8.11 Explicit decomposition of a representation

Let (T, V) be any rep. of a compact group G. Define

$$P_{ij}^{(\mu)} := n_\mu \int_G \overline{M_{ij}^{(\mu)}(g)}\, T(g)\, dg$$

M_{ij}^{μ} w.r.t unitary irreps with ON basis of $V^{(\mu)}$.

$$\boxed{P_{ij}^{(\mu)} P_{kl}^{(\nu)} = \delta^{\mu\nu}\delta_{jk} P_{il}^{(\nu)}}$$

$$T(h) P_{ij}^{\mu} = n_\mu T(h) \int_G dg\, \overline{M_{ij}^{(\mu)}(g)}\, T(g)$$

$$= n_\mu \int_G dg\, \overline{M_{ij}^{(\mu)}(g)}\, T(hg)$$

$$\overset{hg \to g}{=} n_\mu \int_G dg\, \underbrace{\overline{M_{ij}^{(\mu)}(h^{-1}g)}}_{= M_{ki}^{(\mu)}(h)\, \overline{M_{kj}^{(\mu)}(g)}}\, T(g)$$

$$= \sum_k^{n_\mu} M_{ki}^{\mu}(h)\, P_{kj}^{(\mu)}$$

$$T(h) P_i^{\mu j} = \sum_k M_{ki}^{\mu}(h)\, P_k^{\mu j}$$

$\forall \varphi \in V$. $(P_{ij}^{\mu}\varphi \neq 0)$. then

$$\text{span}\{ P_{ij}^{\mu}\varphi,\ i = 1, \cdots n_\mu \} \quad (\text{fix } \mu, j)$$

transforms as (T^μ, V^μ)

8.12 Orthogonality relations of characters; Charater table.

8.12.1 Orthogonality relations —

Recall. a class function on G:

$$f : G \to \mathbb{C}.$$

$f(g) = f(hgh^{-1})$ $\forall g, h \in G$. They span a subspace $L^2(G)^{class} \subset L^2(G)$.

<u>Theorem</u> The characters $\{\chi_\mu\}$ is an orthonormal (ON) basis for the vector space of class functions $L^2(G)^{class}$.

<u>Proof</u>. $\int_G dg\, M^{(\mu)}_{ij}(g)^* M^{(\nu)}_{kl}(g) = \frac{1}{n_\mu}\delta_{\mu\nu}\delta_{ik}\underline{\delta_{jl}}$

set $i=j$. $k=l$ & sum over i,k

$$\Rightarrow \int_G dg\, M^{(\mu)}_{ii}(g)^* M^{(\nu)}_{kk}(g) = \frac{1}{\underline{n_\mu}}\delta_{\mu\nu}\underline{\delta_{ik}}$$

$$\overset{\sum_{i,k}}{\Longrightarrow} \int_G dg\, \chi^\mu(g)^* \chi^\nu(g) = \delta_{\mu\nu}$$

$\Rightarrow \{\chi_\mu\}$ ON set

Completeness ?

$$\forall f \in L^2(G) \xrightarrow[\{M^{\mu}_{ij}\}\text{ complete}]{\text{Peter-Weyl}} f(g) = \sum_{\mu,i,j} \hat{f}^{\mu}_{ij}\, M^{\mu}_{ij}(g)$$

of $f \in L^2(G)^{\text{class}}$ $\quad f(g) = f(hgh^{-1})$

$$\underbrace{\int_G dh\, f(g)}_{=\, f(g)} = \int_G dh\, f(hgh^{-1})$$

$$\int_G f(hgh^{-1})\,dh = \sum_{\mu,i,j} \hat{f}^{\mu}_{ij} \int_G \underbrace{M^{\mu}_{ij}(hgh^{-1})}_{M^{\mu}_{ik}(h)\, M^{\mu}_{kl}(g)\, M^{\mu}_{lj}(h^{-1})}\,dh$$

$$= \sum_{\substack{\mu,i,j\\ k,l}} \hat{f}^{\mu}_{ij}\, M^{\mu}_{kl}(g) \underbrace{\int_G M^{\mu}_{ik}(h)\, M^{\mu\,*}_{jl}(h)\, dh}_{\frac{1}{n_\mu}\delta_{ij}\delta_{kl}}$$

$$= \sum_{\mu,i} \frac{\hat{f}^{\mu}_{ii}}{n_\mu}\, \chi_\mu(g)$$

$$\Rightarrow f(g) = \sum_{\mu,i} \frac{\hat{f}^{\mu}_{ii}}{n_\mu}\, \chi_\mu(g)$$

$\Rightarrow$ $\{\chi_\mu\}$ spans full $L^2(G)^{\text{class}}$.

8.12. Orthogonality relations of characters ; Charater table.

8.12.1 Orthogonality relations —— (cont.)

isotypic decomposition of some rep V.

$$V \cong \oplus a_\mu V^{(\mu)}$$

$$\boxed{\begin{aligned} &\Rightarrow \chi_V = \sum_\mu a_\mu \chi_\mu \\ &a_\mu = \langle \chi_\mu , \chi_V \rangle = \int_G \overline{\chi_\mu(g)}\, \chi_V(g)\, dg \end{aligned}}$$

if $V \cong L^2(G)$ of a finite group.

$$\chi_V(e) = \dim V = |G|$$

$$\chi_V(g \neq e) = 0$$

$$a_\mu = \frac{1}{|G|} \sum_g \overline{\chi_\mu(g)}\, \chi_V(g) = \frac{1}{|G|} \Big(\overbrace{n_\mu \cdot |G|}^{g=e} + \overbrace{0}^{g\neq e} \Big) = n_\mu$$

$$|G| = \sum_\mu a_\mu \dim V^\mu = \sum_\mu n_\mu \cdot n_\mu = \sum_\mu n_\mu^2$$

Projection onto isotypic subspaces

$$P^{\mu}_{ij} := n_\mu \int_G \overline{M^{(\mu)}_{ij}(g)}\, T(g)\, dg$$

$$P^{\mu}_{ij} P^{\nu}_{kl} = \delta_{\mu\nu}\delta_{j,k} P^{\nu}_{il}$$

$$T(h) P^{\mu}_{ij} = \sum_k M^{\mu}_{ki}(h) P^{\mu}_{kj}$$

Define $P^{\mu} := \sum_{i=1}^{n_\mu} P^{\mu}_{ii}$

$$P_\mu := \sum_{i=1}^{n_\mu} P^{(\mu)}_{ii} = \underline{n_\mu \int_G dg\, \overline{\chi_\mu(g)}\, T(g)}$$

$$P_\mu P_\nu = \sum_{i=1}^{n_\mu}\sum_{j=1}^{n_\nu} P^{\mu}_{ii} P^{\nu}_{jj} = \delta_{\mu\nu} \sum_{ij} \delta_{ij} P^{\nu}_{ij} = \delta_{\mu\nu} P_\nu$$

$$(P_\mu^2 = P_\mu)$$

$$P_\mu^+ = n_\mu \int_G \chi_\mu(g)\, T^+(g)\, dg$$

unitary: $\chi_\mu(g) = \sum \lambda_i \quad |\lambda_i| = 1$

$\chi_\mu(g^{-1}) = \sum \lambda_i^{-1} = \sum \overline{\lambda_i}$

$$= n_\mu \int_G \chi_\mu^*(g^{-1})\, T(g^{-1})\, dg$$

$$= P_\mu$$

$\Rightarrow$ projectors onto isotypic subspaces

$$\forall \psi \in V. \quad T(h) \underbrace{P^{\mu}\psi} = T(h) \sum_{i=1}^{n_\mu} P^{(\mu)}_{ii} \psi = \sum_{ki} M^{\mu}_{ki}(h) \underbrace{P^{(\mu)}_{ki} \psi}$$

$$P^{\mu}\psi \in \mathcal{H}^{\mu}$$

$$\mathrm{Tr}(P^{\mu}) = \langle \psi, P^{\mu}\psi \rangle = n_\mu \int_G \underbrace{dg\, \overline{\chi_\mu(g)}\, \chi(g)}_{a_\mu} = n_\mu a_\mu$$

$$= \dim(\mathcal{H}^{\mu} \cong K^{a_\mu} \otimes V^{(\mu)})$$

8.12.2. Character table of finite groups

For finite groups,

we can define a set of class functions

$$\delta_{C_i}(g) = \begin{cases} 1 & g \in C_i \\ 0 & \end{cases}$$

where C_i is a distinct conjugacy class.

$\{\delta_{C_i}\}$ is also a basis for the class functions $L^2(G)^{\text{class}}$

From above. $\{\chi_\mu\}$ is a basis of $L^2(G)^{\text{class}}$

Theorem. The number of conjugacy classes of a finite group G = the number of irreps.

The character table is an $r \times r$ matrix

E $\quad\quad\quad\quad$ $m_r = |C_r|$

	$m_1 C_1$	$m_2 C_2$	$\cdots$	$m_r C_r$
trivial Γ' V^1	$\chi_1(C_1)$	$\chi_1(C_2)$	$\cdots$	
irreps $\rightarrow$ V^2	$\chi_2(C_1)$	$\chi_2(C_2)$	$\cdots$	
$\vdots$	$\vdots$	$\vdots$		$\vdots$
V^r	$\vdots$	$\vdots$		$\chi_r(C_r)$

$$\int_G dg\, \overline{\chi_\mu(g)}\, \chi_\nu(g) = \delta_{\mu\nu} \quad \Longrightarrow$$

$$\frac{1}{|G|} \underset{\underline{\underline{\{C_i\}}}}{\Sigma} \underline{\underline{m_i}}\, \underline{\overline{\chi_\mu(C_i)}}\, \underline{\chi_\nu}(C_i) = \delta_{\mu\nu}$$

define $S_{\mu i} = \sqrt{\dfrac{m_i}{|G|}}\, \chi_\mu(C_i)$ then

$$\sum_{i=1}^{r} S_{\mu i} S^*_{\nu i} = \delta_{\mu\nu}. \qquad S \text{ is a unitary matrix}$$

$$\underline{S \cdot S^\dagger = \mathbb{1}_r}$$

There is a dual orthogonality relation

$$\sum_\mu \overline{\chi_\mu(C_i)}\, \chi_\mu(C_j) = \frac{|G|}{m_i} \delta_{ij}$$

<u>Examples</u>

1. $S_2 \cong \mathbb{Z}_2$

	1	$[(12)]$
1^+	1	1
1^-	1	-1

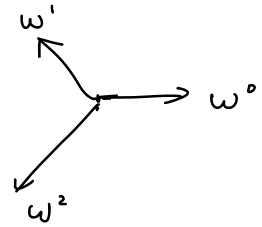
2. $G = \mathbb{Z}_n$ $\qquad \#\{C_j\} = n$

$\#\{\text{irreps}\} = n$

Z_3 : $\rho_m(\bar{j}) = \underline{(\omega_m)^j}$ $\qquad \omega_m = e^{i\frac{2\pi}{3}m}$

$= (\omega_1)^{mj}$ $\qquad \omega = e^{i\frac{2\pi}{3}}$

	$[\bar{0}]$	$[\bar{1}]$	$[\bar{2}]$
ρ_0	1	1	1
ρ_1	1	ω	ω^2
ρ_2	1	ω^2	$\omega^{2\times 2} = \omega$



3. $G = S_3$

σ - 2 cycles $\qquad \tau$ — 3 cycles

$\sigma\tau\sigma = \tau^2 \qquad \tau\sigma\tau^{-1} = \sigma'$

	$[1]$	$3[(12)]$	$2[(123)]$
1^+	1	1	1
1^-	1	-1	1
2	2	A	B
		0	-1

Given a general rep & a character table. How do we find what irreps it reduces into?

① $\mathbb{R}^3$ rep of S_3:

$$1 = \mathbb{1}_3 \qquad (12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (132) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\chi_V = 3 \qquad\qquad 1 \qquad\qquad 0$$

$$a_\mu = \langle \chi_\mu , \chi_V \rangle = \int_G (\chi_\mu(g))^* \chi_V(g)\, dg$$

$$= \frac{1}{|G|} \sum_g \overline{\chi_\mu(g)}\, \chi_V(g)$$

	[1]	3[(12)]	2[(123)]
1^+	1	1	1
1^-	1	-1	1
2	2	0	-1
V	3	1	0

$$a_{1^+} = \frac{1}{6}(3 + 3\times 1 + 2\times 0) = 1$$

$$a_{1^-} = \frac{1}{6}(3 + 3\times(-1) + 2\times 0) = 0$$

$$a_2 = \frac{1}{6}((3\times 2) + 0 + 0) = 1$$

$$\chi_V = \chi_{1^+} + \chi_2$$

$$V \cong V_{1^+} \oplus V_2$$

② Regular rep of S_3. $\dim(L^2(S_3)) = |S_3| = 6$

$$\chi_V(e) = 6$$

$$\chi_V(g \neq e) = 0$$

$$a_\mu = \langle \chi_\mu , \chi_V \rangle = \frac{1}{|G|} \cdot |G| \cdot \chi_\mu(e) = \underline{\dim V^\mu}$$

$$\boxed{L^2(G) \cong \bigoplus_\mu (\dim V^\mu) \cdot V^\mu}$$

4. V a vector space. S_2 permutes on $V \otimes V$.

$$\sigma: \quad v_i \otimes v_j \longmapsto v_j \otimes v_i$$

	1	σ
1^+	1	1
1^-	1	-1

$$\chi_{V\otimes V}(1) = d^2$$

$$\chi_{V\otimes V}(\sigma) = d \quad (\text{only } i=j)$$

$$a_{1^+} = \langle \chi^{1^+}, \chi_{V\otimes V} \rangle = \frac{1}{2} d(d+1)$$

$$a_{1^-} = \langle \chi^{1^-}, \chi_{V\otimes V} \rangle = \frac{1}{2} d(d-1)$$

$$V \otimes V = \frac{1}{2} d(d+1) \cdot V^{1^+} \oplus \frac{1}{2} d(d-1) V^{1^-}$$

$T_{ij}\, v_i \otimes v_j \in V \otimes V$. basis for

symmetric tensors. $\frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i)$

antisymmetric tensors: $\frac{1}{2}(e_i \otimes e_j - e_j \otimes e_i)$

8.13 Decomposition of tensor products of representations.

V carries space of dim n, basis $\{v_1, \cdots v_n\}$

W $\quad m \quad \{w_1, \cdots w_m\}$

$V \otimes W$. dim $n \cdot m$ basis $\{v_i \otimes w_j \ 1 \le i \le n,\ 1 \le j \le m\}$

$$\sum_i a_i v_i \otimes \sum_j b_j w_j = \sum_{ij} a_i b_j \, v_i \otimes w_j$$

G-action $\quad g \cdot (v \otimes w) := (g \cdot v) \otimes (g \cdot w)$

rep. $\quad (T_1 \otimes T_2)(g)(v \otimes w) := T_1(g) \cdot v \otimes T_2(g) w.$

mat. rep. $\quad (M_1 \otimes M_2)(g)_{ia,jb} = [M_1(g)]_{ij} \, [M_2(g)]_{ab}$

character $\quad \chi_{T_1 \otimes T_2} = \chi_{T_1} \cdot \chi_{T_2}$

① particle of spin j_1

j_2 $\quad \Rightarrow V^{j_1} \otimes V^{j_2}$

$\cong \bigoplus a_{j_3} V^{j_3}$

② many-particle system, local Hilbert space

$\mathcal{H}_i$ $\quad$ spin 1/2 fermion $= \{\phi, \uparrow, \downarrow, \uparrow\downarrow\}$

$\mathcal{H} = \bigotimes_i \mathcal{H}_i \Rightarrow \bigoplus_i a_i \mathcal{H}^{\{\alpha\}_i}$ — —

N.S.

$\underline{G} \otimes U(1) \otimes SU(2)$

space group

Let (V_1, T_1) and (V_2, T_2) be two representations with isotypic decompositions (over field k)

$$V_1 = \oplus a_\mu V^\mu \qquad V_2 = \oplus b_\nu V^\nu$$

$$V_1 \otimes V_2 = \oplus a_\mu b_\nu V^\mu \otimes V^\nu$$

$$V^\mu \otimes V^\nu \cong \oplus_\lambda N_{\mu\nu}^\lambda V^\lambda \qquad \left(\oplus D_{\mu\nu}^\lambda \otimes V^\lambda \right)$$

$$\chi_\mu \cdot \chi_\nu = \sum_\lambda N_{\mu\nu}^\lambda \chi_\lambda$$

fusion coefficient

Clebsch-Gordan for $SU(2)$

$$N_{\mu\nu}^\lambda = \langle \chi_\lambda . \chi_\mu \cdot \chi_\nu \rangle$$

for Finite groups

$$N_{\mu\nu}^\lambda = \frac{1}{|G|} \sum_{g \in G} \chi_\mu(g) \chi_\nu(g) \overline{\chi_\lambda(g)}$$

$m_i = |C_i|$

$$= \frac{1}{|G|} \sum_{\{C_i\}} m_i \chi_\mu(C_i) \chi_\nu(C_i) \overline{\chi_\lambda(C_i)}$$

$$N_{\mu\nu}^\lambda = N_{\nu\mu}^\lambda \qquad \left(V^\mu \otimes V^\nu \cong V^\nu \otimes V^\mu \right)$$

Examples. 1. ρ_m of $\mathbb{Z}_N$ $\qquad \rho_m(j) = \left(e^{i\frac{2\pi}{N}m}\right)^j$

$$\rho_m \otimes \rho_n \cong \rho_{m+n}$$

$$N_{mn}^\lambda = \frac{1}{N} \sum_{l} e^{i\frac{2\pi}{N}(m+n)l} e^{-i\frac{2\pi}{N}\cdot\lambda l}$$

$$= \delta_{m+n,\lambda}$$

2. irreps of S_3.

$$V^{1^+} \otimes V^{\mu} \cong \oplus_{\lambda} N_{1^+,\mu}^{\lambda} V^{\lambda}$$

$$N_{1^+,\mu}^{\lambda} = \frac{1}{|G|} \sum m_i \underline{\chi_{\mu}}(C_i) \overline{\underline{\chi_{\lambda}}(C_i)}$$

$$= \delta_{\mu\lambda}$$

$$\bigoplus_{\lambda} \delta_{\mu\lambda} V^{\lambda} = V^{\mu}$$

$$\Rightarrow \underline{V^{1^+} \otimes V^{\mu}} \cong V^{\mu}$$

<u>check</u>

$$V^{-} \otimes V^{-} \cong V^{+}$$

$$V^{-} \otimes V^{2} \cong V^{2}$$

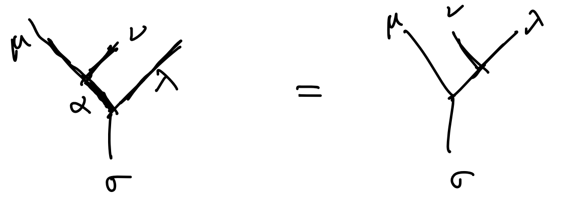
$$V^{2} \otimes V^{2} \cong V^{+} \oplus V^{-} \oplus V^{2}$$

$$\underline{(V^{\mu} \otimes V^{\nu}) \otimes V^{\lambda} \cong V^{\mu} \otimes (V^{\nu} \otimes V^{\lambda})}$$

$$\text{LHS} \cong \oplus_{\alpha} \underline{D_{\mu\nu}^{\alpha}} V^{\alpha} \otimes V^{\lambda}$$

$$\cong \oplus_{\sigma} (\oplus_{\alpha} \underline{D_{\mu\nu}^{\alpha} \otimes D_{\alpha\lambda}^{\sigma}}) \otimes V^{\sigma} \cong \oplus_{\sigma} (\oplus_{\beta} D_{\nu\lambda}^{\beta} \otimes D_{\mu\beta}^{\sigma}) \otimes V^{\sigma}$$

$$\sum_{\alpha} \underline{N_{\mu\nu}^{\alpha}}\, \underline{N_{\alpha\lambda}^{\sigma}} = \sum_{\beta} N_{\mu\beta}^{\sigma} N_{\nu\sigma}^{\beta}$$



"F-move"

digression : "Category theory"

TQFT / anyons / topo. quantum computation

$(X \otimes Y) \otimes (Z \otimes W)$ $\rightarrow$ pentagon relation

(ref. PRB 100, 115147)

Summary of key results

① unitary rep. of compact G.

$$\langle u^{\mu_1}_{i_1 j_1}, u^{\mu_2}_{i_2 j_2} \rangle = \frac{1}{n_\mu} \delta^{\mu_1\mu_2} \delta_{i_1 i_2} \delta_{j_1 j_2}$$

complete, orthogonal basis of $L^2(G)$.

② (Peter-Weyl) $L^2(G) \cong \bigoplus_\mu \mathrm{End}(V^\mu)$

$$i : \bigoplus_\mu \mathrm{End}(V^\mu) \to L^2(G)$$

$$\bigoplus_\mu S_\mu \mapsto \sum_\mu \varphi_{S_\mu}$$

$$\varphi_{S_\mu} := \mathrm{Tr}_{V_\mu}(S\, T(g^{-1}))$$

$\hookrightarrow$ finite G: $\boxed{|G| = \sum_\mu n_\mu^2}$

$(n_\mu = \dim V^\mu)$

③ characters.

$$\int_G \overline{\chi^{(\mu)}(g)}\, \chi^{(\nu)}(g)\, dg = \delta_{\mu\nu}$$

ON basis of $L^2(G)^{\text{class}}$.

④ $V \cong \bigoplus_\mu a_\mu V^{(\mu)}$

$$a_\mu = \int_G \overline{\chi^{(\mu)}}(g)\, \chi_V(g)\, dg = \langle \chi^{(\mu)}, \chi_V \rangle$$

reg. rep. $a_\mu = \langle \chi^\mu, \underline{\chi} \rangle = \frac{1}{|G|} (\dim n_\mu) \cdot |G|$

$= \dim n_\mu$

⑤ # $\underline{\text{irreps}}$ = # conj. class.

		C_i	- - - -
irreps	Γ_1		
	Γ_2		
	$\vdots$		

rows: $\displaystyle \frac{1}{|G|}\sum_{\underline{\{C_i\}}} \underline{|C_i|}\, \chi_\mu(C_i)\, \overline{\chi_\nu(C_i)} = \delta_{\mu\nu}$

columns: $\displaystyle \sum_\mu \overline{\chi_\mu(C_i)}\, \chi_\mu(C_j) = \frac{|G|}{m_i}\,\delta_{ij}$