Recap.

Haar measure . a.k.a. invariant measure/integration

$$\int_G f(hg)\,dg = \int_G f(g)\,dg \qquad (\forall h \in G)$$

left-invariant

SU(2) $\quad g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \quad |\alpha|^2 + |\beta|^2 = 1$

$$g = e^{i\frac{1}{2}\phi\sigma^3} e^{i\frac{1}{2}\theta\sigma^1} e^{i\frac{1}{2}\varphi\sigma^3}$$

$$= \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

$$\begin{cases} \alpha = e^{i\frac{1}{2}(\phi+\varphi)} \cos\frac{\theta}{2} \\ \beta = i e^{i\frac{1}{2}(\phi-\varphi)} \sin\frac{\theta}{2} \end{cases}$$

$$(\phi;\varphi) \sim (\phi, \varphi+4\pi)$$

$$\sim (\phi+4\pi, \varphi)$$

$$\sim (\phi+2\pi, \varphi+2\pi)$$

$\phi \in [0, 2\pi) \quad \theta \in [0, \pi) \quad \varphi \in [0, 4\pi)$

① $d\alpha\, d\bar{\alpha}\, d\beta\, d\bar{\beta} \longrightarrow \left| \frac{\partial(\alpha, \cdots)}{\partial(r, \varphi, \phi, \theta)} \right| dr\, d\varphi\, d\phi\, d\theta$

$\propto \sin\theta$

$$[dg] = C \cdot \sin\theta\, d\theta\, d\phi\, d\varphi$$

$$C = \frac{1}{16\pi^2}$$

② Maurer–Cartan form. $\quad \omega = g^{-1} dg$

$[dg] \propto \mathrm{tr}(\omega^3)$

Non-compact G:

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \,\middle|\, x, y \in \mathbb{R},\ x > 0 \right\}$$

$$\underbrace{\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}}_{h} \underbrace{\begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix}}_{g} = \begin{pmatrix} \underline{xu} & xv+y \\ 0 & 1 \end{pmatrix}$$

left. $du\,dv \to x^2\,du\,dv$

$$[dg] = x^{-2}\,dx\,dy$$

right. $dx\,dy \to \underline{u}\,dx\,dy$ $\quad \underline{(L \neq R)}$

$$[dg] = x^{-1}\,dx\,dy$$

Proposition: (T, V) rep of compact group

inner prod. space

$\Rightarrow (T, V)$ unitarizable.

$\langle , \rangle_1 \Rightarrow$

$$\langle v, w \rangle_2 = \int_G \langle T(g)v, T(g)w \rangle_1 \cdot dg$$

$$\Rightarrow \langle T(g)v, T(g)w \rangle_2 = \langle v, w \rangle_2$$

Regular representation

$G \times G$ acts on G:

$$(g_1, g_2) \longmapsto L(g_1)R(g_2^{-1})$$

$$(g_1, g_2)\cdot g_0 = g_1 g_0 g_2^{-1}$$

$\hookrightarrow$ induced action on $f \in Map(G, \mathbb{C})$

$$[(g_1, g_2) f](g) = f(g_1^{-1} h g_2)$$

$$\{(g_1, g_2)[(g_3, g_4)\cdot f]\}(g)$$

$$= [(g_3, g_4)\cdot f](g_1^{-1} h g_2)$$

$$= f(g_3^{-1} g_1 h g_2 g_4)$$

$$= [(g_1 g_3, g_2 g_4)\cdot f](g)$$

$$L^2(G) = \{ f : G \to \mathbb{C} \mid \int_G |f(g)|^2 dg < \infty \}$$

$$\Uparrow$$

$$\langle f, f \rangle$$

Hilbert space

Definition. $L^2(G)$ regular rep.

$$G \times G \Longrightarrow G \times \{1\}, \quad \{1\} \times G.$$

$L^2(G)$ becomes rep of G.

(T, V) rep of G. basis $\Rightarrow M_n(\mathbb{C})$

$G \times G$ action on $\mathrm{Hom}(V, V) =: \mathrm{End}(V)$

$$(g_1, g_2) \cdot S = T(g_1) \cdot S \cdot T(g_2)^{-1} \qquad S \in \mathrm{End}(V)$$

$G \times G$ rep

$$\iota \quad \mathrm{End}(V) \longrightarrow L^2(G)$$

$$S \longmapsto f_S$$

$$f_S := \mathrm{Tr}_V(S\,T(g^{-1}))$$

$$\begin{tikzcd}
\mathrm{End}(V) \arrow[r, hook, "\iota"] \arrow[d, "T_{\mathrm{End}(V)}"'] & \mathrm{Map}(G, \mathbb{C}) \arrow[d, "T_{\text{regular rep.}}"] \\
\mathrm{End}(V) \arrow[r, hook, "\iota"] & \mathrm{Map}(G, \mathbb{C})
\end{tikzcd}$$

$$\Longleftrightarrow (h_1, h_2) f_S(g) = f_{(h_1, h_2) \cdot S}(g)$$

V with a basis $\{v_i\}$

$$\Rightarrow f_S = M_{ji}(g^{-1})$$

Example. 1 $G = \mu_3 = \{1, \omega, \omega^2\}$ $\omega = e^{i\frac{2}{3}\pi}$

V with basis $\delta_j(\omega^k) = \begin{cases} 1 & j = k \bmod 3 \\ 0 & \text{otherwise.} \end{cases}$

$$(\underline{L(\omega)\cdot\delta_0})(g) = \delta_0(\omega^{-1}g) = \underline{\delta_1}(g)$$

$L(\omega)\delta_1 = \delta_2$

$L(\omega)\delta_2 = \delta_3$

$$L(\omega) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$\top$ reg. rep. from before: $\{e_g\}$

$h \cdot e_g = e_{hg}$

Example 2.

	g_1 e	g_2 a	g_3 b	g_4 c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$V: \langle a^2 = b^2 = (ab)^2 = 1 \rangle$

basis. $\{\delta_{g_i}(g_j)\}$

$$(L(g_2 = a)\delta_{g_i})(g) = \delta_{g_i}(a^{-1}g) = \delta_{ag_i}(g)$$

$L(a)\delta_e = \delta_a$

$L(a)\delta_a = \delta_e$

$L(a)\delta_b = \delta_{ab} = \delta_c$

$L(a)\delta_c = \delta_{ab} = \delta_b$

$$L(a) = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$\underline{\dim V} = |G|$

– Reducible & irreducible representations

direct sum of rep.

$$T_{V\oplus W} = T_V \oplus T_W$$

$$M_{V\oplus W} = \left(\begin{array}{c|c} M_V & 0 \\ \hline 0 & M_W \end{array}\right)$$

We want to "reduce" rep. of large dims.

to rep of smaller dims.

<u>Definition</u>. Let $W \subset V$ be a <u>linear subspace</u>

of carrier space V. of a group representation

$T: G \to GL(V)$

Then W is invariant under T

(i.e. W is an invariant subspace)

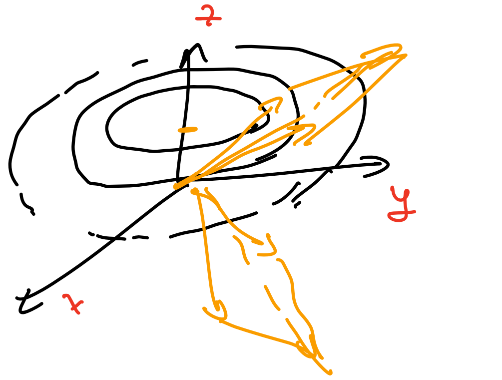
if $\forall g \in G$. $w \in W$

$T(g) w \in W$.

Example.

1. $\{\vec{0}\}$. & V.

2. $\mathbb{R}^3$ under $SO(2)_{\hat{z}}$



The xy plane

$\boxed{(x, y, 0)}$

is an invariant subspace

$(x, y, 1)$ $\quad$ $(x, y, z_0) + (x', y', z_0) = (x+x', y+y', 2z_0)$

$v, w \in W \Rightarrow \alpha v + \beta w \in W \quad \alpha, \beta \in K$

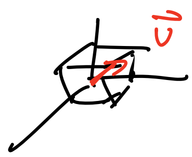
3. canonical rep. of S_n:

$$T(\phi): \vec{e}_i \to \vec{e}_{\phi(i)}$$

$$\vec{v} = \vec{e}_1 + \vec{e}_2 + \cdots + \vec{e}_n$$

$$T(\phi)\vec{v} = T(\phi)\sum_i \vec{e}_i = \sum_i \vec{e}_{\phi(i)} = \vec{v}$$

$\mathbb{R}^3$



$\mathbb{R}^n$: body diagonal of n-cube

4. Matrix rep.

$$M: G \to GL(n, K)$$

M_{ij} as a function: $\quad G \to K$

$$g \mapsto M_{ij}(g)$$

linear span of M_{ij} with fixed $\underline{i}$

$$R_i = \operatorname{span}\{\overrightarrow{M_{ij}},\ j=1,\cdots,n\}$$

$$\underbrace{(R(g)\cdot M_{ij})(h)}_{\text{func. on } G.} = \underline{M_{ij}(hg)}$$

$$= \sum_s \underbrace{M_{sj}(g)}_{\text{coeffs.}} \underbrace{M_{is}(h)}_{\text{func. on } G} \quad (\forall g)$$

$\Rightarrow$ R_i is an invariant subspace.

similarly. $L_j := \operatorname{span}\{M_{ij},\ i=1,\cdots,n\}$

$$(L(g)M_{ij})(h) = M_{ij}(g^{-1}h) = \sum_s M_{is}(g^{-1})M_{sj}(h)$$

L_j is invariant

$\Rightarrow$ $LR = \operatorname{span}\{M_{ij},\ i,j=1,\cdots,n\}$

invariant under $G\times G$ action

$$((g_1,g_2)\cdot f)(h) = f(g_1^{-1}hg_2)$$

<u>Remarks</u>.

1. (T,V) crep. $\exists\, W\subset V$ an invariant subspace then we can restrict T to W. $(T|_W, W)$ is a <u>subrepresentation</u> (T,V).

$$T|_W(g) = \underline{T(g)|_W}$$

we usually write T instead of $T|_W$

2. (T, V) unitary $\Rightarrow$ (T, W) unitary

$$\langle T v_1, T v_2 \rangle = \langle v_1, v_2 \rangle \quad \forall v_i \in \underline{\underline{V}}$$

W

<u>Definition</u>. A representation (T, V) is <u>reducible</u> if there is a proper, nontrivial invariant subspace $W \subset V$. $(W \neq V, 0)$

If V is not reducible, it is an <u>irreducible representation</u> ("<u>irrep</u>")

<u>Remarks</u>.

1. $\forall \underline{v} \in V$. $W = \text{span}\{ \underline{T(g)} v, \forall g \in G \}$

$\forall T(g) v \in W$.

$$T(g') T(g) v = T(g'g) v \in W$$

W is an invariant subspace.

$\not\Leftarrow$

(T, V) irrep $\Rightarrow$ $W = V$.

$\underline{v}$ is called a <u>cyclic vector</u>.

$\underline{S_n}$: $\phi(\vec{e}_1) \to \vec{e}_{\phi(1)}$

2 (T, W) subrep of (T, V)

choose an ordered basis

$\{w_1, \cdots w_k\}$

and complete it to an ordered basis of V

$\{\underline{w_1, \cdots w_k}, \underline{u_{k+1}, \cdots u_n}\}$

$T(g)(w_i) = \underline{(M_{11}(g))_{ji}\, w_j} + \cancel{(M_{21}(g))_{ai}\, u_a}$ 0

$T(g)(u_a) = \underline{(M_{12}(g))_{ja}\, w_j} + \underline{(M_{22}(g))_{ba}\, u_b}$

$$(w, u)\begin{pmatrix} M_{11}(g) & M_{12}(g) \\ M_{21}(g) & M_{22}(g) \end{pmatrix}$$

W invariant $\iff M_{21} = 0$

$$\begin{pmatrix} M_{11}(g_1) & M_{12}(g_1) \\ 0 & M_{22}(g_1) \end{pmatrix}\begin{pmatrix} M_{11}(g_2) & M_{12}(g_2) \\ 0 & M_{22}(g_2) \end{pmatrix}$$

$$= \left(\begin{array}{c|c} M_{11}(g_1)M_{11}(g_2) & M_{11}(g_1)M_{12}(g_2) + M_{12}(g_1)M_{22}(g_2) \\ \hline 0 & M_{22}(g_1)M_{22}(g_2) \end{array}\right)$$

M_{11} is a matrix rep on W.

$(T|_W(g) = T(g)|_W)$

M_{22} is not a rep on $V \setminus W$

If we define a change of basis $\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}$

$$(w, u)\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} = (w, \; wS + u) \equiv (w, u')$$

$$\forall g: \quad \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} M_{11}(g) & M_{12}(g) \\ 0 & M_{22}(g) \end{pmatrix} = \begin{pmatrix} M_{11}(g) & \frac{M_{12}(g) - S M_{22}(g)}{M_{22}(g)} \\ 0 & \end{pmatrix}$$

$S = M_{12}(g) M_{22}(g)^{-1} \quad \forall g \in G.$

this puts a strong restriction on the structure of M.

3. quotien space V/W.

$v_1 \sim v_2$ iff $v_1 - v_2 \in W$.

$$T(g)(v + W) = T(g)(v) + W.$$

$$\begin{aligned} T(g_1)\, T(g_2)(v + W) &= T(g_1)(T(g_2) v + W) \\ &= T(g_1) T(g_2) v + W \\ &= \underbrace{[T(g_1) T(g_2)]}_{= T(g_1 g_2)}(v + W) \end{aligned}$$

define basis of V/W: $u_a + W$.

$$T(g)(u_a) = \underbrace{(M_{12}(g))_{ja}\, w_j}_{\in W} + (M_{22}(g))_{ba}\, u_b$$

M_{22} is the matrix rep on this basis.

- Complete reducibility.

Definition A rep. (T, V) is called completely reducible if it is isomorphic to a direct sum of representitions

$$W_1 \oplus W_2 \oplus \cdots \oplus W_n$$

where W_i's are irrep. Then there is a basis in which the matrix rep. looks like

$$M(g) = \begin{pmatrix} M_{11}(g) & 0 & 0 & 0 \\ 0 & M_{22}(g) & 0 & 0 \\ 0 & 0 & M_{33}(g) & 0 \\ & & & \ddots \end{pmatrix}$$

① a rep. reducible, but not completely reducible
$\Rightarrow$ "indecomposable".

$$\begin{pmatrix} M_{11}(g) & M_{12}(g) \\ 0 & M_{22}(g) \end{pmatrix}$$

② irreps. are completely reducible.

Examples:

1. $G = \mathbb{Z}_2 = \{1, -1\}$ 1-D rep.

trivial $\rho_+(1) = \rho_+(-1) = 1$

$\rho_-(1) = 1, \quad \rho_-(-1) = -1$

2. $G = \mathbb{Z}_2$ 2D rep $\cong S_2 = \{e, \tau\}$

$$M(e) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right)$$

$$M(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tilde{M}(e) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \underline{1} \end{array}\right)$$

$$A = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \tilde{M}(\tau) = A^{-1} M A = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \underline{-1} \end{array}\right)$$

$\rho_+(e) = \rho_+(\tau) = 1$

$\rho_-(e) = 1, \quad \rho_-(\tau) = -1$

$(T, V) \cong \rho_+ \oplus \rho_-$ completely reducible.

3. $G = U(1) = \{ z \in \mathbb{C} \mid |z| = 1 \}$ $V = \mathbb{C}$

$\rho_n(z) = z^n$ for $\forall n \in \mathbb{Z}$.

$\rho_n(z_1 z_2) = (z_1 z_2)^n = z_1^n \cdot z_2^n = \rho_n(z_1)\rho_n(z_2)$

4. Finite-dimensional representations of <u>Abelian</u> groups are completely reducible.

Choose an ordered orthonormal basis.

s.t. all $M(g)$ ($\forall g \in G$) are

commuting unitary matrices

$$M(g_i) M(g_j) = M(g_j) M(g_i) \quad \forall g_i, g_j \in G.$$

$\Rightarrow$ M's can be simultaneously diagonalized.

$$M(g) = \mathrm{diag}\{\lambda_1(g), \lambda_2(g), \cdots \lambda_d(g)\}$$

For $G = U(1)$ any f.d. rep on $V \cong \mathbb{C}^d$

$$M(z) = \mathrm{diag}\{ \rho_{n_1}(z), \rho_{n_2}(z), \cdots \rho_{n_d}(z) \}$$

$$V \cong \rho_{n_1} \oplus \rho_{n_2} \oplus \cdots \oplus \rho_{n_d}$$

$\Rightarrow$ Finite group & compact groups

all irreps are 1D. $\underline{(\text{on } \mathbb{C})}$

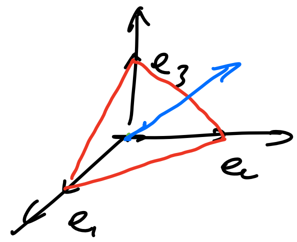
e.g. $SO(2)$ $\quad R(\theta) = \underline{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}$ on $\mathbb{R}$.

$$\to \left(\begin{array}{c|c} e^{i\theta} & 0 \\ \hline 0 & e^{-i\theta} \end{array}\right) \text{ on } \mathbb{C}.$$

5. non abelian. $S_3 \trianglelefteq D_3$

on $\mathbb{R}^3 = \text{span}\{e_1, e_2, e_3\}$

$T(\sigma) e_i = e_{\sigma(i)}$



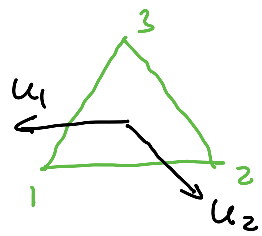
① $u_0 = e_1 + e_2 + e_3$ $\quad W = \{u_0\}$

$T(\sigma) u_0 = u_0 \Rightarrow T|_W = 1_W.$ trivial rep.

② $W^\perp = \text{span}\{u_1, u_2\}$

a. $u_1 = e_1 - e_2$ $\quad \langle u_1, u_0 \rangle = \langle u_2, u_0 \rangle = 0$

$u_2 = e_2 - e_3$



to be continued