HW 07

(b). $\int_{SU(2)} dg\, g_{\alpha\beta} = 0$

$$\int_{SU(2)} dg\, g_{\alpha\beta} g_{\gamma\delta} = \frac{1}{2} \epsilon_{\alpha\gamma} \epsilon_{\beta\delta}$$

$$\alpha = e^{i\frac{1}{2}(\phi+\varphi)} \cos\frac{\theta}{2} \qquad \theta \in [0, \pi)$$

$$\beta = i e^{i\frac{1}{2}(\phi-\varphi)} \sin\frac{\theta}{2} \qquad \phi \in [0, 2\pi)$$

$$\varphi \in [0, 4\pi)$$

$$\int_0^{4\pi} e^{i\frac{1}{2}\varphi} d\varphi = 0$$

$$\phi_{\alpha\beta} \equiv \int dg\, g_{\alpha\beta} = \int dg\, (g_0 g)_{\alpha\beta} = (g_0)_{\alpha\gamma} \cdot \int dg\, g_{\gamma\beta}$$

$$g_0 \cdot \begin{pmatrix} \phi_{0\beta} \\ \phi_{1\beta} \end{pmatrix} = \begin{pmatrix} \phi_{0\beta} \\ \phi_{1\beta} \end{pmatrix} \qquad \forall g_0 \in SU(2)$$

$$g_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow \phi_{0\beta} = \pm \phi_{1\beta} = 0$$

$$(A^{\beta\delta})_{\alpha\gamma} = \int dg\, \underset{=0}{g_{\alpha\beta}} \underset{=0}{g_{\gamma\delta}} = \int dg\, (g_0 g)_{\alpha\beta} (g_0 g)_{\gamma\delta}$$

$$= (g_0)_{\alpha s} \int dg\, g_{s\beta} g_{t\delta} (g_0)_{\gamma t}$$

$$\Rightarrow A^{\beta\delta} = g_0 \cdot A^{\beta\delta} \cdot g_0^T \qquad (\forall g_0)$$

$$g_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \& \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$\Rightarrow A^{\beta\delta} = C_{\beta\delta} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$A^{\alpha\gamma} = C_{\alpha\gamma}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$A_{\alpha\gamma,\beta\delta} = C_{\beta\delta}\,\epsilon_{\alpha\gamma} = C_{\alpha\gamma}\cdot\epsilon_{\beta\delta}$$

$$A_{\alpha\gamma,\beta\delta} = c\cdot\epsilon_{\alpha\gamma}\epsilon_{\beta\delta} \qquad c = \frac{1}{2} \quad \text{explicit calc.}$$

$$g = \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} \qquad g_{\alpha\beta} = \epsilon_{\alpha\alpha'}\epsilon_{\beta\beta'}\bar g_{\alpha'\beta'}$$

$$\int dg\, g_{\alpha\beta}\, g_{\gamma\delta} = \int dg\, \epsilon_{\alpha\alpha'}\epsilon_{\beta\beta'}\bar g_{\alpha'\beta'}\, g_{\gamma\delta}$$

$$= \epsilon_{\alpha\alpha'}\epsilon_{\beta\beta'}\frac{1}{2}\delta_{\alpha'\gamma}\delta_{\beta'\delta}$$

$$= \frac{1}{2}\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}$$

$\frac{1}{n_\mu}$

(c) $$\int_{SU(2)} dg\, g_{\alpha_1\beta_1}\cdots g_{\alpha_n\beta_n} = I = \int dg\, (g_0 g)_{\alpha_1\beta_1}\cdots(g_0 g)_{\alpha_n\beta_n}$$

half α/β 1

① $g_0 = -\mathbb{1}$ $\qquad I = (-1)^n I \xrightarrow{I\neq 0} n$ even

② $g_0 = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$ $\qquad (g_0)_{\alpha\beta} = \delta_{\alpha\beta}\, e^{(-1)^\alpha i\theta}$

$$(g_0 g)_{\alpha\beta} = e^{(-1)^\alpha i\theta} g_{\alpha\beta}$$

left inv. $I = e^{i\theta\sum(-1)^{\alpha_i}} I \xrightarrow{I\neq 0} \sum(-1)^{\alpha_i} = 0 \Rightarrow$ half $\alpha_i = 1$

righ inv. $\qquad$ half $\beta_i = 1$

Recap. Peter – Weyl theorem

compact G.

$$L^2(G) \cong \bigoplus_\mu \mathrm{End}(V^\mu)$$

① compact G unitary irrep f.d.

② $\langle M^\mu_{ij}, M^\nu_{i'j'} \rangle = \frac{1}{n_\mu} \delta_{\mu\nu} \delta_{ii'} \delta_{jj'}$

Complete

③ $\bigoplus_\mu \mathrm{End}(V^\mu) \xrightarrow{\cong} L^2(G)$

$\iota(\bigoplus_i S_i) := \sum_i \varphi_{S_i}$ $\qquad \varphi_S = \mathrm{Tr}_V(S\,T(g))$

↳ Corollary : finite G

$$\boxed{|G| = \sum_\mu n_\mu^2}$$

$\dim V^\mu = n_\mu$

$\mathrm{End}\, V^\mu \cong \mathrm{Mat}_{n_\mu \times n_\mu}$

$|S_3| = 6 = 1^2 + 1^2 + 2^2$ $\qquad \cong D_3$

$\cancel{4 \times 1^2}$

Ortho. relations of χ_μ

$\{\chi_\mu\}$ ON basis of $L^2(G)^{\text{class}} \subset L^2(G)$

$f(g) = f(hgh^{-1}) \quad \forall g, h \in G.$

$$V \cong \bigoplus a_\mu V^{(\mu)}$$

$$a_\mu = \langle \chi_\mu, \chi_V \rangle = \int_G \overline{\chi_\mu(g)}\, \chi_V(g)\, dg$$

Character table $\quad r$ - irreps $\quad r \times r$

	$m_1 C_1$ ($\to E$)	$m_2 C_2$	$\cdots$	$m_r C_r$
trivial $\to V^1$	1	1	$-\ -$	1
$\vdots$				
V^r				

$$\frac{1}{|G|} \sum_{\{C_i\}} m_i\, \overline{\chi_\mu(C_i)}\, \chi_\nu(C_i) = \delta_{\mu\nu}$$

$$\sum_\mu \overline{\chi_\mu(C_i)}\, \chi_\mu(C_j) = \frac{|G|}{m_i} \delta_{ij}$$

S_3

	$[1]$	$3[(12)]$	$2[(123)]$
1^+	1	1	1
1^-	1	-1	1
2	2	0	-1

① $\mathbb{R}^3$ rep of S_3

$$1 = 1_3 \qquad (12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (132) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$\chi \qquad 3 \qquad 1 \qquad 0$

$a_\mu = \langle \chi_\mu, \chi_V \rangle$ $\quad a_{1^+} = 1$

$a_{1^-} = 0$

$a_2 = 1$

$\mathbb{R}^3 \cong \underline{V_{1^+}} \oplus \underline{V_2}$

---

② regular rep of S_3 or any finite G.

$\chi_V(e) = |G|$

$\chi_V(g \neq e) = 0$

$a_\mu = \langle \chi_\mu, \chi_V \rangle = \frac{1}{|G|} \cdot |G| \cdot \underline{\chi_\mu(e)} = \underline{\dim V^\mu}$

$$\boxed{L^2(G) \cong \oplus_\mu \underline{(\dim V^\mu)} \cdot \underline{V^\mu}}$$

$L^2(S_3) \cong 1 \cdot V_{1^+} \oplus 1 \cdot V_{1^-} \oplus 2 \cdot V_2$

4. V a vector space. $\dim V = d$

S_2 permutes on $\underline{V \otimes V}$ $\quad \dim = d^2$

$\underline{\sigma : v_i \otimes v_j \longmapsto v_j \otimes v_i}$

$\chi_{V\otimes V}(1) = d^2$ $\qquad \underline{v_i \otimes v_i}$

$\chi_{V\otimes V}(\sigma) = d$

	1	σ
1^+	1	1
1^-	1	-1

$a_{1^+} = \langle \chi^{1^+}, \chi_{V\otimes V} \rangle = \frac{1}{2} d(d+1)$

$a_{1^-} = \langle \chi^{1^-}, \chi_{V\otimes V} \rangle = \frac{1}{2} d(d-1)$

$$V \otimes V = \tfrac{1}{2}d(d+1)\,V^{1+} \oplus \tfrac{1}{2}d(d-1)\,V^{1-}$$

$T_{ij}\, v_i \otimes v_j \in V \otimes V$, basis

$\begin{cases} \text{symmetric tensors} & \frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i) \\ \text{antisymmetric tensors} & \frac{1}{2}(e_i \otimes e_j - e_j \otimes e_i) \end{cases}$

$\hookrightarrow \quad S_3 = V \otimes V \otimes V \quad ?$

$\longrightarrow \quad S_n = \underbrace{V \otimes V \otimes \cdots \otimes V}_{n} \quad ?$

– Explicit decomposition of a representation

$\hookrightarrow$ How to block diagonalize the rep. ?

isotypic decomposition ?

Let (T, V) be any rep of a compact group G.

$$P_{ij}^{(\mu)} := n_\mu \int_G \overline{(M_{ij}^{(\mu)}(g))}\, T(g)\, dg \in \mathrm{End}(V)$$

$$= \sqrt{n_\mu} \int_G \overline{(\phi_{ij}^{(\mu)}(g))}\, T(g)\, dg$$

$M_{ij}^{(\mu)}$ on ON basis of V^μ.

③

$$P^{(\mu)}_{ij} P^{(\nu)}_{kl} = n_\mu n_\nu \int_{G\times G} \overline{\mathcal{M}^{(\mu)}_{ij}(g_1)}\ \overline{\mathcal{M}^{(\nu)}_{kl}(g_2)}\ T(g_1 g_2) dg_1 dg_2$$

$$= n_\mu n_\nu \int_{G\times G} \overline{\mathcal{M}^{(\mu)}_{ij}(g_1)}\ \overline{\mathcal{M}^{(\nu)}_{kl}(g_1^{-1} g_2)}\ T(g_2) dg_1 dg_2$$

$$= \mathcal{M}^{(\nu)}_{ks}(g_1^{-1})\ \mathcal{M}^{(\nu)}_{sl}(g_2)$$

$$= \overline{\mathcal{M}^{(\nu)}_{sk}(g_1)}\ \mathcal{M}^{(\nu)}_{sl}(g_2)$$

$$= n_\mu n_\nu \int_{G\times G} \overline{\mathcal{M}^{(\mu)}_{ij}(g_1)}\ \mathcal{M}^{(\nu)}_{sk}(g_1) \cdot \overline{\mathcal{M}^{(\nu)}_{sl}(g_2)}\ T(g_2) dg_1 dg_2$$

$$= n_\mu \delta_{\mu\nu} \delta_{is} \delta_{jk} \int_G \overline{\mathcal{M}^{(\nu)}_{sl}(g_2)}\ T(g_2)\, dg_2$$

$$= \sum_s \delta_{\mu\nu} \delta_{is} \delta_{jk} P^{(\nu)}_{sl} = \delta_{\mu\nu} \delta_{jk} P^{(\nu)}_{il}$$

$$P^{(\mu)}_{ij} \cdot P^{(\nu)}_{kl} = \delta_{\mu\nu} \delta_{jk} P^{(\nu)}_{il}$$

Claim. $\forall \psi \in V$. $P^{\mu}_{ij}\psi$ transforms as V^{μ}.

$h \in G$.

$$T(h) P^{\mu}_{ij} = n_\mu T(h) \int_G dg\, \overline{\mathcal{M}^{(\mu)}_{ij}(g)}\ T(g) \leftarrow$$

$$= n_\mu \int_G dg\, \overline{\mathcal{M}^{(\mu)}_{ij}(h^{-1} g)}\ T(g) \leftarrow$$

$$= n_\mu \sum_k \int_G dg\, \mathcal{M}^{(\mu)}_{ki}(h) \cdot \overline{\mathcal{M}^{(\mu)}_{kj}(g)}\ T(g)$$

$$= \sum_k^{n_\mu} \mathcal{M}^{(\mu)}_{ki}(h) P^{(\mu)}_{kj} \qquad (\text{fix } \mu, j)$$

$\forall \psi \in V$. (s.t. $P^{\mu}_{ij}\psi \neq 0$)

$\text{span}\{ P^{\mu}_{ij}\psi \mid i = 1, \cdots n_\mu \}$ (fix $\mu. j$)

transfrms as $(T^\mu \cdot V^\mu)$

$$P^{(\mu)}_{ij} := n_\mu \int_G \overline{M^\mu_{ij}(g)}\, T(g)\, dg$$

$$P_\mu = \sum_{i=1}^{n_\mu} P^\mu_{ii} = n_\mu \int_G \overline{\chi_\mu(g)}\, T(g)\, dg$$

$$P_\mu P_\nu = n_\mu n_\nu \int_{G\times G} \overline{\chi_\mu(g)}\, \overline{\chi_\nu(h)}\, T(gh)\, dg\, dh$$

$$= n_\mu n_\nu \int_{G\times G} \overline{\chi_\mu(g)}\, \overline{\chi_\nu(g^{-1}h)}\, T(h)\, dg\, dh$$

HW. $\int_G \overline{\chi_\mu(g)}\, \overline{\chi_\nu(g^{-1}h)}\, dg = \frac{\delta_{\mu\nu}}{n_\mu} \overline{\chi_\nu(h)}$

$$= \delta_{\mu\nu} \underline{n_\nu \int \overline{\chi_\nu(h)}\, T(h)\, dh} = \delta_{\mu\nu} P_\nu$$

$P_\mu P_\nu = \delta_{\mu\nu} P_\nu$

$\ulcorner$ $P_\mu P_\nu = \sum_i \sum_j P^\mu_{ii} P^\nu_{jj} = \delta_{\mu\nu} P_\nu$ $\lrcorner$

$$P_\mu^\dagger = n_\mu \int_G \chi_\mu(g)\, T^\dagger(g)\, dg \qquad \text{unitary}$$

$$= n_\mu \int_G \underline{\overline{\chi_\mu(g^{-1})}}\, \underline{T(g^{-1})}\, dg$$

$$= P_\mu$$

$$\mathrm{Tr}(P^\mu) = n_\mu \underbrace{\int_G \overline{\chi_\mu(g)}\, \chi(g)\, dg} = n_\mu \cdot a_\mu$$

Examples.

1. $P = \int_G T(g)\, dg$ trivial rep. projector

$$T(h) P = \int_G \underbrace{T(h) T(g)}\, dg = P.$$

$$\int_G T(g)\, dg$$

$T(h)(P\varphi) = P\varphi \quad \forall \varphi. \Rightarrow$ trivial rep

2. S_3 on $\mathbb{R}^3$

HW: $\mathbb{R}^3 \cong V_{(+} \oplus V_2$. [GM] p221

- Block diagonalization of Hamiltonians

$$P_{ij}^{(\mu)} = n_\mu \int_G \overline{M_{ij}^{\mu}(g)}\, T(g)\, dg$$

$\{ \varphi_a \in L^1(G) \}$ G : symmetry group

$$\mathcal{H} \cong \oplus \mathcal{H}^{(\mu)}$$

$$\varphi_{ia}^{\mu j}(x) := P_{ij}^{\mu} \cdot \varphi_a(x)$$

$$\mathcal{H}^{(\mu)} \cong D_\mu \otimes V^\mu$$

$x \in \mathbb{R}^3$

In QM. $|\mu, i, a\rangle = C_i^\mu \int dx\, \varphi_{ia}^{\mu i'}(x) |x\rangle$ (fix i')

$$\langle \mu, i, a | \nu, j, b \rangle = \int dx\, \overline{\varphi_a(x)}\, (P_{ii'}^{\mu})^{+} (P_{ji'}^{\nu})\, \varphi_b(x)$$

$$(P_{ii'}^{\mu})^{+} = P_{i'i}^{\mu}$$

$$= \delta_{\mu\nu} \delta_{ij} \int dx\, \overline{\varphi_a(x)}\, P_{i'i'}^{\nu}\, \varphi_b(x)$$

$\Rightarrow$ states belonging to different irreps are orthogonal

$$\langle \mu, i | \nu, j \rangle = \delta_{\mu\nu} \delta_{ij} \quad \text{normalization.}$$

Given an Hamiltonian H. $[H, T(G)] = 0$

$$\underline{H = T^{+}(g) H T(g)} \qquad (\forall g \in G)$$

$$\langle \mu, i, a | H | \nu, j, b \rangle = \int_G \langle \mu, i, a | T^{+}(g) H T(g) | \nu, j, b \rangle dg$$

$$= \int_G dg \sum_{kl} \overline{M^{\mu}_{ki}(g)} \langle \mu, k, a | H | \nu, l, b \rangle \, M^{\nu}_{lj}(g)$$

$$= \sum_{kl} \frac{1}{n_\mu} \delta_{\mu\nu} \delta_{kl} \delta_{ij} \langle \mu k a | H | \nu l b \rangle$$

$$= \frac{1}{n_\mu} \delta_{\mu\nu} \delta_{ij} \sum_{k}^{n_\mu} \underline{\langle \mu, k, a | H | \mu, k, b \rangle}$$

$\Rightarrow$ H is block diagonal in $\underline{\mu k}$

$$\underline{\langle \mu, i, a | \nu, i, b \rangle = O^{\mu}_{ab}} \rightarrow \text{not guaranteed to be orthogonal}$$

(Gram – Schmidt)

Examples.

1. $G = \mathbb{Z}_2$ as linear operator $\{E, P\}$

$$Px = -x$$

$$E\,\varphi(x) = \varphi(x)$$

$$P\,\varphi(x) = \varphi(P^{-1}x) = \varphi(-x) \qquad (P^2 = E)$$

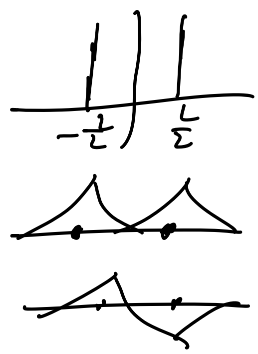
M_{1+}: $M(E) = M(P) = 1$

M_{1-}: $M(E) = 1 \quad M(P) = -1$

$$P^+ = \frac{1}{|G|}\sum_g \overline{M(g)}\,T(g) = \frac{1}{2}(T(E) + T(P))$$

$$P^- = \frac{1}{2}(T(E) - T(P))$$

$$\begin{cases} P^+\varphi(x) = \frac{1}{2}(\varphi(x) + \varphi(-x)) \\ P^-\varphi(x) = \frac{1}{2}(\varphi(x) - \varphi(-x)) \end{cases}$$



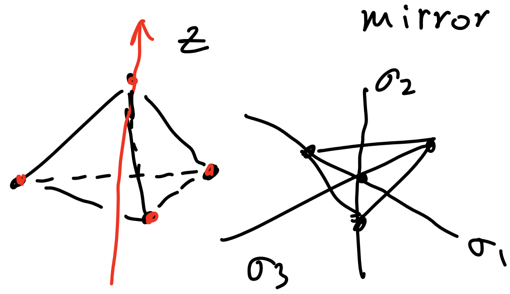
$\hookrightarrow$ even & odd parity solution

Example 2. $G = C_{3v}$ C_3 with additional mirror // principle axis

$C_3 \to (123)$

$\sigma \to (12)$

$C_{3v} \cong S_3$



C_{3v}	E	$2C_3(z)$	$3\sigma_v$
A_1	$+1$	$+1$	$+1$
A_2	$+1$	$+1$	-1
E	$+2$	-1	0