Recap.

$$P_\mu P_\nu = \delta_{\mu\nu} P_\nu \qquad P_\mu^2 = P_\mu$$

$\downarrow$ Group algebra

$$e' = Pe$$

$e'^2 = e'$ primitive idempotent:

cannot $e' = e_1 + e_2$ $(e_i \neq 0)$

$e_1 e_2 = 0$

随金金: $c_i = \sum_{g \in C_i} g$

$$c_i c_j = \sum_{k=1}^{r} C_{ij}^{k} c_k$$

$\Downarrow$ diagonalize c_i's

$$c_i = \sum_{\mu=1}^{r} \underbrace{\lambda_i^\mu}_{\text{eigenvalues}} P^\mu$$

— Representation of S_n

Recall some basics of S_n:

① any two r-cycles $(i_1, i_2, \cdots i_r)$

are conjugate. $\exists \phi(i_a) = j_a$

$$\phi(i_1, i_2, \cdots i_r)\phi^{-1} = (j_1, j_2, \cdots j_r)$$

② denote conjugacy classes of S_n as

$$(1)^{l_1}(2)^{l_2} \cdots (n)^{l_n}. \quad \vec{l} = (l_1, l_2, \cdots l_n)$$

$$\sum_{i=1}^{n} i\, l_i = n$$

③ number of $\{C_i\}$ = "partition function" of n $P(n)$

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idempotents are easy to find for 1D irreps.

① $C = \frac{1}{n!} \sum_{s \in S_n} s \qquad \forall s \in S_n$

$$C \cdot S = S C = C$$

$$C^2 = C$$

The 1D vector space $\{\lambda C\}$ is an irrep

$L(s) \cdot \underline{C} = SC = \underline{C}$ $\qquad$ trivial irrep

② $c = \frac{1}{n!} \sum_{s \in S_n} \mathrm{sgn}(s)$ $\qquad cs = sc = \mathrm{sgn}(s) \cdot c$

$L(s) \cdot c = \underline{\mathrm{sgn}(s) \cdot c}$ $\qquad$ <u>sgn rep</u>

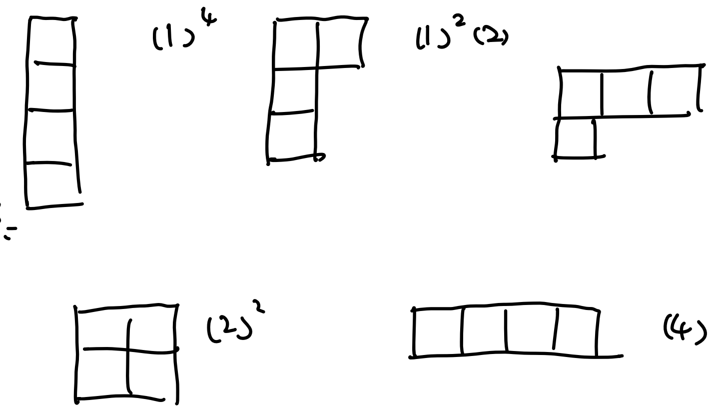
Recall Young diagrams and tableaux:

len. of row

$\lambda_i \geq \lambda_{i+1}$

<u>S_4:</u>

<u>Young diagrams:</u>



Young tableaux:

$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|} \hline 4 & 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$

Standard tableaux:

integers increase within row ($\longrightarrow$)

& column ($\downarrow$)

semistandard:

$\begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$ $\quad (L \leq R)$

Given a tableau T. define permutations

$R(T) :=$ { all row permutations }

$C(T) :=$ { column }

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & & \\ \cline{1-1}\end{array} \leftarrow \quad R(T) = \{ e, (12), (13), (23), (123), (132) \}$$

$$C(T) = \{ e, (14) \}$$

$$R(T) \cap C(T) = \{ e \}$$

Define two elements $P, Q \in R_{S_n} (=: R_n)$

$$P = \sum_{p \in R(T)} p \qquad Q = \sum_{q \in C(T)} \epsilon(q) q$$

$$(\epsilon(q) = \operatorname{sgn}(q))$$

$$c = PQ = \sum_{\substack{p \in R(T) \\ q \in C(T)}} \epsilon(q) \underline{pq}$$

$c \neq 0$: if $p, p' \in R(T)$, $q, q' \in C(T)$

$$pq = p'q' \Rightarrow \underbrace{q(q')^{-1}}_{\in C(T)} = \underbrace{p^{-1} \cdot p'}_{\in P(T)} = e \Rightarrow \begin{array}{l} p = p' \\ q = q' \end{array}$$

<u>Theorem 1</u> : The Young symmetrizer $c = PQ$

corresponding to a tableau T is

"<u>essentially idempotent</u>". The

corresponding invariant subspace $R_n c$

yields an irrep of S_n

① $c^2 = \lambda c$ $(\lambda \in N^+)$

$\lambda^{-1} c$ primitive idempotent

② $c c' = 0$ if T & T' are different partitions

If a Young diagram corresponds to an f-dim irrep.

There are $n!$ $\{Rc\}$, not all linearly independent.

<u>Theorem 2</u> The dimension f of the irrep corresponds to a diagram is the number of standard tableaux. $\{T_i, i = 1 \cdots f\}$

Example :

$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline\end{array}$ $\qquad f^{trivial} = 1$ trivial

$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline\end{array}$ $\qquad f^{sgn} = 1$

Lemma : c_i corresponding to T_i, then

$$\underline{c_i c_j = 0} \quad \text{if } i \neq j$$

$\Rightarrow \{ R_n c_i \}$ are linearly independent

Proof : $\sum x_i c_i = 0 \iff \forall i,\ x_i c_i = 0$

$$\Downarrow$$

$$\sum_i x_i c_i c_j = 0$$

$$\Downarrow$$

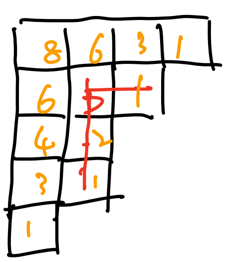
$$x_j c_j^2 = 0 \quad (\forall j) \qquad c_j^2 = \lambda c_j$$

$$\Downarrow$$

$$\lambda x_j c_j = 0$$

$\underline{c^2 = \lambda c} \qquad \lambda = \dfrac{n!}{f}$

f is given "hook length formula"



$h(b) = 4 \qquad f = \dfrac{n!}{\prod_b h(b)} \qquad \lambda = \prod_b h(b)$

Example S_3

diagram	standard tableaux
① $\begin{array}{\|c\|c\|c\|}\hline \ & \ & \ \\ \hline\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 2 & 3 \\ \hline\end{array}$
② $\begin{array}{\|c\|c\|}\hline \ & \ \\ \hline \ \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ $\quad$ $\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$
③ $\begin{array}{\|c\|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$

$$|S_3| = 6 = 1^2 + 2^2 + 1^2$$

① trivial rep.: $\quad P = \sum_{p \in S_3} p \qquad Q = e$

$$f = \frac{n!}{n!} = 1$$

$$\tilde{c} = \frac{1}{n!} c = \frac{1}{6} [e + (12) + (13) + (23) + (123) + (132)]$$

$$R_3 \cdot \tilde{c} = \tilde{c}$$

$$(\sigma \in R_3 (= P_{S_3})\ \sigma \tilde{c} = \tilde{c})$$

2 sgn rep: $\quad P = e \qquad Q = \sum_{g \in S_3} \epsilon(g) \cdot g$

$$\tilde{c} = \frac{1}{6}\left[e - (12) - (13) - (23) + (123) + (132)\right]$$

$$\sigma \in R_3 \qquad \sigma \cdot \tilde{c} = \operatorname{sgn}(\sigma) \cdot \tilde{c}$$

3 $\underset{T_1}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}} \quad \underset{T_2}{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}} \qquad f = \frac{3!}{3} = 2 \qquad \lambda = \frac{n!}{f} = 3$

$$P_1 = e + (12) \qquad P_2 = e + (13)$$

$$Q_1 = e - (13) \qquad Q_2 = e - (12)$$

$$\tilde{c}_1 = \frac{2}{6} P_1 Q_1 = \frac{1}{3}\left[e - (13) + (12) - (132)\right]$$

$$\tilde{c}_2 = \frac{2}{6} P_2 Q_2 = \frac{1}{3}\left[e - (12) + (13) - (123)\right]$$

check $\quad \tilde{c}_i \cdot \tilde{c}_i = \tilde{c}_i \qquad i = 1, 2$

$$\tilde{c}_1 \tilde{c}_2 = \tilde{c}_2 \tilde{c}_1 = 0$$

$R_3 \cdot \tilde{c}_1 : \qquad \tilde{c}_1 = \frac{1}{3}\left[e - (13) + (12) - (132)\right]$

$$e \cdot \tilde{c}_1 = \tilde{c}_1 =: \underline{\underline{v_1}}$$

$$(12) \cdot \tilde{c}_1 = \frac{1}{3}\left[(12) - (132) + e - (13)\right] = \tilde{c}_1 = v_1$$

$$(13) \cdot \tilde{c}_1 = \frac{1}{3}\left[(13) - e + (123) - (23)\right] =: \underline{\underline{v_2}}$$

$$(23) \cdot \tilde{c}_1 = -v_1 - v_2$$

$$(123) \cdot \tilde{c}_1 = v_2 \qquad (132) \cdot \tilde{c}_1 = \underline{-v_1 - v_2}$$

$$\begin{cases} (12)\cdot v_1 = v_1 \\ (12)\cdot v_2 = (12)\cdot(13)\cdot\tilde{c}_1 = (132)\tilde{c}_1 = -v_1 - v_2 \end{cases}$$

$$T[(12)] = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \qquad \chi([12]) = 0$$

$$T[(13)] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$T[(23)] = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$T[(123)] = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \qquad \chi([123]) = -1$$

$T_2 = (23)\cdot T_1$ generates an equivalent irrep.

Example: S_4

$$|S_4| = 4! = 24 = \underset{\text{trivial}}{1^2} + 3^2 + 2^2 + 3^2 + \underset{\text{sgn}}{1^2}$$

(4) $\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline\end{array}$

$(3)(1)$ $\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}$

$(2)(2)$ $\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}$ $\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}$

$(2)(1)^2$ $\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}$

$(1)^4$ $\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}$

Character table of S_4

	e	$6[(12)]$	$3[(12)(34)]$	$8[(123)]$	$6[(1234)]$
V^+	1	1	1	1	1
V^-	1	-1	1	1	-1
① $V^\perp$	3	1	-1	0	-1
② $V^{-\otimes\perp}$	3	-1	-1	0	1
V^2	2	0	2	-1	0

① $V^\perp$: 4 vecs $\hat{e}_i$ $L = \sum \hat{e}_i$ (see previous lectures)

$L^\perp = \{e_1 - e_2, e_2 - e_3, e_1 - e_4\}$

$\chi[(12)] = 1$

$\chi[(12)(34)] = -1$

$\chi[(123)] = 0$

$a_\mu = \langle \chi_\mu, \chi \rangle$

$$\langle \chi^\perp, \chi^\perp \rangle = \frac{1}{24}\left(3^2 + 6 \times 1^2 + 3 \times (-1)^2 + 8 \times 0^2 + 6(-1)^2\right)$$
$$= 1$$

② via tensor product:

$$V^\mu \otimes V^\nu = \bigoplus_\lambda N_{\mu\nu}^\lambda V^\lambda$$

$$\chi_\mu(g)\chi_\nu(g) = \sum_\lambda N_{\mu\nu}^\lambda \chi_\lambda(g)$$

$$V^- \otimes V^\perp =: V^{-\otimes\perp} \qquad \left(\chi^{-\otimes\perp}_{-,\perp} = 1\right)$$

$$\langle \chi_{-\otimes\perp}, \chi_{-\otimes\perp} \rangle = \frac{1}{24}\left(3^2 + 6(-1)^2 + 3(-1)^2 + 8\cdot 0^2 + 6\cdot 1^2\right)$$

$$= 1$$

Further references: Müller.

Symmetry groups and their applications

Chap. 4