Recap:

$N \subset G$.

$gNg^{-1} = N \quad \forall g \in G$.

$N \triangleleft G$.

$\hookrightarrow \quad \phi : G \to G/N$

$g \mapsto g\underline{N}$

N is the kernel of some homomorphism.

$GL(n, K) \xrightarrow{\det} K$

$A \mapsto \det(A)$

$\ker(\det) = SL(n, K)$

$SL \triangleleft GL$

$GL(n, K)/SL(n, K) \cong K^*$

$M \in GL, \quad M = \lambda A \quad (A \in SL)$

$\lambda \in K^*$

$U(n)/SU(n) \cong U(1)$

$O(n)/SO(n) \cong \mathbb{Z}_2$

$\pi : \quad SU(2) \longrightarrow SO(3)$

$\ker \pi = \pm \mathbb{1}_2 \cong \mathbb{Z}_2$

$SU(2)/\mathbb{Z}_2 \cong SO(3)$

$SG. \quad g \in SG \quad g = \{ \underline{\underline{R_\alpha}} \mid \underline{\vec{t}} \}$

$\{R_\alpha | \tau\} \{e | t\} \{R_\alpha | \tau\} = \{ e \mid \underline{R_\alpha t} \}$

$T = \langle \vec{t}_1, \vec{t}_2, \vec{t}_3 \rangle \triangleleft SG$

$SG/T \cong PG \quad$ point group $\subset O(3)$

<u>Simple group</u> : has no nontrivial normal subgroups

$\{1\}, \ G.$

<u>short exact sequence</u> (SES)

$$\underline{1} \xrightarrow{i} G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \xrightarrow{f_3} \underline{1}$$

exactness at G_i : $\operatorname{im} f_i = \ker f_{i+1}$

① $\ker f_1 = \operatorname{im}(i) = \{1_{G_1}\} \qquad$ <u>f_1 injective</u>

② $\operatorname{im} f_1 = \ker f_2$ $\qquad f_3 : G \to 1$

① $\ker f_3 = G_3 = \operatorname{im} f_2 \qquad \underline{f_2 \text{ surjective}}$

$$1 \to \underline{\ker \mu} \xrightarrow{i} G \xrightarrow{\mu} \underline{\operatorname{im} \mu} \to 1$$

$$1 \to \underset{\triangle}{N} \to G \to Q \to 1$$

$N \cong H \triangleleft G \qquad Q = G/N$

" G is an extension of Q by N "

$$1 \to SO(n) \to O(n) \to \mathbb{Z}_2 \to 1$$

$\iff \quad O(n)/SO(n) \cong \mathbb{Z}_2$

$$1 \to \mathbb{Z}_2 \to \underline{SU(2)} \xrightarrow{\pi} \underline{SO(3)} \to 1$$

$\uparrow$ quantum sym. $\qquad \uparrow$ classical symmetry

$$U(R_\pi)U(R_\pi) = \underline{-1} \cdot U(R_{2\pi})$$

$\underline{c(g_1, g_2)} \in U(1)$

Central extension

$$1 \longrightarrow \underline{A} \xrightarrow{i} \underline{E} \xrightarrow{\pi} G \longrightarrow 1$$

1. A abelian

2. $A \subset Z(E)$ $\quad i(a)b = b \cdot i(a) \quad (a \in A,\ b \in E)$

$$1 \to U(1) \longrightarrow U(\mathcal{H}) \longrightarrow U(\mathbb{P}) \to 1$$

$$\mathbb{P}(\mathcal{H}) := (\mathcal{H} \setminus \{0\}) / \underline{\mathbb{C}^*}$$

7. Group actions (cont.)

$\phi : G \times X \to X$

$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x \qquad x \in X \qquad$ left act.

$\qquad\qquad\qquad\qquad\qquad\qquad g_i \in G$

$1_G \cdot x = x.$

① effective: $\forall g \neq 1 \quad \exists x \quad$ s.t. $g \cdot x \neq x$

② transitive: $\forall x, y \in X. \quad \exists g. \quad$ s.t. $y = g \cdot x$

only one orbit

③ free: $\forall g \neq 1 \quad \forall x. \ g \cdot x \neq x$

Def: Stabilizer group / isotropy group

$\mathrm{Stab}_G(x) := \{ g \in G, \ g \cdot x = x \} \subset G$

(G^x)

free $\Leftrightarrow G^x = \{1\} \quad \forall x \in X$

$\mathrm{Fix}_X(g) := \{ x \in X : g \cdot x = x \} \subset X$

(X^g)

free $\Leftrightarrow X^g = \phi$

$$\mathrm{Orb}_G(x) = \{ gx . \ g \in G \}$$

Theorem (Stabilizer – Orbit theorem)

Let X be a G-set. Each left coset of G^x $(x \in X)$ is in a natural 1–1 correspondence with points in $\mathrm{Orb}_G(x)$

There is a natural isomorphism.

$$\varphi : \mathrm{Orb}_G(x) \longrightarrow G/G^x$$
$$g \cdot x \longmapsto g \cdot G^x$$

G-action on G/G^x

$(G \times G/G^x \to G/G^x)$

recall

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi(g)"'] & X' \arrow[d, "\varphi(g)"] \\ X \arrow[r] & X' \end{tikzcd}$$

$$\varphi(gx) = g\,\varphi(x)$$

$$|O_G(x)| = [G : G^x]$$

finite G.

$$= |G| / |G^x|$$

① Well-defined:

$$g x = g' x$$
$$\Leftrightarrow g'^{-1} g x = x$$
$$\Leftrightarrow g'^{-1} g \in G^x$$
$$\Leftrightarrow g G^x = g' G^x$$

② injective. $gG^x = g'G^x$ $\quad (\varphi(gx) = \varphi(g'x))$

$$\Rightarrow g'^{-1} g \in G^x$$
$$\Rightarrow g'^{-1} g x = x$$
$$\Rightarrow g x = g' x$$

Examples:

1. Cosets as right action of H on $X = G$

$$O_H(g) = \{ gh,\ h \in H \} = gH$$

$$\mathrm{Stab}_H(g) (\equiv H^g) = \{ g \cdot h = g \quad h \in H \}$$
$$= \{ 1 \}$$

$$| O_H(g) | = | gH | = | H | / 1 = | H |$$

2. G acts on G by conjugation. $h \in G$

$$O_G(h) = \{ g h g^{-1},\ g \in G \} = C(h)$$

$$\mathrm{Stab}_G(h) = \{ g \in G,\ g h g^{-1} = h \}$$

Definition: The centralizer of h in G

$$C_G(h) := \{ g \in G,\ gh = hg \} = \{ g \in G,\ g h g^{-1} = h \}$$

(1) $C_G(h)$ is a subgroup.

① $e \in C_G(h)$

② $\forall g \in C_G(h) \quad gh = hg$
$\Rightarrow g^{-1} h = h g^{-1}$
$\Rightarrow g^{-1} \in C_G(h)$

③ $\forall g_1, g_2 \in C_G(h) \quad g_1 g_2 h = g_1 h g_2 = h g_1 g_2$
$g_1 g_2 \in C_G(h)$

$|C(h)| = [G : C_G(h)]$

$\uparrow$

# of conj. of h.

H is a subset of G

$C_G(H) := \{ g \in G : gh = hg, \ \forall h \in H \}$

$C_G(G) = Z(G)$

2. G acts on $X = \{$ all subgroups $H \subset G \}$ by conj.

$O_G(H) = \{$ all conjugates of $H \subset G \}$

$Stab_G(H) = \{ g \in G : gHg^{-1} = H \}$

Definition: The normalizer of a subgroup $H \subset G$

$N_G(H) = \{ g \in G : gHg^{-1} = H \}$

① $N_G(H)$: (proof follows $C_G(H)$)

is a subgroup

② $H \triangleleft N_G(H)$ ($H \triangleleft G$: $gHg^{-1} = H, \ \forall g \in G$)

$N_G(H)$ is the largest subgroup of G in which H is normal.

$|O_G(H)| = [G : N_G(H)]$

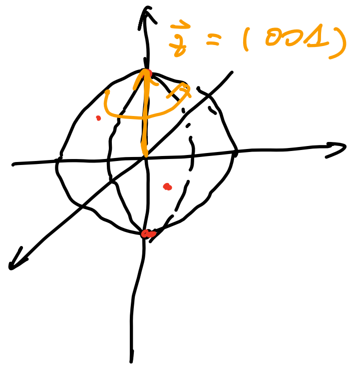
$C_G(H) \subset N_G(H)$

3. $\mathbb{Z}_p$ (p prime) acts on any set X.

$$|O_G(x)| = |G/G^x| \qquad |G^x| = \underline{1} \text{ or } \underline{p}$$

$$|O_G(x)| = p \text{ or } 1$$

4. $SO(3)$ acts on S^2



effective ? ✓

transitive ? ✓

free ? ×

$$\mathrm{Stab}_{SO(3)}(\vec{z}) = \left\{ \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, 0 \le \phi < 2\pi \right\} \cong SO(2)$$

S–O theorem.

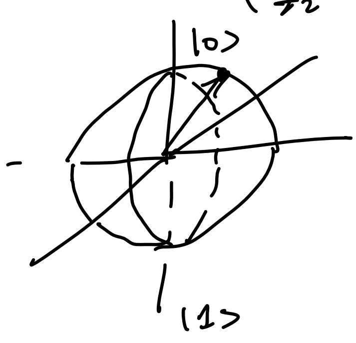
$$\mathrm{Orb}_G(x) \cong G/G^x$$

$\hat{n}$:

$$S^2 \cong SO(3)/SO(2)_{\hat{n}}$$

5. $SU(2)$ acts on the qubit space $\mathbb{C}^2$

$$|\varphi\rangle = \vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$



"Bloch sphere"

$$|\varphi\rangle = \cos\frac{\theta}{2}\,|0\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|1\rangle$$

$$0 \le \theta < \pi$$

$$0 \le \phi < 2\pi$$

rotation: "$\underline{z}\,\underline{y}\,\underline{z}$" Euler angles α, β, γ

$$\underline{e^{-\frac{i}{2}\sigma_z\gamma}}\ \underline{e^{-\frac{i}{2}\sigma_y\beta}}\ \underline{e^{-\frac{i}{2}\sigma_z\alpha}}\ |\varphi\rangle$$

$$\vec{z} = \begin{pmatrix}1\\0\end{pmatrix} = |0\rangle$$

① $e^{-\frac{i}{2}\sigma_z\alpha}|0\rangle = \underline{e^{-\frac{i}{2}\alpha}|0\rangle}$ $\qquad e^{-\frac{i}{2}\sigma_z\alpha} = \begin{pmatrix} e^{-\frac{i}{2}\alpha} & 0 \\ 0 & e^{\frac{i}{2}\alpha}\end{pmatrix}$

② $\begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix}\underline{\begin{pmatrix}1\\0\end{pmatrix}} = \underline{\begin{pmatrix} z \\ -w^* \end{pmatrix}} \quad \xrightarrow{w=0} \quad \begin{pmatrix} z & 0 \\ 0 & z^* \end{pmatrix}$ //

$$\mathrm{Stab}_{SU(2)}(|0\rangle) = \{ e^{-\frac{i}{2}\sigma_z\alpha},\ \underline{\alpha\in[0,2\pi)} \} \cong \underline{U(1)}$$

$$\Rightarrow \underline{S^2} \cong \underline{SU(2)/U(1)}$$

Lemma [(not) Burnside's / orbit counting theorem]

If X is finite G-set (X, G both finite)

then the number of orbits

$$\underline{|\{\text{orbits}\}| = \frac{1}{|G|}\sum_g |X^g|} \quad (X^g = \mathrm{Fix}_X(g) \subset X)$$

Proof: $\sum_g |X^g| = |\{(x,g) \mid g.x = x\}| = \sum_x |G^x|$

$$\text{RHS} = \frac{1}{|G|}\sum_x |G^x| = \sum_x \frac{1}{|O_G(x)|} = \underline{|\{\text{orbits}\}|}$$

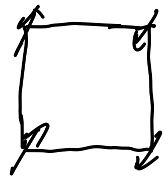
$\uparrow$

$N.\ \frac{1}{N} \times N = 1$

Example:

1. $X = G$ (finite) action = conj.

$$\#\text{ conjugacy classes} = \sum_{g \in G} \frac{|C_G(g)|}{|G|}$$

2. 

"4-spin molecule"

in an external magnetic field

$=: \begin{matrix} + & - \\ + & - \end{matrix}$

$$O_{C_4}\left(\begin{smallmatrix} + & - \\ + & - \end{smallmatrix}\right) = \left\{ \begin{matrix} + & - \\ + & - \end{matrix},\ \begin{matrix} + & + \\ - & - \end{matrix},\ \begin{matrix} - & + \\ - & + \end{matrix},\ \begin{matrix} - & - \\ + & + \end{matrix} \right\}$$

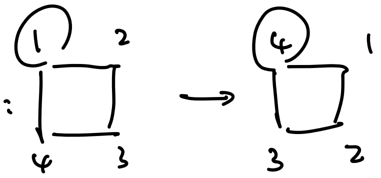
$G = C_4 = \{ R(0), R(\frac{\pi}{2}), R(\pi), R(\frac{3}{2}\pi) \}$ $|G| = 4$

$X = \{ \text{spin config on } \square \}$ $|X| = 2^4 = 16$

$$\#\text{ orbits} = \frac{1}{4}\left(|X^0| + |X^{\frac{\pi}{2}}| + |X^{\pi}| + |X^{\frac{3}{2}\pi}|\right)$$

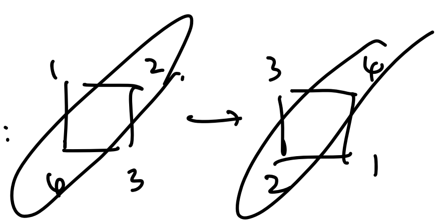
$$= \frac{1}{4}(2^4 + 2 + 4 + 2)$$

$$= 6$$

$\frac{\pi}{2}$: 

$\frac{3}{2}\pi$:

$1 = 2 = 3 = 4$

π: 

$1 = 3$

$2 = 4$

representatives:

$$\left\{ \begin{matrix} + & + \\ + & + \end{matrix},\ \begin{matrix} + & + \\ + & - \end{matrix},\ \begin{matrix} + & + \\ - & - \end{matrix},\ \begin{matrix} + & - \\ - & + \end{matrix},\ \begin{matrix} + & - \\ - & - \end{matrix},\ \begin{matrix} - & - \\ - & - \end{matrix} \right\}$$

orbit $\Longleftrightarrow$ physical config.

$$Z = \sum' e^{-\beta H(\text{config.})}$$

for single spin: $e^{\pm\beta h}$ $\quad p = e^{\beta h}$ $\quad m = e^{-\beta h}$ 8

$$Z = 1 \times p^4 m^0 + 1 \times p^3 m^1 + 2 p^2 m^2$$
$$+ 1 p^1 m^3 + 1 p^0 m^4$$

$\Longrightarrow$ Coloring problem.