Review of basic ideas of rep. theory:

Regular representation: $G \times G$

$$(g_1, g_2) \longmapsto L(g_1) R(g_2^{-1})$$

$$(g_1, g_2) x = g_1 x g_2^{-1} \quad \left(\begin{array}{l} g_i \in G \\ x \in R_G \end{array} \right)$$

now consider L & R : $G \to GL(R_G)$

$\hookrightarrow$ restrict to subgroups $G \times \{1\}$ or $\{1\} \times G$.

LRR: $L(g) \cdot x = g x$

RRR: $R(g) x = x g^{-1}$

$$\underline{L(h) \cdot x} = L(h) \cdot \underline{\sum_g x(g) \cdot g} = \sum_g x(g)(hg) = \sum_g x(h^{-1}g) \cdot g$$

$$\equiv \sum_g [L(h) \cdot x](g) \cdot g$$

(View x also as functions on G. $x: G \to \mathbb{C}$
$g \mapsto x(g)$)

$\Rightarrow$ $[L(h) \cdot x](g) = x(h^{-1} g)$

$\left([R(h) \cdot x](g) = x(g \cdot h) \right)$

Define inner product

$$\langle x, y \rangle = \int_G \overline{x(g)}\, y(g)\, dg$$

$$\stackrel{\text{finite}}{=} \frac{1}{|G|} \sum_g \overline{x(g)}\, y(g)$$

$\Rightarrow \langle L(h)x, L(h)y\rangle = \langle x, y\rangle$ unitary reps

We will use $L(h)$, h, δ_h etc. interchangeably

$$\underline{h} = \sum_g h(g)\cdot g = 1\cdot h \quad \Rightarrow \quad h(g) = \begin{cases} 1 & g=h \\ 0 & \text{otherwise} \end{cases}$$

$\qquad\qquad \shortparallel$

(recover $\underline{\delta_h}$ from before)

$$\underline{\delta_h\cdot\delta_g} = \sum_k \left(\sum_l \delta_h(l)\,\delta_g(l^{-1}\cdot k)\right)\cdot k = 1\cdot(hg) = \underline{\delta_{hg}}$$

$l = h.$

$l^{-1}k = g \quad k = hg$

see h as left action: $\underline{L(h)\delta_g(g')} = \delta_g(h^{-1}\cdot g') = \delta_{hg}(g')$

$$\underline{L(h)\delta_g = \delta_{hg}}$$

group elements can be viewed both as operators and vectors on R_G

Also. expand the class function on R_G:

$$\delta_{C_i}(g) = \begin{cases} 1 & g\in C_i \\ 0 & \text{otherwise} \end{cases}$$

$$\underline{\delta_{C_i}} = \sum_{g\in G} \underline{\delta_{C_i}(g)}\cdot g = \underline{\underline{\sum_{g\in C_i} g}}$$

(or view as class operators C_i)

$\forall h\in G$: $h C_i h^{-1} = \sum_{g\in C_i} h g h^{-1} = \sum_{g'\in C_i} g' = C_i$. $\underline{\underline{C_i \text{ commutes with } \forall h\in G}}$

8.13.2. Projectors onto invariant subspaces

$V = \bigoplus_i W^i$ — invariant subspace.

Suppose. $V = W \oplus W^{\perp}$

Define projector P onto W.

$\forall x \in V$. $x = w + w^{\perp}$ $\quad w \in \underline{W}$, $w^{\perp} \in W^{\perp}$

then $\underline{Px = w}$ $\qquad gw \in W$

$P(gw^{\perp}) = 0$

$\forall g \in G$: $g(Px) = gw = P(\underline{gw}) = Pg(w + \underline{w^{\perp}}) = Pgx$

$\Rightarrow \underline{gP} = Pg$ $\qquad$ P also commutes with $\forall g \in G$.

$\forall x \in R_G$: $Px = \sum_g x(g) \cdot Pg = \underbrace{\sum x(g)}_{x} gPe = xPe$

Define $e' = Pe$: $e'^2 = PePe = P^2e = Pe = e'$ $\quad$ idempotent 幂等元

then the invariant subspace is defined as

$$W = \{ xe' : x \in R_G \} =: R_G \cdot e'$$

$\{ Px : \forall x \in R_G \}$

If $P_1 + P_2 = \mathbb{1} \Rightarrow e = \mathbb{1}e = (P_1 + P_2)e = e_1 + e_2$

$P_1 P_2 = 0 \Rightarrow e_1 e_2 = 0$

irreps: e' is primitive, can not be decomposed into $e_1' + e_2'$ $(e_1' \neq 0, e_2' \neq 0)$

Both C_i and P commutes with $\forall g \in G$. is it possible to find P's onto irreps using C_i?

8.13.3 Construction of character table

We've seen a few character tables for simple groups.

But how do we construct the character tables?

We present an algorithm to obtain them.

If we can find all the projectors onto irreps, or equivalently all the idempotents.

Some ideas:

Recall previously, a Hamiltonian H is an intertwiner. $[H, T(G)] = 0$

The eigenvectors $\{\psi_\mu\}$ span an invariant subspace W of the representation space $L^2(G)$:

$$H\psi_\mu = E_\mu \psi_\mu$$

$$H \underline{T(g)\psi_\mu} = T(g) H \psi_\mu = E_\mu \underline{T(g)\psi_\mu} \qquad \forall g \in G$$

$T(g)\psi_\mu \in W. (\forall g \in G) \Rightarrow W$ is an invariant subspace, i.e. a representation space

$$V \cong \oplus W^\mu$$

If W^μ is still reducible, find another operator that satisfies $[O, T(g)] = 0 \quad (\forall g \in G)$

With a <u>complete set of commuting operators</u> (CSCO), we can achieve a complete reduction of representations / find all irreps!

This is an idea explored systematically by 陈金全 (南大).

① 陈金全. 《群表示论的新途径》

② English translation: Group representation theory for physicists. 2nd. Ed. World Scientific, 2002

③ The representation group and its application to space groups

RMP 57, 211 (1985)

First RMP of PRC.

To illustrate the idea, consider a finite group G.

with r conjugacy classes $[C_i]$ $(i=1,\cdots r)$

$|[C_i]| = m_i$. Correspondingly, r irreps V^μ and characters χ_μ

What operator commutes with all elements of R_G

The center of the group algebra $Z[R_G]$ is spanned by the class operators / functions

$\forall x \in Z(R_G)$: $\quad x g x^{-1} = g$ $\qquad c_i = \sum'_{g \in C_i} g$

They have the following properties:

① $\forall h \in G$. $[c_i, h] = 0$: $h c_i h^{-1} = \sum_{g \in C_i} h g h^{-1} = c_i$

② $\forall i, j$ $[c_i, c_j] = 0$: because of ①

③ closed/complete: $c_i c_j = \sum_{k=1}^{r} c_{ij}^k c_k$, $(c_{ij}^k = c_{ji}^k \in N)$ where c_{ij}^k the class multiplication coefficient, something we can easily compute given a group.

Proof: $\forall h_{i1}, h_{i2} \in C_i$, $\exists g' \in G$, s.t. $h_{i1} = g' h_{i2} g'^{-1}$

$$\sum_{g \in G} g h_{i1} g^{-1} = \sum_g g(g' h_{i2} g'^{-1}) g^{-1} = \sum_g g h_{i2} g^{-1}$$

$m_i = |C_i|$

$g_{ia} \in C_i$

$$\Rightarrow \sum_{g \in G} g c_i g^{-1} = m_i \sum_{g \in G} g g_{ia} g^{-1},$$

$\because$ ①, LHS $= |G| \cdot c_i$

S-O theorem / class eq. $|C(g)| = \dfrac{|G|}{|Z_G(g)|}$

$$\Rightarrow \sum_{g \in G} g g_{ia} g^{-1} = \frac{|G|}{m_i} c_i \quad \Delta$$

One element on LHS, then full class on RHS

$$① \Rightarrow c_i c_j = \frac{1}{|G|} \sum_{g \in G} g(c_i c_j) g^{-1}$$

Any $g \in c_i c_j$, belongs to some C_k, then RHS contains full C_k

$$\Rightarrow \boxed{c_i c_j = \sum_{k=1}^{r} c_{ij}^k c_k} \qquad (*)$$

(Should they be enough for finding all irreps of a group? Some arguments: we've mentioned before that $\{\delta_{C_i}\}$ is a complete basis for $L^2(G)^{class}$, so is $\{\chi_\mu\}$.)

If we can diagonalize some/all C_i's, and decompose them into projectors / find idempotents.

From an algebraic point of view, Eq. (*) provided us with a set of eigen problems.

$$\hat{C}_i \, \delta_{C_j} = \sum_{k=1}^{r} [C^i]_{jk} \, \delta_{C_k}$$

With $\{\delta_{C_i}\}$ an orthogonal basis: of class algebra

(recall inner product $\langle \delta_{C_j}, \delta_{C_k} \rangle = \frac{1}{|G|} \sum_g \delta_{C_j}(g) \delta_{C_k}(g) = \frac{m_j}{|G|} \delta_{jk}$)

Suppose for $\hat{C}_i$ we find its eigenvectors $\{\phi^\mu\}$

$$\hat{C}_i \, \phi^\mu = \lambda_i^\mu \phi^\mu$$

$\lambda^\mu = \lambda^\nu$, or $\phi^\mu \phi^\nu = 0$ ✓

then $\hat{C}_i (\phi^\mu \phi^\nu) = \lambda_i^\mu (\phi^\mu \phi^\nu) \overset{[\hat{C}_i, g]=0}{=} \lambda^\nu (\phi^\mu \phi^\nu)$, i.e. $\phi^\mu \phi^\nu$ is also an eigen vector associated to λ_i^μ. Assuming λ_i^μ is nondegenerate.

then $\phi^\mu \phi^\nu = \alpha_\mu \delta_{\mu\nu} \phi^\mu$, α_μ some constant $\in \mathbb{C}$. (if $\hat{C}_i$ is a CSCO)

Define $P^\mu = \alpha_\mu^{-1} \phi^\mu$, $P^\mu P^\nu = \delta_{\mu\nu} P^\mu$. $\{P^\mu e\}$ are the primitive idempotents of R_G. ↳ projectors onto 1D space

and $C_i = \sum_{\mu=1}^{r} \lambda_i^\mu P^\mu$ is actually a linear combination of projectors onto irreps.

What if there is degeneracy? Find another C_i that splits the degeneracy.

With a complete set of commuting operators (CSCO) one can uniquely determine the P^μ's.

- Note that when restricted to a specific irrep.

$$\underline{\underline{C_i^\mu = \lambda_i^\mu \cdot \mathbb{1}_{V^\mu}}}$$

We can also obtain λ_i^μ by noticing:

$$\begin{cases} \chi_\mu(C_i) = \sum_{g\in C_i} \chi_\mu(g) = m_i \chi([C_i]) \\ C_i \propto \mathbb{1}_{V^\mu} \end{cases}$$

$$\Rightarrow C_i^{(\mu)} = \frac{m_i}{n_\mu} \chi_\mu([C_i]) \cdot \mathbb{1}_{V^\mu} \qquad (n_\mu = \dim V^\mu)$$

i.e. $\lambda_i^\mu = \frac{m_i}{n_\mu} \chi_\mu([C_i])$ remaining two unknowns: $\underline{n_\mu, \chi_\mu}$

$$\frac{1}{|G|} \sum_{C_i} m_i \chi_\mu(C_i) \overline{\chi_\nu(C_i)} = \delta_{\mu\nu} \Rightarrow \frac{1}{|G|} \sum_{C_i} m_i \lambda_i^\mu \overline{\lambda_i^\nu} = \delta_{\mu\nu} \left(\frac{m_i}{n_\mu}\right)^2$$

$$= \langle \lambda_i^\nu, \lambda_i^\mu \rangle$$

$$n_\mu = \frac{m_i}{\sqrt{\langle \lambda_i^\mu, \lambda_i^\mu \rangle}}$$

$$\chi_\mu = \frac{\lambda_i^\mu}{\sqrt{\langle \lambda_i^\mu, \lambda_i^\mu \rangle}}$$

- How to find a minimal CSCO (for different groups) → 陈金全

We will use a possibly "overcomplete" set:

There are in total r linearly independent $\hat{C}_i$'s. We will try to diagonalize all of them.

$C_i^{(\mu)} = \frac{m_i}{n_\mu} \chi_\mu([C_i]) \cdot \mathbb{1}_{V^\mu}$

$$\hookrightarrow \quad \frac{m_i}{n_\mu} \chi_\mu([C_i]) \frac{m_j}{n_\mu} \chi_\mu([C_j]) = \sum_{k=1}^{r} C_{ij}^{k} \frac{m_k}{n_\mu} \chi_\mu([C_k])$$

$$m_i \chi_\mu([C_i]) m_j \chi_\mu([C_j]) = n_\mu \sum_{k=1}^{r} C_{ij}^{k} m_k \chi_\mu([C_k])$$

Now introduce a set of auxiliary variables $\{y^i, i=1,\cdots,r\}$

(so we can differentiate between different C_i's: $C_i \to C_i y^i$)

ΣLHS : $\sum_{i=1}^{r} m_i m_j \chi_\mu([C_i]) \chi_\mu([C_j]) y^i = \sum_{i=1}^{r} (\psi_i y^i) \psi_j \quad (\psi_i := m_i \chi_\mu([C_i]))$

ΣRHS : $\sum_{i=1}^{r} n_\mu \sum_{k=1}^{r} C_{ij}^{k} m_k \chi_\mu([C_k]) y^i = n_\mu \sum_{k=1}^{r} L_j^k \psi_k$

Define $\lambda = \frac{1}{n_\mu} \sum_{i=1}^{r} \psi_i y^i \qquad \left(L_j^k = \sum_i C_{ij}^k y^i\right)$

$$\Rightarrow \quad \sum_{k=1}^{r} L_j^k \psi_k = \lambda \psi_j$$

Solving the eigen problem $\quad (L - \lambda \mathbb{1}) \psi = 0$

and obtain a set of eigenvalues $\{\lambda_\mu\}$

$$(*) \quad \lambda_\mu = \frac{1}{n_\mu} \sum_{i=1}^{r} \overbrace{m_i \chi_\mu([C_i])}^{\psi_i} y^i \qquad \mu = 1, \cdots, r$$

Note if we set $y^j = \delta_{ij}$, we recover our earlier λ_i^μ.

Now recall the orthogonality relation:

$$\frac{1}{|G|}\sum_{C_i} m_i\, \chi_\mu(C_i)\,\overline{\chi_\nu(C_i)} = \delta_{\mu\nu} \qquad \text{(ortho. of rows)}$$

$$\overset{\mu=\nu}{\Rightarrow} \quad \sum_{i=1}^{r} m_i \left|\chi_\mu([C_i])\right|^2 = |G|$$

$$|G| = |\chi_\mu([C_1])|^2 \sum_{i=1}^{r} m_i \left|\frac{\chi_\mu([C_i])}{\chi_\mu([C_1])}\right|^2$$

$$= n_\mu^2 \sum_{i=1}^{r} m_i \left|\frac{\chi_\mu([C_i])}{n_\mu}\right|^2$$

$$\Rightarrow \quad n_\mu = \left[\frac{|G|}{\sum_{i=1}^{r} m_i \underbrace{\left|\frac{\chi_\mu([C_i])}{n_\mu}\right|^2}_{\text{known from above } (*)}}\right]^{\frac{1}{2}}$$

Implementation in practice:

S_3 : E ; $(12), (13), (23)$; $(123), (132)$

① class operators: $C_1 = E$

$C_2 = (12) + (13) + (23)$ $\qquad$ $(12)(13) = (132)$

$C_3 = (123) + (132)$ $\qquad$ $(12)(123) = (1)(23)$

② class multiplication table:

	C_1	C_2	C_3
C_1	C_1	C_2	C_3
C_2	C_2	$\underline{3C_1+3C_3}$	$\underline{2C_2}$
C_3	C_3	$2C_2$	$\underline{2C_1+C_3}$

① explain underlined.
② symmetric ($\because$ abelian)

③ $L_j^k = \sum_i C_{ij}^k y^i$ 3×3 matrix

$$L_1^1 = C_{11}^1 y^1 + C_{21}^1 y^2 + C_{31}^1 y^3 = y^1 + 0 + 0$$

$$L_1^2 = \sum_i C_{i1}^2 y^i = y^2$$

$$L_1^3 = y^3$$

$$L_2^1 = \Sigma C_{i2}^1 y^i = 3y^2$$

$$L_2^2 = \Sigma C_{i2}^2 y^i \qquad L_2^3 = \Sigma C_{i2}^3 y^i$$

$$L_3^1 = \Sigma C_{i3}^1 y^i \qquad L_3^2 = \Sigma C_{i3}^2 y^i$$

$$L_3^3 = \Sigma C_{i3}^3 y^i$$

	C_1	C_2	C_3
C_1	C_1	C_2	C_3
C_2	C_2	$3C_1+3C_3$	$2C_2$
C_3	C_3	$2C_2$	$2C_1+C_3$

$$\hat{L} = \begin{pmatrix} y^1 & y^2 & y^3 \\ 3y^2 & y^1+2y^3 & 3y^2 \\ 2y^3 & 2y^2 & y^1+y^3 \end{pmatrix} \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} y^1 + \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix} y^2 + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix} y^3$$

$$\left\{ \begin{aligned} \lambda_a &= y^1 + 3y^2 + 2y^3 \\ \lambda_b &= y^1 - 3y^2 + 2y^3 \\ \lambda_c &= y^1 + 0y^2 - y^3 \end{aligned} \right.$$

$$\boxed{\lambda_\mu = \sum_{i=1}^{r} \frac{m_i \chi_\mu([C_i])}{n_\mu} y^i}$$

$$n_\mu = \left[\frac{|G|}{\sum_{i=1}^{r} m_i \left| \frac{\chi_\mu([C_i])}{n_\mu} \right|^2} \right]^{\frac{1}{2}}$$

write in cols.

④ $\chi_a = n_a(1, \ 1, \ 1)$ $\qquad n_a = 1$

$\chi_b = n_b(1, \ -1, \ 1)$ $\qquad n_b = 1$

$\chi_c = n_c(1, \ 0, \ -\frac{1}{2})$ $\qquad n_c = \left[\frac{6}{1+3\cdot 0+2\cdot\frac{1}{4}}\right]^{\frac{1}{2}} = 2$

⑤ Character table

	[1]	3[(12)]	2[(123)]
1^+	1	1	1
1^-	1	-1	1
2	2	0	-1

Note that in the solution:

$$\begin{cases} \lambda_a = y^1 \boxed{+3y^2} + 2y^3 \\ \lambda_b = y^1 \boxed{-3y^2} + 2y^3 \\ \lambda_c = y^1 \boxed{+0y^2} - y^3 \end{cases}$$

The eigenvalues of $\hat{C}_2$ is non-degenerate.

This defines a set of unique eigenvectors that diagonalizes all $\hat{C}_i$. Which means $\hat{C}_2$ is a CSCO by itself.

Again, see 陈金全 for details of finding a minimal CSCO.