USING COMPARISONS OF ALTERNATE STRATEGIES TO PROMOTE DISCOURSE IN MATHEMATICS CLASSROOMS

Radha R. Kalathil, University of Illinois, Chicago, 1007 W. Harrison Street, Chicago, IL 60607
kalathil@uic.edu

Abstract: Current mathematics reform calls for greater communication of mathematical thinking in which students share their thinking and ideas with their teachers as well as with their peers in the community. In order to create an environment where mathematical ideas are shared and discussed, teachers are asked to elicit and invite different methods and ideas from students. In addition a key idea in mathematics reform is the need to support discourse in mathematics classrooms. In this paper I examine how comparisons of students' solutions to mathematical tasks were used to foster mathematically significant discourse. Data for this comes from one second-grade classroom in which instruction was based on a reform-based curriculum. I examine how comparisons of students' representations to mathematical problems were used to form discourse patterns and changes in its use over time. I also examine strategies that the teacher used to facilitate the discourse around comparisons.

Introduction

Current mathematics reform in the United States calls for greater communication of mathematical thinking in which students share their thinking and ideas with their teachers as well as with their peers in the community (Ball, 1993; Lampert, 1990; NCTM, 1991, 2000). This model of classroom communication makes explicit that students are expected not just to work on their own ideas and methods. Instead students should critically analyze and examine their peers' strategies and thinking as well (NCTM, 2000). They should ask clarifying questions and push one another's thinking and articulation of ideas.

In order to create an environment where mathematical ideas are shared and discussed, teachers are asked to elicit and invite different methods and ideas from the students (Ball, 1991; Carpenter & Fennema, 1992; Carpenter, Fennema & Franke, 1996; Carpenter, Fennema, Perterson, & Carey, 1988; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; NCTM, 2000). While this is an important goal, simply eliciting different methods may not be enough. In addition, research suggests that students need opportunities to compare and contrast different strategies in order to advance their understandings (Carpenter & Lehrer, 1999). Specifically, researchers claim that finding connections and relationships among ideas helps to structure students' knowledge and facilitates students' ability to apply knowledge in new situations (Carpenter, Fennema, Fuson, Hiebert, Human, & Murray, 1999).

At the same time, research has acknowledged that it is not always easy to implement reform practices (Sherin, 2002). Furthermore, most discussions of the potential benefits of engaging students in comparison focus on the learning process rather than on the teaching process. The goal of this paper, therefore, is to examine the use of comparisons in a reform-based mathematics classroom. In particular, this study explores how comparisons of students' solutions to mathematical tasks were used to foster mathematically significant discourse. The primary research question is, How were comparisons of students' solutions a part of the mathematical discourse in this classroom? In characterizing the discourse around comparisons the study also examines the following two sub-questions: 1. How did students begin to incorporate and use a focus on comparisons within the context of discourse around their represented solutions? 2. What teacher strategies facilitated the advancement of this focus on comparisons?

Prior research emphasizes the potential of comparisons of alternative strategies as a support for student learning. This is based on the way in which comparisons address and promote forms of mental activity, which are critical and from which mathematical understandings emerge (Carpenter & Lehrer, 1999). One important context for comparison of students' methods is whole class discussion. Through discussion students have the opportunity to not just privately compare different approaches, but to publicly discuss and debate ideas. Two major research initiatives have discussed the use of comparisons in classroom environments. The Cognitively Guided Instruction (Carpenter and Fennema, 1992; Fenemma, Franke, Carpenter & Carey, 1993) project emphasizes a classroom culture that supports the development of understanding. One of the key aspects of these classrooms is frequent discussion of

alternate strategies. When children present their strategies to the class, the different artifacts used in the research (e.g. manipulatives) provide a basis for discussing these alternative strategies and identifying potential errors. In Conceptually Based Instruction (Hiebert & Wearne, 1993; Hiebert et.al, 1997) classrooms three kinds of relationships are emphasized in order to promote teaching and learning with understanding: the relationship between what one already knows and new information; the relationship among different ways of representing information; and the relationship among different methods of solving similar problems. In these classrooms, class discussions focus on how different representations can be used and how they are similar and different. As students share and explain their methods and listen to those of others, they are asked to build connections by considering how the methods are alike and different, and the advantages that different methods afford.

While comparisons of alternative strategies has been identified as an important component in mathematics learning, specific ways in which teachers can support this practice has not been a focus of much research. Instead, research has tended to look more broadly at what it means to "teach for understanding", for example. Researchers have acknowledged the key role of the teacher in creating and promoting environments where students have opportunities to learn with understanding (Ball, 1993; Hiebert et.al, 1997; Lampert, 1990; NCTM, 1991, 2000). Second, researchers have articulated key norms that teachers can establish to create classrooms that focus on the development of children's thinking (Cobb, Wood, Yackel, & McNeal, 1992; Hiebert et.al, 1997).

Although researchers have emphasized the changes required for teachers to be able to support teaching for understanding, much of how teachers must do this is underspecified. In contrast, work by Fraivilig, Murphy & Fuson (1999) has identified specific instructional strategies teachers can use to elicit, support and encourage children's thinking while making sure that students' intellectual autonomy is preserved. In particular, Fraivilig, Murphy & Fuson (1999) introduce a framework called Advancing Children's Thinking, which contains three components that are designed to increasingly advance children in their mathematical thinking. Table 1 below provides a brief summary of the three components.

Table 1: Advancing Children's Thinking framework summary

ELICITING (Eliciting children's solution methods) Describes ways in which teachers can provide students with opportunities and necessary encouragement to express their ideas about mathematics.	**SUPPORTING** (Supporting children's conceptual understanding) Describes instructional techniques that support children in carrying out their mathematical solutions.	**EXTENDING** (Extending children's mathematical thinking) Describes ways in which teachers can challenge and extend what children do with their current mathematics thinking.

Here I draw on the above research to specifically examine how teachers can support student discourse around comparisons. I expand on the Advancing Children's Thinking framework by using it to guide an examination of the teacher practices that can support comparisons as a discourse strategy.
The research reported in this study adds to our knowledge and research base about fostering classrooms where students learn with understanding and where discourse is a key element. In particular, it adds to our knowledge of how teachers can learn to implement mathematics reform by closely examining the teacher actions and strategies that support the practice of comparing alternate methods. It does this in two main ways. First, it considers how students and the teacher used comparisons to promote discourse in the classroom. Second, it identifies specific strategies that the teacher used to help students participate in this discourse strategy and presents a framework for understanding teaching in promoting comparisons.

This study contributes to research efforts in a couple of ways. First, it adds to our existing knowledge of classroom discourse by exploring the use of a particular discourse strategy. Second, by closely examining the teacher's actions around the use of comparisons, the study contributes to teacher development and teacher cognition

research efforts. It suggests a number of instructional strategies that teachers can use to advance the notion of comparisons in their own classrooms.

Methodology and Setting

Context

This study took place in a 2nd grade classroom in an urban school. The research is based on the principles of a case-study approach, which allows us to isolate and study in an in-depth manner the issues within a specific context. This is particularly appropriate given the complexity of studying classroom interactions (Cohen, 1990; Fraivillig, 1996; Spillane, 1998, Yin, 1994). The study's teacher, Ms. Aleesha, was in the first year of implementing a reform-based mathematics curriculum and in her fourth year of teaching. The class size was 32 students.

The reform-based curriculum that was being implemented was the Children's Math Worlds curriculum, herein referred to as CMW. CMW places an emphasis on teaching for understanding and on discourse in the classroom. The curriculum also provided frequent opportunities for students to visually represent their solution methods and strategies. In addition, the curriculum encourages students to try different methods and the teacher to elicit these different methods from the students (Fuson et.al, 2000). Students work individually and make visual models of their solution methods. The teacher then calls on two or three students to display these multiple solutions at the board. These methods are discussed as students ask questions and discuss the different ideas.

In a related study (Kalathil, 2004), I examined the nature of discourse in two classrooms that were implementing this curriculum. In that study I identified six distinct discourse structures that were used as students talked about their solutions to various problems. One of the discourse structures identified was the comparison structure. Here a comparison is made among the different solution methods displayed on the board. The comparison could be about the drawings, the methods or the answers obtained. The study also identified three recurrent patterns in the way these discourse structures were used together. One of the interaction patterns identified was *Investigating by Comparing*. In this pattern, interaction can be built by comparing the different solutions displayed. The comparison may be done either in the middle of the discussion or towards the end of the discussion. This study looks closely at the comparison structure and the Investigating by Comparing pattern and examines (a) how this developed over time and (b) the teacher's role in supporting this strategy.

Data collection

During the school year, the mathematics classroom was observed and videotaped using one camera. Observations took place once a week (on average) from September to May, for a total of 22 observations. In addition, detailed field notes were taken at each of these observations. Observation notes contained detailed information concerning (a) general lesson procedures, (b) students' engagement and participation in the activities, (c) key events that occurred, and (d) interactions as the class discussed the different solutions displayed at the board.

Following the observations, the teacher was interviewed. The interviews were conducted soon after the lesson was observed, either right after the lesson, or later on in the same day. These interviews were conducted as input for a larger research project, the Learn-while-teaching project (LWT). The LWT project was intended to inform research on curriculum development and teachers' professional development by supporting and studying teachers' implementation of a reform-based curriculum. The interview questions were designed to reveal teacher learning about mathematics, mathematics teaching and student learning.

Data Analysis

Nine of the 22 lessons were selected for detailed analysis. These lessons were selected to represent a range of mathematical topics addressed across the school year. The lessons were examined, paying particular attention to two aspects of what took place: (a) the role of the teacher and her actions in advancing the discourse strategy of comparisons, and (b) characteristics of how the students used comparisons in their discussions and discourse.

Data analysis was performed using fine-grained analysis of the video, the interview data and field notes. In addition, lesson summaries that were created for the related study (Kalathil, 2004) were considered for analysis. The lesson summaries noted the different discourse structures and interaction patterns that occurred within a given lesson as students talked about their solutions to the problems. The lesson summaries therefore, listed which of the six discourse structures were used within a lesson and when this took place. Using these summaries, the researcher

noted each time in the nine lessons that the comparison structure was used. Here students compare the different solutions presented to solve and represent the given mathematical problem.

Next, each of these comparison segments was then analyzed along three dimensions. (1) I examined the teacher statements and actions in order to characterize the different ways in which the teacher encouraged discourse around comparisons. (2) I characterized the types of comparisons that were explored and the frequency of these comparisons. This provided input concerning the different ways in which students incorporated comparisons in their discourse, as well as the kinds of comparisons and mathematical ideas that were discussed in these comparisons. (3) I examined the lesson summaries to see at what point comparisons were used in a lesson. This provided information on when the class was incorporating the strategy of comparisons in their discourse.

Analysis along the first dimension revealed different ways in which the teacher strategies could be categorized. Using these categories, I reexamined the literature and prior research base to determine how these would fit within the context of research on teaching. The Advancing Children's Thinking framework (Fuson, et al, 2000) appeared to provide a useful way in which to categorize and organize the strategies that had been identified. Specifically, all of the techniques that Ms. Aleesha used to help her students use comparisons fell within the components listed in the Advancing Children's Thinking framework. These are discussed in the results section that follows. These strategies covered aspects of (1) how the teacher elicited or encouraged students in using comparisons of multiple solutions in their discourse, (2) how the teacher supported students' articulation and development of ideas on comparing alternative strategies, and (3) how the teacher facilitated the advancement and extension of students' ideas within discussion in the classroom. To be clear, the particular strategies that are discussed differ from those in the Advanding Children's Thinking framework in that they are specific to Ms. Aleesha's facilitation of comparisons in her classroom. In this way, the results extend the Advancing Children's Thinking framework.

The comparisons were also examined with respect to their mathematical significance. For each comparison identified there were many comments by students concerning aspects of the comparison being discussed. I call these comparison statements. (1) First, I examined if the comparison statement was addressing non-mathematical issues. For example, I investigated whether the comparisons focused on surface features of the represented solutions – for example, of the type "A's drawing has a title, and B's drawing does not." (2) I also examined whether the comparison statement addressed differences in the strategies used. For example was the comparison of the type "A used counting by tens, and B used counting by 1's." (3) Finally I examined if the comparison statement evaluated the different methods against each other. Within this analysis the number of different comparisons students made around each set of solutions were also examined. For example, did students make more than one comparison between the related representations? This part of the analysis also focused on determining if there was any change in the nature of comparisons across the nine lessons.

Finally, the models of the lessons were examined to see if the students began to incorporate comparisons at different point in their discussions across the year. In particular I sought to understand how the students used comparisons as an initial spark for discussion and if this changed across the nine lessons. The results of this analysis are presented in the section that follows.

Results and Discussion

This study explores how comparisons of different solutions to solve and represent mathematical tasks were used by the students and the teacher as a significant discourse strategy in a second grade classroom. The results of this study are organized in two parts. In the first part I identify two broad themes with respect to how the students were able to use comparisons in their discourse. In the second part I examine closely the teacher actions and instructional techniques that advanced the use of the comparison strategy. Here I briefly outline these two parts. The paper will discuss these in detail, along with examples from the data.

Use of comparisons in the classroom

This section describes how the class incorporated comparisons into their discussions of students' solutions to mathematical tasks. Analysis revealed two broad themes with respect to how comparisons were used in the classroom.

First, more mathematically significant comparisons were discussed over time. The nature of the comparisons discussed in class became more mathematically significant over time. Specifically, initial comparisons tended to focus on surface features of the drawings. For example, students focused on whether one of the solutions had a label for the graph or no. As comparisons become a core part of the classroom discourse, the types of comparisons that were discussed began to get increasingly mathematically significant. Students began to compare and discuss the differences in strategies as well as engage in more evaluative discussions, extending the comparisons by critically examining the merits of each method.

A second dimension along which the use of comparisons shifted over time concerns at what point in the lesson comparisons was used. Initially comparisons were used as a wrap-up towards the end of the discussion. Later however, comparisons began to be used increasingly as a spark for further discussion. In other words, comparisons began to be used in the middle of the discussion in order to prompt further discussion. Thus students spent a significant portion of time discussing and exploring these comparisons (as characterized by the fact that the comparisons were in the middle of the segment) and not just briefly considering the alternative strategies at the end of the discussion.

Teacher's strategies for advancing comparisons

In this section I identify and describe instructional strategies used by the teacher to facilitate and encourage students in the discourse practice of comparing different solutions. Table 2 below provides a summary of these strategies. As mentioned earlier, in describing these strategies I use the three components (Eliciting, Supporting and Extending) as described in the Advancing Children's Thinking framework (Fraivilig, Muphy, & Fuson, 1999). The specific techniques used my Ms. Aleesha can be described using many of the sub-categories that are identified in the Advancing Children's Thinking framework.. (Entries that are in bulleted text are specifically unique to Ms. Aleesha's teaching). These strategies are discussed in detail in paper.

Table 2: Ms. Aleesha's strategies for advancing the use of comparisons

Eliciting	Supporting	Extending
Facilitates Students' responding *Elicits many solution methods* • Asks for alternative methods • Thanks students for showing alternative methods • Names specific student methods *Conveys accepting attitude towards students' efforts* • Emphasizes that when a method is different that does not mean it is incorrect *Orchestrates class discussions* *Uses students' explanation*	*Supports describer's thinking* *Assists students in clarifying* • Provides prompts and helps students in articulating and clarifying the comparisons made *Supports listener's thinking* *Demonstrates teacher-selected solution methods without endorsement of a particular method* • Points to unique features of the different methods without endorsing any particular method Supports describer's and listener's thinking *Asks different student to explain* • Asks peers and other students to compare	*Encourages mathematical reflection* *Lists all the solution methods on the chalkboard to promote reflection* • Has students come to the board to display solutions *Goes beyond initial solution methods* *Pushes students to try alternative methods for one problem situation* • Pushes students to describe additional comparisons beyond initial suggestions • Pushes students on the nature of comparisons, pushing them towards exploring more mathematically significant comparisons

for lesson content • Discussion based on students' ideas about comparisons Decides which methods to be discussed • Chooses diverse methods to be discussed in class	students' methods on board • Creates norm of collaborative problem solving and shared intellectual authority	*Promotes use of more efficient solution methods for all students* • Uses the comparison technique to see which method might be easier etc. • Encourages students to learn from differences *Uses students' responses, questions as core lesson* • Uses comparisons to explore significant mathematical content

Implications and future work

This research contributes to our understanding of comparing different methods. In particular it examines the aspect of 'comparing' different methods and strategies within the context of discourse around students' visual representations of solutions to mathematical tasks. It identifies some important characteristics of how students were able to incorporate discussions of comparisons into their discourse. Specifically, students began to use the discourse strategy of comparing different drawings increasingly in their discussions and comparisons came to be used as a spark for further discussion. Students also engaged more and more frequently in discussions of mathematically significant comparisons, addressing strategies and evaluating methods against one another. Thus, students were able effectively to incorporate the use of comparing alternate drawings and strategies in their mathematical discourse.

Using comparisons as a significant discourse strategy also enabled the lesson and discussions to be more responsive to students' ideas and thinking. The ideas brought forward and discussed were not based solely on tasks that were preset or predetermined, and this is an important aspect of teaching and learning with understanding (Carpenter & Lehrer, 1999; Hiebert et.al, 1997). This often resulted in the lesson being adapted according to input and responses from the students.

By closely examining the teacher's actions during discussions of comparisons in the classroom, this study contributes to teacher development and teacher cognition research efforts in important ways. It breaks down the practices of one teacher who was effective in advancing this notion of using comparisons. In doing so, it provides us with instructional strategies that will help teachers advance this notion of comparing different methods in their classrooms. Here I argue for why some of these strategies would prove important in implementing mathematics reform ideas of encouraging student thinking and ideas.

The teacher was able to elicit many different solution methods through different practices. Another important strategy used by the teacher was to emphasize that just because a method was different that did not mean it was wrong. This is especially important within the context of investigating comparisons because it creates the norm that different methods and strategies are useful and valid on their own, and that it is also valuable to compare them. This contributes to the notion of creating a 'safe' environment, where students feel comfortable trying out different approaches, thus bringing their own ideas and knowledge to the environment. It also is important as it reinforces an important reform goal that students understand that there are numerous ways to solve the same problem (NCTM, 1989, 2000).

By discussing comparisons explicitly and by pushing students to go beyond just initial comparisons, the teacher was able to set a norm where discussing alternative strategies was an important aspect of the mathematics learning in this classroom. This helps to build the forms of mental activity necessary for learning with understanding (Carpenter & Lehrer, 1999). This also addresses important aspect of mathematics teaching where the

tasks set by the teacher are meaningful and are such that they encourage students to make sense of the mathematics being learned (Hiebert et.al, 1997).

In conclusion this classroom provides encouragement for mathematics education reform efforts in a few ways. The elicitation processes used by the teacher proved successful and useful in contributing to students' discussions of alternative strategies. This is particularly important, as research has acknowledged that although elicitation is a significant element of mathematics teaching, it is infrequently observed in practice (Cazden, 1988). Students were able to incorporate discussing these different strategies effectively in their discourse, addressing important mathematical ideas. Furthermore, the teacher in this classroom was not a veteran teacher. This was only her fourth year of teaching and her first year of implementing a reform based mathematics curriculum. This study shows that some of the important goals of mathematics reform are achievable and thus provides us with useful tools with which to implement mathematics reform.

The study provides us with an important framework to understand and help teachers in implementing this aspect of mathematics reform teaching. The Advancing Children's Thinking framework was used to detail the practices and techniques that could be used to advance the notion of comparing different solutions. The extensions to the Advancing Children's Thinking framework also provides us with a tool with which to analyze classroom practices.

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