HW 04:

P10. u $\operatorname{tr} u = \alpha$

u^* α^*

$SU(2)$ $g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$ $\overline{(\alpha + \bar{\alpha})} = \alpha + \bar{\alpha}$

$i\sigma_2$ $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

P11. (1) $[G : H] = 2$ $H.$ gH

$\overset{H}{Hg}$

(2) $G/Z(G)$ cyclic $\Rightarrow$ $\langle g Z(G) \rangle$

$\forall a, b \in G.$ $a = g^m z_1$

$b = g^n z_2$

P12. (1) $SL \triangleleft GL$

(2) $[S_n : A_n] = 2$ $\Rightarrow$ $A_n \triangleleft S_n$

$\phi \in A_n$ $\tau \phi \tau^{-1} \in A_n$

P13. ——

Recap. unitarize rep. of a finite group.

$$H = \sum_{g \in G} T^+(g)\, T(g)$$

$$\Rightarrow V^+ H V = \Lambda \rightarrow \text{diag.}$$

$$\Rightarrow \quad A = V \Lambda^{-\frac{1}{2}}$$

$$T(g) = A\, \tilde{T}(g)\, A^+$$

$\tilde{T}(g)$ unitary.

$$\sum_{g} \rightarrow \int dg$$

Haar measure

$$(T_1, V_1), \quad (T_2, V_2)$$
$$n \qquad\qquad m$$

$\Rightarrow$

① $T_1 \oplus T_2$ on $\underbrace{V_1 \oplus V_2}_{n+m}$

$$M_{T_1 \oplus T_2} = \begin{array}{cc} \begin{pmatrix} M_{T_1} & 0 \\ 0 & M_{T_2} \end{pmatrix} & \begin{array}{c} n \\ m \end{array} \end{array}$$

② $T_1 \otimes T_2$ on $\underbrace{V_1 \otimes V_2}_{n \cdot m}$

$$(M_1 \otimes M_2)(g)_{ia,jb} = (M_1(g))_{ij}\, (M_2(g))_{ab}$$

③ on dual space V^*. $V^* : V \to k$

$$(g \cdot v^*)(v_j) = v_i^*(g^{-1} \cdot v_j)$$

$$\Downarrow$$

$$(g v_i^*)(g v_j) = v_i^*(g^{-1} \cdot g v_j) = v_i^* v_j = \delta_{ij}$$

$$T^*(g) = [T(g^{-1})]^{tr}$$

$$M^*(g) = M(g)^{tr,-1}$$

$$M_{T_1 \oplus T_2} : \chi_{T_1 \oplus T_2} = \chi_{T_1} + \chi_{T_2}$$

$$\chi_{T_1 \otimes T_2} = \chi_{T_1} \times \chi_{T_2}$$

Haar measure . invariant measure / integration.

$$\int_G dg\, f(g) = \frac{1}{|G|} \sum_{g \in G} f(g) = \langle f \rangle$$

$f : g \longmapsto f(g) \in k$

$\Uparrow$

$\mathrm{Map}(G, \mathbb{C})$

$$\int_G dg \in \mathrm{Map}(G, \mathbb{C})^*$$

$$f \mapsto \langle f \rangle$$

a measure satisfies left-invariance property.

$$\int_G f(hg)\, dg = \int_G f(g)\, dg \qquad (\forall h \in G)$$

Examples.

$G = \mathbb{R}$

$$\int_G dg\, f(g) = \int_G dg\, f(g+a)$$

$$\Rightarrow \int_G dg\, f(g) = c \int_{-\infty}^{\infty} dx\, f(x)$$

$G = \mathbb{Z}$ $\qquad \int_G dg\, f(g) = \underline{c} \sum_{n \in \mathbb{Z}} f(n)$

$G = \mathbb{R}^*_{>0}$ $\qquad$ "$\times$"

$$\int_{G = \mathbb{R}^*_{>0}} f(g)\, dg = c \int_0^\infty f(x) \frac{dx}{x}$$

$G = GL(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$

$$\int_{GL} f(g)\, dg = c \int f(g)\, |\det g|^{-n} \prod_{ij} dg_{ij}$$

Interlude: topological groups and stuff.

topo. space.

set X,, $T = \{ U_i \mid i \in I \}$

(1) $\phi, X \in T$

(2) $J \subset I$ $\quad \bigcup_{j \in J} U_j \in T$

(3) finite $K \subset I$ $\quad \bigcap_{k \in K} U_k \in T$

U_i open set

$X = \mathbb{R}$

$U_i = (a_i, b_i)$

$f(x)$ cont. derivability.

$f : X \to Y$.

open set $\Leftarrow$ open set | cont. func

<u>Hausdorff</u>. $x, x' \in X$. $\exists$ <u>neighborhood</u>. $U_x, U_{x'}$

$U_x \cap U_{x'} = \phi$.

<u>$x \in U_x \subset N$</u>

Campacity

S . $\{A_j, j \in J\}$ subsets of X.

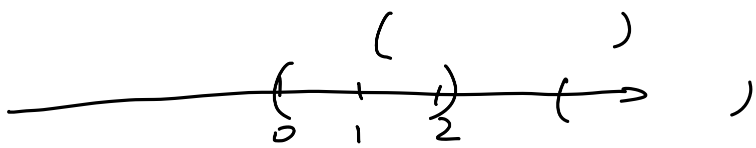
$S \subset \bigcup_{j=J} A_j$

cover of S

$A \in T$. open cover.

compact. every open cover $\{U_i, i \in I\}$

$\exists$ finite $J \subset I$. open cover.



$\{(n, n+2), n \in \mathbb{Z}.\}$

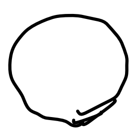
$\mathbb{R}$ is not compact

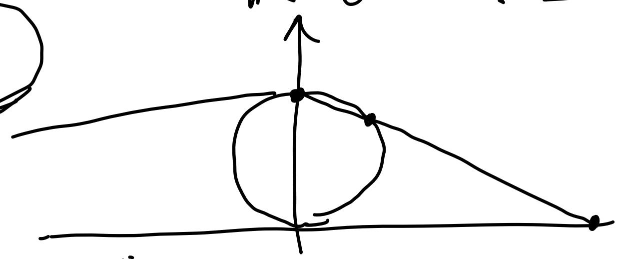
metric space. compactness $\Leftrightarrow$ closed.

$\mathbb{R}^n$. $\mathbb{C}^n$. bounded.

OBC PBC

$\mathbb{R}^n \cup \{\infty\} \cong S^n$





Möbius transformation.

"one-point compactification"

- Haar measure (cont.)

5. $G = U(1) = \{ z \in \mathbb{C}.\ |z| = 1 \}$.

$$\int f(g)\, dg = \frac{1}{2\pi i} \oint f(z) \frac{dz}{z} = \int_0^{2\pi} \frac{d\theta}{2\pi} f(\theta)$$

$$= \underline{1} \qquad\qquad (z = e^{i\theta})$$

6. $SU(2)$. $\quad g \in SU(2)$

$$g = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \qquad |\alpha|^2 + |\beta|^2 = 1$$

$$g = e^{i\frac{1}{2}\phi\sigma^3} e^{i\frac{1}{2}\theta\sigma^1} e^{i\frac{1}{2}\varphi\sigma^3}$$

$$= \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{\frac{i}{2}(\phi+\varphi)} \cos\frac{\theta}{2} \ (= \alpha) & i e^{i\frac{1}{2}(\phi-\varphi)} \sin\frac{\theta}{2} \ ('' \beta) \\ -\beta^* & \alpha^* \end{pmatrix}$$

$\phi \in [0, 2\pi)$

$\theta \in [0, \pi)$

$\underline{\psi \in [0, 4\pi)}$

① $\underline{d\alpha\, d\bar{\alpha}\, d\beta\, d\bar{\beta}}$ $\qquad g \to g_0 g$

$\qquad |\det g| = 1$

(2)

$$d\alpha\, d\bar{\alpha}\, d\beta\, d\bar{\beta} \longrightarrow \overset{J}{\overbrace{\left| \frac{\partial(\alpha, \bar{\alpha}, \beta, \bar{\beta})}{\partial(r, \varphi, \phi, \theta)} \right|}} dr\, d\varphi\, d\phi\, d\theta.$$

$$J|_{r=1} \propto \sin\theta$$

$$\int dg = \int c\, \underbrace{d\varphi\, d\phi}_{2\pi \times 4\pi}\, \underbrace{\sin\theta\, d\theta}_{2}$$

$$c = \frac{1}{16\pi^2}$$

normalized Haar measure.

$$[dg] = \frac{1}{16\pi^2}\, d\varphi\, d\phi\, \sin\theta\, d\theta.$$

② [GM note]. $\frac{1}{16\pi^2}\, d\varphi \wedge d\phi \wedge \sin\theta\, d\theta$

Maurer – Cartan form

$\omega = g^{-1} dg$ invariant under left action

$$(g_0 g)^{-1} d(g_0 g) = g^{-1} \cdot \cancel{(g_0^{-1} g_0)}\, dg = g^{-1} dg \quad \forall g_0 \in G$$

$$[dg] \propto \operatorname{tr}(\omega^3)$$

7. $L \neq R$ for non compact groups

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R},\ x > 0 \right\}$$

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{x} & -\frac{y}{x} \\ 0 & 1 \end{pmatrix} \in G$$

$$\underbrace{\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}}_{g_0 \quad g} \underbrace{\begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix}}_{g \quad g_0} = \underbrace{\begin{pmatrix} xu & xv + y \\ 0 & 1 \end{pmatrix}}_{} \in G.$$

① left-action: $g \longmapsto g_0 \cdot g$

$$du\,dv \longmapsto x^2\, du\,dv.$$

Haar measure: $\underline{x^{-2} dx\,dy}$

② righ-action: $g \longmapsto g \cdot g_0$

$$dx\,dy \longmapsto u\, dx\,dy$$

Haar measure: $\underline{x^{-1} dx\,dy}$

Proposition: If (T, V) is a rep. of a <u>compact group G</u>. V is an inner product space.

$\Rightarrow (T, V)$ is unitarizable.

if T is not unitary w.r.t. the inner product $\langle \cdot , \cdot \rangle_1$. then we can define a new innerproduct

$$\langle v, w \rangle_2 := \int_G \langle T(g) v, T(g) w \rangle_1 \, dg$$

$$\langle T(g) v, T(g) w \rangle_2 = \langle v, w \rangle_2 \qquad \forall v, w \in V.$$

① Finite matrix. rep. $M : G \to GL(n, \mathbb{C})$

M unitary with $\langle \, , \, \rangle_2$ & orthonormal basis $\{u^{(2)}\}$

$\Rightarrow \exists \langle \, , \, \rangle_1$. & orthonormal basis $\{u^{(1)}\}$

$$u^{(1)}_i = \sum_k A_{ki} u^{(2)}_k$$

s.t. $U^{(1)} = A^{-1} U A$. unitary.

(from before: $H = \sum_{g \in G} M^\dagger(g) M(g)$ - -)

- Regular representation

[Recall reg. rep for a finite group.

$$\dim V = |G| = n$$

$$T(g_1) \cdot e_{g_2} = e_{g_1 g_2} \quad]$$

Let G be a group. Then there is a left action of $G \times G$ on G:

$$(g_1, g_2) \longmapsto L(g_1) R(g_2^{-1})$$

$$(g_1, g_2) \cdot g_0 = g_1 g_0 g_2^{-1}$$

Then there is an induced action on $Map(G, \mathbb{C})$

$$\underline{((g_1, g_2) \cdot f)(h) := f(g_1^{-1} h g_2)}$$

which converts the vector space of functions $f : G \to \mathbb{C}$ into a representation space for $G \times G$.

[recall:

$$\hat{\Phi}(g, F)(x) = F(\phi(g^{-1}, x))$$

$$\underline{\hat{\Phi}(g_1,} \ \underline{\hat{\Phi}(g_2}, F))(x) = \hat{\Phi}(g_2, F)(\phi(g_1^{-1}, x))$$

$$= F(\phi(g_2^{-1}, \phi(g_1^{-1}, x)))$$

$$= F(\phi((g_1 g_2)^{-1}, x))$$

$$= \underline{\tilde{\phi}}(g_1 g_2, F)(x) \;]$$

$$[(g_1, g_2)(g_3, g_4) f](h) = [(g_1 g_3, g_2 g_4) \cdot f](h)$$

$$\| \qquad\qquad = f(g_3^{-1} g_1^{-1} h g_2 g_4)$$

$$(g_1, g_2) \cdot [(g_3, g_4) f](h) = [(g_3, g_4) f](g_1^{-1} h g_2)$$

$$= f(g_3^{-1} g_1^{-1} h g_2 g_4)$$

$$G \times G \longrightarrow \mathrm{End}(\{f\})$$

Now, equip G with a left & right invariant <u>Haar measure</u>. consider

$$L^2(G) = \{ f : G \to \mathbb{C} \mid \underbrace{\int_G |f(g)|^2 dg}_{\langle f, f\rangle} < \infty \}$$

(Hilbert space)

<u>$G \times G$</u> action preserves the L^2-property.

<u>Definition</u> The representation $L^2(G)$ is known as the <u>regular representation</u> of G.

If we restrict $G\times G$ to subgroups

$G\times\{1\}$ or $\{1\}\times G$.

Then $L^2(G)$ becomes a representation of $\underline{G}$.

$$(\underline{L(h)}\cdot f)(g) := f(h^{-1}g)$$

(left-regular rep.)

$$(\underline{R(h)\cdot f})(g) := f(g\cdot h)$$

(right-regular rep.)

Note: both $L(h)$. $R(h)$ act on the function space on the <u>left</u>.

Suppose (T, V) is a representation of G.

We can define $G\times G$ action on

$\underline{\mathrm{End}(V) := \mathrm{Hom}(V, V)}$

$\hookrightarrow$ rep. space.

$\forall S \in \mathrm{End}(V)$.

$$(g_1, g_2)\cdot S = T(g_1)\cdot S\cdot T(g_2^{-1})$$

For finite-dim. V we can define a map

$$\iota : \mathrm{End}(V) \longrightarrow L^2(G)$$

$$S \longmapsto f_S$$

$$\underline{f_S := \mathrm{Tr}_V(S\,T(g^{-1}))}$$

which is equivariant (ι is an intertwiner)

$$\begin{tikzcd}
 & S \arrow[dl] \arrow[dr] & \\
(h_1,h_2)\cdot S \arrow[d] & & f_S \arrow[d] \\
f_{(h_1,h_2)\cdot S} & = & (h_1\cdot h_2)\cdot f_S
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{End}(V) \arrow[r, "\iota"] \arrow[d, "T_{\mathrm{End}(V)}"] & \mathrm{Map}(G, \mathbb{C}) \arrow[d, "T_{\text{reg. rep.}}"] \\
\mathrm{End}(V) \arrow[r, "\iota"] & \mathrm{Map}(G, \mathbb{C})
\end{tikzcd}$$

$$\begin{aligned}
(h_1,h_2)\cdot f_S(g) &= f_S(h_1^{-1} g\, h_2) \\
&= \mathrm{Tr}_V(S\,T(h_2^{-1} g^{-1} h_1)) \\
&= \mathrm{Tr}_V(S\,T(h_2)^{-1}\,T(g)^{-1}\,T(h_1)) \\
&= \mathrm{Tr}_V(\underline{T(h_1)\,S\,T(h_2)^{-1}}\,T(g)^{-1}) \\
&= \mathrm{Tr}_V((h_1\cdot h_2)\cdot S\,T(g)^{-1}) \\
&= f_{(h_1,h_2)\cdot S}(g)
\end{aligned}$$

Equip V with an ordered basis $\{v_i\}$

$$T(g)\, v_i = \sum_j M(g)_{ji}\, v_j$$

and take S to be the matrix unit e_{ij}

($[e_{ij}]_{ab} = \delta_{ia}\delta_{jb}$. a bais of $\mathrm{End}(V)$).

$$f_S = \mathrm{Tr}_V(\underline{\underline{S}}\, T(g^{-1}))$$

$$= \mathrm{Tr}_V\big(\textstyle\sum_b \underline{\delta_{ia}\delta_{jb}} M_{bc}(g^{-1})\big)$$

$$= \textstyle\sum_{ac} [\delta_{ia} M_{jc}(g^{-1})]\, \delta_{ac}$$

$$= M_{ji}(g^{-1})$$

$\Rightarrow$ We can view M_{ji} as functions

f_S's are linear combinations of

matrix elements of rep of G.