STA 601/360 Homework 3

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Exercise 1

Part (a)

$$
\begin{aligned}
E(Y \mid \theta) &= \sum_{y\geq 0} yp(y \mid \theta) = \sum_{y\geq 1} yp(y \mid \theta) = \sum_{y\geq 1} y\frac{\theta^y e^{-\theta}}{y!} = \theta e^{-\theta}\sum_{y\geq 1}\frac{\theta^{y-1}}{(y-1)!} \\
&= \theta e^{-\theta}\sum_{y\geq 0}\frac{\theta^y}{y!} = \theta e^{-\theta}e^{\theta} = \theta
\end{aligned}
$$

Part (b)

$$
\begin{aligned}
E(Y^2 \mid \theta) &= \sum_{y\geq 0} y^2 p(y \mid \theta) = \sum_{y\geq 1} y^2 p(y \mid \theta) = \sum_{y\geq 1} y^2\frac{\theta^y e^{-\theta}}{y!} = \theta e^{-\theta}\sum_{y\geq 1} y\frac{\theta^{y-1}}{(y-1)!} \\
&= \theta e^{-\theta}\sum_{y\geq 1}((y-1)+1)\frac{\theta^{y-1}}{(y-1)!} \\
&= \theta e^{-\theta}[\sum_{y\geq 2}\frac{\theta^{y-1}}{(y-2)!} + \sum_{y\geq 1}\frac{\theta^{y-1}}{(y-1)!}] \\
&= \theta e^{-\theta}[\theta\sum_{y\geq 0}\frac{\theta^y}{y!} + \sum_{y\geq 0}\frac{\theta}{y!}] \\
&= \theta e^{-\theta}[\theta e^{\theta} + e^{\theta}] \\
&= \theta^2 + \theta
\end{aligned}
$$

$$
\begin{aligned}
Var(Y^2 \mid \theta) &= E(Y^2 \mid \theta) - (E(Y \mid \theta))^2 \\
&= \theta^2 + \theta - \theta^2 \\
&= \theta
\end{aligned}
$$

Exercise 2

Part (a)

$$
\begin{aligned}
& We\ have: \\
\theta_A &\sim gamma(120, 10) \\
\theta_B &\sim gamma(12, 1) \\
y_{A,1}, ..., y_{A,n} \mid \theta_A &\sim pois(\theta_A) \\
y_{B,1}, ..., y_{B,n} \mid \theta_B &\sim pois(\theta_B) \\
p(\theta \mid y_1, ..., y_n) &= \frac{p(\theta)p(\theta \mid y_1, ..., y_n)}{p(y_1, ...y_n)} \\
p(\theta \mid y_1, ..., y_n) &= \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta} * \prod_{i=1}^{n} \frac{\theta^{y_i}e^{-\theta}}{y_i!} * c(y_1, ..., y_n) \\
&\propto \theta^{a-1}e^{-b\theta}\theta^{\sum y_i}e^{-n\theta} \\
&\propto \theta^{a+\sum y_i - 1}e^{-(b+n)\theta} \\
\implies \theta_A \mid y_{A,1}, ..., y_{A,n_A} &\sim gamma(a_A + \sum y_{i,A},\ b_A + n_A) = gamma(237, 20) \\
\implies \theta_B \mid y_{B,1}, ..., y_{B,n_B} &\sim gamma(a_B + \sum y_{i,B},\ b_B + n_B) = gamma(125, 14) \\
E(\theta_A \mid y_{A,1}, ..., y_{A,n_A}) &= \frac{237}{20} = 11.85 \\
E(\theta_B \mid y_{B,1}, ..., y_{B,n_B}) &= \frac{125}{14} = 8.93 \\
Var(\theta_A \mid y_{A,1}, ..., y_{A,n_A}) &= \frac{237}{20^2} = 0.59 \\
Var(\theta_B \mid y_{B,1}, ..., y_{B,n_B}) &= \frac{125}{14^2} = 0.64
\end{aligned}
$$

```r
qgamma(c(0.025, 0.975), 237, 20)
```

```
## [1] 10.38924 13.40545
```

```r
qgamma(c(0.025, 0.975), 125, 14)
```

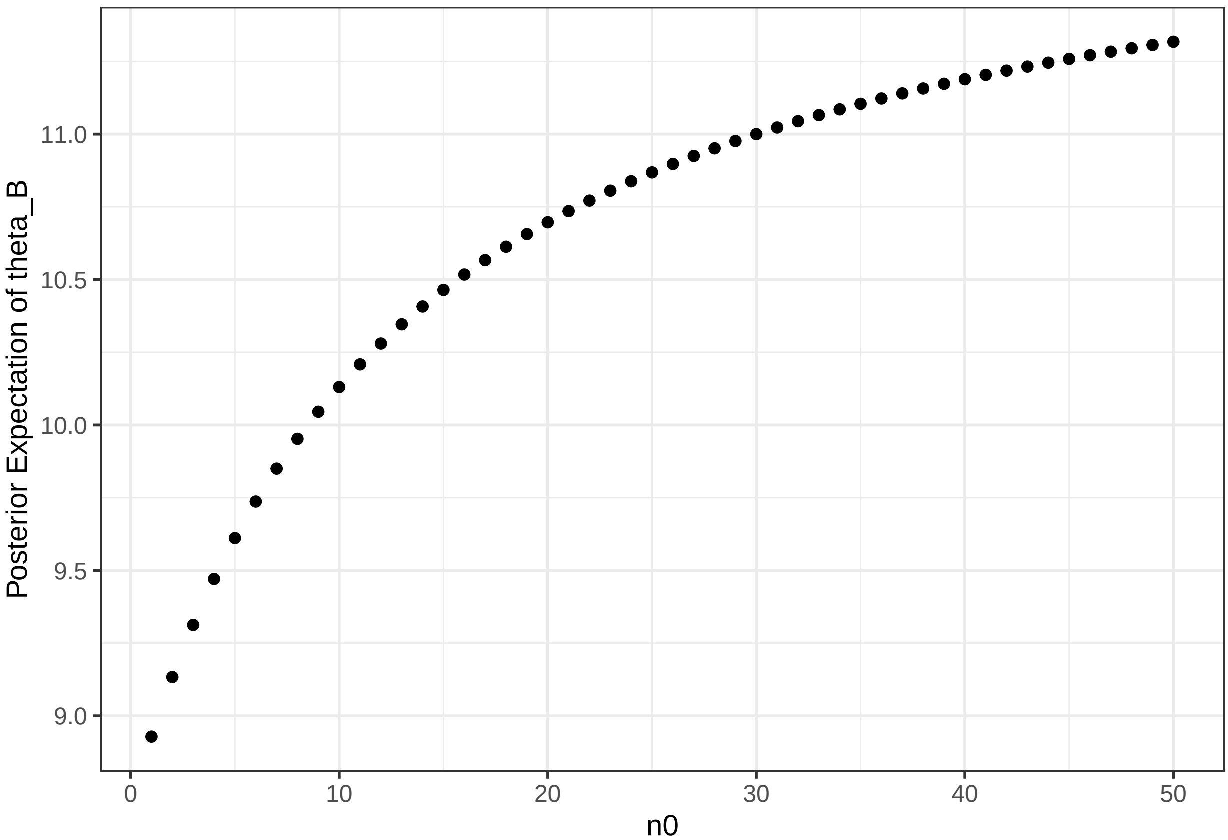
```
## [1]  7.432064 10.560308
```

Part (b)

$$\theta_B \mid y_{B,1}, ..., y_{B,n_B} \sim gamma(12 * n_0 + 113, n_0 + 13)$$

```r
yA = c(12, 9, 12, 14, 13, 13, 15, 8, 15, 6)
yB = c(11, 11, 10, 9, 9, 8, 7, 10, 6, 8, 8, 9, 7)

n0=1:50
post_E=(12*n0+113)/(n0+13)
df_E=data.frame(n0,post_E)
ggplot(df_E, aes(x=n0,y=post_E))+geom_point()+xlab("n0")+ylab("Posterior Expectation of theta_B")
```



- The posterior expectation of θ_A is 11.85. When n0 is about 50, the posterior distribution of θ_B is gamma(713, 63) and thus the posterior expectation of θ_B is close to that of θ_A.

Part (c)

- Type B mice are related to type A mice, indicating knowledge about population A tell us some information about population B. It make sense to have $p(\theta_A, \theta_B) = p(\theta_A) * p(\theta_B)$. Because we get infomation from $y_{A,1}, ..., y_{A,n_A}$ only for posteror distribution but not for prior distribution of θ_A and for the same reason of θ_B. Even if the population of two types of mice related, the prior information of their perameter could be independent.

Exercise 3

Part (a)

$$p(y \mid \phi) = h(y \mid \phi)c(\phi)e^{\phi t(y)} = \frac{2}{\Gamma(a)}(\theta^2)^a y^{2a-1}e^{-\theta^2 y^2}$$

$$\implies \phi = \theta^2, t(y) = -y^2, h(y) = \frac{2}{\Gamma(a)}y^{2a-1}, c(\phi) = \phi^a$$

prior distribution:

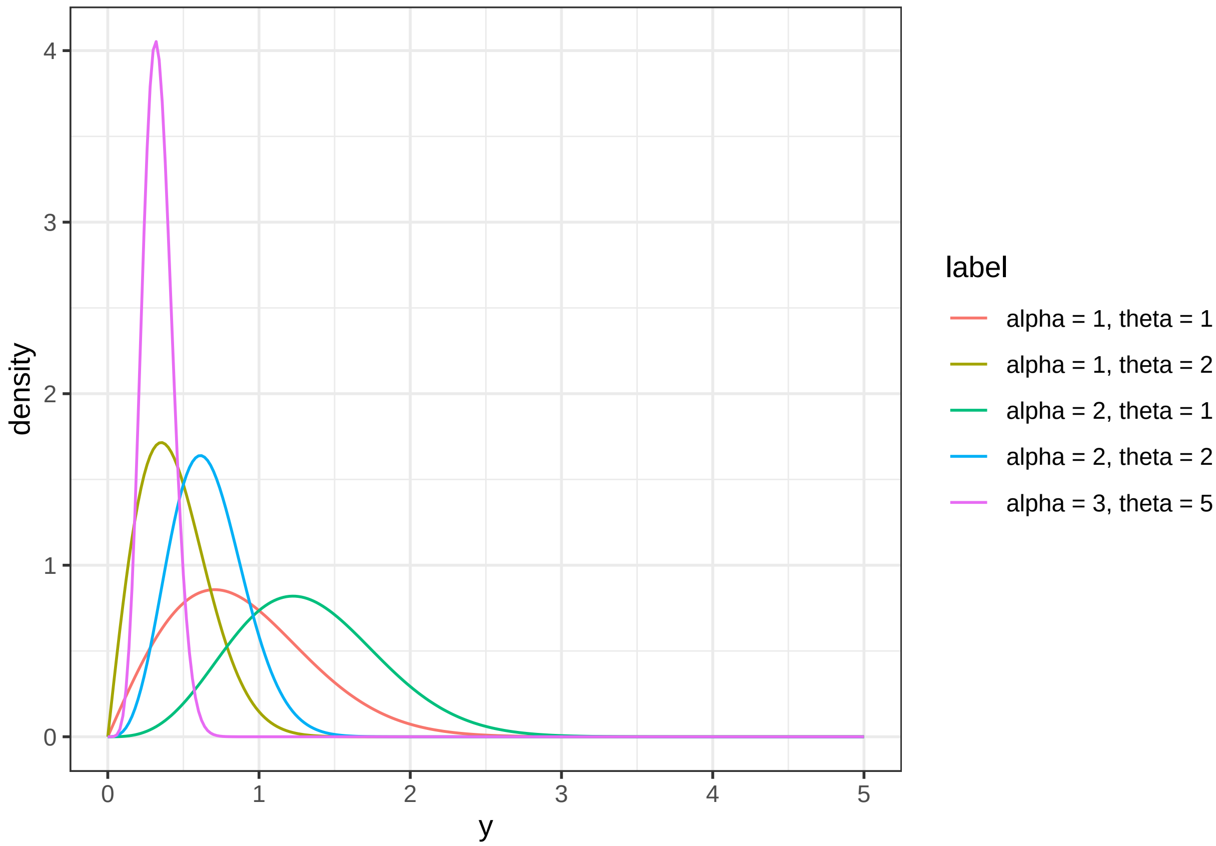
$$p(\phi \mid n_0, t_0) \propto c(\phi)e^{n_0 t_0 \phi} \propto \phi^{a n_0} e^{n_0 t_0 \phi}$$

$$\implies$$

$$p(\theta \mid n_0, t_0) = p(\phi \mid n_0, t_0)\frac{|d\phi|}{|d\theta|} \propto \theta^{2an_0} e^{n_0 t_0 \theta^2} 2\theta \propto \theta^{2an_0+1} e^{n_0 t_0 \theta^2}$$

$$\theta \mid n_0, t_0 \sim Gelenshore(\theta, an_0 + 1, \sqrt{-n_0 t_0})$$

```r
dgalenshore = function(y, a, theta) {
  (2/gamma(a))*theta^(2*a)*y^(2*a-1)*exp(-1*(theta^2)*y^2)
}
y = seq(0, 5, 0.02)
p1 = data.frame(y=y,d=dgalenshore(y, 1, 1),label='alpha = 1, theta = 1')
p2 = data.frame(y=y,d=dgalenshore(y, 1, 2),label='alpha = 1, theta = 2')
p3 = data.frame(y=y,d=dgalenshore(y, 2, 1),label='alpha = 2, theta = 1')
p4 = data.frame(y=y,d=dgalenshore(y, 2, 2),label='alpha = 2, theta = 2')
p5 = data.frame(y=y,d=dgalenshore(y, 3, 5),label='alpha = 3, theta = 5')
plot1 = rbind(p1,p2,p3,p4,p5)
ggplot(plot1, aes(x=y, y=d, group=label, color=label)) + geom_line() + ylab("density")
```



Part (b)

$$
\begin{aligned}
p(\theta \mid y_1, ...y_n) &\propto p(\theta)p(y_1, ...y_n \mid \theta) \\
&\propto \theta^{2an_0+1} e^{n_0 t_0 \theta^2} \theta^{2an} e^{-\theta^2 \sum y_i^2} \\
&\propto \theta^{2an_0+1+2an} e^{-\theta^2 (\sum y_i^2 - n_0 t_0)} \\
\theta \mid y_1, ...y_n &\sim Gelenshore(\theta, an_0 + an + 1, \sqrt{\sum y_i^2 - n_0 t_0})
\end{aligned}
$$

Part (c)

$$
\begin{aligned}
p(\theta_A \mid y_1, ...y_n) &= \frac{2}{\Gamma(an_0 + an + 1)} (\sum y_i^2 - n_0 t_0)^{2(an_0+an+1)} \theta_A^{2(an_0+an+1)-1} e^{-\theta_A^2 (\sum y_i^2 - n_0 t_0)} \\
p(\theta_B \mid y_1, ...y_n) &= \frac{2}{\Gamma(an_0 + an + 1)} (\sum y_i^2 - n_0 t_0)^{2(an_0+an+1)} \theta_B^{2(an_0+an+1)-1} e^{-\theta_B^2 (\sum y_i^2 - n_0 t_0)} \\
\frac{p(\theta_A \mid y_1, ...y_n)}{p(\theta_B \mid y_1, ...y_n)} &= (\frac{\theta_A}{\theta_B})^{2(an_0+an+1)-1} e^{-(\theta_A^2 + \theta_B^2)(\sum y_i^2 - n_0 t_0)}
\end{aligned}
$$

- A sufficient statistic is $\sum y_i^2$.

Part (d)

$$
E(\theta \mid y_1, ...y_n) = \frac{\Gamma(an_0 + an + 1) + \frac{1}{2}}{\sqrt{\sum y_i^2 - n_0 t_0} \Gamma(an_0 + an + 1)}
$$

Part (e)

$$
\begin{aligned}
p(\tilde{y} \mid y_1, ..., y_n) &= \int_0^\infty p(\tilde{y}, \theta \mid y_1, ..., y_n) d\theta \\
&= \int_0^\infty p(\tilde{y} \mid \theta, y_1, ..., y_n) p(\theta \mid y_1, ..., y_n) d\theta \\
&= \int_0^\infty p(\tilde{y} \mid \theta) p(\theta \mid y_1, ..., y_n) d\theta \\
&= \int_0^\infty \frac{2}{\Gamma(a)} \theta^{2a} \tilde{y}^{2a-1} e^{-\theta^2 \tilde{y}^2} \frac{2}{\Gamma(an_0 + an + 1)} (\sqrt{\sum y_i^2 - n_0 t_0})^{2(an_0+an+1)} \theta^{2(an_0+an+1)-1} e^{-(\sum y_i^2 - n_0 t_0)\theta^2} d\theta \\
&= \frac{4}{\Gamma(a)\Gamma(an_0 + an + 1)} \tilde{y}^{2a-1} (\sum y_i^2 - n_0 t_0)^{(an_0+an+1)} \int_0^\infty \theta^{2a+2(an_0+an+1)-1} e^{-(\tilde{y}^2 + \sum y_i^2 - n_0 t_0)\theta^2} d\theta \\
&\because \int_0^\infty \frac{2}{\Gamma(a)} \theta^{2a} y^{2a-1} e^{-\theta^2 y^2} dy = 1 \\
&\therefore \int_0^\infty y^{2a-1} e^{-\theta^2 y^2} dy = \frac{\Gamma(a)}{2\theta^{2a}} \\
&\implies \int_0^\infty \theta^{2a+2(an_0+an+1)-1} e^{-(\tilde{y}^2 + \sum y_i^2 - n_0 t_0)\theta^2} d\theta = \frac{\Gamma(an_0 + an + 1 + a)}{2(\tilde{y}^2 + \sum y_i^2 - n_0 t_0)^{an_0+an+1+a}}, \\
&= \frac{4}{\Gamma(a)\Gamma(an_0 + an + 1)} \tilde{y}^{2a-1} (\sum y_i^2 - n_0 t_0)^{(an_0+an+1)} \frac{\Gamma(an_0 + an + 1 + a)}{2(\tilde{y}^2 + \sum y_i^2 - n_0 t_0)^{an_0+an+1+a}} \\
&= \frac{2\Gamma(an_0 + an + 1 + a)}{\Gamma(a)\Gamma(an_0 + an + 1)} \frac{\tilde{y}^{2a-1}}{(\tilde{y}^2 + \sum y_i^2 - n_0 t_0)^a} \left(\frac{\sum y_i^2 - n_0 t_0}{\tilde{y}^2 + \sum y_i^2 - n_0 t_0}\right)^{an_0+an+1}
\end{aligned}
$$

Exercise 4

Part (a)

$$
\theta_1 \mid \sum_{i=1}^{100} Y_i = 57 \sim Beta(58, 44)
$$
$$
\theta_2 \mid \sum_{i=1}^{50} Y_i = 30 \sim Beta(31, 21)
$$

```
set.seed(20)
theta1 = rbeta(10000, 58, 44)
theta2 = rbeta(10000, 31, 21)
mean(theta1 < theta2)
```

```
## [1] 0.6243
```

Exercise 5

Part (a)

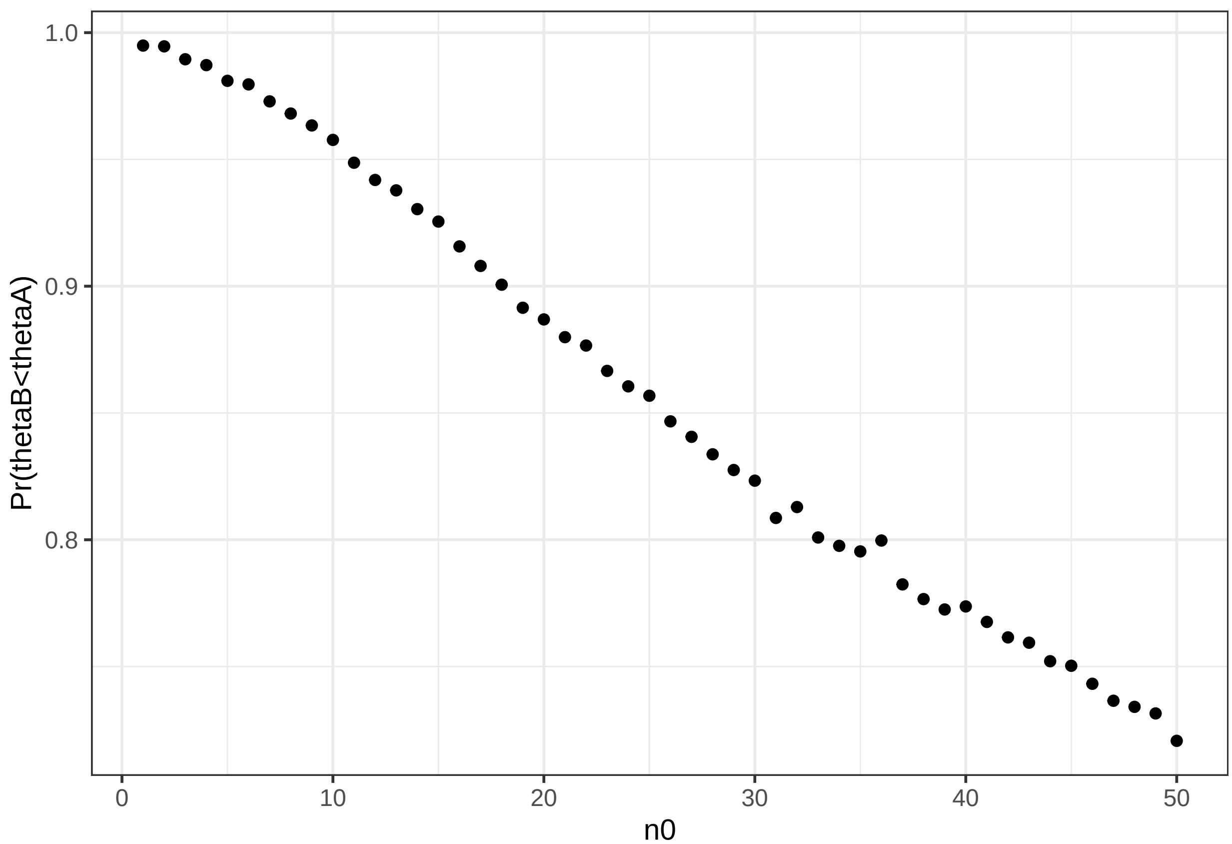
$$
\theta_A \mid y_A \sim Gamma(237, 20)
$$
$$
\theta_B \mid y_B \sim Gamma(125, 14)
$$

```r
set.seed(20)
thetaA = rgamma(10000, 237, 20)
thetaB = rgamma(10000, 125, 14)
mean(thetaB < thetaA)
```

```
## [1] 0.9949
```

Part (b)

```r
set.seed(20)
n0 = 1:50
thetaA1=rgamma(10000, 237, 20)
mc1=seq(length(n0))
for(n in n0){
  thetaB1=rgamma(10000, (12*n)+113, n+13)
  mc1[n]=mean(thetaB1<thetaA1)
}
dfmc1=data.frame(n0, mc1)
ggplot(dfmc1, aes(x=n0,y=mc1))+geom_point()+xlab("n0")+ylab("Pr(thetaB<thetaA)")
```



- The bigger n0 is, the less sensitive the conclusions about the event $\theta_B < \theta_A$ are.

Part (c)

```r
set.seed(20)
n0 = 1:50
thetaA2=rgamma(10000, 237, 20)
YA=rpois(10000, thetaA2)
mc2=seq(length(n0))
for(n in n0){
  thetaB2=rgamma(10000, (12*n)+113, n+13)
  YB=rpois(10000, thetaB2)
  mc2[n]=mean(YB<YA)
}
dfmc2=data.frame(n0,mc2)
ggplot(dfmc2, aes(x=n0,y=mc2))+geom_point()+xlab("n0")+ylab("Pr(Y_B<Y_A)")
```