STA 512 HW2 Due Jan 29, 2020

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1. $Y|\theta \sim N(\theta, \sigma^2)$

$$p(Y|\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\frac{(y-\theta)^2}{\sigma^2}\right\}$$

$\theta \sim N(\mu, \tau^2)$

$$p(\theta) = \frac{1}{\sqrt{2\pi\tau^2}} \exp\left\{-\frac{1}{2}\frac{(\theta-\mu)^2}{\tau^2}\right\}$$

$$p(Y,\theta) = p(Y|\theta)p(\theta)$$

$$= \frac{1}{2\pi\sigma\tau} \exp\left\{-\frac{1}{2}\left[\left(\frac{y-\theta}{\sigma}\right)^2 + \left(\frac{\theta-\mu}{\tau^2}\right)^2\right]\right\}$$

$$p(Y) = \int_{-\infty}^{+\infty} p(Y,\theta)\,d\theta$$

$$= \frac{1}{2\pi\sigma\tau} \int \exp\left\{-\frac{1}{2}\left[\left(\frac{y^2}{\sigma^2}+\frac{\mu^2}{\tau^2}\right) + \left(\theta^2\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right) - 2\theta\left(\frac{y}{\sigma^2}+\frac{\mu}{\tau^2}\right)\right)\right]\right\} d\theta$$

$$= \frac{1}{2\pi\sigma\tau} \exp\left\{-\frac{1}{2}\left(\frac{y^2}{\sigma^2}+\frac{\mu^2}{\tau^2}\right)\right\} \int \exp\left\{-\frac{1}{2}\left[\theta^2\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right) - 2\theta\left(\frac{y}{\sigma^2}+\frac{\mu}{\tau^2}\right)\right]\right\} d\theta$$

We can use kernel, let $a = \frac{1}{\sigma^2}+\frac{1}{\tau^2}$, $b = \frac{y}{\sigma^2}+\frac{\mu}{\tau^2}$

We have,

$$\int \frac{1}{\sqrt{2\pi/a}} \exp\left\{-\frac{1}{2}\frac{(\theta-b/a)^2}{1/a}\right\} d\theta = 1$$

$$\int \sqrt{\frac{\frac{1}{\sigma^2}+\frac{1}{\tau^2}}{2\pi}} \exp\left\{-\frac{1}{2}\left[\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right)\theta^2 - 2\theta\left(\frac{y}{\sigma^2}+\frac{\mu}{\tau^2}\right) + \frac{\left(\frac{y}{\sigma^2}+\frac{\mu}{\tau^2}\right)^2}{\frac{1}{\sigma^2}+\frac{1}{\tau^2}}\right]\right\} d\theta = 1$$

$$\int \exp\left\{-\frac{1}{2}\left[\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right)\theta^2 - 2\theta\left(\frac{1}{\sigma^2}+\frac{\mu}{\tau^2}\right)\right]\right\} d\theta = \sqrt{\frac{2\pi}{\frac{1}{\sigma^2}+\frac{1}{\tau^2}}} \exp\left\{\frac{1}{2}\frac{\left(\frac{1}{\sigma^2}+\frac{\mu}{\tau^2}\right)^2}{\frac{1}{\sigma^2}+\frac{1}{\tau^2}}\right\}$$

Then, $$p(Y) = \frac{1}{\sqrt{2\pi\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right)}\,\sigma\tau} \exp\left\{-\frac{1}{2}\left(\frac{y^2}{\sigma^2}+\frac{\mu^2}{\tau^2} - \frac{\left(\frac{y}{\sigma^2}+\frac{\mu}{\tau^2}\right)^2}{\frac{1}{\sigma^2}+\frac{1}{\tau^2}}\right)\right\}$$

$$\propto N(\mu, \tau^2+\sigma^2)$$

$$p(\theta|Y) = \frac{p(Y,\theta)}{p(Y)} = \frac{\frac{1}{2\pi\sigma\tau} \exp\left\{-\frac{1}{2}\left[\frac{(y-\theta)^2}{\sigma^2} + \frac{(\theta-\mu)^2}{\tau^2}\right]\right\}}{\frac{1}{\sqrt{2\pi(\tau^2+\sigma^2)}} \exp\left\{-\frac{1}{2}\frac{(y-\mu)^2}{\tau^2+\sigma^2}\right\}}$$

2. if r.v. X and Y are independent, $x, y \in R$

$$p_{X,Y}(x, y) = Pr(X = x, Y = y)$$
$$= Pr(X = x)\, Pr(Y = y)$$
$$= p_X(x)\, p_Y(y)$$

if $p_{X,Y}(x, y) = p_X(x)\, p_Y(y)$. $x, y \in R$

We take $A \subset R$, $B \subset R$

for continuous case:

$$Pr(X \in A, Y \in B) = \int_B \int_A p_{X,Y}(x, y)\, dx\, dy$$
$$= \int_B \int_A p_X(x)\, p_Y(y)\, dx\, dy$$
$$= \int_A p_X(x)\, dx \int_B p_Y(y)\, dy$$
$$= Pr(X \in A)\, Pr(Y \in B)$$

$\Rightarrow$ X and Y are independent

for discrete case:

$$Pr(X \in A, Y \in B) = \sum_B \sum_A p_{X,Y}(x, y)$$
$$= \sum_B \sum_A p_X(x)\, p_Y(y)$$
$$= \sum_A p_X(x) \sum_B p_Y(y)$$
$$= Pr(X \in A)\, Pr(Y \in B)$$

$\Rightarrow$ X and Y are independent.

3. if r.v. X, Y are independent, $x, y \in \mathbb{R}$

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} = \frac{Pr(X=x, Y=y)}{Pr(Y=y)}$$

$$= \frac{Pr(X=x)\, Pr(Y=y)}{Pr(Y=y)}$$

$$= Pr(X=x) = p_X(x)$$

if $p_{X|Y}(x|y) = p_X(x)$, $x, y \in \mathbb{R}$,

We take $A \subset \mathbb{R}$, $B \subset \mathbb{R}$

in continuous case:

$$Pr(X \in A, Y \in B) = \int_A \int_B p_{X,Y}(x,y)\, dx\, dy$$

$$= \int_A \int_B p_{X|Y}(x|y)\, p_Y(y)\, dx\, dy$$

$$= \int_A \int_B p_X(x)\, p_Y(y)\, dx\, dy$$

$$= Pr(X \in A)\, Pr(Y \in B)$$

$\Rightarrow$ X and Y are independent

in discrete case:

$$Pr(X \in A, Y \in B) = \sum_A \sum_B p_{X,Y}(x,y)$$

$$= \sum_A \sum_B p_{X|Y}(x,y)\, p_Y(y)$$

$$= \sum_A \sum_B p_X(x)\, p_Y(y)$$

$$= \sum_A p_X(x) \sum_B p_Y(y)$$

$$= Pr(X \in A)\, Pr(Y \in B)$$

$\Rightarrow$ X and Y are independent

4. $U \in R$, $V \in R$, g and h are invertible

$Pr(U=u, V=v)$

$= Pr(g(X)=u, h(Y)=v)$

$= Pr(X=g^{-1}(u), Y=h^{-1}(v))$

because X and Y are independent

$= Pr(X=g^{-1}(u)) Pr(Y=h^{-1}(v))$

$= Pr(g(X)=u) Pr(h(Y)=v)$

$= Pr(U=u) Pr(V=v)$

$\Rightarrow$ U and V are independent

5. $Y_1 \cdots Y_n \overset{iid}{\sim} p(y)$, cdf is $F_Y(y)$

① $Y_{(1)} = \min\{Y_1 \cdots Y_n\}$

$$F_{Y_{(1)}}(y) = Pr(Y_{(1)} \le y) = 1 - Pr(Y_{(1)} > y)$$
$$= 1 - \prod_{i=1}^{n} Pr(Y_i > y)$$
$$= 1 - \prod_{i=1}^{n} (1 - Pr(Y_i \le y))$$
$$= 1 - \prod_{i=1}^{n} (1 - F_Y(y))$$
$$= 1 - (1 - F_Y(y))^n$$

$$p_{Y_{(1)}}(y) = \frac{d F_{Y_{(1)}}(y)}{dy} = n(1 - F_Y(y))^{n-1} \, p(y)$$

② $Y_{(n)} = \max\{Y_1 \cdots Y_n\}$

$$F_{Y_{(n)}}(y) = Pr(Y_{(n)} \le y)$$
$$= \prod_{i=1}^{n} Pr(Y_i \le y)$$
$$= (F_Y(y))^n$$

$$p_{Y_{(n)}}(y) = \frac{d F_{Y_{(n)}}(y)}{dy} = n F_Y(y)^{n-1} \, p(y)$$

6. (a) $F_{Y_{(1)}}(y) = 1 - [1 - F(y)]^n$

$p_{Y_{(1)}}(y) = n(1 - F(y))^{n-1} p(y)$

$p(y) = \lambda e^{-\lambda y}$, $F_Y(y) = 1 - e^{-\lambda y}$

$F_{Y_{(1)}}(y) = 1 - e^{-n\lambda y}$

$p_{Y_{(1)}}(y) = n\lambda e^{-n\lambda y}$

$Y_{(1)} \sim \exp(1/\lambda n)$

(b) $Y \in [0, 1]$

$p(y) = 1$, $F_Y(y) = y$

$F_{Y_{(1)}}(y) = 1 - (1 - y)^n$

$p_{Y_{(1)}}(y) = n(1-y)^{n-1}$

(c) Y follows a discrete distribution with pdf

$p(y) = \theta(1-\theta)^{y-1}$ for $Y \in \{1, 2, \cdots\}$

$F_Y(y) = \sum_{y' \le y} p(y') = \sum_{y=1}^{n} \theta(1-\theta)^{y-1} = 1 - (1-\theta)^y$

$F_{Y_{(1)}}(y) = 1 - (1-\theta)^{ny}$

$p_{Y_{(1)}}(y) = \Pr(Y_{(1)} = y)$

$= \Pr(Y_{(1)} \le y) - \Pr(Y_{(1)} \le y-1)$

$= (1 - (1-\theta)^{ny}) - (1 - (1-\theta)^{n(y-1)})$

$= (1-\theta)^{ny}((1-\theta)^{-n} - 1)$

7. (a) $Y \in (0, \infty)$

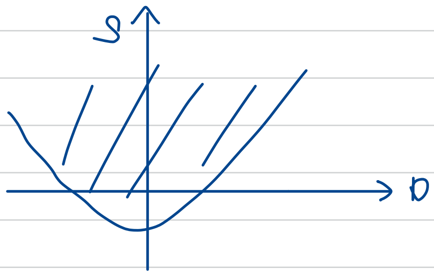
$$Pr(Y_i \le y) = \int_0^y p(t)\, dt$$

$$= \int_0^y \tfrac{1}{\lambda} e^{-t/\lambda}\, dt$$

$$= 1 - e^{-y/\lambda}$$

(b) $S + D = 2Y_1 \quad S \in (-D, +\infty)$

$S - D = 2Y_2 \quad S \in (D, +\infty)$



(c) $Y_1 \cdots Y_n \overset{iid}{\sim} p(y)$

$$p_{Y_1 Y_2}(y_1, y_2) = p_{Y_1}(y_1)\, p_{Y_2}(y_2) = \left(\tfrac{1}{\lambda}\right)^2 e^{-\frac{y_1+y_2}{\lambda}}$$

let $X = \begin{pmatrix} Y_1 + Y_2 \\ Y_1 - Y_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$

$$Y' = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \frac{X_1+X_2}{2} \\ \frac{X_1-X_2}{2} \end{pmatrix}$$

$$p_X(x) = p_{Y'}(y') \left| \frac{\partial Y'}{\partial X} \right| = \frac{1}{2\lambda^2} e^{-\frac{y_1+y_2}{\lambda}} = \frac{1}{2\lambda^2} e^{-\frac{x_1}{\lambda}}$$

$\Rightarrow \quad p_{S,D}(s,d) = \frac{1}{2\lambda^2} e^{-s/\lambda} \qquad |d| < s$

$p_S(s) = \int_{-s}^{s} p_{S,D}(s,d)\,dd$

$= \frac{s}{\lambda^2} e^{-s/\lambda}$

$0 < s < \infty$

$p_{D|S}(d|s) = \frac{p_{S,D}(s,d)}{\quad} = \frac{1}{2s}$

(d) $\quad p_D(d) = \int_0^\infty p_{S,D}(s,d)\,ds$

$\because \; s > |d|$

if $d \geq 0$, $s \in (d, \infty)$, $p_D(d) = \int_d^\infty p_{S,D}(s,d)\,ds$

$= \frac{1}{2\lambda} e^{-d/\lambda}$

if $d < 0$, $s \in (-d, \infty)$, $p_D(d) = \int_{-d}^\infty p_{S,D}(s,d)\,ds$

$= \frac{1}{2\lambda} e^{d/\lambda}$