A Tempest in a Teapot Is but a Drop in the Ocean: Action–Objects in Analogical Mathematical Reasoning

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Abstract: We discuss a brief transcribed excerpt from a task-based interview with Li, an 11.5-year-old participant in a design-based research study of probabilistic cognition pertaining to the binomial. We investigate whether and how Li made sense of the behavior of an unfamiliar computer-based artifact—the diminishing proportional impact of successive random samples on the overall shape of a dynamically accumulating outcome distribution. Li constructed two informal analogical situations as multimodal discursive means of concretizing, elaborating, and communicating his emerging understanding of the artifact's behavior. These non-routine utterances shifted the discourse to an explicitly embodied, imagistic space bearing unique affordances for negotiated epistemic syntheses of phenomenological and technological constructions of quantitative relations. Microgenetic analysis suggests that Li's presymbolic notion was not a static magnitude but an intensive-quantity "action–object"; he subsequently unpacked this dynamical "*a*/*b*" qualia into its constitutive "*a*" and "*b*" elements. We reflect on implications of this counter-curricular sequence for educational design.

Background: Student Analogy as Researcher Opportunity

We are interested in the phenomenon of mathematics learning. We conceptualize mathematics learning as the process in which an individual builds meaning for mathematical artifacts, such as notions, semiotic devices, and procedures. We research this learning process in an attempt to understand what it is that students do when they build meaning for mathematical artifacts, what teachers do to support this process, and what instructional materials may best serve this process. In the current study, we examine one student's metaphorical reasoning as he attempts to construct meaning for the behavior of an unfamiliar mathematical artifact. In particular, we seek to determine the student's initial ontology of the artifact as indicating his pre-articulated phenomenological resources that could plausibly serve as proto-mathematical. So doing, we question the warrant of some epistemological assumptions underlying traditional mathematics curriculum.

Our comments in this essentially theoretical paper should be taken as no more than conjectural. Nevertheless, an appeal of the paper would be the potentially effective theoretical fit, and hence methodological fit, between the dynamical nature of the particular artifact at the center of the student's inquiry and our interest in the dynamics of multimodal mathematical reasoning. This fit enables us to elaborate on Phenomenology tenets and Learning-Sciences conjectures regarding the precedence of unreflective perception of action over analytic construction of concepts, as we investigate the purchase of these conjectures on our empirical data. Ultimately, we hope to draw tentative conclusions aligned with parallel research efforts so as to offer an intellectual space for conversation informing future studies.

Probability—the mathematical content selected for this study—is amenable for our investigation of tension between phenomenological and cultural constructions of situations and artifacts. That is, the unique epistemological "mode" created by situations involving uncertainty, and in particular the challenges of articulating such uncertainty symbolically, appears to impel students to seek non-symbolical discursive genres as means of expressing their intuitive quantitative reasoning (e.g., Rubin & Hammerman, 2007). The consequent protracted discursive interim from the embodied to the symbolical is conducive for examining the microgenesis of meaning. Namely, we conjecture that the metaphors generated by this study's focal student served him as more than vehicles for communicating "ready-made" coherent notions. Rather, we believe that the cognitive–discursive actions of evoking and "grammarizing" these metaphors were instrumental for the student's *initial* articulation of his reasoning. If this conjecture bears out through further research, it would present a number of implications for mathematics-education theory and practice. In terms of theory, we would suggest that idiosyncratic metaphors could enable diverse individual students to ground challenging mathematical constructs in their prior quantitative experiences, both formal and informal. In terms of practice, we would encourage teachers to cultivate socio-mathematical classroom norms of discourse in which such metaphorical constructions are approved and possibly even solicited.

Metaphor as a Unique Semiotic Means of Objectification

When students encounter a new situation and attempt to determine what it is, they are tacitly attempting to determine what it is *like*. This reasoning process is usually opaque to others as well as, perhaps, to the students. Usually, it is only when the students subsequently respond in non-normative ways to problem situations that we assume that their underlying conceptualizations of the situation are non-normative. By stating their metaphors explicitly, though, students create opportunities to reflect more concretely on their emerging conceptual system for a given mathematical subject matter content as well as to receive targeted formative assessment and guidance in learning this content. We conjecture that instructional discourse around metaphor plays a unique role in students' bridging between tacit and cultural constructions of quantitative situations. Through analyzing a student's metaphorical constructions, we hope to promote research into this conjecture. Our proposal to endorse metaphorical reasoning as central to the practice of mathematicians and, hence, of mathematics-education researchers, agrees with recent emerging interest in "embodied" or "multimodal" forms of quantitative reasoning (Abrahamson, 2009b; Edwards, Radford, & Arzarello, 2009; Lemke, 1998; Presmeg, 2006) as well as with converging testimonies and theory pertaining to the explorative and desultory process of mathematical learning and discovery (Lakatos, 1976; Schoenfeld, 1991; Thompson, 1993). Furthermore, demonstrating pivotal roles of multimodal mathematical reasoning would add to the ongoing conversation among cognitive scientists regarding a looming paradigm shift from amodal to modal conceptualization of human reasoning (see Barsalou, 2008; but cf. Dove, 2009).

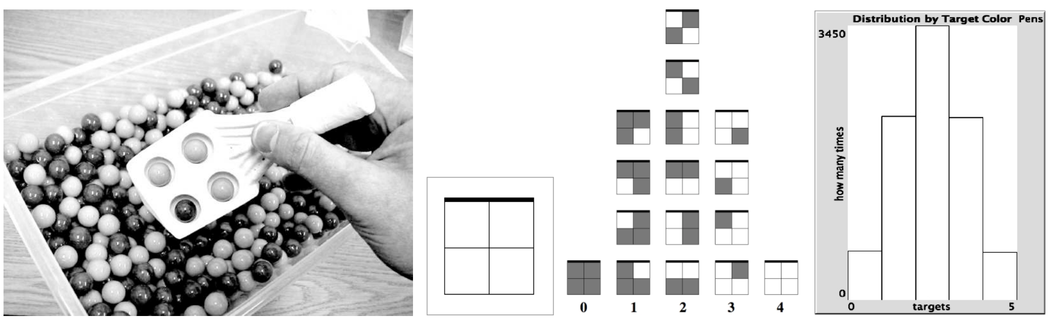
We view mathematical learning as a dialogic, distributed, reflexive, and emergent process. Students learn by building personal meanings for mathematical artifacts. They do so by appropriating these artifacts as significant means for multimodal semiotic objectification of their presymbolic notions. It is thus that designed and guided participation in artifact-based social activities mediates disciplinary knowledge as internalized discursive praxis (Mariotti, 2009; Radford, 2008; Sfard, 2002). And yet, students' naïve presymbolic notions, judgments, implicit heuristics, and inferences for quantitative situations are sometimes more sophisticated than they can initially express (e.g., Gelman & Williams, 1998), particularly when students who are not yet fluent in proportional constructs attempt to express notions of perceptually privileged intensive quantities (Piaget, 1952; Stroup, 2002). Thus, it is important that educators create opportunities for students to avail of their capacity for pre-articulated quantitative reasoning, such as by making available in the learning environment mathematical artifacts that accord with structure information students tacitly seek (Abrahamson, 2009b). In this negotiation of meanings for mathematical artifacts (Cobb & Bauersfeld, 1995), metaphor serves as a powerful discursive engagement, because it explicitizes—for the student, teacher, classroom, *and researcher*—the student's (idiosyncratic) reasoning.

That said, for this study we do not take a philosophical or cognitive position as to how students first develop these metaphors or how they evoke them in situ as inferential discursive mechanisms (cf. diSessa, 1983; Halliday, 1993; Lakoff & Núñez, 2000; see in Ortony, 1993). Instead, we treat the evocation of metaphor at a larger analytic granularity, as a species of abductive inference amid encounter with a novel problem (e.g., Prawat, 1999). Moreover, we are committed to framing students' reasoning as situated not in a solipsistic void but as an integral aspect of discourse (Sfard, 2002). Particularly, we are inspired by Radford's (e.g., 2008) view of mathematics learning as guided discursive processes in which students appropriate semiotic artifacts as means for objectifying their presymbolic notions. From this semiotic–cultural perspective, we wish to examine the hypothesis that metaphor, too, is a means of objectification and to explore its unique properties, roles, and prospects within institutionalized practices of mathematical learning. We propose that students recall personal experiential gestalts as a means of reifying the emerging coherence they sense in mathematical artifacts yet initially lack formal constructs or vocabulary to express.

Source Data and Research Questions

The transcribed excerpt that constitutes the source data for our study is from an interview with Li, an 11.5 year-old male student ranked by his mathematics teachers as high-achieving. Li was one of 28 Grades 4–6 study participants from a private suburban K–8 school in the greater San Francisco East Bay area. The context of these 1-hr. interviews was a design-based research study of late-elementary and middle-school students' intuitive notions pertaining to the mathematical study of probability as well as the potential roles that a set of mixed-media technologies of our design could play in enabling these students to leverage any such naïve notions toward understanding mathematical formulations of random phenomena, specifically the binomial. Using a flexible protocol, we conducted semi-structured clinical interviews (Ginsburg, 1997). The sessions were videotaped for subsequent analysis, and selected episodes were transcribed. The research team employed collaborative qualitative microgenetic analysis techniques (Siegler & Crowley, 1991) as well as grounded theory (Glaser & Strauss, 1967), through which emergent insights were articulated and iteratively crosschecked against the entire data corpus as a means of consolidating, investigating, and developing new constructs (diSessa & Cobb, 2004). Instances of student elaborate metaphorical reasoning were extremely rare (only four identified within a total of 26 hours of data) yet appear to be uniquely informative. In particular, students' gestures therein enable us to speculate on their conceptualizations of proportionality and probability (e.g., Alibali, Bassok, Olseth, Syc, & Goldin-Meadow, 1999).

Figure 1 overviews four central objects and target artifacts that were used, constructed, and/or automatically generated during the interview. The marbles scooper is used to draw a sample of exactly four ordered marbles from an "urn" of mixed green and blue marbles (see Figure 1a). This hypergeometric experiment—*during* each 4-marbles sample draw, marbles are not replaced—approximates the binomial due to the ratio of *n* (4 marbles) to the content of the urn (hundreds of marbles). In the interview, we first ask the participant to guess what would be the experimental results of scooping. Next, the participant is guided to use crayons and a pile of stock-paper cards each bearing a blank 2-by-2 matrix (Figure 1b), so as to create the sample space of the experiment and assemble it in the form of the *combinations tower* (see Figure 1c). Later, the dyad engages computer-based simulations of the same experiment (e.g., see interface fragment—a histogram—in Figure 1d). Here, we examine one student's reaction to one feature of one artifact—Li's reaction to the diminishing proportional impact of samples in the "4-Blocks" simulation.[1]



a. *b.* *c.* *d.*

Figure 1. Materials used in the study—theoretical and empirical embodiments of the 2-by-2 mathematical object: (a) The marbles scooper; (b) a template for performing combinatorial analysis; (c) the combinations tower—a distributed sample space of the marbles-scooping experiment; and (d) an actual experimental outcome distribution produced by "4-Blocks," a computer-based simulation of this probability experiment.

The computer-based model "4-Blocks," built in NetLogo (Wilensky, 1999), simulates the marbles-box probability experiment. 4-Blocks includes a virtual 2-by-2 array (see Figure 2, next page). When a virtual sample is taken, each of the four cells in this array is randomly assigned either green or blue coloration. For example, the sample may be three green and one blue cells in any of the four possible orders (hence, "3g1b"). The model computes the number of green cells in the array, such as 3, and this value is supplemented to a list (that by default is not seen by the user). Immediately, the interface reflects this new result by "bumping up" the appropriate histogram column, such as the second-from-right column, by a vertical extent commensurate with one unit. A feature of 4-Blocks is that its histogram updates dynamically even as the experiment is running. (To interact with the 4-Blocks model as well as to view the video clip discussed herein, visit http://edrl.berkeley.edu/publications/conferences/ICLS/Abrahamson-ICLS2010/.)

The vertical extent of a unit is dynamically calibrated to the histogram's maximum *y*-axis value (e.g., this *y*-value is 43, in Figure 2). That is, the histogram has an "autoplot" feature: when a category column "grows" such that it is about to exceed the maximum *y*-value, this value increases by one unit so as to accommodate the impending growth. The column thus remains as tall as it was, "glued" to the top of the histogram frame, while other columns appear to sag down a unit. Consequently, as the simulation is running, the vertical extent of a single unit keeps diminishing. Thus, when the maximum *y*-value is 10 (the default initial value), a unit is 1/10 of the height of the frame, resulting in the perception of a major upward jolt on the screen when a sample is taken. In Figure 2, a unit would be smaller, at 1/43 of the height. When the maximum *y*-value is relatively large, such as 1,000, a unit is only 1/1,000 of the histogram, resulting in a minute perceptual change. Moreover, at 10,000, an upward motion may demand finer calibration than the pixels could accommodate, so that very often the columns would not register any perceivable change at all.

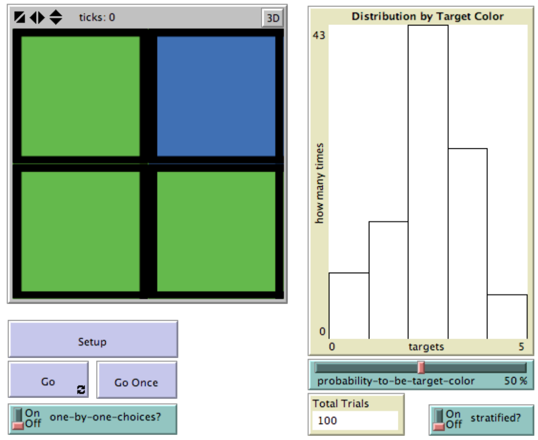


Figure 2. The 4-Blocks NetLogo simulation of the marbles-box probability experiment after 100 samples.

This phenomenon—the diminishing perceptual impact of samples as the experiment runs—is a potentially useful feature of the NetLogo modeling-and-simulation environment, because of its inherent capacity to gradually shift the user's attention away from the impact of each haphazard sample and toward the distribution gestalt, which the user might thus objectify as an aggregate property of the simulated phenomenon (Wilensky, 1997). Moreover, the absence of numerals along the histogram's *x*- and *y*-axes appears to impact the user's primary construction of relations among the histogram columns: rather than constructing these relations as mediated through symbolically notated absolute values and calculated differences among them, the user can construct them proportionately based on immediate perceptions of relative heights and/or areas. Thus, in each and every long-term run of the simulation, the user may experience opportunities to witness and reflect on the gradual yet ineluctable emergence of the distribution shape, such as the 1-4-6-4-1 shape of the marbles-scooping binomial experiment set at $p = .5$ (Figure 1d); with guidance, users may be able to construe this shape as objectifying their presymbolic sense of the phenomenon's aggregate characteristics (e.g., the expected plurality of 2g2b, see Abrahamson, 2009b).

By and large, such was the case with all our 28 participants, including Li. Yet how would Li make sense of the histogram? In particular, would Li perceive the impact of each sample upon the distribution as proportional to the quantity of aggregated outcomes? If so, what resources might this student, who has not studied ratio-and-proportion formally, bring to bear in making sense of this phenomenon of diminishing perceptual impact, and how might he articulate his inferences? From an instructional-design perspective, is the technologically sophisticated semiotic system built into the autoplotting histogram helping or hindering Li's learning?; is this histogram "cognitively ergonomic" (see in Abrahamson, 2009b), and if so, which phenomenological resources could it tacitly cue and how might Li articulate these notions mathematically?

Analysis: Analogy-Based Unpacking of Intensive-Quantity Action–Objects

In this section we present and discuss a 41-second videographed excerpt and annotated transcription of a conversation fragment between Li and the interviewer, in which Li reasons analogically about the samples' diminishing perceptual impact on the histogram shape. The excerpt begins when Li and Dor have already negotiated the construction of the sample space, have assembled it in the form of the combinations tower, have discussed relations between the marbles-box experiment and the combinations tower, and have been working on 4-Blocks. Doing so, Li draws on non-mathematical phenomena as contexts for his quantitative reasoning. Li's rhetorical strategy is to build an argument by comparing two extreme cases of sampling: (a) at the beginning of the experiment, when a relatively small number of outcomes has accumulated so that each supplementary sample causes quite a "splash"; and (b) well into the experiment, when a considerable number of outcomes has accumulated, so that any additional sample causes but a "ripple." His first context—the glass/lake analogy—will present this contrast in full, and then the second context—batting averages—elides the first of the two cases in pragmatic enthymeme and states only the second case. Just prior to these analogies, though, Li anticipates that the emergent experimental frequency distribution will resemble the combinations tower in shape; he will then monitor this emergence to evaluate his prediction.

Dor: So what do you think might be the shape of the **columns**, as **it** goes up?
Li: [Gesturing to the combinations tower] Something like this.
Dor: Well let's see. [activates experiment, explains autoplotting] So what's happening?

Li: [Gesturing to the distribution] **It's hovering at around this** [gestures to combinations tower]…. [20 seconds later, when 5,000 samples have been drawn] **Look, it's** [the frequency distribution] **almost exactly like this** [the combinations tower]…. [30 seconds later] **Now they're** [columns] **moving less** [inaudible].

Dor: Why is **it** moving less?

Embedded in the interviewer's questions is an implicit linguistic passage from "the columns" to "it." Li picks up this cue and accordingly refers to the columnar frequency distribution with the pronoun "it." Irrespective of the fullness of his understanding at this point, Li's appropriation of the interviewer's singular pronoun as reference to the column collective suggests he is construing the distribution as an intact, if amorphous, object. Soon after, Li again frames the aggregate motion as "plural" behavior. Below, this *motion* is about to become the *object* of discussion. Namely, the histogram's figural change is first described with a verb ("moving") but will soon become a noun ("splash" vs. "ripple") (cf. Bakker, 2007, for a similar case of Peircean 'hypostatic abstraction' in statistical analysis). What is unique about these nouns is that they are inherently about change. That is, the object at the core of Li's proto-mathematical reasoning is not a static magnitude, such as the measured vertical displacement of the histogram bars (an '*a*' element) or the cumulative number of samples (a '*b*' element)—Li's tacit phenomenological primitive in this dynamical experience is a synthetic a priori—an intact action-based *a*/*b* change-over-time intensive-quantity unit (cf. Stroup, 2002). Thus, technological affordances of visualization media—the dynamical autoplotting histogram—may reverse mathematics learning sequences from the traditional magnitudes-before-relation to relation-before-magnitudes. Li's ensuing metaphorical outburst, in which he deftly unpacks the intact *a*/*b* change–object into its *a* and *b* constituents, we submit, implies that we should take pause, as mathematics educators, to consider the possibility that students learning intensive quantities, such as in the mathematics of change, may avail of trajectories that go counter to traditional curricular sequencing. It is thus that computers may "restructurate" mathematical content (Wilensky & Papert, 2009).

Yet what phenomenological resources will Li bring to bear in explicating the change–object? Whereas the NetLogo histogram, when the model is run, foregrounds change as the salient object of attention and may thus render the mathematics of change more accessible to learners, the autoplotting feature may conceivably challenge students due to its ostensible phenomenological aberration. That is, in "real life," we may muse, growing aggregates actually occupy greater space. Intriguingly, however, natural visual perception is a perspectival experience, in which retinal prints of objects depend on their distance:

Li: /2 sec/ **Because**…/3 sec/ [gazes up to the wall] **the larger number**…/3 sec/ uhh… ["checks in" with the interviewer] /2 sec/ [rapidly] **Like if you have a little glass** [iconic gesture: LH cups a glass in natural position near body; gazes at glass] **of water and you drop a marble in** [iconic: LH uncups, rises, drops marble], **it's gonna be... there's gonna be, like, a splash** [LH, palm up, abrupt vertical rise], **but if you have a big giant lake** [LH, palm up, drawn back and up above head to encircle expansive lake; RH, in jacket pocket, budges to complete circumference], **and you throw a marble in** [LH catapults marble, then scratches nose]**, there's just gonna be a ripple** [joins LH thumb and index, lowers hand to chest height on right-side of embodied space; taps fingers together, possibly marking marble's contact with surface, then inscribes smooth horizontal line across to the left; hand opens]. **It's a, it**… [gestures to histogram, orients gaze and pivots body towards it]

Dor: **Oh, ok. Like each individual additional sample** [LH gyrates swiftly, iterating addenda] **is causing less of a** [LH & RH "contain" combinations-tower outline]**…**

Li: [cuts in mid sentence] **Yeah, it's like a batting average in baseball. If you've already had five hundred at-bats**(2) [LH opens, shifts slightly to the left, palm up, "holding" the 500 at-bats, then relaxes]**, and then you get out one more time, it's not going to make it go down that much—it'll make it go down like two points** [LH pinches thumb and index, as per "ripple," as if to subtract two points]**, or**…

Dor: Ok, so as we go along, each successive sample causes less of a commotion.

Li: **Yeah.** [Gazes back at the computer screen] **Look, they're just barely moving.**

The conversation, above, pertained to variation in the effect of a single sample on the proportional distribution of experimental outcomes: as outcomes aggregate, the proportional impact of each additional sample diminishes. Viewed on the computer screen, the self-compressing histogram converges on the 1-4-

6-4-1 distribution, eventually portraying the dynamic aggregation as perceptually static, albeit the constantly updating maximal *y*-axis value reveals the process as additively active. Li appears to understand this principle. However, this high-achieving 6th-grader's limited fluency with proportional constructs does not enable him to capture the process with appropriate vocabulary, such as "proportion" and its cognates, or with suitable mathematical constructs and arithmetic operations for conceptualizing and treating rational numbers, as witnessed in his aborted attempt, "Because the larger number…" Consequently, Li evokes a situation analogous to the dynamic artifact yet evaluated as more conducive to objectifying the principle. Li thus unpacks "splash > ripple" as "one-marble-impact : size-of-glass > one-marble-impact : size-of-lake."

The key physical dimension underlying the splash-vs.-ripple articulated comparison is the vertical height of the water displacement in each. And yet, would not a marble thrown into a lake produce a vertical displacement of water that is at least as large if not larger than in a glass? What is at stake here, it appears, is not the absolute physical size of phenomena but their pre-rationalized perturbations to the perceptual field. Thus, ratio is pre-built into perspective as an optical calibration. Moreover, equally sized distal events are normalized by their containers, because these perceptions' phenomenological circumstances are such that events occurring in larger containers are physically farther from the viewer and therefore retinally smaller. Li's spontaneous gestures inscribe the glass and lake *as they would be perceived in the unreflective phenomenology of lived experience*, so that the lake occupies little more optical canvas than the glass does upon Li's "visuo–spatial sketchpad" (see Baddeley & Hitch, 1974).[(3)] Thus, Li can compare two intensive quantities—the splash and the ripple—on the basis of their retinal magnitude alone (see Figure 3, below).[(4)]

Glass Splash Lake Ripple

Figure 3. Dominant gestures in Li's glass-vs.-lake comparison analogy, with arrows as action overlays.

In summary of Li's first analogy, his argumentation responded to the interviewer's request for an explanation regarding the patterned behavior of a dynamical mathematical object—an electronic histogram that is confined in its total interface real-estate yet monitors an ever-aggregating vertically surging outcome distribution. Li's analogical constructions, however, *shifted the dyad into an imagistic discursive space, wherein larger aggregates are in fact embodied in larger objects*. Therein, Li compared two intact experiential events, both a priori intensive quantities; only through engaging in thinking-for-speaking (Slobin, 1996) did Li unpack each of these quantities "cubistically" (Nemirovsky & Ferrara, 2009) such that its respective constituents were "liberated" (Bamberger & Ziporyn, 1991) into *a* and *b* magnitudes so that each event was reconstructed as an *a*/*b* qualitative quotient. This unpacking is anecdotally enhanced by Li's essentially unidextrous sequential gesturing, which radicalizes the discursive caveat of linear utterance.

In selecting a baseball context for his second analogy, Li evokes a situation that is more conducive to mathematizing his argument. First, monitoring quantitative aspects of this popular activity is a familiar, culturally appropriate practice. Second, the baseball example is more ontologically consistent than the water example: in the glass-vs.-lake analogy, the perturbation was caused by a marble dropped into water and thereafter the marble is ignored, whereas in the baseball scenario, the perturbation was caused by the player performing an at-bat, which is thereafter added to the denominator of the "batting average" (and to the numerator, contingent on how the at-bat played out). Third, the baseball analogy offers a pre-enumerated set of discrete units, 500 at bats, whereas the water analogy would demand measurement of the continuous body before further calculation could be performed. Finally, a batting average is an a priori proportional construct, so that this analogy is much nearer to the probability context—culturally, semantically, semiotically, and arithmetically—than the water-surface analogy. Thus, in selecting the baseball analogy, Li also steers the dialogue from the rich imagistic detour, which the interviewer had countenanced, back to the institutionally normative genre of discussing quantitative reasoning numerically. Using numbers as precise signifiers of quantity, in the baseball analogy, Li need not communicate magnitudes gesturally—indeed, comparison of the vast "big giant lake" gesture to the modest "500 at-bats" cupping as well as, analogically, comparison of the lake-impact gesture to the "-2" (read: "-.002") gesture demonstrate pragmatic contraction in iconicity from the first to the second context (see Radford, 2008).

Having completed his two-analogy argumentation sequence (see Figure 4, below), Li looped the dyad back to the phenomenon under inquiry. Presumably, the interlocutors, who renew their co-attention to the histogram, do so with newly shared professional vision established in analogical discursive space.(5)

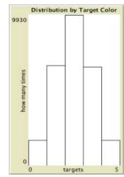
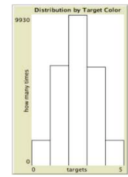
Impact	MAJOR			BARELY DISCERNABLE		
Context	**Before**	**Perturb**	**Effect**	**Before**	**Perturb**	**Effect**
Marbles-Scooping Experiment		+1 sample →			+1 sample →	
	↕		↕	↕		↕
Marble in Water	Cup	throw marble	"Splash"	Lake	throw marble	"Ripple"
				↕		↕
Batting Average	–	–	–	Batting average at 500 at-bats (x/500)	One out	Batting average at 501 at-bats (x/501)

Figure 4. Analogizing the proportional impact of an addend as a function of the size of the aggregate.

Li's imagistic analogical detour suggests that he needed to step back from the sophisticated semiotic device and reconstruct a phenomenological scaffold by which to perform epistemic adjustment that then enabled him numerical objectification of his presymbolic image of proportional convergence. Implicit to this process of mathematization was re-describing multimodal phenomenology in conventional techno–scientific form. This bridging task, an aspect of meta-representational capacity (diSessa & Sherin, 2000), is cognitively non-trivial. We implicate the technologically sophisticated self-adjusting histogram as complicit to the tension between Li's tacit and cultural presentation of quantitative information: by perceptually foregrounding proportional change over additive change, the autoplotting histogram offers a cognitively ergonomic engagement of the mathematics of change, yet to avail of this restructurated entry to the disciplinary practice, students require guided opportunities to negotiate embodied and inscribed constructions of focal phenomena. Metaphorical reasoning is one means of accomplishing this negotiation.

Conclusions

Whereas there is a certain logical appeal to thinking of an object as somehow cognitively simpler than an action (object + motion), the phenomenological perspective perceives actions as prior to objects—objects need to be deliberately pulled out of experience as transcendental to unreflective action. Analysis of Li's imagistic construction revealed that he was drawing on phenomenological gestalts that are experientially a priori to mathematically articulated notions of ratio. Understanding such processes could inform the work of mathematics educators. In particular, educators who reason about mathematics logically but not phenomenologically are liable to introduce a/b constructs by initially presenting a and b as separate elements and only then calculating their quotient and suggesting meanings for this number. Yet such would-be scaffolding may sometimes hamper rather than optimize students' opportunities to draw on their informal reasoning, particularly their tacit intensive-quantity gestalts, in constructing mathematical notions. Instructional design solutions, we believe, lie in between, in creating opportunities for students to negotiate phenomenological qualia and their mathematical reconstructions (Abrahamson, 2009a, 2009b, 2009c).(6)

Endnotes

(1) Elsewhere, we furnish detailed explanation of the design motivation and rationale and report on findings and design modifications throughout the iterated study cycles (Abrahamson, 2009c). Therein, we also report on the earlier part of Li's interview, focusing on his "semiotic leap" from presymbolic to articulated notions.

(2) A "batting average" is the cumulative ratio of hits (successful batting) to at-bats (opportunities to do so).

(3) It could have been fascinating to witness how Li would then inscribe these images using pencil and paper.

(4) The elongated horizontal extent of the ripple gesture does not map onto any *mathematical* analog—it is not the a, the b, or the a/b—yet it is what one would see/feel, so it is gestured. Paradoxically, this mathematically excessive magnitude could constitute a disservice to the comparison, because it inflates the a. Yet this "soothing" gesture may be contributing a *holistic* diminutive affect that thus in fact mitigates the ripple in comparison with the splash.

(5) The interviewer never appeared perturbed by dimensions of embodiment, discourse, and inscription implicit to Li's utterance, because he shares with Li tacit cognitive mechanisms for engaging multimodal reasoning.

(6) The data analyzed herein were collected in a study supported by an NAE/Spencer Postdoctoral Fellowship 2005-6. The ideas we present herein are the result of the Seeing Chance group's collaborative work, foremost Mike Bryant

References

Abrahamson, D. (2009a). Embodied design. *Educational Studies in Mathematics, 70*(1), 27-47.

Abrahamson, D. (2009b). Orchestrating semiotic leaps from tacit to cultural quantitative reasoning. *Cognition and Instruction, 27*(3), 175-224.

Abrahamson, D. (2009). A student's synthesis of tacit and mathematical knowledge as a researcher's lens on bridging learning theory. In M. Borovcnik & R. Kapadia (Eds.), Research and developments in probability education [Special Issue]. *International Electronic Journal of Mathematics Education, 4*(3), 195-226. Accessed Jan. 1, 2010 at http://www.iejme.com/032009/main.htm.

Alibali, M. W., et al. (1999). Illuminating mental representations through speech and gesture. *Psych Science, 10*, 327-333.

Baddeley, A. D., & Hitch, G. J. L. (1974). Working memory. In G. A. Bower (Ed.), *The psychology of learning and motivation* (Vol. 8, pp. 47-89). NYC: Academic Press.

Bakker, A. (2007). Diagrammatic reasoning and hypostatic abstraction in stats. ed. *Semiotica, 164*(1/4), 9-29.

Bamberger, J., & Ziporyn, E. (1991). Getting it wrong. *The World of Music, 34*(3), 22-56.

Barsalou, L. W. (2008). Grounded cognition. *Annual Review of Psychology, 59*, 617-645.

Cobb, P., & Bauersfeld, H. (Eds.). (1995). *The emergence of mathematical meaning*. Hillsdale, NJ: LEA.

diSessa, A. A. (1983). Phenomenology and the evolution of intuition. In D. Gentner & A. Stevens (Eds.), *Mental models* (pp. 15-33). Hillsdale, NJ: LEA.

diSessa, A. A., & Cobb, P. (2004). Ontological innovation and the role of theory in design experiments. *Journal of the Learning Sciences, 13*(1), 77-103.

diSessa, A. A., & Sherin, B. (2000). Meta-representation. *J. of Mathematical Behavior, 19*, 385-398.

Dove, G. (2009). Beyond perceptual symbols. *Cognition, 110*(3), 412-431.

Edwards, L., Radford, L., & Arzarello, F. (2009). Gestures and multimodality in the construction of mathematical meaning [Special Issue]. *Educational Studies of Mathematics, 70*(2).

Gelman, R., & Williams, E. (1998). Enabling constraints for cognitive development and learning. In D. Kuhn & R. Siegler (Eds.), *Cognition, perception and language* (Vol. 2, pp. 575-630). NY: Wiley.

Ginsburg, H. P. (1997). *Entering the child's mind*. NYC: Cambridge University Press.

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory.* Chicago: Aldine.

Halliday, M. A. K. (1993). Towards a language-based theory of learning. *Linguistics and Ed., 5*, 93-116.

Lakatos, I. (1976). *Proofs and refutations*. New York: Cambridge University Press.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from*. NYC: Basic Books.

Lemke, J. L. (1998). Multiplying meaning: Visual and verbal semiotics in scientific text. In J. R. Martin & R. Veel (Eds.), *Reading science* (pp. 87-113). London: Routledge.

Mariotti, M. A. (2009). Artifacts and signs after a Vygotskian perspective. *ZDM, 41*, 427-440.

Nemirovsky, R., & Ferrara, F. (2009). Mathematical imagination and embodied cognition. *ESM, 70*(2), 159-174.

Ortony, A. (Ed.). (1993). *Metaphor and thought*. Cambridge: Cambridge U. Press.

Piaget, J. (1952). *The child's conception of number*. London: Routledge.

Prawat, R. S. (1999). Dewey, Peirce, and the learning paradox. *Amer Ed Res J, 36*, 47-76.

Presmeg, N. (2006). Research on visualization in learning and teaching mathematics. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psych. of math. ed.* (pp. 205-235). Rotterdam: Sense.

Radford, L. (2008). Iconicity and contraction. *ZDM, 40*(1), 83-96.

Rubin, A., & Hammerman, J. (2007, Aug.). *Soup or stew: Metaphors for the relationship between samples and populations.* Paper presented at SRTL-5, Warwick, UK.

Schoenfeld, A. H. (1991). On mathematics as sense-making. In J. F. Voss, D. N. Perkins & J. Segal (Eds.), *Informal reasoning and education* (pp. vii-xvii). NJ: LEA.

Sfard, A. (2002). The interplay of intimations and implementations. *JLS, 11*(2&3), 319-357.

Siegler, R. S., & Crowley, K. (1991). The microgenetic method. *Amer Psy, 46*(6), 606-620.

Slobin, D. I. (1996). From "thought and language" to "thinking to speaking". In J. Gumperz & S. C. Levinson (Eds.), *Rethinking linguistic relativity* (pp. 70-96). Cambridge: Cambridge U. Press.

Stroup, W. M. (2002). Understanding qualitative calculus. *Int J of Comp for Math Learning, 7*(2), 167-215.

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *ESM, 3*, 165-208.

Wilensky, U. (1997). What is normal anyway?: Therapy for epistemological anxiety. *ESM, 33*(2), 171-202.

Wilensky, U. (1999). *NetLogo*. Northwestern U, Evanston, IL: CCL http://ccl.northwestern.edu/netlogo/.

Wilensky, U., & Papert, S. (2009). *Restructurations*. (Manuscript in preparation).