Mathematical Manipulatives as Designed Artifacts: the Cognitive, Affective, and Technological Dimensions

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Abstract. Mathematical manipulatives—tangible objects with a pedagogical purpose—have become popular tools in mathematics education. But typically, the notion of a "manipulative" carries with it a number of additional assumptions: that these objects are designed for elementary (as opposed to advanced) mathematics instruction; that they have little in the way of emotional meaning for their users; and that they are relatively simple, "low-tech" objects. In this paper we challenge these assumptions. Drawing on our experiences in two related projects in educational computing, we suggest that manipulatives may be designed for advanced mathematical topics; that they may offer creative (and thus affectively important) opportunities for students; and that they may be designed in ways that accompany or incorporate computational media.

1. Introduction: Mathematical Manipulatives as Objects of Design

Mathematical manipulatives—broadly speaking, tangible objects with an explicit pedagogical purpose—have a venerable presence in American elementary education. Early examples, such as the blocks used by Maria Montessori [Montessori 1912] and the well-known Cuisenaire rods [P1], have over time been succeeded by a still larger set of beautiful objects: Geoboards [P2], Polydrons [P3], and so forth. Undeniably, many mathematics teachers express positive views of such manipulatives (for instance, the NCTM Standards document recommends their use in K-4 classrooms ([NCTM 1989], p. 17)). Still, as [Resnick & Omanson 1987] have observed in arithmetic classrooms, children do not always succeed in linking the use of manipulatives to the mathematical concepts or operations that these objects are supposed to illuminate; while [Ball 1992], describing the popularity of manipulatives in mathematics education, writes that they "have become part of educational dogma: Using them helps students; not using them hinders students. There is little open, principled debate about the purposes of using manipulatives and their appropriate role in helping students learn."

This paper is intended to contribute to the discussion whose absence is noted by Ball. The continuing popularity of manipulatives suggests that mathematics teachers do find value in them; but the occasional cautionary voices suggest in turn that the use of manipulatives is hardly a guarantee of educational success. Creating high-quality mathematical manipulatives is, in fact, a design task worthy of study in its own right. This paper will explore several usually unheralded aspects of this design task, with a view toward articulating what we feel are important themes and principles in the construction of mathematical manipulatives. Our discussion will focus on several interrelated "dimensions" of manipulative design:

- The *cognitive* dimension (i.e., what types of learning or reflection the manipulatives are intended to promote);
- The *technological* dimension (i.e., how—or whether—the advent of new technologies may open up intriguing new directions in manipulative design);
- The *social/affective* dimension (i.e., how manipulatives function in the social or personal lives of both children and adults).

In exploring these ideas, we hope not only to inform (and share our own experiences with) the creators of novel manipulatives, but also to suggest new avenues of assessment and observation—new ways of interpreting the uses of manipulatives in both school and home settings.

The observations in this paper derive primarily from two related research efforts that we have pursued at the University of Colorado. The following (second) section outlines these two projects and describes the types of manipulatives employed: briefly, the first project centered on the use of manipulatives to illustrate important concepts in higher mathematics and functional programming, while the second has focused on allowing students to design and create customized polyhedral models through the use of a computer program. The third section—the heart of this paper—will enumerate issues regarding the creation and use of manipulatives that have come to light in our own experiences within these two projects. Some issues are more closely associated with the first project, and some with the second; but in total, they frame a coherent perspective regarding the pedagogical value of mathematical manipulatives. In the fourth section, we summarize this perspective and outline what we feel are promising directions for future research in the design and use of manipulatives.

2. Two Research Projects Involving the Use of Manipulatives: A Summary

2.1 Understanding Higher-Order Functions

The first project [DiBiase 1995a, DiBiase 1995b, DiBiase & Eisenberg 1995] was geared towards studying students' abilities to master *higher-order functions* [Abelson *et al.* 1985, Eisenberg *et al.* 1987]—a notoriously difficult concept whose explicit treatment is typically first presented in college algebra (or functional programming) courses. Eighteen students, ranging from 10-21 years of age, volunteered for the project. During the course of the 10 week (average) curriculum, students used a computer application named *SchemePaint* [Eisenberg 1995] to write graphics programs employing higher-order functions. At the same time, concrete manipulatives—literally, small felt pillows—representing data objects in the programming language were available as illustrations of this somewhat abstract concept. Pillows of distinct colors (representing distinct data types) and associated textual contents were used to represent data objects such as numbers and procedures. The basic idea behind the construction of these manipulatives was to respond to the specific difficulties and misconceptions that students tend to exhibit in dealing with the notion of functional data objects; that is, the physical representation needed to embody valid characteristics of functional data while at the same time contradicting the most commonly-encountered misconceptions.

2.2 HyperGami

The second project involves the use of a program named *HyperGami*[Eisenberg & Nishioka in press] to introduce students to the domain of solid geometry. The central notion behind HyperGami is that it permits students to create solid (polyhedral) shapes, viewing them in three-dimensional form on the computer screen; these shapes may then be "unfolded" by the system to produce two-dimensional patterns (or *folding nets*) which in turn may be decorated by a variety of means; and finally, the newly-decorated folding nets may be sent as output to a color printer, after which they may be cut out and folded into paper polyhedral models.

Several features of HyperGami deserve special emphasis in this brief outline. First, the system—like the aforementioned SchemePaint—is a *programmable application*, combining elements of direct manipulation interfaces (e.g., for choosing colors and viewing solids) with a "domain-enriched" version of the MacScheme programming language [P4]. Second, the system does not limit users to a pre-defined set of polyhedral models, but rather allows users to create new, customized polyhedra (often employing "classical" shapes as starting points). [Fig. 1] shows an example of this notion: here, a tetrahedron is truncated at each of its (four) vertices, then "stretched" along the z-axis to produce a new polyhedron; this shape is then unfolded to produce a net, which is decorated with solid colors and patterns; the resulting two-dimensional may now be printed out and folded into a model. As a final point it should be noted that, in practice, HyperGami is used not merely to create abstract mathematical models, but as a tool for creating more elaborate paper sculptures. [Fig. 2] illustrates the idea: here, a paper turtle has been created from several polyhedral "building blocks": e.g., the turtle's shell is a portion of a truncated dodecahedron, his eyes are icosahedra, his tail is a stretched tetrahedron, and so forth.

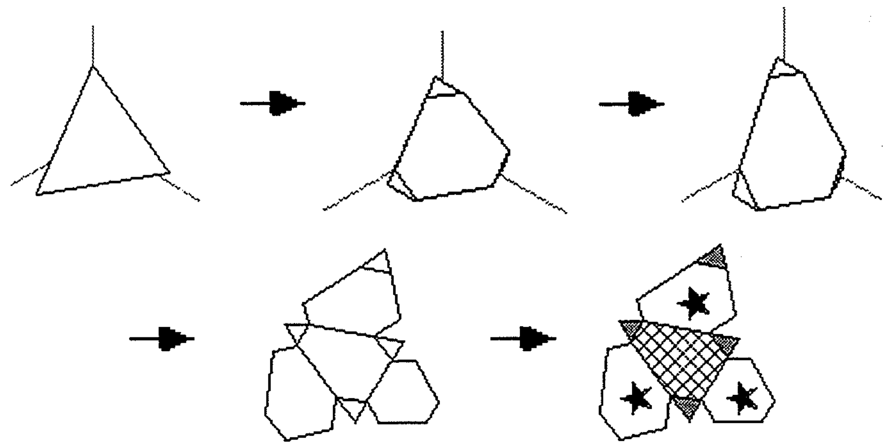

Figure 1: (*Top row*) A tetrahedron is truncated, then stretched. (*Bottom row*) A folding net is produced for the new shape; this net is then decorated with solid colors, patterns, and turtle graphics. The decorated shape may now be printed and folded into the desired solid.

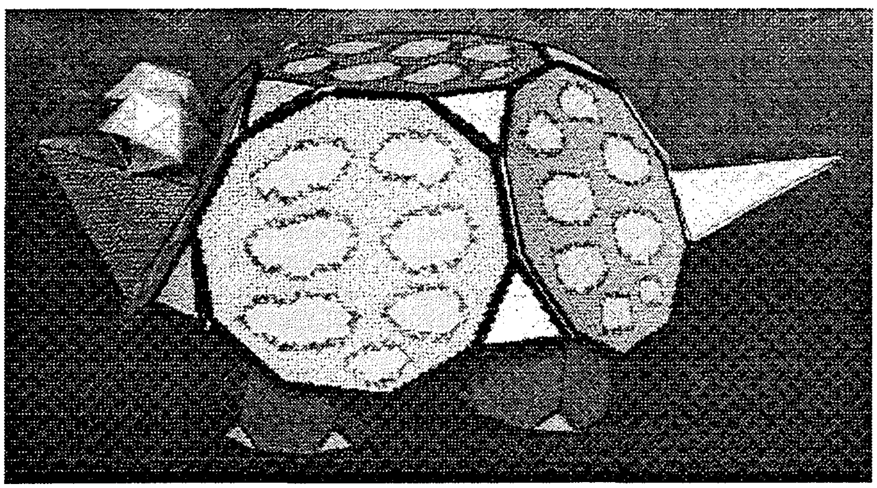

Figure 2: A polyhedral turtle created using HyperGami's customized shapes.

HyperGami has been employed on a long-term tutorial basis with sixteen (volunteer) elementary- and middle-school students during the past three semesters; these students have typically worked with us for approximately one semester at a time on a weekly or biweekly basis, either in individual tutorials or in pairs. In the course of this work, children have created a wide variety of polyhedral models and sculptures with the software, including (e.g.) stellated forms, a personalized polyhedral "die", a paper brontosaurus, Christmas ornaments, polyhedral chocolate candies (using the HyperGami forms as molds), and many others. (See [Eisenberg & Nishioka in press] for more description of students' work with the system.)

3. Issues in the Creation and Use of Mathematical Manipulatives

The previous section outlined two projects that have, for us, served as springboards for thinking about mathematical manipulatives as designed artifacts. In fact neither of these projects originated as an *explicit* investigation into manipulative design; but within both projects, a number of interesting (and often unexpected) issues have emerged that have caused us to rethink our received notions on this subject. In this section we describe the most important of these issues, focusing in turn on the cognitive, affective, and technological "dimensions" defined earlier.

3.1 Cognitive Issues

Self-constructed manipulatives

Typically, mathematical manipulatives are designed and marketed as fully-formed objects—number blocks, balancing scales, wooden solids—that are pre-assembled and ready for students to use. In slightly less pre-digested form, one sometimes finds manipulative "kits" (e.g., plastic pieces with which to assemble certain types of polyhedra); while these involve some measure of creative participation on the part of the student, the set of "constructible" objects is still relatively constrained. In contrast, our experiences with HyperGami have suggested that—rather than presenting students with fully-formed objects, or kits of pieces—there may be special value in providing students with *tools* for constructing their own personalized manipulatives. Thus, rather than presenting students with a geometric construction kit, the HyperGami software allows students to create a (potentially infinite) set of never-before-seen polyhedra, and to decorate those polyhedra in ways that are mathematically revealing (e.g., by using geometric patterns to highlight certain symmetry operations) or simply aesthetically appealing.

There are several potentially beneficial consequences to this notion of providing students with "manipulative creation tools" rather than with manipulatives themselves. One of these consequences—the fact that the created objects have status as artistic creations, and thus may be invested with pride and affection by their creators—will be discussed shortly, as a reflection of the "social dimension" of manipulative design. Another possible benefit arises from allowing the student to participate not merely in the *use* of manipulatives but in their *creation*, and thus to have the opportunity to reflect upon the creative act of designing mathematical objects. Indeed, the writings of mathematicians such as [Fomenko 1994] and [Hadamard 1949] imply that for some mathematicians, it is the ability to create or tailor personalized visual representations of abstract concepts (rather than simply the ability to recall the representations of others) that proves crucial to mathematical understanding. (See also the quote from Sfard later in this paper.) If this is true, then students should arguably have practice in crafting—and not merely employing—manipulatives. More broadly, this suggestion reflects the general outlook of "constructionism" as advocated by Papert and his colleagues [Papert 1991]—an educational philosophy that stresses the value of allowing learners to participate in the creation of mathematically rich objects and representations.

Of course, not all manipulatives need be created by students; and there are undeniable benefits to the use of traditional "pre-assembled" manipulatives. Such pre-assembled objects may be especially convenient or labor-saving for teachers; or perhaps the construction of these objects by students might involve such a high level of skill, or tolerance for drudgery, as to render the entire enterprise counterproductive. Nonetheless—as with all the issues described in this paper—our purpose in discussing this particular issue is not to advocate that all mathematical manipulatives must be student-constructed, but rather to advocate that this is one of the possibilities that designers ought to routinely consider.

Manipulatives for Higher Mathematics

Traditionally, mathematical manipulatives are regarded as "children's objects", rather like stuffed animals; indeed, virtually every commercially available manipulative is explicitly targeted at elementary school students. This fact ought to be surprising: since manipulatives are so popular with elementary school teachers, one might ask why the "culture" of manipulative use has not spread to high schools and universities. Indeed, the question is particularly vexing in the light of the difficulties that students of higher mathematics reportedly encounter: why are there no manipulatives designed for students of calculus, or non-Euclidean geometry, or field theory, or Fourier transforms?

The answer may lie, partially, in the "technological dimension" to be discussed shortly: that is, it may be the case that constructing adequate manipulatives for some of these advanced topics requires technological sophistication that is only now beginning to emerge. Or one might argue that the answer lies in the "social dimension": since manipulatives are associated with children's work, older students are reluctant to employ them. Notwithstanding the circular element in this last argument, it is our belief that mathematical manipulatives may in fact be especially valuable in presenting advanced mathematical topics. This conviction is largely a result of our experience in the "higher-order functions" project: for some students, even the presence of a simple tangible "function object" apparently assisted in understanding the use of (e.g.) functions as arguments to other functions [DiBiase 1995a]. [Sfard 1992], in a discussion of students' difficulties with the concept of function, writes:

> *The Ideal mathematician*, according to Davis and Hersh (1983, p. 35), *studies objects whose existence is unsuspected by all except a handful of his fellows.* Even the strangest abstract entities, while scrutinized and manipulated, seem to the mathematician as unquestionably real as the pen with which he or she writes papers... Shortly after being introduced to [the notion of function, the student] is expected to analyze and manipulate the new entity with a confidence which can only be achieved by those who can treat it as if it were a real thing. Many of our students, however, seem to be lacking this ability. [Italics in the original.]

Sfard's observations, together with our own experiences, suggest that this ability to "reify" the notion of a function may be supported by the growth of a "culture of manipulatives" for these concepts. Plausibly, the same argument applies to other abstract mathematical concepts—transforms, groups, and so forth.

3.2 Social/Affective Issues

Manipulatives as "social currency"
In the previous discussion, we alluded to the value of permitting students not merely to use manipulatives, but to create them. One of the themes that emerges from this notion is that student-created mathematical manipulatives—such as HyperGami polyhedra—may take on far richer social roles than do traditional manipulatives. Repeatedly we have seen students employing their HyperGami constructions as elements in their social lives: one 13-year old girl placed her newly-created polyhedron on display at home; a 13-year-old boy gave one of his more elaborate constructions as a gift to an adult; a 10-year-old girl made a polyhedron as a wrapper for a Christmas treat for her father; an 11-year-old girl gave a nickname to one of her polyhedral creations. (Indeed, we ourselves have used HyperGami sculptures as gifts and displays in much the same way; and if truth be told, several of our own constructions have nicknames as well!)

Such tales are of course "anecdotal evidence"—and their sheer profusion and variation defy easy statistical summary—but in our experience there is a noticeable consistency to the way in which HyperGami constructions have attained what we have called "social currency" in the lives of our students and ourselves. Constructions are exhibited, played with, donated, photographed, anthropomorphized. It is important to point out, in this context, that very little in the mathematical experience of students ever attains this status: one cannot (e.g.) give a completed workbook page or number-block construction as a gift. It is thus precisely because student-constructed manipulatives have a personalized, unique, and aesthetic side that they are capable of acting as much more than "mere" mathematical illustrations. And there is still another side to this discussion: because some HyperGami constructions have been created by teams or groups of students (e.g., one pair of girls created, in partnership, a mutually-decorated great stellated dodecahedron), the resulting objects take on still another sort of "social currency"—joint ownership. Such objects become symbols, for each contributor, of the participation and personalities of his or her colleagues.

Our observations of the uses of HyperGami constructions echo in part the discussion provided by [Csikszentmihalyi 1993] in his study of the manner in which people invest physical objects with personal importance:

> "Artifacts help objectify the self... [they] reveal the continuity of the self through time, by providing foci of involvement in the present, mementos and souvenirs of the past, and signposts to future goals. ... [they] give concrete evidence of one's place in a social network as symbols (literally, the joining together) of valued relationships. In these... ways things stabilize our sense of who we are; they give a permanent shape to our views of ourselves that otherwise would quickly dissolve in the flux of consciousness." (p. 23)

For designers of mathematical manipulatives, the crucial message here is that manipulatives are, first and foremost, physical objects; and as such, they ought to be (though rarely are) capable of taking on the roles that Csikszentmihalyi describes.

Age- and gender-related aspects of manipulative design
As noted earlier in this paper, mathematical manipulatives are generally associated with use by younger children; thus the problem of creating manipulatives that may be used by students of widely varying ages rarely arises in this traditional context. However, our recommendation for introducing manipulatives into higher mathematics

implies, in turn, that we must begin to examine the ways in which manipulatives are used not only by elementary school students, but also by teenagers and adults.

These reflections have arisen, in large part, from our observations of students in the higher-order-function project—in particular, how students of different ages responded to the tangible "function objects" created for that project. Broadly speaking, students below the age of 12 enjoyed using the function manipulatives (one turned manipulative use into his own game). Students above the age of 16 acknowledged the role of the manipulatives in learning the concept of higher order functions, but were generally uninterested in extended use of the objects. Between these demarcation points—between the ages of 12 and 16—we observed that students were extremely resistant to the use of manipulatives, particularly when their work was likely to be observed by peers. In one scenario, for instance, three tenth-graders worked with our curriculum as a group; but a noticeably higher level of effort was exhibited when one or two students were absent. In another scenario, a tenth-grade (male) student responded to the manipulatives with the exclamation, "Oh, not the puffy objects. I hate the puffy objects."

HyperGami constructions have met with a similar pattern of response in our experience: elementary and middle-school students (up to about the age of 13) and adults (beyond the age of about 18) have responded positively to the constructions; but high school students—especially boys—have been noticeably less interested.

Again, these are anecdotal observations; and they have emerged gradually, over time spent working within our two projects. These experiences have sensitized us, however, to the issue of designing manipulatives with an eye toward students' sense of dignity (for lack of a better term). It is a common observation (cf. [Eckert 1989], [Csikszentmihalyi *et al.* 1993]) that students of high school age in particular are acutely sensitive to how they are perceived by their peers; designing educational objects for students at this age is thus not simply a matter of tailoring the manipulatives' content to higher-level subjects, but of tailoring their style and appearance so as not to appear embarrassingly childlike. Similarly, the distinct "cultures" of male and female students at this age may result in certain styles of manipulative design being perceived (whether rightly or not) as inappropriate (the male tenth-grader's aversion to "puffy" felt objects may be illustrative here); designers of manipulatives for older students may discover that it is important to create "male-appropriate" and "female-appropriate" versions of manipulatives (or to make special efforts to create "gender-neutral" manipulatives)—an issue that does not come up with objects made for younger children. Clearly these are topics that merit much longer discussion; for now, our point is again that mathematical manipulatives are not merely *physical* and *educational* but also *social* objects—objects that reflect the social identities of their users—and must be designed with a perspective that goes beyond their role as cognitive aids.

3.3 Technological Issues

Traditionally, mathematical manipulatives in the classroom are envisioned as relatively "low-tech" objects—e.g., wooden blocks, or plastic tiles. (It is interesting to note, for instance, that the authors of the NCTM standards, while advocating the use of calculators and computers in mathematics education, do not to our knowledge ever describe these instruments as "manipulatives". As an illustration, see the discussion in [NCTM 89], pp. 67-8: here, manipulatives, calculators, and computers are all recommended as classroom materials, but are classified as separate bulleted items.) Our experience in both projects suggests the value of a more catholic view of manipulatives, in which "high-" and "low-tech" qualities are combined. In the case of the higher-order-functions project, tangible manipulatives were used to represent computational objects; and computational processes enacted with the manipulatives were then compared with running programs. In HyperGami, the structure of students' activity is one in which abstract computational work—the design of new polyhedra—is followed by periods of patient craftwork—the assembling of models. The very same abstract "object", then, is experienced both in its computational and tangible form in the course of a single project. It is precisely this sort of *varied* activity, moving between representations, that may prove an important element of manipulative design generally; embedding manipulatives within (partly) technological environments may serve to alleviate the sorts of difficulties in integrating mathematical representations noted earlier in the realm of arithmetic by Resnick and Omanson. Similarly, our advocacy of student-designed manipulatives reflects a belief in the value of computational tools such as HyperGami; the advent of more (and more powerful) tools of this nature should, over time, render manipulative design into something more closely approaching a craft for children and adults, rather than an educational industry.

4. Summary; Future Directions

In this paper, we have conveniently labelled our observations as corresponding to the cognitive, social/affective, and technological "dimensions" of manipulative design. Life is, of course, not so easily compartmentalized; and in fact, the themes discussed in this paper are deeply interrelated. A move toward incorporating technology into mathematics education leads naturally toward providing computational "construction kits" such as HyperGami; it likewise leads toward the notion of designing manipulatives for computational topics such as higher-order functions. In turn, our first efforts in these directions alert us both to unexpected roles that manipulatives can play—as sculptures, gifts, or souvenirs—and to unexpected problems in designing manipulatives—e.g., that they may embarrass their older users. And we are led still further, into considering what it may mean to design manipulatives for topics in higher mathematics.

Over time, a richer definition of "manipulative" may be in order in mathematics education. If the role of a "manipulative" is to embody a mathematical idea, visibly and interactively, within a tangible object, then such objects may well be designed through computation, or reflect mathematical ideas derived from computation. Indeed, as in the case of Resnick's programmable LEGO/Logo "creatures" [Resnick 1993], mathematical manipulatives may themselves include computational elements built within them. Treating manipulative design as a technologically-enriched craft leads to still other directions of work—for instance, we have recently used HyperGami polyhedra as molds for materials such as wax and plaster, and have incorporated them into home-made mobiles, science-kit objects (e.g., a kaleidoscope and balancing toy), and paper machines. Going still further, the placement of manipulative design tools into new social settings afforded by technology (most notably the World Wide Web) could have major consequences for the "affective dimension" of manipulative design; for instance, if a student can publish her own design for a mathematical object to a wide community, she may be more inclined to think of that object as an artistic performance (and at the same time, she may be more likely to find a supportive peer group for her work). And finally, the notion of "mathematical manipulative", in this view, may come to reflect a wider perspective than one that focuses on designing pre-assembled products for teaching individual concepts to young children. Rather, we may see manipulatives as objects that embody both abstract and concrete notions, and both "adult" and "children's" mathematics—and ultimately, perhaps, objects that combine the cognitive benefits of a scientific demonstration with the emotional resonance of a birthday present.

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Products
[P1] Cuisenaire Co. of America. *Cuisenaire rods*. White Plains, NY.
[P2] Cuisenaire Co. of America. *Geoboards.*White Plains, NY.
[P3] Cuisenaire Co. of America. *Polydrons*. White Plains, NY.
[P4] LightShip Software. *MacScheme*. Palo Alto, CA.