# Semester review

$$\begin{tikzcd} & \text{G T} \arrow[dl] \arrow[dr] & \\ \text{A. Groups} & & \text{B. Group rep.} \end{tikzcd}$$

## A. Groups

1 Def. of groups $(G, m, I, e)$

① set $G$.

② $\underline{m}: G \times G \longrightarrow G$ $\Big\}$ closure

③ $\underline{I}: G \longrightarrow G$

④ $e \in G$. $g \cdot e = e \cdot g = g$ $\quad \underline{m}(g, \underline{I}(g)) = e$

○ $\underline{m}, \underline{I}$ : $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ "+" "×" ×

○ finite vs. infinite : $|G| \overset{?}{<} \infty$

○ abelian vs non-abelian $ab \overset{?}{=} ba$

semidirect & direct product

a. semi. $H \rtimes_\alpha G$ $\quad \alpha: \mathrm{Aut}(H)$

$$(h_1, g_1) \cdot_\alpha (h_2, g_2) := (h_1 \underline{\alpha_{g_1}(h_2)}, g_1 g_2)$$

b. $\alpha$ trivial $(h_1, g_1)\cdot(h_2, g_2) = (h_1h_2, g_1g_2)$

**subgroups** $H \subset G$

$\left.\begin{array}{ll} \underline{m}: & H\times H \to H \\ \underline{I}: & H \to H \end{array}\right\}$

$\{e\}$, $G$. proper $H \neq G$.

↳ $H \triangleleft G$. : $gHg^{-1} = H$

↳ simple group.

↳ **centralizer**

$C_G(h) = \{g \in G.\ gh = hg\}$

$C_G(H) = \{g \in G,\ gh = hg.\ \forall h \in H\}$

↳ $C_G(G) =: \underline{Z(G)}$

**normalizer**:

$N_G(H) := \{g \in G.\ \underline{gHg^{-1} = H}\}$

$C_G(H) \subseteq N_G(H)$

**GL(n, k)**:

**subgroups**: ① SL

O. SO

U. SU

$\}$ $\det U = ?$

② $\underline{A^T J A = J}$

$\hookrightarrow O(p,q)$ $\quad J_{pq} = \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix}$

$\rightarrow O(1,3)$

symplectic $\quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

<u>group presentation</u>.

$$G = \langle \underbrace{g_1 \cdots g_n}_{\text{generator}} \mid \underbrace{R_1 \cdots R_r}_{\text{relations}} \rangle$$

$\mu_N : \langle A \mid A^N = 1 \rangle$

$D_n . \langle A . B \mid A^n = B^2 = (AB)^2 = 1 \rangle$

$\mathbb{Z} : \langle 1 \rangle$

2. <u>Homomorphism & isomorphism</u> :

$\varphi : G \to G'$

$$\begin{array}{ccc} G \times G & \xrightarrow{m} & G \\ {\scriptstyle \varphi\times\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ G' \times G' & \xrightarrow{m'} & G' \end{array}$$

$\varphi(g_1) \cdot_{G'} \varphi(g_2) = \varphi(g_1 \cdot_{G} g_2)$

$\hookrightarrow \begin{cases} \varphi(e) = e' \\ \varphi(g^{-1}) = \varphi(g)^{-1} \end{cases}$

$$\ker\varphi = \{ g\in G : \varphi(g) = 1_{G'} \}$$

$$\mathrm{im}\,\varphi = \varphi(G)$$

Ex. ① $\pi : SU(2) \longrightarrow SO(3)$

$$\ker\pi = \{\pm 1\}$$

$$\mathrm{im}\,\pi = SO(3)$$

② $T : G \longrightarrow GL(V)$

isomorphism : hom + (1-1 & onto invertible)

1-1: $\ker\varphi = \{e\}$

onto: $\varphi(G) = G'$

$\hookrightarrow$ $G = G'$ : $\mathrm{Aut}(G)$

Ex. ① $\mu_N \cong \mathbb{Z}_N$

② $GL(V) \cong GL(n, K)$

matrix rep : $T : G \longrightarrow GL(n, K)$

$$T(g)\hat{e}_i = T(g)_{ji}\,\hat{e}_j$$

$\hookrightarrow$ $T \cong T'$ $\quad \exists S \in GL(n, K)$

$$T'(g) = S\,T(g)\,S^{-1} \quad (\forall g\in G)$$

3. Group action $G$ on $X$

$$\Phi : G \to S_X := \{ X \xrightarrow{f} X \ , \ f \text{ invertible} \}$$

$$g \mapsto \phi(g, \cdot)$$

$$\Phi_g(x) = \phi(g, x) =: g \cdot x$$

$$g_2 \cdot (g_1 \cdot x) = (g_1 g_2) x$$

↳ orbits $\quad \mathrm{Orb}_G(x) = \{ g \cdot x \ , \ g \in G \}$

① $x \sim y$ iff $g \cdot x = y$

② partition of $G$

$$O_G(x) = O_G(x') \quad \text{or} \quad O_G(x) \cap O_G(x') = \emptyset$$

set $X/G$

↳ fixed points $\quad \mathrm{Fix}_X(g) = \{ \underset{x \in X}{\underline{gx = x}} \} \subset \underline{X}$

↳ Stabilizer $\quad \mathrm{Stab}_G(x) = \{ g \in G : \underline{g \cdot x = x} \} \subset G$

group action is :

1. effective : $\mathrm{Fix}_X(g \neq e) \neq X$
2. transitive : $\mathrm{Orb}_G(x) = X$
3. free : $\mathrm{Fix}_X(g \neq e) = \emptyset$

Theorem (Stab - orbit)

$$O_G(x) \xrightarrow{\cong} G/G^x$$

$$g \cdot x \longmapsto g \cdot G^x$$

finit $G$: $|O_G(x)| = [G : G^x]$

Ex. $SO(3)$ on $S^2$

$$S^2 \cong SO(3)/SO(2)$$

$$S^2 \cong SU(2)/U(1)$$

4. $G$ action on $G$.

① $H \subset G$. right action on $G$

$$gH = \{ gh : h \in H \}$$

$\hookrightarrow$ $\underline{g_1 H} = g_2 H$ $\quad g_1 H \cap g_2 H = \emptyset$

(Langrange) ~~Finite~~ Finite $G$.

$$|G| / |H| = [G : H]$$

② action by $\underline{\text{conjugacy}}$

$h \sim h'$ if $h' = g h g^{-1}$

$$C(h) = \{ g h g^{-1} : g \in G \} = h^G$$

$$|C(g)| = [G : \underline{C_G(g)}]$$

$\hookrightarrow$ $\mathrm{Stab}_G(g)$

Finite $G$: $|C(g)| = \dfrac{|G|}{|C_G(g)|}$

$+ \ \Sigma' |C(g)| = |G|$

$\Rightarrow$ class eq.

$$|G| = \sum_{\{C_g\}} \frac{|G|}{|C_G(g)|}$$

↪ ① $|G| = p^n \rightarrow Z(G) \neq \{e\}$

② (Cauchy)

$p \mid |G| \Rightarrow \exists g.$ order $p$

↪ <u>class function</u> : $f$ on $G$

$$f(g g_0 g^{-1}) = f(g_0) \quad \forall g, g_0 \in G$$

↪ <u>$\chi_T(g) = Tr\ T(g)$</u>

5. morphisms of $G$-spaces / equivariant map

$f : X \rightarrow X'$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Phi(g)"'] & X' \arrow[d, "\Phi'(g)"] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

## 6. $S_n$

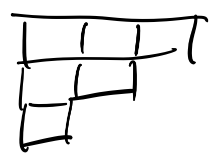
$$\begin{pmatrix}1&2&3&4\\2&3&1&3\end{pmatrix} =: (1243)$$

① $\phi \in S_n$ : unique cycle decomposition

② r-cycles conjugate

$\hookrightarrow$ conj. class labeled as eg

$$\vec{\lambda} = \{3, 2, 1\}$$

Young diagram  $S_6$

$$\mathrm{sgn}: S_n \longrightarrow \mathbb{Z}_2$$

$$\phi \longmapsto \mathrm{sgn}(\phi) := (-1)^{n-t}$$

$A_n \triangleleft S_n$ $\left(\longleftarrow H \subset G.\ [G:H] = 2\right)$

$$|A_n| = \frac{1}{2}|S_n|$$

## 7. quotient groups

$N \triangleleft G$ : $G/N$

$$(g_1 \cdot N)\cdot(g_2 \cdot N) := (g_1 g_2) N$$

$$\mu: G \to G/N$$

$$g \longmapsto gN$$

$\ker \mu = N$

Theorem: $G/\ker\mu \cong \operatorname{im}\mu$

Ex. $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n \quad (\mu : i \mapsto i + n\mathbb{Z})$

SES: $1 \longrightarrow \ker\mu \longrightarrow G \xrightarrow{\mu} \operatorname{im}\mu \longrightarrow 1$

ES: $\longrightarrow G_{i-1} \xrightarrow{f_{i-1}} G_i \xrightarrow{f_i} G_{i+1}$

$$\ker f_i = \operatorname{im} f_{i-1}$$

$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$

- $N \cong H \triangleleft G$
- $Q \cong G/H$

$1 \longrightarrow \mathbb{Z}_2 \longrightarrow SU(2) \xrightarrow{\pi} SO(3) \longrightarrow 1$

$A \subset Z(E) \qquad E \qquad G$

# B. Group rep.

1. Def. ① $G \to GL(V) \cong GL(n, K)$

$$g \mapsto T(g) \longmapsto M(g)$$

② equivalent rep

$$\begin{tikzcd} V_1 \arrow[r, "A"] \arrow[d, "T_1(g)"'] & V_2 \arrow[d, "T_2(g)"] \\ V_1 \arrow[r] & V_2 \end{tikzcd}$$

$$\Rightarrow T_2(g) = A T_1(g) A^{-1}$$

③ unitary.

$$\langle \underline{U(g)} w, \underline{U(g)} v \rangle = \langle w, v \rangle$$

↪ compact / finite

2. Haar measure:

$$f : G \to \mathbb{C}$$

$$\int_G dg : f \mapsto \langle f \rangle$$

$$\int_G f(hg) dg = \int_G f(g) dg \qquad \text{left-inv.}$$

$G$. finit / compact   left = right

Ex. $\mathbb{R}$,   $\mathbb{R}^*_{>0}$   $\int \frac{dx}{x}$

$SU(2)$   $\int \frac{1}{16\pi^2} d\psi \, d\phi \, \sin\theta \, d\theta$

unitarization :

$$\underline{\langle v, w \rangle_2} := \int_G \langle T(g)v, T(g)w \rangle_1 \, dg$$

3. <u>Regular rep</u> $f \in Map(G, \mathbb{C})$

$$\textcircled{?} \ \underline{[(g_1, g_2) f](h) = f(g_1^{-1} h g_2)}$$

$$( (g_1, g_2) \cdot g_0 = g_1 g_0 g_2^{-1} )$$

$$\underline{G \times G} \longrightarrow End(\{f\})$$

$$L^2(G) = \{ f : G \to \mathbb{C} \mid \int_G |f(g)|^2 dg < \infty \}$$

$$\int_G |f(g)|^2 dg = \langle f, f \rangle$$

(Hilbert space)

$\hookrightarrow$ $\underline{G \times \{1\}}$ or $\{1\} \times G$

$$\hookrightarrow \quad \underline{\delta_g}(g') = \begin{cases} 1 & g' = g \\ 0 & \text{otherwise} \end{cases}$$

$$g_1 \cdot \delta_{g_2} = \delta_{g_1 g_2}$$

4. reducible & irreducible.

$\exists W \subset V$ invariant subspace

completely : $V \cong \oplus W^{\mu}$

Ex. ① Abelian.

② $S_n$ rep on $\{\vec{e}_i,\ i=1,\dots,n\} =: V$

$$V = W \oplus W^{\perp}$$

$W \ni \{\sum \hat{e}_i\}$

③ $L^2(G)$

④ non-compact $\underline{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}}$ $x$

↪ isotypic decomposition

$$V \cong \bigoplus_{\mu} \underline{\mathrm{Hom}_G(V^{\mu}, V)} \otimes V^{\mu}$$

$$\cong \bigoplus_{\mu} K^{a_{\mu}} \otimes V^{\mu}$$

$$=: \bigoplus_{\mu} a_{\mu} \underline{V^{\mu}} \qquad a_{\mu} = \dim_K \mathrm{Hom}_G(V^{\mu}, V)$$

## 5. Schur's lemma

$$\begin{array}{ccc} V_1 & \xrightarrow{A} & V_2 \\ {\scriptstyle T_1(g)}\downarrow & & \downarrow{\scriptstyle T_2(g)} \\ V_1 & \xrightarrow{A} & V_2 \end{array}$$

$V_1, V_2$. irreps

① $A = 0$. or an isomorphism

② $V_1 = V_2 = V$ (complex vec. space)

$\underline{A(v) = \lambda v}$ $(\lambda \in \mathbb{C})$

For physics: $\mathcal{H}$

$$[H, T(G)] = 0$$

$$\mathcal{H} \cong \bigoplus_{\mu} \underbrace{\mathrm{Hom}_G(\mathcal{H}^{\mu}, \mathcal{H})}_{\circlearrowleft \, \underline{D^{\mu}}} \otimes \mathcal{H}^{\mu}$$

$$H \cong \underline{H^{\mu}} \otimes 1_{\mathcal{H}^{\mu}}$$

$$\tilde{H} = S H S^{-1} = \begin{pmatrix} \underline{\boxed{\phantom{x}}}^{\,e_1} & 0 & & 0 \\ & \boxed{\phantom{x}}^{\,e_2} & 0 & \\ \phi & & \boxed{\phantom{x}} & \\ \bar{u} & & & \boxed{\phantom{x}} \\ & & & & \boxed{\phantom{x}} \end{pmatrix}$$

6. Pontryagin dual. $S$ abelian

$$\hat{S} = \mathrm{Hom}(S, U(1)) \ni \chi_1, \chi_2$$

$$(\chi_1 \cdot \chi_2)(s) := \chi_1(s)\, \chi_2(s)$$

$$\hat{\hat{S}} = \mathrm{Hom}(\hat{S}, U(1))$$

(P.vK.) $S \in LCA$ : $\hat{\hat{S}} \cong S$

| $S$ | $\hat{S}$ |
|---|---|
| $\mathbb{Z}_n$ | $\mathbb{Z}_n$ |
| $\mathbb{R}$ | $\mathbb{R}$ |
| $U(1)$ | $\mathbb{Z}$ |
| $\mathbb{Z}$ | $U(1)$ |

$\hookrightarrow \quad \Gamma \cong \mathbb{Z}^d$

$$\hat{\Gamma} \cong U(1)^d \longrightarrow BZ.$$

$$\underline{L_\gamma \psi(x)} = \psi(x+\gamma) \qquad \underline{k} \sim k + 2n\pi$$

$$\hookrightarrow = \chi_{\underline{k}}(\gamma)\psi(x) \qquad \gamma \in \Gamma$$

$$\begin{cases} \psi(x) = e^{2\pi i k x} u_k(x) \\ u_k(x) = u_k(x+\gamma) \end{cases}$$

## 7. Peter — Weyl theorem & orthogonal relations

$$L^2(G) \cong \oplus_\mu \mathrm{End}(V^\mu)$$

finite $G$ : $|G| = \sum_\mu n_\mu^2$

① $\langle \underline{M^{\mu_1}_{i_1 j_1}}, M^{\mu_2}_{i_2 j_2} \rangle = \frac{1}{n_\mu} \delta^{\mu_1 \mu_2} \delta_{i_1 i_2} \delta_{j_1 j_2}$

② $\iota : \oplus_\mu \mathrm{End}(V^\mu) \longrightarrow L^2(G)$

$$\oplus_i S_i \longmapsto \underline{\sum_i \psi_{S_i}}$$

$\hookrightarrow$ $\{\chi_\mu\}$ ON basis of $L^2(G)^{\text{class}}$

$$\int_G dg\, \overline{\chi_\mu(g)}\, \chi_\nu(g) = \delta_{\mu\nu}$$

$\hookrightarrow$ character table. row & column

↪ projectors $P^{\mu}_{ij} = n_\mu \int_G \overline{u^{\mu}_{ij}(g)}\, T(g)\, dg$

$$P^{\mu} = n_\mu \int_G \overline{\chi_\mu(g)}\, T(g)\, dg$$

$$P^{\mu} P^{\nu} = \delta_{\mu\nu} P^{\nu}$$

↪ ① general finite group

"class operator":

$$\hat{C}_i = \sum_{\mu=1}^{r} \lambda_i^{\mu} P^{\mu}$$

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② $S_n$

$C = PQ$

↳ Schur - Weyl duality

$$V^{\otimes n} \cong \oplus D^{\lambda} \otimes V^{\lambda}$$

reps $G$ ← irreps of $S_n$

$V$ irrep. → $D^{\lambda}$ irrep

$V = \mathbb{C}^d$ → irrep $GL(d, \mathbb{C})$

8. induced rep. $\Psi \in \mathrm{Map}(G, V)$

$$(g, h)\,\Psi(g_0) = \rho(h)\,\Psi(g^{-1} g_0 h)$$

Ind $V$ = fixed points $\{1\} \times H$

$$= \{ \Psi : G \to V \mid \Psi(g h^{-1}) = \rho(h)\,\Psi(g) \}$$

$$\begin{array}{ccc} V_c & \xrightarrow{\rho_{\mathrm{Ind}}(g)} & V_{gC} =: C' \\ \downarrow {\scriptstyle ev_c} & & \downarrow {\scriptstyle ev_{c'}} \\ V & \xrightarrow[\rho_V(g_{c'}^{-1} g g_c)]{} & V \end{array}$$

$C = C'$

$$\chi_{\mathrm{Ind}} = \sum_{gC = C} \chi_V(g_c^{-1} g g_c)$$

$$\mathrm{Ind}\, V \cong \bigoplus V_c \qquad \dim = [G : H] \dim V$$