STA 601/360 Homework2

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Exercise 1

Hoff 3.1

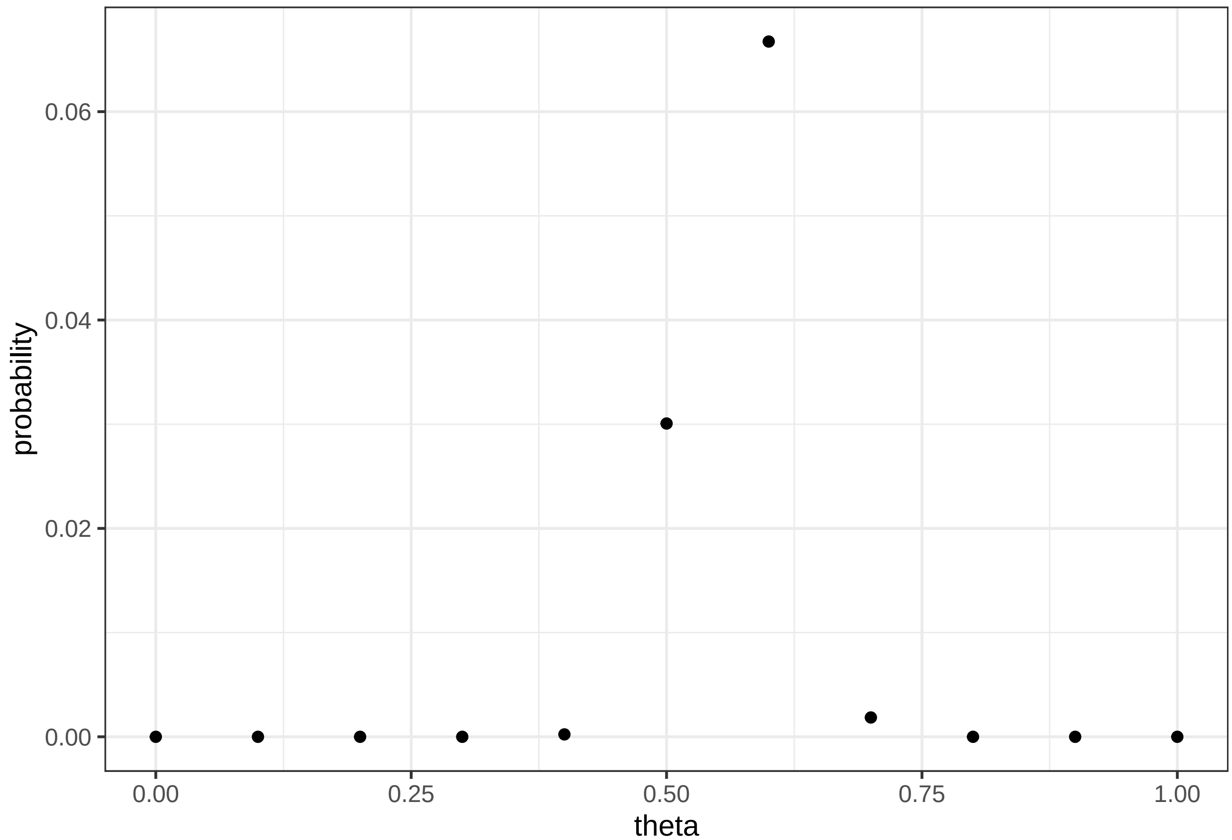
Part (a)

$$Pr(Y_1 = y_1, Y_2 = y_2, ..., Y_{100} = y_{100} \mid \theta) = \theta^{\sum_{i=1}^{n} y_i}(1-\theta)^{100-\sum_{i=1}^{n} y_i}$$
$$Pr(\sum_{i=1}^{n} Y_i = y \mid \theta) = \binom{n}{y}\theta^{y}(1-\theta)^{100-y}$$

Part (b)

$$Pr(\sum Y_i = 57 \mid \theta) = \binom{100}{57}\theta^{57}(1-\theta)^{43}$$

```r
theta1 <- seq(0,1,0.1)
b_df = data.frame(
  theta = theta1 ,
  probability = choose(100,57)*(theta1^57)*((1-theta1)^43)
)
ggplot(b_df,aes(x = theta, y = probability)) + geom_point()
```

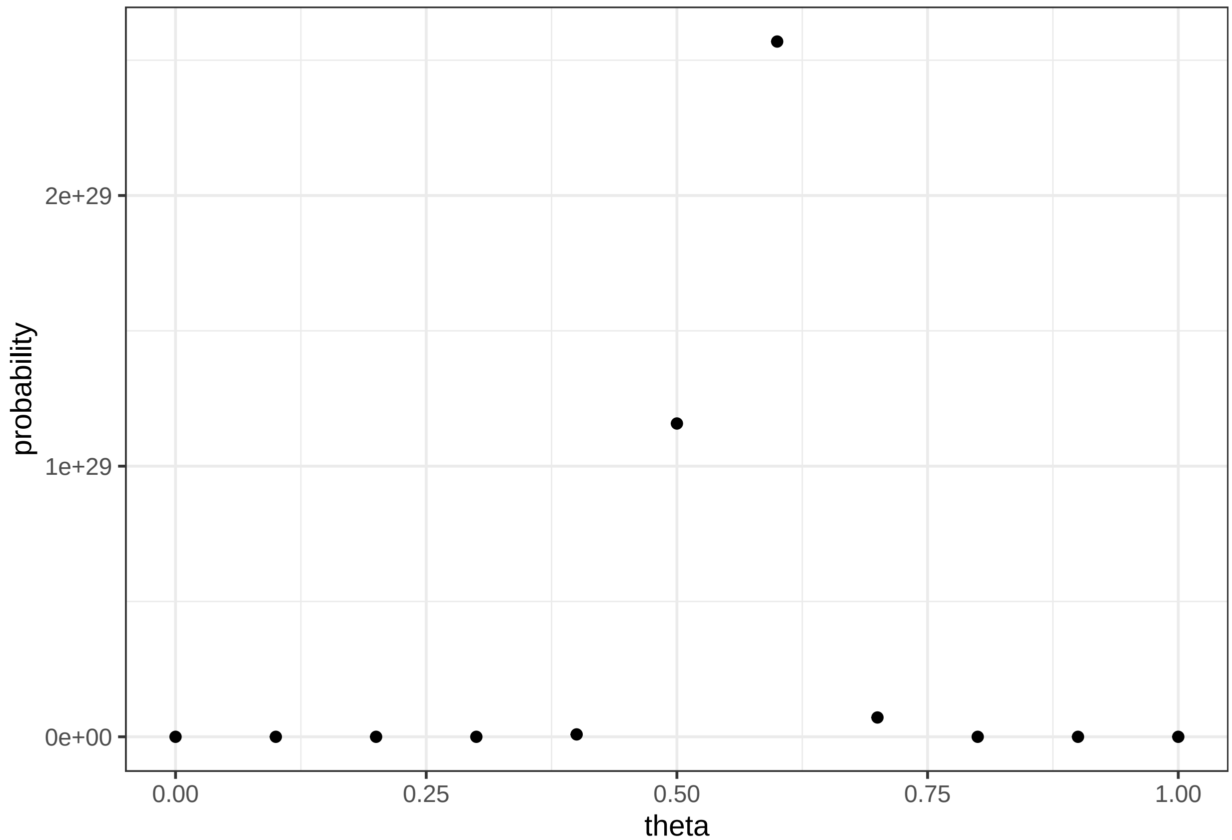


Part (c)

$$p(\theta \mid \sum_{i=1}^{100} Y_i = y) = \frac{p(\sum_{i=1}^{100} Y_i = y \mid \theta) p(\theta)}{p(\sum_{i=1}^{100} Y_i = y)}$$

$$\implies p(\theta \mid \sum_{i=1}^{100} Y_i = 57) = \frac{\binom{100}{57} \theta^{57} (1-\theta)^{43}}{\frac{\Gamma(58)\Gamma(44)}{\Gamma(102)}}$$

```
c_df = data.frame(
  theta = theta1,
  probability = choose(100,57)*gamma(102)*(theta1^57)*((1-theta1)^43)/(gamma(58)*gamma(44))
)
ggplot(c_df,aes(x = theta, y = probability)) + geom_point()
```

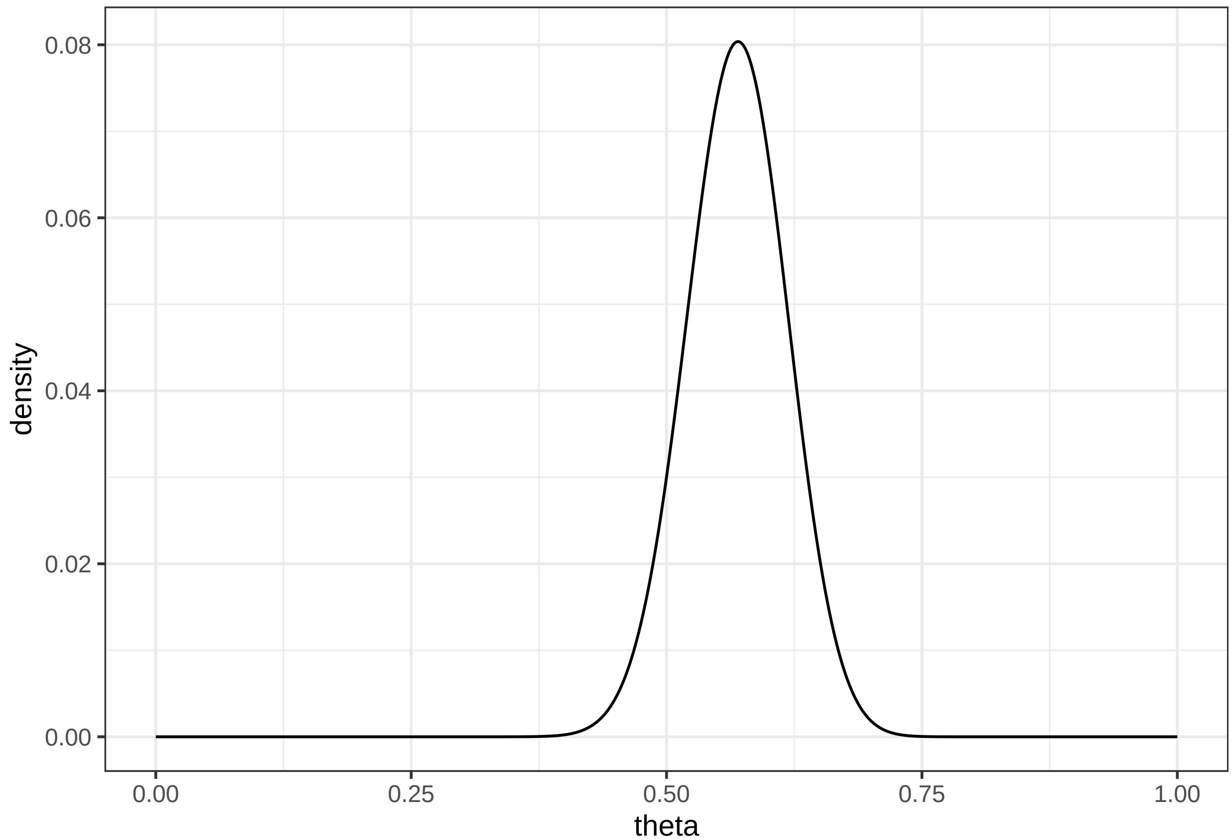


Part (d)

$$p(\theta) = 1$$

$$p(\theta)Pr(\sum_{i=1}^{100} Y_i = 57 \mid \theta) = \binom{100}{57}\theta^{57}(1-\theta)^{43}$$

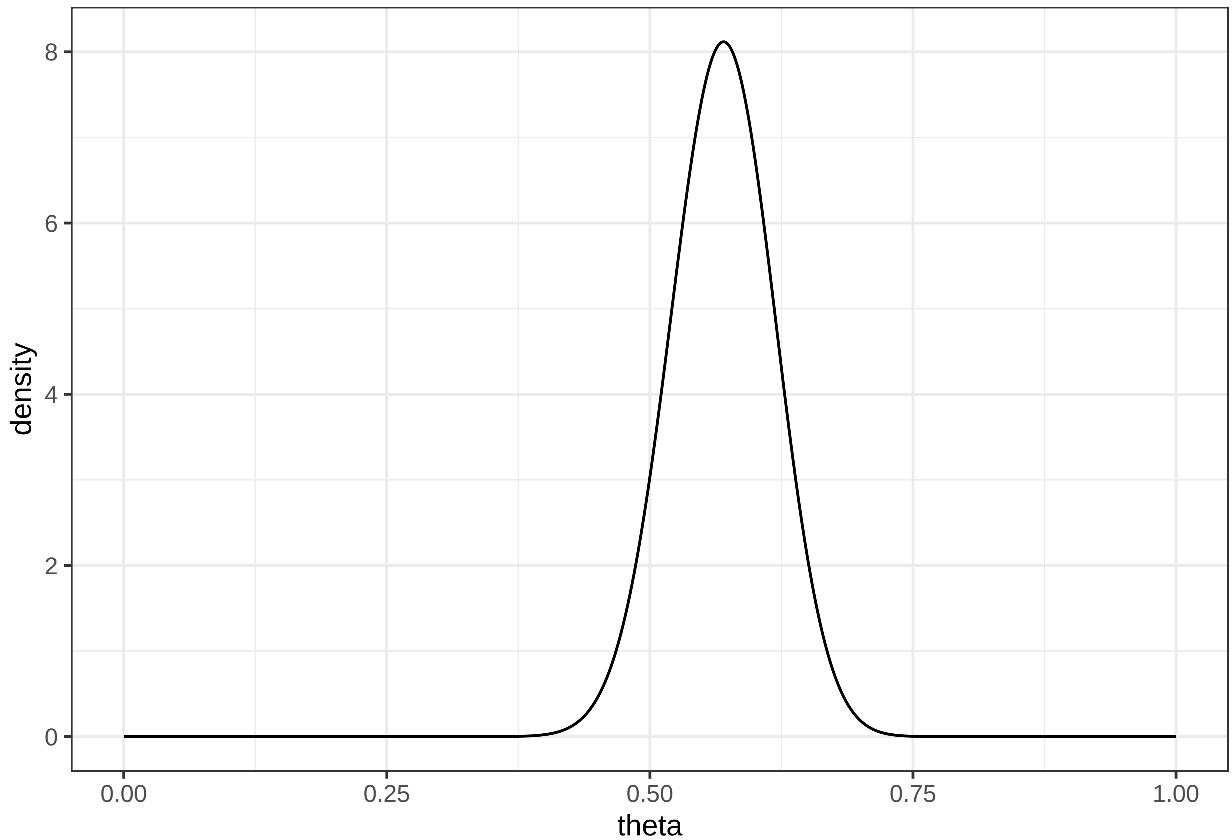
```
d_df = data.frame(
  theta = seq(0, 1, by = 0.001),
  density = choose(100,57)*(seq(0, 1, by = 0.001)^57)*((1-seq(0, 1, by = 0.001))^43)
)
ggplot(d_df, aes(x = theta, y = density)) + geom_line()
```



Part (e)

$$p(\theta \mid \sum_{i=1}^{100} Y_i = y) \sim Beta(58, 44)$$

```r
e_df = data.frame(
  theta = seq(0, 1, by = 0.001),
  density = dbeta(seq(0, 1, by = 0.001), 58, 44)
)
ggplot(e_df, aes(x = theta, y = density)) + geom_line()
```

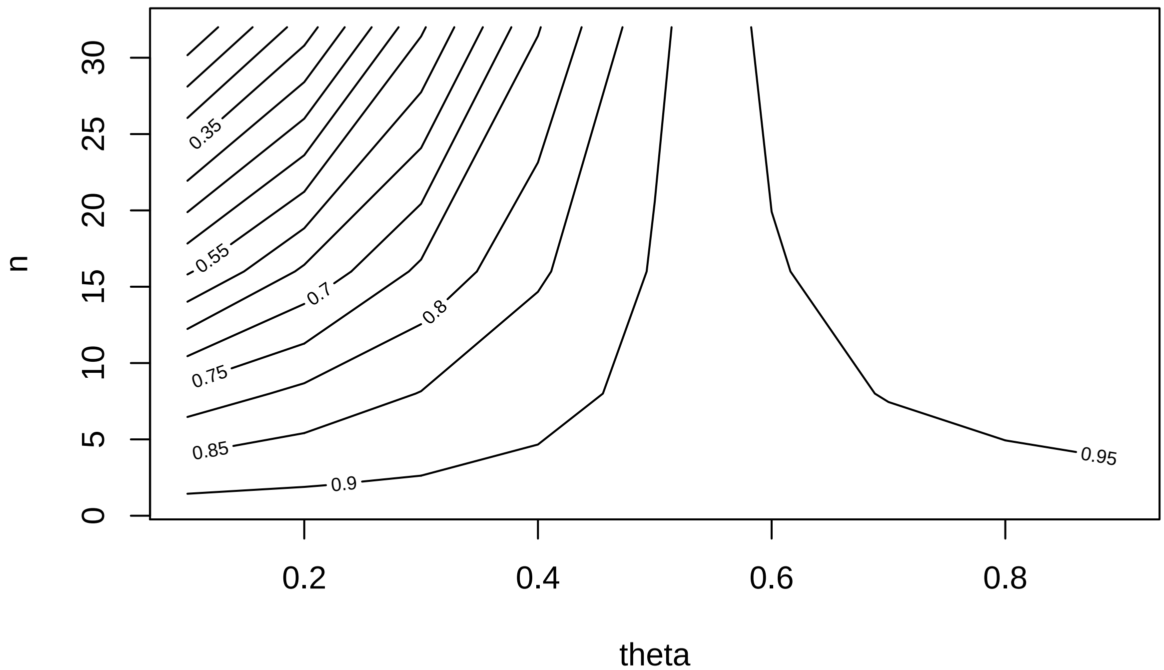


The plots of b and c have the same shape and different scale, b'scale is bigger than c's. The plots of d and e have the same shape and different scale, d's scale is smaller than e's. This is because the posterior distribution $p(\theta \mid \sum_{i=1}^{100} Y_i = y)$ is equal to $p(\sum_{i=1}^{100} Y_i = 57 \mid \theta)$ divided by something that is smaller than 1 and does not depend on θ.

Exercise 2

Hoff 3.2

```
theta0 <- seq(0.1, 0.9, 0.1)
n0 <- c(1, 2, 8, 16, 32)
a <- outer(theta0, n0)
b <- outer((1-theta0), n0)
p <- matrix(0,nrow=9,ncol=5)
for(i in 1:9){
  for(j in 1:5){
    p[i,j]=1-pbeta(0.5,a[i,j]+57,b[i,j]+100-57)
  }
}
contour(x=theta0, y=n0, z=p, levels=seq(0.1,1,0.05), xlab="theta", ylab="n")
```



The plot gives contours of the posterior propability $Pr[\theta > 0.5 \mid \sum Y = 57]$ and indicates that people with weak prior beliefs or high prior expectations are more certain that the policy Z's support rate is above 0.5. A high degree of certain (95%) is only achieved by people who already thought the support rate is higher. And people should believe that $\theta > 0.5$ based on the data that $\sum Y = 57$.