Mathematics Teachers' Abilities to Use and Make Sense of Drawn Representations

Chandra Hawley Orrill, Susan Sexton, Soo-Jin Lee, Cheryl Gerde
Learning & Performance Support Lab, University of Georgia
611 Aderhold Hall, Athens, GA 30602
Email: corrill@uga.edu, sextons@uga.edu, sjjh0314@uga.edu, cgerde@uga.edu

Abstract: This qualitative study considers middle grades' mathematics teachers' abilities to make sense of drawn representations of fraction and decimal operations. Our interest was in understanding how teachers interpret these drawings and whether they can flexibly move within and between the various drawings. Our findings focus issues of relevant mathematics that emerged as well as teachers' abilities to move between and within the various drawings of interest. The conclusions tie our findings to the need for more professional development to support teachers in using drawn representations.

Introduction

The National Council of Teachers of Mathematics (NCTM, 2000) and others (e.g., Kilpatrick, Swafford, & Findell, 2001) have suggested that engaging students in mathematics through multiple representations is powerful and necessary to support students' development of mathematical understandings. Flexibility in utilizing representations is a primary characteristic of competent problem solvers (e.g., Dreyfus & Eisenberg, 1996). This flexibility supports conversations about students' understandings (NCTM, 2000; Goldin, 2002) and supports connection-making among mathematical concepts (Lesh, 1987; NCTM, 2000). Despite the widespread belief that using drawn or physical representations to support student learning is critical, teachers generally do not use them in their classrooms. We hypothesize that this underutilization is, at least in part, a product of teachers' lack of comfort in using and interpreting drawn representations. We further hypothesize that this lack of comfort relates to a critical gap in teachers' mathematical knowledge for teaching (Sowder, Philipp, Armstrong, & Schappelle, 1998). That is, one or more pieces of the ideal teacher's "knowledge package" (Ma, 1999) are weak. The knowledge package a teacher needs for effective teaching of mathematics includes knowing content deeply, knowing content conceptually, and knowing the connections among ideas as well as "the representations for and the common student difficulties with particular ideas" (Ball, Lubienski, and Mewborn, 2001, p. 448).

To explore our hypothesis that an incomplete knowledge package may be at play in teachers' use (or lack thereof) of representations, we sought to understand whether and how teachers could interpret representations and how they relate the drawn representations to the mathematics that they teach. In this study, we consider 12 teachers' responses to questions about seven different assessment items that required them to make sense of drawn representations of fraction and decimal operations. These included area and number lines models for various operations appropriate to sixth and seventh grade content.

This work addresses some important gaps in the literature. First, much of the research on multiple representations has been conducted in the area of algebra with emphasis on student learning (Rider, 2004) with little attention paid to representations in other mathematical strands. Second, work done on representations has often focused on translating between disparate representations such as moving from a graph to a verbal description (e.g., Gagatsis & Shiakalli, 2004), rather than translating between drawn representations. Furthermore, while the research abounds on student conceptual understanding of rational numbers, the literature has yet to provide adequate knowledge of teacher understanding and use of multiple representations, specifically the use of drawn representations in the domain of rational numbers.

Literature Review

As a way of understanding the field in terms of teachers' abilities to interpret drawn representations, we provide an abbreviated overview of two areas of the literature. First, we discuss mathematical knowledge for teaching because it is critical to the assessment used in the study and because it provides a framework for thinking about teacher knowledge. Then, we discuss the existing literature on teacher knowledge of fractions and decimals. The overview presented here is not meant to be exhaustive, rather it is meant to situate this work in the domain of existing research.

Mathematical Knowledge for Teaching

For the purposes of this study, we focus on the knowledge construct called *mathematical knowledge for teaching* (MKT; Ball, 2003) which has been defined as being similar to Shulman's *pedagogical content knowledge* (Shulman, 1986) but with additional emphasis on the specific knowledge a teacher needs to teach particular content. In short, it is the knowledge that is uniquely necessary for teaching students mathematics. A teacher with stronger MKT has the knowledge and skills to not only introduce content, but also to interpret student work and support students in moving from their current mathematical understandings to new understandings by providing them with opportunities to make connections between and among mathematical ideas. This is critical because using mathematical representations may open the classroom to more novel student approaches to solving problems, thus, creating a situation in which the teacher needs to interpret and respond to novel student thinking on the fly. MKT is a recent addition to the field in terms of its conceptualization, however, major efforts to develop instruments for measuring MKT have already taken place. The *Learning Mathematics for Teaching* (SII/LMT, 2004) assessment is a widely used instrument focused on MKT. In recent research, the LMT developers were able to show not only that MKT is important, but that differences among teacher LMT scores at the third grade level were as strongly related to students' achievement as was the students' SES level (Hill, Rowan, & Ball, 2004). This strongly suggests that understanding the extent to which teachers have MKT is important for understanding their likely ability to impact student performance.

This is particularly important to the current study for several reasons. The most pressing reason is that the assessment items used for the teacher interviews included some from the LMT as well as additional items that were written by our own research team using the same approaches as those in the LMT. The questions included in the interviews provided teachers with one or more drawn representations for a fraction or decimal operation and asked the teacher to interpret different aspects of the representations. For example, some of the questions simply asked the teachers about what the drawing was showing them. Other questions provided sample student drawings and asked the teacher to determine whether the approaches used by the hypothetical students were appropriate. Many of the teachers who took the assessment indicated that the test was difficult and very different from other assessments they had taken. However, many of them also claimed to enjoy the opportunity to engage with the problems in the assessment.

Teacher Knowledge of Fractions and Decimals

While there is a rich body of research on students' understandings about rational numbers and operations with them in the mathematics education literature, there is a distinct shortage of research on teacher knowledge in this area. It is generally known that many elementary and middle school teachers are not strong in their knowledge of fractions and fraction operations (e.g., Ma, 1999; Ball, Lubienski, & Mewborn, 2001). However, teacher thinking about fractions and decimals has generally not been as carefully explored as student thinking about these topics (e.g., Behr, Khoury, Harel, Post, & Lesh, 1997; Steffe, 2002).

There are relatively few studies related to deep understanding of teacher knowledge. In one such study, Izsák (2008) examined a teacher's knowledge of fraction multiplication as she interpreted students' work with drawn representations and used drawn representations to teach fraction operations in the classroom. Izsák analyzed the teacher's work utilizing Steffe's (2002) notion of *unit structure*. This allowed analysis of teachers' abilities to attend to *referent units* – that is their ability to keep track of what the whole was in each operation. While this is relatively straightforward in addition and subtraction situations because both parts of the problem relate to the same unit (i.e., whole), it is more difficult in fraction multiplication and division because the answer refers to different units (i.e., a part of a part). For example, in a problem such as $\frac{1}{2}-\frac{1}{3}$, the referent unit for both numbers is a whole (e.g., I have one-half pan of brownies and I take away brownies that equal one-third of the entire pan; See Figure 1). However, in a problem such as $\frac{1}{2}\times\frac{1}{3}$, you are considering a quantity that is one-half of the part that is one-third of a whole (e.g., you have one-third of a pan of brownies and you want to take half of that piece of brownies away; See Figure 2).

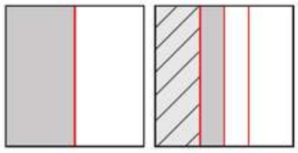

Figure 1. One-half of a pan minus one-third of a pan.

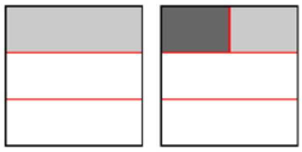

Figure 2. One-half of one-third of a pan of brownies.

In his study, Izsák found that this teacher could adapt linear or area representations in her teaching because she had flexibility in her ability to identify the referent units. However, the study uncovered that the participant found the intended use of drawn representation unclear. Thus, the study indicated that at least some teachers can use drawn representations flexibly, but that the reasons for using them may not be apparent to the teachers, therefore the teachers do not capitalize on the opportunities representations present.

In another study, Izsák, Tillema, and Tunç-Pekkan (2008) explored both teacher and student knowledge of fraction addition using number lines in one classroom. In this study, the researchers conducted interviews with the teacher and with several pairs of students in the class in which they were asked to solve problems and to explain their understanding of different classroom events as prompted by video of the classroom. From detailed analysis of interactions between the teacher and one student, Izsák et al. determined that drawings can lead to miscommunication between teachers and students. Each of the different strategies the teacher and her student used to solve problems confounded each other's understanding of fractions. Specifically, there were several cases where the teacher imposed her own thinking about fraction addition because she was unable to understand the student's approach to coordinating units to solve fraction addition problems. Further, and relevant to the current study, the teacher clearly only valued number line representations as stepwise replacements for the addition algorithm. That is, according to the teacher, representations were used to model an answer rather than as a tool to solve problems. Together, Izsák's studies suggest that MKT needs to include not only teachers' recognition of different kinds of drawn representations for solving problems, but also a clearer understanding of when and why they should be used as well as how students might use them.

Methods

The data reported in the present study emerged from a larger study of middle school teacher professional development for which a rational numbers teacher assessment was developed and piloted with middle school teachers. In the validation effort, over two-hundred teachers took the assessment and approximately 25 were interviewed about selected items on the assessment. As part of the assessment validation effort, these interviews were conducted as a means for understanding how teachers were approaching solving the items on the assessment. The test items were taken, with permission, from the University of Michigan's LMT assessment or were created by the project team. The content of the questions focused solely on fraction, decimal, ratio and proportions concepts, although their form varied between verbal, numeric, and drawn representations. The focus in this study was on 12 teachers' interviews that included discussion of their approach to up to seven rational number items that focused on drawn representations. All of the teachers were interview on at least five of the items of interest to the present study. The teachers were selected because they were the total population interviewed on the two pilot forms of the assessment (13 additional teachers were interviewed to validate the final version of the assessment form and one teacher's data were removed from consideration due to technical difficulties). The teachers each completed the multiple-choice assessment on paper, then were asked to talk about the test items in these interviews, which occurred within three days of the original assessment administration. In all cases, the teachers were given their test paper as a prompt for thinking about how they answered each item.

The participants were from three school districts (one rural and two urban). They were selected for interviews based on their district (i.e., to maximize our personnel resources, we only interviewed in a small number of districts, though the overall sample for the test itself was a national one) and their willingness to participate. Because of our selection criteria, we were able to obtain a sample of teachers that included one special education teacher and 11 regular mathematics teachers. The participants ranged in experience from second-year teachers with undergraduate degrees in Middle Grades Education to teachers with alternative certification to teachers with 20 or more years of experience. Several of the teachers had masters degrees or higher. The sample included four men, four African-Americans, and five rural teachers. Some of the teachers had experience with the standards-based *Connected Mathematics Program* (Lappan, Fey, Fitzgerald, Friel, Phillips, 2002) that promotes the use of representations (as laid out by NCTM) while others had primarily used more traditional teaching materials.

The twelve teachers interviewed were videotaped with two cameras (one focusing on the written work and hand gestures, the other on the entire view of the interviewer and interviewee). These videos were mixed into one file, thus creating a *restored view* (Hall, 2000) and then transcribed verbatim. The transcripts, thus, became the primary data source. However, the videos were used as a secondary source when clarification was needed on a gesture or statement.

The initial analysis focused on uncovering the primary categories in the data. Each transcript was analyzed by at least two members of the research team using emergent codes and memoing (Strauss & Corbin, 1998) to find emergent themes, which were used to develop pools of meaning in the participants' discussions (Coffey & Atkinson, 1996). These pools of meaning represent the researcher identified context to which the comments belong. Once the initial coding was complete, we capitalized on our opportunity to triangulate across researchers (Denzin, 1989) by discussing our individual findings within and across participants to develop the consensus findings for the study. We also saw looking across research participants as a source of triangulation for our research (Denzin, 1989). As the findings began to emerge, further analysis was used to make sense of the emerging trends. To this end, each researcher focused on particular emergent findings to make sense of them, again relying on the other team members to identify inconsistencies in the analysis.

Findings

In our analysis of how and whether these middle school teachers were able to make sense of drawn representations of fraction operations, we found that teachers relied on four approaches to solving problems with drawn representations. Further, we found that the teachers struggled to move between different representations and, at times, even within a single drawn representation when it was used in a different kind of problem. Finally, we saw that teachers struggled with their flexibility in interpretation on a number of problems. This limited flexibility was sometimes related to mathematical issues such as constraints in flexibility with referent units while other times it was related to particular issues participants were having with specific representations.

Approaches to Solving Problems with Representations

Four key approaches emerged as teachers explained their approaches to these problems. One approach was focused on looking for a set of perceived requisite features of the diagram with little or no attention paid to the mathematics of interest. A second approach was for teachers to work from the solution of the problem to find the representation that best showed that response. In other words, they used representations as a means to illustrate their solution rather than as a means to find a solution. Third, some teachers used the process of solving the problem as a basis for selecting the best answer. The fourth approach involved the teachers applying measurement approaches to solving the problems.

Identifying Requisite Parts

This approach was pervasive for some teachers. They tended to describe their selections in terms of identifying the parts of the diagrams they were attending to. For example, in a situation such as that shown in Figure 2, a teacher using this approach would note that she could clearly see the one-third, the one-half, and the "answer", thus the problem would be counted as correct. This is an interesting approach because it assumes that the representations somehow contain specific stepwise elements that are required for their correctness. When viewing the representations this way, the teachers were less likely to accept representations with slight modifications as being correct. For example, in Figure 3, a teacher using this approach would accept (a) because it shows an area that is $\frac{1}{4}$ designated by vertical bars with $\frac{2}{3}$ indicated by the horizontal bars. The answer is the shaded area where $\frac{1}{4}$ and $\frac{2}{3}$ cross. However, a few teachers did not accept (b) which shows $\frac{2}{3}$ of a square that is $\frac{1}{4}$ of the whole and several rejected (c) which shows $\frac{2}{3}$ of $\frac{2}{8}$. While several of the teachers acknowledged that (c) relied on equivalent fractions, they did not accept the model because it did not meet the requirement of each piece being present exactly as it is in the statement of the problem. This, of course, is important because teachers need to be able to make sense of student work – which is not predictable and prescribed – on the fly in the classroom.

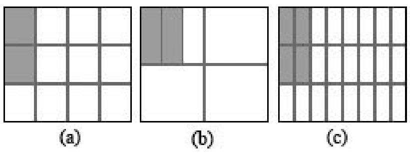

Figure 3. Examples of Multiplication with Area Models.[1]

Looking for the Diagram that Matches a Solution

Seven of the interviewed teachers solved at least some of the items with an algorithm before they selected their responses. In fact, three used that approach throughout the assessment. This resulted in the representation serving only as a model of the answer. This is to be expected, to an extent, given that most mathematics teachers are experts in working with traditional algorithm-based approaches to mathematics. On the assessment, however, we constructed items so that teachers needed more than rules to select the correct answer – they needed deep understanding of the concept (Lesh, Post, & Behr, 1987). In fact, in most cases, the assessment provides the correct answer as part of the question stem because we are interested in understanding how teachers interpret the situation rather than their ability to solve the problem. This led some teachers to accept answers that illustrated correct quantities derived in inappropriate ways. For example, on one number-line fraction multiplication item, one answer choice number line modeled multiplication and the next modeled subtraction, but both had the same quantity highlighted as a solution. Teachers who looked for a diagram to match the solution reported that either of these representations was appropriate because they both showed the correct answer – and the teachers were drawn to answers rather than processes.

Using the Process of Solving to Select Solutions

This was the least common of the four approaches taken by the teachers who adhered to these approaches. In this approach, when the teachers were asked to find the correct representation of a problem such as $\frac{2}{5} \div \frac{1}{7} = \frac{14}{5}$, they stated that their decision for selecting the correct diagram was based on what they know about fraction division – that is, they knew that they needed to “flip and multiply”, therefore they looked for a solution that showed a length of $\frac{2}{5}$ repeated seven times. While rare, this approach was the one used by most of the participants who responded to this item correctly. We speculate that this was true because adults have been taught to automatically invert and multiply the divisor in the fraction division, and if teachers view drawing as a representations of algorithm, then this approach is sensible.

Measuring to Find a Solution

Unlike the other three approaches, this approach seemed to be a last resort in that participants only used it in cases where they were visibly unable to use any of their other approaches to select an answer. In all we saw more than six teachers use this approach at least one time. In the measurement approach, the teachers relied on some aspect of measurement to determine the answer. In its most pronounced form, this included teachers drawing particular lengths on their papers and comparing them to the lengths on the diagrams to determine whether an answer was feasible. In it’s more subtle form, teachers would report that they had selected answers because the shaded areas were equal to other drawings or because the lengths all “looked” the same. Specifically, we saw six teachers try to use this method on the single decimal multiplication item – which was an item most of the teachers were unable to find appropriate answers for. Of those six, two used this approach on fraction multiplication with area and three with the number line multiplication problems.

The analysis on teacher approaches to representations revealed that they did not activate their conceptual knowledge of fraction operations when making sense of these drawings. This may be due to teachers’ deep-rooted procedural knowledge of approaching fractions. We had hoped to see teachers use the representations in place of algorithms – particularly in those cases where the teachers were using classroom materials that capitalized on representations. Clearly, this analysis raises questions about the reasonable expectations for using drawn representations and about the need for professional development to support teachers in thinking about representations in different ways.

Translating and Transforming

Consistent with earlier studies (e.g., Gagatsis & Shiakalli, 2004), we found that teachers had varying abilities to “translate”[3], that is, move from one representation to another (e.g., number line to area model). Further, we found that knowledge of a single representation was not an indicator of a universal ability with that representation as our participants struggled to “transform” or move between different instances of the same representation. For example, few of the teachers were able to successfully reason about an area model for decimal multiplication, but all were able to reason about at least one area model for fraction multiplication. For our participants, the representation for modeling fraction division and decimal multiplication presented the most difficulty. Participants generally made explicit comments about their lack of understanding of those particular representations. Overall, participants’ inattention to referent units, their lack of flexibility with a model as representing more than one fraction simultaneously, or their general unfamiliarity with a particular representation were the three main reasons for the challenges they demonstrated while working with these drawn representations. In cases where the participant was especially confused about a particular representation, s/he typically demonstrated inattentiveness to units or demonstrated a loss of mathematical reasoning.

Challenges in Transforming

All participants exhibited difficulty when presented with variations of a single representation. How those difficulties manifested varied from participant to participant. Issues with transforming were most pronounced in items that considered area models for fraction multiplication, fraction division, and decimal multiplication. Most participants (11 out of 12) commented or exhibited familiarity with the most basic area model for fraction multiplication (See Figure 3a). When asked to extend their knowledge of the array model for fraction multiplication to division or decimal multiplication, nearly all participants (11 out of 12) exhibited some type of challenge in their mathematical thinking.

For example, one participant expressed familiarity with area model in fraction multiplication because she had experienced it in her undergraduate work and in her teaching materials. However, when asked to determine a correct answer for a division problem using arrays, the participant selected the one that most closely modeled multiplication rather than division. In her rationale, she explained that she was selecting the one with recognizable attributes. For example, in a problem such as $\frac{3}{4} \div \frac{1}{8}$, the participant would have incorrectly selected Figure 4a over Figure 4b because it had shaded overlap much like the multiplication problems with which she was familiar. In her interview, the participant described her rationale, saying, "I guess on this one my thinking was, kind of like an area model, where you look for the overlap and there are two pieces that overlap on this one [Figure 4a]. And that's why I felt it showed the answer."

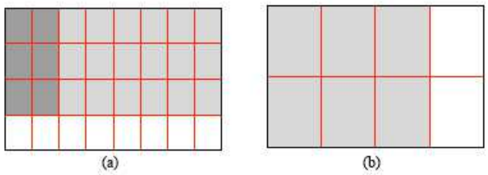

Figure 4. Examples of Division with Area Model.

Challenges in Translating

As discussed above, most participants were familiar with the area model for multiplication. However, 10 out of 12 participants experienced difficulty when translating between an area model for an arithmetic operation (e.g., fraction multiplication) to the number line model of the same operation. For example, the participants who showed some level of facility in their use of the area model for fraction multiplication were familiar with the representation shown in Figure 3a. However, when faced with a number line representation for the same concept, these participants incorrectly selected a representation that showed fraction subtraction but had a computational result equivalent to that of fraction multiplication[3]. Only one teacher showed greater facility with number lines than area models.

In short, our analysis of teachers' abilities to move within and between different representations indicated that teachers' understanding of the representations appears to be contextualized. Perhaps because they perceive the representations as either illustrations of a solution or as stepwise procedures to be used in the same way traditional algorithms are used, these participants were not always able to translate their understanding of a representation from one situation to another. The less familiar they were with a drawn situation, the more likely they were to struggle with translating it. In the extreme cases, our participants failed to make mathematical connections even when they had already demonstrated adequate knowledge to interpret the drawings.

Flexibility of Identifying Referent Units

The theme of flexibility was important in our effort to understand how the teachers were interpreting drawn representations. In our analysis we uncovered a general lack of flexibility related to referent units. Such flexibility is is critical to understanding the representation. Because these problems all focused on fractions and decimals, teachers who were unable to flexibly move between different interpretations of the units struggled to make sense of the problems. In one simple example, the teachers were provided with a problem similar to Figure 5. Most of the teachers recognized this problem as showing a common mistake students make and readily identified that one interpretation of the problem is that the first square is one-fourth which is added to a rectangle equal to three-fourths. In that case, the answer would be four-fourths or one rectangle completely colored in. However, three teachers were unwilling and two more were only tentatively willing to accept that the small square could be one whole and that the resulting shaded rectangle would then be equal to four wholes. We interpreted this as a limitation in the teachers' flexibility with units.

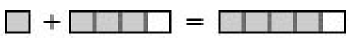

Figure 5. A fraction addition problem.

We saw this same limited ability to understanding multiple interpretations of a single drawing in one fraction division problem where teachers were unable to accept that a large square was equal to anything other than one or five (they were being asked to interpret a division response of 5/3 which was acceptably modeled in the diagram). In all, only two teachers were able to accept that problem as correct, one was tentative and the other nine were unable to conceptualize the diagram as showing anything other than one or five.

We also saw limitations in flexibility in the decimal multiplication problem, which showed only the portion of the area being modeled and not the unit. All twelve participants struggled with this item and only three of them successfully determined that the drawing provided in the item did not include any rectangles with an area equal to one. Interestingly, based on our analyses, we posit that the participants' difficulties on this problem were linked to their inability to make sense of a drawing that did not show the unit. While nine teachers explicitly referred to the representation as showing the area – with one side equal to 1.5 and the other equal to .5 – when asked about the grey shaded area, the participants shifted from this area view of the diagram to interpret the grey area as being equal to five. They explained this was true because the grey area was five tall and one wide.

Figure 6. An area model for decimal multiplication.

Our participants' lack of flexibility in identifying and interpreting units was somewhat surprising and is clearly problematic in terms of their abilities to use drawn representations in their classrooms. When given the opportunity, middle grades students draw myriad novel representations and the teachers need to be able to interpret them on the fly. Otherwise, like the teacher in Izsák's (2008) study, they will redirect students to work the problems the "right" way rather than support them in developing their own mathematical understanding.

Conclusion

This study considered how twelve middle grades teachers made sense of drawn representations, specifically number lines and area models, of fraction and decimal operations. Our interest in this was driven by the calls for teachers to use such representations in their classrooms as tools for promoting student communication about mathematical ideas and connections between mathematical ideas. In our own work, we have seen teachers use representations in a variety of ways, but had little understanding of what the teachers think the representations show or how they work. This study allowed us to explore this issue.

While the findings presented here indicate there are a number of deficits in these teachers' understandings of fraction and decimal representations, the problem cannot be seen as belonging exclusively to the teachers. After all, many of them have never had opportunities to explore mathematics with drawn representations. Above all, this study raises a number of issues to be addressed in professional development. For example, professional development for supporting teachers in learning to use representations clearly needs to focus on interpreting and generating a variety of models rather than simply focusing on how to draw one "right" model of each problem. In order to be empowered to interpret representations, teachers need to be given the opportunity to connect their own mathematical knowledge to the representation and, perhaps, even build new understandings about area and length as it relates to drawn representations.

If we work from the premise that using representations is good for students in developing mathematical connections (Lesh, 1987; NCTM, 2000), then most certainly representations are equally important for teachers in their professional learning. Professional development needs to support teachers in being able to take what they know (e.g., how to determine area) and apply it in novel situations to make sense of them.

Endnotes

(1) All questions for this assessment are secure, therefore all illustrations provided in the paper are examples meant to give a sense of the mathematics and representation rather than the actual question from the assessment.
(2) We borrowed the terms "translate" and "transform" from the work of Lesh, Post, and Behr, 1987.
(3) Fractions $\frac{1}{a}$ and $\frac{1}{b}$ where $b-a=1$ and a, b are real numbers would yield: $\frac{1}{a}-\frac{1}{b}=\frac{b-a}{ab}=\frac{1}{ab}$.

References

Ball, D. (2003). What mathematical knowledge is needed for teaching mathematics? Secretary's Summit on Mathematics, U.S. Department of Education, February 6, 2003; Washington, D.C. Retrieved October 13, 2007 from http://www.ed.gov/inits/mathscience

Ball, D., Lubienski, S., & Mewborn, D. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.) *Handbook of research on teaching* (4th ed., pp. 433–456). New York: Macmillan.

Behr, M. J., Khoury, H. A., Harel, G., Post, T., & Lesh, R. (1997). Conceptual units analysis of preservice elementary school teachers' strategies on a rational number as operator task. *Journal for Research in Mathematics Education, 28*(1), 48-69.

Coffey, A., & Atkinson, P. (1996). *Making sense of qualitative data*. Thousand Oaks, CA: Sage Publications, Inc.

Denzin, N. K. (1989). *The research act: A theoretical introduction to sociological methods.* Englewood Cliffs, NJ: Prentice Hall.

Dreyfus, T. & Eisenberg, T. (1996). On different facets of mathematical thinking. In R.J. Sternberg & T. Ben-Zeev (Eds.), *The nature of mathematical thinking*, (pp. 253–284). Mahwah, NJ: Erlbaum.

Gagatsis, A. & Shiakalli, M. (2004). Ability to translate from one representation of the concept of function to another and mathematical problem solving. *Educational Psychology, 24*(5), 645–657.

Goldin, G. (2002). Representation in mathematical learning and problem solving. In L. English (Ed.), *Handbook of international research in mathematics education* (pp. 197–218). Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.

Hall, R. (2000). Videorecording as theory. In Kelly, A. E., & Lesh, R. A. (Eds.) *Handbook of research design in mathematics and science education* (pp. 647–664). Mahwah, NJ: Erlbaum.

Hill, H. C., Rowan, B., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *Elementary School Journal, 105*(1), 11.

Izsák, A. (2008). Mathematical knowledge for teaching fraction multiplication. C*ognition and Instruction, 26*, 95-143.

Izsák, A., Tillema, E., & Tunç-Pekkan, Z. (2008). Teaching and learning fraction addition on number lines. *Journal for Research in Mathematics Education, 39*(1), 33-62*.*

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N. & Phillips, E. D. (2002). *Connected mathematics program*. Glenview, IL: Prentice Hall.

Lesh, R. (1987). The evolution of problem representations in the presence of powerful conceptual amplifiers. In C. Janvier (Ed.), *Problems of representation in teaching and learning mathematics* (pp. 197-206). Hillsdale, NJ: Erlbaum.

Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.) *Problems on representation in the teaching and learning of mathematics* (pp. 33–40). London: Lawrence Erlbaum Associates, Publishers.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Kilpatrick, Swafford, & Findell (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Rider, R. (2004). The effect of multi-representational methods on students' knowledge of function concepts in developmental college mathematics. Doctoral Dissertation, North Carolina State University.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

SII/LMT (Study of Instructional Improvement/Learning Mathematics for Teaching) (2004). *Learning mathematics for teaching*. Ann Arbor, MI: Consortium for Policy Research in Education.

Sowder, J., Philipp, R., Armstrong, B., & Schappelle, B. (1998). *Middle-grade teachers' mathematical knowledge and its relationship to instruction: A research monograph*. New York: State University of New York Press.

Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge. *Journal of Mathematical Behavior*, *102*, 1–41.

Strauss, A. & Corbin, J. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (2nd ed.). Thousand Oaks, CA: Sage.

Acknowledgments

The work reported here is supported by the National Science Foundation under grant number REC-0633975. The results reported here are the opinions of the authors and may not reflection those of NSF. The authors wish to thank the Does it Work research team for their support and particularly Andrew Izsák and Daniel Brink for their help in conducting the interviews analyzed for this report.