STA 532 HW1 Due Jan 22, 2020

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1. For continuous case:

we know that

$$P(T=y) = F(y) - \sup_{y'<y} F(y') = F(y) - F(y) = 0$$

if $a \geq b$, then

① $Pr(T \in (a,b]) = 0$

② $Pr(T \in (a,b)) = 0$

③ $Pr(T \in [a,b]) = 0$

else, we have

① $Pr(T \in (a,b]) = Pr(T \leq b) - Pr(T \leq a) = F(b) - F(a)$

② $Pr(T \in (a,b)) = Pr(T < b) - Pr(T \leq a)$

$= Pr(T < b) + Pr(T = b) - Pr(T \leq a)$

$= F(b) - F(a)$

③ $Pr(T \in [a,b]) = Pr(T \leq b) - Pr(T < a)$

$= Pr(T \leq b) - Pr(T < a) - Pr(T = a)$

$= F(b) - F(a)$

For discrete case:

we know that

$$P(Y=y) = F(y) - \sup_{y'<y} F(y')$$

if $a \geq b$, then

① $Pr(Y \in (a, b]) = 0$

② $Pr(Y \in (a, b)) = 0$

③ if $a > b$, $Pr(Y \in [a, b]) = 0$

if $a = b$, $Pr(Y \in [a, b]) = Pr(Y = a) = F(a) - \sup_{y'<a} F(y')$

else, we have

① $Pr(Y \in (a, b]) = Pr(Y \leq b) - Pr(Y \leq a) = F(b) - F(a)$

② $$\begin{aligned} Pr(Y \in (a, b)) &= Pr(Y < b) - Pr(Y \leq a) \\ &= Pr(Y < b) + Pr(Y = b) - Pr(Y = b) - Pr(Y \leq a) \\ &= F(b) - (F(b) - \sup_{y'<b} F(y')) - F(a) \\ &= \sup_{y'<b} F(y') - F(a) \end{aligned}$$

③ $$\begin{aligned} Pr(Y \in [a, b]) &= Pr(Y \leq b) - Pr(Y < a) \\ &= Pr(Y \leq b) - Pr(Y < a) - Pr(Y = a) + Pr(Y = a) \\ &= F(b) - F(a) + (F(a) - \sup_{y'<a} F(y')) \\ &= F(b) - \sup_{y'<a} F(y') \end{aligned}$$

2. (a) $W = g(Y) = -\log Y$,

$y \in (0,1)$, $y = e^{-w}$, then g is monotonic, $w \in (-\infty, +\infty)$

$$p_W(w) = \left| \frac{d e^{-w}}{dw} \right| p_Y(e^{-w}) = e^{-w}$$

(b) $W = g(Y) = 1/Y$

$y \in R$, $y = 1/w$, then g is not monotonic at $y = 0$

if $y < 0$,

$$p_W(w) = \left| \frac{d\, 1/w}{dw} \right| \cdot p_Y(1/w) = \frac{1}{2(w^2+1)}, \quad w \in (0, +\infty)$$

if $y > 0$,

$$p_W(w) = \left| \frac{d\, 1/w}{dw} \right| \cdot p_Y(1/w) = \frac{1}{2(w^2+1)}, \quad w \in (-\infty, 0)$$

(c) $W = g(Y) = e^Y$

$Y \sim N(0,1)$, $y = \log w$, then g is monotonic, $w \in (0, +\infty)$

$$p_W(w) = \left| \frac{d \log w}{dw} \right| p_Y(\log w) = \frac{1}{w} \frac{\exp(-\frac{1}{2}(\log w)^2)}{\sqrt{2\pi}}$$

(d) $W = g(Y) = Y^2$

$Y \sim t_v$, g is not monotic, $W \in [0, +$

$F_W(w) = Pr(W \le w) = Pr(Y^2 \le w) = Pr(-\sqrt{w} \le Y \le \sqrt{w})$

$= F_Y(\sqrt{w}) - F_Y(-\sqrt{w})$

$$p_W(w) = \frac{\partial F_W(w)}{\partial w} = \frac{\partial F_Y(\sqrt{w})}{\partial w} - \frac{\partial F_Y(-\sqrt{w})}{\partial w}$$

$$= p_Y(\sqrt{w}) \left| \frac{\partial \sqrt{w}}{\partial w} \right| - p_Y(-\sqrt{w}) \left| \frac{\partial(-\sqrt{w})}{\partial w} \right|$$

$$= \frac{\left(1 + \frac{w}{v}\right)^{-\frac{v+1}{2}} \Gamma\left(\frac{v+1}{2}\right)}{w^{-\frac{1}{2}} \sqrt{v\pi}\, \Gamma\left(\frac{v}{2}\right)}$$

3. (a) $F_u(u) = Pr(U \le u) = Pr(F_Y(Y) \le u)$
$= Pr(Y \le F_Y^{-1}(u)) = F_Y(F_Y^{-1}(u))$
$= u$

(b) $F_X(x) = Pr(X \le x) = Pr(F_Y^{-1}(U) \le x)$
$= Pr(U \le F_Y(x)) = F_u(F_Y(x))$
$= F_Y(x)$

(c) qnorm(runif(n))

$P_u(u) = 1$
From the derivation above we can see that a continuous strictly increasing CDF follows a uniform distribution.

We can first use runif() to generate values from uniform distribution, then use qnorm() to simulate a normal distribution.

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7. $\int \Pr(X \in A \mid Y = y)\, p_Y(y)\, dy$

$$= \int \frac{\Pr(X \in A, Y = y)}{p_Y(y)}\, p_Y(y)\, dy$$

$$= \int \Pr(X \in A, Y = y)\, dy$$

$$= \iint_A p_{X,Y}(x, y)\, dx\, dy$$

$$= \int_A p_X(x)\, dx$$

$$= \Pr(X \in A)$$

We sum up all the probabilities, for all possible values of $y \in Y$ of both $Y = y$ and $x \in A$ under the circumstance of $Y = y$ happening together, we can get the probability of $X \in A$ happening. This is an analogy to the discrete case.

For the continuous case, we can divide the value domain of Y to very small divisions, then sum up all the divisions and get the probability of $X \in A$.

5. $\lim_{\varepsilon \to 0} Pr(X \in A \mid Y \in B_\varepsilon)$

$$= \lim_{\varepsilon \to 0} \frac{Pr(X \in A, Y \in B_\varepsilon)}{Pr(Y \in B_\varepsilon)}$$

$$= \lim_{\varepsilon \to 0} \frac{\int_A \int_{y-\varepsilon}^{y} p_{X,Y}(x,t)\, dx\, dt}{\int_{y-\varepsilon}^{y} p_Y(t)\, dt}$$

$$\lim_{\varepsilon \to 0} \int_A \int_{y-\varepsilon}^{y} p_{X,Y}(x,t)\, dx\, dt$$

$$= \varepsilon \cdot \int_A \lim_{\varepsilon \to 0} \frac{\int_{y-\varepsilon}^{y} p_{X,Y}(x,t)\, dt}{\varepsilon}\, dx$$

$$= \varepsilon \cdot \int_A \lim_{\varepsilon \to 0} \frac{F_{X,Y}(x,y) - F_{X,Y}(x, y-\varepsilon)}{\varepsilon}\, dx$$

$$= \varepsilon \cdot \int_A p_{X,Y}(x,y)\, dx$$

$$\lim_{\varepsilon \to 0} \int_{y-\varepsilon}^{y} p_Y(t)\, dt$$

$$= \varepsilon \cdot \lim_{\varepsilon \to 0} \frac{\int_{y-\varepsilon}^{y} p_Y(t)\, dt}{\varepsilon}$$

$$= \varepsilon \cdot \lim_{\varepsilon \to 0} \frac{F_Y(y) - F_Y(y-\varepsilon)}{\varepsilon}$$

$$= \varepsilon \cdot p_Y(y)$$

$$\Rightarrow \lim_{\varepsilon \to 0} Pr(X \in A \mid Y \in B_\varepsilon)$$

$$= \frac{\varepsilon \int_A p_{X,Y}(x,y)\, dx}{\varepsilon\, p_Y(y)} = \int_A \frac{p_{X,Y}(x,y)}{p_Y(y)}\, dx$$

$$= \int_A p_{X|Y}(x|y)\, dx$$

6. $Y|X \sim \text{Gamma}(c, x)$

$p_{Y|X}(y|x) = x^c \, y^{c-1} \, e^{-xy} / \Gamma(c)$

$X \sim \text{Gamma}(a, b)$

$p_X(x) = b^a \, x^{a-1} \, e^{-bx} / \Gamma(a)$

$$\Rightarrow \; p_{X,Y}(x, y) = p_{Y|X}(x, y) \times p_X(x)$$

$$= x^{a+c-1} \, e^{-xy-bx} \, y^{c-1} \frac{b^a}{\Gamma(a)\Gamma(c)}$$

$$\Rightarrow \; p_Y(y) = \int_0^{+\infty} p_{X,Y}(x, y) \, dx$$

$$= \int_0^{+\infty} x^{a+c-1} \, e^{-xy-bx} \, y^{c-1} \frac{b^a}{\Gamma(a)\Gamma(c)} \, dx$$

$$= y^{c-1} \frac{b^a}{\Gamma(a)\Gamma(c)} \int_0^{+\infty} \underbrace{x^{a+c-1} \, e^{-xy-bx}}_{\text{kernel of Gamma}(a+c,\, y+b)} \, dx$$

we have $\int_0^{+\infty} x^{a+c-1} \, e^{-xy-bx} \, (y+b)^{a+c} / \Gamma(a+c) \, dx = 1$

$$\Rightarrow \int_0^{+\infty} x^{a+c-1} \, e^{-xy-bx} \, dx = \frac{\Gamma(a+c)}{(y+b)^{a+c}}$$

$$p_Y(y) = y^{c-1} \frac{b^a}{\Gamma(a)\Gamma(c)} \frac{\Gamma(a+c)}{(y+b)^{a+c}}$$

$$\Rightarrow \; p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

$$= \frac{x^{a+c-1} \, e^{-xy-bx} \, \cancel{y^{c-1}} \, \cancel{b^a}}{\cancel{\Gamma(a)} \, \cancel{\Gamma(c)}} \times \frac{\cancel{\Gamma(a)} \cancel{\Gamma(c)} (y+b)^{a+c}}{\cancel{y^{c-1}} \, \cancel{b^a} \, \Gamma(a+c)}$$

$$= \frac{x^{a+c-1} \, e^{-xy-bx} \, (y+b)^{a+c}}{\Gamma(a+c)}$$