8.15 Schur-Weyl duality and irreps of $GL(d, K)$

Refs: ① § 11.16 of Moore

② Fulton & Harris. Chap 6.

In a general physical system, the full representation space is given by tensor product of single-particle Hilbert spaces $\mathcal{H}_n = \otimes^n \mathcal{H}_1$. So naturally we want to understand how it decomposes into smaller irreps. Now we try to understand the structure of $V^{\otimes n}$, where V is a general rep. ($V = K^d$, $K = \mathbb{R}, \mathbb{C}$)

We start simple, with $V \otimes V$. It forms a natural rep of S_2:

$$\sigma \equiv (12): \quad v_i \otimes v_j \longmapsto v_j \otimes v_i$$

Define Young symmetrizers $\quad c_+ = e + (12) \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$

$$c_- = e - (12) \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$$

$$c_+ V^{\otimes 2} = \text{span}\{ v_i \otimes v_j + v_j \otimes v_i \} =: \text{Sym}^2 V$$

$$c_- V^{\otimes 2} = \text{span}\{ v_i \otimes v_j - v_j \otimes v_i \} =: \Lambda^2 V$$

$$V^{\otimes 2} \cong \text{Sym}^2 V \oplus \Lambda^2 V \cong D^+ \otimes \mathbb{1}^+ \oplus D^- \otimes \mathbb{1}^- \quad \text{isotypic decomposition}$$

$$\dim = \frac{d(d+1)}{2} \qquad \frac{d(d-1)}{2}$$

Any element $t \in V^{\otimes 2}$ is given by a rank-2 tensor:

$$t = \sum_{ij} a_{ij}\, v_i \otimes v_j \qquad (a \in K^{d^2})$$

S_2 can be seen as equally acts on the tensor a:

$$\sigma \cdot t = \sum_{ij} a_{ij}\, v_j \otimes v_i = \sum_{ij} a_{ji}\, v_i \otimes v_j$$

i.e. $(\sigma \cdot a)_{ij} = a_{ji}$

Now consider V the rep of G, some internal sym, then $V \otimes V$ is naturally a rep of G.

$$T^{V\otimes V}_{(g)}(v_1 \otimes v_2) = T(g)\, v_1 \otimes T(g)\, v_1$$

on tensor: $T(g) \cdot t = \sum_{ij} a_{ij}\, (T(g)\, v_i \otimes T(g)\, v_j)$

$$= \sum_{\substack{ij\\kl}} a_{ij}\, M_{ki} M_{lj}\; v_k \otimes v_l$$

i.e. $(g \cdot a)_{kl} = \sum_{ij} M(g)_{k\underline{i}}\, M(g)_{l\underline{\underline{j}}}\, a_{ij}$

(contracts the column index)

Now. a very useful observation:

the action of G and S_n commutes on $V^{\otimes n}$

$$\left\{\begin{aligned} g \cdot \sigma\; v_i \otimes v_j &= g \cdot v_j \otimes v_i = \sum_{kl} M_{kj} M_{li}\; v_k \otimes v_l \\ \sigma \cdot g\; v_i \otimes v_j &= \sigma \sum_{kl} M_{li} M_{kj}\; v_l \otimes v_k = \sum_{kl} M_{kj} M_{li}\; v_k \otimes v_l \end{aligned}\right.$$

$V^{\otimes n}$ is a rep of $G \times S_n$

What's the significance?

We can perform isotypic decomposition of $V^{\otimes n}$ as

$$V^{\otimes n} \cong \oplus_\lambda D^\lambda \otimes R_\lambda.$$

λ a partition of n. i.e. labels an irrep. of S_n

$D^\lambda = \mathrm{Hom}_{S_n}(R_\lambda. V^{\otimes n})$ is the degeneracy space/multiplicity space

spanned by all linear maps from R_λ into $V^{\otimes n}$

that commute with S_n action

($T \in D^\lambda$. $T(\sigma \cdot r) = \sigma \cdot T(r)$)

Schur – Weyl duality theorem : (Fulton & Harris for proofs)

$$V^{\otimes n} \cong \bigoplus_\lambda D_\lambda \otimes R_\lambda$$

R_λ are the irreps of S_n

$D_\lambda = \mathrm{Hom}_{S_n}(R_\lambda. V^{\otimes n})$ the degeneracy space.

The representations D_λ are irreducible representations of $GL(d. K)$ (and its subgroups)

All irreps can be found by varying n

Thus. to construct D_λ. as we haven seen earlier. can be done by decomposition into irreps of S_n. which can be done using Young symmetrizers.

$$V^{\otimes n} \cong \bigoplus_T P(T) V^{\otimes n}$$

$\{T\}$ all <u>standard tableaux</u>

Example. Spin-0 and 1 rep of $SU(2)$

Consider $G = SU(2)$ and S_2

$V = \{|+\rangle, |-\rangle\}$

$V^{\otimes 2} = \{ |s_1\rangle \otimes |s_2\rangle , \ s_i \in V \}$ $\quad$ dim $= 4$

$$V^{\otimes 2} \cong W_1 \otimes P^+ \oplus W_0 \otimes P^-$$

$$W_1 = \mathrm{Sym}^2 V = \{ \overset{|1,1\rangle}{|+\rangle \otimes |+\rangle} , \overset{|1,0\rangle}{\tfrac{1}{\sqrt{2}}(|+\rangle \otimes |-\rangle + |-\rangle \otimes |+\rangle)} , \overset{|1,-1\rangle}{|-\rangle \otimes |-\rangle} \}$$

$$W_0 = \Lambda^2 V = \{ \underset{|0,0\rangle}{\tfrac{1}{\sqrt{2}}(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle)} \}$$

Now consider the group action of $g \in SU(2)$ on V.

$$g = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \qquad |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{cases} g|+\rangle = \alpha|+\rangle + \beta|-\rangle & (= g_{11}|+\rangle + g_{21}|-\rangle) \\ g|-\rangle = -\bar{\beta}|+\rangle + \bar{\alpha}|-\rangle \end{cases}$$

$$g|1,1\rangle = g|+\rangle \otimes g|+\rangle = \alpha^2 |++\rangle + \alpha\beta(|+-\rangle + |-+\rangle) + \beta^2 |--\rangle$$

$$= \alpha^2 |1,1\rangle + \sqrt{2}\alpha\beta |1,0\rangle + \beta^2 |1,-1\rangle$$

$$g|1,-1\rangle = \bar\beta^2 |{+}{+}\rangle - \bar\alpha\bar\beta(|{+}{-}\rangle + |{-}{+}\rangle) + \bar\alpha^2 |{-}{-}\rangle$$

$$= \bar\beta^2 |1,1\rangle - \sqrt{2}\,\bar\alpha\bar\beta\, |1,0\rangle + \bar\alpha^2 |1,-1\rangle$$

$$g|1,0\rangle = \frac{1}{\sqrt{2}}\left(g|+\rangle \otimes g|-\rangle + g|-\rangle \otimes g|+\rangle \right)$$

$$= \frac{1}{\sqrt{2}}\Big(-2\alpha\bar\beta\, |{+}{+}\rangle + (|\alpha|^2 - |\beta|^2)(|{+}{-}\rangle + |{-}{+}\rangle)$$
$$+ 2\bar\alpha\beta\, |{-}{-}\rangle \Big)$$

$$= -\sqrt{2}\,\alpha\bar\beta\, |1,1\rangle + (|\alpha|^2 - |\beta|^2)|1,0\rangle + \sqrt{2}\,\bar\alpha\beta\, |1,-1\rangle$$

$$D^1(g) = \begin{array}{c} \\ \\ \end{array}\begin{array}{ccc} |1,1\rangle & |1,0\rangle & |1,-1\rangle \\ \end{array}$$
$$D^1(g) = \begin{pmatrix} \alpha^2 & -\sqrt{2}\alpha\bar\beta & \bar\beta^2 \\ \sqrt{2}\alpha\beta & |\alpha|^2 - |\beta|^2 & -\sqrt{2}\,\bar\alpha\bar\beta \\ \beta^2 & \sqrt{2}\,\bar\alpha\beta & \bar\alpha^2 \end{pmatrix} \qquad \text{Wigner-}D^1 \text{ matrix}$$

Later: $\mathrm{Sym}^n(\mathbb{C}^2)$ are irreps of $SU(2)$ defined by the trivial irrep of S_n. $\dim = \binom{n+d-1}{n} \overset{d=2}{=} n+1$

For $W_0 = \{\frac{1}{\sqrt{2}}(|{+}{-}\rangle - |{-}{+}\rangle)\} \equiv |0,0\rangle$ Scalar/trivial

$$g\cdot|0,0\rangle = \frac{1}{\sqrt{2}}(|\alpha|^2 |{+}{-}\rangle - |\beta|^2 |{-}{+}\rangle$$
$$- (|\alpha|^2 |{-}{+}\rangle - |\beta|^2 |{+}{-}\rangle)$$
$$= \frac{1}{\sqrt{2}}(|{+}{-}\rangle - |{-}{+}\rangle) = |0,0\rangle$$

$$g|+\rangle = \alpha|+\rangle + \beta|-\rangle$$
$$g|-\rangle = -\bar\beta|+\rangle + \bar\alpha|-\rangle$$

$\Rightarrow$ Tensors of definite symmetries (obtained via Young symmetrizers) transform as irreps of $GL(d,K)$.

Example . $V^{\otimes 3} \overset{\text{span}}{=} \{ v_i \otimes v_j \otimes v_k \}$

S_3	$[()]$	$3[(12)]$	$2[(123)]$
1^+	1	1	1
1^-	1	-1	1
2	2	0	-1

$\chi([()]) = d^3$

$\chi([(12)]) = d^2$ $\qquad \underline{v_i \otimes v_i \otimes v_k}$

$\chi([(123)]) = d$

$$a_{1^+} = \langle \chi_{1^+}, \chi \rangle = \frac{1}{6}(d^3 \times 1 + d^2 \times 3 + d \times 2) = \underline{\frac{1}{6} d(d+1)(d+2)}$$

$$a_{1^-} = \langle \chi_{1^-}, \chi \rangle = \frac{1}{6}(d^3 - 3d^2 + 2d) = \frac{1}{6} d(d-1)(d-2)$$

$$a_2 = \langle \chi_2, \chi \rangle = \frac{1}{6}(2d^3 - 2d) = \frac{1}{3} d(d+1)(d-1)$$

① $\boxed{1|2|3}$ $\qquad C = PQ = e + (12) + (13) + (23) + (123) + (132)$

$$C \cdot V^{\otimes 3} = \text{span} \{ \sum_\sigma v_{\sigma(i)} \otimes v_{\sigma(j)} \otimes v_{\sigma(k)} \}$$

$$= \text{Sym}^3 V$$

$$t = \Sigma\, a_{ijk}\, v_i \otimes v_j \otimes v_k$$

$$\sigma \cdot t = \Sigma\, a_{ijk}\, v_{\sigma(i)} \otimes v_{\sigma(j)} \otimes v_{\sigma(k)}$$

$$= \Sigma\, a_{\sigma^{-1}(i)\sigma^{-1}(j)\sigma^{-1}(k)}\, v_i \otimes v_j \otimes v_k$$

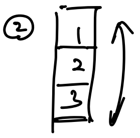
$$\Rightarrow (\sigma \cdot a)_{ijk} = a_{\sigma^{-1}(i)\sigma^{-1}(j)\sigma^{-1}(k)}$$

$$\underline{(a_S)_{ijk} = \sum_\sigma a_{\sigma^{-1}(i)\sigma^{-1}(j)\sigma^{-1}(k)} = \sum_\sigma a_{\sigma(i)\sigma(j)\sigma(k)}}$$

$$\Rightarrow (a_S)_{jik} = (a_S)_{ijk}$$

$$(\sigma a_S)_{ijk} = (a_S)_{ijk}$$

②



$$c = e - (12) - (13) - (23) + (123) + (132)$$

$$(a_\wedge)_{ijk} = \sum_\sigma sgn(\sigma)\, a_{\sigma^{-1}(i), \sigma^{-1}(j), \sigma^{-1}(k)}$$

$$\underline{(a_\wedge)_{jik}} = (\tau(ij)\, a_\wedge)_{ijk}$$

$$= \sum_\sigma \underline{\tau(ij)}\, sgn(\sigma)\, a_{\underline{\sigma^{-1}(i)}, \underline{\sigma^{-1}(j)}, \sigma^{-1}(k)}$$

$$= \sum_\sigma sgn(\sigma)\, a_{\underline{\sigma^{-1}(j)}, \underline{\sigma^{-1}(i)}, \sigma^{-1}(k)}$$

$$= \sum_\sigma sgn(\underline{\sigma\, \tau(ij)})\, a_{\underline{\sigma^{-1}(i)}, \sigma^{-1}(j), \underline{\sigma^{-1}(k)}}$$

$$= -\sum_\sigma sgn(\sigma)\, a_{\sigma^{-1}(i), \sigma^{-1}(j), \sigma^{-1}(k)}$$

$$= \underline{-(a_\wedge)_{ijk}}$$

if $d = 2$: $i, j. k \in \{1, 2\}$

$$a_{1,1,2} = -a_{1,1,2} = 0$$

$\Rightarrow$ all elements $a_{ijk} = 0$

$V = k^d$. the irrep corresponding to a Young diagram is 0 of d is smaller than the number of rows of the Young diagram.

③ $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$ $\quad c_{(2,1)} = (e+(12))(e-(13)) = e+(12)-\underline{(13)}-\underline{(132)}$

$$c_{(2,1)}V^{\otimes 3} = \text{span}\{v_i\otimes v_j\otimes v_k + \underline{v_j}\otimes \underline{v_i}\otimes v_k - v_{\underline{k}}\otimes v_j\otimes \underline{v_i} - \underline{v_k\otimes v_i\otimes v_j}\}$$

$$(a_2)_{ijk} = a_{ijk} + a_{\underline{ji}k} - a_{\underline{k}j\underline{i}} - a_{\underline{jki}}$$

$$\begin{tikzcd} i \arrow[r] & k \arrow[r] & j \arrow[ll, bend right] \end{tikzcd}$$

$$\left(\begin{array}{l} \sigma: v_i\otimes v_j\otimes v_k \longrightarrow v_{\sigma(i)}\otimes v_{\sigma(j)}\otimes v_{\sigma(k)} \\ a_{ijk} \longrightarrow a_{\sigma^{-1}(i)\,\sigma^{-1}(j)\,\sigma^{-1}(k)} \end{array} \right)$$

↘

$$\begin{tikzcd} i \arrow[rr, bend left] & k \arrow[l] & j \arrow[l] \end{tikzcd}$$

$$\begin{cases} (a_2)_{ijk} + (a_2)_{jki} + (a_2)_{kij} = 0 & - A \\ (a_2)_{\underline{i}j\underline{k}} = -(a_2)_{\underline{k}j\underline{i}} & - B \end{cases}$$

$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$: $B \longrightarrow (a_2)_{ijk} = -(a_2)_{jik}$

In physics. $\text{Sym}^n V$ for bosons $\lambda = (n)$

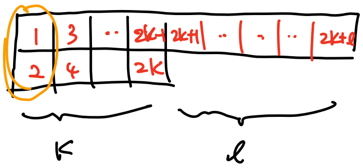
$\Lambda^n V$ for fermions $\lambda = (1, 1 \dots 1)$

other partitions : parastatistics

2. $G = SU(2) \subset GL(2, \mathbb{C})$ irreps

We consider Young diagrams with at most 2 rows.

$T =$



The corresponding Young symmetrizer.

$$c_T = P_T \underline{\underline{Q_T}}$$

$$c_T \cdot v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n} \quad (i_m \in \{1, 2\}) \qquad \left(v_{i_1} \wedge v_{i_2} := v_{i_1} \otimes v_{i_2} - v_{i_2} \otimes v_{i_1} \right)$$

$$= P_T \, (\underline{v_{i_1} \wedge v_{i_2}}) \otimes (\underline{v_{i_3} \wedge v_{i_4}}) \otimes \cdots \otimes (v_{i_{2k-1}} \wedge v_{i_{2k}})$$

$$Q_T = \prod_{i=1}^{k} e - (2i-1, 2i) \qquad \otimes v_{i_{2k+1}} \otimes \cdots \otimes v_{i_{2k+\ell}}$$

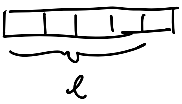
$v_{i_{2j-1}} \wedge v_{i_{2j}} \neq 0$ iff $i_{2j-1} \neq i_{2j}$ $\quad v_1 \wedge v_2$ or $v_2 \wedge v_1$

The <u>non-zero</u> images of c_T is

$$c_T \bigotimes_{j=1}^{n} v_{i_j} = \underline{P_T \left[\bigotimes^{k} (v_1 \wedge v_2) \right]} \otimes v_{i_{2k+1}} \otimes \cdots \otimes v_{i_{2k+\ell}}$$

$$= (-1)^{k} \bigotimes_{i}^{k} (v_1 \wedge v_2) \otimes P_{T'} (v_{i_{2k+1}} \otimes \cdots \otimes v_{i_{2k+\ell}})$$

T':



$V^{\otimes n}$ as rep of $SU(2)$.

$u \in SU(2)$ acts on $v_1 \wedge v_2$

$$
\begin{aligned}
u \cdot (v_1 \wedge v_2) &= u\,(v_1 \otimes v_2 - v_2 \otimes v_1) \\
&= \sum_{ij} u_{i1} u_{j2}\, v_i \otimes v_j - \sum_{ij} u_{i2} u_{j1}\, v_i \otimes v_j \\
&= (u_{11} u_{12} - u_{12} u_{11})\, v_1 \otimes v_1 + \\
&\quad (u_{11} u_{22} - u_{12} u_{21})\, v_1 \otimes v_2 + \\
&\quad (u_{21} u_{12} - u_{22} u_{11})\, v_2 \otimes v_1 + \\
&\quad (u_{21} u_{22} - u_{22} u_{21})\, v_2 \otimes v_2 \\
&= (\det u)\, v_1 \wedge v_2
\end{aligned}
$$

$$u^{\otimes n}\Big(C_T \bigotimes_j^n v_{i_j}\Big) = \big(\overset{1}{\cancel{\det u}}\big)^k (-1)^k \bigotimes_i^k (v_1 \wedge v_2) \otimes u^{\otimes l} P_{T'} (v_{i_{2k+1}} \otimes \cdots \otimes v_{i_{2k+l}})$$

$u \in SU(2)$ acts non-trivially only on $\underline{P_{T'}(v_{i_{2k+1}} \otimes \cdots \otimes v_{i_n})}$

$\Rightarrow$ irreps of $SU(2)$ is in one-to-one correspondence with Young diagrams of a <u>single row</u> of l boxes

Dimension of the irrep.

$d=2$: $\binom{l+d-1}{l} = \binom{l+1}{l} = l+1$

$\mathrm{span}\{ v_{i_1} \otimes \cdots \otimes v_{i_l} \}$
$i_1 \le i_2 \le \cdots \le i_l$
$\dim = l+1$

in physics, $l = 2j$ "spin-j representation of $SU(2)$"

$\Rightarrow$ irreps : $\mathrm{Sym}^l V$. $V \cong \mathbb{C}^2$ the fundamental rep.