Reconceptualizing Mathematical Learning Disabilities: A Diagnostic Case Study

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Abstract: Mathematical learning disabilities (MLDs) research aims to define, diagnose, and remediate the cognitive impairments that characterize MLDs. Unfortunately little progress has been made towards these goals, because of unresolved methodological issues involved in distinguishing general low math achievement from low math achievement due to MLDs. To address these methodological issues, I conducted a small-scale exploratory study of MLD, which relied on several data sources to ensure that a student's low achievement was due to persistent and debilitating difficulties with mathematics that cannot be attributed to non-cognitive sources. I conducted four weekly video-taped one-on-one tutoring sessions with students focused on the topic of fractions. In this paper I present a case study of one student with MLD. Analysis indicated that the student understood mathematical representations of fractional quantity in atypical ways and this alternative understanding was problematic for the development of more advanced fraction concepts.

Introduction

Lisa counts on her fingers to solve addition problems, cannot add fractions, and has just failed an arithmetic course. Lisa is a 19-year-old college student. Despite her best efforts and years of tutoring, she has not mastered elementary mathematics. Unfortunately, research on mathematical learning disabilities (MLDs) offers little insight to help understand why students, like Lisa, experience persistent, pervasive, and debilitating difficulties with mathematics. How are the difficulties experienced by students with MLD qualitatively different than those experienced by all students? What are the sources of the student's difficulties? How can researchers distinguish students with MLDs from students with low mathematics achievement? These difficult questions remain unanswered and are the focus of this exploratory study.

Research on MLD is in its infancy, as compared to research on more general learning disabilities (Fletcher, Lyon, Fuchs, & Barnes, 2008; Mazzocco 2007). Learning disability research has traditionally focused on language-based learning problems, which involve difficulties with processing phonemes (the smallest unit of sound), and therefore provides little insight into the cognitive difficulties students experience in mathematics (Fletcher et al., 2008). MLDs are typically understood as core cognitive deficits, which result in a student's failure to learn or remember mathematical procedures or concepts (Geary, 2004). However, there is currently no consensus operational definition of MLDs and no diagnostic instruments available to accurately identify students with MLDs (Murphy, Mazzocco, Hanich, & Early, 2007). Consequently, researchers rely on low math achievement test scores to classify students, which does not address the myriad reasons, besides MLDs, a student could have performed poorly on the test. Given the definitional and methodological issues hindering this field, I adopt an alternative theoretical and methodological approach to the study of MLDs.

In my research I adopt a *difference*, rather than *deficit*, understanding of disability and leverage advances made in language-based learning disability research for both subject identification and analytic foci. I ensure that the students, not only meet the low-achievement criteria established by other researchers, but that the student's low-achievement is likely due to a cognitive rather than environmental factor. I define MLD by building upon innovative subject classification methodologies used in learning disability research, which require that students demonstrate (1) low achievement (for my purposes, in mathematics), (2) no evidence of confounding factors explaining that low achievement, and (3) failure to respond to an instructional intervention (Fletcher et al., 2008). In this case study, I conducted a detailed qualitative analysis of one student as she engaged in the process of learning over several tutoring sessions with a particular focus on the student's ability to conceptualize, represent, and work with numbers, specifically fractions. Just as the capacity to process phonemes is fundamental to reading competency, numbers are the fundamental building block of mathematics (Dehaene 1997, Landerl, Bevan & Butterworth, 2004). Numbers, or quantities, get represented in a variety of modalities (e.g., orally: "three-fourths", numerically "3/4", and graphically: area model for 3/4, see Figure 1). Fluency with these representational systems is a crucial part of what it means to do and understand mathematics (Ball, 1993; Lesh, Post, & Behr, 1987; NCTM Principles and Standards, 2000). This suggests that students' fluency with number, quantity, and representations is a crucial component of mathematical development and a productive analytic lens through which to explore MLDs.

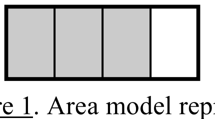

Figure 1. Area model representation of 3/4.

This exploratory case study focuses on identifying characteristics of MLDs. I conducted detailed diagnostic analyses of tutoring sessions focusing on the difficulties the student experienced, the kinds of atypical understandings she demonstrated, and her use of representations. This research addresses the following questions:

1) What is the nature of the student's difficulties during the tutoring sessions?
2) What are the characteristics of the student's atypical understanding of mathematical concepts over the course of the tutoring sessions?
3) How does the student use and understand mathematical representations?

Analysis indicated that the student experiences persistent difficulties with fraction equivalence and operations. Additionally, the student demonstrated atypical understandings of representations of fractional quantity, which hindered her ability to make sense of more complex fraction concepts.

Prior Research on Mathematical Learning Disabilities

Prior research on MLDs has made limited progress towards understanding this complex phenomenon due to a lack of a consensus definition for what constitutes a MLD. Currently, it is a widely accepted practice to use achievement test scores below a given cut-off (typically the 25th percentile) to identify students with MLDs (Murphy et al., 2007; Geary, 2004). Various critiques of this approach have been identified. The cut-off score used is arbitrarily selected (Francis et al., 2005), and can range from the 8th percentile to the 46th percentile for studies of MLDs (Gersten, Clarke, & Mazzocco, 2007; Swanson & Jerman, 2006). This has led to variability in the profile of students classified as having MLDs, suggesting researchers are not all studying a common phenomenon (Murphy et al., 2007). In addition, researchers rarely attempt to control for factors that are well-established as correlated with low-achievement. Hanich, Jordan, Kaplan, and Dick (2001) report that studies of MLDs have an over-representation of minority, poor, and non-native English speaking students in the MLD group. This signifies that the proxy of low achievement does not adequately address the complex social components at play in the operational definition of MLDs. Given the lack of adequate subject identification methods, it can be argued that researchers are studying *general characteristics of low math achievement rather than MLDs*. To address this methodological hurdle, I propose adopting an alternative theoretical perspective and methodological approach.

Theoretical Approach

Rather than adopting the traditional conceptualization of MLDs as cognitive *deficits* (Geary, 2004), I conceptualize MLDs in terms of cognitive *difference.* A deficit understanding of MLDs orients researchers to identifying the ways in which a group of students classified as having MLDs are in some way deficient when compared to a typically achieving group. Therefore, any statistically significant group difference is assumed to be consequential and indicative of the students' MLDs. In contrast, a *difference* understanding of MLDs orients researchers to identifying the nature of the individual student's qualitative differences. Instead of asking: "by how much is the MLD group deficient?" this approach asks: "what differences are consequential for an individual with a MLD and how are they consequential?" Recent MLD research supports the idea that qualitative differences exhibited by students with MLDs may ultimately be the key to differentiating these learning disabilities from low math achievement (Mazzocco, Devlin, & McKenney, 2008). Therefore, I adopt a difference model of MLDs, building on a Vygotskian perspective of disability, in which a student with a disability "is not simply a child less developed than his peers but is a child who has developed differently" (Vygotsky, Knox, Stevens, Rieber, & Carton, 1993 p. 30). I argue that the nature of a student's disability is revealed through an analysis of *qualitative* differences in how the student is reasoning over time, rather than in a quantification of deficits in knowledge measured at one point in time.

Fractions

For a number of reasons, fractions provide a mathematically rich terrain to explore how students with MLDs make sense of mathematical concepts. First, MLDs research to date has focused almost exclusively on students' difficulties remembering math facts (e.g., 4+5=9; Gersten, Jordan, & Flojo, 2005; Swanson & Jerman, 2006). Equating MLDs with difficulty learning math facts reduces mathematical cognition to efficient and accurate production of an answer. This finding does little to explain the range of difficulties students may have across the variety of mathematical domains (Dowker, 2005; Geary, 2005). Second, the kinds of difficulties that typically achieving students experience when learning fraction concepts have been extensively documented in math cognition research (see Lamon, 2007 for a review). It is therefore possible to classify the difficulties demonstrated by students with MLDs as typical or atypical. Considering MLDs in the context of fractions allows for the exploration of procedural, conceptual, and representational difficulties that students with a MLD may experience in this more complex mathematical domain.

Methods
This case study was drawn from a larger study that involved an extensive subject recruitment and classification process to ensure that each student's mathematical difficulties were due to an MLD and not other factors. Low achievement in math was considered a necessary but not a sufficient criterion for a MLD classification. In addition, subjects classified as having MLDs did not exhibit any evidence of other factors that could explain their low achievement and when given a sufficient instructional intervention, the student failed to show learning gains. Figure 2 provides an overview of each phase of this project including: initial recruitment, data collection, subject classification, and diagnostic analysis. Each will be discussed in turn.

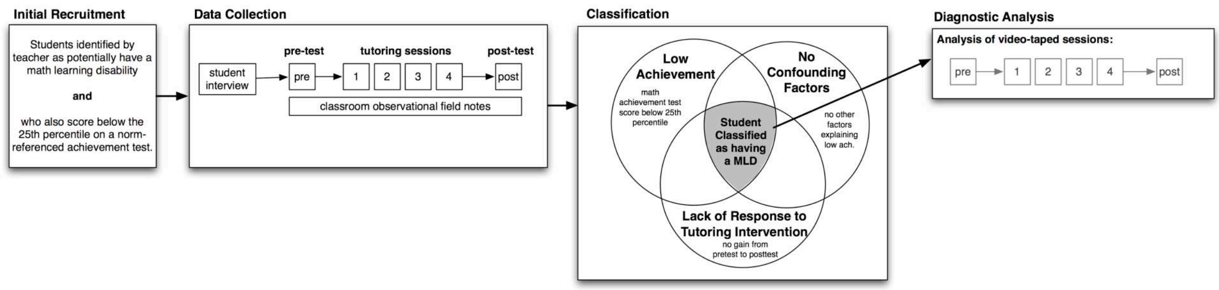

Figure 2. Overview of the design of the study.

Initial Recruitment
Subjects were recruited from several school sites, including: a public middle school, a private high school, and a community college. Math teachers from these schools were asked to identify students who they believe may have a MLD. For those students identified, only students who met the canonical MLDs low-achievement qualification (scoring below the 25th percentile on their most recent math achievement test) were invited to participate in the study.

Data Collection

Student Interview
A 30-minute interview was conducted with the student before the administration of the pretest. The interview questions attempted to elicit the student's characterization of his/her difficulties (e.g., what about learning math is hard for you?), perceived level of effort (e.g., do you tend to complete all your math homework?) and available resources (e.g., what do you do when you don't know how to answer a math problem?)

Pretest / Posttest
A videotaped semi-structured clinical interview-style pretest and identical posttest was administered to all subjects. The assessment was designed to cover all fraction concepts targeted in the tutoring sequence (see Table 1 for exemplar problems). The pretest was administered the week before the commencement of tutoring sessions and the posttest was administered the week following the completion of the tutoring sessions. The student's answers and explanations were scored according to a rubric and total scores were calculated for each of the target concepts.

Tutoring Sessions
Four hour-long weekly videotaped tutoring sessions were conducted with the subjects focusing on fraction concepts. These sessions were designed based on prior research on the teaching and learning of fraction concepts and have been piloted and refined. In each session a sequence of problems was posed to the student, each problem was intended to build upon prior problems and provoke conversations about important math content. Table 1 provides details of the targeted concepts and exemplar problems from the pretest/posttest and tutoring session problems. Each of the problems had follow-up questions and/or counter suggestions. To refine the tutoring protocol (pretest, tutoring sequence, and posttest), it was administered to fifth-grade students with no learning difficulties. The fifth-grade students demonstrated substantial gains from pretest to posttest and therefore, this instructional tutoring sequence was considered to be an adequate learning environment.

Table 1: Target fraction concepts, exemplar problems from pretest/posttest and tutoring sessions, and references for related research.

Concept	Details	Exemplar Pretest/Posttest Problem	Exemplar Tutoring Session Problem	Related Research
Fraction Representation and Interpretation	A fraction represents a single number rather than two independent numbers, which involves determining the number of equal parts of a unit out of the total number of equal parts.	Can you circle the pictures that you think are the same as 1/2? [*representations include number lines, discrete pieces, area models. Both 1/2 and equivalent fractions (e.g. 2/4) are included along with common student errors.*]	Without writing down any numbers can, you draw a picture of 3/4 so that other people would know that it's a picture of 3/4?	Ball (1993) Lesh et al. (1987) Mack (1990) Post et al. (1985) Behr et al. (1992) Saxe et al. (2005)
Fraction Comparison	Fraction size depends on relationship between two numbers.	Which is bigger or are they equal? [*2 area models are presented based on Armstrong & Larson*]	Put these cards in order from least to greatest. (Card values: 0, 1/1000, 1/2, 2/3, 8/12, 9/10, 7/7, 1, 1 2/5, 3/2)	Armstrong & Larson (1995) Post et al., (1985)
Equivalent Fractions	Many different fraction names designate the same amount	Can you come up with a fraction equivalent to: ½, 1/3, 2/5, 8/12?	Using a whole sheet of paper, draw 2/3 of a cake. Figure out how you can cut the cake again so you still have even pieces. What has change and what has stayed the same?	Lamon (1996) Lamon (2006) Lamon (2007) Post et al. (1985) Ni (2001) Kamii & Clark(1995)
Fraction Addition	Addition and subtraction of fractions requires common denominators	1/2+1/4=	1/2+1/3 = How would you solve this problem using pictures?	Mack (1992) Steffe (2003)

Classroom Observations

Classroom observations were conducted concurrently with the tutoring sessions. Field notes documented any evidence of the student's attention or behavior issues, level of effort, and performance relative to peers. This was an exclusionary data source intended to capture evidence of students who have non-cognitive issues, which could explain their poor performance.

Subject Classification

Several data sources were drawn upon to determine if the student's low achievement could be explained by other factors. Those factors highly correlated with low math achievement were considered confounding factors for the purposes of this study. These included: attention or behavior problems, lack of English fluency, insufficient resources, math anxiety (Ashcraft, Krause, & Hopko, 2007; Chatterji, 2005; Diversity in Mathematics Education, 2007; Zentall, 2007). Classroom observations, interviews, and the videotaped pretest, tutoring sessions, and posttest were used to determine if any students exhibited or reported potential confounding factors. I do not assume that MLD cannot co-occur with these factors, but that disentangling the effect of the confounding factor from the effect of MLD is beyond the scope of this study, and therefore any students demonstrating these factors were excluded. The goal was not to identify all students with MLDs, but to ensure that all students included in the larger study had an MLD. In addition, this tutoring protocol attempts to rule out the possibility that the student's history of low-achievement was simply due to poor instruction. If the student *did* show gains from pretest to posttest, it suggests that prior instruction may have been insufficient and an underlying cause of the student's low achievement. Conversely, if the student *did not* show gains from pretest to posttest, this suggests that poor instruction was not the primary cause of the student's low achievement and indicates that the student likely has an MLD. For example, Figure 3 displays a comparison of a fifth grade student's pretest-posttest change as compared to that of a community college case study student (Lisa), who I classify as having a MLD. The contrast between the student's scores (as measured by slope from pretest to posttest) is striking, providing strong substantiation of an MLD classification for Lisa.

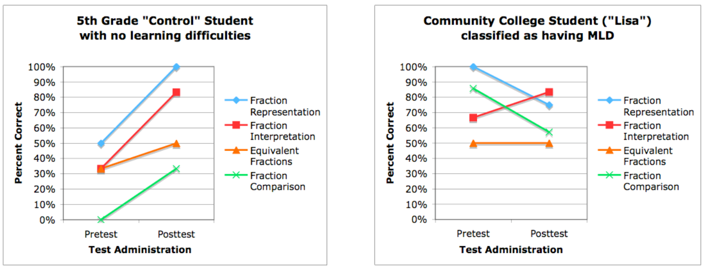

Figure 3. Comparison of a pretest/posttest scores for a typically-achieving fifth grade student and a community college student who is classified a having a MLD.

Diagnostic Analytic Approach

All videotapes of the sessions (pretest, tutoring sequence, and posttest) were transcribed and all student work was scanned. A microgenetic bottom-up analysis of the whole data corpus for each student was conducted (see Schoenfeld, Smith, & Arcavi, 1993, for an example of this kind of analysis applied to graphing). This kind of analysis involved iterative passes through the data in an attempt to generate analytic categories that capture the nature of the student's understanding (including difficulties, atypical understandings, and uses of representations). The generation and refinement of the analytic categories involved a process of compiling the instances of a proposed kind of understanding, specifying inclusion and exclusion criteria, generating a hypothesis about the understanding, and returning to the entire data corpus (both transcripts and videos) to support or reject the hypothesis. Analysis focused on identifying the kinds of student understanding that resulted in difficulties or appeared to be consequential for how the student was making sense of the mathematics. Some of these student understandings were well documented as common problems in the math cognition literature (e.g., thinking that 1/100 is larger than 1/10 because 100 is larger than 10) and were therefore not the focus of my analysis. The rationale for focusing on these atypical understandings was an attempt to evaluate what, beyond the canonical difficulties, resulted in the student's failure to learn during the tutoring sessions.

Results

The focal case study subject, Lisa, was considered to have a MLD because she exhibits low achievement, which was not attributable to other factors, and because she failed to respond to the tutoring instruction. Lisa had just completed her first year of community college, and her math achievement test score was in the lowest 25th percentile. Interviews and observations revealed that there were no identifiable confounding factors; Lisa was a white upper-middle class, native English speaker with no attention or behavior problems. Lisa's overall score did not improve from prettest to posttest, and for some fraction topics, her score was lower at the time of the posttest (see Figure 3).

Diagnostic Analysis

Why does Lisa fail to learn from the tutoring sessions? A detailed analysis of all sessions (pretest, tutoring sessions, and posttest) indicates that Lisa experienced difficulties with comparing fractions, adding fractions, and generating equivalent fractions throughout all of the sessions including the posttest. Her understanding of fractional quantity appeared to be unstable, in that she operated on representations of fractions as if they represented a partitioning or removal activity as opposed to a fractional quantity. This alternative understanding of fraction representations appears to derail Lisa's ability to build an understanding of fraction equivalence, comparisons, and operations.

A detailed analysis of Lisa's use and understanding of area model representations was conducted. Area models are a common pedagogical representation for fractions, in which a shape is partitioned into evenly-sized pieces corresponding to the denominator, and then the number of pieces corresponding to the numerator is shaded. For example, an area model for 7/12 would be drawn by partitioning a shape into 12 even pieces and then shading 7 of those pieces. Lisa, however, understood area models as if they signified an *action* as opposed to a *quantity*. This understanding has two primary manifestations. First, the shading of area models was understood as the amount taken away, rather than indicative of the fractional quantity. Second, Lisa understood "one-half" as an *action* of halving rather than the quantity of 1/2. These understandings were considered atypical because they have not been documented in the math cognition literature, and were not experienced by any of the fifth-grade subjects used to validate the tutoring protocol. In addition, this alternative understanding proved to be highly consequential for the student's ability to make sense of more complex fraction concepts.

Understanding Area Models as Representing an Amount "Gone"

Lisa understood the shading of area models as the amount removed, rather than indicative of the fractional quantity. Lisa's focus on the removal action as opposed to quantity led to inconsistently between her construction and interpretation of area model representations. In the following example, she was asked to compare the fractions 7/12 and 1/2. She correctly constructed a drawing of 7/12 by partitioning the shape into 12 pieces and shading 7 of those pieces. However, she incorrectly determined that 7/12 is *smaller* than 1/2 and when I asked her to explain she referred to the shaded pieces as "gone" and began attending to the five pieces "left."

a)  b) 

Figure 4. Reproduction of the student's drawing of 7/12, in which she (a) partitioned a shape into 12 pieces, and then (b) colored 7 of the pieces.

L: I mean, ok, so let's say that this is the cake (gestures back and forth over entire shape– Figure 4b) and seven pieces are gone (makes sweeping motion over the shaded pieces of the shape).
…
K: Which one do I have more for?
L: Um. The half.

Although she drew a canonical area model (using shading to represent the numerator) she subsequently interpreted the shaded region of area models as being "gone". This disconnection between her correct construction and incorrect interpretation was not easily remediated and persisted throughout the tutoring sessions to the posttest. I argue that, for Lisa, the shading represented the *action* of removal rather than the fractional *quantity*. In this example, her ability to make sense of the fraction comparison task was compromised by her atypical conception of the mathematical representation. Throughout the tutoring sessions, this understanding of fractional representations in terms of action appeared to be highly consequential and detrimental to her ability to build a more complete understanding of fraction equivalence, fraction comparison, and fraction addition.

Understanding "one-half" as Partitioning Rather Than a Quantity

Lisa often represented the fraction one-half as a partition rather than a quantity. For example, in the posttest she drew one-half by "halving" the shapes but not by designating a quantity of one-half (see Figure 5). When asked about what part of the drawing was one-half, she pointed to or redrew the line darker.

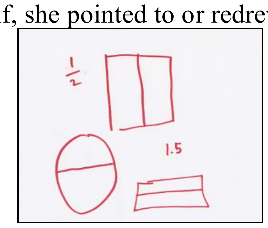

Figure 5. The written work of the student's answer to the posttest item "draw or write one-half".

This action orientation to area model representations of quantity appeared to be consequential to her inability to develop a robust understanding of equivalent fraction and fraction addition concepts. For example, at the end of one tutoring session she was attempting to write in her journal about what she had learned. During the tutoring session she had been correctly generating equivalent fractions by repartitioning area models (see Figure 6).

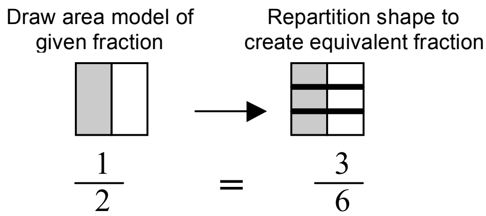

Figure 6. Illustration of the process used to create equivalent fractions during the tutoring session.

Although she had been successfully using area models to create equivalent fractions during this session, when she attempted to summarize what she had learned with an example, she incorrectly determined that 1/2 equaled 1/6. She began by drawing a square and partitioning it into two pieces (without shading). She then added two new horizontal partitions. However, because her drawing did not involve shading of 1/2, she incorrectly determined that 1/2 was equivalent to 1/6.

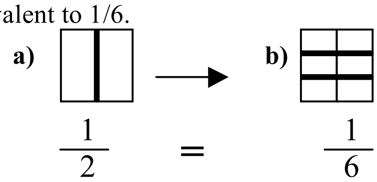

Figure 7. Recreation of Lisa's attempt to construct an equivalent fraction for 1/2 which omitted shading.

L: Ok, so like, what we were doing, if you have like one-half (draws shape and draws vertical line, see Figure 7a), and then like, cut it like that, (draws partitions, see figure 7b) it stll stays the same. It is just cut into different sections then.
K: So, how many sections is it cut into?

L: Six. So one-sixth?

After representing 1/2 as the partitioning line rather than by shading one-half of the shape, she produced a repartitioned area model with no shading. She correctly determined that the shape was divided into six pieces and guessed that the equivalent fraction for 1/2 was 1/6. This recording of an example, intended to be the culminating activity for this tutoring session, was undermined by her unconventional representation of one-half as a partition rather than a quantity. Lisa's answers and gestures throughout the tutoring sessions suggested that she understood one-half to be represented by the act of partitioning rather than a resulting quantity. Lisa's understanding of both the shaded region as signifying an amount taken away and her understanding of one-half as a partition, involved what proved to be a detrimental orientation to the area model representation as *action* rather than a *quantity*.

Conclusion

Lisa's atypical understanding, representation, and manipulation of quantities appeared to be a fundamental source of her difficulties, which interfered with her ability to develop a foundational fraction concepts of equivalence and fraction operations. The two primary manifestations: understanding shading as taken away and understanding one-half as a partition as opposed to a quantity, have appeared in two other students in the larger study who are classified as having MLD. This suggests that the atypical understanding of representations of quantities may be a powerful finding and a productive avenue to consider in the design of diagnostic tools. Given this kind of diagnostic understanding of a student's difficulties, it is possible to begin designing and testing remediation tools, which help the student compensate for these difficulties. This case study explored the nature of MLDs, by identifying the kinds of problems a student with an MLD exhibits beyond the canonical problems typically experienced by students when learning mathematics. Future work should attempt to capitalize on the existence of these kinds of qualitative differences exhibited by students with MLD, rather than relying upon math achievement tests for classificatory purposes. Ultimately the goal of MLD research should not be to label students and classify their deficiencies, but to understand their differences and help them learn to compensate.

References

Armstrong, B., & Larson, C. N. (1995). Students' use of part-whole and direct comparison strategies for comparing partitioned rectangles. *Journal for Research in Mathematics Education, 26*(1), 2-19.

Ashcraft, M. H., Krause, J. A., & Hopko, D. R. (2007). Is math anxiety a mathematical learning disability? In D. B. Berch & M. M. M. Mazzocco (Eds.), *Why is Math So Hard for Some Children?: The Nature and Origins of Mathematical Learning Difficulties and Disabilities* (pp. 329-348). Baltimore: Paul H Brookes.

Ball, D. L. (1993). Halves, pieces, and twoths: Constructing and using representational contexts in teaching fractions. In T. P. Carpenter, E. Fennema, & T.A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 157-195). Hillsdale, NJ: Erlbaum.

Behr, M. J., Harel, G., Post, T., Lesh, R. (1992). Rational number, ratio, and proportion. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296-333). New York: Macmillan Publishing.

Chatterji, M. (2005). Achievement gaps and correlates of early mathematics achievement: Evidence from the ECLS K–first grade sample. Education Policy Analysis Archives, 13(46).

Diversity in Mathematics Education (DiME) Center for Learning and Teaching (2007). Culture, race, power, and mathematics education. In F. K. Lester (Ed.) Second Handbook of Research on Mathematics Teaching and Learning (pp. 405-434). Charlotte, NC: Information Age Publishing, Inc.

Dowker, A. (2005). Early Identification and Intervention for Students With Mathematics Difficulties. *Journal of learning disabilities*, *38*(4), 324-332.

Fletcher, J. M., Lyon, G. R., Fuchs, L. S., Barnes, M. A. (2007). *Learning disabilities: From Identification to intervention.* New York, NY, US: Guilford Press.

Geary, D.C. (2004). Mathematics and learning disabilities. *Journal of Learning Disabilities, 37,* 4-15.

Geary, D. C. (2005). Role of Cognitive Theory in the Study of Learning Disability in Mathematics. *Journal of learning disabilities, 38*(4), 305-307.

Gersten, R., Clarke, B., & Mazzocco, M. M. M. (2007). Historical and Contemporary Perspectives on Mathematical Learning Disabilities. In D. B. Berch & M. M. M. Mazzocco (Eds.), *Why is Math So Hard for Some Children?: The Nature and Origins of Mathematical Learning Difficulties and Disabilities* (pp. 7-27). Baltimore: Paul H Brookes.

Gersten, R., Jordan, N. C., & Flojo, J. R. (2005). Early Identification and Interventions for Students With Mathematics Difficulties. *Journal of learning disabilities, 38*(4), 293-304.

Hanich, L. B., Jordan, N. C., Kaplan, D., & Dick, J. (2001). Performance Across Different Areas of Mathematical Cognition in Children With Learning Difficulties. *Journal of educational psychology, 93*(3), 615-626.

Kamii, C., & Clark, F. B. (1995). Equivalent fractions: Their difficulty and educational implications. *The Journal of Mathematical Behavior, 14*(4), 365-378.

Landerl, K., Bevan, A., & Butterworth, B. (2004). Developmental dyscalculia and basic numerical capacities: A study of 8-9-year-old students. *Cognition, 93(*2). 99-125.

Lamon, S. J. (1996). The development of unitizing: Its role in children's partitioning strategies. *Journal for Research in Mathematics Education, 27*(2), 170-193.

Lamon, S. J. (2006). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers* (2nd ed.). Mahwah, NJ: Lawrence Erlbaum Associates.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F.K. Lester, Jr. (Ed.) *Second handbook of research on mathematics teaching and learning* (pp. 629-668). Charlotte, NC: Information Age Pub Inc.

Lesh, R., Post, T., Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 33-40). Hillsdale, NJ: Erlbaum.

Mack, N. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education, 21,* 16-32.

Mazzocco, M. M. M. (2007). Defining and differentiating mathematical learning disabilities and difficulties. In D. B. Berch, & M. M. M. Mazzocco (Eds.) *Why is math so hard for some children?: The nature and origins of mathematical learning difficulties and disabilities* (pp. 29-48). Baltimore, MD: Paul H Brookes Publishing.

Mazzocco, M.M.M., Devlin, K.T. & McKenney, S. J. (2008). Is it a Fact? Timed Arithmetic Performance of Children With Mathematical Learning Disabilities (MLD) Varies as a Function of How MLD is Defined', *Developmental Neuropsychology, 33:*3, 318-344.

Murphy MM, Mazzocco MMM, Hanich L, & Early MC. (2007). Cognitive characteristics of children with mathematics learning disability (MLD) vary as a function of the cut-off criterion used to define MLD. *Journal of Learning Disabilities.* 40:458-478

National Council of Teachers of Mathematics. (2000). Principles and standards for school mathematics. Reston, VA: Author.

Ni, Y. (2001). Semantic domains of rational numbers and the acquisition of fraction equivalence. *Contemporary Educational Psychology, 26*(3), 400-417.

Post, T. R., Wachsmuth, I., Lesh, R., & Behr, M. J. (1985). Order and equivalence of rational numbers: A cognitive analysis. *Journal for Research in Mathematics Education, 16*(1), 18-36.

Saxe, G. B., Taylor, E. V., McIntosh, C., & Gearhart, M. (2005). Representing fractions with standard notation: A developmental analysis. *Journal for Research in Mathematics Education, 36*(2), 137-157.

Schoenfeld, A. H., Smith, J. P., Arcavi, A. (1993). Learning: The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed) *Advances in Instructional Psychology.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Steffe, L. P. (2003). Fractional commensurate, composition, and adding schemes learning trajectories of Jason and Laura: Grade 5. *The Journal of Mathematical Behavior. Special Issue: Fractions, Ratio and Proportional Reasoning, Part B, 22*(3), 237-295.

Swanson, H, L. (2007). Cognitive Aspects of Math Disabilities. In D. B. Berch, & M. M. M. Mazzocco (Eds.) *Why is math so hard for some children?: The nature and origins of mathematical learning difficulties and disabilities* (pp. 133-146). Baltimore, MD: Paul H Brookes Publishing.

Swanson, H. L., & Jerman, O. (2006). Math Disabilities: A Selective Meta-analysis of the Literature. *Review of Educational Research, 76*(2), 249-274.

Vygotsky, L. S., Knox, J. E., Stevens, C. B., Rieber, R. W., & Carton, A. S. (1993). *The collected works of L. S. Vygotsky, Vol. 2: The fundamentals of defectology (abnormal psychology and learning disabilities).* New York, NY, US: Plenum Press.

Zentall, S. S. (2007). Math performance of students with ADHD: Cognitive and behavioral contributors and interventions. In D. B. Berch & M. M. M. Mazzocco (Eds.), *Why is Math So Hard for Some Children?: The Nature and Origins of Mathematical Learning Difficulties and Disabilities* (pp. 219-244). Baltimore: Paul H Brookes.

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