STA 512 HW6 Due Wed 3/4 Yi Mi, ym154

1. We have $S_n^2 = \frac{1}{n-1}\sum(Y_i - \bar{Y})^2$

$S_0^2 = \frac{1}{n}\sum(Y_i - \mu)^2$

$S_1^2 = \frac{1}{n}\sum(Y_i - \bar{Y})^2$

We know that ① $S_n^2 = \frac{n}{n-1} S_1^2$

② $S_1^2 = \frac{1}{n}\sum(Y_i - \bar{Y})^2 = \frac{1}{n}\sum(Y_i - \mu - (\bar{Y} - \mu))^2$

$= \frac{1}{n}\left[\sum(Y_i-\mu)^2 - 2\sum(Y_i-\mu)(\bar{Y}-\mu) + \sum(\bar{Y}-\mu)^2\right]$

$= \frac{1}{n}\left[\sum(Y_i-\mu)^2 - 2(\bar{Y}-\mu)^2 \cdot n + n(\bar{Y}-\mu)^2\right]$

$= \frac{1}{n}\left[\sum(Y_i-\mu)^2 - n(\bar{Y}-\mu)^2\right]$

$= \frac{1}{n}\sum(Y_i-\mu)^2 - (\bar{Y}-\mu)^2$

$= S_0^2 - (\bar{Y}-\mu)^2$

For $S_0^2$, $E(S_0^2) = E\left(\frac{1}{n}\sum(Y_i-\mu)^2\right) = \frac{1}{n}\sum E(Y_i-\mu)^2 = \frac{1}{n}\sum\sigma^2 = \sigma^2$

$Var(S_0^2) = Var\left(\frac{1}{n}\sum(Y_i-\mu)^2\right) = \frac{1}{n^2}\sum Var(Y_i-\mu)^2$

By CLT, $S_0^2 \xrightarrow{d} N(\sigma^2, Var(Y_i-\mu)^2/n)$

For $S_1^2$, $S_1^2 = S_0^2 - (\bar{Y}-\mu)^2$ Let $A = \sqrt{Var(Y_i-\mu)^2}$

$S_1^2 - \sigma^2 = S_0^2 - \sigma^2 - (\bar{Y}-\mu)^2$

$\sqrt{n}\,\frac{S_1^2-\sigma^2}{A} = \sqrt{n}\,\frac{S_0^2-\sigma^2}{A} - \frac{1}{\sqrt{n}}\frac{\sigma^2}{A}\left(\frac{\bar{Y}-\mu}{\sigma/\sqrt{n}}\right)^2$

We know that,

① $\sqrt{n}\,\frac{S_0^2-\sigma^2}{A} \xrightarrow{d} N(0,1)$

and, $E(\bar{Y}) = E\left(\frac{\sum Y_i}{n}\right) = \frac{\sum E(Y_i)}{n} = \mu$

$Var(\bar{Y}) = Var\left(\frac{\sum Y_i}{n}\right) = \frac{\sum Var(Y_i)}{n^2} = \frac{\sum \sigma^2}{n^2} = \frac{\sigma^2}{n}$

$Pr(|\bar{Y}-\mu| > \varepsilon) < \frac{\sigma^2}{n\varepsilon} \to 0$ as $n \to \infty$

$\Rightarrow \bar{Y} \xrightarrow{P} \mu$

② thus, $\frac{1}{\sqrt{n}} \frac{\sigma^2}{A} \left(\frac{\bar{Y}-\mu}{\sigma/\sqrt{n}}\right)^2 \to 0$

thus, $\sqrt{n}\,\frac{S_1^2-\sigma^2}{A} \xrightarrow{d} N(0,1)$

$S_1^2 \xrightarrow{d} N(\sigma^2, Var(Y_i-\mu)^2/n)$

For $S_n^2$, $S_n^2 = \frac{n}{n-1} S_1^2$

$$\sqrt{n}\,\frac{S_n^2-\sigma^2}{A} = \frac{n}{n-1}\sqrt{n}\,\frac{S_0^2-\sigma^2}{A} + \sqrt{\frac{n}{(n-1)^2}}\,\frac{\sigma^2}{A}$$

we know that,

① $\frac{n}{n-1} \xrightarrow{P} 1$

② $\sqrt{n}\,\frac{S_0^2-\sigma^2}{A} \to N(0,1)$

③ $\sqrt{\frac{n}{(n-1)^2}}\,\frac{\sigma^2}{A} \xrightarrow{P} 0$

thus,

$\sqrt{n}\,\frac{S_n^2-\sigma^2}{A} \xrightarrow{d} N(0,1)$

$S_n^2 \xrightarrow{d} N(\sigma^2, Var(Y_i-\mu)^2/n)$

where $Var(Y_i-\mu)^2 = \gamma - (\sigma^2-\mu^2)^2$

$S_n^2$ is CAN estimator of $\sigma^2$

2. (1) $\hat{\theta} = \bar{W} = \frac{\sum W_i}{n} = \frac{\sum Y_i/X_i}{n}$

$E(\hat{\theta}) = E\left(\frac{\sum Y_i/X_i}{n}\right) = \frac{\sum E(Y_i/X_i)}{n} = \frac{\sum \theta X_i/X_i}{n} = \theta$

$\text{Bias}(\hat{\theta}, \theta) = E(\hat{\theta}) - \theta = \theta - \theta = 0$

$\text{Var}(\hat{\theta}) = \text{Var}\left(\frac{\sum Y_i/X_i}{n}\right) = \frac{\sum \text{Var}(Y_i)/X_i^2}{n^2} = \frac{\sum \sigma^2/X_i^2}{n^2} = \frac{\sigma^2}{n^2}\sum \frac{1}{X_i^2}$

(2) $L(\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\frac{(Y_i - \theta X_i)^2}{\sigma^2}\right\}$

$= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum (Y_i - \theta X_i)^2\right\}$

$\ell(\theta) = \log L(\theta) = -\frac{1}{2} n \log 2\pi - n\log\sigma - \frac{1}{2\sigma^2}\sum (Y_i - \theta X_i)^2$

$\frac{\partial \ell(\theta)}{\partial\theta} = -\frac{1}{2\sigma^2}\left[-2\sum X_i(Y_i - \theta X_i)\right] = 0$

$\Rightarrow \hat{\theta}_{MLE} = \frac{\sum X_i Y_i}{\sum X_i^2}$

$E(\hat{\theta}_{MLE}) = E\left(\frac{\sum X_i Y_i}{\sum X_i^2}\right) = \frac{\sum X_i E Y_i}{\sum X_i^2} = \frac{\sum X_i \theta X_i}{\sum X_i^2} = \theta$

$\text{Bias}(\hat{\theta}_{MLE}, \theta) = E(\hat{\theta}_{MLE}) - \theta = 0$

$\text{Var}(\hat{\theta}_{MLE}) = \text{Var}\left(\frac{\sum X_i Y_i}{\sum X_i^2}\right) = \frac{\sum X_i^2 \text{Var}(Y_i)}{(\sum X_i^2)^2} = \frac{\sum X_i^2 \sigma^2}{(\sum X_i^2)^2} = \frac{\sigma^2}{\sum X_i^2}$

(3) $MSE(\hat{\theta}) = Bias(\hat{\theta}, \theta) + Var(\hat{\theta}) = \frac{\sigma^2}{n^2} \Sigma \frac{1}{x_i^2}$

$MSE(\hat{\theta}_{MLE}) = \frac{\sigma^2}{\Sigma x_i^2}$

let $y_i = \frac{1}{x_i^2}$ , $z_i = x_i^2$ , $y_i = \frac{1}{z_i}$

$\Rightarrow MSE(\hat{\theta}) = \frac{\sigma^2}{n^2} \Sigma y_i = \frac{\sigma^2}{n} E y_i = \frac{\sigma^2}{n} E \frac{1}{z_i}$

$MSE(\hat{\theta}_{MSE}) = \frac{\sigma^2}{n E z_i}$

let $g(a) = \frac{1}{a}$ , we know that $E(g(x)) \geq g(E(x))$ for convex $g(a)$

so, $E(\frac{1}{z_i}) \geq 1/E(z_i)$

$\Rightarrow MSE(\hat{\theta}) \geq MSE(\hat{\theta}_{MLE})$

$\hat{\theta}_{MLE}$ is better

(4) the lager the $x_i$, the smaller the MSE for both estimators

so I recommend larger $\mu_x$ and $\sigma_x^2$

identify $c(y)$, $\theta$, $t(y)$. $A(\theta)$

3. (a)

i. $p(y) = \binom{n}{y} p^y (1-p)^{n-y}$

$c(y) = \binom{n}{y}$

$\exp\{\log(p^y (1-p)^{n-y})\} = \exp\{ y\log p + n\log(1-p) - y\log(1-p)\}$

$= \exp\{ y \log\frac{p}{1-p} + n\log(1-p)\}$

$\Rightarrow \theta = \log\frac{p}{1-p}$, $t(y) = y$, $A(\theta) = -n\log(1-p)$

ii. $p(y) = \mu^y e^{-\mu}/y!$

$c(y) = \frac{1}{y!}$

$\exp\{\log \mu^y - \mu\} = \exp\{y\log\mu - \mu\}$

$\Rightarrow \theta = \log\mu$, $t(y) = y$. $A(\theta) = \mu$

iii. $p(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}\}$

$e^{\log\frac{1}{\sigma}}$

$e^{-\log\sigma}$

$c(y) = \frac{1}{\sqrt{2\pi}}$

$\exp\{-\frac{1}{2\sigma^2}(y^2 - 2\mu y + \mu^2) - \log\sigma\} = \exp\{-\frac{1}{2\sigma^2}y^2 + \frac{\mu}{\sigma^2}y - (\frac{\mu^2}{2\sigma^2} + \log\sigma)\}$

$\Rightarrow \theta = \begin{bmatrix} -\frac{1}{2\sigma^2} \\ \frac{\mu}{\sigma^2} \end{bmatrix}$, $t(y) = \begin{bmatrix} y^2 \\ y \end{bmatrix}$. $A(\theta) = \frac{\mu^2}{2\sigma^2} + \log\sigma$

iv. $p(y) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} y^{a-1} (1-y)^{b-1}$

$$\exp\left\{\log\left[\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} y^{a-1} (1-y)^{b-1}\right]\right\}$$

$$= \exp\left\{ \log\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} + (a-1)\log y + (b-1)\log(1-y) \right\}$$

$$= \exp\left\{ \log\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} + a\log y + b\log(1-y) - \log y(1-y) \right\}$$

$$\Rightarrow \theta = \begin{bmatrix} a \\ b \end{bmatrix}, \quad t(y) = \begin{bmatrix} \log y \\ \log(1-y) \end{bmatrix}, \quad A(\theta) = \log\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$$

$$c(y) = \log\frac{1}{y(1-y)}$$

(b) $L(\theta) = \prod_{i=1}^{n} p(y|\theta) = \prod_{i=1}^{n} c(y)\exp\{\theta^T t(y_i) - A(\theta)\}$

$$\ell(\theta) = \log L(\theta) = \sum_{i=1}^{n} \log c(y_i) + \sum_{i=1}^{n} \theta^T t(y_i) - nA(\theta)$$

$$= \theta^T \Sigma t(y_i) - nA(\theta) + \text{constant}$$

(c) $\frac{\partial \ell(\theta)}{\partial \theta} = \Sigma t(y_i) - n\frac{\partial A(\theta)}{\partial \theta} = 0$

$$A'(\theta) = \frac{\Sigma t(y_i)}{n}$$

(d) $\frac{\partial p(y|\theta)}{\partial\theta} = \frac{\partial}{\partial\theta}\left[c(y)\exp\{\theta^T t(y) - A(\theta)\}\right]$

$= c(y)\exp\{\theta^T t(y) - A(\theta)\}\left[t(y) - A'(\theta)\right]$

$\int \frac{\partial p(y|\theta)}{\partial\theta}\, dy = \int p(y|\theta)\left[t(y) - A'(\theta)\right] d$

$= \int p(y|\theta)\, t(y)\, dy - \int p(y|\theta)\, A'(\theta)\, dy$

$= E(t(y)) - E(A'(\theta))$

when $\theta = \hat{\theta}_{MLE}$

$\int \frac{\partial p(y|\theta)}{\partial\theta}\, dy = 0 = E(t(y)) - A'(\hat{\theta}_{MLE})$

$E(t(y)) = A'(\hat{\theta}_{MLE}) = \frac{\Sigma t(y_i)}{n}$

which is the same with the former equation.

4.(a) $L(\theta) = \prod_{i=1}^{n} p(\theta) = \prod_{i=1}^{n} \theta^{y_i}(1-\theta)^{1-y_i}$

$\ell(\theta) = \Sigma\left[y_i \log\theta + (1-y_i)\log(1-\theta)\right]$

$= \log\theta \, \Sigma y_i + \log(1-\theta)(n-\Sigma y_i)$

$\frac{\partial \ell(\theta)}{\partial\theta} = \frac{\Sigma y_i}{\theta} - \frac{n-\Sigma y_i}{1-\theta} = 0$

$\Rightarrow \hat{\theta}_{MLE} = \frac{\Sigma y_i}{n}$

(b) $L(\psi) = \prod_{i=1}^{n} \left(e^{\psi}/(1+e^{\psi})\right)^{y_i} \left(1/(1+e^{\psi})\right)^{1-y_i}$

$\ell(\psi) = \Sigma\left[ y_i \log\frac{e^{\psi}}{1+e^{\psi}} + (1-y_i)\log\frac{1}{1+e^{\psi}}\right]$

$= (\psi - \log(1+e^{\psi}))\Sigma y_i - \log(1+e^{\psi})(n-\Sigma y_i)$

$\frac{\partial \ell(\psi)}{\partial\psi} = \Sigma y_i - \frac{\Sigma y_i}{1+e^{\psi}}\cdot e^{\psi} - \frac{n-\Sigma y_i}{1+e^{\psi}}\cdot e^{\psi}$

$= \Sigma y_i - \frac{n\cdot e^{\psi}}{1+e^{\psi}} = 0$

$\Rightarrow \hat{\psi}_{MLE} = \log\frac{\Sigma y_i}{n-\Sigma y_i}$

$\frac{e^{\hat{\psi}_{MLE}}}{1+e^{\hat{\psi}_{MLE}}} = \frac{\Sigma y_i/(n-\Sigma y_i)}{1+\Sigma y_i/(n-\Sigma y_i)} = \frac{\Sigma y_i}{n} = \hat{\theta}_{MLE}$

5. $\mathcal{P}_1 = \{f_\theta(y) : \theta \in \Theta\}$ $\quad\quad$ $\mathcal{P}_2 = \{g_\varphi(y) : \varphi \in \Psi\}$

$L(\theta) = \prod_{i=1}^{n} f_\theta(y_i)$

$l(\theta) = \log L(\theta) = \Sigma \log f_\theta(y_i) = \Sigma \log g_{h(\theta)}(y_i)$

$\frac{\partial l(\theta)}{\partial \theta} = \Sigma \frac{1}{f_\theta(y_i)} \frac{\partial f_\theta(y_i)}{\partial \theta} = 0$

$L(\varphi) = \prod_{i=1}^{n} g_\varphi(y_i)$

$l(\varphi) = \log L(\varphi) = \Sigma \log g_\varphi(y_i)$

$\frac{\partial l(\varphi)}{\partial \varphi} = \Sigma \frac{1}{g_\varphi(y_i)} \frac{\partial g_\varphi(y_i)}{\partial \varphi} = 0$

we know $f_\theta(y) = g_{h(\theta)}(y)$

So $\hat{\varphi}_{MLE} = h(\hat{\theta}_{MLE})$