

§ 10. orthogonality relations of matrix elements
of reps ; Peter-Weyl theorem.

Recall : ① Basics of rep. rep.

$$L^2(G) = \{ f : G \rightarrow \mathbb{C} \mid \int_G |f(g)|^2 dg < \infty \}$$

is a unitary $G \times G$

② V a rep. $\text{End}(V) := \text{Hom}(V, V)$ is
also a unitary rep of $G \times G$.

$$S \in \text{End}(V) : (g_1, g_2) \cdot S = T(g_1) \cdot S \cdot T(g_2)^{-1}$$

$$\iota : \text{End}(V) \longrightarrow L^2(G)$$

$$S \longmapsto \text{Tr}_V(S T(g)^{-1}) := \varphi_S$$

matrix unit $\underline{e_{ij}} \longmapsto \underline{M_{ij}^{T(g)^{-1}}} = M(g^{-1})_{ji}$

by a simple extension:

$$\begin{aligned} \iota : \bigoplus_{\mu} \text{End}(V^{\mu}) &\longrightarrow L^2(G) \\ \bigoplus_i S_i &\longmapsto \sum_i \varphi_{S_i} \end{aligned}$$

Peter-Weyl theorem: G compact. Then there is an isomorphism of $G \times G$ representations

$$L^2(G) \cong \bigoplus_{\mu} \text{End}(V^{\mu})$$

where we sum over the distinct isomorphism class of each irrep exactly once.

Peter-Weyl theorem is the consequence of two statements.

1. Let (V, T) be a unitary irrep of a compact group G on a complex vector space V .

Then V is finite dimensional.

Proof: pick a nonzero $v \in V$. Define, for $w \in V$.

$$L(w) = \int_G df \langle T(f)v, w \rangle T(f)v$$

L is an operator $V \rightarrow V$

$$\begin{aligned} L(T(h)w) &= \int_G df \langle T(f)v, \pi(h)w \rangle T(f)v \\ &= \int_G df \langle T(h^{-1}f)v, w \rangle T(f)v \\ &\stackrel{h^{-1}f \rightarrow f}{=} \int_G df \langle T(f)v, w \rangle T(hf)v \\ &= Tw \int_G df \langle T(f)v, w \rangle T(f)v \\ &= T(h) \cdot L(w) \end{aligned}$$

L is an intertwiner $\therefore L(Tg) = Tg \cdot L \quad \forall g \in G$.

Schur's lemma $\Rightarrow L = \lambda 1_V, \quad \lambda \in \mathbb{C}$.

$$\langle v, L(v) \rangle = \int_G dg |\langle Tg, v, v \rangle|^2$$

$$\hookrightarrow \lambda \|v\|^2$$

$$\lambda = \frac{1}{\|v\|^2} \int_G dg \langle Tg, v, v \rangle$$

$$\text{Tr}(L) = \sum_i \langle v_i, L(v_i) \rangle$$

$$= \int_G dg \langle Tg, v, v_i \rangle \langle v_i, Tg, v \rangle$$

$$= \int_G dg |\langle v_i, Tg, v \rangle|^2$$

$$= \int_G dg \|Tg, v\|^2$$

$\hookrightarrow \|v\|^2$ due to unitary

$$= \|v\|^2 \text{vol}(G) < \infty$$

$$\lambda \cdot \dim V = \|v\|^2 \text{vol}(G)$$

$$\hookrightarrow \dim V = \text{vol}(G) \frac{\|v\|^4}{\int_G |\langle v, Tg, v \rangle|^2 dg}$$

2. Let G be a compact group. The Hermitian inner product on $L^2(G)$

$$\langle \varphi_1, \varphi_2 \rangle := \int_G \varphi_1^*(g) \varphi_2(g) dg$$

with normalized Haar measure, s.t. the

$$\text{volume of } G \int_G dg = 1.$$

$$L^2(G) \cong \bigoplus_n a_n V^n$$

$\stackrel{=}{=} \text{Let } \{V^n\} \text{ be a set of representations of distinct isomorphism classes of unitary irreps.}$

(Because of statement 1). For each V^n

choose an orthonormal (ON) basis $w_i^{(\mu)}$.

$$i=1, \dots, n_\mu. \quad n_\mu = \dim V^{(\mu)}$$

$$T^{(\mu)}(g) w_i^{(\mu)} = \sum_{j=1}^{n_\mu} M_{ji}^{(\mu)}(g) w_j^{(\mu)}$$

$M_{ij}^{(\mu)}$ form a complete orthogonal set of functions on $L^2(G)$.

$$\langle M_{i_1, j_1}^{(\mu_1)}, M_{i_2, j_2}^{(\mu_2)} \rangle = \frac{1}{n_\mu} \delta^{\mu_1, \mu_2} \delta_{i_1, i_2} \delta_{j_1, j_2}$$

Proof. $\forall A: V^\mu \rightarrow V^\nu$ a linear transf.

$$\tilde{A} := \int_G T^\nu(g) A T^\mu(g^{-1}) dg$$

$$T^\nu(h) \tilde{A} = \int_G T^\nu(hg) A T^\mu(g^{-1}) dg$$

$$\stackrel{g \mapsto h^{-1}g}{=} \int_G T^\nu(g) A T^\mu((h^{-1}g)^{-1}) dg$$

$$= \left(\int_G T^\nu(g) A T^\mu(g)^{-1} dg \right) T^\mu(h)$$

$$= \tilde{A} T^\mu(h)$$

\tilde{A} is an intertwiner

$$\begin{array}{ccc} V^\mu & \xrightarrow{\tilde{A}} & V^\nu \\ \downarrow T^\mu & & \downarrow T^\nu \\ V^\mu & \xrightarrow{\tilde{A}} & V^\nu \end{array}$$

By Schur's lemma. $\tilde{A} = \delta_{\mu\nu} \hat{A}$. $\hat{A} = \underline{\underline{C_A 1_\nu}}$

Assign a basis for V^μ and V^ν

$$[\tilde{A}]_{ia} = \underline{\delta_{\mu\nu} C_A \delta_{ia}} = \int_{\mathcal{G}} dg [\mu^\nu(g) A \mu^\mu(g^{-1})]_{ia}$$

$$= \sum_{i', a'} \int_{\mathcal{G}} dg \underline{\mu_{ii'}^\nu(g) A_{i'a'} \mu_{a'a}^\mu(g^{-1})} \quad (*)$$

set $\mu = \nu$, $i = a$, and take the trace.

$$nC_A = \sum_{i, i', a'} \int_{\mathcal{G}} dg \mu_{ii'}^\mu(g) A_{i'a'} \mu_{a'i}^\mu(g^{-1})$$

$$= \int_{\mathcal{G}} dg \text{Tr} \left(\overbrace{\mu^\mu(g)}^{\text{Tr}} A \overbrace{\mu^\mu(g^{-1})}^{\text{Tr}} \right)$$

$$= \int_{\mathcal{G}} dg (\text{Tr} A) = \text{Tr} A$$

$$\Rightarrow \underline{C_A = \frac{1}{n_\mu} \text{Tr} A}$$

Now take A to be the matrix unit e_{jk}
 ($\text{Tr} e_{jk} = \delta_{jk}$).

insert into (*)

$$\sum_{i, a'} \int_{\mathcal{G}} dg \mu_{ii'}^\nu(g) \overset{[e_{jk}]_{i'a'}}{\parallel} \mu_{a'a}^\mu(g^{-1}) \overset{\frac{\delta_{jk}}{n_\mu}}{\parallel} = \frac{\text{Tr} e_{jk}}{n_\mu} \delta_{\mu\nu} \delta_{ia}$$

$$\Rightarrow \int_{\mathcal{G}} dg \underline{\mu_{ij}^\nu(g)} \underline{\mu_{ka}^\mu(g^{-1})} = \frac{1}{n_\mu} \delta_{\mu\nu} \delta_{ia} \delta_{jk}$$

$$\Downarrow$$

$$[\mu^\mu(g)^+]_{ka} = \underline{\mu_{ak}^\mu(g)}$$

$$\Rightarrow \langle \mu_{ak}^\mu, \mu_{ij}^\nu \rangle = \frac{1}{n_\mu} \delta_{\mu\nu} \delta_{ia} \delta_{jk}$$

$$\Rightarrow \langle \mu_{i_1, j_1}^{\mu_1}, \mu_{i_2, j_2}^{\mu_2} \rangle = \frac{1}{n_\mu} \delta^{\mu_1 \mu_2} \delta_{i_1, i_2} \delta_{j_1, j_2}$$

We have shown that $\{M_{ij}^\mu\}$ is a set of orthogonal functions on $L^2(G)$.

basis \Leftarrow completeness?

Let W be the subspace spanned by $\{M_{ij}^\mu\}$.

\Rightarrow The orthogonal complement W^\perp is also a unitary rep. of $G \times G$.

\Rightarrow decomposable into unitary irreps V^μ

$\{f_j\}_{j=1}^{n_\mu}$ transforms as V^μ under right regular rep.

$$R(g)f_j = \sum_k M(g)_{kj}^\mu f_k$$

$$\underline{f(hg)} = \sum_k M(g)_{kj}^\mu f_k(h)$$

$$\stackrel{h=1}{\Rightarrow} f(g) = \sum_k f_k(1) \underline{M_{kg}^\mu(f)} \quad (\forall g \in G)$$

$f \in W$ contradiction with the assumption $f \in W^\perp$

$$\Rightarrow W^\perp = 0$$

[if with left reg. rep.

$$L(g) f_j = \sum \mu^h(g)_{kj} f_k$$

$$\underline{f(g^{-1}h)} = \sum \mu^h(g)_{kj} f_k(h)$$

$$\begin{aligned} h=1 \Rightarrow \underline{f(g)} &= \sum \mu^1(g^{-1})_{kj} f_k(1) \\ &= \sum \underline{\overline{\mu^1(g)}_{jk}} f_k(1) \end{aligned}$$

$\{ \overline{\mu^1}_{ij} \}$ is another set
of orthogonal basis \rfloor

$\Rightarrow \{ \mu^1_{ij} \}$ is complete.

so now we know $\underline{\bigoplus_{\mu} \text{End } V^{\mu} \cong L^2(G)}$

Corollary for finite groups.

$L^2(G)$ of $\dim |G|$:

$$\underline{\delta_a(g)} = \begin{cases} 1 & g=a \\ 0 & \end{cases}$$

$$(g \delta_a = \delta_{ga})$$

$$\forall f: G \rightarrow \mathbb{C}. \quad f = \sum_{g \in G} f(g) \delta_g$$

$$\text{End}(V^{\mu}) \cong \text{Mat}_{n_{\mu} \times n_{\mu}}(\mathbb{C}) \quad \underline{\text{eij}}$$

$$\dim_{\mathbb{C}}(\text{End}(V^{\mu})) = n_{\mu}^2$$

$$\Rightarrow |G| = \sum_{\mu} n_{\mu}^2$$

Examples 1. S_3 $|S_3| = 6$

$$6 = 1 \times 6 \quad \times \text{ abelian}$$

$$= 1 + 1 + 2^2 \quad \checkmark$$

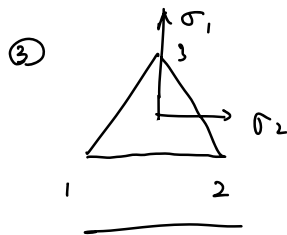
$$L(S_3) \cong \Gamma_{\text{trivial}} \oplus \Gamma_{\text{sgn}} \oplus 2\Gamma_2$$

(+) (-) (2)

① $\mu^+(\phi) = 1 \quad \forall \phi \in S_3$

② $\mu^-(\phi) = 1 \quad \phi \in \{(), (123), (132)\} = A_3$

$\mu^-(\phi) = -1 \quad \phi \in \{(12), (13), (23)\}$



$$\mu^{(2)}(12) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mu^{(2)}(13) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$\mu^{(2)}(123) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\langle \mu_{ij}^{\mu}, \mu_{i'j'}^{\nu} \rangle = \frac{1}{n_{\mu}} \delta_{\mu\nu} \delta_{ii'} \delta_{jj'}$$

a. $\langle \mu^+, \mu^- \rangle = 0$

b. $\langle \mu^+, \mu_{11}^{(2)} \rangle = \frac{1}{6} \sum \mu_{11}^{(2)}(\phi) = \frac{2}{6} (1 - \frac{1}{2} - \frac{1}{2}) = 0$

c. $\langle \mu_{11}^{(2)}, \mu_{11}^{(2)} \rangle = \frac{2}{6} (1 + \frac{1}{4} + \frac{1}{4}) = \frac{1}{2} = \frac{1}{n_{\mu}}$

$$2. \quad G = \mathbb{Z}_2 = \{ \sigma \mid \sigma^2 = 1 \}$$

$$\varphi \in L^2(G) = \{ \text{Map}(G, \mathbb{C}) \}$$

$$\varphi(1) = \varphi_+ \in \mathbb{C}$$

$$\varphi(\sigma) = \varphi_-$$

$$L^2(G) \cong \mathbb{C}^2$$

$$\mathbb{Z}_2 \text{ irreps } \rho_{\pm}(\sigma) = \pm 1 \quad V_{\pm} \cong \mathbb{C}$$

$$\begin{cases} \mu^+(1) = \mu^-(1) = 1 \\ \mu^+(\sigma) = 1 \quad \mu^-(\sigma) = -1 \end{cases}$$

$\{ \mu^+, \mu^- \}$ or basis of $L^2(\mathbb{Z}_2)$

$$\Rightarrow \varphi = \frac{\varphi_+ + \varphi_-}{2} \mu^+ + \frac{\varphi_+ - \varphi_-}{2} \mu^-$$

$$\text{Previously: } T(\sigma)$$

$$\left(\begin{aligned} P_{\pm} &= \frac{1}{2} (\mu^{\pm}(1) \cdot 1 + \mu^{\pm}(\sigma) T(\sigma)) \\ &= \frac{1}{2} (1 \pm T(\sigma)) \end{aligned} \right) \quad P_{\pm} = \frac{1}{2} (1 \pm T(\sigma)) \text{ is of the}$$

$$\text{is form: } P_{\pm} = \int_G \overline{\mu^{\pm}(g)} T(g) dg \quad (\text{later})$$

$$3. \quad G = U(1) \quad (\hat{G} = \mathbb{Z})$$

$$(p_n, v_n): \quad p_n(z) = z^n \quad n \in \mathbb{Z}. \quad \left(= \underline{e^{i\theta n}} \right) \\ v_n \cong \mathbb{C} \quad \theta \in [0, 2\pi)$$

$$\langle p_{n_1}, p_{n_2} \rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} (\overline{p_{n_1}(\theta)})^* p_{n_2}(\theta) = \delta_{n_1, n_2} \\ e^{i\theta(n_1 - n_2)}$$

$$\{ p_n = e^{i\theta n} \} \text{ or basis: } \psi = \sum_n \alpha_n p_n$$

$$\alpha_n = \int_{U(1)} p_n^* \psi(g) dg$$

4. S_4 ? $|S_4| = 24 = \underbrace{1+1}_{\text{trivial}} + \overset{\text{sgn}}{1} + 3^2 + 13$?
standard

S_4	e (IIII)	(12) II	$(12)(34)$ II	(123) III	(1234) IIII
P_{triv}	1	1	1	1	1
P_{sgn}	1	-1	1	1	-1
P_{std}	3	1	-1	0	-1
$P_{\text{sgn}} \otimes P_{\text{std}}$	3	-1	-1	0	1
P_2	2				

is $13 = 2^2 + 3^2$, or 1×9 or else?

(Problem 15 from HW)

$$[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}, \quad [G, G] \trianglelefteq G.$$

$$\rho: G \rightarrow \mathbb{C}^*$$

$$\rho([g_1, g_2]) = \rho(g_1 g_2 g_1^{-1} g_2^{-1}) = \rho(e) = 1 \quad \text{trivial}$$

$$g[G, G] \in G/[G, G] \Rightarrow \rho(g[G, G]) = \rho(g)$$

distinct 1D rep of G = distinct rep of $G/[G, G]$

$$G/[G, G] \text{ abelian} \Rightarrow \text{all irreps 1D}$$

$$\# \text{ characters} = \# \text{ conj. classes}$$

$$= |G/[G, G]|$$

$$[G, G] = A_n \Rightarrow |G/[G, G]| = 2.$$

\Rightarrow two distinct 1D irreps

8.11 Explicit decomposition of a representation

Let (T, V) be any rep. of a compact group G . Define

$$\underline{P_{ij}^{(\mu)}} := n_\mu \int_G \overline{\mu_{ij}^{(\mu)}(g)} T(g) dg$$

$\mu_{ij}^{(\mu)}$ w.r.t unitary irreps with ON basis of $V^{(\mu)}$

$$\boxed{P_{ij}^{(\mu)} P_{kl}^{(\nu)} = \delta^{\mu\nu} \delta_{jk} P_{il}^{(\nu)}}$$

$$\begin{aligned} T(h) P_{ij}^{(\mu)} &= n_\mu T(h) \int_G dg \overline{\mu_{ij}^{(\mu)}(g)} T(g) \\ &= n_\mu \int_G dg \overline{\mu_{ij}^{(\mu)}(g)} T(hg) \\ &\stackrel{hg \rightarrow g}{=} n_\mu \int_G dg \overline{\mu_{ij}^{(\mu)}(h^{-1}g)} T(g) \\ &\quad \mu_{ki}^{(\mu)}(h) \overline{\mu_{kj}^{(\mu)}(g)} \\ &= \sum_k \mu_{ki}^{(\mu)}(h) P_{kj}^{(\mu)} \end{aligned}$$

$$T(h) P_i^{(\mu j)} = \sum_k \mu_{ki}^{(\mu)}(h) P_k^{(\mu j)}$$

$\forall \varphi \in V$. $(P_{ij}^{(\mu)} \varphi \neq 0)$. then

$$\underline{\text{span } \{ P_{ij}^{(\mu)} \varphi, i=1, \dots, n_\mu \} \text{ (fix } \mu, j \text{)}}$$

transforms as $(T^{(\mu)}, V^{(\mu)})$

8.12. Orthogonality relations of characters ;

Character table.

8.12.1 Orthogonality relations —

Recall - a class function on G :

$$f: G \rightarrow \mathbb{C}.$$

$f(g) = f(hgh^{-1}) \quad \forall g, h \in G$. They span
a subspace $L^2(G)^{\text{class}} \subset L^2(G)$.

Theorem The characters $\{\chi_\mu\}$ is an
orthonormal (ON) basis for the
vector space of class functions $L^2(G)^{\text{class}}$.

Proof. $\int_G dg M_{ij}^{(\mu)}(g)^* M_{kl}^{(\nu)}(g) = \frac{1}{n_\mu} \delta_{\mu\nu} \delta_{ik} \delta_{jl}$

Set $i=j, k=l$ & sum over i, k

$$\Rightarrow \int_G dg M_{ii}^{(\mu)}(g)^* M_{kk}^{(\nu)}(g) = \frac{1}{n_\mu} \delta_{\mu\nu} \delta_{ik}$$

$$\stackrel{\sum_{i,k}}{\Rightarrow} \int_G dg \chi_\mu(g)^* \chi_\nu(g) = \delta_{\mu\nu}$$

$\Rightarrow \{\chi_\mu\}$ ON set

Completeness ?

$$\forall f \in L^2(G) \xrightarrow[\{\mu_{ij}^\mu\} \text{ complete}]{\text{Peter-Weyl}} f(g) = \sum_{\mu, i, j} \hat{f}_{ij}^\mu \mu_{ij}^\mu(g)$$

$$\text{of } f \in L^2(G)^{\text{class.}} \quad f(g) = f(hgh^{-1})$$

$$\int_G dh f(g) = \int_G dh f(hgh^{-1})$$

$$\stackrel{u}{=} f(g)$$

$$\begin{aligned} \int_G f(hgh^{-1}) dh &= \sum_{\mu, i, j} \hat{f}_{ij}^\mu \int_G \mu_{ij}^\mu(hgh^{-1}) dh \\ &\quad \downarrow \\ &\quad \mu_{ik}^\mu(h) \mu_{kl}^\mu(g) \mu_{lj}^\mu(h^{-1}) \\ &= \sum_{\substack{\mu, i, j \\ k, l}} \hat{f}_{ij}^\mu \mu_{kl}^\mu(g) \int_G \mu_{ik}^\mu(h) \mu_{jl}^{\mu*}(h) dh \\ &\quad \underline{\qquad \qquad \qquad} \quad \underline{\qquad \qquad \qquad} \\ &\quad \qquad \qquad \frac{1}{n_\mu} \delta_{ij} \delta_{kl} \\ &= \sum_{\mu, i} \frac{\hat{f}_{ii}^\mu}{n_\mu} \chi_\mu(g) \end{aligned}$$

$$\Rightarrow f(g) = \sum_{\mu, i} \frac{\hat{f}_{ii}^\mu}{n_\mu} \chi_\mu(g)$$

$$\Rightarrow \{\chi_\mu\} \text{ spans full } L^2(G)^{\text{class.}}$$