STA 601 Homework1

Yi Mi

04 September, 2019

Exercise 1

Full conditions: Let X,Y,Z be random variables with joint density (discrete or continuous) p(x,y,z) ∝f(x,z)g(y,z)h(z). Show that a) p(x|y,z)∝f(x,z), i.e. p(x|y,z) is a function of x and z; b) p(y|x,z)∝g(y,z), i.e. p(y|x,z) is a function of y and z; c) X and Y are conditionally independent given Z.

Part (a)

$$
\begin{aligned}
&\text{When X, Y, Z are discrete,}\\
&p(x \mid y,z) = \frac{p(x,y,z)}{p(y,z)} \propto \frac{f(x,z)g(y,z)h(z)}{\sum_{x\in X} f(x,z)g(y,z)h(z)} = \frac{f(x,z)}{\sum_{x\in X} f(x,z)}\\
&\text{When X, Y, Z are continuous,}\\
&p(x \mid y,z) = \frac{p(x,y,z)}{p(y,z)} \propto \frac{f(x,z)g(y,z)h(z)}{\int_{x\in X} f(x,z)g(y,z)h(z)} = \frac{f(x,z)}{\int_{x\in X} f(x,z)}\\
&\implies p(x \mid y,z) \propto f(x,z)
\end{aligned}
$$

Part (b)

$$

$$
\begin{aligned}
&\text{When X, Y, Z are discrete,}\\
&p(y \mid x,z) = \frac{p(x,y,z)}{p(x,z)} \propto \frac{f(x,z)g(y,z)h(z)}{\sum_{y\in Y} f(x,z)g(y,z)h(z)} = \frac{g(y,z)}{\sum_{y\in Y} g(y,z)}\\
&\text{When X, Y, Z are continuous,}\\
&p(y \mid x,z) = \frac{p(x,y,z)}{p(x,z)} \propto \frac{f(x,z)g(y,z)h(z)}{\int_{y\in Y} f(x,z)g(y,z)h(z)} = \frac{g(y,z)}{\int_{y\in Y} g(y,z)}\\
&\implies p(y \mid x,z) \propto g(y,z)
\end{aligned}
$$

$$

Part (c)

When X, Y, Z are discrete,

$$
\begin{aligned}
p(x \mid z) =& \frac{p(x, z)}{p(z)} = \frac{\sum_{y \in Y} p(x, y, z)}{\sum_{x \in X} \sum_{y \in Y} p(x, y, z)} \\
=& \frac{h(z) f(x, z) \sum_{y \in Y} g(y, z)}{h(z) \sum_{x \in X} f(x, z) \sum_{y \in Y} g(y, z)} \\
=& \frac{f(x, z)}{\sum_{x \in X} f(x, z)}
\end{aligned}
$$

When X, Y, Z are continuous,

$$
\begin{aligned}
p(x \mid z) =& \frac{p(x, z)}{p(z)} = \frac{\int_{y \in Y} p(x, y, z)}{\int_{x \in X} \int_{y \in Y} p(x, y, z)} \\
=& \frac{h(z) f(x, z) \int_{y \in Y} g(y, z)}{h(z) \int_{x \in X} f(x, z) \int_{y \in Y} g(y, z)} \\
=& \frac{f(x, z)}{\int_{x \in X} f(x, z)} \\
& \implies p(x \mid z) \propto f(x, z) \\
& \text{meanwhile, } p(x \mid y, z) \propto f(x, z) \\
& \implies \text{Y gives no additional info about X beyond that in knowing Z.} \\
& \implies \text{X and Y are conditionally independent given Z.}
\end{aligned}
$$

Exercise 2

Conditional independence: Suppose events A and B are conditionally independent given C, which is written $A \perp B \mid C$. Show that this implies that $A^c \perp B \mid C$, $A \perp B^c \mid C$, and $A^c \perp B^c \mid C$, where A^c means “not A.” Find an example where $A \perp B \mid C$ holds but $A \perp B \mid C^c$ does not hold.

Part (a)

$$
\begin{aligned}
A \perp B \mid C \implies P(A \mid C) &= P(A \mid B, C) \\
\therefore P(A^c \mid C) &= 1 - P(A \mid C) \\
&= 1 - P(A \mid B, C) \\
&= P(A^c \mid B, C) \\
\implies A^c \perp B \mid C &
\end{aligned}
$$

Part (b)

$$
\begin{aligned}
A \perp B \mid C \implies P(B \mid C) &= P(B \mid A, C) \\
\therefore P(B^c \mid C) &= 1 - P(B \mid C) \\
&= 1 - P(B \mid A, C) \\
&= P(B^c \mid A, C) \\
\implies A \perp B^c \mid C &
\end{aligned}
$$

Part (c)

$$
\begin{aligned}
A \perp B \mid C \implies A \perp B^c \mid C \implies P(A \mid C) &= P(A \mid B^c, C) \\
\therefore P(A^c \mid C) &= 1 - P(A \mid C) \\
&= 1 - P(A \mid B^c, C) \\
&= P(A^c \mid B^c, C) \\
\implies A^c \perp B^c \mid C &
\end{aligned}
$$

Part (d)

There is a bag containing 4 balls, 2 of which are red and 2 are yellow. Now randomlly select 2 balls one by one from the bag. Let A be the event that the first ball is red, B be the event that the second ball is red and C be the event that the 2 selected balls have same color.

$$
\begin{aligned}
P(A \mid C) &= \frac{P(A, C)}{P(C)} = \frac{1/4}{2/4} = \frac{1}{2} \\
P(A \mid B, C) &= \frac{P(A, B, C)}{P(B, C)} = \frac{1/4}{2/4} = \frac{1}{2} \\
P(A \mid C) &= P(A \mid B, C) \implies A \perp B \mid C \text{ holds.}
\end{aligned}
$$

$$
\begin{aligned}
P(A \mid C^c) &= \frac{P(A, C^c)}{P(C^c)} = \frac{1/4}{2/4} = \frac{1}{2} \\
P(A \mid B, C^c) &= \frac{P(A, B, C^c)}{P(B, C^c)} = \frac{0}{1/4} = 0 \\
P(A \mid C) &\neq P(A \mid B, C^c) \implies A \perp B \mid C^c \text{ does not hold.}
\end{aligned}
$$

Exercise 3

There are three coins in a bag; two fair coins (probability of heads = probability of tails) and one fake coin (probability of heads = 1). a. You reach in and select one coin at random and throw it in the air. What is the probability that it lands on heads? b. You reach in and select one coin at random and throw it in the air and get heads. What is the probability that it is the fake coin?

Part (a)

$$
\begin{aligned}
P(heads \mid fair) &= \frac{1}{2} \\
P(heads \mid fake) &= 1
\end{aligned}
$$

$$
\begin{aligned}
P(heads) &= P(heads \mid fair) * P(fair) + P(heads \mid fake) * P(fake) \\
&= \frac{1}{2} * \frac{2}{3} + 1 * \frac{1}{3} \\
&= \frac{2}{3}
\end{aligned}
$$

Part (b)

$$
\begin{aligned}
P(fake \mid heads) &= \frac{P(heads \mid fake)P(fake)}{P(heads)} \\
&= \frac{1 * \frac{1}{3}}{\frac{2}{3}} \\
&= \frac{1}{2}
\end{aligned}
$$

Exercise 4

Monty Hall Problem! There are three doors on stage and a host hides a prize behind one of the doors and goats behind the other two doors. A contestant picks one of the three doors (say door 1) and then the game show host opens one of the remaining two doors (say door 3), making sure to reveal a goat. Should the contestant switch to door 2? Prior knowledge: The prize originally had 1/3 probability of being behind any of the three doors. The host’s model: he knows which door has the prize. If he has to choose between a door with a goat and one with a prize, he chooses the one with the goat. If he has to choose between two doors with goats, he picks one of the doors with probability 1/2.

$$
\begin{aligned}
&\text{if stick to door 1,} \\
&P(finally\, prize) = P(originally\, prize) = \frac{1}{3} \\
\text{if switch to door 2,}& \\
P(finally\, prize) &= P(originally\, prize) * 0 + P(originally\, goat) * 1 \\
&= \frac{1}{3} * 0 + \frac{2}{3} * 1 \\
&= \frac{2}{3}
\end{aligned}
$$