Recap:

$C_G(g) := \{h \in G : hg = gh\} = \{h \in G \mid hgh^{-1} = g\}$

$|C(h)| = [G : C_G(h)]$

finite $|G|$: $\Rightarrow$ $|G| = \sum_{\{C(g)\}} \dfrac{|G|}{|C_G(g)|}$ "class equation"

$\Leftrightarrow$ $|G| = |Z(G)| + \sum_{\substack{\{C(g)\} \\ g \notin Z(G)}} \dfrac{|G|}{|C_G(g)|}$

Theorems.

① $|G| = p^n$ (p prime)

$\Rightarrow$ $Z(G) \neq \{1\}$

$|G| = 2^3$ $\qquad Z(Q) = \{\pm\} \cong \mathbb{Z}_2$

$Z(\mathbb{Z}_8) = \mathbb{Z}_8$

($|G| = p^2$ $\qquad Z(G) = G$.)

② Cauchy:

$p \mid |G|$. (p prime) $\Rightarrow \exists g \in G$. order p

Representation theory

① V vector space. over K.

$$GL(V) \quad Aut(V) \qquad V \to V.$$

② rep. of G.

$$T: G \to GL(V)$$
$$g \mapsto T(g)$$

$(\underline{V}, \underline{T})$ denotes a rep.

$$T(g_1)T(g_2) = T(g_1 g_2)$$

equip V with $\{e_1 \cdots e_n\}$ n-dim

$$GL(V) \cong GL(n, K)$$

$$T(g)\hat{e}_i = \sum_j M(g)_{ji}\hat{e}_j$$

$$T(g_1)T(g_2) = T(g_1 g_2) \Leftrightarrow M(g_1)M(g_2) = M(g_1 g_2)$$

Examples :

1. regular rep.

$\dim V = |G| = n$.

$$g \to e_g$$

$$T(g_1)\, e_{g_2} = e_{g_1 g_2}$$

2. G on X

$$x \longrightarrow gx$$

$$T(g)\, e_{\underline{x}} = e_{\underline{gx}}$$

permutation rep.

3. $G = \mathbb{Z}, \mathbb{R}, \mathbb{C}$

1-dim $\quad T: G \longrightarrow GL(\mathbb{C})$

$$n \longmapsto a^n \qquad a \in \mathbb{C}^*$$

$$m+n \longrightarrow a^m \cdot a^n$$

2-dim $\quad T: G \longrightarrow GL(2, k)$

$$n \longrightarrow \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

$$m+n \longrightarrow \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & m+n \\ 0 & 1 \end{pmatrix}$$

Def. intertwiner A

$$\begin{tikzcd} V_1 \arrow[r, "A"] \arrow[d, "T_1(g)"'] & V_2 \arrow[d, "T_2(g)"] \\ V_1 \arrow[r, "A"] & V_2 \end{tikzcd} \qquad T_1(g) \cdot A = A \cdot T_2(g)$$

Def equivalent rep. $T_1(g) \cong T_2(g)$

$(T_1, V_1) \cong (T_2, V_2)$

A invertible. $\quad A : V_1 \longrightarrow V_2$

an isomorphism

$$T_2(g) = A\, T_1(g)\, \underset{=}{A^{+}}$$

Def unitary rep.

V inner product space. $\langle \cdot , \cdot \rangle$

$$\langle U(g)v,\, U(g)w \rangle = \langle v, w \rangle \quad \forall\, v, w \in V \quad \forall\, g \in G.$$

QM. $|\langle \phi | \varphi \rangle|^2$

Wigner theorem $\Rightarrow$ Sym. in QM $\begin{cases} \text{unitary} \\ \text{antiunitary} \end{cases}$

$$[\underset{=}{U}, \underset{=}{H}] = 0.$$

– unitary representations (cont.)

<u>Definition</u> : If a rep. (V, T) is equivalent to a unitary rep. then it is said to be unitarizable.

To unitarize a rep. of a <u>finite group</u> :

Let $T(g)$ be a (non-unitary) rep.

$$H = \sum_{g \in G} T^{+}(g)\, T(g)$$ Hermitian & positive def.

$$T^{+}(h) H T(h) = \sum_{g} T^{+}(h) T^{+}(g) T(g) T(h)$$

$$= \sum_{g} T^{+}(gh) T(gh)$$

$$= H$$

$$V^{+} H V = \Lambda = \mathrm{diag}(\lambda_1 \cdots \lambda_n) \qquad (\forall \lambda_i > 0)$$

Define $\tilde{T}(g) = \Lambda^{\frac{1}{2}} V^{+} T(g) V \Lambda^{-\frac{1}{2}}$

$$\tilde{T}^{+}(g) \tilde{T}(g) = \Lambda^{-\frac{1}{2}} V^{+} T^{+}(g) \underline{V \Lambda^{\frac{1}{2}} \Lambda^{\frac{1}{2}} V^{+}} T(g) V \Lambda^{-\frac{1}{2}}$$

$$= \Lambda^{-\frac{1}{2}} V^{+} \underline{T^{+}(g)\, H\, T(g)} V \Lambda^{-\frac{1}{2}}$$

$$= \Lambda^{-\frac{1}{2}} \underbrace{V^{+} H V}_{\Lambda} \Lambda^{-\frac{1}{2}}$$

$$= \mathbb{1}$$

$\Rightarrow T(g) = A\tilde{T}(g)A^{-1} \qquad A = V\Lambda^{-\frac{1}{2}}$

$\Rightarrow$ reps. of finite groups are unitarizable.

(hand-waving)

Compact groups

$\mathbb{C}^m$. compact $\Leftrightarrow$ closed. bounded.

$U(n) = \{ A \in GL(n, \mathbb{C}) \mid A^{+}A = 1 \} \subset \mathbb{C}^{n^2}$

$\sum_j (A^{+})_{ij} A_{ji} = 1$

$\Rightarrow \sum_j (A^{*})_{ij} A_{ij} = \sum_j |A_{ij}|^2 = 1$

$\Rightarrow |A_{ij}| \leq 1 \qquad \forall ij$

$\sum_g \longrightarrow \int dg$ "some" measure

$\Rightarrow$ Reps of finite groups & compact groups are unitarizable.

Compact:

SU, O, SO.

quaternions

$Sp(n) = \{ T \in \mathrm{Mat}(n, \mathbb{H}) : T^{*}T = \mathbb{1} \}$

$q = t + xi + yj + zk$

$\bar{q} = t - xi - yj - zk$

$Sp(1) = \{ q \in \mathbb{H} : |q| = 1 \} = \{ t + xi + yj + zk : t^2 + x^2 + y^2 + z^2 = 1 \}$

$\cong S^3$

$Sp(1) \cong SU(2)$

$Sp(n) \cong U(2n) \cap Sp(2n, \mathbb{C})$

<u>non-compact</u>:

$O(1, d)$, $SL(2, \mathbb{C})$

$Sp(2n, \mathbb{K}) \rightarrow \begin{pmatrix} 1 & B \\ 0 & P \end{pmatrix}$

$GL(n, \mathbb{K})$

- Direct Sum, tensor product, and dual

(T_1, V_1) and (T_2, V_2) are two reps of dims n, m. resp.

each with basis $\{v_1, \cdots v_n\}$, $\{w_1, \cdots w_m\}$

① $V_1 \oplus V_2$: vector space of dim $n+m$

with basis

$\{(v_1, 0), (v_2, 0) \cdots (v_n, 0)$

$(0, w_1), (0, w_2) \cdots (0, w_m)\}$

② $V_1 \otimes V_2$: vector space of dim nm

with basis

$\{v_i \otimes w_j : 1 \le i \le n, \ 1 \le j \le m\}$

$$\left(\sum_i a_i v_i\right) \otimes \left(\sum_j b_j w_j\right) = \sum_{ij} a_i b_j \, v_i \otimes w_j$$

③ dual vector space: $V^* := \mathrm{Hom}(V, K)$

linear maps : $V \to K$.

$\dim V^* = \dim V = n$.

with basis v_i^* $\quad v_i^*(v_j) = \delta_{ij}$

Same for representations:

① on $V_1 \oplus V_2$:

G-action: $g \cdot (v, w) := (g \cdot v, g \cdot w)$

rep. $[(T_1 \oplus T_2)(g)](v \oplus w) := T_1(g) \cdot v \oplus T_2(g) \cdot w$

mat. rep.

$$M_{T_1 \oplus T_2}(g) = \left(\begin{array}{c|c} M_{T_1(g)} & 0 \\ \hline 0 & M_{T_2(g)} \end{array}\right)$$

② on $V_1 \otimes V_2$:

$g \cdot (v \otimes w) := (g \cdot v) \otimes (g \cdot w)$

$[(T_1 \otimes T_2)(g)](v \otimes w) := T_1(g) v \otimes T_2(g) w.$

$(M_1 \otimes M_2)(g)_{ia, jb} = [M_1(g)]_{ij} [M_2(g)]_{ab}$

③ on V^*:

$$\underline{(g\cdot v_i^*)}(v_j) = \underline{v_i^*}\,(\underline{g^{-1}\cdot v_j})$$

(induced G-action on function space)

"natural pairing"

$$\underline{(g\cdot v_i^*)}(\underline{g v_j}) = v_i^*(g^{-1} g\cdot v_j) = v_i^*(v_j) = \underline{\delta_{ij}}$$

$$T(g): V \to V \quad : \quad v \mapsto T(g)v$$

$$T^*(g): V^* \to V^* \qquad v^* \mapsto T^*(g)v^*$$

$$\delta_{ij} = v_i^* v_j = \sum_k T^*(g)_{ki} v_k^* \sum_l T(g)_{lj} v_l$$

$$= \sum_{kl} T^*(g)_{ki} T(g)_{lj} \overbrace{\cancel{(v_k^* v_l)}}^{\delta_{kl}}$$

$$= \sum_l T^*(g)_{li} T(g)_{lj} = \delta_{ij}$$

$$\Leftrightarrow T^*(g) = [T(g^{-1})]^{tr}$$

matrix rep. $M^*(g) = [M(g^{-1})]^{tr}$

— Characters

For any finite-dim. rep. $T: G \to \mathrm{Aut}(V)$ of any group G. define the <u>character of the rep.</u> denoted χ_T. It is a function on the group:

$$\chi_T : G \longrightarrow k$$

$$\chi_T(g) := \mathrm{Tr}_V(T(g))$$

1. independent of basis choice.

2. equivalent $\Leftrightarrow$ same character func.

$$\chi_T(h^{-1}gh) = \chi_T(g) \qquad \text{"class function"}$$

3. $M_{T_1 \oplus T_2}(g) = \begin{pmatrix} M_{T_1}(g) & 0 \\ 0 & M_{T_2}(g) \end{pmatrix}$

$$\chi_{T_1 \oplus T_2} = \chi_{T_1} + \chi_{T_2}$$

4. $(M_1 \otimes M_2)(g)_{ia, jb} = (M_1(g))_{ij} (M_2(g))_{ab}$

$$M_1 \otimes M_2 = \begin{pmatrix} m_1^{11} M_2 & \cdots & \\ & m_1^{22} M_2 & \\ \vdots & & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} m_1^{11} m_2^{11} & & \cdots & \\ & m_1^{11} m_2^{22} & & \\ & & \ddots & \\ \vdots & & & m_1^{22} m_2^{11} \\ & & & & \ddots \end{pmatrix}$$

$$\chi_{T_1 \otimes T_2} = \chi_{T_1} \cdot \chi_{T_2}$$

5. $T_2(g) = A\, T_1(g)\, A^{-1}$

$$\mathrm{Tr}(T_2(g)) = \mathrm{Tr}(A\, T_1(g) A^{-1})$$

$$= \mathrm{Tr}(T_1(g))$$

6. $\chi(1) = n$. dim $V = n$.

$|\chi(g)| \le n$ $(g \ne 1)$

$\chi(g^{-1}) = \chi(g)^{\text{conj}}$.

- <u>Haar measure</u> (or invariant integration measure)

Consider G as a <u>measure space</u>.

[measure space: a set X. and a collection of subsets $\mathcal{B}$. ① includes ϕ. ② closed under complement. countable union. countable intersection]

$(X, \mathcal{B})$

A measure. is then a function

$$\mu : \mathcal{B} \longrightarrow \mathbb{R}_+$$

① $\mu(\phi) = 0$

② $\mu(\cup_k E_k) = \sum_k \mu(E_k)$

↳ countable unions of disjoint sets.

Consider a function $f : G \to \mathbb{C}$.

($f \in \text{Map}(G, \mathbb{C})$)

$$\underbrace{\int_G dg\, f(g)}_{} = \underset{\text{finite group}}{\frac{1}{|G|}\sum_{g\in G} f(g)} = \langle f \rangle$$

$\int_G dg \in (\mathrm{Map}(G, \mathbb{C}))^*$ dual space

$\int_G dg : f \to \langle f \rangle$

require that it satisfies the left-invariant condition:

Haar measure. $\int_G f(hg)\,dg = \int_G f(g)\,dg \quad (\forall h \in G)$

[finite group: $\underline{\frac{1}{|G|}\sum_g \underline{f(g)} = \frac{1}{|G|}\sum_g \underline{f(hg)}}$]

From a group action perspective:

G acts on itself on left.

$L_h : g \mapsto hg$

$(L_h^* f)(g) := f(hg)$

weight

$$\sum_{g\in G} \rho(g)\underline{f(g)} = \sum_{g\in G} \rho(g)\underline{f(hg)} = \sum_{g\in G} \rho(h^{-1}g) f(g)$$

$\forall h.\ \rho(g) = \rho(h^{-1}g) \to \underline{\rho(g)}$ constant.

$\Rightarrow$ For a <u>finite group</u>. left & right invariant measures are unique up to overall scale.

Compact Lie groups:

L: $\int_G f(hg)\,dg = \int_G f(g)\,dg \Rightarrow d(h^{-1}g) = dg$

R: $\int_G f(gh)\,dg = \int_G f(g)\,dg \Rightarrow d(gh^{-1}) = dg$

$(\forall h \in G)$

$\Rightarrow$ For a general topological group G.

(that is locally compact & Hausdorff.).

there exists a unique left invariant measure (up to scale). similarly also a unique right invariant measure.

$\boxed{\text{Left} \neq \text{right}}$ (not necessarily the same)

Example:

1. $G = \mathbb{R}$.

$\int_G dg\, f(g) = c\int_{-\infty}^{\infty} dx\, f(x)$ $\quad c$ constant.

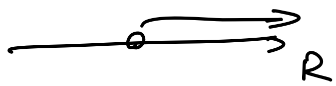
(Lebesgue integration)

$\int_{-\infty}^{\infty} dx\, f(x+a) = \int_{-\infty}^{\infty} dx\, f(x)$

2. $G = \mathbb{Z}$

$$\int_G dg\, f(g) = c \sum_{n \in \mathbb{Z}} f(n)$$

3. $G = \mathbb{R}^*_{>0}$



with multiplication.

$$\int_G f(g)\, dg = c \int_0^\infty f(x)\, dx/x$$

$\forall a \in \mathbb{R}^*_{>0}$

$$\int_0^\infty f(ax) \frac{dx}{x} = \int_0^\infty f(x) \frac{d(x/a)}{x/a} = \int_0^\infty f(x) \frac{dx}{x}$$

4. $G = GL(n, \mathbb{R})$

g_{ij} coordinates on the open domain

$\{ g \mid \det g \neq 0 \} \subset M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$

Euclidean measure $\prod_{ij} dg_{ij}$

$$g \longmapsto g_0 g = g'$$

$$\prod_{ij} dg'_{ij} \longmapsto \left| \frac{\partial(g'_{11}, \cdots g'_{nn})}{\partial(g_{11}, \cdots g_{nn})} \right| \prod_{ij} dg_{ij}$$

$$g'_{ij} = \sum_k (g_0)_{ik} g_{kj}$$

$$\frac{\partial g'_{ij}}{\partial g_{kl}} = (g_0)_{ik} \underline{\underline{\delta_{jl}}}$$

$$= \left| \begin{array}{c} \\ k,l=1 \\ k,l=2 \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} i,j=1 & i,j=2 & & \end{array} \\ \left(\begin{array}{c|c|c|c} g_0 & & & \\ \hline & g_0 & & \\ \hline & & g_0 & \\ \hline & & & \ddots \end{array} \right) \end{array} \right| \prod_{ij} dg_{ij}$$

$$= |\det g_0|^n \prod_{ij} dg_{ij}$$

⇒ Haar measure;

$$\int_{GL(n,\mathbb{R})} f(g)\,dg = c\int_{\det g \neq 0} f(g)\,|\det g|^{-n} \prod_{ij} dg_{ij}$$

$$\int f(g_0 g)\,|\det g|^{-n} \prod_{ij} dg_{ij}$$

$$= \int f(g)\,|\det g_0^{-1} g|^{-n} \underline{\prod_{ij} d(g_0^{-1} g)_{ij}}$$

$$= \int f(g)\,|\det g|^{-n} \cancel{|\det g_0|^{n}\,|\det g_0|^{-n}} \prod_{ij} dg_{ij}$$

$$= \int f(g)\,|\det g|^{-n} \prod_{ij} dg_{ij}$$