Teachers' Understanding of Partitioning When Modeling Fraction Arithmetic

Chandra Hawley Orrill, University of Georgia, LPSL – 611 Aderhold Hall, Athens, GA 30602, corrill@uga.edu
Andrew Izsák, San Diego State University, Dept. of Mathematics & Statistics, 5500 Campanile Drive, San Diego, CA 92182, aizsak@sciences.sdsu.edu
Erik Jacobson, Zandra de Araujo, University of Georgia, LPSL – 611 Aderhold Hall, Athens, GA 30602
Email: erikdjacobson@gmail.com, dearaujo@uga.edu

Abstract: We analyze how middle grades teachers in a professional development program reasoned about fraction arithmetic using length and area models. We discovered that teachers' abilities to partition length and area quantities were critical. In particular, we focus on ways that teachers' used multiplication factor/product relationships, distributive reasoning, and levels of units. Our findings contribute to the learning sciences' growing understanding of what teachers must know in order to use drawn representations effectively in instruction.

Theoretical Issue & Significance

External representations are important for organizing, recording, and communicating relationships in both mathematics (NCTM, 2000) and science (National Research Council, 1996). Given this importance, it is natural that representations would hold an important place in discussion of teacher knowledge. When introducing the construct of pedagogical content knowledge, Shulman (1986) emphasized knowledge of students' thinking about particular topics, typical difficulties that students have, and *representations* that make mathematical ideas accessible to students. More recently, Ball and colleagues (e.g., Ball, Thames, & Phelps, 2008) have developed the notion of mathematical knowledge for teaching (MKT) that refines Shulman's categories. MKT emphasizes the mathematics that teachers use to accomplish tasks central to their practice—for instance, using curricular materials judiciously, *choosing and using representations*, skillfully interpreting and responding to students' work, and designing assessments. Clearly, using representations is important for teachers.

Despite scholars' acceptance of the importance of PCK and the inclusion of external representations in theoretical perspectives on teacher knowledge, educational researchers do not know much about what teachers can do with drawn representations of problem situations and what they need to learn in order to use drawn representations effectively in instruction. The present study examines the knowledge that teachers need for reasoning about fraction arithmetic with drawn models. Fractions are an essential foundation for the study of algebra and functions that are used widely in mathematics and the sciences.

Research on teachers' knowledge of fractions for teaching has concentrated on fraction division. One finding that cuts across studies is that teachers can confuse situations that call for dividing by a fraction with ones that call for dividing by a whole number or multiplying by a fraction (e.g., Ball, 1990; Ma, 1999). Ma not only reported such constraints but also began to unpack PCK by indentifying *knowledge packages* that consist of numerous subtopics that are connected to one another and that support the teaching of a larger topic. To illustrate with the example most relevant to the present study, Ma described a knowledge package for fraction division in terms of the meanings of addition, multiplication, and division with whole numbers; the concepts of inverse operation, fraction, and unit; and the meaning of multiplication with fractions (p. 77). Other researchers (e.g., Behr, Khoury, Harel, Post, & Lesh, 1997; Izsák, 2008) have focused on the conceptual units that teachers form when using drawn models and manipulatives to reason about fraction arithmetic. This requires a more fine-grained perspective on knowledge than the subtopics Ma used to describe knowledge packages. In the present study, we extend this more fine-grained work by considering the knowledge teachers need to partition (e.g., subdivide) quantities when reasoning about fraction arithmetic. Understanding the fine-grained knowledge required to use drawn representations in instruction can inform more effective learning experiences for teachers.

Methodological Approach

We collected data for this study during a 15-week professional development course on fractions, decimals, and proportions offered to 14 middle grades teachers. The course emphasized solving problems in small groups and comparing solutions during whole-class discussions. Using drawn representations (e.g., number line and area models) to solve fractions problems was a central theme for the course. Software that allowed flexible manipulative of area models was often used by the teachers to support their reasoning. The teachers came from one large, urban school district. The course met once a week, and each session lasted three hours. We videotaped all sessions using two video cameras, one to capture the participants and one to capture their inscriptions. We analyzed the videos to identify the key mathematical challenges that teachers encountered. Through this analysis we discovered that teachers often have trouble partitioning quantities in ways that lead to problem solutions.

Findings, Conclusions, & Implications

We found that teachers' abilities to partition quantities were critical to their success in reasoning with drawn representations of fraction arithmetic. Specifically, we identified three aspects of partitioning that mattered. First, teachers' ability to anticipate factor/product pairs was vital. Teachers who did not invoke this whole-number knowledge were unable to anticipate how they might use subdivisions on a number line or in an area model. To illustrate, if a number line was partitioned into thirds and teachers needed to find fourths, many teachers did not understand that partitioning into 12ths would show thirds and fourths at the same time.

Second, teachers varied in their abilities to think about partitions distributively. In one example, teachers were asked to share two candy bars evenly between five people. The teachers partitioned each candy bar into five pieces but struggled with the question of how much of one candy bar each person got. They determined that two people got 2/5 of one bar, two people got 2/5 of the second bar, and the fifth person got 1/5 from each bar. The teachers recognized that 2/5 and 1/5 + 1/5 were the same numerically, but about half the class was unable to reconcile that getting 2/5 of one bar or 1/5 from each bar were the same.

Third, we noted that the teachers often resisted thinking in terms of nested levels of units. Research on students' multiplicative reasoning (e.g., Steffe, 1994), which includes fractions, has indentified nested levels of units as essential and challenging for students. To illustrate, one teacher was trying to use drawings to demonstrate why $2/3 \div 1/4 = 8/3$. She drew a rectangle, partitioned it into thirds, and shaded two of the thirds. She then discarded one of the thirds and explained that the problem only asked about 2/3. She divided each of the remaining thirds into four pieces and said that these eight pieces corresponded to the numerator in the numerical answer and that the thirds gave the denominator in the answer. Her interpretation of fourths in this answer was incorrect, and a fundamental difficulty appeared to be that she did not maintain correct relationships among fourths, nested in thirds, nested in the whole. (A correct explanation would describe 8/3 and the number of fourths that are in 2/3.) More generally, teachers who simplified tasks by ignoring "missing" parts struggled to make sense of their answers because they had eliminated the units to which the answers referred.

Our findings suggest that teachers need to develop a set of understandings related to partitioning quantities if they are to reason with drawn representations of fraction arithmetic effectively. Furthermore, teachers must be able to reason with drawn representations for themselves before they can use similar drawings in instruction. More generally, although researchers have investigated teachers' understanding of fraction arithmetic, most studies have not examined teachers' knowledge at a sufficiently fine grain-size to specify with precision what teachers must know in order to use representations in instruction. By uncovering the knowledge teachers need to use to interpret and reason with representations and drawings, we will be better able to provide the professional learning opportunities necessary to support teachers.

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Acknowledgments

The work reported here was supported by the National Science Foundation under grant number REC-0633975. The results reported here are the opinions of the authors and may not reflect those of the NSF. Chandra Orrill can be reached at University of Massachusetts Dartmouth, Kaput Center for Design & Innovation in STEM Education, 200 Mill Rd, Suite 150B, Fairhaven, MA 02719, corrill@umassd.edu.