The Role of Definition in Supporting Mathematical Activity

Marta Kobiela, Rich Lehrer, Vanderbilt University, Department of Teaching and Learning
0230 GPC, 230 Appleton Place, Nashville, TN 37203-5721
Email: marta.a.kobiela@vanderbilt.edu, rich.lehrer@vanderbilt.edu

Abstract: Definition is a form of mathematical activity that is fundamental to the profession's emphasis on conjecture, theorem and proof, yet is often treated in school as rote memory. We describe an alternative approach in which sixth-graders invented and revised definitions during investigations of space and geometry. Defining led to contest about taken-for-granted intuitions. A mathematical system emerged as relations among defined objects were constructed and extended, first by conjecture and later by theorem and proof.

Introduction

Often, in mathematics classrooms, definitions are learned by mere naming and memorizing. Not only is this practice problematic for students' conceptual development (Vinner, 1991), but it also denies students opportunities for mathematically richer work. Here, we describe an alternative approach, in which sixth-grade students (ages 11,12) created and refined mathematical definitions of objects featured in their school's curriculum. We conjectured that the activity of definition would encourage analysis of important qualities of these mathematical objects and would afford opportunities for developing *relations* among multiple objects. Reasoning about relations constitutes a mathematical system, and a focus on system is the departure point for other forms of important mathematical work, such as questioning, conjecturing, and proving.

The sixth grade class's work with definition was conducted within a curriculum unit on polygons (Connected Mathematics Project). Our emphasis on space and geometry stemmed from the prospects afforded by spatial reasoning for a general mathematics education: "Geometry, broadly conceived, can help students connect with mathematics, and geometry can be an ideal vehicle for building what we call a 'habits-of-mind perspective' (Goldenberg, Cuoco, & Mark, 1998, p. 3)." Goldenberg et al. (1998) stress that a habit-of-mind entails a close coupling between a specific form of mathematical practice and the grounds for knowing. For example, proofs should explain even as they secure the foundations of a particular conjecture. Hence, a habit-of-mind reflects epistemic disposition as well as a particular form of activity.

Our design for instruction capitalized on students' everyday experiences and conceptions of space, especially bodily motion, and on everyday forms of argument, especially propensities to categorize and classify. For example, we anchored students' learning about polygons to paths that they walked (Abelson & diSessa, 1980; Lehrer, Randle, & Sancilio, 1989) and related familiar properties of polygons, such as "straight" sides to experiences of unchanging direction while walking. Working from these embodied forms of activity, we cultivated students' dispositions toward posing questions and making conjectures. We privileged forms of explanation that were oriented toward the general and that appealed to mathematical system.

Given this support for reasoning about the mathematics of space, we aimed to support student authority to create their own definitions as a mathematical community. Here we trace initial mathematical explorations that emerged as students pursued one deceptively simple question: "What is a polygon?" The teacher asked this question to get a sense of what students had learned about polygons from a week of previous activity with their classroom teacher. We illustrate how the dynamic of definition unsettled what students took for granted (e.g., straight) and supported the development of a mathematical system. We then illustrate how the initial work of defining supported later forms of mathematical activity, as we describe how students elevated a conjecture about the number of diagonals in a polygon to the status of a theorem.

Method

Participants (n =18, 10 male, 11-12 years of age) attended an urban school serving primarily underrepresented youth in the southeastern region of the United States, where 60 to 80-percent of the school qualifies for free or reduced lunch each year. One of us served as the primary classroom instructor for mathematics during the school year. Mathematics class was conducted for 1.5 hours twice each week. Each lesson was videotaped and digitally rendered. Field notes were taken of whole group interactions in order to contextualize the video recordings and serve as a platform for reflection to inform the next day's instruction. Although our choice of mathematical topics was informed by the school's grade-level standards for mathematics, the conduct of any particular class was informed by our interpretations of students' questions and by our judgments of their current levels of understanding. The latter were informed by classroom interaction, by the results of periodic assessments and by summaries written in the students' journals of their understandings and experiences.

The analysis was informed by our history as participant observers and as designers to focus on themes that emerged during the course of our year-long involvement with the class. We employed comparative methods of interpretation to guide our analysis of the unfolding of student thinking, as suggested by the field notes and episodes of video. We watched video of lessons and generated preliminary ideas about what we noticed about the class's developing definitions. We then followed up on selected classroom lessons to refine or refute our conjectures about the emergence and history of use of particular definitions, focusing on episodes at the start of the year and several months later. We traced the development of definitions through the first six lessons by parsing whole class activity into segments of "definitional activity," which included instances where a mathematical object (e.g., polygon) was being defined or its definition contested in some form (e.g., is an circle a polygon?). We then identified links between one definitional activity and another, that is, where discussion about one mathematical object spurred discussion of another. For example, when defining polygon as "sides and angles and closed," a question arose of what constitutes a "side." We used these references in classroom conversation to generate an image of the mathematical system developed within the span of these initial lessons.

Results

Definition was an avenue for developing mathematical qualities of space, especially fundamental notions such as straight, angle, side and closure. Definitions began with contested claims about an aspect of space. Embodied activity served as a starting point and definition as means for resolving these contested claims about aspects such "straight" and "bent." Definitions were made increasingly public via practices of representation. For example, rotating his body different amounts (e.g. 90-degrees, 120-degrees), the teacher asked the students to describe the turns on paper, using words and/or drawings. Students were selected to present their representations while others were asked to describe, compare and contrast them. The discussion highlighted differences in how the drawings showed relative amounts of turn and ultimately supported a conception of angle as rotation. As definitions of these qualities of space were refined and elaborated, they were increasingly interconnected and related.

Explorations of central concepts in Euclidean space began by pursuing the definition of a polygon. Students' initial definitions of polygon introduced new mathematical objects: side, angle and regular polygon, all of which were contested. Students increasingly became more central participants in discussions about definitions. For instance, during the third class, when a student posed a football-shaped object as a possible polygon (in response to the teacher's question of whether a polygon can have two sides), another group of students responded with the question, "What's a side, people?" During the first six lessons analyzed, students constructed systems of relations among defined qualities. For instance, straight participated in subsequent definition of vertex. Moreover, all of these were considered from multiple perspectives: as dynamic paths created by motion (a local view), and as structures emerging from this dynamic activity (a global view). Hence, definitions increasingly reflected these multiple perspectives on shape and form.

These central concepts served as building blocks for other important forms of mathematical activity. For instance, later in the year the definition of diagonal set the stage for the development of a theorem relating the number of sides of a polygon to the number of diagonals. The theorem, that the number of diagonals of a polygon is a function of the number of sides [$(n^2 - 3n)/2$, where n represents the number of sides], originated in small-group investigations where students first noticed and described empirical patterns. The students' justifications of the pattern were supported by one student's introduction of the diagonal as a *path* in which the direction of connection did not matter. This embodied definition helped students conceptualize the number of vertices that could be "reached" from a single vertex as (n-3). Because there were n vertices, they reasoned that the number of "reachable" vertices was n x (n-3). But, as the student had earlier pointed out, reaching via a diagonal was a special kind of path, one that did not preserve direction. Hence, the expression *must be* $n(n-3)/2$. This expression of generalization was built from the particular, but its structure relied on embodied perspectives of a polygon's diagonal, sides, and vertices. It also was an argument of necessity, an important mathematical habit-of-mind. Although humble, definition and classification has traditionally resulted in fruitful mathematical pursuit (Senechal, 1990), and this historic trend was evident in the mathematical activity of the classroom too.

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