Reconceptualizing Measurement

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Abstract: This paper proposes a new conceptual model of measurement, that includes qualitative ways of thinking about characteristics. Previous mathematics and science education research and policy documents have treated measurement as solely quantitative: absolute (a number of standard units) or relative (measurement with non-standard units or defining the size of one object in terms of another). Theoretical arguments from information science, statistics, and developmental psychology are employed to support the inclusion of qualitative ways of thinking in measurement. The model includes two qualitative ways of thinking: ordering and grouping, for a total of four ways of thinking. Given that assessment tasks can be posed for each way of thinking, the model implies that there can be 16 types of strategy. Strategies are illustrated with examples from student interviews. The model is also shown to account well for strategies previously reported. Implications for educational practice and research are suggested.

Introduction

Measurement is recognized by US standards documents as a "content strand" (National Council of Teachers of Mathematics [NCTM], 2000) or "domain" (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGACBP & CCSSO], 2010). Yet American students fare worse on measurement than on other mathematical topics in international comparisons (National Center for Education Statistics [NCES], 1996). Thus, improved measurement education is needed in the US.

Measurement has traditionally been thought of as quantitative: assigning numbers to characteristics (Joram, Subrahmanyam, & Gelman, 1998; NCTM, 2000. Yet we can think about characteristics of objects in ways that do not involve numbers; for instance, determining which of two aligned objects extends further or realizing that an object will not fit in a given container.

Our new conceptual model of measurement contemplates two quantitative and two qualitative ways of thinking about characteristics. *Absolute* measurements involve a number of standard units, e.g., the average height of US adult males is 1.75 meters. *Relative* measurements describe one object in terms of another, e.g, a brick is 11 times heavier than an 8-oz can of juice. The qualitative ways of thinking are grouping and ordering. *Grouping*, or classifying, results in objects that are alike in a certain characteristic being placed in the same group, and objects that are not alike in different groups (Inhelder & Piaget, 1969). *Ordering*, or seriation, is "the product of a set of asymmetrical transitive relations connected in series" (Inhelder & Piaget, 1969, pp. 5–6). By comparing pairs of objects, and using transitivity, a learner can construct an ordered sequence. This conceptual model substantially broadens the scope of measurement in the context of education, and provides a solid foundation on which better instruction, curriculum, and assessment for the teaching and learning of measurement can be built.

Theoretical framework

Theoretical support for including qualitative ways of thinking

Measurement has traditionally been thought of as "the assignment of a numerical value to an attribute of an object, such as the length of a pencil" (NCTM, 2000, p. 44). A comprehensive review of measurement estimation proposed that "skilled estimators move back and forth between written or verbal linear measurements and representations of their corresponding magnitudes on a mental number line" (1998, p. 413), in doing so mapping "the route from number to quantity and back" (Joram et al., 1998). More recently, Jones and colleagues proposed that measurement estimation requires knowledge of units of measurement and understanding of measurement tools (Jones, Gardner, Taylor, Forrester, & Andre, 2012, p. 173). However, there are sound theoretical reasons from several fields to include qualitative ways of thinking about characteristics under the rubric of measurement.

According to *information theory*, measurement is a process for reducing uncertainty about an attribute of a phenomenon, based on observations (Hubbard, 2010). Saying that object X is smaller than a cat but larger than an ant – ordering - clearly provides information about that object. Knowing that an *E. coli* bacterium falls in the group of objects too small to see with the unaided eye – grouping - gives some idea about its length. In *statistics theory*, variables – which refer to characteristics - can be of different levels: nominal, ordinal, interval, and ratio (Stevens, S. S., 1946). Ordering and grouping are the basis of nominal and ordinal scales. Piaget and Inhelder's

(1971) *developmental theory* proposes that children construct the idea of space. The “sub-logical operations” stage allows them to realize that the whole is greater than the part. The next stage involves non-metric “extensive” quantity, allowing for comparison between parts, which enables ordering and grouping. The final stage involves “metric” quantities, which involve relative and absolute size.

Tasks and strategies

From a learning sciences perspective, students’ strategies and approaches to solving problems are of critical interest. Thus, it is important to distinguish tasks (assigned to the student) from strategies (selected and employed by the student). This distinction of task and strategies leads to 16 basic combinations of task and strategy. These 16 combinations will be illustrated with empirical student data below.

Data sources

The data that illustrate students’ ways of thinking are drawn from a corpus of over 100 interviews conducted with students in grades six through undergraduate and two PhD science experts. Participants included 32 middle school and 33 high school students from a diverse, low SES public school district; 31 students come from the middle and high school grades of a private, mid-high SES, non-denominational school; and ten undergraduates and two professors with PhDs in science from a public research university. All three schools are in the Midwestern USA. Students were interviewed by the first author, doctoral students, and PhDs, based on an interview protocol. The interview protocol was developed following a construct-centered design process (Shin, Stevens, S. Y., & Krajcik, 2010). The methodology followed in developing the interview protocol and interviewing students is described in detail elsewhere (Delgado, 2009; Delgado, Stevens, Shin, & Krajcik, 2015). Tasks included ordering, grouping, relative size, and absolute size questions in the context of length. Length is particularly important because it can serve as a gateway to the understanding of weight, area, volume, or other less tangible variables (Joram et al., 1998). The ordering and grouping tasks involved ten objects: atom, molecule, virus, mitochondrion, red blood cell, pinhead, ant, human, mountain, and Earth. Students were asked to think aloud when carrying out the tasks, and then asked follow-up questions on their process. They were next asked to estimate the size of an atom, a red blood cell, a human, and the Earth relative to the pinhead, then to estimate the absolute size of the four objects after being told that the diameter of the pinhead was around 1 mm or 1/16 of an inch, while thinking out loud.

Results

Strategies for the ordering task

Ordering task solved through ordering alone: One seventh grade student (1006) correctly ordered atom < molecule < mitochondrion < cell, saying that atoms make up molecules, a mitochondrion is made up of molecules, and mitochondria make up cells. She used part-whole relationships to establish pairwise comparisons that she subsequently connected through transitivity.

Ordering through grouping. A ninth grader (0052) said he knew that a pinhead is bigger than a cell because you can see the pinhead (but not the cell). He used the groups of visible – non-visible objects to accurately make one pairwise comparison.

Ordering through relative size. One 10th grade student (0080) used estimates of how many atoms lined up would make up the size of a red blood cell, a pinhead, and a human to order these objects: “Like 10 atoms make up a cell, and like 100 a pinhead, and probably like 1000, or probably way more than that [the human]…”

Ordering through absolute size. Student 3002, an undergraduate, characterized the objects in terms of the unit most convenient to express their sizes: she associated the human with meters, mountains with kilometers, and the Earth with thousands of kilometers.

Strategies for the grouping task

Grouping through ordering. One 11th grade student (0089) made groups by first ordering, then pairing the largest and smallest object (atom, Earth), the second largest and second smallest (mountain, molecule), etc. He explained that he was grouping “for balance”. More conventional groups could also be made by first grouping, then dividing into two or more groups.

Grouping through grouping alone. Pre-existing classifications were employed by many in order to group by size, for example visualization tool ranges. An undergraduate (Z46) stated: “The easiest division would be naked eye observation, so you’d have these five [atom, molecule, virus, mitochondrion, red blood cell] and these five [pinhead, ant, human, mountain, Earth].”

Grouping through relative size. The PhD made groups by considering the powers of ten that separated adjacent objects in the ordered sequence. He grouped together objects that differed by only one power of ten (pin and ant), and separated objects that different by more than one power of ten.

Grouping through absolute size. An undergraduate (3002) grouped by the unit that could be used most conveniently to express the size of each object, ending up with a nanometer (billionths of meter) group (atom, molecule); a micrometer (millionths of meter) group (virus, mitochondrion, cell); a millimeter group (pinhead, ant); a meter group (human); a kilometer group (mountain); and a thousands of kilometers group (Earth).

Strategies for the relative size estimation task

Relative size through ordering. Some students seemed to use ordering in estimating relative size. One seventh grader (#1004) had little confidence in her estimate for the atom relative to the pinhead, but explained that she used her response for the relative size for cell to generate a larger number of times smaller.

Relative size through grouping. The reasoning of a ninth grader (0046) shows her use of grouping – the set of objects too small to see – to get some idea of relative size: "[Cells] are really small. You can't really see them unless you've got like a microscope…A million [times smaller than head of a pin] because you can't see it."

Relative size through relative size alone. One seventh grade student (1006) used iteration to estimate successively larger parts of the body:

> I was comparing the size of the pin to my thumb, and that'd be about like 100 pins, then to make up my whole thumb instead of just the edge would be maybe like 300, and then for my palm it'd be like 1000, then to make my whole body it'd be 1 million.

Relative size through absolute size. We observed several students and the professors using the known height of the human and the diameter of the pinhead (given) to calculate the relative size of the human.

Strategies for the absolute size estimation task

Absolute size estimation through ordering. We did not find instances of students using ordering to aid their absolute size estimation, without also quantifying. However, one could estimate the thickness of DNA if one knew that it is larger than an atom (0.1 nm) but smaller than the diameter of the hemoglobin protein (5.5 nm).

Absolute size estimation through grouping. One seventh grader (8018) used grouping by instrument required to generate absolute size estimates for atom and cell: "Ok, let's see, an atom, a pinhead is 1 mm and an atom is *microscopic* [calculations]…it's like a millionth, more than a millionth of a mm."

Absolute size estimation through relative size. One undergraduate (Z46) explicitly used his estimates of relative size for atom compared to a pinhead (one million times smaller) to estimate the absolute size of the atom: "The nm is 1 billionth of a m… Then I guess, according to this, the atom would have to be 1 nm, if I were to base this estimate of [absolute] size on the relative."

Absolute size estimation through absolute size alone. Students frequently recalled the height of a human, thus employing absolute way of thinking for an absolute measurement task, and one professor recalled the accurate value of approximately 4000 mi for the radius of the Earth.

Discussion and conclusion

This paper presented a conceptual model for measurement learning that includes both quantitative and qualitative ways of thinking about measurement, justified with theory from three different fields. It thus broadens the field of measurement learning beyond quantitative conceptions of measurement. Fifteen of the 16 types of strategies suggested by our model were identified in interviews, and we made a hypothetical strategy for the remaining case.

In the mathematics education literature, measuring with non-standard units (relative) is often seen as a developmental step towards formal measurement with standard units (abstract)(NCTM, 2000; NGACBP & CCSSO, 2010; National Research Council [NRC], 2012). However, we posit that the four ways of thinking are useful at all stages. Joram and colleagues propose that learners possess a (quantitative) mental measurement line with *ordered* objects anchored by obejcts of known absolute size (1998). We also know that humans make sense of their world through classification (Kosslyn, 1980).

The literature on measurement and measurement estimation identified nine distinct strategies for the estimation of absolute size. Six of these depend solely on absolute strategies: recalling an absolute size (Paritosh & Klenk, 2006), measuring with an actual instrument (Clements & Stephan, 2004; Lehrer, 2003; Szilagyi, Clements, & Sarama, 2013; Wiedtke, 1990), measuring with a mental instrument (Joram et al., 1998; Szilagyi et al., 2013), unit iteration using standard units (Castillo, 2006; Joram et al., 1998; Szilagyi et al., 2013), squeezing (Castillo, 2006; Joram et al., 1998), and using a landmark of similar size (Castillo, 2006; Joram et al., 1998; Lang, 2001; Paritosh & Klenk, 2006). The remaining three depend on relative size: measurement with non-standard units (Castillo, 2006; Joram et al., 1998); using a part of known size and the number of parts (Joram et al., 1998);

and decomposing a whole and estimating a part (Castillo, 2006; Joram et al., 1998; Lang, 2001; Paritosh & Klenk, 2006). The model presented here provides a framework with which to classify and organize the strategies previously identified, and additionally suggests several other strategies.

By explicitly describing ways of thinking as well as strategies, this paper paves the way for the development of new curriculum, instruction, and assessment of measurement that provides additional strategies and can support conceptual understanding through the purposeful linking across the ways of thinking.

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