Problem 25 (From the lecture)

Verify the following statements from the lecture:

(1) The actions of S_n and a group G on $V^{\otimes n}$ as defined in the lecture commute.

(2) The tensor coefficients in the irrep corresponding to the Young tableau

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

satisfy $a_{ijk} + a_{jki} + a_{kij} = 0$ and $a_{ijk} = -a_{jik}$.

(Optional) Problem 26 (Irreps of SU(N))

As discussed in the lecture, the Schur-Weyl duality dictates that distinct irreps of SU(N) are given by different Young diagrams with at most N rows. Using the Young diagrams, the decomposition of tensor product of irreps $T_1 \otimes T_2$ can be computed in a pictorial way as described below:

(1) assign distinct labels to boxes in each row of T_2, e.g. $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$

(2) attach boxes of T_2 row by row to T_1 in all possible ways that result in another semistandard Young tableau.

(3) after attaching each row, one should keep only those tableaux that when reading *from right to left while going from top to bottom*, the sequence of numbers attached should contain at least as many 1's as 2's, 2's as 3', etc at any point.

Below is an example:

$$\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \cong \begin{array}{|c|c|c|c|}\hline & & 1 & 1 \\ \hline & 2 & \multicolumn{2}{c}{} \\ \cline{1-2}\end{array} \oplus \begin{array}{|c|c|c|c|}\hline & & 1 & 1 \\ \hline & \multicolumn{3}{c}{} \\ \cline{1-1} 2 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|c|c|}\hline & & 1 \\ \hline & 1 & 2 \\ \hline\end{array} \oplus$$

$$\begin{array}{|c|c|c|}\hline & & 1 \\ \hline & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|c|c|}\hline & & 1 \\ \hline & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} 1 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|c|c|}\hline & & 1 \\ \hline & \multicolumn{2}{c}{} \\ \cline{1-1} 1 & \multicolumn{2}{c}{} \\ \cline{1-1} 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|c|}\hline & \\ \hline & 1 \\ \hline 1 & 2 \\ \hline\end{array} \oplus \begin{array}{|c|c|}\hline & \\ \hline & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \tag{1}$$

Note that the tableau $\begin{array}{|c|c|c|c|}\hline & & 1 & 2 \\ \hline & 1 & \multicolumn{2}{c}{} \\ \cline{1-2}\end{array}$ is not allowed because it violates condition (3).

(1) Perform the decomposition for the following tensor product for SU(3):

$$\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$$

(2) The dimension of the irrep is given by $D = \Pi_b \frac{(N+d(b))}{h(b)}$, where the multiplication goes over all boxes b in the diagram. $h(b)$ is the hook length of box b, and $d(b)$ defined in the following example diagram

0	1	2	3
−1	0	1	2
−2	−1	0	1

Check that the dimensions match in the decomposition you just performed. For more details consult textbooks, e.g. Jones, *Groups, representations and physics, 2nd Ed., IoP 1998.*