8.8 Schur's lemmas

Lemma 1. Let G be any group. Let V_1, V_2 be vector spaces over any field k, s.t. they are carrier spaces of irreps of G.

If $A : V_1 \to V_2$ is an intertwiner between these two irreps, then A is either zero or an isomorphism of representations.
(invertible)

⌈ recall an intertwiner is a morphism of G-actions

$$\begin{tikzcd} V_1 \arrow[r, "A"] \arrow[d, "T_1(g)"'] & V_2 \arrow[d, "T_2(g)"] \\ V_1 \arrow[r, "A"] & V_2 \end{tikzcd}$$

$$T_2(g) A = A T_1(g)$$ ⌋

Proof. $\ker A := \{ v_1 \in V_1 \mid A(v_1) = 0 \}$

$\operatorname{im} A := \{ v_2 \in V_2 \mid \exists\, v_1 \in V_1,\ \text{s.t. } v_2 = A(v_1) \}$

A an intertwiner, then

① $v_1 \in \ker A$. $\quad A(T_1(g)\cdot v_1) = T_1(g)(Av_1) = 0$

$T_1(g)\, v_1 \in \ker A,\ \forall g \in G$

$\Rightarrow \ker A$ is an invariant subspace (of V_1)

② $v_2 \in \operatorname{im} A \quad T_2(g)\, v_2 \overset{\exists v_1}{=} T_2(g) A \cdot v_1 = A(T_1(g)\cdot v_1) \in \operatorname{im} A.$

$\Rightarrow \operatorname{im} A$ inv. subspace. (of V_2)

V_1 is an irrep. $\Rightarrow$ $\ker A$ either 0 or V_1

if $\ker A = V_1$ then $A = 0$ (a)

else $\ker A = 0$, A is then injective. (b.1)

($AV_1 = AV_2 \Rightarrow A(V_1 - V_2) = 0 \Rightarrow V_1 - V_2 \in \ker$)

which means $\operatorname{im} A$ cannot be $0 \Rightarrow \operatorname{im} A = V_2$

A is also surjective (b.2)

$\Rightarrow$ A is an isomorphism

skip

Now, set $V_1 = V_2 = V$. Then all $A: V \to V \in \operatorname{End}(V) := \operatorname{Hom}_G(V, V)$

form a endomorphism ring $(+, \times)$

$$\begin{cases} (A_1 \cdot A_2) v = A_1 \cdot (A_2 \cdot v) \\ (A_1 + A_2) v = A_1 v + A_2 v \end{cases}$$

Shur's lemma, A is invertible. a multiplication inverse is defined. $(AA^{-1} = 1)$

$\Rightarrow$ division ring / algebra. (non com. $\Rightarrow$ skew field, commutative $\Rightarrow$ field)

Examples: $\mathbb{R}$. $\mathbb{C}$. $\mathbb{H} \cong \operatorname{span}\{\mathbb{1}, i\sigma^k\}$
$\hookrightarrow$ quaternions

Lemma 2: Suppose (T, V) is an irrep of G and V a finite-dim. complex vector space.

$A \in \operatorname{Hom}_G(V, V)$ $A: V \to V$ an $\mathbb{C}$-linear intertwiner. $(A\,T(g) = T(g) A, \forall g \in G)$

Then A is proportional to the identity transformation:

$$A(v) = \lambda v \quad (\lambda \in \mathbb{C})$$

Proof. $\exists v$. s.t. $Av = \lambda v$. i.e. there is always a nonzero eigenvector. it follows from the fact the $p(x) = \det(x\mathbb{1} - A)$ always has a root in $\mathbb{C}$. (fundamental theorem of algebra)

Then the eigenspace $C = \{w : Aw = \lambda w\}$ is non zero.

$$A\,T(g)\,w = T(g)\,A w = \lambda\, T(g)\, w \qquad \forall g \in G$$

$\Rightarrow$ C is an invariant subspace

$\overset{\text{irrep}}{\Rightarrow}$ $C = V$. $\Rightarrow$ $A = \lambda \mathbb{1}$

Remarks.

1. If V_2 is completely reducible as $V_2 = W_1 \oplus W_2$

$$\mathrm{Hom}_G(V_1, W_1 \oplus W_2) \cong \mathrm{Hom}_G(V_1, W_1) \oplus \mathrm{Hom}_G(V_1, W_2)$$

$$\mathrm{Hom}_G(V^{(\mu)}, V) \cong \mathrm{Hom}_G\Big(V^{(\mu)}, \bigoplus_\nu K^{a_\nu} \otimes V^{(\nu)}\Big)$$

$$= \bigoplus_\nu K^{a_\nu} \otimes \underbrace{\mathrm{Hom}_G(V^{(\mu)}, V^{(\nu)})}_{\lambda\,\delta_{\mu\nu} \text{ by Schur's 1. lemma}}$$

$$= \underline{\underline{K^{a_\mu}}} \otimes \underbrace{\mathrm{Hom}_G(V^{(\mu)}, V^{(\mu)})}_{\propto \mathbb{C}. \text{ 2.nd lemma}}$$

($\because$ $\mathrm{Hom}_G(V_1, V_2 \otimes V_3)$

$= V_2 \otimes \mathrm{Hom}_2(V_1, V_3)$

if G acts trivially on V_2)

K^{a_μ} is the linear space of G-invariant maps from $V^{(\mu)} \to V$. They can be thought as intertwiners.

There is a canonical equivariant map

$$\mathrm{Hom}_G(V^{(\mu)}, V) \otimes V^{(\mu)} \longrightarrow V$$

$$A \otimes v \longmapsto A(v) \in V.$$

and the isomorphism

$$\underset{=}{\oplus} \underbrace{\mathrm{Hom}_G(V^{(\mu)}, V)}_{\cong K^{a_\mu}} \otimes V^{(\mu)} \cong V$$

2. 1 is directly related to block diagonalization of Hamiltonians.

$\mathcal{H}$ the Hilbert space is a representation of some symmetry group G. and completely reducible

$$\mathcal{H} \cong \oplus_\mu \mathcal{H}^{(\mu)}$$

$$\mathcal{H}^{(\mu)} := D_\mu \otimes V^{(\mu)}$$

H is a Hamiltonian : $H : \mathcal{H} \to \mathcal{H}$.

is an intertwiner. (commutes with G)

By schur's lemma,

$$H \cong \oplus_\mu \underbrace{H^{(\mu)}}_{\text{Hermitian operators on } D_\mu \text{ not determined by symmetry}} \otimes \overbrace{1_{V^{(\mu)}}}^{\text{Shur's lemma}}$$

We are familiar with this:

For H with certain symmetry, with a suitable basis transformation / choice

$$SHS^{-1} = \begin{pmatrix} H_{11} & 0 & 0 & \\ 0 & H_{22} & 0 & \\ 0 & 0 & H_{33} & \\ & & & \ddots \end{pmatrix}$$

Block diagonal, with blocks labeled by some "quantum number".

If an operator O, $[O, G] = 0$

$$O = \oplus_\mu O^{(\mu)} \times 1_{V^{(\mu)}}$$

$$\langle \varphi_1, O\varphi_2 \rangle = 0 \quad \text{if} \quad \varphi_1 \in \mathcal{H}^{(\mu)} \quad \varphi_2 \in \mathcal{H}^{(\nu)} \quad (\mu \neq \nu)$$

<u>Example.</u> $\mathbb{Z}_2$ action on $\mathcal{H}$.

$T^2 = 1$ $\qquad$ $|1\rangle$ $\quad$ $|2\rangle$

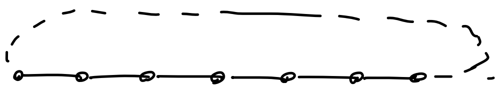
$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

$$= \mathbb{K}^n \otimes \rho_+ \oplus \mathbb{K}^n \otimes \rho_-$$

$$\rho_+ = \{\tfrac{1}{\sqrt{2}}(|1\rangle + |2\rangle)\} \qquad \rho_- = \{\tfrac{1}{\sqrt{2}}(|1\rangle - |2\rangle)\}$$

generalize $T^2=1$ $\Rightarrow$ 1D tight-binding model

$$\hat H = -t\sum_{\langle ij\rangle} \overset{|i\rangle\langle j|}{a_i^\dagger a_j} + h.c.$$



$$G = C_N = \langle T \mid T^N = 1\rangle$$

$$\tilde T = \mathrm{diag}\{e^{ik_1}, e^{ik_2}, \dots e^{ik_N}\}$$

$$\mathcal{H} = \oplus \mathcal{H}^l \qquad \mathcal{H}^l = \sum_j e^{ik_l\cdot j}|j\rangle$$

$$\tilde T \sum e^{ik_l\cdot j}|j\rangle = \sum_j e^{ik_l j}|j+1\rangle$$
$$= e^{-ik_l}\sum_j e^{ik_l(j+1)}|j+1\rangle$$

$$\langle k_l|H|k_m\rangle = -2t\cos k_l\,\delta_{lm}$$

in general

2nd lemma

$$H_{(\mu_1,i_1,\alpha_1),(\mu_2,i_2,\alpha_2)} = \delta_{\mu_1\mu_2}\,\delta_{i_1 i_2}\,h_{\alpha_1\alpha_2}$$

$i_j = 1,\dots,n_\mu$

1st lemma

8.9 Pontryagin duality ~~skipped~~