Choosing integration methods when solving differential equations

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Abstract: There are two common types of solution methods for solving simple integrals: using integration constants or using limits of integration. We use the resources framework to model student solution methods. Preliminary results indicate both problematic and meaningful intersections of physical meaning and mathematical formalism when solving linked integrals.

Students solving separable first order differential equations in typically use two methods: an integration constant ("+C") or limits of integration. The two methods are similar, involving anti-derivatives and boundary conditions. But, we have found that students using the integration constant method rarely find a physically complete solution. We use a resources framework (Hammer, 1996, 2000; Sabella & Redish, 2007; Sayre & Wittmann, 2008; Sherin, 1996) to give a fine-grain, "knowledge-in-pieces" analysis of student reasoning (diSessa, 1988, 1993).

Consider the question asked in Figure 1. Students are given a separable differential equation and a set of unusual boundary conditions. A typical solution of the integrals in time and velocity requires that one carry out indefinite integration on each. In the "+C" method, the matched equations require a velocity of 366 m/s at 0 s. In the limits method, the time integral runs from t = 0 s to some unspecified time, *t*, and the velocity integral runs from 366 m/s to some undefined velocity*, v*. For all the mathematical similarity of the two, our students rarely use limits. We have asked this question (with and without some steps filled in) in several settings, and present examples from three student groups to illustrate problems they have with integration limits and how we might help them.

A group is working on the following problem.

A bullet fired horizontally has a muzzle velocity of 366 m/s and experiences a $-cv^2$ air resistance. Find an equation that describes the horizontal velocity of the bullet with respect to time.

A student writes:

$$\sum F_x = -cv^2 = m\frac{dv}{dt}$$

$$-\frac{c}{m}dt = \frac{dv}{v^2}$$

What would you do next?

Figure 1. Integrating a separable differential equation, phrased as an air resistance problem.

Groups A and B both favored solving the differential equation using integration constants – a member of Group B called this the "normal" solution method. Both groups found the appropriate antiderivatives, used the boundary conditions to find values for the integration constants, and solved the integrals. (Though this contradicts our statement that students rarely give complete responses, we note that individual and group behavior on the problem seems very different.) Only when pressed did students in Groups A and B attempt to solve the integrals using the limits method. In Group A, Wes, Derek, and Heather discussed which limits to use.

> Wes: So, what are we going to call this? (indicates the upper limit of integration on the *dv* integration) V-one, I mean v-naught? Or 366 period? [His limits are then from 0 m/s to 366 m/s.]
> (later) Derek: The limits of integration are wrong. You need a variable in there. You have zero to *t* over here, but you have two functions, or two single numbers over there. You need a variable instead…. You want to replace zero with a *v*. [His limits are then from *v* to 366 m/s.]
> (later) Derek: Actually, I have it backwards. 366 on the bottom and *v* on the top I think. [correct.]

In Group B, students used physical reasoning about the final velocity of the bullet (0 m/s) to choose limits. Ben said "So you go from v-initial, which is 366, to zero," but Kent challenged with "At the current time, no matter what the time is, is the velocity automatically zero if it's not initial?" From this, Ben and Ned constructed the right limits, "What he's doing is, you're saying is you go from v-initial to some v. It doesn't go to zero, it goes to some velocity." The language of "going to" is worth noting and was repeated by many students using limits correctly.

For both groups, we observe individual actions: choosing limits, and finding the values of those limits by extracting this information from the boundary conditions. Each of these steps was separate and distinct for students.

In Group C, Max and Phil debated the use of the integration constants and integration limits methods. In their variation of the problem, no boundary conditions had been given. These needed to be determined using appropriate assumptions and reading out information from the problem statement. Max and Phil had already solved the velocity integral using only indefinite integration, without using limits or an integration constant. (As a result, the functional dependence of their final solution would turn out incorrect.) When solving the time integral, Phil advocated the use of integration constants and Max the use of integration limits.

> Max: No, this is where you should be going just plain *t*, or *t*-naught to *t*, or zero to *t*, because initial time is when you throw it to some time, so just go to *t* instead of *t* plus *c*. <pause> Because then your integration of time would probably be going from zero, when you throw it, to *t*, some time later. [During this, his hands move choppily from a point close to his body to one far away.]
> Phil: Hmmmmm… or you could do t plus t-naught. You need a constant though.
> (later) Phil: I just can't imagine integrating without having an extra constant. I guess if you said t-naught is zero, so, I mean, yeah.
> Max: but t-naught is the initial time
> Phil: But you don't know that's zero.

Max begins by applying limits, but takes some time explaining his choice of limits to properly connect the mathematics and the physics. Note that Max accompanied his description of the integral "going from" one variable to the next by a specific gesture: chopping motions as his hands figuratively moved as if down a number line. His gestures suggest to us that the use of upper limits of undefined variables accompanies a view of integration not just as a formal algebraic antiderivative (as it was with Derek in Group A or Phil), but as something that covers a range. Max correctly interpreted the physical situation. In contrast, Phil needed to be convinced to find a value for his integration constant, and resists adding physical meaning to it. Their discussion never resolved itself completely.

We summarize some of the individual difficulties that students have with this problem. They often do not match integration limits across the equals sign. They rarely independently use a variable upper limit (an unspecified *v* and time *t*, again matched across the equals sign). And, they often over-interpret final states (such as v = 0 m/s).

We summarize students' productive use of procedural resources in Figure 2. Each of the resources should be activated for a complete solution. The network of resources used when solving with integration constants is shown with solid arrowheads. The network of resources used when solving with integration limits is shown with open arrowheads. Note that Phil, for example, does not activate all the resources in the graph, leaving out extract boundary condition and find value until pushed by Max. Students in Groups A and B used all resources in the graph.

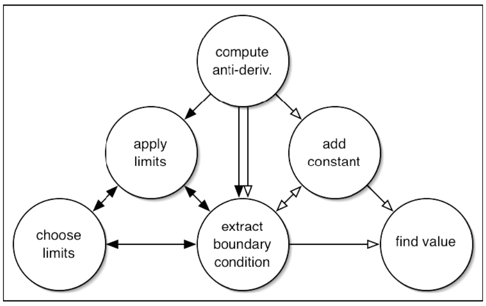

Figure 2. Procedural resources available for the limits (solid arrowhead) and the +C (open arrowheads) methods of solving differential equations. Each network can be thought of as part of an epistemic game.

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