Recap: symmetric group

S_n permutations of $X = \{1, \dots, n\}$

$\phi = \begin{pmatrix} 1, 2 & & n \\ p_1 & p_2 & p_n \end{pmatrix}$ $\phi(i) = p_i$

$(\phi_i \phi_j)\phi_k = \phi_i(\phi_j \phi_k)$

$\phi.$ $1 \to 2$
$2 \to 3$
$\vdots$
$r \to 1$

r-cycle
$(1 2, \dots r)$

2-cycle (ij) transposition.

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = (1 2 3 4)$

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} = (1)(2 4 3) = (2 4 3)$

$(2 4 3) = (4 3 2) = (3 2 4)$

$(2 3 4)(5 6) = (5 6)(2 3 4)$

$(2 3 4)^{-1} = (4 3 2)$

Theorem $\phi \in S_n$ decomposition into disjoint cycles.

$\to$ complete fact. $\underbrace{(1)}_{3}\overbrace{(1)(4 3 2)}^{2}$

Cayley's theorem:

G. isomorphic to a subgroup of S_G

$|G| = n$. $\quad G \cong H \subset S_n$

$L(h) : G \to G$

$L(h)\cdot g \mapsto h\cdot g$

$L(h_1)\cdot L(h_2) = L(h_1 h_2)$

$L : G \to \underline{\mathrm{im}\, G} \subset S_G$

$h \mapsto L(h)$ $\quad$ isomorphism

$\mu_N \cong \mathbb{Z}_N \cong A_3 \subset S_3$

$D_4 \quad |D_4| = 8 \quad D_4 \cong H \subset S_8$

$D_4 \cong H \subset S_4$

Cayley's table / multi. table

$G = V$

	1	2	3	4
	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$$\phi(a) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$T(a) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

① $(ij)(jk)(ij) = (ik) = (jk)(ij)(jk)$

② $((ij))^2 = 1$

③ $(ij)(kl) = (kl)(ij)$ $\quad \{i,j\} \cap \{k,l\} = \phi$

transpos. generate $\underline{\phi \in S_n}$

$$\underline{(1, 2, \cdots r) = (1\, r)(1\, r-1) \cdots (1\, 2)}$$

$$\Updownarrow$$

$$\underline{\sigma_i = (i, i+1)}$$

$$\Updownarrow$$

$$\underline{\langle (12), (12 \cdots n) \rangle}$$

τ transpos $\quad sgn(\tau) = -1$ $\quad \underline{n-1}$

$$\left(\begin{array}{l} \phi = \sigma_1 \cdots \sigma_t \in S_n \text{ complete fact.} \\ sgn(\phi) = (-1)^{n-t} \end{array} \right)$$

$$sgn(\tau\phi) = -sgn(\phi)$$

$$sgn(\phi_1\phi_2) = sgn(\phi_1)\, sgn(\phi_2)$$

$$S_n \longrightarrow \mathbb{Z}_2$$

$$\phi \longmapsto sgn(\phi) = \pm 1 \qquad (\epsilon_{ijk})$$

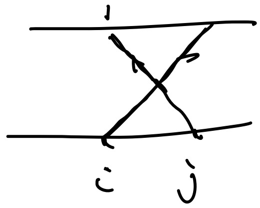
$A_n \subset S_n$ alternating group.

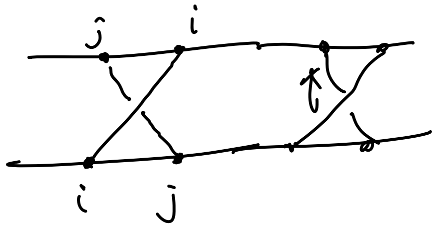
$\{$ even permutation $\}$

$|A_n| = \frac{1}{2}|S_n| = \frac{1}{2}n!$

- Symmetric group & the braiding group

$\tilde{\sigma}_i = (i \quad i+1)$

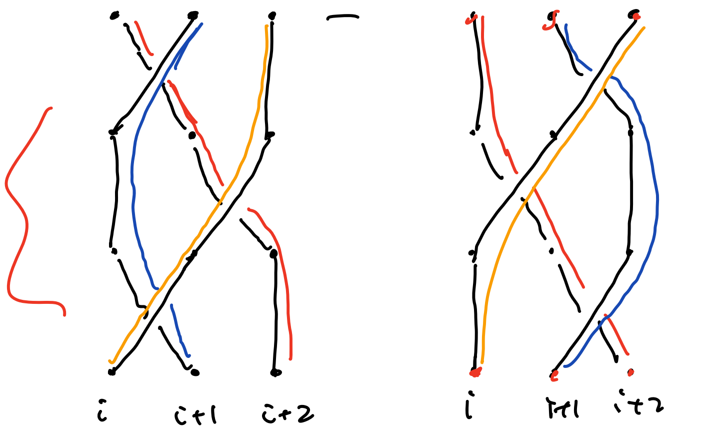


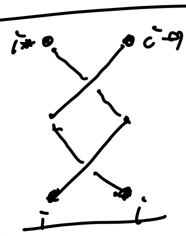


① $\tilde{\sigma}_i \tilde{\sigma}_j = \tilde{\sigma}_j \tilde{\sigma}_i \qquad |i-j| \geq 2$

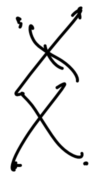
Topological Quantum Comp.

② $\tilde{\sigma}_i \tilde{\sigma}_{i+1} \tilde{\sigma}_i = \tilde{\sigma}_{i+1} \tilde{\sigma}_i \tilde{\sigma}_{i+1}$





$\sigma^n \neq 1$



$= 1$

$B_n := \langle \tilde{\sigma}_1 \cdots \tilde{\sigma}_{n-1} \mid \tilde{\sigma}_i \tilde{\sigma}_j \tilde{\sigma}_i^{-1} \tilde{\sigma}_j^{-1} = 1, \ |i-j| \geq 2$

$\tilde{\sigma}_i \tilde{\sigma}_{i+1} \tilde{\sigma}_i = \tilde{\sigma}_{i+1} \tilde{\sigma}_i \tilde{\sigma}_{i+1} \rangle$

$S_n := \langle \sigma_1 \cdots \sigma_{n-1} \mid \sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1} = 1, \ |i-j| \geq 2$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

$$\langle \underline{\sigma_i^2 = 1} \rangle$$

$$\phi : \quad B_n \longrightarrow S_n$$

$$\tilde{\sigma}_i \longmapsto \sigma_i \quad = \mathbb{1}$$

$\operatorname{Ker} \phi$?

6 Cosets & conjugacy

6.1. Cosets and Lagrange theorem.

Definition Let $H \subset G$ be a subgroup. The set

$$gH := \{gh,\ h \in H\} \qquad (g \in G)$$

is called a left-coset of H

(right-coset $Hg = \{hg,\ h \in H\}$)

g is a representative of gH.

Example: $G = \mathbb{Z}$ $\quad H = n\mathbb{Z}$

$$g + H = \{g + n \cdot r,\ r \in \mathbb{Z}\}$$
$$= \{i \in \mathbb{Z}.\ i = g \bmod n\}$$

$n = 2 \qquad gH = \{H,\ H+1\}$

Example. $G = S_3 \quad H = \{1.\ (12)\} \cong S_2 \cong \mathbb{Z}_2$
$\subset S_3$

$S_3 = \{1,\ (12),\ (13),\ (23)$
$(123),\ (132)\}$

gH: $1 \cdot H = H$

$(12) \cdot H = \{(12),\ 1\} = H$

$(13)\ H = \{(13),\ (13)(12) = (123)\}$

$(23)\ H = \{(23),\ (132)\}$

$(123)H = \{(123), (13)\} = (13)\cdot H$

$(132)H = \{(132), (23)\} = (23)\cdot H$

Theorem (left) cosets are either identical or disjoint

Proof: $g \in g_1H \cap g_2H$

$g = g_1h_1 = g_2h_2 \quad . \quad \exists\, h_1, h_2 \in H$

$g_1 = g_2h_2h_1^{-1} = g_2h_3$

$\Rightarrow g_1H = g_2H$

Left cosets define an equivalence relation

$g_1 \sim g_2$ if $\exists\, h \in H$. s.t. $g_1 = g_2h$

Theorem (Lagrange). If H is a subgroup of a finite group G.

$|H|$ divides $|G|$

Proof: $|g_iH| = |H| \quad g_i \in G$

$G = \bigcup_{i=1}^{m} g_iH$

$\Big\}\ |G|/|H| = m$

m is the number of disjoint cosets.

Corollary: $|G| = p$ is a prime. $\Rightarrow$

G is a cyclic group.

Proof. $\forall g \in G$ $(g \neq 1)$

$H = \{1, g, g^2 \ldots\}$

$|G|/|H| \in \mathbb{Z}$ $\quad |H| = p = |G|$

$\underline{H = G}$

Corollary: (Fermat's little theorem)

p is a prime

a a integer

$a^p = a \mod p$

Definition: G a group. H a subgroup. The set of left cosets in G is denoted G/H

It is the set of orbits under the right group action of H on G.

(G/H is referred to as homogeneous space). The cardinality of this set is the index of H in G

denoted $[G : H] = \underline{\underline{|G|/|H|}}$.

Example. 1 $G = S_3$ $\quad H = \{1, (12)\}$

$|S_3| = 6$

$G/H = \{H, (13)H, (23)H\}$

$= \{H, (123)H, (132)H\}$

$[G : H] = 6/2 = 3$

2. $G = \langle \omega \mid \omega^{2n} = 1 \rangle$

$H = \langle \Omega \mid \Omega^{n} = 1 \rangle$

$[G : H] = 2 \qquad G/H = \{H, \omega H\}$

3. $G = A_4 \qquad |A_4| = \frac{1}{2} 4! = 12$

$H = \{1, (12)(34)\} \cong \mathbb{Z}_2 \qquad |H| = 2$

$[G : H] = 12/2 = 6 \qquad 12 = 6 \times 2$

4 $G = A_4$. $\quad ?\ \exists H$. s.t. $[G : H] = 2$

$H \subset G$.

$g \in G$. $g \notin H \qquad [G : H] = 2$

$G/H = \{H, gH\} \qquad gH = H$

$\Updownarrow$

$gh_1 = h_2$

$\Updownarrow$

$g = h_2 h_1^{-1} \in H$

if $g^2 H = gH \qquad g^2 h_1 = g h_2 \qquad g = h_2 h_1^{-1} \in H$ ✗

$\Rightarrow g^2H = H \quad \underline{g^2 \in H}$

$g \in G \ (\notin H) \Rightarrow g^2 \in H$

$g \in H \Rightarrow g^2 \in H$

$\underline{\forall g \in G : \ g^2 \in H}$

$G = A_4$

$(123)(123) = (132) \rightarrow$ all 3-cycles are $\underline{\text{squares}}$

$\left| \underline{\begin{matrix} (123) & (132) & (124) & (142) \\ (134) & (143) & (234) & (243) \end{matrix}} \right| = \underline{8} \qquad \underline{|H| = 6}$

Converse of Lagrange theorem is in general not true.

Theorem (Sylow's first theorem) p prime p^k divides $|G|$ for a non negative integer k. $\Rightarrow \exists H \subset G.\ |H| = p^k$.

Example. 1. $S_3 \qquad |S_3| = 6 = \underline{2} \times \underline{3}$

$|H| = 2 : \ S_2 \cong \mathbb{Z}_2$

$|H| = 3 : \ A_3 \cong \mathbb{Z}_3$

2 Quaternion group

$$|Q| = 8 = 2^3$$

$|H| = 2$: $\{\pm 1\} \cong \mathbb{Z}_2$

4 : $\{1, -1, i, -i\} \cong \mathbb{Z}_4$

$\{1, -1, j, -j\}$

$\{1, -1, k, -k\}$

- 6.2 Conjugacy

Definition.

a) a group element h is conjugate to h' if $\exists\, g \in G$. s.t. $h' = ghg^{-1}$

b) conjugacy defines "$\sim$"

$$\left\{ \begin{array}{l} h \sim h \\ h_1 \sim h_2 \iff h_2 \sim h_1 \\ h_1 \sim h_2,\ h_2 \sim h_3 \Rightarrow h_1 \sim h_3 \end{array} \right)$$

The conjugacy class under this relation is called the conjugacy class (of h)

$$C(h) := \{ghg^{-1},\ g \in G\} \ (=: h^G)$$

c) $H \subset G$ subgroup. its conjugate

$gHg^{-1} := \{ghg^{-1} : h \in H\}$ is also a

subgroup. (sometimes denoted H^g)

① $e \in H^g$

② $(gh_1g^{-1})(gh_2g^{-1}) = g\underbrace{(h_1h_2)}_{\in H}g^{-1} \in H^g$

③ $I(ghg^{-1}) = gh^{-1}g^{-1} \in H^g$

Examples:

1. $(a_1a_2)(a_3a_4a_5) \sim (b_1b_2)(b_3b_4b_5)$

$\tau(a_1) = b_1$

$\tau(a_2) = b_2$

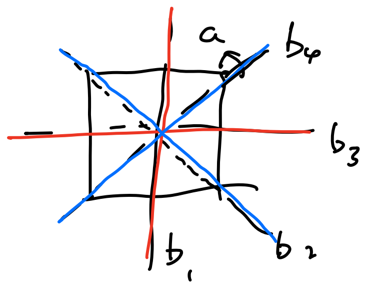
$\tau(a_3) = b_3$

$\vdots$

$\tau(a_1a_2)(a_3a_4a_5)\tau^{-1} = (b_1b_2)(b_3b_4b_5)$

permutations are conjugate if they have the same cycle decomposition structure.

2 $D_4 := \langle a\, b : a^4 = b^2 = (ab)^2 = 1 \rangle$



$\underline{cb_1c^{-1}} = b_3$ $\quad$ $\underline{b_1 \sim b_3}$

$cb_2c^{-1} = b_4$ $\quad$ $\underline{b_2 \sim b_4}$

conjugacy generalizes the notion of

similarity of matrices $\quad$ $(T'_{(\beta)} = \underline{S}\, \underline{T_{(\beta)}} S^{-1})$

Examples in $GL(n,K)$:

1 $G = U(n) := \{ A \in M_n(\mathbb{C}) \mid AA^\dagger = 1_n \}$

$\forall u \in U(n) \quad \exists g \in U(n)$, s.t.

$\underline{gug^{-1}} = \underline{\mathrm{diag}(z_1, \cdots z_n)} \quad (|z_i| = 1)$

$\underline{(z_1 \cdots z_n)} \in U(1)^n$ labels different conjugacy classes ?

$A(\phi)^{-1} \mathrm{diag}(z_1 \cdots z_n) A(\phi) = \mathrm{diag}(z_{\phi(1)}, z_{\phi(2)}, \cdots z_{\phi(n)})$

$gug^{-1} = \underline{\mathrm{diag}(\{z_i\})}$

$[A(\phi)^{-1} g] u [A(\phi)^{-1} g]^{-1} = A(\phi)^{-1} \mathrm{diag}(\{z_i\}) A(\phi)$

$= \underline{\mathrm{diag}(\{z_{\phi(i)}\})}$

same conjugacy class up to arbitrary permutation.

$\Rightarrow$ $\underline{U(1)^n / S_n}$ label conj. class.

2. general $g \in GL(n, \mathbb{C})$. not necessarily diagonalizable.

Define the characteristic polynomial

$$P_A(x) := \det(x\mathbb{1} - A)$$

$$
\begin{aligned}
P_{gAg^{-1}}(x) &= \det(x\mathbb{1} - gAg^{-1}) \\
&= \det[g(x\mathbb{1} - A)g^{-1}] \\
&= \det(x\mathbb{1} - A) = \underline{P_A(x)}
\end{aligned}
$$

<u>Definition</u> A <u>class function</u> on a group is a function f on G. s.t.

$$\underline{f(g g_0 g^{-1}) = f(g_0)} \quad \forall g, g_0 \in G$$

1 For a matrix rep. define <u>character</u> of the representation

$$\underline{\chi_T(g) := \operatorname{Tr} T(g)}$$

<u>Definition</u> Two homomorphisms $\varphi_i : G_1 \longrightarrow G_2$ are conjugate if $\exists\, g_2 \in G_2$ s.t.

$$\varphi_2(g_1) = g_2 \varphi_1(g_1) g_2^{-1}$$

matrix rep. $T : G \longrightarrow GL(n, k)$

$$\underline{T' = S T S^{-1}} \qquad S \in GL(n, k)$$

$\Rightarrow$ "equivalent representation"

$$\chi(T') = \chi(T)$$