STA 532 HW7 Due 3/25 Yi Mi ym154

1. (a) the moment generating function of Xi is

$$M_{X_i}(t) = e^{t\mu + \frac{1}{2}\sigma^2 t^2} = E(e^{tX_i})$$

$$E(Y_i) = E(e^{X_i}) = M_X(1) = e^{\mu + \frac{1}{2}\sigma^2}$$

$$Var(Y_i) = EY_i^2 - (EY_i)^2 = M_X(2) - M_X(1)^2$$

$$= e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2}$$

$$E(\bar{Y}) = E(\Sigma Y_i / n) = \Sigma E(Y_i)/n = e^{\mu + \frac{1}{2}\sigma^2}$$

$$Var(\bar{Y}) = Var(\Sigma Y_i / n) \overset{iid}{=} \Sigma Var(Y_i)/n^2 = \frac{1}{n}(e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2})$$

(b)
$$f_Y(y) = f_X(\ln y) \left|\frac{dx}{dy}\right|$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{y} \exp\left\{-\frac{1}{2}\frac{(\ln y - \mu)^2}{\sigma^2}\right\}$$

$$L(\mu, \sigma^2) = \Pi f_Y(y_i)$$

$$\ell(\mu, \sigma^2) = \Sigma \log f_Y(y_i) = -\frac{1}{2\sigma^2}\Sigma(\ln y_i - \mu)^2 - \frac{1}{2}\log 2\pi\sigma^2 + C$$

$$\frac{\partial \ell(\mu, \sigma^2)}{\partial \mu} = \frac{\Sigma \ln y_i}{\sigma^2} - \frac{n}{\sigma^2}\mu = 0 \Rightarrow \hat{\mu} = \frac{\Sigma \ln y_i}{n}$$

$$\frac{\partial \ell(\mu, \sigma^2)}{\partial \sigma^2} = 0 \Rightarrow \hat{\sigma}^2 = \frac{\Sigma(\ln y_i - \hat{\mu})^2}{n}$$

the MLE of a function is the function of the MLE)

thus, $\hat{\phi}_{MLE} = e^{\hat{\mu} + \frac{1}{2}\hat{\sigma}^2} = \exp\left\{\Sigma \ln y_i / n + \frac{1}{2n}\Sigma(\ln y_i - \hat{\mu})^2\right\}$

let $\phi = g(\mu, \sigma^2) = e^{\mu + \frac{1}{2}\sigma^2}$

$$I_n(\mu, \sigma^2) = \begin{bmatrix} \frac{n}{\sigma^2} & \frac{\Sigma \ln y_i - n\mu}{(\sigma^2)^2} \\ \frac{\Sigma \ln y_i - n\mu}{(\sigma^2)^2} & \frac{\Sigma (\ln y_i - \mu)^2}{(\sigma^2)^3} + \frac{1}{2(\sigma^2)^2} \end{bmatrix}$$

$$\nabla g = \begin{pmatrix} \frac{\Sigma \ln y_i}{\sigma^2} - \frac{n}{\sigma^2}\mu \\ \frac{\Sigma (\ln y_i - \mu)^2}{2(\sigma^2)^2} - \frac{1}{2\sigma^2} \end{pmatrix}$$

$$J_n = I_n(\mu, \sigma^2)^{-1}$$

$$\hat{se}(\hat{\phi}_{MLE}) = \sqrt{(\hat{\nabla} g)^T \hat{J}_n (\hat{\nabla} g)}$$

$$Var(\bar{Y}) \geq \hat{Var}(\hat{\phi}_{MLE})$$

2. (a) $L(a,b) = \prod \frac{\beta^a}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$

$\ell(a,b) = n\alpha\log\beta - n\log\Gamma(\alpha) + (\alpha-1)\Sigma\log x_i - \beta\Sigma x_i$

$\hat{a}_{MLE} = \arg\max_a \ell(a,b)$

$\hat{b}_{MLE} = \arg\max_b \ell(a,b)$

(b) $I_n(a,b) = -\begin{bmatrix} E(H_{aa}) & E(H_{ab}) \\ E(H_{ba}) & E(H_{bb}) \end{bmatrix}$

$= \begin{bmatrix} n\frac{\partial^2 \log\Gamma(a)}{\partial a^2} & -\frac{n}{b} \\ -\frac{n}{b} & \frac{na}{b^2} \end{bmatrix}$, $\frac{\partial^2\log\Gamma(a)}{\partial a^2}$ is trigamma

let $J_n(a,b) = I_n^{-1}(a,b) = \frac{1}{\frac{n^2 a}{b^2}\frac{\partial^2\log\Gamma(a)}{\partial a^2} - \frac{n^2}{b^2}} \begin{bmatrix} \frac{na}{b^2} & \frac{n}{b} \\ \frac{n}{b} & \frac{n\partial^2\log\Gamma(a)}{\partial a^2} \end{bmatrix}$

$(\hat{a}_{MLE}, \hat{b}_{MLE}) \sim N[(a,b), \hat{J}_n(\hat{a},\hat{b})]$

(c) let $g(a,b) = \frac{a}{b} = \mu$

$\hat{se}(\hat{\mu}) = [(\hat{\nabla} g)^T \hat{J}_n (\hat{\nabla} g)]^{1/2}$

$\nabla g = \begin{pmatrix} \frac{1}{b} \\ -\frac{a}{b^2} \end{pmatrix}$

$\Rightarrow \hat{se}(\hat{\mu}) = \left[-\frac{an}{b^4}\left[1 - a\cdot\frac{\partial^2\log\Gamma(a)}{\partial a^2}\right]\cdot \frac{1}{\frac{n^2 a}{b^2}\frac{\partial^2\log\Gamma(a)}{\partial a^2} - \frac{n^2}{b^2}}\right]^{1/2}$

$= \sqrt{\frac{a}{nb^2}}$

$\hat{\mu} \sim N(\frac{a}{b}, \frac{a}{nb^2})$

$$Var(\bar{Y}) = \frac{Var(Y_i)}{n} = \frac{a}{b^2 \cdot n}$$

$$Var(\hat{\mu}_{MLE}) = Var(\bar{Y})$$

④

3. (a) $E[\log p(Y \mid \mu, \sigma^2)]$

$$= E\left[-\frac{1}{2}\left(\log 2\pi\sigma^2 + \frac{(Y-\mu)^2}{\sigma^2}\right)\right]$$

$$= -\frac{1}{2}\log 2\pi\sigma^2 - \frac{1}{2}\frac{EY^2}{\sigma^2} + \frac{\mu}{\sigma^2}EY - \frac{1}{2}\frac{\mu^2}{\sigma^2}$$

$$EY = \frac{a}{b}, \quad EY^2 = Var(Y) + (EY)^2 = \frac{a}{b^2} + \frac{a^2}{b^2} = \frac{a+a^2}{b^2}$$

$$= -\frac{1}{2}\log 2\pi\sigma^2 - \frac{1}{2}\frac{a+a^2}{b^2\sigma^2} + \frac{\mu}{\sigma^2}\frac{a}{b} - \frac{1}{2}\frac{\mu^2}{\sigma^2}$$

$$\hat{\mu} = \arg\max_{\mu} E = \frac{a}{b}$$

$$\hat{\sigma}^2 = \arg\max_{\sigma^2} E = \frac{a}{b^2}$$

(b) $\hat{\mu}_{MLE} \xrightarrow{P} \frac{a}{b}$

$\hat{\sigma}^2_{MLE} \xrightarrow{P} \frac{a}{b^2}$

For Normal Assumption

(c) $\hat{SE}(\hat{\mu}_N) = \sqrt{\hat{I_n}(\mu_N)^{-1}}$

$$I_n(\mu_N) = -E\left(\frac{\partial^2 \Sigma \log f(\mu_N)}{\partial \mu_N^2}\right) = \frac{n}{\sigma^2}$$

$$\Rightarrow \hat{SE}(\hat{\mu}_N) = \sqrt{\frac{\hat{\sigma}^2}{n}} = \sqrt{\frac{a}{nb^2}}$$

For Gamma Assumption, $\hat{\mu}_G = \frac{a}{b}$

$$\hat{SE}(\hat{\mu}_G) = \sqrt{\frac{a}{nb^2}}$$

(d) The wrong assumption does not influence the statistical inference when sample size is big enough.

4. (a) A:

$$L_A(\theta) = \prod_{i=1}^{n} f(x_i | \theta)$$

B:

$$L_B(\theta) = \prod_{i=1}^{n} f(g^{-1}(Y_i) | \theta)$$

(b) $\hat{\theta}_A = \hat{\theta}_B$

$\hat{Var}(\hat{\theta}_A) = \hat{Var}(\hat{\theta}_B)$, as g is invertible

5.

(a) $\hat{\theta} = 2\bar{Y}$

$Var(\hat{\theta}) = \frac{4}{n} \cdot var(Y) = \frac{1}{3n}\theta^2$

(b) $\ell(\theta) = \Sigma \log \frac{1}{\theta} = n \log \frac{1}{\theta}$

$\hat{\theta}_{MLE} = \arg\max_{\theta} \ell(\theta)$

let $0 \le Y_{(1)} \le Y_{(2)} \le \cdots \le Y_{(n)} \le \theta$

thus, $\hat{\theta}_{MLE} = Y_{(n)} = \max\{Y_1 \cdots Y_n\}$

(c) $Pr(\hat{\theta}_{MLE} \le y) = Pr(Y_1 \le y, Y_2 \le y, \cdots Y_n \le y)$

$= Pr(Y \le y)^n$

$= (\frac{y}{\theta})^n$ for y in $[0, \theta]$

pdf : $n y^{n-1} / \theta^n$

(d) $E(\hat{\theta}_{MLE}) = \int_0^\theta n \cdot y^{n-1} / \theta^n \cdot y \, dy = \frac{1}{\theta^n} \frac{n}{n+1} y^{n+1} \Big|_0^\theta = \frac{n}{n+1}\theta$

$Var(\hat{\theta}_{MLE}) = \int_0^\theta n \cdot y^{n-1} / \theta^n \cdot y^2 \, dy - [E(\hat{\theta}_{MLE})]^2$

$= \frac{n}{n+2}\theta^2 - (\frac{n}{n+1}\theta)^2$

$= \theta^2 [\frac{n}{n+2} - (\frac{n}{n+1})^2]$

$MSE(\hat{\theta}_{MLE}) = Var(\hat{\theta}_{MLE}) + [E(\hat{\theta}_{MLE}) - \theta]^2$

$= \theta^2 [\frac{n}{n+2} + \frac{1-n}{1+n}]$

$= \theta^2 \frac{2}{(n+2)(n+1)}$

$$E(\hat{\theta}) = 2E(\bar{Y}) = 2\frac{\sum E(Y)}{n} = \theta$$

$$Var(\hat{\theta}) = \frac{1}{3n}\theta^2$$

$$MSE(\hat{\theta}) = \frac{1}{3n}\theta^2$$

$$\frac{MSE(\hat{\theta}_{MLE})}{MSE(\hat{\theta})} = \frac{6n}{(n+1)(n+2)} \leq 1$$

$\Rightarrow$ $\hat{\theta}_{MLE}$ performs better than $\hat{\theta}$.