

8.10. Orthogonality relations of matrix elements
of reps; Peter-Weyl theorem.

Recall: ① Basics of rep. rep.

$$L^2(G) = \{ f : G \rightarrow \mathbb{C} \mid \int_G |f(g)|^2 dg < \infty \}$$

is a unitary $G \times G$

② V a rep. $\text{End}(V) := \text{Hom}(V, V)$ is also a unitary rep of $G \times G$.

$$S \in \text{End}(V) : \underbrace{(g_1, g_2) \cdot S = T(g_1) \cdot S \cdot T(g_2)^{-1}}$$

$$\iota : \text{End}(V) \longrightarrow L^2(G)$$

$$S \longmapsto \overline{\text{Tr}_v(S T(g))} := \varphi_S$$

$$\text{matrix unit } e_{ij} \longmapsto M_{ij}^{T^{-1}} = \langle T(g^{-1})_j, i \rangle$$

by a simple extension:

$$\iota : \bigoplus_{\mu} \underline{\text{End}(V^\mu)} \longrightarrow \underline{L^2(G)}$$

$$\bigoplus_i S_i \longmapsto \sum_i \underline{\varphi_{S_i}}$$

Peter-Weyl theorem. If G compact. Then there is an isomorphism of $G \times G$ representations

$$L^2(G) \cong \bigoplus_{\mu} \text{End}(V^{(\mu)})$$

where we sum over the distinct isomorphism class of each irrep exactly once.

Peter-Weyl theorem is the consequence of two statements.

1. Let (V, T) be a unitary irrep of a compact group G on a complex vector space V .

Then V is finite dimensional.

Proof: pick a nonzero $v \in V$. Define, for $w \in V$.

$$L(w) = \int_G df \langle T(f)v, w \rangle T(f)v$$

L is an operator $V \rightarrow V$

$$\begin{aligned} L(T(h)w) &= \int_G df \langle T(f)v, T(h)w \rangle T(f)v \\ &= \int_G df \langle T(h^{-1}f)v, w \rangle T(f)v \\ &\stackrel{h^{-1}f \rightarrow f}{=} \int_G df \langle T(f)v, w \rangle T(hf)v \\ &= Tw \int_G df \langle T(f)v, w \rangle T(f)v \\ &= T(h) \cdot L(w) \end{aligned}$$

L is an intertwiner $\Rightarrow LT(h) = T(h)L \quad \forall h \in G$.

Schur's lemma $\Rightarrow L = \lambda \mathbb{1}_V \quad \lambda \in \mathbb{C}$

$$\begin{aligned} \langle v, L(v) \rangle &= \int_G dg |\langle T(g)v, v \rangle|^2 \\ \hookrightarrow \lambda &\equiv \lambda \|v\|^2 \quad \lambda = \frac{\int_G dg \langle T(g)v, v \rangle}{\int_G dg} \\ \text{Tr}(L) &= \sum_i \langle v_i, L(v_i) \rangle \\ &= \sum_i \int_G dg \langle T(g)v_i, v_i \rangle \langle v_i, T(g)v_i \rangle \\ &= \sum_i \underbrace{\int_G dg |\langle v_i, T(g)v_i \rangle|^2}_{\|T(g)v_i\|^2} \\ &= \int_G dg \|T(g)v\|^2 \\ &\qquad \text{"}\|v\|^2 \text{ due to unitary"} \\ &= \|v\|^2 \text{ vol}(G) < \infty \end{aligned}$$

$$\lambda \cdot \dim V = \|v\|^2 \text{ vol}(G)$$

$$\hookrightarrow \boxed{\dim V = \text{vol}(G) \frac{\|v\|^4}{\int_G |\langle v, T(g)v \rangle|^2 dg}}$$

2. Let G be a compact group. The Hermitian inner product on $L^2(G)$

$$\langle \varphi_1, \varphi_2 \rangle := \int_G \varphi_1^*(g) \varphi_2(g) dg$$

with normalized Haar measure. s.t. the volume of G $\int_G dg = 1$.

$L^2(G) \cong \bigoplus_{k=1}^n \alpha^k V^{(k)}$
 \Rightarrow Let $\{V^{(k)}\}$ be a set of representations of distinct isomorphism classes of unitary irreps.

(Because of statement 1). For each $V^{(k)}$

choose an orthonormal (ON) basis $w_i^{(\mu)}$.

$$i=1, \dots, n_\mu, \quad n_\mu = \dim V^{(\mu)}$$

$$T^{(\mu)}(g) w_i^{(\mu)} = \sum_{j=1}^{n_\mu} M_{ji}^{(\mu)}(g) w_j^{(\mu)}$$

$\parallel M_{ij}^{(\mu)}$ form a complete orthogonal set of functions on $L^2(G)$.

$$\langle M_{i_1, j_1}^{(\mu_1)}, M_{i_2, j_2}^{(\mu_2)} \rangle = \frac{1}{n_\mu} \delta^{\mu_1, \mu_2} \delta_{i_1, i_2} \delta_{j_1, j_2}$$

Proof. $\forall A: V^\mu \rightarrow V^\nu$ a linear transf.

$$\tilde{A} := \int_G T^\nu(g) A T^\mu(g^+) dg$$

$$T^\nu(h) \tilde{A} = \int_G T^\nu(hg) A T^\mu(g^+) dg$$

$$= \int_G T^\nu(g) A T^\mu((h^{-1}g)^+) dg$$

$$= \left(\int_G T^\nu(g) A T^\mu(g)^+ dg \right) T^\mu(h)$$

$$= \tilde{A} T^\mu(h)$$

\tilde{A} is an intertwiner

$$\begin{array}{ccc} V^\mu & \xrightarrow{\tilde{A}} & V^\nu \\ \downarrow T^\mu & & \downarrow T^\nu \\ V^\mu & \xrightarrow{\hat{A}} & V^\nu \end{array}$$

By Schur's lemma. $\tilde{A} = \delta_{\mu\nu} \hat{A}$. $\hat{A} = \underline{c_A} \mathbf{1}_V$

Assign a basis for V^μ and V^ν

$$[\tilde{A}]_{ia} = \underbrace{\delta_{\mu\nu} C_A \cdot \delta_{ia}}_{(*)} = \int_G dg [\mu^\nu(g) A \mu^\mu(g^{-1})]_{ia}$$

$$= \sum_{i,a'} \underbrace{\int_G dg M_{ii'}^\nu(g) A_{i'a'} M_{a'a}^\mu(g^{-1})}_{(*)}$$

Set $\mu = \nu$, $i = a$. and take the trace.

$$\begin{aligned} nC_A &= \sum_{i,i',a'} \int_G dg M_{ii'}^\mu(g) A_{i'a'} M_{a'i}^\mu(g^{-1}) \\ &= \int_G dg \text{Tr} (\overbrace{\mu^\mu(g)}^{\text{Tr } A} A \overbrace{\mu^\mu(g^{-1})}^{\text{Tr } A}) \\ &= \int_G dg (\text{Tr } A) = \text{Tr } A \\ \Rightarrow C_A &= \underbrace{\frac{1}{n_\mu} \text{Tr } A}_{(*)} \end{aligned}$$

Now take A to be the matrix unit e_{jk}
 $(\text{Tr } e_{jk} = \delta_{jk})$.

$$\begin{aligned} \text{insert into } (*) &\\ \sum_{i'a'} \int_G dg M_{ii'}^\nu(g) \underbrace{(\delta_{j'i} \delta_{ka'})}_{\substack{(*) \\ \text{Tr } e_{jk}}} M_{a'a}^\mu(g^{-1}) &= \frac{\text{Tr } e_{jk}}{n_\mu} \delta_{\mu\nu} \delta_{ia} \delta_{jk} \\ \Rightarrow \int_G dg \underbrace{\mu_{ij}^\nu(g)}_{\substack{\downarrow \\ [M_{ik}^\mu(g)]_{ka}^+}} \underbrace{\mu_{ka}^\mu(g^{-1})}_{\substack{\downarrow \\ [M_{ak}^\mu(g)]_{ka}}} &= \frac{1}{n_\mu} \delta_{\mu\nu} \delta_{ia} \delta_{jk} \\ \Rightarrow \langle M_{ak}^\mu, M_{ij}^\nu \rangle &= \frac{1}{n_\mu} \delta_{\mu\nu} \delta_{ia} \delta_{jk} \\ \Rightarrow \langle M_{i_1,j_1}^{\mu_1}, M_{i_2,j_2}^{\mu_2} \rangle &= \frac{1}{n_\mu} \delta_{\mu_1\mu_2} \delta_{i_1,i_2} \delta_{j_1,j_2} \end{aligned}$$

We have shown that $\{M_{ij}^\mu\}$ is a set of orthogonal functions on $L^2(G)$,

basis \Leftrightarrow completeness ?

Let \underline{W} be the subspace spanned by $\{M_{ij}^\mu\}$.

\Rightarrow The orthogonal complement \underline{W}^\perp is also a unitary rep. of $G \times G$.

\Rightarrow decomposable into unitary irreps V^μ

$\{f_j\}_{j=1}^{n_\mu}$ transforms as V^μ under right regular rep.

$$R(g) f_j = \sum M(g)_{kj}^\mu f_k$$

$$f(hg) = \sum M(g)_{kj}^\mu f_k(h)$$

$$\stackrel{h=1}{\Rightarrow} f(g) = \sum_k f_k(1) \underbrace{M_{kg}^\mu(g)}_{f \in W^\perp} \quad (\forall g \in G)$$

$f \in W$ contradiction with the assumption
 $f \in W^\perp$

$$\Rightarrow W^\perp = 0$$

[if with left reg. rep.

$$L(f) f_j = \sum \mu^k (f)_{kj} f_k$$

$$\underline{f(g^{-1}h)} = \sum \mu^k (f)_{kj} f_k(h)$$

$$\begin{aligned} h \vdash \Rightarrow \underline{f(g)} &= \sum \mu^k (g^{-1})_{kj} f_k(1) \\ &= \sum \overline{\mu^k(g)}_{jk} f_k(1) \end{aligned}$$

$\{ \overline{\mu^k}_{ij} \}$ is another set

of orthogonal basis ↴

$\Rightarrow \{ \mu^k_{ij} \}$ is complete.

so now we know $\bigoplus_{\mu} \underline{\text{End } V^{\mu}} \cong \underline{L^*(G)}$

Corollary for finite groups.

$L^*(G)$ of $\dim |G|$:

$$\underline{\delta_a}(f) = \begin{cases} 1 & f=a \\ 0 & \text{else} \end{cases}$$

$$(g \delta_a = \delta_{ga})$$

$$f: G \rightarrow \mathbb{C} \quad f = \sum_{g \in G} f(g) \delta_g$$

$$\underline{\text{End } (V^{\mu})} \cong \text{Mat}_{n_{\mu} \times n_{\mu}}(\mathbb{C})$$

$$\dim_{\mathbb{C}} (\text{End } (V^{\mu})) = n_{\mu}^2$$

$$\Rightarrow |G| = \sum_{\mu} n_{\mu}^2$$

Examples 1. S_3 , $|S_3| = 6$

$$6 = 1 \times 6 \quad \text{x. abelian}$$

$$= 1 + 1 + 2^2 \quad \checkmark$$

$$L(S_3) \cong \Gamma_{\text{trivial}} \oplus \Gamma_{\text{sgn}} \oplus 2\Gamma_2$$

$$(+) \quad (-) \quad (2)$$

$$\textcircled{1} \quad M^+(\phi) = 1 \quad \forall \phi \in S_3$$

$$\textcircled{2} \quad M^-(\phi) = 1 \quad \phi \in \{(1), (123), (132)\} = A_3$$

$$M^-(\phi) = -1 \quad \phi \in \{(12), (13), (23)\}$$

\textcircled{3}

$$M^{(2)}(12) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$M^{(2)}(13) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$M^{(2)}(23) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\langle M_{ij}^\mu, M_{i'j'}^\mu \rangle = \frac{1}{n_\mu} \delta_{\mu\mu} \delta_{ii'} \delta_{jj'},$$

$$\text{a. } \langle M^+, M^- \rangle = 0$$

$$\text{b. } \langle M^+, M^{(2)}_{11} \rangle = \frac{1}{6} \sum M^{(2)}_{11}(\phi) = \frac{2}{6} (1 - \frac{1}{2} - \frac{1}{2}) = 0$$

$$\text{c. } \langle M^{(2)}_{11}, M^{(2)}_{11} \rangle = \frac{2}{6} (1 + \frac{1}{4} + \frac{1}{4}) = \frac{1}{2} = \frac{1}{n_\mu}$$

$$2. G = \mathbb{Z}_2 = \langle \sigma | \sigma^2 = 1 \rangle$$

$\varphi \in L^2(G) = \{ \text{Map } (G, \mathbb{C}) \}$

$$\varphi(1) = \varphi_+ \in \mathbb{C}$$

$$\varphi(\sigma) = \varphi_-$$

$$L^2(G) \cong \mathbb{C}^2$$

$$\mathbb{Z}_2 \text{ irreps } \rho_{\pm}(\sigma) = \pm 1 \quad V_{\pm} \cong \mathbb{C}$$

$$\left\{ \begin{array}{l} M^+(1) = M^-(1) = 1 \\ M^+(\sigma) = 1 \quad M^-(\sigma) = -1 \end{array} \right.$$

$$\Rightarrow \varphi = \frac{\varphi_+ + \varphi_-}{2} M^+ + \frac{\varphi_+ - \varphi_-}{2} M^-$$

{ M^+, M^- } on basis

of $L^2(\mathbb{Z}_2)$

$$\text{Previously : } T(\sigma)$$

$$\left(\begin{array}{l} P_{\pm} = \frac{1}{2} (M^{\pm} \mathbb{I} + M^{\pm}(\sigma) T(\sigma)) \\ = \frac{1}{2} (1 \pm T(\sigma)) \end{array} \right) \quad P_{\pm} = \frac{1}{2} (1 \pm T(\sigma)) \quad \text{is of the} \\ \text{is form : } P_{\pm} = \int_{\mathbb{R}} \overline{M^{\pm}(g)} T(g) dg \quad (\text{later})$$

$$3. G = U(1) \quad (\hat{G} = \mathbb{Z})$$

$$(P_n, V_n) : \quad P_n(z) = z^n \quad n \in \mathbb{Z} \quad (= \underbrace{e^{izn}}_{\theta \in [0, 2\pi]})$$

$$V_n \cong \mathbb{C}$$

$$\langle P_{n_1}, P_{n_2} \rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} (\overline{P_{n_1}(\theta)})^* P_{n_2}(\theta) = \delta_{n_1, n_2} \\ e^{i\theta(n_1 - n_2)}$$

$$\{P_n = e^{i\theta n}\} \text{ on basis : } \Psi = \sum_n \alpha_n P_n$$

$$\alpha_n = \int_{U(1)} P_n^* \Psi(g) dg$$

$$4. S_4 ? \quad |S_4| = 24 = 1 + 1 + \underbrace{3^2}_{\text{trivial}} + 13 ?$$

sgn
standard
11

S_4	e (1111)	(12) $\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}$	(12)(34) $\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix}$	(123) $\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix}$	(1234) $\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{smallmatrix}$
P_{triv}	1	1	1	1	1
P_{sgn}	1	-1	1	1	-1
$P_{\text{std.}}$	3	1	-1	0	-1
$P_{\text{sgn}} \otimes P_{\text{std.}}$	3	-1	-1	0	1
P_2	2				

$$\text{is } 13 = 2^2 + 3^2, \text{ or } 1 \times 9 \text{ or else?}$$

(Problem 15 from HW)

$$[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}, \quad [G, G] \trianglelefteq G.$$

$$\rho: G \rightarrow \mathbb{C}^\times$$

$$\rho([g_1, g_2]) = \rho(g_1 g_2 g_1^{-1} g_2^{-1}) = \rho(e) = 1. \quad \text{trivial}$$

$$g[G, G] \in G/[G, G] \Rightarrow \rho(g[G, G]) = \rho(g)$$

distinct 1D rep of G = distinct rep of $G/[G, G]$

$G/[G, G]$ abelian \Rightarrow all irreps 1D

characters = # conj. classes

$$= |G/[G, G]|$$

$$[G, G] = A_n \Rightarrow |G/[G, G]| = 2.$$

\Rightarrow two distinct 1D irreps

8.11 Explicit decomposition of a representation

Let (T, V) be any rep. of a compact group G . Define

$$\underline{P}_{ij}^{(\mu)} := n_\mu \int_G \overline{\lambda_{ij}^{(\mu)}(g)} T(g) dg$$

$\lambda_{ij}^{(\mu)}$ w.r.t unitary irreps with ON basis of V^μ .

$$\boxed{\underline{P}_{ij}^{(\mu)} \underline{P}_{kl}^{(\nu)} = \delta^{\mu\nu} \delta_{jk} \underline{P}_{il}^{(\nu)}}$$

$$\begin{aligned} T(h) \underline{P}_{ij}^{\mu} &= n_\mu T(h) \int_G dg \overline{\lambda_{ij}^{(\mu)}(g)} T(g) \\ &= n_\mu \int_G dg \underbrace{\overline{\lambda_{ij}^{(\mu)}(g)}}_k T(hg) \\ &\stackrel{hg \rightarrow g}{=} n_\mu \int_G dg \frac{\overline{\lambda_{ij}^{(\mu)}(h^{-1}g)} T(g)}{\lambda_{ki}^{(\mu)}(h) \lambda_{kj}^{(\mu)}(g)} \\ &= \sum_k^n \lambda_{ki}^{(\mu)}(h) \underline{P}_{kj}^{(\mu)} \end{aligned}$$

$$T(h) \underline{P}_i^{\mu j} = \sum_k \lambda_{ki}^{(\mu)}(h) \underline{P}_k^{\mu j}$$

$\forall \varphi \in V$. $(\underline{P}_{ij}^{\mu} \varphi \neq 0)$. then

$$\underline{\text{span}} \{ \underline{P}_{ij}^{\mu} \varphi, i=1, \dots, n_\mu \} \quad (\text{fix } \mu, j)$$

transforms as (T^μ, V^μ)

8.12. Orthogonality relations of characters ;

Character table.

8.12.1 Orthogonality relations —

Recall - a class function on G :

$$f: G \rightarrow \mathbb{C}.$$

$f(g) = f(hgh^{-1}) \quad \forall g, h \in G$. They span
a subspace $L^2(G)^{\text{class}} \subset L^2(G)$.

Theorem The characters $\{x_\mu\}$ is an
orthonormal (ON) basis for the
vector space of class functions $L^2(G)^{\text{class}}$.

$$\text{Proof. } \int_G df M_{ij}^{(\mu)}(f) M_{kl}^{(\nu)}(f) = \frac{1}{n_\mu} \delta_{\mu\nu} \delta_{ik} \delta_{jl}$$

Set $i=j$, $k=l$ & sum over j, k

$$\Rightarrow \int_G df M_{ii}^{(\mu)}(f)^* M_{kk}^{(\nu)}(f) = \frac{1}{n_\mu} \delta_{\mu\nu} \underline{\delta_{ik}}$$

$$\stackrel{\sum_{i,k}}{\Rightarrow} \int_G df x^\mu(f)^* x^\nu(f) = \delta_{\mu\nu}$$

$\Rightarrow \{x_\mu\}$ ON set

Completeness ?

$$\forall f \in L^2(G) \xrightarrow[\text{if } M_{ij}^\mu \text{ complete}]{\text{Peter-Weyl}} f(g) = \sum_{\mu, i, j} \hat{f}_{ij}^\mu M_{ij}^\mu(f)$$

$$\text{of } f \in L^2(G)^{\text{class.}} \quad f(g) = f(hgh^{-1})$$

$$\int_G dh f(g) = \int_G dh f(hgh^{-1}) \\ \xrightarrow{=} {}^h f(g)$$

$$\int_G f(hgh^{-1}) dh = \sum_{\mu, i, j} \hat{f}_{ij}^\mu \int_G M_{ij}^\mu(hgh^{-1}) dh \\ \downarrow \\ M_{ik}(h) M_{kl}^\mu(g) M_{lj}^\mu(h^{-1}) \\ = \sum_{\mu, i, j} \hat{f}_{ij}^\mu M_{kl}^\mu(g) \int_G M_{ik}(h) M_{jl}^\mu(h^{-1}) dh \\ \frac{1}{n_\mu} \delta_{ij} \delta_{kl}$$

$$= \sum_{\mu, i} \frac{\hat{f}_{ii}^\mu}{n_\mu} X_\mu(g)$$

$$\Rightarrow f(g) = \sum_{\mu, i} \frac{\hat{f}_{ii}^\mu}{n_\mu} X_\mu(g)$$

$\Rightarrow \{X_\mu\}$ spans full $L^2(G)^{\text{class.}}$