Recap : Cosets & conjugacy.

Def. $H \subset G$.

$$gH := \{ gh , h \in H \} \subset G.$$

left-coset of H

("G" $= H$ "X" $= G$ group action

perspective)

$G = \mathbb{Z}$. $H = 2\mathbb{Z}$.

$gH = \{ H. \quad 1 + H \}$

$G = S_3$ $H = \{ 1, (12) \} \cong S_2$

$g \in G$. $gH = \{ H, \quad \{ (13), (123) \}, \quad \{ (23), (132) \}$

(left) cosets   : $\underline{identical}$ or $\underline{disjoint}$

Theorem (Lagrange) $H \subset$ finite G.

$$|H| \mid |G| |G| = m|H|$$

$|G| =$ prime. $\longrightarrow$ G cyclic

$$\langle g \rangle = \underline{\{ 1, g, g^2 \cdot \quad \} = G}$$

index of a subgroup

$$[G : H] := |G/H| = |G|/|H|$$

$$G = A_4 \qquad \nexists H \text{ s.t. } [G:H] = 2$$

$$|H| = 6.$$

$\Rightarrow$ converse of Lagrange Theorem

is in general <u>not true</u>.

$\Rightarrow$ Special case: (Sylow)

$$p \text{ prime} \qquad p^k \mid |G|$$

$$\exists H, \quad |H| = p^k$$

Conjugacy

$$h' = g h g^{-1} \qquad \exists g \in G$$

$$h' \sim h \qquad \text{equivalence relation}$$

$$\hookrightarrow \qquad \text{class}$$

$$C(h) := \{ g h g^{-1} : g \in G \} \ (=: h^G)$$

H subgroup $\longrightarrow$ gHg^{-1} subgroup

S_n.. same cycle decomposition $\iff$ conjugate

$\hookrightarrow$ "equivalent representation"

$$T' = STS^{-1}$$

$GL(n,k)$:

diagonalizable: $\underline{U(1)^N/S_n}$

un~

$$P_A(x) = \det(x \mathbb{1} - A)$$

class function

$$f(g\,g_0\,g^{-1}) = f(g_0) , \quad \forall g, g_0 \in G.$$

$$\chi_T(g) := \underline{\mathrm{Tr}\, T(g)}$$

— 6.3. Conjugacy classes in S_n

Recall the underlined cycle decomposition of $g \in S_n$

$(i_1 , i_2 \cdots i_k)$ is a k-cycle, then

$$g(i_1 , i_2 \cdots , i_k) g^{-1} \quad \text{is also a } k\text{-cycle.}$$

if $g(i_a) = j_a \quad a = 1, \cdots k.$ then

$$g(i_1 , i_2 , \cdots i_k) g^{-1} = (g(i_1) \, g(i_2) \cdots g(i_k))$$
$$= j_1 j_2 \cdots j_k$$

$\Rightarrow$ Any two k-cycles are conjugate.

$$(12)((123)(12)^{-1} = (12)(123)(12)$$
$$\Big\{ \quad = (12)(13) = (132)$$

$$(213) = (132)$$

$\hookrightarrow$ permutations are conjugate iff

they have the same cycle decomposition.

$\hookrightarrow$ label the conjugacy classes by

their cycle decomposition. denoted as

$C(\vec{l})$, $\vec{l} = (l_1 , \cdots l_n)$ $\qquad l_i$ number of i-cycles

$$\boxed{n = \sum_{j=1}^{n} j \, l_j}$$

$$\phi = (12)(34)(678)(11,12) \in S_{12}$$

$$= (12)(34)(5)(678)(9)(10)(11,12)$$

$$\ell_1 = 3 \qquad \ell_2 = 3 \qquad \ell_3 = 1 \quad, \quad \ell_{\geq 4} = 0$$

$$\vec{\ell} = (3\ 3\ 0\ 0\ 0\ -\ .\ )$$

Def <u>partition of n</u> : decomposition of n

into a sum of nonnegative integers.

The number of <u>distinct</u> partitions

of n is called the "<u>partition function</u>"

of n, denoted $P(n)$

$\hookrightarrow$ conjugacy classes of S_n

$\quad\quad \Longleftrightarrow \quad P(n)$

Example : S_4 $\qquad\qquad (g^n = 1)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \searrow$

partition	cycle decomp.	g.	$|C(g)|$	order of g
$4 = 1+1+1+1$	$(1)^4$	1	1	1
$4 = 1+1+2$	$(1)^2(2)$	(ab)	$C_4^2 = 6$	2
$4 = 1+3$	$(1)(3)$	(abc)	$C_4^3 \times 2 = 8$	3
$4 = 2+2$	$(2)^2$	$(ab)(cd)$	$C_6^2 / 2 = 3$	2
$4 = 4$	(4)	$(abcd)$	6	4

$$|S_4| = 4! = 24 \quad = \quad 1 + 6 + 8 + 3 + 6 = 24 \quad \checkmark$$

Young diagram

$1+1+1+1$:

$(1)^4$

λ_i – length of i-th row

$\lambda_i \geq \lambda_{i+1}$

$2+1+1$

$(1)^2(2)$

$(2)^2$
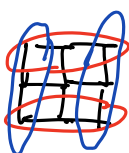
| 1 | 2 | 3 | 4 |

Young tableau

| 2 | 1 | 4 | 3 |

$3+1$

$(1)(3)$

4

(4)

graphic representation.

$\rightarrow$ define permutation within column

$\rightarrow$ Young tableau

important for
e.g. constructing
irreducible representations

Example :   a collection of harmonic oscillators

$$h_j = \hbar \omega_j (a_j^\dagger a_j + \tfrac{1}{2}) \quad (\omega_j = j\omega_0)$$

$$H - E_0 = \sum_{j=1}^{\infty} \hbar \omega_0 j (a_j^\dagger a_j + \tfrac{1}{2})$$

System with a fixed energy $E_N = N \hbar \omega_0$

$$|P_{\vec{\ell}}\rangle = \frac{1}{\sqrt{\ell_1! \ell_2! \cdots}} (a_1^+)^{\ell_1} (a_2^+)^{\ell_2} \cdots (a_n^+)^{\ell_n} |0\rangle$$

$$N = \sum j \ell_j$$

$P(n) \longrightarrow$ degeneracy of the states

— 6.4 Normal subgroup & Quotient groups

Definition A subgroup $N < G$ is called a
normal subgroup (invariant subgroup.
self-conjugate subgroup)

$$g N g^{-1} = N \qquad \forall\, g \in G$$

denoted as $N \lhd G$.

↪ center of a group. $Z(G)$

$\forall z \in Z(G) ; \quad zg = gz \iff gzg^{-1} = z$
$\forall f \in G$

$g N g^{-1} = N \;\not\Rightarrow\; g n g^{-1} = n \qquad n \in N$

$\Leftarrow$

$Z(G) \lhd G$

Examples:

1 $H \subset$ abelian group G. $\forall h \in H$

$$g h g^{-1} = (g g^{-1}) h = h$$

→ All subgroups of an Abelian group
are normal.

2. Homomorphisms

$$\phi: G \longrightarrow G'$$

$\ker(\phi)$ is a normal subgroup of G.

$\forall k \in \ker(\phi) \qquad \phi(k) = 1$

$$\phi(gkg^{-1}) = \phi(g)\,\phi(k)\,\phi(g^{-1}) = \phi(g)\,\phi(g^{-1})$$

$$= \phi(g)\phi(g)^{-1} = 1$$

$$\Rightarrow gkg^{-1} \in \ker(\phi)$$

$$\Rightarrow g\,\ker(\phi)\,g^{-1} = \ker(\phi) \qquad \forall g \in G$$

$$\Rightarrow \ker(\phi) \vartriangleleft G.$$

Theorem   If $N \vartriangleleft G$, then the set of left cosets $G/N = \{gN, g \in G\}$ has natural group structure. with group multiplication defined as :

$$(g_1 N)\cdot(g_2 N) := (g_1 g_2)N$$

G/N. "quotient group" / "factor group"

$$(g_1 N) \cdot (g_2 N) = g_1 (g_2 g_2^{-1}) N g_2 N$$

$$= (g_1 g_2)(\underline{\underline{g_2^{-1} N g_2}}) N \overset{=N}{}$$

$$= (g_1 g_2) N$$

for a general $H \subset G$, it doesn't hold:

$$S_3 = \{ e, \ (12), (13), (23),$$
$$(123), (132) \}$$

$H = \{e, (12)\}$ not a normal subgroup

$$g_1 = (23) \qquad g_2 = (23)^{-1} = (321) = (132)$$

$$h_1 = (12) = h_2$$

$$(g_1 n_1)(g_2 n_2) = (13)(23) = (213) = \underline{\underline{(132)}} \notin H$$

$$\neq \qquad (g_1 g_2) H = H$$

? order of $G/N = [G : N]$

Corollary: If $N \triangleleft G$, then the natural map

$$\overline{\phi} : G \longrightarrow G/N$$

$$g \longmapsto gN$$

is a surjective homomorphism.

$$\ker \overline{\phi} = N$$

① $\phi(g_1) \phi(g_2) = \phi(g_1 g_2)$

$\Longleftrightarrow (g_1 N)(g_2 N) = (g_1 g_2)N$

② $\phi(g) = gN = N. \implies g \in N.$

$\ker \phi = N$

$\implies$ Every normal subgroup is the kernel of some homomorphism.

Excemple:

1. $n\mathbb{Z} := \langle n \rangle \triangleleft \mathbb{Z}$

$= \{ \cdots -2n, -n, 0, n, 2n \cdots \}$

$\mathbb{Z}/n\mathbb{Z} = \{ i + n\mathbb{Z}, 0 \leq i \leq n-1 \} = \mathbb{Z}_n$

$$\phi: \quad \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$$
$$i \longmapsto i + n\mathbb{Z}.$$

$$\ker(\phi) = n\mathbb{Z}.$$

quotient group is not a subgroup

$\mathbb{Z}$: no finite subgroup except for $\{0\}$

$\mathbb{Z}_n$: finite group $|\mathbb{Z}_n| = n$.

$\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is not a subgroup of $\mathbb{Z}$.

$\left(\begin{array}{l} \text{happens sometimes:} \\ \qquad G = \mathbb{Z}_6 \quad N = \mathbb{Z}_2 \\ \qquad \mathbb{Z}_6/\mathbb{Z}_2 \cong \mathbb{Z}_2 \end{array}\right)$

2. $S_3 = \{ 1, (12), (13), (23) \qquad |S_3| = 6$
$\qquad\qquad (123), (132) \}$

$A_3 = \{ 1, (123), (132) \} \subseteq S_3 \qquad |A_3| = 3$

$(12)(123)(12)^{-1} = (132) \in A_3$

$(12)(132)(12)^{-1} = (123) \in A_3$

$' \qquad \vdots$

$\Rightarrow A_3 \triangleleft S_3$

$H \subset G. \quad [G:H] = 2 \Rightarrow H \triangleleft G \quad (HW)$

$[S_3 : A_3] = 2 \Rightarrow A_3 \triangleleft S_3$

3. $D_4 = \langle a, b \mid a^4 = b^2 = (ab)^2 = 1 \rangle$ $a: \frac{\pi}{2}$ rotation

    $b:$ reflection

$D_4 = \langle e, a, a^2, a^3, b, ab, a^2b, a^3b \rangle$ $|D_4| = 8$

① $\underset{\cdot}{aba^{-1}} = a^2b \iff ba^{-1} = ab \iff b^{-1}a^{-1} = ab$

$\underset{b^{-1}}{\overset{\lor}{}}$ $\iff (ab)^2 = 1$

$\underline{\{e, b, a^2b, a^2\}}$ $|\ | = 4$

② $a(ab)a^{-1} = a^3b$

$\underline{\{e, ab, a^3b, a^2\}}$ $|\ | = 4$

③ $\underline{\{e, a, a^2, a^3\}}$ $|\ | = 4$

④ $\underline{\{e, a^2\}} = Z(G)$ $|\ | = 2$

⑤. $N = \{e, a^2, b, a^2b\} \cong D_2 \cong V \quad (A = a^2 \ B = b)$

$= \langle A, B \mid A^2 = B^2 = (AB)^2 = 1 \rangle$

$D_4 / D_2 = \{D_2, AD_2\} \cong Z_2$

② $N = \{e, ab, a^2, a^3b\} \cong D_2$ $(A = a^2, B = a^3b)$

$\qquad = \langle A, B \mid A^2 = B^2 = (AB)^2 = 1 \rangle$

③ $N = \{e, a, a^2, a^4\} \cong \mathbb{Z}_4$

$\qquad D_6/_{\mathbb{Z}_4} = \{\mathbb{Z}_4, b\mathbb{Z}_4\} \cong \mathbb{Z}_2$

④ $N = \{e, a^2\} = Z(D_4) \cong \mathbb{Z}_2$

$\qquad D_4/_{\mathbb{Z}_2} = \{\mathbb{Z}_2, a\mathbb{Z}_2, b\mathbb{Z}_2, ab\mathbb{Z}_4\} \cong D_2$

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \quad (\neq \mathbb{Z}_4)$

$\qquad |G| = 4: \qquad G \cong \mathbb{Z}_4 \qquad \underline{a^4 = 1}$

$\qquad \qquad \qquad \qquad G \cong D_2 \cong V \qquad \underline{a^2 = b^2 = (ab)^2 = 1}$

$\qquad (a\mathbb{Z}_2)(a\mathbb{Z}_2) = a^2\mathbb{Z}_2 = -\mathbb{Z}_2 = \mathbb{Z}_2$

$\boxed{G/Z(G) \text{ is cyclic} \Longrightarrow G \text{ is Abelian}}$ HW