Recap. Group algebra

1. group elements : operators/actions & vectors/objects

$$\mathbb{C}[G] = \{ \sum_g \alpha_g \cdot g \mid \alpha_g \in \mathbb{C} \}$$

$$\begin{cases} x+y = \sum_g [x(g) + y(g)] \cdot g \\ \alpha x = \sum_g [\alpha x(g)] \cdot g \\ x \cdot y = \sum_g x(g)\, y(g^{-1}k) \cdot k \end{cases}$$

$$L(h) \cdot x \overset{\Delta}{=} L(h) \sum x(g) \cdot g = \sum x(g)(hg) = \sum_g x(h^{-1}g)\, g$$

$$\Rightarrow [L(h) \cdot x](g) = x(h^{-1}g)$$

$$x = \delta_g \Rightarrow h \cdot \delta_g = \delta_{hg}$$

2. Define class operators

$$C_i = \sum_{g \in C_i} g \quad \in \mathrm{End}_G(\mathbb{C}[G])$$

then $[C_i, g] = 0 \quad (\forall g \in G)$. intertwiners

All $\{C_i\}$ commute $\Rightarrow$ can be simultaneously diagonalized. We can represent them within the center of $\mathbb{C}[G]$: span $\{C_i\}$

$$\boxed{\hat{C}_i \cdot \hat{C}_j = \sum_k D_{ij}^k \hat{C}_k} \qquad D_{ij} \text{ are the}$$

recall matrix rep:

$$g \cdot e_i = \sum_j M_{ji}(g) e_j$$

$$\uparrow \quad \uparrow$$

$$\hat{C}_j \; C_i = \sum_k D(j)_{ki} C_k$$

what are the eigenvectors?

$$\hat{C}_j \vec{\phi}^{(\mu)} = \lambda_\mu \vec{\phi}^{(\mu)}$$

We haven shown that assuming nondegeneracy

What if there is degeneracy? Find another C_i that splits the degeneracy.
$\Rightarrow$ We will see later that there is no degeneracy if all $\{C_i\}$ are diagonalized. (CSCO-I)

$$\hat{C}_j \vec{\phi}^\mu \vec{\phi}^\nu = (\lambda^\mu \vec{\phi}^\mu)\vec{\phi}^\nu = \phi^\mu(\lambda^\nu \vec{\phi}^\nu) = \lambda^\mu \delta_{\mu\nu} \vec{\phi}^\mu \vec{\phi}^\nu$$

$\{\vec{\phi}^\mu\}$ behave like projectors!

$$\hat{C}_i \vec{\phi} = \lambda_i \vec{\phi} \quad \text{then} \quad \vec{\phi} = \sum_{i=1}^{r} \alpha_i C_i$$

$$C_i \vec{\phi} = C_i \sum_j \alpha_j C_j = \sum_j \alpha_j C_i C_j = \sum_{jk} \alpha_j D_{ij}^k C_k$$

$$\equiv \lambda_i \vec{\phi} = \lambda_i \sum_j \alpha_j C_j = \sum_k (\lambda_i \alpha_k) C_k$$

$$\Rightarrow \sum_j \alpha_j D_{ij}^k = \lambda_i \alpha_k$$

$$\vec{\phi}^2 = \sum_{ij} \alpha_i \alpha_j C_i C_j = \sum_{ijk} \alpha_i \alpha_j D_{ij}^k C_k = \sum_k \left(\sum_{ij} \alpha_i \alpha_j D_{ij}^k\right) C_k$$

$$= \sum_k \left[\sum_i \alpha_i (\lambda_i \alpha_k)\right] C_k = \underbrace{\sum_i \lambda_i \alpha_i}_{a} \underbrace{\sum_k \alpha_k C_k}_{\vec{\phi}} \propto \vec{\phi}$$

$\tilde{\vec{\phi}} = \frac{1}{a} \vec{\phi}$, then $\tilde{\phi}^2 = \tilde{\phi}$ idempotent.

We will look at these eigenvectors more closely.

8.13.3 Construction of character tables (cont.)

We know that C_i's are intertwiners, which means when restricted to a specific irrep:

$$\underline{\underline{C_i^\mu = \lambda_i^\mu \cdot \mathbb{1}_{V^\mu}}} \qquad \text{(Schur's lemma)}$$

Taking trace/character.

$$\text{LHS} = \mathrm{Tr}_{V^\mu} \sum_{g \in C_i} g = m_i \, \chi^\mu([C_i])$$

$$\text{RHS} = \lambda_i^\mu \cdot n_\mu$$

$$\Rightarrow \boxed{\lambda_i^\mu = \frac{m_i}{n_\mu} \chi^\mu([C_i])}$$

These are actually the eigen values. Why?

$$C_j C_i = \sum [D(j)]_{ki} C_k$$

then taking the characters on V^μ

wrong in class.

$$\chi_\mu(C_j)\chi_\mu(C_i) = \sum_k (D_j)_{ki} \chi_\mu(C_k)$$

$$= \lambda_i^\mu \chi_\mu(C_j)$$

i.e. $\sum_k (D_j)_{ki} \chi_\mu(C_k) = \lambda_j^\mu \chi_\mu(C_i)$

take $\vec{x}_\mu = \begin{pmatrix} \chi_\mu(C_1) \\ \chi_\mu(C_2) \\ \vdots \\ \chi_\mu(C_r) \end{pmatrix}$ then

$D^T(j) \vec{x}_\mu = \lambda_j \vec{x}_\mu$ in matrix form.

$$\mathrm{Tr}_\mu(C_i C_j) = \mathrm{Tr}_\mu(C_i \cdot \lambda_j^\mu \mathbb{1}_{V^\mu})$$

$$= \lambda_j^\mu \, \mathrm{Tr}_\mu(C_i)$$

$$= \lambda_j^\mu \, m_i \, \chi_\mu([C_i])$$

$$\sum_k \mathrm{Tr}_\mu([D_i]_{kj} C_k) = \sum_k [D_i]_{kj} \, m_k \, \chi_\mu([C_k])$$

$$\Rightarrow \sum_k (D_i)_{kj} \vec{x}_\mu = \lambda_j^\mu \vec{x}_\mu$$

$$\vec{x}_\mu = \begin{pmatrix} m_1 \chi_\mu([C_1]) \\ m_2 \chi_\mu([C_2]) \\ \vdots \\ m_r \chi_\mu([C_r]) \end{pmatrix}$$

remaining two unknowns: $\underline{n_\mu, \chi_\mu}$

$$\frac{1}{|G|}\sum_{C_i} m_i \chi_\mu(C_i)\overline{\chi_\nu(C_i)} = \delta_{\mu\nu}$$

$$\Rightarrow \frac{1}{|G|}\sum_{C_i} m_i \lambda_i^\mu \overline{\lambda_i^\nu} = \delta_{\mu\nu}\left(\frac{m_i}{n_\mu}\right)^2 \quad (\equiv \langle \lambda_i^\nu, \lambda_i^\mu \rangle)$$

$$\Rightarrow \begin{cases} n_\mu = \dfrac{m_i}{\sqrt{\langle \lambda_i^\mu, \lambda_i^\mu \rangle}} \\ \chi_\mu = \dfrac{\lambda_i^\mu}{\sqrt{\langle \lambda_i^\mu, \lambda_i^\mu \rangle}} \end{cases}$$

Now, back to the assumption about no degeneracy

once diagonalized all class operators:

$$\boxed{\lambda_i^\mu = \frac{m_i}{n_\mu}\chi^\mu([C_i])}$$

$$\vec{\lambda}^\mu = \{\lambda_1^\mu, \lambda_2^\mu \cdots \lambda_r^\mu\}, \quad \vec{\lambda}^\nu = \{\lambda_1^\nu, \lambda_2^\nu, \cdots \lambda_r^\nu\}$$

if $\underline{\forall j}$, $\lambda_j^\mu = \lambda_j^\nu$, then

$$\frac{m_j}{n_\mu}\chi^\mu([C_j]) = \frac{m_j}{n_\nu}\chi^\nu([C_j])$$

$\Rightarrow$ which means two characters are proportional to each other cannot be because they should be orthogonal!

Diagonalizing all class operators $\Rightarrow$ full knowledge of $\underline{\chi_\mu, n_\mu}$

see 陈金全 for examples of finding / using a minimal CS Co-I (subset of class operators).

We will just diagonalize all operators.

$c_i^{(\mu)} = \frac{m_i}{n_\mu}\chi_\mu([C_i])\cdot 1_{V^\mu}$

$$\hookrightarrow \quad \frac{m_i}{n_\mu}\chi_\mu([C_i])\,\frac{m_j}{n_\mu}\chi_\mu([C_j]) = \sum_{k=1}^{r} C_{ij}^{k}\,\frac{m_k}{n_\mu}\chi_\mu([C_k])$$

$$\underline{m_i\chi_\mu([C_i])\,m_j\chi_\mu([C_j]) = n_\mu\sum_{k=1}^{r} C_{ij}^{k}\,m_k\chi_\mu([C_k])}$$

Now introduce a set of auxiliary variables $\{y^i, i=1,\cdots,r\}$

(So we can differentiate between different C_i's: $C_i \to C_i y^i$)

$$\sum \text{LHS}: \quad \sum_{i=1}^{r} m_i m_j \chi_\mu([C_i])\chi_\mu([C_j])\,y^i = \sum_{i=1}^{r}(\psi_i y^i)\psi_j \qquad (\psi_i := m_i\chi_\mu([C_i]))$$

$$\sum \text{RHS}: \quad \sum_{i=1}^{r} n_\mu \sum_{k=1}^{r} C_{ij}^{k}\, m_k\chi_\mu([C_k])\,y^i = n_\mu\sum_{k=1}^{r} L_j^k\,\psi_k$$

Define $\lambda = \frac{1}{n_\mu}\sum_{i=1}^{r}\psi_i y^i$ $\qquad (L_j^k = \sum_i C_{ij}^k y^i)$

$$\Rightarrow \quad \sum_{k=1}^{r} L_j^k\,\psi_k = \lambda\,\psi_j$$

Solving the eigen problem $(L - \lambda\mathbb{1})\psi = 0$

and obtain a set of eigenvalues $\{\lambda_\mu\}$

$$(*) \quad \underline{\lambda_\mu = \frac{1}{n_\mu}\sum_{i=1}^{r}\overbrace{m_i\chi_\mu([C_i])}^{\psi_i}\,y^i} \qquad \mu = 1,\cdots,r$$

Note if we set $y^j = \delta_{ij}$, we recover our earlier λ_i^μ.

Now recall the orthogonality relation:

$$\frac{1}{|G|}\sum_{C_i} m_i \chi_\mu(C_i)\overline{\chi_\nu(C_i)} = \delta_{\mu\nu} \quad \text{(ortho. of rows)}$$

$$\overset{\mu=\nu:}{\Rightarrow} \sum_{i=1}^{r} m_i |\chi_\mu([C_i])|^2 = |G|$$

$$|G| = |\chi_\mu([C_1])|^2 \sum_{i=1}^{r} m_i \left|\frac{\chi_\mu([C_i])}{\chi_\mu([C_1])}\right|^2$$

$$= n_\mu^2 \sum_{i=1}^{r} m_i \left|\frac{\chi_\mu([C_i])}{n_\mu}\right|^2$$

$$\Rightarrow n_\mu = \left[\frac{|G|}{\sum_{i=1}^{r} m_i \left|\frac{\chi_\mu([C_i])}{n_\mu}\right|^2}\right]^{\frac{1}{2}}$$

← known from above (*)

Implementation in practice:

S_3 : E ; $(12), (13), (23)$; $(123), (132)$

① class operators: $C_1 = E$

$C_2 = (12)+(13)+(23)$ $\quad (12)(13) = (132)$

$C_3 = (123)+(132)$ $\quad (12)(123) = (1)(23)$

② class multiplication table:

	C_1	C_2	C_3
C_1	C_1	C_2	C_3
C_2	C_2	$\underline{3C_1+3C_3}$	$\underline{2C_2}$
C_3	C_3	$2C_2$	$\underline{2C_1+C_3}$

① explain underlined.
② symmetric (∵ abelian)

③ $L_j^k = \sum_i C_{ij}^k y^i$ 3×3 matrix

$$L_1^1 = C_{11}^1 y^1 + C_{21}^1 y^2 + C_{31}^1 y^3 = y^1 + 0 + 0$$

$$L_1^2 = \sum_i C_{i1}^2 y^i = y^2$$

$$L_1^3 = y^3$$

$$L_2^1 = \sum C_{i2}^1 y^i = 3y^2$$

$$L_2^2 = \sum C_{i2}^2 y^i \qquad L_2^3 = \sum C_{i2}^3 y^i$$

$$L_3^1 = \sum C_{i3}^1 y^i \qquad L_3^2 = \sum C_{i3}^2 y^i$$

$$L_3^3 = \sum C_{i3}^3 y^i$$

	C_1	C_2	C_3
C_1	C_1	C_2	C_3
C_2	C_2	$3C_1+3C_3$	$2C_2$
C_3	C_3	$2C_2$	$2C_1+C_3$

$$\hat{L} = \begin{pmatrix} y^1 & y^2 & y^3 \\ 3y^2 & y^1+2y^3 & 3y^2 \\ 2y^3 & 2y^2 & y^1+y^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} y^1 + \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix} y^2 + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix} y^3$$

$$\boxed{\lambda_\mu = \sum_{i=1}^{r} \frac{m_i \chi_\mu([C_i])}{n_\mu} y^i}$$

$$\begin{cases} \lambda_a = y^1 + 3y^2 + 2y^3 \\ \lambda_b = y^1 - 3y^2 + 2y^3 \\ \lambda_c = y^1 + 0y^2 - y^3 \end{cases}$$

$$n_\mu = \left[\frac{|G|}{\sum_{i=1}^{r} m_i \left| \frac{\chi_\mu([C_i])}{n_\mu} \right|^2} \right]^{\frac{1}{2}}$$

write in cols.

④ $\chi_a = n_a(1, 1, 1)$ $\qquad n_a = 1$

$\chi_b = n_b(1, -1, 1)$ $\qquad n_b = 1$

$\chi_c = n_c(1, 0, -\frac{1}{2})$ $\qquad n_c = \left[\frac{6}{1+3\cdot 0+2\cdot\frac{1}{4}}\right]^{\frac{1}{2}} = 2$

⑤ Character table

	$[1]$	$3[(12)]$	$2[(123)]$
1^+	1	1	1
1^-	1	-1	1
2	2	0	-1

Note that in the solution:

$$\begin{cases} \lambda_a = y^1 \boxed{+3y^2} + 2y^3 \\ \lambda_b = y^1 \boxed{-3y^2} + 2y^3 \\ \lambda_c = y^1 \boxed{+0y^2} - y^3 \end{cases}$$

The eigenvalues of $\hat{C}_2$ is non-degenerate.

This defines a set of unique eigenvectors that diagonalizes all $\hat{C}_i$. Which means $\hat{C}_2$ is a CSCO by itself.

$$\begin{array}{lllll} & C_1 & C_2 & C_3 & \quad () \quad (12) \quad (123) \\ \chi_{\mu_1} = n_{\mu_1}(& \frac{1}{1}, & \frac{3}{3}, & \frac{2}{2}) & = (1, 1, 1) \\ \chi_{\mu_2} = n_{\mu_2}(& \frac{1}{1}, & \frac{-3}{3}, & \frac{2}{2}) & = (1, -1, 1) \\ \chi_{\mu_3} = n_{\mu_3}(& \frac{1}{1}, & \frac{0}{3}, & -\frac{1}{2}) & = (2, 0, -1) \end{array}$$

$$\chi^{\mu}_i = n_\mu \frac{\lambda^\mu_i}{m_i}$$

$m_1 = 1 \quad m_2 = 3 \quad m_3 = 2$

↪ normalization:

$$\begin{aligned} n_{\mu_1} = n_{\mu_2} = 1 \\ n_{\mu_3} = 2 \end{aligned} \quad \Leftarrow \quad \begin{cases} \langle \chi_{\mu_1}, \chi_{\mu_1} \rangle = \frac{1}{6} n_{\mu_1}^2 \cdot 6 = 1 \\ \langle \chi_{\mu_2}, \chi_{\mu_2} \rangle = \frac{1}{6} n_{\mu_2}^2 \cdot 6 = 1 \\ \langle \chi_{\mu_3}, \chi_{\mu_3} \rangle = \frac{1}{6} n_{\mu_3}^2 (1 + 0 + \frac{1}{4} \times 2) \\ \qquad = \frac{1}{4} n_{\mu_3}^2 = 1 \end{cases}$$

Projectors:

$$\hat{C}_i \hat{C}_j = \sum_k [D_i]_{jk} C_k$$

$$\hat{C}_i \cdot \phi_\mu = \lambda_i^\mu \phi_\mu \qquad \phi_\mu = \sum_i \phi_\mu(C_i) C_i \equiv \phi_\mu^i C_i$$

$$\sum_j \phi_\mu^j \hat{C}_i \hat{C}_j = \lambda_i^\mu \sum_k \phi_\mu^k C_k$$

$$\Rightarrow \sum_{jk} \phi_\mu^j [D_i]_{jk} C_k = \lambda_i^\mu \sum_k \phi_\mu^k C_k$$

$$\Rightarrow \sum_k \left(\sum_j (D_i^T)_{kj} \phi_\mu^j \right) C_k = \sum_k \lambda_i^\mu \phi_\mu^k \cdot C_k$$

$$\Rightarrow \sum_j (D_i^T)_{kj} \phi_\mu^j = \lambda_i^\mu \phi_\mu^k$$

$$\sum_j (D_i^T - \lambda_i^\mu \delta_{jk}) \phi_\mu^j = 0$$

ϕ_μ are eigenvectors of D_i^T with basis $\{C_1, C_2, C_3\}$

$$D_2^T = \begin{pmatrix} & 3 & \\ 1 & & 2 \\ & 3 & \end{pmatrix}$$

$\lambda_2^{\mu_1} = 3 \qquad \phi_{\mu_1} \propto (1, 1, 1)^T \leftarrow \chi_\mu(i)$

$\lambda_2^{\mu_2} = -3 \qquad \phi_{\mu_2} \propto (1, -1, 1)^T$

$\lambda_2^{\mu_3} = 0 \qquad \phi_{\mu_3} \propto (2, 0, -1)^T$

$$P_{\mu_1} = \alpha_{\mu_1}(C_1 + C_2 + C_3)$$

	C_1	C_2	C_3
C_1	C_1	C_2	C_3
C_2	C_2	$3C_1 + 3C_3$	$2C_2$
C_3	C_3	$2C_2$	$2C_1 + C_3$

$$P_{\mu_1}^2 = \alpha_{\mu_1}^2 (C_1^2 + C_2^2 + C_3^2 + 2C_1C_2 + 2C_1C_3 + 2C_2C_3)$$

$$= \alpha_{\mu_1}^2 (\underline{C_1} + \underline{3C_1} + \underbrace{3C_3} + \underline{2C_1} + \underbrace{C_3} + \underbrace{2C_2} + \underbrace{2C_3} + \underbrace{4C_2})$$

$$= 6\alpha_{\mu_1}^2 (C_1 + C_2 + C_3) = \alpha_\mu (C_1 + C_2 + C_3) \equiv P_{\mu_1}$$

$$\alpha_{\mu_1} = \frac{1}{6}$$

$$P_{\mu_1} = \frac{1}{6}(C_1 + C_2 + C_3)$$

$$P_{\mu_2} = \frac{1}{6}(C_1 - C_2 + C_3)$$

$$P_{\mu_3} = \frac{1}{3}(2C_1 - C_3)$$

$$P_{\mu_1} P_{\mu_2} \propto C_1^2 + C_3^2 + 2C_1C_3 - C_2^2 = C_1 + 2C_1 + C_3 + 2C_3 - (3C_1 + 3C_3)$$
$$= 0$$

$$P_{\mu_1} P_{\mu_3} \propto (C_1 + C_2 + C_3)(2C_1 - C_3) = 2C_1^2 - C_1C_3 + 2C_1C_2 - C_2C_3$$
$$+ 2C_1C_3 - C_3^2$$

$$= 2C_1 - C_3 + 2C_2 - 2C_2 \quad = 0$$
$$+ 2C_3 - (2C_1 + C_3)$$

$$\hat{C}_2 P_{\mu_1} = \frac{1}{6}(C_1C_2 + C_2^2 + C_2C_3)$$

$$= \frac{1}{6}(C_2 + 3C_1 + 3C_3 + 2C_2) = 3 \cdot \frac{1}{6}(C_1 + C_2 + C_3)$$

$$\equiv \frac{m_2}{n_{\mu_1}} \chi_{\mu_1}([C_2]) \cdot P_{\mu_1}$$

$$(12) P_{\mu_1} = (12) \cdot \frac{1}{6}(e \pm (12) \pm (23) \pm (13) + (123) + (132))$$

$$(12) P_{\mu_2} = \frac{1}{6}((12) \pm e \pm (123) \pm (132) + (23) + (13))$$

$$= \chi_{\mu_1} \cdot P_{\mu_1}$$

$$T(h) P^{\mu} = \sum_{i,k=1}^{n_\mu} M^{\mu}_{ki}(h) P^{\mu}_{ki}$$

$$= (-1) \cdot P_{\mu_2} = \chi_{\mu_2} \cdot P_{\mu_2}$$

$$P_{\mu_3} = P^{11}_{\mu_3} + P^{22}_{\mu_3} \qquad P^{\mu}_{ij} P^{\mu}_{kl} = \delta_{jk} P^{\mu}_{il}$$

$$T(h) P^{\mu}_{ij} = \sum_{k=1}^{n_\mu} M^{\mu}_{ki}(h) P^{\mu}_{kj}$$

$$\boxed{\{P^{\mu}_{kj},\ k = 1, \cdots n_\mu\}}$$

$$\Rightarrow P^{11} \cdot P^{22} = 0 \qquad P^{\mu}_{11} P^{\mu}_{21} = 0$$

what if $\quad P^{11} = e - (13) + (12) - (132)$?

$$P^{22} = e - (12) + (13) - (123)$$

satisfy the orthogonality relation $\left\{ \begin{array}{l} P^{11}_{\mu_3} P^{22}_{\mu_3} = \delta_{12} P^{12}_{\mu_3} = 0 \\ P^{11}_{\mu_3} + P^{22}_{\mu_3} = P_{\mu_3} \end{array} \right.$

in principle. find more commuting operators to lift degeneracies on the group space R_G

(CSCO − II , CSCO − III)

use projectors from subgroups:

S_2 :

	e	(12)
$\underline{1}$	1	1
$-\underline{1}$	1	−1

	C_1	C_2
C_1	C_1	C_2
C_2	C_2	C_1

P^k_{2j} : $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\lambda = \pm 1$

$P_1' = \frac{1}{2}(e + (12))$

$P_2' = \frac{1}{2}(e - (12))$

$G \supset G_1$

$P^{\nu}_{\nu_1} = P^{\nu} P^{\nu_1}$

$\hookrightarrow \quad \underset{-1}{P^2_1} = P^2 \underset{P'^{-1}}{P'^{1}} = \frac{1}{6}(2e - (123) - (132))(e \underline{+} (12))$

$= \frac{1}{6}(2e \underline{+} 2(12) - (123) \mp (13) - (132) \mp (23))$

$\left\{ \begin{array}{l} P^2 = P_1' + P_{-1}' \checkmark \\ P_1' P_{-1}' = 0 \end{array} \right.$

$\underset{S_3}{\underset{\uparrow}{C_2}} + \underset{S_2}{\underset{\uparrow}{C_2'}}$ CSCO − II

$(12)\, P^2_{\pm 1} = \pm P^2_{\pm}$