Recap:

- Projection operators

$$P_{ij}^{(\mu)} = n_\mu \int_G \overline{M_{ij}^{\mu}(g)}\, T(g)\, dg$$

$$P_{ij}^{(\mu)} P_{kl}^{(\nu)} = \delta_{\mu\nu}\delta_{jk} P_{il}^{(\nu)}$$

$$\Downarrow$$

$$T(h) P_{ij}^{\mu} = \sum_{k}^{n_\mu} M_{ki}^{\mu}(h) P_{kj}^{(\mu)}(g) \qquad \text{fix } (\mu, j)$$

$$\left(T(h)\hat{e}_i = \sum M_{ki}\hat{e}_j \right)$$

$$\Rightarrow \operatorname{span}\{ P_{ij}^{\mu}\varphi \mid i = 1, \dots, n_\mu \} \Rightarrow (T^{\mu}, V^{\mu})$$

$$\operatorname{Tr}(P_{ij}^{\mu})$$

$$\Downarrow$$

$$P_\mu P_\nu = \delta_{\mu\nu} P_\nu \qquad (P_\mu^2 = P_\mu)$$

↪ "idempotent"

$$P_\mu^{+} = P_\mu$$

$$\operatorname{Tr}(P_\mu) = n_\mu \cdot a_\mu$$

$\mathbb{R}^3$ rep S_3

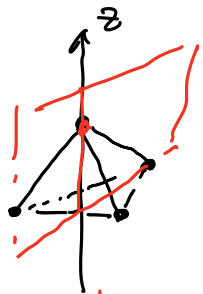
$$\mathbb{R}^3 \cong V_{1+} \oplus V_2$$

Character table for point group C_{3v}

	C_{3v}	E	$2C_3$ (z)	$3\sigma_v$	linear functions, rotations	quadratic functions	cubic functions
$1^+ \rightarrow$	A_1	+1	+1	+1	z	x^2+y^2, z^2	z^3, $x(x^2-3y^2)$, $z(x^2+y^2)$
$1^- \rightarrow$	A_2	+1	+1	-1	R_z	-	$y(3x^2-y^2)$
$2 \rightarrow$	E	+2	-1	0	(x, y) (R_x, R_y)	(x^2-y^2, xy) (xz, yz)	(xz^2, yz^2) $[xyz, z(x^2-y^2)]$ $[x(x^2+y^2), y(x^2+y^2)]$

Source: http://symmetry.jacobs-university.de/cgi-bin/group.cgi?group=403&option=4

$G = C_{3v}$



$C_3(z) \rightarrow (123)$

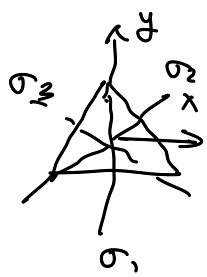
$\sigma \rightarrow (12)$

$C_{3v} \cong S_3$

Symmetry operation on $(x, y, z) \rightarrow (x', y', z')$

$$M(E) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M(C_3) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M(C_3^2) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M(\sigma_1) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$M(\sigma_2) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M(\sigma_3) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{T(g)\varphi(r) = \varphi(T(g)^{-1} \cdot r)}$$

$C_3^{-1}\cdot(x,y,z)^T$ ①

$$\varphi^{A_1,A_2} \equiv P_{A_1}\varphi = \frac{1}{6}\Big[\underbrace{\varphi(x,y,z)}_{T(E)} + \underbrace{\varphi\left(-\tfrac{1}{2}x+\tfrac{\sqrt3}{2}y,\ \tfrac{\sqrt3}{2}x-\tfrac{1}{2}y,\ z\right)}_{T(C_3)} +$$

$$\varphi\left(-\tfrac{1}{2}x-\tfrac{\sqrt3}{2}y,\ \tfrac{\sqrt3}{2}x+\tfrac{1}{2}y,\ z\right) \qquad T(C_3^2)$$

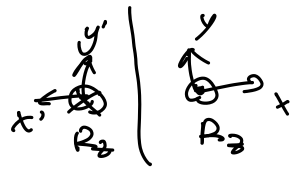
$$\left.\begin{aligned} &\pm\ \varphi(-x,\ y,\ z) \\ &\pm\ \varphi\left(\tfrac{1}{2}x+\tfrac{\sqrt3}{2}y,\ \tfrac{\sqrt3}{2}x-\tfrac{1}{2}y,\ z\right) \\ &\pm\ \varphi\left(\tfrac{1}{2}x-\tfrac{\sqrt3}{2}y,\ -\tfrac{\sqrt3}{2}x-\tfrac{1}{2}y,\ z\right)\Big] \end{aligned}\right\} T(\sigma_i)$$

linear functions: $\varphi(x,y,z) = \alpha x + \beta y + \gamma z$

$\varphi^{A_1} = z$

$\varphi^{A_2} = 0 \longrightarrow$ consider $R_z = \hat{x}\times\hat{y}$

R_z



E

$\varphi_i^{E,j}$: $\varphi(x,y)$	$\varphi_1^{E,1}$	$\varphi_2^{E,1}$	$\varphi_1^{E,2}$	$\varphi_2^{E,2}$
x	x	y	0	0
y	0	0	x	y
z/R_z	0	0	0	0

linear basis for E : (x,y)

or (R_x, R_y)

Mulliken symbols :

A / B : 1-dim. symmetric / antisymmetric w.r.t. principle rotations. $\chi(C_n) = \pm 1$

E : 2-dim "entartet" degenerate.

T : 3-dim

G : 4-dim

H : 5-dim

subscripts :

1 / 2 : symmetric / anti-symm. w.r.t vertical mirror plane $\chi(\sigma_v) = \pm 1$

g / u : "grade / ungrade" even / odd w.r.t. inversion $\chi(i) = \pm 1$

primes : ' / '' : sym / asym. w.r.t. horizontal mirror plane

Summary of key results :

① unitary irrep. of compact G.

$$\langle M^{\mu_1}_{i_1 j_1} , M^{\mu_2}_{i_2 j_2} \rangle = \frac{1}{n_\mu} \delta^{\mu_1 \mu_2} \delta_{i_1, i_2} \delta_{j_1, j_2}$$

complete orthogonal basis of $L^2(G)$

② (Peter-Weyl) $L^2(G) \cong \bigoplus_\mu \mathrm{End}(V^\mu)$

$$\iota : \bigoplus_\mu \mathrm{End}(V^\mu) \longrightarrow L^2(G)$$

$$\bigoplus_\mu S_\mu \longmapsto \sum_\mu \varphi_{S_\mu}$$

$$\varphi_S := \mathrm{Tr}_V(S\,T(g))$$

is an isomorphism

↪ Corollary : finite G

$$|G| = \sum_\mu n_\mu^2$$

$$|S_3| = 6 = 1^2 + 1^2 + 2^2$$

③ ortho. of characters:

$$\int_G \overline{\chi^\mu(g)}\, \chi^\nu(g)\, dg = \delta_{\mu\nu}$$

ON basis of $L^2(G)^{\mathrm{class}}$

④ $V \cong \bigoplus_\mu a_\mu U^\mu$

$$a_\mu = \int_G \overline{\chi^\mu(g)}\, \chi_V(g)\, dg \equiv \langle \chi^\mu, \chi_V \rangle$$

Reg. rep: $\chi_v(e) = |G|$

$$\Rightarrow a_\mu = \langle \chi^\mu, \chi \rangle = \frac{1}{|G|} \underbrace{\dim V^\mu}_{\chi^\mu(e)} \cdot \underbrace{|G|}_{\chi_v(e)} = \dim V^\mu$$

$$L^2(G) \cong \bigoplus_\mu (\dim V^\mu) V^\mu$$

⑤ orthogonality. of character tables.

rows: $\frac{1}{|G|} \sum_{C_i} |C_i| \chi_\mu(C_i) \overline{\chi_\nu(C_i)} = \delta_{\mu\nu}$

columns: $\sum_\mu \overline{\chi_\mu(C_i)} \chi_\mu(C_j) = \frac{|G|}{m_i} \delta_{ij}$

- <u>Tensor products of representations</u>.

Recall the tensor product of reps:

$$\begin{cases} V & \text{carrier space, } \dim n \quad \{v_1, v_2, \cdots v_n\} \\ W & m \quad \{w_1, w_2, \cdots w_m\} \end{cases}$$

$V \otimes W$: $\underline{\dim n \cdot m}$ basis $\{ v_i \otimes w_j, \ 1 \le i \le n, \ 1 \le j \le m \}$

$$\sum_i a_i v_i \otimes \sum_j b_j w_j = \sum_{ij} a_i b_j \, v_i \otimes w_j$$

G action: $g(v \otimes w) = (g \cdot v) \otimes (g \cdot w)$

$$[(T_1 \otimes T_2)(g)](v \otimes w) = T_1(g) \cdot v \otimes T_2(g) w$$

$$\chi_{T_1 \otimes T_2} = \chi_{T_1} \cdot \chi_{T_2}$$

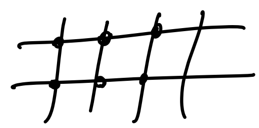
Tensor products appear naturally in physics: e.g. multiparticle states from single particle ones.

1. particles of angular momentum j_1 & j_2
   spin

   e.g. superconductivity

   $$\tfrac{1}{2} \otimes \tfrac{1}{2} \to 0 \oplus 1$$

2. many-body problem.



each site has a local Hilbert space

$$\mathcal{H}_i = \text{span}\{\phi, |\uparrow\rangle, |\downarrow\rangle, |\uparrow\downarrow\rangle\}$$

$$\underline{\mathcal{H} = \bigotimes_i^n \mathcal{H}_i} \qquad \dim(\mathcal{H}) = \underline{4^n}$$

$\hookrightarrow$ <u>in general</u>

$$G \otimes U(1) \otimes SU(2)$$

$\uparrow$

crystal

1. what are the irreps ?
2. <u>symmetry breaking</u> / phase transition

$$G \to \text{subgroup } H.$$

Let (T_1, V_1) and (T_2, V_2) be two representations with isotypic decompositions

$$V_1 = \bigoplus_\mu a_\mu V^\mu \quad . \quad V_2 = \bigoplus_\nu b_\nu V^\nu$$

$$\underline{V_1 \otimes V_2} = \bigoplus a_\mu b_\nu \underline{V^\mu \otimes V^\nu}$$

↳ in general reducible

$$V^\mu \otimes V^\nu \cong \bigoplus_\lambda \overbrace{\mathrm{Hom}_G(V^\lambda, V^\mu \otimes V^\nu)}^{D^\lambda_{\mu\nu}} \otimes V^\lambda =: \bigoplus_\lambda N^\lambda_{\mu\nu} V^\lambda$$

$$\mathrm{Hom}_G(V^\lambda, V^\mu \otimes V^\nu) \cong \bigoplus_{\lambda'} \mathrm{Hom}_G(V^\lambda, D^{\lambda'}_{\mu\nu} \otimes V^{\lambda'})$$

$$\cong \bigoplus_{\lambda'} D^{\lambda'}_{\mu\nu} \otimes \underbrace{\mathrm{Hom}_G(V^\lambda, V^{\lambda'})}_{\delta_{\lambda\lambda'} 1_\lambda}$$

$$= D^\lambda_{\mu\nu}$$

$$N^\lambda_{\mu\nu} = \dim_K \mathrm{Hom}_G(V^\lambda, V^\mu \otimes V^\nu) = \dim_K D^\lambda_{\mu\nu}$$

Finite groups : $N^\lambda_{\mu\nu}$ a non-negative integer.

SU(2) : "Clebsh-Gordan coefficient"

in general : "fusion coefficient."

Take the character on both sides:

$$\chi_\mu \cdot \chi_\nu = \sum_\lambda N^\lambda_{\mu\nu} \chi_\lambda$$

ortho. of character ; take inner product

$$N_{\mu\nu}^{\lambda} = \langle \chi_\lambda , \chi_\mu \chi_\nu \rangle$$

Finite groups:

$$N_{\mu\nu}^{\lambda} = \frac{1}{|G|} \sum_{g\in G} \chi_\mu(g)\, \chi_\nu(g)\, \overline{\chi_\lambda(g)}$$

$$= \frac{1}{|G|} \sum_{\{C_i\}} m_i\, \chi_\mu(C_i)\, \chi_\nu(C_i)\, \overline{\chi_\lambda(C_i)}$$

$m_i = |C_i|$

easy to see that $N_{\mu\nu}^{\lambda} = N_{\nu\mu}^{\lambda}$

Define: $S_{\mu i} := \sqrt{\frac{m_i}{|G|}}\, \chi_\mu(C_i)$

$$N_{\mu\nu}^{\lambda} = \sum_i \frac{S_{\mu i} S_{\nu i} \overline{S_{\lambda i}}}{S_{1 i}} \qquad S_{1i} = \sqrt{\frac{m_i}{|G|}} \cdot 1$$

$\hookrightarrow$ trivial rep.

"Verlinde formula" in CFT.

Example. 1. ρ_m rep of $\mathbb{Z}/N\mathbb{Z}$.

$$\rho_m \otimes \rho_n \cong \rho_{m+n}$$

$$\rho_m(\bar{l}) = e^{i2\pi m l/N}$$

$$S_{\mu i} = \sqrt{\frac{1}{|G|}}\, e^{i2\pi \mu i/N}.$$

$$\sum \frac{S_{\mu i} S_{\nu i} \overline{S_{\lambda i}}}{\sqrt{\frac{1}{|G|}}} = 1 = N_{\mu\nu}^{\lambda}$$

2. irreps of S_3 $(g \in S_3)$

V^+: $\rho^+(g) = 1$

V^-: $\rho^-(g) = \mathrm{sgn}(g) = \pm 1$

V^2: 2-dim irrep

	$[1]$	$3[(12)]$	$2[(123)]$
1^+	1	1	1
1^-	1	-1	1
2	2	0	-1

$$N_{\mu\nu}^{\lambda} = \frac{1}{|G|} \sum_{\{C_i\}} m_i \chi_\mu(C_i) \chi_\nu(C_i) \overline{\chi_\lambda(C_i)}$$

$$N_{1^+,\nu}^{\lambda} = \frac{1}{|G|} \sum \underline{m_i \chi_\nu(C_i) \overline{\chi_\lambda(C_i)}}$$

$$= \delta_{\nu\lambda}$$

① $V^+ \otimes V^\nu \cong \oplus_\lambda \delta_{\nu\lambda} V^\lambda = V^\nu$

HW

② $V^- \otimes V^- \cong V^+$

③ $V^- \otimes V^2 \cong V^2$

④ $V^2 \otimes V^2 \cong V^+ \oplus V^- \oplus V^2$

Now consider tensor product of 3 irreps.

$$(V^\mu \otimes V^\nu) \otimes V^\lambda = V^\mu \otimes (V^\nu \otimes V^\lambda)$$

$$\cong \left(\bigoplus_\alpha D^\alpha_{\mu\nu} \otimes V^\alpha\right) \otimes V^\lambda \qquad \cong V^\mu \otimes \left(\bigoplus_\beta D^\beta_{\nu\lambda} \otimes V^\beta\right)$$

$$\cong \bigoplus_\kappa \left(\bigoplus_\alpha D^\alpha_{\mu\nu} \otimes D^\kappa_{\alpha\lambda}\right) \otimes V^\kappa \qquad \cong \bigoplus_\kappa \left(\bigoplus_\beta D^\beta_{\nu\lambda} \otimes D^\kappa_{\mu\beta}\right) \otimes V^\kappa$$

$$\sum_\alpha N^\alpha_{\mu\nu} N^\kappa_{\alpha\lambda} = \sum_\beta N^\kappa_{\mu\beta} N^\beta_{\nu\lambda}$$

To put representations into a broader context, they can be described using

<u>Category theory</u>.

(representations belong to fusion category
tensor category)

<u>Category</u> C $\begin{cases} ob(C) \\ hom(X, Y) \quad X, Y \in ob(C) \end{cases}$

① $id_X \in hom(X, X)$

② Composition: $hom(Y, Z) \times hom(X, Y) \to hom(X, Z)$

$$(g, f) \mapsto g \circ f$$

$$(h \circ g) \circ f = h \circ (g \circ f)$$

Example: Group

$g \cdot f$

Monoidal category

1. $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$

2. $\mathbb{1} \in \mathcal{C}$

3. $(x \otimes y) \otimes z \xrightarrow{\alpha_{x,y,z}} x \otimes (y \otimes z)$ ↗ associator

   (isomorphism)

4. $1 \otimes x \xrightarrow{\lambda_x} x \quad , \quad x \otimes 1 \xrightarrow{\rho_x} x$

---

fusion / tensor category.

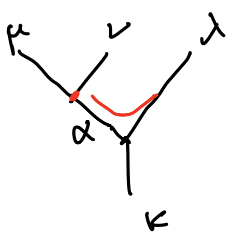
Above example:

$$(V^\mu \otimes V^\nu) \otimes V^\lambda \quad = \quad V^\mu \otimes (V^\nu \otimes V^\lambda)$$

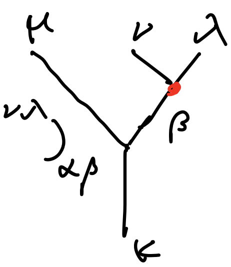
$$\cong \left(\oplus_\alpha D^\alpha_{\mu\nu} \otimes V^\alpha\right) \otimes V^\lambda \qquad \cong V^\mu \otimes \left(\oplus_\beta D^\beta_{\nu\lambda} \otimes V^\beta\right)$$

$$\cong \oplus_\kappa \left(\oplus_\alpha D^\alpha_{\mu\nu} \otimes D^\kappa_{\alpha\lambda}\right) \otimes V^\kappa \qquad \cong \oplus_\kappa \left(\oplus_\beta D^\beta_{\nu\lambda} \otimes D^\kappa_{\mu\beta}\right) \otimes V^\kappa$$

String diagram :



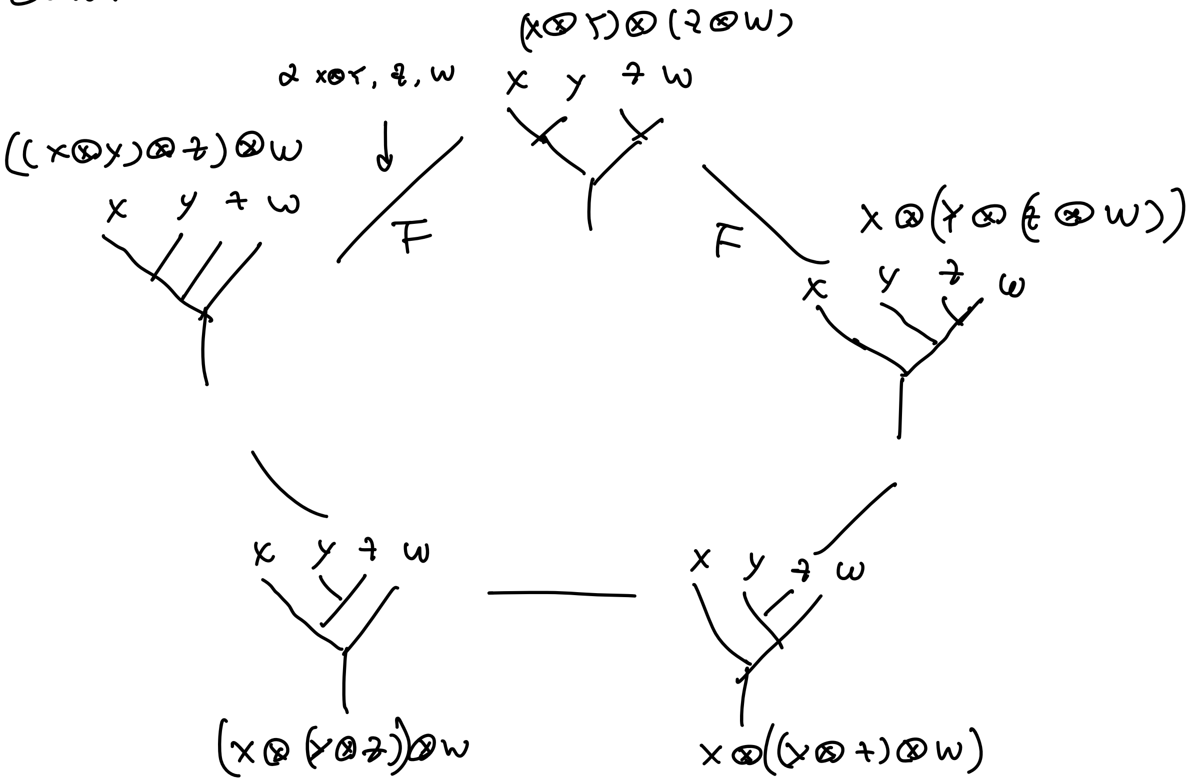
$= \sum_\beta (F^{\mu\nu\lambda}_\kappa)_{\alpha\beta}$



"F-move"

$\alpha_{\mu,\nu,\lambda}$

pentagon coherence relation:



Add braiding : b_{xy} $\quad x \otimes y \longrightarrow y \otimes x$.

braided monoidal category : anyons

knots