P20. Real rep: $M_{\bar{T}}(g) = S M_T(g) S^{-1} \quad \forall g \quad (*)$

$$\bar{T}(g)\cdot \bar{v_i} = [M_{\bar{T}}(g)]_{ji}\, \bar{v_j} = \overline{M^*_{\bar{T}}(g)_{ji}\, v_j}$$

$$\equiv \overline{T(g)\, v_i} = \overline{M_T(g)_{ji}\, v_j}$$

$$\Rightarrow M_T(g) = M^*_{\bar{T}}(g)$$

$$\overset{(*)}{\Rightarrow} M^*_T(g) = S M_T(g) S^{-1}$$

P21 . (1) $(\tilde{T}(g)\cdot\phi)(v) = T_W(g)\cdot\phi(T_V(g^{-1})v)$

$$[\tilde{T}(g_1)(\tilde{T}(g_2)\,\phi)](v) = T_W(g_1)\cdot(\tilde{T}(g_2)\phi)\,(T_V(g_1^{-1})\cdot v)$$

$$= T_W(g_1)\, T_W(g_2)\,\phi\,(T_V(g_2^{-1})\, T_V(g_1^{-1})\, v)$$

$$= T_W(g_1 g_2)\,\phi\,(T_V((g_1 g_2)^{-1})\, v)$$

$$= [\tilde{T}(g_1 g_2)\,\phi](v)$$

(2) $V^* := \mathrm{Hom}(V, K) \cong V^* \otimes K$ $\quad$ T_W acts trivially on K.

Rep. in (1) becomes

$$(T^*(g)\, v_i^*)(v_j) = v_i^*(T(g)^{-1}\cdot v_j)$$

which is exactly the dual rep.

(3) V with basis $\{v_i\}$. W $\{w_a\}$

$$\mathrm{Hom}(V, W) \simeq \mathrm{Mat}_{m\times n}(\mathbb{C})$$

$$(\tilde{T}(g)\cdot\phi)(v) = T_W(g)\cdot\phi(T_V(g^{-1})v)$$

take $\phi = e_{ai}$, $e_{ai}(v_j) = w_a\delta_{ij}$ $\quad T v_j = \sum M_{ij} v_i$

$$\begin{aligned}
\forall v_j :\quad [\tilde{T}(g)e_{ai}](v_j) &= T_W(g)\Big\{e_{ai}\big(\sum_k [M(g)^{-1}]_{kj} v_k\big)\Big\} \\
&= T_W(g)\cdot\Big(\sum_k [M(g)^{-1}]_{kj}\, e_{ai}(v_k)\Big) \\
&= T_W(g)\Big(\sum_k [M(g)^{-1}]_{kj}\, w_a \delta_{ik}\Big) \\
&= T_W(g)\cdot [M(g)^{-1}]_{ij}\, w_a \\
&= [M(g)^{-1}]_{ij} \sum_b M(g)_{ba}\, w_b \\
&= \sum_b [M(g)]_{ba} [M(g)^{-1}]_{ij}\, e_{bj}(v_j) \\
&= \sum_b [M(g)]_{ba} [M(g)^{tr,-1}]_{ji}\, e_{bj}(v_j)
\end{aligned}$$

$$\Rightarrow \tilde{T}(g)e_{ai} = \sum_b [M(g)]_{ba} [M(g)^{tr,-1}]_{ki}\, e_{bk}$$

P22. $\quad \langle v, w\rangle = \dfrac{1}{|G|}\sum_g \langle T(g)v, T(g)w\rangle$