Recognizing and Supporting Perseverance in Mathematical Problem-Solving via Conceptual Thinking Scaffolds

Joseph DiNapoli, Montclair State University, dinapolij@montclair.edu
Emily K. Miller, West Chester University, emiller@wcupa.edu

Abstract: Perseverance, or initiating and sustaining productive struggle in the face of obstacles, is integral for the learning of mathematics. Yet, the nature of such struggle is uncomfortable in the moment and often avoided for some students. This study aims to interrogate the phenomenon of learning through a lens of perseverance by investigating the effect of scaffolding mathematics tasks on students' perseverance process during problem-solving. The findings illustrate that prompting students to conceptualize a mathematical situation prior to problem-solving can encourage re-initiating and re-sustaining mathematically productive effort upon reaching a perceived impasse, despite reported discomfort. These results suggest specific methods by which in-the-moment perseverance, and thus learning mathematics with understanding, can be supported.

Perseverance, or initiating and sustaining in-the-moment productive struggle in the face of one or more mathematical obstacles, setbacks, or discouragements, is a constructive process by which understandings are developed. The notions of withstanding dissonance and overcoming obstacles have long been recognized as key learning processes (Dewey, 1910; Festinger, 1957; Polya, 1971). Such processes are especially relevant for making meaning of mathematics because students develop their understandings through productive struggle, or as they wrestle with ideas that are within reach, but not yet well formed (Kapur, 2010, 2011; Hiebert, 2013; Hiebert & Grouws, 2007; Warshauer, 2014). Additionally, reconciling times of significant uncertainty (i.e., a perceived impasse) is critical for mathematics learning. The processes of struggle to approach, reach, and make continued progress despite a perceived impasse put forth cognitive demands upon the learner that are conducive for development of conceptual ideas (Collins, Brown, & Newman, 1988; VanLehn et al., 2003; Zaslavsky, 2005). As such, encouraging student perseverance has been made explicit as a demand for reform to help improve the teaching and learning of mathematics for all students (CCSS, 2010; Kilpatrick, Swafford, & Findell, 2001; NCTM, 2014). This demand has widened the assumptions about what can constitute mathematical learning. The process of perseverance, long viewed as solely an affective or supporting component to mathematical learning, can now be simultaneously viewed as an outcome measure, alongside more standard summative achievement (DiNapoli, 2018).

Supporting student perseverance during mathematical problem-solving

Despite widespread educational support around the notion of struggling with challenging mathematics, both students and teachers can be reluctant to engage in and offer opportunities for perseverance (DiNapoli & Marzocchi, 2017). Consequently, several recent research efforts have sought to make explicit classroom practices that support or constrain student perseverance with challenging mathematics (Bass & Ball, 2015; DiNapoli, 2019; Kapur & Bielaczyc, 2012; Stein & Lane, 1996; Kapur, 2010, 2011; Sengupta-Irving & Agarwal, 2017; Sorto, McCabe, Warshauer, & Warshauer, 2009; Warshauer, 2014). For instance, Warshauer's (2014) exploration of productive struggle found that teachers used probing questions and encouragement as effective methods to nurture perseverance at times of immense student struggle, but had great difficultly providing consistent support for all classroom students at the same time. Bass and Ball (2015) explored the nature of perseverance by implementing classroom tasks with familiar entry points, yet a complex structure (i.e., low-floor/high-ceiling tasks). The researchers observed children leveraging these opportunities to persevere in their efforts despite challenge and seemingly make mathematical progress. The authors urge for future research to document the student perspective around such perseverance and if it was indeed productive. Kapur (2010) found that providing consistent opportunities for students to persevere with unfamiliar mathematical tasks encouraged more variability in problem-solving strategies and greater learning gains, compared to providing consistent opportunities to engage with more procedural mathematics. All students, however, were unlikely to make additional attempts at solving a problem after reaching an impasse – an area of concern. Collectively, these studies introduce ideas for how perseverance can be operationalized, and suggestions for future research to carefully study perseverance from the student point of view to better understand how it can be supported.

Additionally, in a recent journal series on nurturing perseverant problem solvers, several teaching practices were identified and described as advantageous. These included encouraging independent student thinking by restricting teacher reassurance feedback during problem-solving (Bieda & Huhn, 2017; Housen, 2017), scaffolding students' experiences with tasks through assessing questions, advancing questions, and judicious telling (Freeburn & Arbaugh, 2017), and scaffolding student engagement with mathematics through establishing a culture of guiding, exploratory self-questioning (Kress, 2017). These studies offer insight into effective teacher moves for supporting student perseverance with mathematical tasks, but also call attention to logistical concerns about teachers providing targeted support for each and every student (Warshauer, 2014).

One methodological approach to bypass these logistical concerns is to embed similar scaffold supports into the mathematical tasks themselves to better ensure each student has an opportunity to engage with them. Anghileri (2006) explains different levels of scaffolds that can be applied to tasks to help students problem-solve. Most relevant to supporting student perseverance are embedded conceptual thinking scaffolds, which provide opportunities for students to conceptualize the situation by making connections from their prior mathematical knowledge to the task at hand and mapping out their own strategies for problem-solving. Such conceptualization scaffolds provide a structure for thinking and acting that can organize a problem-solving plan coming entirely from a student's own ideas. Moreover, these scaffolds align with Polya's (1971) stages of problem-solving through which learners approach a task. While stages 3 and 4 help describe the actions of perseverance, it is stages 1 and 2 that can theoretically support those actions by encouraging students to conceptualize the mathematical situation at hand (see Figure 1).

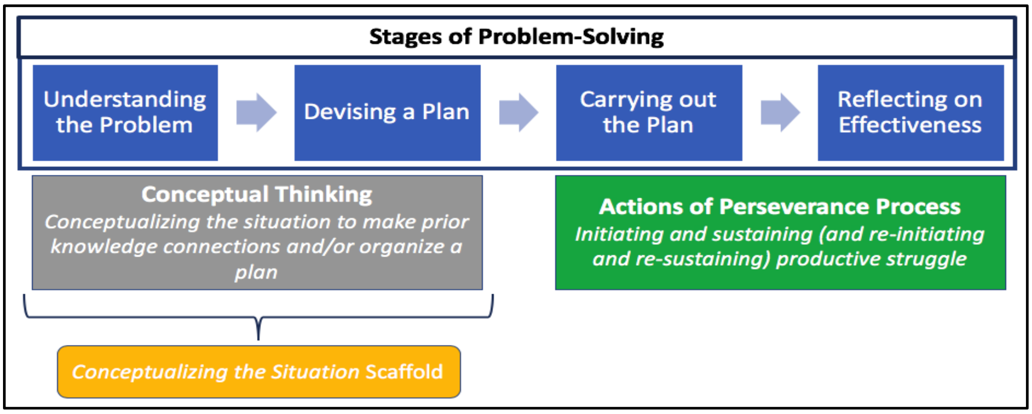

Figure 1. Scaffolding perseverance in problem-solving.

There is ample work that shows how conceptualization scaffolds encourage initial engagement with challenging mathematical tasks: when students have opportunities to make connections to what they already know and record all of their ideas and plans prior to the actual execution of problem-solving strategies, they are better suited to initiate and sustain their engagement with the task (e.g., Hmelo-Silver & Barrows, 2006). However, it is less clear how conceptualization scaffolds affect student perseverance upon reaching a perceived impasse – a moment that could be leveraged for key conceptual learning gains (VenLehn et al., 2003). If scaffolds are to support student perseverance during problem-solving, they must not only support initiating and sustaining productive struggle at the outset, but also support re-initiating and re-sustaining productive struggle after a student encounters a significant setback and is uncertain about how to continue. Therefore, via the student point of view, this study addressed how initially engaging with conceptualization scaffolds embedded at the start of low-floor/high-ceiling mathematics tasks supported perseverance, especially after a perceived impasse.

Methodology

The participants were 30 ninth-grade students from two algebra classes with the same teacher in a Mid-Atlantic state. These participants were purposely chosen to have demonstrated, via pretest, the prerequisite knowledge necessary to initially engage with each task included in the study. Out of a possible 30 pretest points, all participants scored between 27-30 points. (1)

To collect data, each participant was observed engaging with five mathematical tasks, one per week. These tasks were rated as analogous, expert-level tasks by the Mathematics Assessment Project (MAP) because of their low-floor/high-ceiling structure and required generalization objectives. Each task had two parts. For each participant, three tasks were randomly chosen to be scaffolded, and two tasks were randomly chosen to be non-scaffolded. The conceptualization scaffold embedded into the scaffolded tasks read: "*Before you start, what mathematical ideas or steps do you think might be important for solving this problem? Write down your ideas in detail*." Each participant worked on these set of five tasks in a random order. Two of the five tasks will be discussed in this paper. "Cross Totals" asked students to generalize rules about how to arrange the integers 1-9 in a symmetric cross such that equal horizontal and vertical sums would be possible or not possible. "Triangular Frameworks" asked students to generalize rules about how to build different triangles using the triangle inequality theorem if the longest side was even or odd in length. All tasks are available for view at map.mathshell.org.

This study prioritized the student point of view of their perseverance, thus there were many opportunities for participants to make their in-the-moment perspectives explicit during problem-solving (Ericsson & Simon, 1993). Think-aloud interviews were conducted for each participant while they worked on each task, and video-reflection interviews were conducted immediately after they decided to stop working (see Webel, 2013). Additionally, once a participant had engaged with all five tasks (and thus all five think-aloud interviews and video-reflection interviews), an exit interview was conducted to give each participant an opportunity to comment on their overall experience working on the five tasks. In all, 11 interviews with each participant were conducted, or 330 interviews in all for this study, totaling over 120 hours of problem-solving/interview data.

The Three-Phase Perseverance Framework (3PP) (see Table 1) was developed and used to operationalize the perseverance process in this context. It was designed to reflect perspectives of concept (Dolle, Gomez, Russell, & Bryk, 2013; Middleton, Tallman, Hatfield, & Davis, 2015), problem-solving actions (Pólya, 1971; Schoenfeld & Sloane, 2016; Silver, 2013;), self-regulation (Baumeister & Vohs, 2004; Carver & Scheier, 2001; Zimmerman & Schunk, 2011), and making and recognizing mathematical progress (Gresalfi & Barnes, 2015). The 3PP considers first if the task at hand warrants perseverance for a participant (the Entrance Phase), considers next the ways in which a participant initiates and sustains productive struggle (the Initial Attempt Phase), and considers last the ways in which a participants re-initiates and re-sustains productive struggle, if they reached an impasse as a result of their initial attempt (the Additional Attempt Phase). A participant was determined to have reached a perceived impasse if they affirmed they were substantially stuck and unsure how to continue (VenLehn et al., 2003). A participant was deemed to have initiated effort if they expressed an intent to engage with the task via problem-solving. Sustaining effort was based on the participant taking action upon their intent to engage with the task via problem-solving. Mathematical productivity was determined based on the extent to which the participant perceived themselves as better understanding the mathematical situation as a result of their efforts. To capture the student point of view during engagement with tasks, coding decisions (or whether or not certain engagement constituted evidence of perseverance) depended on student cues from all interviews.

Table 1: Three-phase perseverance framework (3PP)

Entrance Phase	
Clarity (C-0)	Objectives were understood
Initial Obstacle (IO-0)	Expressed or implied that a solution pathway was not immediately apparent
Initial Attempt Phase	
Initiated Effort (IE-1)	Expressed intent to engage with task
Sustained Effort (SE-1)	Used problem-solving heuristics to explore task
Outcome of Effort (OE-1)	Made mathematical progress toward a solution
Additional Attempt Phase (after *n* perceived impasse(s))	
Re-initiated Effort (IE-n+1)	Expressed intent to re-engage with task
Re-sustained Effort (SE-n+1)	Used problem-solving heuristics to explore task
Outcome of Effort (OE-n+1)	Made additional mathematical progress toward a solution

Note 1. n = the number of perceived impasses

Note 2. All 30 participants in this study surpassed the Entrance Phase and proceeded to the Initial Attempt Phase.

A point-based analysis with the 3PP was used to help inform deeper investigation of the ways in which participants persevered on scaffolded and non-scaffolded tasks. Each participants' experiences with each task were analyzed using the framework, and each component in the Initial Attempt and Additional Attempt Phases were coded as 1 or 0, as affirming evidence or otherwise, respectively. (2) Coding decisions were informed by interview data alone. For instance, participants earned a 1 for an Outcome of Effort component only if they affirmed perceiving mathematical progress; such decisions were not based on author perceptions. Since each task had two parts, with three phrases each for the initial and additional attempt per part, there were 12 framework components to consider, per participant, per task. Thus, 3PP scores ranged from 0 to 12, depicting minimal to optimal demonstrated perseverance in this context, respectively. Participants did not need to completely solve the task to earn a 12, they just had to exhibit perseverance in all 3PP components. Once 3PP scores were determined for all participants' experiences with all tasks, paired *t*-tests were conducted to compare 3PP scores on scaffolded and non-scaffolded tasks. Also, interviews were inductively coded to reveal why participants persevered differently on scaffolded tasks compared to on non-scaffolded tasks. Each interview was coded by two independent coders. Inter-rater reliability was 93%.

Results

Since the number of scaffolded tasks differed from the number of non-scaffolded tasks, each participant was assigned a re-scaled score from 0 to 1 for each type of task, calculated as the proportion of the maximum available points (36 for the three scaffolded tasks and 24 for the two non-scaffolded tasks; each 12 points per task) the participants received. Participants demonstrated more evidence of perseverance when working on scaffolded tasks compared to on non-scaffolded tasks, in general, as evidenced by significantly greater re-scaled perseverance scores ($M_S = 0.6278$, $SD_S = .25$, $M_{NS} = 0.3444$, $SD_{NS} = .28$; $t(29) = 8.453$, $p < .001$) (see Table 2), as well as by participant reports. Differences in participants' perseverance in the Additional Attempt Phase of the 3PP under the two conditions were even greater than this overall difference. Participants demonstrated more evidence of perseverance after encountering a perceived impasse while working on scaffolded tasks compared to on non-scaffolded tasks. This was evident by significantly greater re-scaled Additional Attempt Phase perseverance scores ($M_S = 0.4759$, $SD_S = .32$, $M_{NS} = 0.1667$, $SD_{NS} = .30$; $t(29) = 6.222$, $p < .001$) (see Table 2), as well as by participant reports. All participants affirmed in the Entrance Phase that they understood the objectives, but did not know how to solve each task. Also, all participants reported a perceived impasse as a result of their engagement with each task.

Table 2: Three-phase perseverance scores

	Re-Scaled Perseverance Scores (Maximum of 1 after rescaling)		Re-Scaled Additional Attempt Perseverance Scores (Maximum of 1 after rescaling)	
Task Type	Mean	*n*	Mean	*n*
Scaffold	0.6278	90 (3 tasks × 30 participants)	0.4759	90 (3 tasks × 30 participants)
Non-Scaffold	0.3444	60 (2 tasks ×30 participants)	0.1667	60 (2 tasks ×30 participants)
	Difference: $t(29) = 8.453$, $p < 0.001$		*Difference:* $t(29) = 6.222$, $p < 0.01$	

The most prevalent difference between participants' perseverance on scaffolded and non-scaffolded tasks was whether and how they re-initiated and re-sustained a productive additional attempt at solving the problem. This means that while working on scaffolded tasks, participants often continued to productively struggle toward a solution after reaching a perceived impasse. This was not often the case after reaching an impasse while working on non-scaffolded tasks. When considering the specific Additional Attempt Phase components of re-initiating, re-sustaining, and outcome of effort, participants working on scaffolded tasks demonstrated more aspects of perseverance compared to their work on non-scaffolded tasks (see Table 3).

Table 3: Perseverance frequencies in additional attempt phase

Component	On Scaffolded Tasks	On Non-Scaffolded Tasks
Re-Initiated Effort	65/90 (72%)	15/60 (25%)
Re-Sustained Effort	59/90 (66%)	15/60 (25%)
Productive Outcome of Effort	48/90 (53%)	10/60 (17%)

While talking about their engagement with the five problem-solving tasks in the study, all participants reported a general positive effect of their preliminary conceptualizing work prompted by the scaffolds on their mathematical engagement. Related, none of the participants engaged in any noticeable preliminary conceptualizing work on tasks that did not prompt it (i.e., their non-scaffolded tasks). Most of the participants explicitly mentioned in their interviews that the reason they did not engage in such preliminary work on non-scaffolded tasks was because the task did not specifically ask them to do so, even though they recognized such work was helpful to them. Several more specific themes emerged from the analysis of all interviews that helped explain why participants found it easier to persevere on scaffolded tasks compared to on non-scaffolded tasks, especially after reaching an impasse (e.g., the scaffolds helped participants re-visit a different idea, re-conceptualize the situation after a mistake, stay organized, and/or establish momentum to stay interested and engaged).

Illustrative case: James's experiences with a scaffolded and non-scaffolded task

To illustrate how perseverance was better supported in scaffolded tasks compared to non-scaffolded tasks, especially by helping participants make an additional attempt, consider participant James's experiences with Cross Totals and Triangular Frameworks. For James, Cross Totals was his fourth overall task and his third and final scaffolded task. Triangular Frameworks was his fifth and final task, the second of his two non-scaffolded tasks. James passed through the 3PP Entrance Phase on both tasks by affirming he understood the objectives (*C-0*), but had "no idea what to do" (*IO-0*). James earned 3PP scores of 6 in his Initial and Additional Attempt Phases, resulting in a maximum overall score of 12 for his work on Cross Totals. On Triangular Frameworks, James earned 3PP scores of 6 and 0 in his Initial and Additional Attempt Phases, respectively, resulting in an overall score of 6, with the most notable difference being the lack of an additional attempt at solving after reaching an impasse on Triangular Frameworks.

On Cross Totals, James began his work with the scaffold prompt, brainstorming about the parameters of possible and impossible cross totals and that the middle number would be included in both horizontal and vertical sums. After recording his conceptualization ideas, James started his initial attempt at solving the problem stating he would try to "find some possible ones first, and those might connect to impossible ones." He initiated (IE-1) and sustained (SE-1) his effort by guessing and checking different arrangements of integers along two lines representative of the lines of the cross, with one integer in the middle of both lines, and finding their sums. James noticed he was not having success with his plan, and affirmed he was at an impasse when he said "I'm stuck. This is harder than I thought" (first perceived impasse, n = 1). During his video-reflection, James revealed "I felt stressful here because I never learned this and there was not any way I would know how to do it. I definitely panicked." He did, however, believe he made some mathematical progress toward both objectives by figuring out "that you couldn't just throw numbers in [the cross], you had to think about big numbers and small numbers" (OE-1).

After James revealed he was at an impasse during his think-aloud, he paused and admitted "I don't know what to do now." Frustrated, he started looking around at his papers in front of him and eventually pointed to his list of mathematical ideas under the scaffold prompt and said, "Well I haven't used [the middle number] yet really. Something might be special about the middle number." During his video-reflection, he explained his point of view during these moments, "I had kind of forgotten that I wrote all this stuff down. So I saw again that middle number idea. It was kind of like my life-preserver…it kept me from giving up."

When James revisited his original conceptualization of this problem, prompted by the embedded scaffold, he was propelled forward into making an additional attempt at solving. He thought-aloud about his plan to re-initiate his effort (IE-2) by studying the middle number in possible and impossible cross totals. He began re-sustaining his effort (SE-2) by changing his point of view, a different heuristic, and examined the given example. He noticed that "this one has 9 in the middle to get a 27, they are balancing the big and small numbers... something about the evens and odds." James wrote down his observations around the provided example and went on to explore his own examples (see Figure 2) and solve the problem (OE-2). During his think-aloud interview, he concluded and defended his solution:

> All solutions are odd, like the middle can't be even to do a cross total... I know that has to be true because if you have an even in the middle, like, you add up all the numbers 1-9 and you get 45. But you have to count the middle one twice, so if it's like 4, that gets you 49. But 49 can't give you a cross total because if you cut it in half you get a 0.5. You need to add an odd number twice so it cuts in half perfectly.

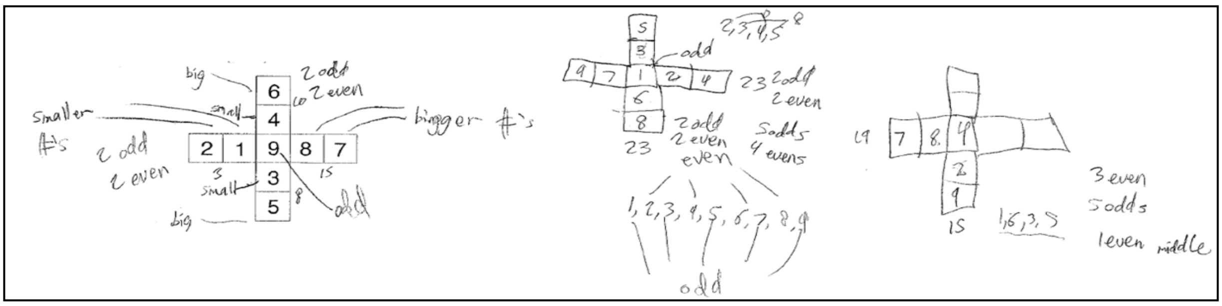

Figure 2. James' work in additional attempt phase on cross totals.

On Triangular Frameworks, one of his non-scaffolded tasks, James did not record his initial conceptualization of the mathematics. He started his initial attempt toward both objectives by saying "I'll try to find some other evens and odds that work out." He initiated (*IE-1*) and sustained (*SE-1*) his effort by guessing and checking different triangles, the same initial heuristic he used in Cross Totals the week before. James examined the given example of six frameworks made with a longest side of 7m, and made mathematical progress (*OE-1*) when he built four frameworks with a longest side of 6m, writing "6-5-4, 6-5-3, 6-5-2, 6-4-3" on his paper. James concluded, "It looks like it's one less for odd or two less for even, so the rules might be that." Before James wrote his rules, he said, "Wait, let me look at something." Clearly troubled, James went on to write "5-4-3" and "5-4-2" on his paper. Then, seemingly ignoring what he had just done, he wrote his rules about the situation (rules that were incorrect), that if the longest side, c, is odd he can make c-1 frameworks, and if c is even he can make c-2 frameworks. Finally, he said "I'm done" and stopped working without making an additional attempt. While reviewing the video of these moments, James admitted that he was at an impasse (first perceived impasse, $n = 1$) during the latter stages of his first attempt. He said, "I was looking at if 5 was the longest side and my rule didn't work! There should have been four of them but there were only two, 5-4-3 and 5-4-2…so I panicked and pretended like it worked." When asked why, he said, "I panicked when it didn't work. I didn't even know where to start to fix it. So I really wanted to stop."

James' experiences with Cross Totals and Triangular Frameworks are illustrative of the ways in which participants persevered while working on their scaffolded tasks compared to on their non-scaffolded tasks. Similarly to his engagement on Cross Totals, James leveraged his initial conceptualization work while working on his two other scaffolded tasks to help make a quality additional attempt at solving despite encountering frustrating impasses. On the other hand, similar to his work on Triangular Frameworks, James did not make an additional attempt at solving the other non-scaffolded task; he cited overwhelming stress and feeling disorganized after encountering a setback as the primary reason he did not persevere. During his exit interview, James shared his perspective on how initially attending to conceptualizing the mathematical situation had a positive effect on his engagement for all scaffolded tasks:

> You have to have a plan of what you can try to do…Like the ones that made me write down what I thought first. That was really good. I used that a lot because sometimes you forget where you're going in a problem, it's like chaos – like on [Triangular Frameworks]. But on other ones, like [Cross Totals] I got stuck but had a way out of it. It seems easier with that because you don't have to think of a way out when you're mad or stuck or something.

For James, and for almost all participants in this study, responding first to the conceptualization scaffold served as a "life-preserver" of sorts later, when participants were "panicked" and most tempted to give up. In these moments, the conceptual thinking recorded after engaging with the scaffold prompt acted as an organizational toolbox from which to draw a fresh mathematical idea, or a new connection between ideas, to use to help re-engage with the task upon impasse and to continue to productively struggle to make sense of the mathematical situation. Participants were persevering in problem-solving cyclically, with each additional attempt as a new opportunity to productively struggle supported by their own conceptual ideas (see Figure 3). Without recording their conceptual thinking on non-scaffolded tasks, participants felt lost and frustrated after a setback and often gave up without making an additional attempt at solving.

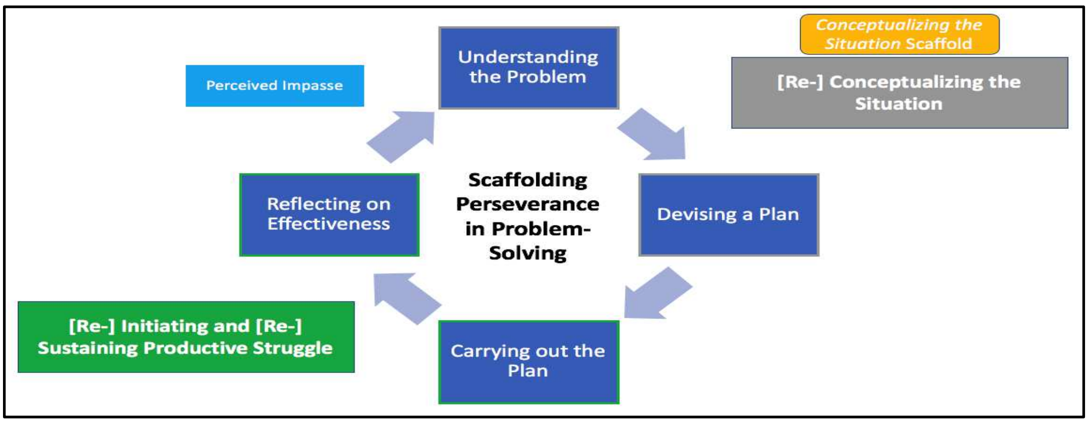

Figure 3. Rethinking scaffolding perseverance in problem-solving.

Discussion and conclusion

This study prioritized process over outcome regarding what counts as evidence of mathematical learning, illustrated by the operationalization of perseverance during problem-solving as an analytic tool. Prior research on supporting student perseverance with mathematical tasks was limited to a focus on the low-floor/high-ceiling structure of the task itself (e.g., Bass & Ball, 2015), teacher moves that nurtured independent student thinking (e.g., Freeburn & Arbaugh, 2017), and using conceptualization scaffolds to encourage initial engagement (e.g., Hmelo-Silver & Barrows, 2006), but did not examine in detail the student perspective around moments when perseverance is necessary, especially upon perceived moments of impasse. Addressing classroom logistical concerns by embedding scaffolds directly into low-floor/high-ceiling tasks (Anghileri, 2006), this study extends prior research by investigating from the student point of view how conceptualization scaffolds help students persevere re-initiate and re-sustain productive struggle after reaching an impasse. The results showed positive effects of prompting students to record their initial conceptualization of a mathematical situation prior to problem-solving, as demonstrated by more evidence of perseverance on scaffolded tasks as compared to non-scaffolded tasks. Arguably the most notable effect occurred when students reached an impasse while problem-solving, because often they were able to leverage their initial conceptualization to overcome that obstacle and continue to make progress. This implies such scaffolding can help students persevere past impasses and continue to make progress, if provided the opportunity to explore mathematics as a discipline of activity, where both process and product are valued (Schoenfeld & Sloane, 2016). Greater efforts are needed to establish attending to conceptual thinking as a normative and welcomed school practice during all problem-solving, not just when prompted (Warshauer, 2014). Each of these recommendations depend on embracing a process-over-outcome paradigm, that is, to value the perseverance in problem-solving process as evidence of learning itself, as opposed to relying solely on outcome achievement measures.

Endnotes

(1) An independent samples *t*-test showed no significant difference between pre-test scores for students in different classes ($t = -0.8393$, $p = .4084$).

(2) Only one additional attempt at solving the problem in the Additional Attempt Phase was analyzed. A participant could have continued their effort in the Additional Attempt Phase – making a second, third, or fourth additional attempt, and so on – but such efforts were rare and not considered here.

References

Anghileri, J. (2006). Scaffolding practices that enhance mathematics learning. *Journal of Mathematics Teacher Education, 9*(1), 33-52.

Bass, H. & Ball, D. L. (2015). *Beyond "you can do it!": Developing mathematical perseverance in elementary school*. http://www.spencer.org/collected-papers-april-2015

Baumeister, R. F., & Vohs, K. D. (Eds.) (2004). *Handbook of self-regulation: Research, theory, and applications.* New York: Guilford.

Bieda, K. N. & Huhn, C. (2017). Investigating problem-solving perseverance using lesson study. *Mathematics Teacher, 111*(3), 207-212.

Carver, C. S., & Scheier, M. F. (2001). *On the self-regulation of behavior*. Cambridge, UK: Cambridge University Press.

Collins, A., Brown, J. S., & Newman, S. E. (1988). Cognitive apprenticeship: Teaching the craft of reading, writing and mathematics. Thinking: *The Journal of Philosophy for Children, 8*(1), 2-10.

Common Core State Standards (CCSS). (2010). C*ommon Core State Standards for Mathematics*. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Dewey, J. (1910). *How we think.* Boston: Heath.

DiNapoli, J. (2018). *Supporting secondary students' perseverance for solving challenging mathematics tasks.* (Publication No. 19716/23958) [Doctoral dissertation, University of Delaware]. ProQuest Dissertations Publishing.

DiNapoli, J. (2019). Persevering toward what? Investigating the relationship between ninth-grade students' achievement goals and perseverant actions on an algebraic task. *International Electronic Journal of Mathematics Education, 14*(3), 435-453. DOI: https://doi.org/10.29333/iejme/5747

DiNapoli, J. & Marzocchi, A. S. (2017). Productive struggle: What we can learn from working with pre-service teachers. *The ComMuniCator, 41*(4), 10-13.

Dolle, J. R., Gomez, L. M., Russell, J. L., & Bryk, A. S. (2013). More than a network: Building professional communities for educational improvement. *National Society for the Study of Education, 112*(2), 443-463.

Ericsson, K. A. & Simon, H. (1993). *Protocol analysis: Verbal reports as data*. Cambridge: MIT Press.

Festinger, L. (1957). *A theory of cognitive dissonance*. Evanston, IL: Row, Peterson.

Freeburn, B, & Arbaugh, F. (2017). Supporting productive struggle with communication moves. *Mathematics Teacher, 111*(3), 176-181.

Gresalfi, M. S., & Barnes, J. (2015). Designing feedback in an immersive videogame: supporting student mathematical engagement. *Educational Technology Research and Development*, *64*(1), 1-22.

Hiebert, J. (Ed.). (2013). *Conceptual and procedural knowledge: The case of mathematics*. New York, NY: Routledge.

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. *Second Handbook of Research on Mathematics Teaching and Learning*, *1*, 371-404.

Hmelo-Silver, C. E., & Barrows, H. S. (2006). Goals and strategies of a problem-based learning facilitator. *Interdisciplinary Journal of Problem-Based Learning, 1*(1), 21-39.

Housen, M. (2017). Guess the number of …. *Mathematics Teacher*, *111*(3), 198-199.

Kapur, M. (2010). Productive failure in mathematical problem solving. *Instructional Science, 38*(6), 523-550.

Kapur, M. (2011). A further study of productive failure in mathematical problem-solving: Unpacking the design components. *Instructional Science, 39*(4), 561–579.

Kapur, M., & Bielaczyc, K. (2012). Designing for productive failure. *Journal of the Learning Sciences, 21*(1), 45-83.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics.* Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education, National Research Council. Washington, DC: National Academy Press.

Kress, N. E. (2017). 6 essential questions for problem solving. *Mathematics Teacher, 111*(3), 190-196.

Middleton, J. A., Tallman, M., Hatfield, N., & Davis, O. (2015). Taking the severe out of perseverance: Strategies for building mathematical determination. In N. Alpert (Ed.), *The Collected Papers*. Chicago, IL: Spencer Foundation. Article Available Online [http://www.spencer.org/collected-papers-april-2015].

Kress, N. E. (2017). 6 essential questions for problem solving. *Mathematics Teacher, 111*(3), 190-196.

NCTM. (2014). Principles to actions: Ensuring mathematical success for all. Reston, VA: NCTM.

Pólya, G. (1971). *How to solve it* (2nd ed.). Princeton, NJ: Princeton University Press.

Schoenfeld, A. H., & Sloane, A. H. (Eds.). (2016). *Mathematical thinking and problem solving*. New York, NY: Routledge.

Sengupta-Irving, T. & Agarwal, P. (2017). Conceptualizing perseverance in problem-solving as collective enterprise, *Mathematical Thinking and Learning, 19*(2), 115-138.

Silver, E. A. (Ed.). (2013). *Teaching and learning mathematical problem-solving: Multiple research perspectives*. Routledge.

Sorto, M. A., McCabe, T., Warshauer, M., & Warshauer, H. (2009). Understanding the value of a question: An analysis of a lesson. *Journal of Mathematical Sciences & Mathematics Education, 4*(1), 50-60.

Stein, M. K. & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. *Educational Research and Evaluation, 2*, 50–80.

VanLehn, K., Siler, S., Murray, C., Yamauchi, T., & Baggett, W. (2003). Why do only some events cause learning during human tutoring? *Cognition & Instruction, 21*(3), 209–249.

Warshauer, H. K. (2015). Productive struggle in middle school mathematics classrooms. *Journal of Mathematics Teacher Education*, *18*(4), 375-400.

Webel, C. (2013). High school students' goals for working together in mathematics class: Mediating the practical rationality of studenting. *Mathematical Thinking and Learning, 15*(1), 24-57.

Zaslavsky, O. (2005). Seizing the opportunity to create uncertainty in learning mathematics, *Educational Studies in Mathematics, 60*, 297-321.

Zimmerman, B. J., & Schunk, D. H. (Eds.) (2011). *Handbook of self-regulation of learning and performance.* New York: Routledge.