Recap. Homomorphism & isomorphism

$\varphi : G \longrightarrow G'$. s.t. $\forall g_1, g_2 \in G$

$$\varphi(\underline{m}(g_1, g_2)) = \underline{m}'(\varphi(g_1), \varphi(g_2))$$

$$\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$$

$$\varphi(e) = e'$$

$$\varphi(g^{-1}) = \varphi(g)^{-1}$$

Homomorph.

$$\begin{array}{ccc} G \times G & \xrightarrow{m} & G \\ \varphi\times\varphi \downarrow & & \downarrow \varphi \\ G' \times G' & \xrightarrow{m'} & G' \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{I} & G \\ \varphi \downarrow & & \downarrow \varphi \\ G' & \xrightarrow{I'} & G' \end{array}$$

① φ injective : $\varphi(g) = e'$ iff $g = e$ → bijective

$\Longleftrightarrow \varphi(g_1) = \varphi(g_2) \Rightarrow g_1 = g_2$

② φ surjective $\forall g' \in G'$ $\exists g \in G$ $\varphi(g) = g'$

→ Isomorphism

isomorphic groups are the same

φ iso.: $\underline{G} \xrightarrow{\varphi} \underline{G}$ $\quad Aut(G)$

kerenel & image

$\varphi . \ G \to H$

$K = \ker \varphi = \{ g \in G : \mu(g) = 1_H \}$

$\operatorname{im} \varphi = \varphi(G)$

$K \subset G$ subgroup

$\operatorname{im} \varphi \subset H$ subgroup

φ iso. $\quad K = \{ 1_G \} \to$ injectivity

$\operatorname{im} \varphi = H \to$ surjectivity

$\mu_N \cong \mathbb{Z}_N \qquad \varphi(\bar{r}) := e^{i \frac{2\pi}{N} r}$

$\mathbb{Z}_N \to \mu_N$

$\pi : SU(2) \to SO(3) \qquad u \in SU(2), \ \pi(u) \in SO(3)$

$u \vec{x} \cdot \vec{\sigma} u^{+} := \underbrace{\sum (\pi(u)_{ij} x_j) \cdot \sigma_i}_{\equiv \vec{y}}$

$\pi(u_1 u_2) = \pi(u_1) \pi(u_2)$

$\det (u \vec{x} \cdot \vec{\sigma} \cdot u^{T}) = -\vec{x}^2$

$\Rightarrow \vec{x}^2 = \vec{y}^2$

$\det (\vec{y} \cdot \vec{\sigma}) = -\vec{y}^2$

$\underline{\pi(u) = \pi(-u)}$

$\pi(u) \in O(3) \qquad \underline{\det A = 1}$

"double cover"

$u \approx 1 \to \pi(u) \approx 1$

$1 \vec{x} \cdot \vec{\sigma} \cdot 1 = \vec{y} \cdot \vec{\sigma} \qquad \underline{\vec{y} = \vec{x}} \qquad \pi(u) \in SO(3)$

$$T : G \longrightarrow \underline{GL(n, K)} \simeq GL(V) \quad V = K^n$$

n-dim vector space

$V = \{\vec{e}_1, \vec{e}_2, \dots \vec{e}_n\}$

"matrix representation"

$$T(g_1) T(g_2) = T(g_1 g_2)$$

$$T'(g) = \underline{S}\, T(g) S^{-1} \qquad e_i = \sum_{j=1}^{n} S_{ji} e'_j$$

μ_3

$$T(1) = \mathbb{1}_3$$

$$T(\omega) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$T(\omega^2) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

D_4

30.09.22

4. Group actions on sets

Definition. Given any set X, the set of permutations

$$S_X := \{ X \xrightarrow{f} X : f \text{ 1-1 \& onto (invertible) } \}$$
(bijective)

is a group under composition $m(f_1, f_2) := f_1 \circ f_2$

$$\begin{tikzcd} X \arrow[r, shift left, "f_2"] \arrow[rr, bend right, "f_1 \circ f_2"'] & X \arrow[l, shift left, "f_2^{-1}"] \arrow[r, shift left, "f_1"] & X \arrow[l, shift left, "f_1^{-1}"] \end{tikzcd}$$

Definition. A (left) group action Φ of G is a homomorphism:

$$\Phi : G \longrightarrow S_X$$
$$g \longmapsto \phi(g, \cdot)$$

($\phi(g,\cdot) : G \times X \to X$ $\quad \phi(g,x) \in X \ (\forall x \in X)$

$$\phi(g_1, \phi(g_2, x)) = \phi(g_1 g_2, x) \qquad \cancel{(R: x.g)}$$

$$\phi(1_G, x) = x \quad (\forall x \in X)$$

$$\phi(g, \phi(g^{-1}, x)) = \phi(g g^{-1}, x) = \phi(1_G, x) = x$$

$$g \cdot x := \phi(g, x)$$

$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$$

if a set X has a group action by a group G

X : G-set.

Example: 1. $X = G$ action by <u>multiplication</u>

def: $g \cdot x = gx \in \underline{G} = X \quad x \in X = G$.

① $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$

2. $X = G$ action by <u>conjugation</u>

def: $g \cdot x = g x g^{-1} \in G = X$

① $g_1 \cdot (g_2 \cdot x) = g_1 \cdot (g_2 x g_2^{-1}) = g_1 \, g_2 \, x g_2^{-1} g_1^{-1}$

$= (g_1 g_2) x (g_1 g_2)^{-1}$

$= (g_1 g_2) \cdot x$

② $e \cdot x = e \cdot x \cdot e^{-1} = x$

G. is an abelian group $\quad g \cdot x = g x g^{-1}$

$= g g^{-1} x = x \quad \forall g \in G$

3. $GL(n, K) =: G$. $\quad X = K^n$. $\quad v = (v_1, \cdots v_n)^T \in X$

$A \cdot \vec{v} = \sum_j A_{ij} v_j$

matrix rep. a group action on

the carrier space <u>V</u>

4 <u>Space group</u> acts on $\mathbb{R}^3$

$SG : \{g|\tau\}$ g: point group elements

proper & improper rotations

reflections etc.

τ : translation

def: $\{g|\vec{\tau}\}\cdot\vec{r} = R_g\vec{r} + \vec{\tau}$ R_g : 3×3 on $\mathbb{R}$

$$\underline{\{g_1|\vec{\tau}_1\}}\,\underline{\{g_2|\vec{\tau}_2\}}\,\vec{r} = \{g_1|\vec{\tau}_1\}(R_2\vec{r}+\vec{\tau}_2)$$
$$= R_1(R_2\vec{r}+\vec{\tau}_2)+\vec{\tau}_1$$
$$= R_1R_2\vec{r} + (R_1\vec{\tau}_2+\vec{\tau}_1)$$
$$= \underline{\{g_1g_2|R_1\vec{\tau}_2+\vec{\tau}_1\}}$$

$$\{g|\vec{\tau}\} = \left(\begin{array}{c|c} 1 & 0 \\ \hline \vec{\tau} & \underset{3\times3}{R_g} \end{array}\right) \qquad \vec{r} \to \begin{pmatrix} 1 \\ \vec{r} \end{pmatrix}$$

$$\{g_1|\vec{\tau}_1\}\{g_2|\vec{\tau}_2\} = \begin{pmatrix} 1 & 0 \\ \tau_1 & R_1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \tau_2 & R_2 \end{pmatrix}$$

$$= \left(\begin{array}{c|c} 1 & 0 \\ \hline R\tau_2+\tau_1 & R_1R_2 \end{array}\right) \equiv \{g_1g_2|R_1\vec{\tau}_2+\vec{\tau}_1\}$$

Definition (orbits) Let X be a G-set

the orbit of G through a point $x \in X$ is the set

$$O_G(x) \; \left(\mathrm{Orb}_G(x) \right) := \{ y : \exists g \in G, \text{ s.t. } y = g \cdot x \}$$
$$\equiv \{ g \cdot x \mid \forall g \in G \}$$

$x \sim y$ iff $\exists g \in G$. s.t. $g \cdot x = y$

(i) reflexive: $x \sim x$ $\Rightarrow (e \cdot x = x)$

(ii) symmetric: $x \sim y \Leftrightarrow y \sim x$ $\quad y = g \cdot x$, $x = g^{-1} \cdot y$

(iii) transitive: $x \sim y$, $y \sim z \Rightarrow x \sim z$

$y = g_1 x$, $z = g_2 y$ $\Rightarrow z = (\underbrace{g_2 g_1}_{\in G}) x$

$O_G(x)$ are equivalence classes under group action.

Distinct orbits of G partition X:

① $\forall x \in O_G(x)$ $\quad \forall x \in X$

② $O_G(x_1) \cap O_G(x_2) \neq \emptyset \ni x \Rightarrow O_G(x_1) = O_G(x_2)$

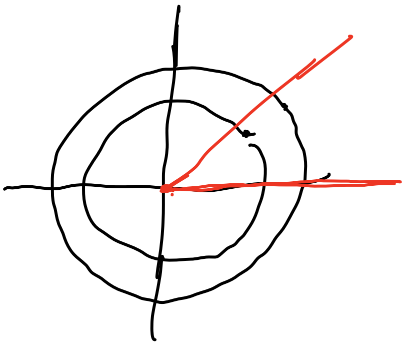
X is covered by disjoint orbits

The set of orbits is denoted X/G

Examples

1. $G = SO(2, \mathbb{R})$ on $\mathbb{R}^2$

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\phi - y\sin\phi \\ x\sin\phi + y\cos\phi \end{pmatrix} \equiv \begin{pmatrix} x' \\ y' \end{pmatrix}$$

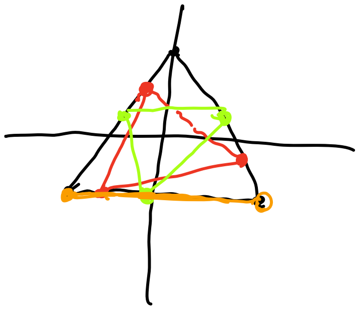


$\mathbb{R}^2 / SO(2)$: $O_G(x)$

$x \in [0, +\infty)$

2. $X = \Delta$ $\quad G = C_3 = \{ R(0), R(2\pi/3), R(4\pi/3) \}$

$\cong \mathbb{Z}_3$ or μ_3 $\quad \subset SO(2, \mathbb{R})$



Δ / C_3 :

3. SG on $\mathbb{R}^3$: orbits "Wyckoff positions"

4. $G = GL(n, K)$ $\quad X = K^n$ n-dim vector space on K

orbits. $O_0 = \{\vec{0}\}$

$O_x = \{ \vec{x} \in K^n \mid \vec{x} \neq 0 \}$

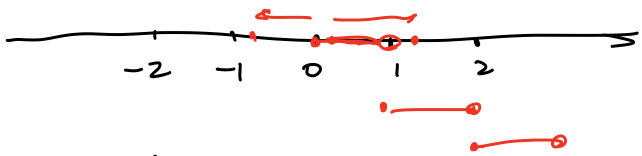
5. $G = \langle g \rangle \cong \mathbb{Z}$ $\qquad g^m \cdot g^n = g^{m+n} \longrightarrow m+n$

$G \longrightarrow g$

$O_x = \{ g^n \cdot x ,\ n \in \mathbb{Z} \}$

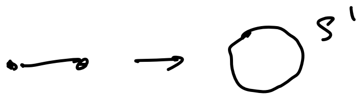
$\mathbb{Z}$ on $\mathbb{R}$:
$\shortparallel$ $\quad$ $\shortparallel$
G $\quad$ X

$n : x \longmapsto x + n$



$\mathbb{R}/\mathbb{Z}$. $\quad x \in [0, 1) \cong S^1$

$x \in \mathbb{R}/\mathbb{Z}$. $\qquad p(x) := e^{2\pi i x}$



6. $G = \mathbb{Z}_2 = \{ e, \sigma \}$ $\quad (\sigma^2 = e)$ $\quad$ acts on $\mathbb{R}^{n+1}$

$$\sigma \cdot (x^1, \dots x^{n+1})^T = (x^1, \dots x^p, \underline{-x^{p+1}, \dots -x^{p+q}})^T$$

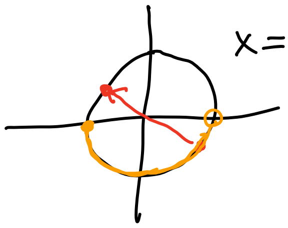
$$\sigma = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix} \qquad p+q = n+1$$

$\sum_i (x^i)^2 - 1 = 0$ is preserved. $\quad$ n-dim S^n

Consider $p = 0$ $\quad q = n+1$

$\sigma \cdot \vec{x} = -\vec{x}$

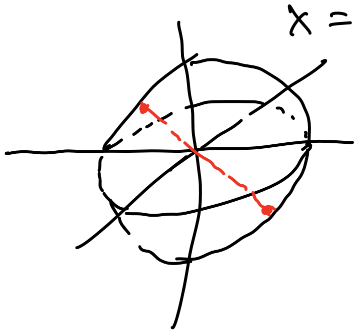
$n=1$ $X = S^1$

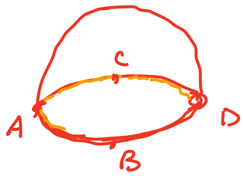


$\Longrightarrow$

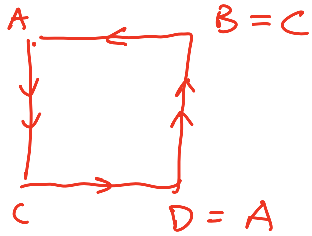
$= S^1$

$n=2$ $X = S^2$

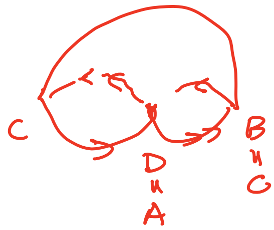




$AC = DB$



$\rightarrow$



$\rightarrow$

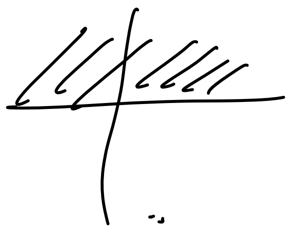
$S^n / \mathbb{Z}_2 = \mathbb{RP}^n$ $\quad \mathbb{RP}^1 = S^1.$ $\quad \mathbb{RP}^n \neq S^n \ (n \geq 2)$

7. $G = \underline{SL}(2, \mathbb{R}) := \{ A \in M_2(\mathbb{R}) \mid \det A = 1 \}$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad ad - bc = 1$$

acts on complex upper plane

$$\mathcal{H} = \{ \tau \mid \operatorname{Im} \tau > 0 \}$$



$$A \cdot \tau := \frac{a\tau + b}{c\tau + d}$$ "Mobius transformation"

① $A_1 \cdot A_2 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \underline{\begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}}$

② $A_1 \cdot (A_2\tau) = A_1 \cdot \left(\frac{a_2\tau + b_2}{c_2\tau + d_2} \right) = \frac{a_1(\quad) + b_1}{c_1(\quad) + d_1}$

$$= \frac{a_1(a_2\tau + b_2) + b_1(c_2\tau + d_2)}{c_1(a_2\tau + b_2) + d_2(c_2\tau + d_2)}$$

i. $A_1(A_2\tau) = (A_1A_2)\tau$

ii. $e = \mathbb{1}_2$

iii. A^{-1}

$$\underline{\mathrm{Im}(g\tau)} = \mathrm{Im}\left(\frac{a\tau + b}{c\tau + d} \right) = \frac{\mathrm{Im}[(a\tau + b)(c\tau + d)^*]}{|c\tau + d|^2}$$

$$= \frac{\mathrm{Im}\,[(a\tau + b)(c\tau^* + d)]}{|\quad|^2}$$

$\mathrm{Im}\,\tau^* = -\,\mathrm{Im}\,\tau$

$$= \frac{\mathrm{Im}(\cancel{ac|\tau|^2} + ad\tau - bc\tau^* \cancel{+bd})}{|\quad|^2}$$

$g \in SL(2,\mathbb{R})$ $$= \frac{\mathrm{Im}\,(\overset{1}{\cancel{ad - bc}})\tau}{|\quad|^2}$$

$$= \frac{\mathrm{Im}\,\tau}{|c\tau + d|^2}$$

$g \in SL(2,\mathbb{C})$ does not preserve the upper plane.

Definition. Let X, X' be two G-spaces

An <u>equivariant map</u> $f : X \to X'$

satisfy

$$f(\underline{g \cdot x}) = \underline{g} \cdot \underline{f(x)} \qquad \forall x \in X$$
$$\forall g \in G$$

f "morphism of G-spaces"

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Phi(g)"'] & X' \arrow[d, "\Phi'(g)"] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

$$f(\phi(g.x)) = \phi'(g, f(x))$$

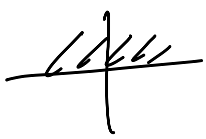
Example.

1 $G = \mathbb{Z}_2$ acts on $\mathbb{R}^{n+1}$

$M : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ equivariant iif

$$M = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array} \right) \qquad \begin{array}{l} A \in M_p(\mathbb{R}) \\ D \in M_q(\mathbb{R}) \end{array}$$

2. $SL(2, \mathbb{R})$ on $\mathbb{H}$



$e : \mathbb{H} \to \overline{\mathbb{H}}$

$c(\tau) \mapsto \bar{\tau}$ equivariant map

3. $\mathbb{Z}$ on $\mathbb{R}$.

$$\phi_n : x \longrightarrow x + n \qquad n \in \mathbb{Z}$$

$f : \mathbb{R} \to \mathbb{R}$ equivariant iff.

$f(x) = x + \alpha$ for some $\alpha \in \mathbb{R}$

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{f} & \mathbb{R} \\ \phi \downarrow & & \downarrow \phi \\ \mathbb{R} & \xrightarrow{f} & \mathbb{R} \end{array} \qquad \frac{f(x) + n_1 = f(x + n_1)}{f(x) = x + \alpha.}$$

Group actions on sets $\rightarrow$ induce actions on associated function space.

X, Y two G-sets. $\mathcal{F}[X \to Y]$ is the set of functions from X to Y.

$\phi : G \times X \longrightarrow X$ left G-action.

We can define a corresponding G-action $\tilde{\phi}$ on $\mathcal{F}[X \to Y]$

$$\tilde{\phi}(g, F)(x) := F(\phi(g^{-1}, x)) \in \mathcal{F}[X \to Y]$$

$$(\underbrace{g \cdot F}_{\tilde{\phi}})(x) = F(\underbrace{g^{-1} \cdot x}_{\phi})$$

$$1.\quad (1_G \cdot F)(x) = F(1_G^{-1}\, x) = F(x) \quad \checkmark$$

$$2.\quad [g_1\cdot(g_2\cdot F)](x) = (g_2\cdot F)(g_1^{-1}\cdot x)$$
$$= F[g_2^{-1}\cdot(g_1^{-1}\cdot x)] = F[(g_1 g_2)^{-1}\cdot x] \quad \checkmark$$
$$= [(g_1\cdot g_2)\cdot F](x)$$

<u>Example</u> $G: \{g|\vec{\tau}\}$ on $\mathbb{R}^3$

$F[\mathbb{R}^3 \to \mathbb{C}] \longrightarrow \varphi(\vec{r})$ wavefunction

$$\underline{(g\cdot\varphi)(\vec{r}) = \varphi(g^{-1}\cdot\vec{r})}$$

$g = \{e|\vec{\tau}\}$