde{\chi} \rangle = \frac{1}{6}(3^2 + 3\cdot\epsilon^2 + 2\cdot 0) = 2$$

$$\mathrm{Ind}_{S_2}^{S_3}(V(\epsilon)) = V(\epsilon) \oplus W_2$$

$$\uparrow \qquad \uparrow$$

$$\dim \quad 1 + 2 = 3$$

- Frobenius reciprocity

$$\mathrm{Rep}\, H \underset{\mathrm{Res}_H^G}{\overset{\mathrm{Ind}_H^G}{\rightleftarrows}} \mathrm{Rep}\, G.$$

Theorem (Frobenius)

Let V be a rep of H. $(H \subset G)$

W be a rep of G

then

$$\mathrm{Hom}_G(W, \mathrm{Ind}_H^G V) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G W, V)$$

Next lecture

$$\mathrm{Ind}_H^G V \cong \bigoplus W^\mu \otimes \underline{\mathrm{Hom}_G(W^\mu, \mathrm{Ind}_H^G V)}$$

$$\bigoplus W^\mu \otimes \underline{\mathrm{Hom}_H(\mathrm{Res}_H^G W^\mu, V)}$$

$$\langle \chi_{\mathrm{Ind}_H^G V}, \chi_W \rangle = \langle \chi_V, \chi_{\mathrm{Res}_H^G W} \rangle$$