Recap. Pontrjagin dual.

Abelian S $\quad \hat{S} = \mathrm{Hom}(S, U(1))$

S	$\hat{S}$
$\mathbb{R}$	$\mathbb{R}$
$U(1)$	$\mathbb{Z}$
$\mathbb{Z}$	$U(1)$
$\mathbb{Z}/n\mathbb{Z}$	$\mathbb{Z}/n\mathbb{Z}$

PvK. $\quad S \in LCA \quad \hat{\hat{S}} = S$

$\Gamma \cong \mathbb{Z}^d$.

$\chi_{\bar{k}}(\gamma) = \exp(2\pi i\, k\cdot\gamma) \qquad \bar{k} = k + \Gamma^\vee$

↳ wavefunction

$L_\gamma \varphi(x) = \varphi(x+\gamma) = \chi_{\bar{k}}(\gamma)\,\varphi(x)$

↳ Bloch theorem

$u_k(x+\gamma) = u_k(x)$

$\varphi(x) = e^{2\pi i kx} u_k(x)$

Ortho. relations of matrix elements of rep.

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Peter – Weyl theorem.

① Compact G. unitary irrep (V, T) is finite dimensional

$$\underline{A(w) = \int_G dg \langle T(g)v, w\rangle \underline{T(g)v}.}$$

② $\langle M^{\mu}_{ij}, M^{\nu}_{i'j'}\rangle = \frac{1}{n_\mu}\delta_{\mu\nu}\delta_{ii'}\delta_{jj'}$ ✓

B. completeness ?

span $\{M^{\mu}_{ij}\} = W \subset L^2(G)$

$W^{\perp} \longrightarrow \bigoplus_{\mu} V^{\mu}$

$\underline{f_i \in W^{\perp}}$ transforms as V^{μ}

$R(g)f_j = \sum_k \underline{M(g)^{\mu}_{kj}} f_k$

right regular rep $R(g)f\cdot(h) = f\cdot(hg)$

$\underline{f_j(g)} = \sum_k \underline{f_k(1)}\, \underline{M^{\mu}_{kj}(g)}$ $(\forall g \in G)$

$\hookrightarrow f_j \in W$

$\Rightarrow W = L^2(G)$

left reg. rep

$$L(g)\cdot f_j = f_j(g^{-1}h) = \sum M^{\mu}_{kj}(g) f_k(h)$$

$h=1$

$$\Rightarrow \underline{f(g)} = \sum M^{\mu}(g^{-1})_{kj} f_k(1)$$

$$= \sum f_k(1) \underline{M^{\mu}(g)^{*}_{jk}} \qquad \{M^{*}\}$$

$\{M^{\mu}_{ij}\}$ is complete.

$\Rightarrow$ $\{M^{k}_{ij}\}$ is an orthogonal basis

$\phi^{k}_{ij} = \sqrt{n}\, M^{k}_{ij}$ ON basis.

Peter-Weyl theorem:

$$L^2(G) \underset{=}{\overset{u}{\cong}} \oplus_{\mu} \mathrm{End}(V^{(\mu)})$$

Recall:

$$\mathrm{End}(V) \hookrightarrow L^2(G)$$

$$S \longmapsto \mathrm{Tr}_V(S\,T(g)) := \Psi_S$$

$$e_{ij} \longmapsto M^{tr,-1}_{ij}$$

$$\bigoplus_\mu \mathrm{End}\, V^\mu \xrightarrow{\iota} L^2(G)$$

$$\iota\left(\bigoplus_i s_i\right) := \sum_i \varphi_{s_i}$$

injective + surjective $\rightarrow$ isomorphism

Corollary finite G.

$$|G| = \sum_\mu n_\mu^2$$

LHS: $\dim L^2(G) = |G|$

RHS: $\bigoplus_\mu \mathrm{End}(V^\mu)$ $\qquad \dim(\mathrm{End}(V^\mu)) = n_\mu^2$

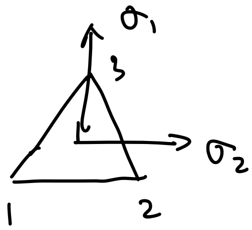
(HW7) S_3 irreps

$$|S_3| = 6 = \underset{\substack{\uparrow \\ \text{trivial}}}{1^2} + \underset{\substack{\uparrow \\ \mathrm{sgn}(\phi)}}{1^2} + 4 \qquad 4 = 1+1+1+1 = 2^2$$

① trivial: $M^+(\phi) = 1 \quad \forall \phi \in S_3$

② sgn: $M^-(\phi) = 1 \quad \phi \in \{(\,), (123), (132)\}$

$M^-(\phi) = -1 \quad \phi \in \{(12), (13), (23)\}$

③ 

$$M^{(2)}((12)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$M^{(2)}((13)) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$M^{(2)}((123)) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\begin{cases} M_{11}^{(2)}(\,) = M_{11}^{(2)}(12) = 1 \\ M_{11}^{(2)}(13) = M_{11}^{(2)}(23) = -\frac{1}{2} \\ M_{11}^{(2)}(123) = M_{11}^{(2)}(132) = -\frac{1}{2} \end{cases}$$

① $\langle M^+, M^- \rangle = 0 \qquad \delta^{\mu\nu} \qquad \mu = + \qquad \nu = -$

② $\langle M^+, M_{11}^{(2)} \rangle = \frac{1}{6} \sum_g M_{11}^{(2)}(g) = \frac{2}{6}(1 - \frac{1}{2} - \frac{1}{2}) = 0$

③ $\langle M_{11}^{(2)}, M_{11}^{(2)} \rangle = \frac{2}{6}(1 + \frac{1}{4} + \frac{1}{4}) = \frac{1}{2} = \frac{1}{n_\mu}$

Interlude: finding irreps of S_3

a. trivial $\quad$ 1-dim

b. $\mathrm{sgn}(\phi)$ $\quad$ 1-dim.

$A_3 \subset S_3 \qquad A_3 = \langle \tau \rangle \qquad \tau$ - 3-cycle.

W an arbitrary representation spanned by eigenvectors of τ

$$W = \oplus V_i \qquad V_i = \mathbb{C} v_i$$

$$\underline{\tau v_i = \lambda_i v_i = \omega^i v_i} \qquad \begin{matrix} (\omega^3 = 1) \\ \tau^3 = 1 \end{matrix}$$

Conjugate by a transposition σ.

$$\sigma \tau \sigma = \tau^2 \Leftrightarrow \tau \sigma = \sigma \tau^2$$

$$\tau(\sigma(v)) = \sigma\tau^2(v) = \omega^{2i}\sigma(v)$$

① $\omega^i \neq 1$ $\quad v, \sigma(v)$ linearly independent

$\Rightarrow$ 2D rep

② $\omega^i = 1$

if $\begin{cases} \sigma(v) = v. & \text{trivial} \\ \sigma(v) = -v & \text{sgn}. \end{cases}$

if $\sigma(v), v$. linearly independent

$\{\sigma(v)+v\} \cong$ trivial

$\{\sigma(v)-v\} \cong$ sgn

Remarks.

1. later with character theory

reg. rep.

$$V \cong \bigoplus (\dim V^\mu) \cdot V^\mu$$

2. How many 1D irreps?

Answer: index of $G' = [G, G]$

$$[g_i, g_j] = g_i g_j g_i^{-1} g_j^{-1}$$

1D reps $\quad T([g_i, g_j]) = T(g_i) T(g_j) T(g_i^{-1}) T(g_j^{-1})$

$$= 1$$

trivial

$$\begin{tikzcd} G \arrow[r, "T"] \arrow[d, "\pi"'] & \mathbb{C}^* \\ G/G' \arrow[ur, "\overline{T}"'] & \end{tikzcd}$$

$T: G \to \mathbb{C}^*$

$\overline{T}: G/G' \to \mathbb{C}^*$

$gG' \mapsto \overline{T}(gG')$

$T = \overline{T} \cdot \pi$

$\pi: G \to G/G'$

$g \mapsto gG'$

G/G' abelian $\Rightarrow$ distinct irrep $= |G/G'|$

$|G/G'|$ index $[G:G']$

Further examples

1 $G = \mathbb{Z}_2 = \langle \sigma \mid \sigma^2 = 1 \rangle$

$\varphi: G \to \mathbb{C}$

$\varphi(1) = \varphi_+ \in \mathbb{C}$ $\qquad L^2(\mathbb{Z}_2) \cong \mathbb{C}^2$

$\varphi(\sigma) = \varphi_-$

$\mu^+(1) = \mu^+(\sigma) = 1$

$\mu^-(1) = 1 \;, \; \mu^-(\sigma) = -1$

$$\varphi = \frac{\varphi_+ + \varphi_-}{2} \mu^+ + \frac{\varphi_+ - \varphi_-}{2} \mu^-$$

$$\left\{ \begin{aligned} \varphi(1) &= \frac{\varphi_+ + \varphi_-}{2} + \frac{\varphi_+ - \varphi_-}{2} = \varphi_+ \\ \varphi(\sigma) &= \frac{\varphi_+ + \varphi_-}{2} - \frac{\varphi_+ - \varphi_-}{2} = \varphi_- \end{aligned} \right.$$

2 $G = \mathbb{Z}/n\mathbb{Z} = \langle \omega \mid \omega^n = 1 \rangle \qquad \omega = e^{\frac{2\pi}{n} i}$

all irreps 1-dim. $\quad V = \mathbb{C}$

$$\rho_m(\omega) = \omega^m \qquad (m \in \mathbb{Z})$$

$$\mathcal{M}^{(m)}(\omega^j) = \omega^{mj} = e^{\frac{2\pi i}{n}(m \cdot j)}$$

$$\frac{1}{|G|} \sum_{g \in G} \mathcal{M}^{(m_1)}(g)^* \mathcal{M}^{(m_2)}(g) = \delta_{(m_1 - m_2) \bmod n,\, 0}$$

$$\begin{cases} \Psi = \sum_m \hat{\Psi}_m \mathcal{M}^{(m)} & \in \{ f : \mathbb{Z}_n \to \mathbb{C} \} \\ \underline{\hat{\Psi}_m} = \langle \mathcal{M}^{(m)}, \Psi \rangle = \int_{\mathbb{Z}_n} (\mathcal{M}^{(m)}(g))^* \Psi(g)\, dg \end{cases}$$

$$\Psi : G \to \mathbb{C}$$

$$\hat{\Psi} = (\mathcal{F}\Psi) : \hat{G} \to \mathbb{C}$$

3. $\quad G = U(1) = \{ z \in \mathbb{C} \mid |z| = 1 \}$

(ρ_n, V_n): $\quad \rho_n(z) = z^n \quad n \in \mathbb{Z}. \qquad z = e^{i\theta}$

$$V_n \cong \mathbb{C}$$

$\underline{\{ e^{in\theta} \}}$ are ON basis.

$\Rightarrow$ L^2-function $\Psi(\theta)$ on the circle can be expanded as

$$\Psi = \sum_{\{\rho_n\}} \hat{\Psi}_n \underline{\rho_n}$$

– orthogonal relations of characters

character tables

Class function on G:

$$f : G \to \mathbb{C}$$

$$f(g) = f(hgh^{-1}), \quad \forall g, h \in G$$

$$L^2(G)^{class} \subset L^2(G) \qquad \text{a subspace.}$$

<u>Theorem</u>. The characters $\{\chi_\mu\}$ is an orthonormal basis for the vector space of class functions $L^2(G)^{class}$.

<u>Proof</u>. $$\underline{\int_G [dg]\, \overline{M_{ij}^{(\mu)}(g)}\, M_{kl}^{(\nu)}(g) = \frac{1}{n_\mu} \delta_{\mu\nu} \delta_{ik} \delta_{jl}}$$

$i = j$, $k = l$

$$\int_G [dg]\, \overline{M_{ii}^{(\mu)}(g)}\, M_{kk}^{(\nu)}(g) = \frac{1}{n_\mu} \delta_{\mu\nu} \delta_{ik}$$

$$\Rightarrow \int_G [dg]\, \overline{\chi_\mu(g)}\, \chi_\nu(g) = \sum_{ik} \frac{1}{n_\mu} \delta_{\mu\nu} \delta_{ik}$$

$$= \delta_{\mu\nu}$$

$\Rightarrow \{\chi_\mu\}$ is ON set.

Completeness :

$$\forall f \in L^2(G) \xrightarrow{P.W} f(g) = \sum_{\substack{\mu, \\ i,j}} \hat{f}^{\mu}_{ij}\, u^{\mu}_{ij}(g)$$

$f \in L^2(G)^{\text{class}}$.

$$f(g) = f(hgh^{-1})$$

$$\underline{f(g) \equiv \int_G dh\, f(g)} = \int_G dh\, f(hgh^{-1})$$

RHS.
$$\int_G f(hgh^{-1})\, dh = \sum_{\mu,i,j} \hat{f}^{\mu}_{ij} \int_G \underline{u^{\mu}_{ij}(hgh^{-1})}\, dh$$

$$\| $$

$$u^{\mu}_{ik}(h)\, u^{\mu}_{kl}(g)\, \underline{u^{\mu}_{lj}(h^{-1})}$$

$$= \sum_{\substack{\mu,i,j \\ k l}} \hat{f}^{\mu}_{ij}\, \underline{u^{\mu}_{kl}(g)} \underbrace{\int_G u^{\mu}_{ik}(h)\, \overline{u^{\mu}_{jl}(h)}\, dh}_{\frac{1}{n_\mu}\delta_{ij}\underline{\delta_{kl}}}$$

$$L^2(G)^{\text{class}} \ni \underline{f(g)} = \underline{\sum_{\mu,i} \frac{\hat{f}_{ii}}{n_\mu} \chi_\mu(g)}$$

$\Rightarrow \{\chi_\mu\}$ span full $L^2(G)^{\text{class}}$.

isotypic decomposition

$$V \cong \oplus\, a_\mu V^\mu \implies \chi_V = \sum_\mu a_\mu \chi_\mu$$

$$a_\mu = \underline{\langle \chi_\mu, \chi_V \rangle} = \int_G \overline{(\chi_\mu(g))}\, \chi_V(g)\, dg.$$

isomorphism class of a rep. of

a compact group is completely

determined by its character function.

– Character table of finite groups

For finite groups, we can define a set of class functions

$$\delta_{C_i}(g) = \begin{cases} 1 & g \in C_i \\ 0 & \text{else} \end{cases}$$

C_i : different conj. classes.

$\{\delta_{C_i}\}$ is a basis of $L^2(G)^{\text{class}}$

Theorem. The number of conjugacy classes of a finite group G is the same as the number of irreducible representations of G.

The character table is a $r \times r$ matrix

$r = \#$ of irreps $= \#$ conj. class

E

	$m_1 C_1$	$m_2 C_2$	$\cdots$	$m_r C_r$
trivial → V^1	$\chi_1(C_1) = 1$	$\chi_1(C_2) = 1$	$\cdots$	$\chi_1(C_r) = 1$
V^2	$\chi_2(C_1)$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$			
V^r	$\chi_r(C_1)$	$\cdots$	$\cdots$	$\cdots$

$$m_i = |C_i|$$

$$\int_G [dg]\, \overline{\chi_\mu(g)}\, \chi_\nu(g) = \delta_{\mu\nu}$$

$\Downarrow$ finite G

$$\frac{1}{|G|} \sum_{\{C_i\}} m_i\, \underline{\overline{\chi_\mu(C_i)}}\, \chi_\nu(C_i) = \delta_{\mu\nu}$$

$$S_{\mu i} = \sqrt{\frac{m_i}{|G|}}\, \chi_\mu(C_i)$$

$$\sum_{i=1}^{r} S_{\mu i} \overline{S_{\nu i}} = \delta_{\mu\nu} \Leftrightarrow S \cdot S^{+} = 1_{r\times r}$$

$$(S^+ S)_{ij} = \sum_{\mu=1}^{r} \overline{S_{\mu i}}\, S_{\mu j} = \delta_{ij}$$

$$\sum_{\mu} \overline{\chi_\mu(C_i)}\, \chi_\mu(C_j) = \frac{|G|}{m_i} \delta_{ij}$$

Example.

1. $G = S_2 = \{1, (12)\}$

	$[1]$	$[(12)]$
1^+	1	1
1^-	1	-1

2. $G = \mathbb{Z}/n\mathbb{Z}$ $\quad (\hat{\mathbb{Z}}_n = \mathbb{Z}_n)$

irrep. $\rho_m \quad m \in \mathbb{Z}/n\mathbb{Z}$.

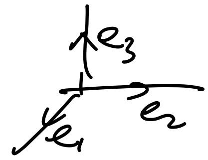
$n=3$: $\rho_m(\bar{j}) = \omega^{mj}$, $\omega = e^{i\frac{2\pi}{3}}$

	$[\bar{0}]$	$[\bar{1}]$	$[\bar{2}]$
ρ_0	1	1	1
ρ_1	1	ω	ω^2
ρ_2	1	ω^2	ω

3. $G = S_3$

	$[1]$	$3[(12)]$	$2[(123)]$	
1^+	1	1	1	$1\times1 + 3(1\times-1)$
1^-	1	-1	1	$+2(1\times1) = 0$
2	2	0	-1	$1\times2 + 2\times(1\times-1) = 0$

$\mathbb{R}^3$ rep of S_3



$$1 = \mathbb{1}_3 \qquad (12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (132) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\chi_{\mathbb{R}^3} = \quad 3 \qquad\qquad 1 \qquad\qquad 0$$

$$a_\mu = \langle \chi_\mu, \chi_V \rangle = \frac{1}{|G|} \sum_g \overline{\chi_\mu(g)}\, \chi_V(g)$$

	[1]	3 [(12)]	2 [(123)]
1^+	1	1	1
1^-	1	-1	1
2	2	0	-1

$$a_{1^+} = \frac{1}{6}(3 + 3\times 1 + 0) = 1$$

$$a_{1^-} = \frac{1}{6}(3 + 3\times(-1) + 0) = 0$$

$$a_2 = \frac{1}{6}(3\times 2 + 0 + 0) = 1$$

$$\mathbb{R}^3 \cong V_{1^+} \oplus V_2$$