Recap : Cosets & conjugacy.

Def. $H \subset G$.

$gH := \{ gh, h \in H \} \subset G$.

left-coset of H

("G" = H "X" = G group action perspective)

$G = \mathbb{Z}$. $H = 2\mathbb{Z}$.

$gH = \{ H. \ 1+H \}$

$G = S_3$ $H = \{1, (12)\} \cong S_2$

$g \in G$. $gH = \{ H, \ \{(13), (123)\}, \ \{(23), (132)\} \}$

(left) cosets : identical or disjoint

Theorem (Lagrange) $H \subset$ finite G.

$|H| \mid |G|$ $|G| = m|H|$

$|G|$ = prime. $\rightarrow$ G cyclic

$\langle g \rangle = \underline{\{1, g, g^2 \cdots \} = G}$

index of a subgroup

$$[G:H] := |G/H| = |G|/|H|$$

$G = A_4$ $\nexists H$ s.t. $[G:H] = 2$

$|H| = 6$.

$\Rightarrow$ Converse of Lagrange Theorem is in general not true.

$\Rightarrow$ special case. (Sylow)

p prime $p^k \mid |G|$

$\exists H$, $|H| = p^k$

Conjugacy

$h' = ghg^{-1}$ $\exists g \in G$

$h' \sim h$ equivalence relation

$\hookrightarrow$ class

$$C(h) := \{ghg^{-1} : g \in G\} \;(=: h^G)$$

H subgroup $\longrightarrow$ gHg^{-1} subgroup

S_n.. same cycle decomposition $\Leftrightarrow$ conjugate

$\hookrightarrow$ "equivalent representation"

$$T' = STS^{-1}$$

$GL(n, k)$:

diagonalizable: $\underline{U(1)^N}/\underline{S_n}$

un~

$$P_A(x) = \det(\lambda 1 - A)$$

<u>class function</u>

$$f(g g_0 g^{-1}) = f(g_0) \ , \quad \forall g, g_0 \in G.$$

$$\chi_T(g) := \underline{\mathrm{Tr}\, T(g)}$$

– 6.3. Conjugacy classes in S_n

Recall the cycle decomposition of $g \in S_n$

$(i_1 i_2 \cdots i_k)$ is a k-cycle. then

$g(i_1, i_2 \cdots, i_k) g^{-1}$ is also a k-cycle.

if $g(i_a) = j_a$ $a = 1, \cdots k$. then

$$g(i_1, i_2, \cdots i_k) g^{-1} = (g(i_1)\ g(i_2) \cdots g(i_k))$$
$$= j_1 j_2 \cdots j_k$$

$\Rightarrow$ Any two k-cycles are conjugate.

$$(12)((123))(12)^{-1} = (12)(123)(12)$$
$$= (12)(13) = (132)$$

$$(213) = (132)$$

↪ permutations are conjugate iff they have the same cycle decomposition.

↪ label the conjugacy classes by their cycle decomposition. denoted as

$C(\vec{l})$. $\vec{l} = (l_1, \cdots l_n)$ l_i number of i-cycles

$$\boxed{n = \sum_{j=1}^{n} j\, l_j}$$

$\phi = (12)(34)(678)(11,12) \in S_{12}$

$= (12)(34)(5)(678)(9)(10)(11,12)$

$l_1 = 3 \quad l_2 = 3 \quad l_3 = 1, \quad l_{\geq 4} = 0$

$\vec{l} = (3\ 3\ 1\ 0\ 0\ 0 \ldots)$

<u>Def</u> <u>partition of n</u> : decomposition of n into a sum of nonnegative integers.

The number of <u>distinct</u> partitions of n is called the "<u>partition function</u>" of n, denoted $P(n)$

↳ conjugacy classes of S_n

$\Leftrightarrow \quad p(n)$

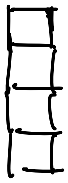
Example: S_4

$(g^n = 1)$

partition	cycle decomp.	g.	$\lvert C(g) \rvert$	order of g
$4 = 1+1+1+1$	$(1)^4$	1	1	1
$4 = 1+1+2$	$(1)^2(2)$	(ab)	$C_4^2 = 6$	2
$4 = 1+3$	$(1)(3)$	(abc)	$C_4^3 \times 2 = 8$	3
$4 = 2+2$	$(2)^2$	$(ab)(cd)$	$C_4^2/2 = 3$	2
$4 = 4$	(4)	$(abcd)$	6	4

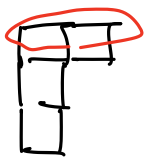
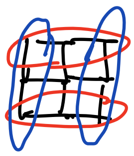
$$|S_4| = 4! = 24 \doteq 1+6+8+3+6 = 24 \quad \checkmark$$

Young diagram

$1+1+1+1$:  $(1)^4$

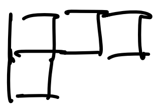
λ_i - length of i-th row

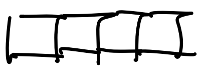
$\lambda_i \geq \lambda_{i+1}$

$2+1+1$  $(1)^2(2)$  $(2)^2$

1	2	3	4
2	1	4	3

Young tableau

$3+1$  $(1)(3)$

4  (4)

graphic representation

→ define permutation within column

→ Young tableau

important for

e.g. constructing

irreducible representations

Example : a collection of harmonic oscillators

$$h_j = \hbar \omega_j \left(a_j^{+} a_j + \tfrac{1}{2} \right) \qquad \left(\omega_j = j\,\omega_0 \right)$$

$$H - E_0 = \sum_{j=1}^{\infty} \hbar \omega_0 \, j \left(a_j^{+} a_j + \tfrac{1}{2} \right)$$

system with a fixed energy $E_0 = N \hbar \omega_0$

$$|\varphi_{\vec{l}}\rangle = \frac{1}{\sqrt{l_1!\, l_2! \cdots}} (a_1^{+})^{l_1} (a_2^{+})^{l_2} \cdots (a_n^{+})^{l_n} |0\rangle$$

$$N = \sum j\, l_j$$

$p(n) \rightarrow$ degeneracy of the states

– 6.4 Normal subgroup & Quotient groups

Definition A subgroup $N < G$ is called a

normal subgroup (invariant subgroup.

self-conjugating subgroup)

$$gNg^{-1} = N \qquad \forall g \in G$$

denoted as $N \triangleleft G$.

↪ center of a group. $Z(G)$

$\forall z \in Z(G)$: $zg = gz \Leftrightarrow gzg^{-1} = z$

$\forall g \in G$

$$gNg^{-1} = N \not\Rightarrow gng^{-1} = n \qquad n \in N$$

$$\Leftarrow$$

$$Z(G) \triangleleft G$$

Examples:

1 $H \subset$ abelian group G. $\forall h \in H$

$$ghg^{-1} = (gg^{-1})h = h$$

→ All subgroups of an Abelian group are normal.

2. Homomorphisms

$$\phi : G \longrightarrow G'$$

$\ker(\phi)$ is a normal subgroup of G.

$\forall k \in \ker(\phi) \quad \phi(k) = 1$

$$\phi(gkg^{-1}) = \phi(g)\cancel{\phi(k)}^{1}\phi(g^{-1}) = \phi(g)\phi(g^{-1})$$
$$= \phi(g)\phi(g)^{-1} = 1$$

$\Rightarrow gkg^{-1} \in \ker(\phi)$

$\Rightarrow \underline{g \ker(\phi) g^{-1} = \ker(\phi)} \quad \forall g \in G$

$\Rightarrow \underline{\ker(\phi) \triangleleft G.}$

<u>Theorem</u> If $N \triangleleft G$. then the set of left cosets $G/N = \{gN, g \in G\}$ has natural group structure. with group multiplication defined as:

$$(g_1N)\cdot(g_2N) := (g_1g_2)N$$

G/N. "quotient group" / "factor group"

$$(g_1 N)\cdot(g_2 N) = g_1(g_2 g_2^{-1}) N g_2 N$$

$$= (g_1 g_2)\underbrace{(g_2^{-1} N g_2)}_{}\overset{= N}{} N$$

$$= (g_1 g_2) N$$

for a general $H \subset G$, it doesn't hold:

$$S_3 = \{ e, (12), (13), (23), (123), (132) \}$$

$H = \{ e, (12) \}$ not a normal subgroup

$g_1 = (123) \qquad g_2 = (123)^{-1} = (321) = (132)$

$h_1 = (12) = h_2$

$$(g_1 h_1)(g_2 h_2) = (13)(23) = (213) = \underline{(132)} \notin H$$

$$\neq \quad (\underline{g_1 g_2}) H = H$$

? order of $G/N = [G : N]$

Corollary: If $N \triangleleft G$, then the natural map

$$\phi: G \to G/N$$
$$g \mapsto gN$$

is a surjective homomorphism.

$$\ker \phi = N$$

① $\underline{\phi(g_1)\phi(g_2) = \phi(g_1 g_2)}$

$\Leftrightarrow$ $\underline{(g_1 N)(g_2 N) = (g_1 g_2)N}$

② $\phi(g) = gN = N. \Rightarrow g \in N.$

$\underline{\ker \phi = N}$

$\Rightarrow$ Every <u>normal subgroup</u> is the kernel of <u>some homomorphism</u>.

<u>Example</u>:

1. $n\mathbb{Z} := \langle n \rangle \triangleleft \mathbb{Z}$

$= \{\cdots -2n, -n, 0, n, 2n, \cdots\}$

$\mathbb{Z}/n\mathbb{Z} = \{ i + n\mathbb{Z} \quad 0 \le i \le n-1 \} = \mathbb{Z}_n$

$$\phi: \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$$
$$i \longmapsto i + n\mathbb{Z}$$

$$\ker(\phi) = n\mathbb{Z}$$

quotient group is not a subgroup

$\mathbb{Z}$: no finite subgroup except for $\{0\}$

$\mathbb{Z}_n$: finite group $|\mathbb{Z}_n| = n$.

$\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is not a subgroup of $\mathbb{Z}$

(happens sometimes:

$G = \mathbb{Z}_6 \quad N = \mathbb{Z}_2$

$\mathbb{Z}_6/\mathbb{Z}_2 \cong \mathbb{Z}_2$)

2. $S_3 = \{ 1, (12), (13), (23)$ $\qquad |S_3| = 6$

$(123), (132) \}$

$A_3 = \{ 1, (123), (132) \} \subset S_3 \qquad |A_3| = 3$

$(12)(123)(12)^{-1} = (132) \in A_3$

$(12)(132)(12)^{-1} = (123) \in A_3$

$\vdots$

$\Rightarrow A_3 \triangleleft S_3$

$H \subset G$. $[G : H] = 2 \Rightarrow H \triangleleft G$ (HW)

$[S_3 : A_3] = 2 \Rightarrow A_3 \triangleleft S_3$

3. $D_4 = \langle a, b \mid a^4 = b^2 = (ab)^2 = 1 \rangle$

a : $\frac{\pi}{2}$ rotation

b : reflection

$D_4 = \langle e, a, a^2, a^3, b, ab, a^2b, a^3b \rangle$ $|D_4| = 8$

① $aba^{-1} = a^2b \Leftrightarrow ba^{-1} = ab \Leftrightarrow b^{-1}a^{-1} = ab$

$\overset{\vee}{b^{-1}}$ $\Leftrightarrow (ab)^2 = 1$

$\{e, b, a^2b, a^2\}$ $|\ | = 4$

② $a(ab)a^{-1} = a^3b$

$\{e, ab, a^3b, a^2\}$ $|\ | = 4$

③ $\{e, a, a^2, a^3\}$ $|\ | = 4$

④ $\{e, a^2\} = Z(G)$ $|\ | = 2$

①. $N = \{e, a^2, b, a^2b\} \cong D_2 \cong V$ $(A = a^2\ B = b)$

$= \langle A, B \mid A^2 = B^2 = (AB)^2 = 1 \rangle$

$D_4 / D_2 = \{D_2, \cancel{AD_2}\} \cong Z_2$

correction: should be aD_2

② $N = \{e, ab, a^2, a^3b\} \cong D_2$ $(A = a^2,\ B = a^3b)$

$= \langle A, B \mid A^2 = B^2 = (AB)^2 = 1 \rangle$

③ $N = \{e, a, a^2, a^4\} \cong Z_4$

$D_4/Z_4 = \{\underline{Z_4}, \underline{bZ_4}\} \cong Z_2$

④ $N = \{e, a^2\} = Z(D_4) \cong Z_2$

$D_4/Z_2 = \{\underline{Z_2, aZ_2, bZ_2, abZ_2}\} \cong D_2$ $(\neq \cancel{Z_4})$

$|G| = 4$: $G \cong Z_4$ $\underline{a^4 = 1}$

$G \cong D_2 \cong V$ $\underline{a^2 = b^2 = (ab)^2 = 1}$

$(aZ_2)(aZ_2) = a^2 Z_2 = Z_2 = Z_2$

$\boxed{G/Z(G) \text{ is cyclic} \Longleftrightarrow G \text{ is Abelian}}$ HW