Predictions of Integer Model Affordances Pan Out for Short-Term, but Not Longer-Term, Learning of Initial Integer Constructs

Julie Nurnberger-Haag, Kent State University, jnurnber@kent.edu

Abstract: Arithmetic with negative numbers is difficult to learn, so world-wide many instructional methods have been developed for this topic. Teachers often use multiple integer instructional models within the same unit, yet little is understood about the impact of each. Thus, this study compared student learning in authentic classrooms randomly assigned to one of two models, each of which was predicted to better support learning of particular constructs. Preliminary results with 70 fifth-grade students suggested that in the short term each model fostered better learning of the predicted construct, yet findings suggest longer-term knowledge was better supported by students who used the particular number line model. These findings were interpreted using embodied cognition to contribute to the growing recognition that how students physically move during instruction influences learning.

Keywords: mathematics, negative numbers, embodied cognition, off-line cognition

Introduction

It is commonly known across the globe that students have difficulty calculating with negative numbers. Three initial constructs, which in the United States curricular standards are scheduled for sixth grade, are sums of additive inverses ($-a + a=0$, where $a>0$), ordering integers and opposite operation (National Governors Association, 2010). The crucial opposite operation construct requires some additional explanation. Knowing -3 and 3 are located at the position the same distance from zero but on the opposite side on a number line is a structural meaning of the idea of opposite (Vlassis, 2008). Moreover, students must also conceive of a negative sign symbol as an operation sign, specifically meaning "take the opposite of" (Ryan & Williams, 2007; Vlassis, 2008). Most subsequent mathematics courses, for example, require fluent use of this meaning (e.g., –Y as "the opposite of Y" or $-(-4x - 5)$ as "the opposite of negative four X minus five").

How varied models support or interfere with learning aspects of integer operations have only recently begun (Nurnberger-Haag, 2018; Tsang, Blair, Bofferding, & Schwartz, 2015). Integer models are commonly classified into a) cancellation and b) number line (Küchemann, 1981). Cancellation models are defined by the conceptual idea of two quantities that cancel each other regardless of tool, whereas a number line model is defined by the representational tool used (Küchemann, 1981; Nurnberger-Haag, 2018). The most common cancellation model is a chip model in which two colors are used to represent positive or negative quantities (van de Walle et al., 2010). I define instructional models as the patterns of words and accompanying physical motions in relation to representational tools such as number lines or physical chips. This research uses the theoretical perspectives of embodied cognition, namely that how people move influences how they think (e.g., Antle, 2011; Barsalou, 2008; Lakoff & Nunez, 2000; Wilson, 2002). Consequently, this study investigated the prediction that due to affordances of how students move and think with different integer instructional models, these models would differentially influence learning of integer constructs.

A common way to use black (positive) and white (negative) chips for example to act out -7 + 3 would begin with seven white chips and put in three black chips. The students would then match up each of three white chips with the 3 black chips to cancel that quantity leaving -4. The design of this chip model is based on the sum of additive inverses being zero, such that students using this model experience constant repetition and application of this idea. Thus, the chip model should better help students learn the additive inverse construct. In contrast, this model may not support the construct of ordering numbers, because the magnitude of visible chips is larger even when the numerical value is smaller than another (e.g., -14 is smaller than +7 but 14 white chips visually looks like a larger quantity than 7 black chips).

The number line tool orders integers from least to greatest, so any number line model might be predicted to better support ordering numbers as a construct. Neither typical number line models nor chip models were designed to convey the construct of an opposite operation (Nurnberger-Haag, 2007). A particular number line model, referred to here as the walk model, was designed in order to fit this operational meaning as well as the primary operations of addition, subtraction, multiplication, and division (Nurnberger-Haag, 2007). With this model, students physically "turn the opposite direction" for subtraction and negative signs that require an operational meaning rather than the structural/positional meaning (Nurnberger-Haag, 2007; Vlassis, 2008). Students are also encouraged to use this specific language to cooccur with physical movements "turn the opposite direction" rather than simply "go in the other direction" or "turn around." For example, for -7 - -3, students stand next to the point -7 on a large number line, turn the opposite direction twice (once for the subtraction sign and once for the negative sign of -3) and walk three, resulting in an answer of -4.

It is commonly understood that typical classroom instruction of integer arithmetic in the United States lasts several weeks and within any classroom, teachers often use multiple and varied models, methods, games, or contexts (although studies to document these practices are needed). If educators better understand which models support certain learning objectives and why, then instruction for these difficult topics might be better advised. Thus, this report asked: When a single model is used to learn integer arithmetic, does the model foster better student learning of the predicted integer construct?

Method

In a larger pre-post-delayed post study, learning of integer arithmetic with a chip or number line model was compared by finding schools that allowed an eight-lesson mini-unit on integer arithmetic to be taught to their students the year before the school began this formal instruction. This resulted in two school districts participating, both of which had 45% of students in these grades receiving free and reduced lunch (which in the United States is used as a proxy for socioeconomic status). All fifth-grade classes of one district during second semester and all sixth-grade classes during the first semester of a second district (Nurnberger-Haag, 2018) participated. All eight classes across sites were randomly assigned to learn integer arithmetic with either a chip or number line model such that half of the classes learned with each model (Nurnberger-Haag, 2018). By the US Standards, fifth grade is the year before integers are typically introduced (National Governors Association, 2010), so this preliminary analysis focuses on those fifth-grade students who completed all three assessment timepoints (N=70). As is typical of classroom instruction, posttests were administered the day after instruction ended. Delayed posttests were administered five weeks later. The classroom teacher taught geometry content during this delay time-period and avoided reviewing integer operations.

All classes were authentic, intact classes in which students attended their regularly scheduled math course with the teacher present, but the teacher did not assist with instruction to allow the researcher to maintain parallel instruction across classes. The researcher instructed all classes in eight parallel lessons differing only due to model. The two chip classes used black and white chips to represent positive and negative numbers respectively. The one walk class emphasized the "-" symbol as meaning "turn the opposite direction" whereas traditional number line models treat subtraction as backward motion (Nurnberger-Haag, 2007). During instruction students used the assigned model to learn to add (including, but not limited to sums of additive inverses), subtract, multiply and divide negative numbers and represent opposite numbers using the structural meaning (not operations).

All problems students experienced during instruction differed from test items, so that test items for the constructs of ordering numbers and sums of additive inverses were near transfer items. The ordering numbers construct was assessed with five questions. Three of the items asked students to circle either the greatest or least number of three choices, and two other items were open-ended asking students to provide a number less than the stated negative number. Strong ordering knowledge was defined as scoring at least four out of five on these items. The sums of additive inverses construct question asked students whether they agreed with one student who said an expression such as -8 + (-7 + 7) does not give the same answer as -8 + (-5 + 5) or whether they agreed with another student who said these were the same answer. Students were asked to explain their reasoning. For this report strong sums of additive inverse knowledge was defined as those students who either stated something about the sum of two opposites being zero as part of their rationale that these expressions were equivalent, or students who calculated to determine the equivalency.

The opposite operations construct was assessed as a far transfer construct. In other words, instruction avoided opposite operation problems to eliminate the possibility that students who memorized notational patterns would be indistinguishable from those who understood the meaning of the negative sign as an operator. Multiple phases of piloting and revision with more than 300 students were conducted to create an item to assess whether students who were not yet versed in algebraic notation could think of a negative sign as the opposite of a quantity, that is –(X) where X is a negative number, yet without using a variable in the test item. Due to space in this proceeding the following is a shortened description of the item used: "(negative number)" was presented in blue font and "–(negative number)" in green and students were asked to describe how the blue and green quantities were different. At pretest, none of these fifth-grade students correctly answered this item. Strong opposite operation knowledge at post and delayed posttest was defined as stating a non-negative solution.

Due to the distributions of scores, sample size of this analysis, and the fact that teachers make instructional decisions based on percent accuracy rather than statistical analyses, this preliminary analysis that would be publicly available for researchers as well as teachers reports descriptive statistics to discuss trends.

Statistical analyses on the full data set (approximately N=170) will be ready to share in the presentation along with qualitative examples of student reasoning to illustrate embodied expressions in their reasoning.

Results

To assess learning of each construct, students who were already at ceiling at pretest were removed from analysis. Table 1 displays the percent of students who learned with each model that demonstrated strong knowledge of sums of additive inverses, ordering numbers, and opposite operations at post and delayed posttest. The model that was predicted to foster better learning of each construct had the higher percent of students with strong knowledge at posttest. In contrast, the trends differed at delayed posttest five weeks later. More students who learned with the chip model tended to regress across all constructs from post to delayed test, whereas some students who learned with the walk-it-off model tended to continue to gain knowledge during this same delay.

Table 1: Percent of students with strong knowledge of each construct at post and delayed posttest timepoints

	Near Transfer				Far Transfer	
	Sums of Additive Inverses		Ordering		Opposite Operation	
	Post	Delay	Post	Delay	Post	Delay
Chip	40%	32%	35.3%	29.4%	2%	--
Walk	20%	33.3%	50%	58.3%	15%	27%

Note. Sums of Additive Inverses (*n*=25 chips and *n*=15 walk), Ordering numbers (*n*=17 chips and *n*=12 walk) and Opposite Operation (chips *n*=44, walk *n*=26)

The chip model was predicted to foster strong sums of additive inverse knowledge. Although twice as many students who learned with chips than walk had strong knowledge of sums of additive inverses at posttest (see Table 1), this advantage disappeared by the delayed posttest. Decay of learning is typically expected with a delayed test; however, only the chip model group reflected decay, whereas during the same five-week delay students who learned with the walk model gained understanding of the sums of additive inverses.

The walk model was expected to foster better ordering knowledge, because it is an ordered number line model. At pretest more than half of the students in each model condition were at ceiling with strong ordering knowledge. Once those students were removed to assess learning of ordering knowledge, as predicted, the walk model fostered more students to have strong ordering knowledge at posttest. Again, a greater proportion of walk students demonstrated stronger knowledge at delayed test than at posttest, whereas the proportion of chip students revealed some decay of ordering knowledge by delayed test.

The walk model was specifically designed to foster opposite operation knowledge (Nurnberger-Haag, 2007). This construct was assessed with a far transfer task for reasons described in the methods. In spite of there being 1.7 times as many students who learned with the chip model (n=44) than with this number line (n=26), at posttest just one student (*n*=2%) who used chips could conceive of an expression with a negative symbol as having a non-negative solution but then failed to do so after the five-week delay. In contrast, 15% of the students who learned with the walk-it-off model were able to provide a non-negative solution to the post-test task, indicating far transfer of using the negative sign as an operation. Moreover, note in Table 1 the trend that students who learned with the walk-it-off model gained knowledge over the delay period such that almost twice as many walk students demonstrated opposite operation knowledge at delayed posttest than at posttest.

In summary, regardless of whether students learned with a chip or walk model, five weeks after instruction a similar proportion of students demonstrated strong sums of additive inverses knowledge. Twice the percent of walk students compared to chips students had gained strong ordering knowledge at the delayed timepoint. Moreover, more than one-fourth of the walk students conceived of the negative sign as symbolizing a quantity that was not negative, whereas the chip model failed to foster any student to demonstrate this understanding.

Conclusions

If conclusions had been drawn solely from post-test analyses, then it would be reasonable to conclude that each model differentially benefitted student learning in ways that confirm what researchers and teachers would expect (i.e., to teach sums of additive inverses use a Chip model; to teach Ordering and Opposite Operation the Walk model is better). The longer-term knowledge of students, however, is the most important indicator of instructional practice. For longer-term learning under the conditions of withholding review of integer arithmetic, the walk-it-off number line model seems to have been more beneficial across both near and far transfer constructs. Such a claim is supported by the trend that the strength of the chip model group's knowledge tended to decay over time across all constructs, whereas the walk group continued to gain strength over the same time period.

One perspective from embodied cognition to aide interpretation of the surprising trends is that even off-line cognition is embodied (Wilson, 2002). In particular, "tasks with consistent mapping between stimulus and response can be automatized, but tasks with varied mapping cannot (Schneider & Shiffren, 1977)" (Wilson, 2002, p. 634). The walk model consistently maps written symbols of an expression to embodied actions (Nurnberger-Haag, 2007; Nurnberger-Haag, 2018). For the problems -6 - -3 and -3 - -6 for instance, students treat the first number as a position on the number line, turn the opposite direction for the subtraction sign and also for the negative sign of the second number, then walk the distance designated by the magnitude of the second number. The process is the same for both problems. In contrast, the chip model requires students to move the chips in ways that vary for each problem. For example, -6 - -3 and -3 - -6 are both subtraction problems so the ultimate step to solve it with chips involves literally taking chips away to model subtraction. The first problem only requires that step (start with 6 white chips and literally take 3 white chips away). In contrast, -3 - -6 requires several extraneous physical motions that contradict the operation of the problem, because there are not enough white chips to remove six, so students must add or physically put in sufficient equal quantities of white and black chips to maintain the value of the expression in order to physically take away six white chips (see Nurnberger-Haag, 2018 for a detailed analysis of integer subtraction in terms of embodied cognition). Additional analyses of existing data should be conducted to better understand the embodied reasons for these learning outcomes. Moreover new studies should be designed to replicate these trends and uncover how students' motions with the walk model might better support off-line integer learning, whereas moving physical chips might yield unstable knowledge that is more susceptible to decay. Furthermore, new studies that assess additional timepoints would be valuable to replicate these trends and test knowledge retention, gains, or decay across longer time periods, particularly after a summer vacation. Such research is needed to influence mathematics education research and practice to recognize that mathematics learning is embodied, and to suggest ways to teach that are more consistent with student cognition.

References

Antle, A. N. (2011). Balancing justice: Comparing whole body and controller-based interaction for an abstract domain. *International Journal of Arts and Technology, 6*(4), 388-409.

Barsalou, L. W. (2008). Grounded cognition. *Annual Review of Psychology, 59*, 617-645.

Küchemann, D. (1981). Positive and negative numbers. In K. M. Hart (Ed.), *Children's understanding of mathematics: 11–16* (pp. 82–87). London, UK: John Murray.

Lakoff, G., & Núñez, R.E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards mathematics.* Washington, DC: Authors.

Nurnberger-Haag, J. (2018). Take it away or walk the other way? Finding positive solutions for integer subtraction. L. Bofferding and N. Wessman-Enzinger, Eds., *Exploring the integer addition and subtraction landscape: Perspectives on integer thinking.* Cham, Switzerland: Springer.

Nurnberger-Haag, J. (2007). Integers made easy: Just walk it off! *Mathematics Teaching in the Middle School, 13*(2), 118-121.

Ryan, J., & Williams, J. (2007). *Children's mathematics 4-15: Learning from errors and misconceptions.* Berkshire, England: Open University Press.

Tsang, J. M., Blair, K. P., Bofferding, L., & Schwartz, D. L. (2015). Learning to "see" less than nothing: Putting perceptual skills to work for learning numerical structure. *Cognition and Instruction, 33*, 154-197.

Van de Walle, J., Karp, K., & Bay-Williams, J. (2010). *Elementary and middle school mathematics: Teaching developmentally* (7th ed.). New York: Allyn and Bacon.

Vlassis, J. (2008). The role of mathematical symbols in the development of number conceptualization: The case of the minus sign. *Philosophical Psychology, 21*(4), 555-570.

Wilson, M. (2002). Six views of embodied cognition. *Psychonomic Bulletin & Review*, *9*(4), 625-636.

Acknowledgments

This research was supported by a National Academy of Education/Spencer Foundation Dissertation Fellowship and Michigan State University Dissertation Research Expense. The data reflects an expansion from a poster focused on a single construct presented at the 2018 International Conference of the Learning Sciences.