Recap

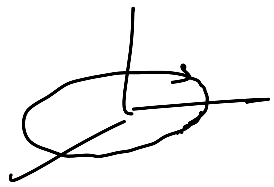
1. invariant subspace. $W \subset V$.

$\forall v \in W$. $\forall g \in G$.

$T(g) \cdot v \in W$.

$\left(\forall v, u \in W. \quad \alpha v + \beta u \in W. \quad \alpha, \beta \in K \right)$

$\mathbb{R}^3$ under $SO(2)$



$z \neq 0$ plane $\neq W$

$\mathbb{R}^n$. S_n.

$T(\phi) \vec{e}_i = \vec{e}_{\phi(i)}$

$\vec{v} = \sum \vec{e}_i$ invariant. subspace 1D

M_{ij} elements of mat. rep.

$f : G \to K$

$R_i = \{ M_{ij}, \text{ fix } i, \ j = 1, \cdots n \}$

$L_i = \{ \quad \text{fix } j. \ i = 1, \cdots n \}$

$R_i L_i \to G \times G$.

V has an invariant subspace. W.

$$(T, V) \xrightarrow{\text{restrict}} (T|_W, W)$$

(T, W) subrep.

reducible rep. V has a proper, nontrivial W. $(W \neq V,\ (W \neq 0))$

not reducible $\rightarrow$ irreducible "irrep"

$\forall v \in V$. $\operatorname{span}\{T(g)\cdot v, \forall g \in G\} = W$.

$$T(g_1)(T(g_2)v) = T(g_1 g_2)v.$$

V irreducible $\Rightarrow$ v cyclic

(T, W). W $V \backslash W$.

$$M(g) = \begin{pmatrix} \boxed{M_{11}(g)} & M_{12}(g) \\ 0 & M_{22}(g) \end{pmatrix}$$

complete reducibility:

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$$

$$M(g) = \begin{pmatrix} M_{11}(g) & & \\ & M_{22}(g) & \\ & & \ddots \end{pmatrix}$$

$G = \mathbb{Z}_2$. one-dim. $\rho_+(1) = \rho_+(-1) = 1$

$\rho_-(1) = 1 \quad \rho_-(-1) = -1$

two-dim $M(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$M(-1) = M((12)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

diag $\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$(T, V) \cong \rho_+ \oplus \rho_-$

$G = U(1) \qquad \rho_n(z) = z^n \qquad \forall n \in \mathbb{Z}$.

$\rho_n(z_1 z_2) = (z_1 z_2)^n = \rho_n(z_1)\rho_n(z_2)$

F.D. reps of Abelian. completely reducible.

$M(g_1) M(g_2) = M(g_2) M(g_1)$

$\Rightarrow$ simultaneously diag.

$M(z) = \text{diag}\{\lambda_1(z), \lambda_2(z), \cdots \lambda_d(z)\}$

$U(1) \qquad V \cong \mathbb{C}^d$.

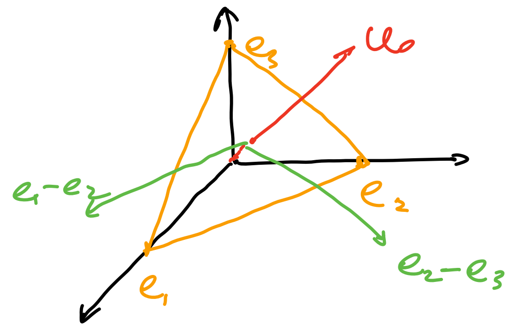
$V \cong \rho_{n_1} \oplus \rho_{n_2} \oplus \cdots \oplus \rho_{n_d}$.

$SO(2)$. $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \overset{\text{on } \mathbb{C}}{=} \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}$

irreducible $\mathbb{R}^2$.

reducible & irreducible reps (cont.)

Examples 5. $S_3 \cong D_3$



① $u_0 = e_1+e_2+e_3$

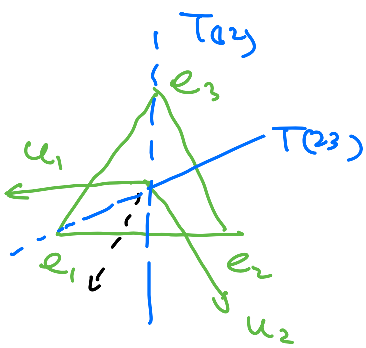
$T(\sigma)u_0 = \Sigma e_{\sigma(i)} = \Sigma e_i = u_0$

$W = \{u_0\}$

$T|_W = 1_W$. trivial rep.

② $W^\perp = \text{span}\{u_1, u_2\}$

a. $\begin{cases} u_1 = e_1 - e_2 \\ u_2 = e_2 - e_3 \end{cases}$ $\langle e_1-e_2, e_1+e_2+e_3 \rangle = 0$



$T((12)) \cdot u_1 = -u_1$

$T((12)) \cdot u_2 = u_1 + u_2$

$M((12)) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$

$M((12)) \cdot M((12)) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

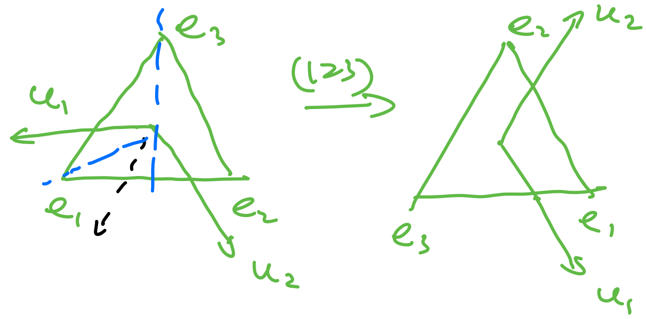
$T((23))\, u_1 = u_1 + u_2$

$T((23))\, u_2 = -u_2$

$M((23)) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$

$T((123))\cdot u_1 = u_2$

$T((123))\cdot u_2 = -u_1 - u_2$



$$M((123)) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \Rightarrow M^3_{(123)} = \mathbb{1}_2$$

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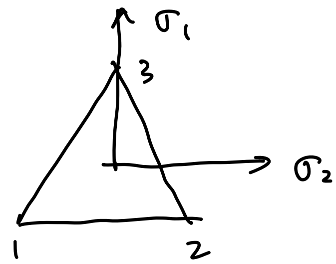
b.



$T[(23)]\,\sigma_1 = -\frac{1}{2}\sigma_1 + \frac{\sqrt{3}}{2}\sigma_2$

$T[(23)]\,\sigma_2 = \frac{\sqrt{3}}{2}\sigma_1 + \frac{1}{2}\sigma_2$

$$M[(23)] = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \qquad \det M = -1$$

$$M[(123)] = R(\tfrac{2}{3}\pi) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \qquad \det M = 1$$

$$\mathbb{R}^3 \cong \underset{\{u_0\}}{W} \oplus \underset{\substack{\{u_1, u_2\} \\ \{\sigma_1, \sigma_2\}}}{W^\perp}$$

6. $\mathbb{R}^3$. S_3 $\Longrightarrow$ $\mathbb{R}^n$. S_n

$u_0 = \sum e_i$ $\quad L = \{u_0\}$ invariant.

$L = \{x \sum e_i, x \in \mathbb{R}\}$ $\quad 1$

$L^\perp = \{\sum x_i e_i \mid \sum x_i = 0, x_i \in \mathbb{R}\}$ $\quad n-1$

$\Rightarrow$ L & $L^\perp$ both irreducible.

$V \cong \underset{\Delta}{L} \oplus \underset{\Delta}{L^\perp}$

$\ulcorner$ $U \subset L^\perp$ invariant. subspace.

$u = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n \in U$

WLOG. $x_1 \neq x_2$ $\quad$ ($x_i = x_0 \Rightarrow u = 0$)

$$u - \tau(12) \cdot u = x_1 e_1 + x_2 e_2 - (x_1 e_2 + x_2 e_1)$$
$$= (x_1 - x_2)(e_1 - e_2) \in U$$

$\Rightarrow e_1 - e_2 \in U$

$(123 \cdots n)^{i-1} u \Rightarrow e_i - e_{i+1} \in U$

$U = \text{span}\{e_i - e_{i+1}, i = 1, \cdots n-1\} = L^\perp.$ $\lrcorner$

⑥

7. a. $\rho(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ $x \in \mathbb{R}$, or $\mathbb{C}$.

$$\left(\begin{array}{c|c} 1 & x \\ \hline 0 & 1 \end{array}\right)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha + \beta x \\ \beta \end{pmatrix}$$

. $\left\{\begin{pmatrix} \alpha \\ 0 \end{pmatrix}\right\}$ invariant subspace.

b. $B(y) \in SO_0(1,1)$

$$\left\{ B(y) = \begin{pmatrix} \cosh y & \sinh y \\ \sinh y & \cosh y \end{pmatrix}, -\infty < y < \infty \right\}$$

$$B(y_1) B(y_2) = B(y_1 + y_2)$$

$$T(B(y)) = \left(\begin{array}{c|c} 1 & y \\ \hline 0 & 1 \end{array}\right)$$

c. $A \in GL(n, K)$

$$T(A) = \begin{pmatrix} 1 & \log|\det A| \\ 0 & 1 \end{pmatrix}$$

$$T(A)T(B) = \begin{pmatrix} 1 & \log|\det A| + \log|\det B| \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \log|\det AB| \\ 0 & 1 \end{pmatrix} = T(AB)$$

8. semidirect product $H \rtimes_\alpha G$.

Γ direct product. $H \times G$ $h_i \in H$. $g_i \in G$

$$(h_1, g_1)(h_2, g_2) = (\underline{h_1 \cdot_H h_2}, g_1 \cdot_G g_2)$$

$$(h_1, g_1)\cdot(h_2, g_2) = (h_1 \underline{\alpha_{g_1}(h_2)}, g_1 g_2)$$

α: G action of G on H.

direct product $\Rightarrow$ trivial action

a.

$G = GL(n, K)$ $H = Mat_n(K)$

$$(h_1, g_1)(h_2, g_2) := (\underline{h_1 + g_1 h_2 g_1^{tr}}, \underline{g_1 g_2})$$

$$T(h, g) := \underbrace{\begin{pmatrix} g & h g^{tr,-1} \\ 0 & g^{tr,-1} \end{pmatrix}}$$

$$\begin{pmatrix} g_1, & h_1 g_1^{tr,-1} \\ 0, & g_1^{tr,-1} \end{pmatrix}\begin{pmatrix} g_2 & h_2 g_2^{tr,-1} \\ & g_2^{tr,-1} \end{pmatrix} = \begin{pmatrix} g_1 g_2 & (h_1 + g_1 h_2 g_1^{tr})(g_1 g_2)^{tr,-1} \\ 0 & (g_1 g_2)^{tr,-1} \end{pmatrix}$$

b, $\{\underline{\alpha} | \underline{\vec{\tau}}\}$ G Euclidean group. / SG

$\in O(3)$ translation

$\alpha \in PG$.

$\vec{\tau} \in$ translation subgroup.

$$\{\alpha | \vec{\tau}\} \cdot \vec{r} = R(\alpha)\vec{r} + \vec{\tau}$$

$$\{\alpha_1|\vec{\tau}_1\}\{\alpha_2|\vec{\tau}_2\}\vec{r} = \{\alpha_1|\vec{\tau}_1\}(R(\alpha_2)\vec{r} + \vec{\tau}_2)$$
$$= R(\alpha_1)R(\alpha_2)\vec{r} + (R(\alpha_1)\vec{\tau}_2 + \vec{\tau}_1)$$
$$= \{\alpha_1\alpha_2 | R(\alpha_2)\vec{\tau}_2 + \vec{\tau}_1\}\cdot\vec{r}$$

$$(h_1, g_1)\cdot(h_2, g_2) = (h_1\,\alpha_{g_1}(h_2),\ g_1 g_2)$$

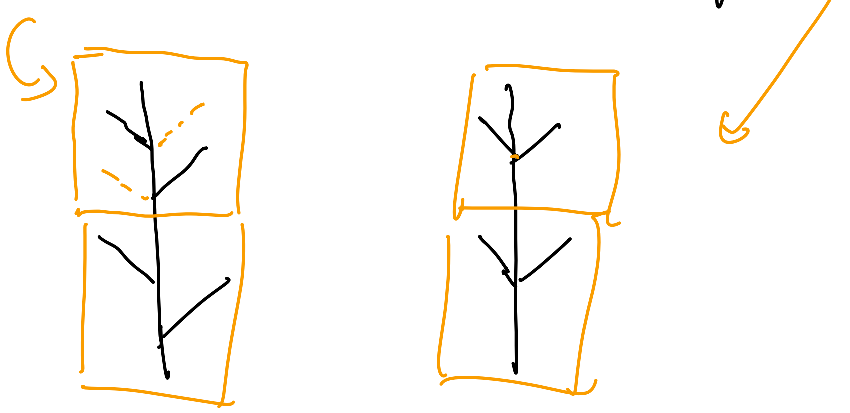
Mat. rep. $\left(\begin{array}{c|c} \boxed{R} & \tau \\ \hline 0 & 1 \end{array}\right)$

Note: $H \rtimes_\alpha G = \underline{T \rtimes_\alpha PG}$

symmorphic space groups

nonsymmorphic space groups.

glide reflection



Space groups:

$$\{R_\alpha|\tau\} = \{R_\alpha|\vec{R}\cdot n + \tau_\alpha\} = \{E|\vec{R}\cdot n\}\{R_\alpha|\tau_\alpha\}$$

① $\exists$ origin s.t. $\forall \alpha \in PG.\ \tau_\alpha = 0 \Rightarrow$ symmorphic

(PG is a subgroup of SG)

② otherwise nonsymmorphic.

Proposition. Let (T, V) be a unitary rep. on an inner product space V.

$W \subset U$ is an invariant subspace.

$W^{\perp}$ is an invariant subspace.

$(W^{\perp} = \{ y \in V \mid \langle y, x \rangle = 0 \quad \forall x \in W \})$

Proof: $\forall g \in G$. $y \in W^{\perp}$ $x \in W$.

$$\langle T(g) y, x \rangle = \langle y, T(g)^{+} x \rangle$$

$$= \langle \underbrace{y}_{\in W^{\perp}}, \underbrace{T(g^{-1}) x}_{\in W} \rangle = 0$$

$\Rightarrow T(g) y \in W^{\perp}$, $\forall g \in G$.

$\Rightarrow W^{\perp}$ is an invariant subspace.

Corollaries:

1. F.D. uni. rep. are always completely reducible.

V — irreducible ✓

$\hookrightarrow V \cong W \oplus W^{\perp}$

$\Rightarrow \oplus W_i$

2. $\left\{\begin{array}{l}\text{compact groups} \\ \text{finite}\end{array}\right.$ $\Rightarrow$ unitarizable.

$\Rightarrow$ completely reducible

3. finite G. regular reps $L^2(G)$

are completely reducible

$$\left(\begin{array}{c} \dim(L^2(G)) = |G| \\ L_g \delta_h = \delta_{gh} \end{array} \right)$$

4. Compact G. $L^2(G)$ infinite dimensional

Peter-Weyl theorem

$$L^2(G) \cong \underline{\oplus_\mu \mathrm{End}(V^{(\mu)})}$$

finite dimensional irreps

5. $\begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}$ noncompact groups

finite-dim reps are nonunitary

may have infinite-dim unitary reps.

- Isotypic components

G. irreps are countable.

choose a representative $(T^{(\mu)}, V^{(\mu)})$

for each isomorphism class of irreps.

$$V \cong \bigoplus_{\mu} \bigoplus_{i=1}^{a_\mu} V^{(\mu)}$$

a_μ is the # of times $V^{(\mu)}$ appears

in the decomposition

$\bigoplus_{i=1}^{a_\mu} V^{(\mu)}$ is called the isotypic component

of V (associated to μ)

Example.
$T: U(1) \to U(N)$

$$T(z) = \mathrm{diag}(e_j, e_j, e_j \cdots e_l, e_l, \cdots)$$

$$(T, V) \cong \bigoplus_{j \in \mathbb{Z}} n_j e_j$$