P 14. $D = \{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, z = e^{i\theta} \} \cong U(1)$

(a) $d \in D$ $\quad u = \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} \in SU(2)$ $\quad |\alpha|^2 + |\beta|^2 = 1$

$$u d u^{-1} = \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & \bar z \end{pmatrix} \begin{pmatrix} \bar\alpha & -\beta \\ \bar\beta & \alpha \end{pmatrix}$$

$$= \begin{pmatrix} |\alpha|^2 z + |\beta|^2 \bar z & \alpha\beta(-z + \bar z) \\ \bar\alpha\bar\beta(-z + \bar z) & |\alpha|^2 \bar z + |\beta|^2 z \end{pmatrix} \in D$$

$\Rightarrow \alpha = 0$ or $\beta = 0$

$$N_{SU(2)}(D) = \underline{\{ \begin{pmatrix} z & 0 \\ 0 & \bar z \end{pmatrix}, z \in U(1) \}} \cup \{ \begin{pmatrix} 0 & -\bar z \\ z & 0 \end{pmatrix}, z \in U(1) \}$$

$$= \underline{D} \cup \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} D$$

(b) $N_{SU(2)}(D)/D = \{ \begin{pmatrix} z & 0 \\ 0 & \bar z \end{pmatrix} D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} D,$

$$\begin{pmatrix} 0 & -\bar z \\ z & 0 \end{pmatrix} D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} D \} \cong \mathbb{Z}_2$$

(c) $$\begin{pmatrix} \alpha & 0 \\ 0 & \bar\alpha \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & \bar z \end{pmatrix} \begin{pmatrix} \bar\alpha & 0 \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & \bar z \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\bar\alpha \\ \alpha & 0 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & \bar z \end{pmatrix} \begin{pmatrix} 0 & \bar\alpha \\ -\alpha & 0 \end{pmatrix} = \begin{pmatrix} \bar z & 0 \\ 0 & z \end{pmatrix}$$

(d) should at least contain $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. and some

$a = \begin{pmatrix} 0 & z \\ -\bar z & 0 \end{pmatrix}$. then it contains $a^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

& $a^3 = \begin{pmatrix} 0 & -\bar{z} \\ \bar{z} & 0 \end{pmatrix}$. it's not isomorphic to $\mathbb{Z}_2$.

($N_{SU(2)}(D)/D$ is not a subgroup of $SU(2)$ or $N_{SU(2)}(D)$)

P15.

G-set X. $\phi : G \longrightarrow S_X$

(a) effective $\iff$ ϕ injective, i.e. $\phi(g) = 1$ iff $g = 1$

$\forall g \neq 1.\ \exists x$ s.t. $gx_1 = x_2 \neq x_1 \iff \forall g \neq 1.\ \phi(g)$ is a nontrivial permutation

$\phi(g) \neq 1$

(b) $\{g_i\}$ are ineffective $\forall g \in G$

$g_i g x = g_i \cdot x' = x'$ $\quad (\forall x \in X)$

$g g_i x = g x = x'$

$\Rightarrow g_i g = g g_i \quad \forall g \in G$

trivial to show $\{g_i\}$ is a group

$\Rightarrow \quad H = \{ g_i : g_i x = x \ \forall x \in X \} \trianglelefteq G$

(c) define the action $G/H \times X \longrightarrow X$

$(gH) \cdot x := gx$

$\forall x \in X$, s.t $(gH)x = x \iff gx = x \iff g \in H \iff gH = H = 1_{G/H}$

P16. X a finite G set.

G - action transitive $\Rightarrow$ one orbit $= X$

Burnside's lemma $\Rightarrow$ $|G| = \sum_{g \in G} |X^g|$

If all g's have fixed points. $\sum_{g\in G} |X^g| \geq \sum_{g \in G} 1 = |G|$

equality holds iff $\forall g.\ |X^g| = 1$

But $|X^e| = |G| > 1$ $\left(\because \underbrace{|G| = |X|\,|G_x|}_{\text{stabilizer-orbit theorem}} \geq |X| > 1 \right)$

$\Rightarrow |X^g| = 0$ for some g.