Recap. Group algebra

1. group elements , operators/actions & vectors/objects

$$\mathbb{C}[G] = \left\{ \sum_g \alpha_g \cdot g \mid \alpha_g \in \mathbb{C} \right\}$$

$$\begin{cases} x + y = \sum_g [x(g) + y(g)] \cdot g \\ \alpha x = \sum_g [\alpha x(g)] \cdot g \\ x \cdot y = \sum_g x(g) \, y(g^{-1}k) \cdot k \end{cases}$$

$$L(h) \cdot x = L(h) \sum x(g) \cdot g = \sum x(g) \, (hg) = \sum_g x(h^{-1}g) \, g$$

$$\Rightarrow [L(h) \cdot x](g) = x(h^{-1}g)$$

$$x = \delta_g \quad \Rightarrow \quad h \cdot \delta_g = \delta_{hg}$$

2. Define class operators

$$C_i = \sum_{g \in C_i} g \quad \in End_G (\mathbb{C}[G])$$

then $[C_i , g] = 0 \ (\forall g \in G)$, intertwiners

All $\{C_i\}$ commute $\Rightarrow$ can be simultaneously diagonalized. we can represent them within the center of $\mathbb{C}[G]$: span $\{C_i\}$

$$\boxed{\hat{C}_i \cdot \hat{C}_j = \sum_k D_{ij}^k \, \hat{C}_k}$$

D_{ij} are the

recall matrix rep:

$$g \cdot e_i = \sum_j M_{ji}(g) e_j$$

$$\hat{C}_j \quad C_i = \sum_k D(j)_{ki} C_k$$

what are the eigenvectors?

$$\hat{C}_j \vec{\phi}^{(\mu)} = \lambda_\mu \vec{\phi}^{(\mu)}$$

We haven shown that assuming nondegeneracy

What if there is degeneracy? Find another
C_i that splits the degeneracy.

$\Rightarrow$ We will see later that there is no degeneracy

if all $\{C_i\}$ are diagonalized. (CSCO-I)

$$\hat{C}_j \vec{\phi}^\mu \vec{\phi}^\nu = (\lambda^\mu \vec{\phi}^\mu) \vec{\phi}^\nu = \vec{\phi}^\mu (\lambda \vec{\phi}^\nu) = \lambda^\mu \delta_{\mu\nu} \vec{\phi}^\mu \vec{\phi}^\nu$$

$\{\vec{\phi}^\mu\}$ behave like projectors!

$$\hat{C}_i \vec{\phi} = \lambda_i \vec{\phi} \quad \text{then} \quad \vec{\phi} = \sum_{i=1}^r \alpha_i C_i$$

$$C_i \vec{\phi} = C_i \sum_j \alpha_j C_j = \sum_j \alpha_j C_i C_j = \sum_{jk} \alpha_j D_{ij}^k C_k$$

$$= \lambda_i \vec{\phi} = \lambda_i \sum_j \alpha_j C_j = \sum_k (\lambda_i \alpha_k) C_k$$

$$\Rightarrow \sum_j \alpha_j D_{ij}^k = \lambda_i \alpha_k$$

$$\vec{\phi}^2 = \sum_{ij} \alpha_i \alpha_j C_i C_j = \sum_{ijk} \alpha_i \alpha_j D_{ij}^k C_k = \sum_k (\sum_{ij} \alpha_i \alpha_j D_{ij}^k) C_k$$

$$= \sum_k (\sum_i \alpha_i (\lambda_i \alpha_k)) C_k = \sum_i \lambda_i \alpha_i \underbrace{\sum_k \alpha_k C_k}_{\vec{\phi}} \propto \vec{\phi}$$

$$\underbrace{\qquad}_{\alpha}$$

$$\vec{\tilde{\phi}} = \frac{1}{\alpha} \vec{\phi}, \quad \text{then} \quad \vec{\tilde{\phi}}^2 = \vec{\tilde{\phi}} \quad \text{idempotent.}$$

We will look at these eigenvectors more closely.

8.13.3 Construction of character tables (cont.)

We know that C_i's are intertwiners, which means when restricted to a specific irrep:

$$c_i^\mu = \lambda_i^\mu \cdot \mathbb{1}_{V^\mu} \qquad \text{(Schur's lemma)}$$

Taking trace/character.

$$\text{LHS} = \text{Tr}_{V^\mu} \sum_{g \in C_i} g = m_i \, \chi^\mu([C_i])$$

$$\text{RHS} = \lambda_i^\mu \cdot n_\mu$$

$$\Rightarrow \boxed{\lambda_i^\mu = \frac{m_i}{n_\mu} \chi^\mu([C_i])}$$

These are actually the eigen values. Why?

$$C_j \, C_i = \sum [D(j)]_{ki} \, C_k$$

then taking the characters on V^μ

$$\chi_\mu(C_j)\,\chi_\mu(C_i) = \sum_k (D_j)_{ki}\, \chi_\mu(C_k)$$

$$= \lambda_i^\mu \, \chi_\mu(C_j)$$

i.e. $\sum_k (D_j)_{ki}\, \chi_\mu(C_k) = \lambda_j^\mu \, \chi_\mu(C_i)$

take $\vec{\chi}_\mu = \begin{pmatrix} \chi_\mu(C_1) \\ \chi_\mu(C_2) \\ \vdots \\ \chi_\mu(C_r) \end{pmatrix}$ then

$$D(j)^T \, \vec{\chi}_\mu = \lambda_j \, \vec{\chi}_\mu \quad \text{in matrix form.}$$

$$\text{Tr}_\mu(C_i \, C_j) = \text{Tr}_\mu(C_j \cdot \lambda_i^\mu \mathbb{1}_{V^\mu})$$

$$= \lambda_i^\mu \, \text{Tr}_\mu(C_j)$$

$$= \lambda_i^\mu \, m_j \, \chi([C_j])$$

$$\sum_k \text{Tr}_\mu([D_i]_{kj} C_k) = \sum_k [D_i]_{kj} \, m_k \, \chi_\mu([C_k])$$

$$\Rightarrow \quad D_i^T \, \vec{\chi}_\mu = \lambda_i^\mu \, \vec{\chi}_\mu$$

$$\vec{\chi}_\mu = \begin{pmatrix} m_1 \, \chi_\mu([C_1]) \\ m_2 \, \chi_\mu([C_2]) \\ \vdots \\ m_r \, \chi_\mu([C_r]) \end{pmatrix} \quad \begin{array}{l}\text{up to}\\ \text{normalization}\end{array}$$

remaining two unknowns: n_μ, x_μ

$$\frac{1}{|G|} \sum_{C_i} m_i\, X_\mu(C_i)\overline{X_\nu(C_i)} = \delta_{\mu\nu}$$

$$\Rightarrow \frac{1}{|G|} \sum_{C_i} m_i\, \lambda_i^\mu\, \overline{\lambda_i^\nu} = \delta_{\mu\nu}\left(\frac{m_i}{n_\mu}\right)^2 \quad \left(\equiv \langle \lambda_i^\nu . \lambda_i^\mu \rangle\right)$$

$$\Rightarrow \begin{cases} n_\mu = \dfrac{m_i}{N\langle \lambda_i^\mu . \lambda_i^\mu \rangle} \\[3mm] X_\mu = \dfrac{\lambda_i^\mu}{\sqrt{\langle \lambda_i^\mu . \lambda_i^\mu \rangle}} \end{cases}$$

Now, back to the assumption about no degeneracy once diagonalized all class operators:

$$\boxed{\lambda_i^\mu = \frac{m_i}{n_\mu} X^\mu([C_i])}$$

$$\vec{\lambda}^\mu = \{ \lambda_1^\mu, \lambda_2^\mu \cdots \lambda_r^\mu \}, \quad \vec{\lambda}^\nu = \{ \lambda_1^\nu, \lambda_2^\nu \cdots \lambda_r^\nu \}$$

if $\underline{\forall j}$. $\lambda_j^\mu = \lambda_j^\nu$. then

$$\frac{m_j}{n_\mu} X^\mu([C_j]) = \frac{m_j}{n_\nu} X^\nu([C_j])$$

$\Rightarrow$ which means two characters are proportional to each other cannot be because they should be orthogonal..!

Diagonalizing all class operators $\Rightarrow$ full knowledge of $\underline{X_\mu, n_\mu}$

see 陈金金 for examples of finding / using a minimal CS Co-I (subset of class operators).

We will just diagonalize all operators.

$$c_i^{(\mu)} = \frac{m_i}{n_\mu} \chi_\mu([C_0]) \cdot \psi_{\nu^\mu}$$

$$\longrightarrow \quad \frac{m_i}{n_\mu} \chi_\mu([C_i]) \, \frac{m_j}{n_\mu} \chi_\mu([C_j]) = \sum_{k=1}^{r} c_{ij}^k \frac{m_k}{n_\mu} \chi_\mu([C_k])$$

$$m_i \chi_\mu([C_i]) \, m_j \chi_\mu([C_j]) = n_\mu \sum_{k=1}^{r} c_{ij}^k \, m_k \chi_\mu([C_k])$$

Now introduce a set of auxiliary variables $\{ y^i, \, i=1,\cdots,r \}$

(So we can differentiate between different c_i's: $c_i \rightarrow c_i \, y^i$)

$\sum$ LHS : $\displaystyle\sum_{i=1}^{r} m_i m_j \, \chi_\mu([C_i]) \chi_\mu([C_j])) \, y^i = \sum_{i=1}^{r} (\psi_i \, y^i) \psi_j \quad (\psi_i := m_i \chi_\mu([C_i]))$

$\sum$ RHS : $\displaystyle\sum_{i=1}^{r} n_\mu \sum_{k=1}^{r} c_{ij}^k \, m_k \chi_\mu([C_k]) \, y^i = n_\mu \sum_{k=1}^{r} L_j^k \psi_k$

$\qquad$ Define $\quad \lambda = \dfrac{1}{n_\mu} \displaystyle\sum_{i=1}^{r} \psi_i \, y^i \qquad\qquad\qquad \left(L_j^k = \sum_i c_{ij}^k \, y^i \right)$

$\qquad \Longrightarrow \displaystyle\sum_{k=1}^{r} L_j^k \psi_k = \lambda \psi_j$

$\qquad$ Solving the eigen problem $\quad (L - \lambda \mathbb{1}) \psi = 0$

$\qquad$ and obtain a set of eigenvalues $\{ \lambda_\mu \}$

(✱) $\quad \lambda_\mu = \dfrac{1}{n_\mu} \displaystyle\sum_{i=1}^{r} \overbrace{m_i \chi_\mu([C_i])}^{\psi_i} \, y^i \qquad\qquad \mu = 1, \cdots, r$

Note if we set $y^j = \delta_{ij}$, we recover our earlier λ_i^μ.

Now recall the orthogonality relation:

$$\frac{1}{|G|} \sum_{C_i} m_i\, \chi_\mu(C_i)\, \overline{\chi_\nu(C_i)} = \delta_{\mu\nu} \qquad (\text{ortho. of rows})$$

$$\overset{\mu=\nu}{\Longrightarrow} \sum_{i=1}^{r} m_i \left| \chi_\mu([C_i]) \right|^2 = |G|$$

$$|G| = \left| \chi_\mu([C_1]) \right|^2 \sum_{i=1}^{r} m_i \left| \frac{\chi_\mu([C_i])}{\chi_\mu([C_1])} \right|^2$$

$$= n_\mu^2 \sum_{i=1}^{r} m_i \left| \frac{\chi_\mu([C_i])}{n_\mu} \right|^2$$

$$\Longrightarrow \quad n_\mu = \left[\frac{|G|}{\sum_{i=1}^{r} m_i \left| \frac{\chi_\mu([C_i])}{n_\mu} \right|^2} \right]^{\frac{1}{2}}$$

↖ known from above (✱)

Implementation in practice:

$$S_3: \quad E\ ;\ (12),(13),(23)\ ;\ (123),(132)$$

① class operators: $C_1 = E$

$$C_2 = (12) + (13) + (23) \qquad (12)(13) = (132)$$

$$C_3 = (123) + (132) \qquad (12)(123) = (1)(23)$$

② class multiplication table:

	C_1	C_2	C_3
C_1	C_1	C_2	C_3
C_2	C_2	$3C_1 + 3C_3$	$2C_2$
C_3	C_3	$2C_2$	$2C_1 + C_3$

① explain underlined.
② symmetric (∵ abelian)

③ $L_j^k = \sum_i C_{ij}^k y^i$ 3×3 matrix

$L_1^1 = C_{11}^1 y^1 + C_{21}^1 y^2 + C_{31}^1 y^3 = y^1 + 0 + 0$

$L_1^2 = \sum_i C_{i1}^2 y^i = y^2$

$L_1^3 = y^3$

$L_2^1 = \sum C_{i2}^1 y^i = 3y^2$

$L_2^2 = \sum C_{i2}^2 y^i$ $L_2^3 = \sum C_{i2}^3 y^i$

$L_3^1 = \sum C_{i3}^1 y^i$ $L_3^2 = \sum C_{i3}^2 y^i$

$L_3^3 = \sum C_{i3}^2 y^i$

	C_1	C_2	C_3
C_1	C_1	C_2	C_3
C_2	C_2	$3C_1 + 3C_3$	$2C_2$
C_3	C_3	$2C_2$	$2C_1 + C_3$

$$\hat{L} = \begin{pmatrix} y^1 & y^2 & y^3 \\ 3y^2 & y^1 + 2y^3 & 3y^2 \\ 2y^3 & 2y^2 & y^1 + y^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} y^1 + \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix} y^2$$

$$+ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix} y^3$$

$\begin{cases} \lambda_a = y^1 + 3y^2 + 2y^3 \\ \lambda_b = y^1 - 3y^2 + 2y^3 \\ \lambda_c = y^1 + 0\, y^2 - y^3 \end{cases}$

$\boxed{\lambda_\mu = \sum_{i=1}^{r} \dfrac{m_i \chi_\mu([C_i])}{n_\mu} y^i}$

$n_\mu = \left[\dfrac{|G|}{\sum_{i=1}^{r} m_i \left| \dfrac{\chi_\mu([C_i])}{n_\mu} \right|^2} \right]^{\frac{1}{2}}$

write in cols. ↓

④ $\chi_a = n_a (\, 1, \quad 1 \quad, \quad 1)$ $n_a = 1$

$\chi_b = n_b (\, 1, \quad -1 \quad, \quad 1)$ $n_b = 1$

$\chi_c = n_c (\, 1, \quad 0, \quad -\tfrac{1}{2})$ $n_c = \left[\dfrac{6}{1 + 3 \cdot 0 + 2 \cdot \frac{1}{4}} \right]^{\frac{1}{2}} = 2$

⑤ Character table

	[1]	3[(12)]	2[(123)]
1^+	1	1	1
1^-	1	-1	1
2	2	0	-1

Note that in the solution:

$$\begin{cases} \lambda_a = y^1 \boxed{+ 3y^2} + 2y^3 \\ \lambda_b = y^1 \boxed{- 3y^2} + 2y^3 \\ \lambda_c = y^1 \boxed{+ 0y^2} - y^3 \end{cases}$$

The eigenvalues of $\hat{C}_2$ is non-degenerate.
This defines a set of unique eigenvectors
that diagonalizes all $\hat{C}_i$. Which means $\hat{C}_2$ is
a CSCO by itself.

$$\begin{aligned} &\quad\quad\quad C_1 \quad\quad C_2 \quad\quad C_3 \quad\quad\quad () \quad (12) \quad (123) \\ &\chi_{\mu_1} = n_{\mu_1}\left(\frac{1}{1}, \frac{3}{3}, \frac{2}{2}\right) = (1, 1, 1) \\ &\chi_{\mu_2} = n_{\mu_2}\left(\frac{1}{1}, \frac{-3}{3}, \frac{2}{2}\right) = (1, -1, 1) \\ &\chi_{\mu_3} = n_{\mu_3}\left(\frac{1}{1}, \frac{0}{3}, -\frac{1}{2}\right) = (2, 0, -1) \end{aligned}$$

$$\chi_i^\mu = n_\mu \frac{\lambda_i^\mu}{m_i}$$

$m_1 = 1 \quad m_2 = 3 \quad m_3 = 2$

$\hookrightarrow$ normalization:

$$\langle \chi_{\mu_1}, \chi_{\mu_1} \rangle = \frac{1}{6} n_{\mu_1}^2 \cdot 6 = 1$$

$$\begin{cases} \langle \chi_{\mu_1}, \chi_{\mu_1} \rangle = \frac{1}{6} n_{\mu_1}^2 \cdot 6 = 1 \\ \langle \chi_{\mu_2}, \chi_{\mu_2} \rangle = \frac{1}{6} n_{\mu_2}^2 \cdot 6 = 1 \\ \langle \chi_{\mu_3}, \chi_{\mu_3} \rangle = \frac{1}{6} n_{\mu_3}^2 (1 + 0 + \frac{1}{4} \times 2 \\ \quad\quad\quad = \frac{1}{4} n_{\mu_3}^2 = 1 \end{cases}$$

$$\begin{aligned} n_{\mu_1} &= n_{\mu_2} = 1 \\ n_{\mu_3} &= 2 \end{aligned} \quad \Longleftarrow$$

Projectors:

$$\hat{C}_i \cdot \hat{C}_j = \sum_k [D_i]_{jk} C_k$$

$$\hat{C}_i \cdot \phi_\mu = \lambda_i^\mu \phi_\mu \qquad\qquad \phi_\mu = \sum_i \phi_\mu(C_i) C_i$$
$$= \phi_\mu^i C_i$$

$$\sum_j \phi_\mu^j \hat{C}_i \hat{C}_j = \lambda_i^\mu \sum_k \phi_\mu^k C_k$$

$$\Rightarrow \sum_{jk} \phi_\mu^j [D_i]_{jk} C_k = \lambda_i^\mu \sum_k \phi_\mu^k C_k$$

$$\Rightarrow \sum_k \left(\sum_j (D_i^T)_{kj} \phi_\mu^j \right) C_k = \sum_k \lambda_i^\mu \phi_\mu^k \cdot C_k$$

$$\Rightarrow \sum_j (D_i^T)_{kj} \phi_\mu^j = \lambda_i^\mu \phi_\mu^k$$

$$\sum_j (D_i^T - \lambda_i^\mu \delta_{jk}) \phi_\mu^j = 0$$

ϕ_μ are eigenvectors of D_i^T with basis $\{ C_1, C_2, C_3 \}$

$$D_2^T = \begin{pmatrix} & 3 \\ 1 & & 2 \\ & 3 & \end{pmatrix}$$

$$\lambda_2^{\mu_1} = 3 \qquad \phi_{\mu_1} \propto (1,1,1)^T \leftarrow \chi_\mu(i)$$
$$\lambda_2^{\mu_2} = -3 \qquad \phi_{\mu_2} \propto (1,-1,1)^T$$
$$\lambda_2^{\mu_3} = 0 \qquad \phi_{\mu_3} \propto (2,0,-1)^T$$

$$P_{\mu_1} = \alpha_{\mu_1}(C_1 + C_2 + C_3)$$

	C_1	C_2	C_3
C_1	C_1	C_2	C_3
C_2	C_2	$3C_1 + 3C_3$	$2C_2$
C_3	C_3	$2C_2$	$2C_1 + C_3$

$$P_{\mu_1}^2 = \alpha_{\mu_1}^2 (C_1^2 + C_2^2 + C_3^2 + 2C_1 C_2 + 2C_1 C_3 + 2C_2 C_3)$$

$$= \alpha_{\mu_1}^2 (\underline{C_1} + 3C_1 + 3C_3 + 2C_1 + C_3$$
$$+ 2C_2 + 2C_3 + 4C_2)$$

$$= 6\alpha_{\mu_1}^2 (C_1 + C_2 + C_3) = \alpha_\mu (C_1 + C_2 + C_3)$$
$$= P_{\mu_1}$$

$$\alpha_{\mu_1} = \frac{1}{6}$$

$$P_{\mu_1} = \tfrac{1}{6}(C_1 + C_2 + C_3)$$

$$P_{\mu_2} = \tfrac{1}{6}(C_1 - C_2 + C_3)$$

$$P_{\mu_3} = \tfrac{1}{3}(2C_1 - C_3)$$

$$P_{\mu_1} P_{\mu_2} \propto C_1^2 + C_3^2 + 2C_1C_3 - C_2^2 = C_1 + 2C_1 + C_3 + 2C_3 - (3C_1 + 3C_3)$$
$$= 0$$

$$P_{\mu_1} P_{\mu_3} \propto (C_1 + C_2 + C_3)(2C_1 - C_3) = 2C_1^2 - C_1C_3 + 2C_1C_2 - C_2C_3$$
$$+ 2C_1C_3 - C_3^2$$

$$= 2C_1 - C_3 + 2C_2 - 2C_2 \qquad = 0$$
$$+ 2C_3 - (2C_1 + C_3)$$

$$\hat{C}_2 \, P_{\mu_1} = \tfrac{1}{6}(C_1 C_2 + C_2^2 + C_2 C_3)$$

$$= \tfrac{1}{6}(C_2 + 3C_1 + 3C_3 + 2C_2) = 3 \cdot \tfrac{1}{6}(C_1 + C_2 + C_3)$$

$$\equiv \frac{m_2}{n_{\mu_1}} \chi_{\mu_1}([C_2]) \cdot P_{\mu_1}$$

$$(12)\, P_{\mu_1} = (12) \cdot \tfrac{1}{6}(e \pm (12) \pm (23) \pm (13) + (123) + (132))$$

$$(12)\, P_{\mu_2} = \tfrac{1}{6}((12) \pm e \pm (123) \pm (132) + (23) + (13))$$

$$= \chi_{\mu_1} \cdot P_{\mu_1}$$

$$= (-1) \cdot P_{\mu_2} = \chi_{\mu_2} \cdot P_{\mu_2}$$

$$T(h)\, P^\mu = \sum_{i,k=1}^{n_\mu} M_{ki}^\mu(h)\, P_{ki}$$

$$P_{\mu_3} = P_{\mu_3}^{11} + P_{\mu_3}^{22} \qquad P_{ij}^\mu P_{kl}^\mu = \delta_{jk} P_{il}^\mu$$

$$T(h)\, P_{ij}^\mu = \sum_{k=1}^{n_\mu} M_{ki}^\mu(h)\, P_{kj}^\mu$$

$$\Rightarrow P^{11} \cdot P^{22} = 0 \qquad P_{11}^\mu \, P_{21}^\mu = 0$$

$$\boxed{\{P_{kj}^\mu \,,\; k = 1, \cdots n_\mu\}}$$

what if $\quad P^{11} = e - (13) + (12) - (132) \quad$?

$$P^{22} = e - (12) + (13) - (123)$$

satisfy the orthogonality relation $\begin{cases} P_{\mu_3}^{11} P_{\mu_3}^{22} = \delta_{12} P_{\mu_3}^{12} = 0 \\ P_{\mu_3}^{11} + P_{\mu_3}^{22} = P_{\mu_3} \end{cases}$

in principle. find more commuting operators
to lift degeneracies on the group space R_G
(CSCO – II , CSCO – III)

use projectors from subgroups:

S_2 :

	e	(12)
1	1	1
-1	1	-1

	C_1	C_2
C_1	C_1	C_2
C_2	C_2	C_1

$P_{2j}^k : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\lambda = \pm 1$

$P_1' = \frac{1}{2}(e + (12))$

$P_2' = \frac{1}{2}(e - (12))$

$G \supset G_1$

$P_{\nu_1}^\nu = P^\nu P^{\nu_1}$

$\hookrightarrow P_1^2 = P^2 P_1' = \frac{1}{6}(2e - (123) - (132))(e \underset{-}{+} (12))$

$\underset{\overset{|}{-1}}{} \quad \underset{P'^{-1}}{}$

$= \frac{1}{6}(2e \underset{-}{+} 2(12) - (123) \underset{+}{-} (13) - (132)$

$\underset{+}{-}(23))$

$\begin{cases} P^2 = P_1' + P_{-1}' \quad \checkmark \\ P_1' P_{-1}' = 0 \end{cases}$

$C_2 + C_2' \qquad$ CSCO – II

$\qquad \uparrow \qquad \uparrow$

$\qquad S_3 \qquad S_2$

$(12) \, P_{\pm}^2 = \pm P_{\pm}^2$