Interactional Achievement of Shared Mathematical Understanding in a Virtual Math Team

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Abstract: Learning mathematics involves specific forms of social practice. In this paper, we describe socially situated, interactional processes involved with collaborative learning of mathematics in a special online collaborative learning environment. Our analysis highlights the methodic ways group members enact the affordances of their situation (a) to visually explore a mathematical pattern, (b) to co-construct shared mathematical artifacts, (c) to make visible the meaning of the construction, (d) to translate between graphical, narrative and symbolic representations and (e) to coordinate their actions across multiple interaction spaces, while they are working on open-ended math problems. In particular, we identify key roles of referential and representational practices in the co-construction of deep mathematical group understanding. The case study illustrates how mathematical understanding is built and shared through the online interaction.

Introduction

Developing pedagogies and instructional tools to support learning math with understanding is a major goal in mathematics education (NCTM, 2000). A common theme among various characterizations of mathematical understanding in the math education literature involves *constructing relationships* among mathematical facts and procedures (Hiebert & Wearne, 1996). In particular, math education practitioners treat recognition of connections among *multiple realizations* of a math concept encapsulated in various inscriptional forms as evidence of deep understanding of that subject matter (Kaput, 1998; Sfard, 2008; Healy & Hoyles, 1999). For instance, the concept of function in the modern math curriculum is introduced through its graphical, narrative, tabular, and symbolic realizations. Hence, a deep understanding of the function concept is ascribed to a learner to the extent he/she can demonstrate how seemingly different graphical, narrative, and symbolic forms are interrelated as realizations of each other in specific problem-solving circumstances that require the use of functions. On the other hand, students who demonstrate difficulties in realizing such connections are considered to perceive actions associated with distinct forms as isolated sets of skills, and hence are said to have a shallow understanding of the subject matter (Carpenter & Lehrer, 1999).

Multimodal interaction spaces—which typically bring together two or more synchronous online communication technologies such as text-chat and a shared graphical workspace—have been widely employed in CSCL research and in commercial collaboration suites such as Elluminate and Wimba to support collaborative learning activities of small groups online (Dillenbourg & Traum, 2006; Soller, 2004; Suthers et al., 2001). The way such systems are designed as a juxtaposition of several technologically independent online communication tools not only brings various *affordances* (i.e. possibilities-for and/or constraints-on actions), but also carries important interactional consequences for the users (Cakir, Zemel & Stahl, 2009; Suthers, 2006; Dohn 2009). Providing access to a rich set of modalities for action allows users to demonstrate their reasoning in multiple semiotic forms. Nevertheless, the achievement of connections that foster the kind of mathematical understanding desired by math educators is conditioned upon team members' success in devising shared methods for coordinated use of these rich resources.

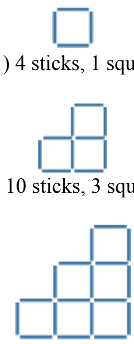
Although CSCL environments with multimodal interaction spaces offer rich possibilities for the creation, manipulation, and sharing of mathematical artifacts online, the interactional organization of mathematical meaning-making activities in such online environments is a relatively unexplored area in CSCL and in math education. In an effort to address this gap, we have designed an online environment with multiple interaction spaces called Virtual Math Teams (VMT), which allows users to exchange textual as well as graphical contributions online (Stahl, 2009). The VMT environment also provides additional resources, such as explicit referencing and special awareness markers, to help users coordinate their actions across multiple spaces. Of special interest to researchers, this environment includes a Replayer tool to replay a chat session as it unfolded in real time and inspect how students organize their joint activity to achieve the kinds of connections indicative of deep understanding of math.

In this paper we focus on the practical methods through which VMT participants achieve the kinds of connections across multiple semiotic modalities that are often taken as indicative of deep mathematical understanding. We take the math education practitioners' account of what constitutes deep learning of math as a starting point, but instead of treating understanding as a mental state of the individual learner that is typically inferred by outcome measures, we argue that deep mathematical understanding can be located in the practices of collective multimodal reasoning displayed by teams of students through the sequential and spatial organization of their actions. In an effort to study the practices of multimodal reasoning online, we employ an ethnomethodological case-study approach and investigate the methods through which small groups of students coordinate their actions across multiple interaction spaces of the VMT environment as they collectively construct, relate and reason with multiple forms of mathematical artifacts to solve an open-ended math problem. Our analysis has identified key roles of referential and representational practices in the co-construction of deep mathematical understanding.

Data & Methodology

The excerpts we analyze in this paper are obtained from a problem-solving session of a team of three upper-middle-school students who participated in the VMT Spring Fest 2006. This event brought together several teams from the US, Singapore, and Scotland to collaborate on an open-ended math task on combinatorial patterns. Students were recruited anonymously through their teachers. Members of the teams generally did not know each other before the first session. Neither they nor we knew anything about each other (e.g., age or gender) except chat handle and information that may have been communicated during the sessions. Each group participated in four sessions during a two-week period, and each session lasted over an hour. Each session was moderated by a Math Forum staff member; the facilitators' task was to help the teams when they experienced technical difficulties, not to instruct or participate in the problem-solving work. Figure 6 below shows a screenshot of the VMT Chat environment that hosted these online sessions.

During their first session, all the teams were asked to work on a particular pattern of squares made up of sticks (see Figure 1 below). For the remaining three sessions the teams were asked to come up with their own shapes, describe the patterns they observed as mathematical formulae, and share their observations with other teams through a wiki page. This task was chosen because of the possibilities it afforded for many different solution approaches ranging from simple counting procedures to more advanced methods, such as the use of recursive functions and exploring the arithmetic properties of various number sequences. Moreover, the task had both algebraic and geometric aspects, which would potentially allow us to observe how participants put many features of the VMT software system into use. The open-ended nature of the activity stemmed from the need to agree upon a new shape made by sticks. This required groups to engage in a different kind of problem-solving activity as compared to traditional situations where questions are given in advance and there is a single "correct" answer—presumably already known by a teacher. We used a traditional problem to seed the activity and then left it up to each group to decide the kinds of shapes they found interesting and worth exploring further (Moss & Beatty, 2006; Watson & Mason, 2005).



(1) 4 sticks, 1 square

(2) 10 sticks, 3 squares

(3) 18 sticks, 6 squares

N	Sticks	Squares
1	4	1
2	10	3
3	18	6
4	?	?
5	?	?
6	?	?
...	...	...
N	?	?

Session I

1. Draw the pattern for N=4, N=5, and N=6 in the whiteboard. Discuss as a group: How does the graphic pattern grow?
2. Fill in the cells of the table for sticks and squares in rows N=4, N=5, and N=6. Once you agree on these results, post them on the VMT Wiki
3. Can your group see a pattern of growth for the number of sticks and squares? When you are ready, post your ideas about the pattern of growth on the VMT Wiki.

Sessions II and III

1. Discuss the feedback that you received about your previous session.
2. **WHAT IF?** Mathematicians do not just solve other people's problems - they also explore little worlds of patterns that they define and find interesting. Think about other mathematical problems related to the problem with the sticks. For instance, consider other arrangements of squares in addition to the triangle arrangement (diamond, cross, etc.). **What if** instead of squares you use other polygons like triangles, hexagons, etc.? Which polygons work well for building patterns like this? How about 3-D figures, like cubes with edges, sides and cubes? What are the different methods (induction, series, recursion, graphing, tables, etc.) you can use to analyze these different patterns?
3. Go to the VMT Wiki and share the most interesting math problems that your group chose to work on.

Figure 1: Task description for Spring Fest 2006

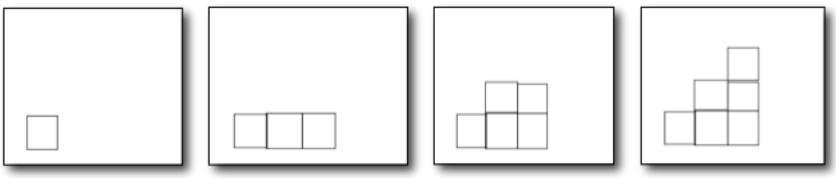
Studying the collective meaning-making practices enacted by the users of CSCL systems requires a close analysis of the process of collaboration itself (Stahl, Koschmann & Suthers, 2006; Koschmann, Stahl & Zemel, 2007). In an effort to investigate the organization of interactions across the dual-interaction spaces of the VMT environment, we consider the small group as the unit of analysis (Stahl, 2006), and we apply the methods of Ethnomethodology (EM) (Garfinkel, 1967; Livingston, 1986) and Conversation Analysis (CA) (Sacks, 1962/1995; ten Have, 1999) to conduct case studies of online group interaction. Our work is informed by studies of interaction mediated by online text-chat with similar methods (Garcia & Jacobs, 1998; O'Neill & Martin, 2003), although the availability of a shared drawing area and explicit support for deictic references in our online environment as well as our focus on mathematical practice significantly differentiate our study from theirs.

The goal of Conversation Analysis is to make explicit and describe the normally tacit commonsense understandings and procedures group members use to organize their conduct in particular interactional settings. Commonsense understandings and procedures are subjected to analytical scrutiny because they “enable actors to recognize and act on their real world circumstances, grasp the intentions and motivations of others, and achieve mutual understandings” (Goodwin & Heritage, 1990, p. 285). Group members’ shared competencies in organizing their conduct not only allow them to produce their own actions, but also to interpret the actions of others (Garfinkel & Sacks, 1970). Since members enact these understandings and/or procedures in their situated actions, researchers can discover them through detailed analysis of members’ sequentially organized conduct (Schegloff & Sacks, 1973).

We subjected our analysis of VMT data to intersubjective agreement by conducting numerous CA data sessions (ten Have, 1999). During the data sessions we used the VMT Replayer tool, which allows us to replay a VMT chat session as it unfolded in real time based on the timestamps of actions recorded in the log file. The order of actions—chat postings, whiteboard actions, awareness messages—we observe with the Replayer as researchers exactly matches the order of actions originally observed by the users. This property of the Replayer allows us to study the sequential unfolding of events during the entire chat session. In short, the VMT environment provides us a perspicuous setting in which the mathematical meaning-making process is *made visible* as a joint practical achievement of participants that is “observably and accountably embedded in collaborative activity” (Koschmann, 2001, p. 19).

Analysis

The following sequence of drawing actions (Figures 2 to 6 below) is observed at the beginning of the very first session of a team in the VMT environment. Shortly after a greeting episode, one student, Davidcyl, begins to draw a set of squares on the shared whiteboard. He begins by drawing three squares that are aligned horizontally with respect to each other, which is made evident through his careful placement of the squares side by side (see Figure 2 below). Then he adds two more squares on top of the initial block of three, which introduces a second layer to the drawing. Finally, he adds a single square on top of the second level, which produces the stair-step shape displayed in the last frame of Figure 2. Note that he builds the pattern row-by-row here.



6:24:30 6:24:37 6:24:48 6:24:50

Figure 2: First stages of Davidcyl's drawing activity.

Next, Davidcyl starts adding a new column to the right of the drawing (see Figure 3). He introduces a new top level by adding a new square first, and then he adds 3 more squares that are aligned vertically with respect to each other and horizontally with respect to existing squares (see second frame in Figure 3). Then he produces a duplicate of this diagram by using the copy/paste feature of the whiteboard (see the last frame in Figure 3). Here, he builds the next iteration by adding a new column to the previous stage, starting the new column by making visible that it will be one square higher than the highest previous column.

Afterwards, Davidcyl moves the pasted drawing to an empty space below the copied diagram. As he did earlier, he adds a new column to the right of the prior stage to produce the next stage. This time he copies the entire 4th column, pastes a copy next to it, and then adds a single square on its top to complete the new stage (Figure 4). Next, Davidcyl produces another shape in a similar way by performing a copy/paste of his last drawing, moving the copy to the empty space below, and adding a new column to its right (see Figure 5). Yet, this time the squares of the new column are added one by one, which may be considered as an act of counting. In Figure 4, the new column is explicitly shown to be a copy of the highest column plus one square. In Figure 5, the number of squares in the new column are counted individually, possibly noting that there are N of them. The likelihood that the counting of the squares in the new column is related to the stage, N, of the pattern is grounded by Davidcyl's immediately subsequent reference to the diagrams as related to "*n=4,5,6*".

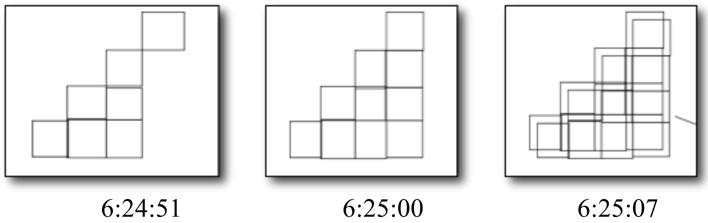


Figure 3: Davidcyl introduces the 4th column and pastes a copy of the whole shape.

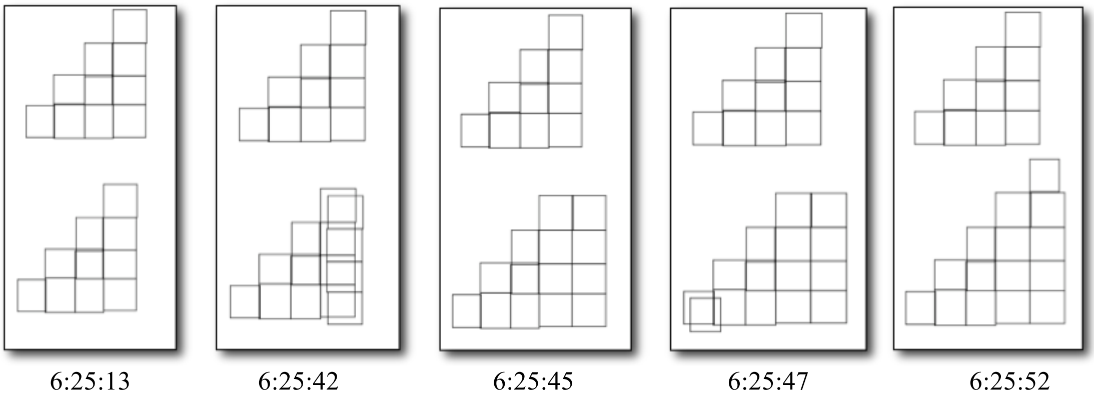


Figure 4: Davidcyl uses copy/paste to produce the next stage of the pattern

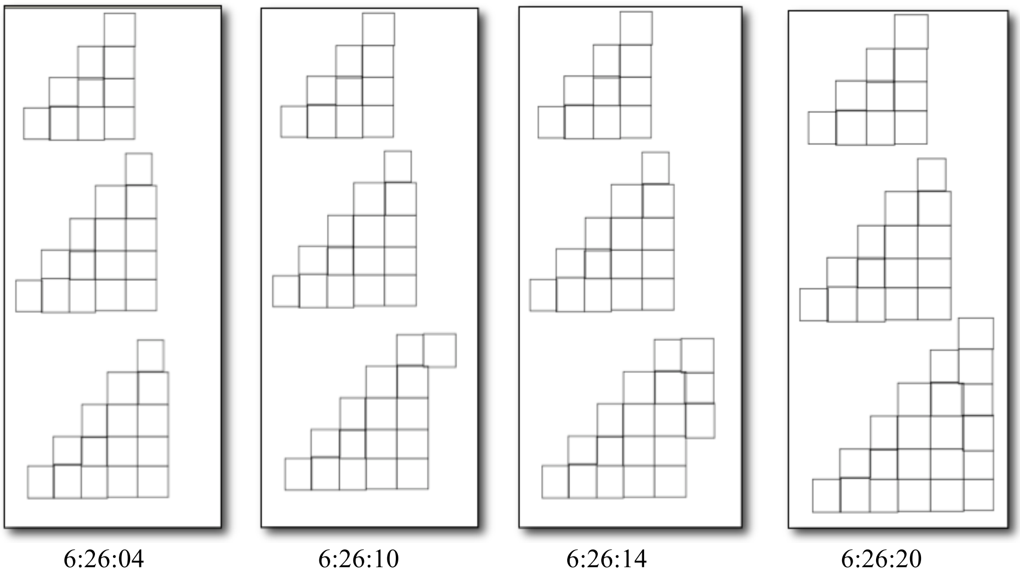


Figure 5: Davidcyl's drawing of the 6th stage

Shortly after his last drawing action at 6:26:20, Davidcyl posts a chat message stating, "*ok I've drawn n=4,5,6*" at 6:26:25. Figure 6 shows the state of the interface at this moment. The "*ok*" at the beginning of the message could be read as some kind of a transition move (Beach, 1995). The next part "I've drawn" makes an explicit verbal reference to his recent (indicated by the use of past perfect tense) drawing actions. Finally, the expression "*n=4,5,6*" provides an *algebraic gloss* for the drawings, which specifies how those drawings should be seen or treated. Once read in relation to the task description, Davidcyl's recent actions across both spaces can be treated as a response to the first bullet under session 1, which states "Draw the pattern for N=4, N=5, and N=6 in the whiteboard" (see Figure 1 above for the task description). The discussion that immediately followed Davidcyl's drawings and his last chat statement is displayed in Table 1 below.

Davidcyl's posting at line 26 is stated as a declarative, so it can be read as a claim or assertion. The references to "*n*" (i.e., not to a particular stage like 2nd or 5th) invoke a variable as a gloss for referring to the features of the general pattern. Moreover, the use of the clause “more…than” suggests a comparison between two things, in particular the two cases indexed by the phrases “*nth pattern*” and *“(n-1)th pattern*” respectively. Hence, Davidcyl’s posting can be read as a claim about how the number of squares changes between the $(n-1)^{th}$ and n^{th} stages of the pattern at hand. The two cases compared in the posting correspond to two consecutive stages of the staircase pattern. Davidcyl’s prior drawing work included similar transitions among pairs of particular stages. For instance, while he was drawing the 4^{th} stage, he added a column of 4 new squares to the right of the 3^{rd} stage. Hence, Davidcyl’s narrative uses the drawings for particular cases as a resource to index the properties of the general pattern, which is implicated in the regularity/organization projected by his prior drawing actions.

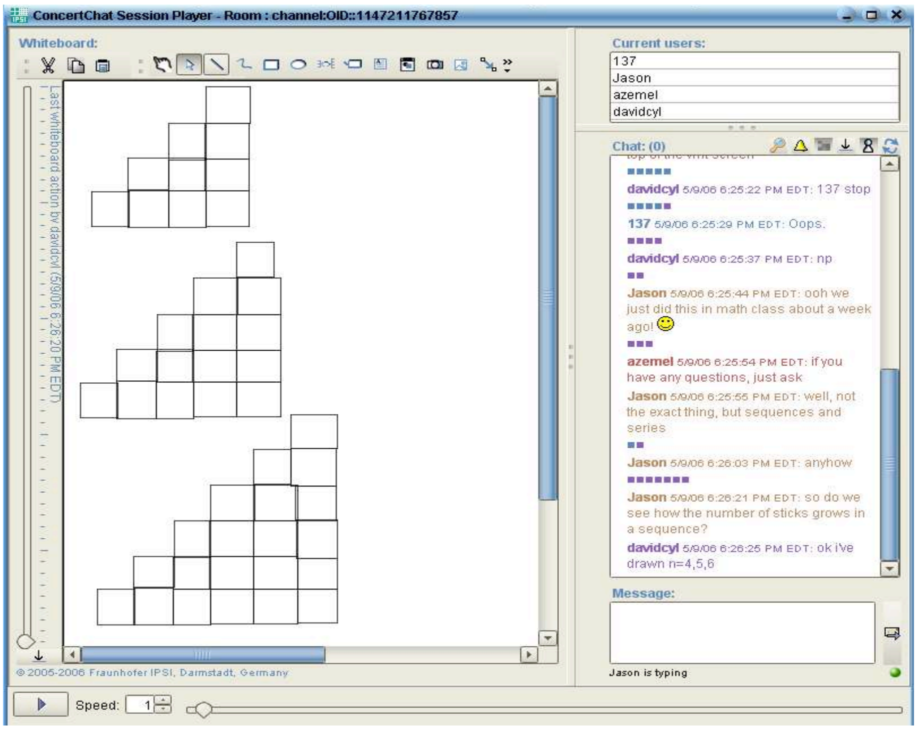


Figure 6: The state of the VMT environment when Davidcyl posted “ok I’ve drawn n=4,5,6” at 6:26:25.

Table 1: Chat discussion following the drawing activity

Chat Index	Time Start Typing	Time of Posting	Author	Content
26	18:27:13	18:27:32	davidcyl	the nth pattern has n more squares than the (n-1)th pattern
	18:27:30	18:27:47	137	*[137 has fully erased the chat message]*
	18:27:47	18:27:52	137	*[137 has fully erased the chat message]*
27	18:27:37	18:27:55	davidcyl	basically it's 1+2+..+(n-1)+n for the number of squares in the nth pattern
	18:27:57	18:27:57	137	*[137 has fully erased the chat message]*
28	18:28:02	18:28:16	137	so n(n+1)/2
29	18:27:56	18:28:24	davidcyl	and we can use the gaussian sum to determine the sum: n(1+n)/2
30	18:28:27	18:28:36	davidcyl	137 got it

In the next line, Davidcyl elaborates on his description by providing a summation of integers that accounts for the number of squares required to form the n^{th} stage. In particular, the expression “*1+2+..+(n-1)+n*” describes a method to count the squares that form the n^{th} stage. Since Davidcyl made his orientation to columns explicit through his prior drawing work while he methodically added a new column to produce a next stage, this expression can be read as a *formulation* of his column-by-column counting work in algebraic form. In other words, Davidcyl achieves a (*narrative*) transition from the *visual* to the *algebraic*, which is informed by his methodic construction of specific stages of the staircase pattern that allowed him to isolate relevant components of the general pattern and derive a systematic counting method.

As Davidcyl composes a next posting, 137 posts a so-prefaced math expression at line 28, "*So n(n+1)/2*" that (a) shows 137 has been attending to the organization of Davidcyl’s ongoing exposition, (b) displays 137's recognition of the next problem-solving step projected by prior remarks, (c) offers an algebraic realization of the procedure described by Davidcyl, and (d) call on others to assess the relevance and validity of his claim. Davidcyl's message at line 29 (which is produced in parallel with line 28 as indicated by the typing times) is a more elaborate statement that identifies how his prior statements, if treated as a Gaussian sum, yield the same expression that 137 put forward at line 28 (viz. "n(n+1)/2"). Given that 137 anticipated Davidcyl's Gaussian sum, Davidcyl announces in the very next posting that "*137 got it,*" which recognizes the relevance of 137's posting at that particular moment in interaction, and treats 137's coordinated contribution as an act of understanding.

Discussion

Given the characterization of deep mathematical understanding in the math education literature, methodic ways through which participants coordinate their actions across the whiteboard and chat spaces are of particular interest to our investigation of mathematical understanding or meaning making at the small-group level. The episode we analyzed above includes a situation where a user, who has been active in the whiteboard, moves on to the other interaction space and posts a message referring to his prior drawing work. The chat message sequentially followed the drawings, and hence presumed their availability as a shared referential resource, so that the interlocutors can make sense of what is possibly referred to by the indexical expression "*n=4,5,6*" included in the posting. Davidcyl's explicit orientation to timing or sequencing is further evidenced by his use of the past perfect tense and his temporal positioning of the message immediately after the final step of the drawing. Moreover, the chat posting reflexively gave further specificity to the prior drawing work by informing everyone that the diagrams should be seen as specific cases of the staircase pattern described in the task description. This suggests that temporal proximity among actions can serve as an interactional resource/cue for the participants to treat those actions in reference to each other, especially when the actions are performed across different interaction spaces. In short, Davidcyl has demonstrated a *method* that one can call *verbal referencing*, which is employed by VMT users when they need to communicate to each other that a narrative/symbolic account needs to be read in relation to a whiteboard object.

Davidcyl's use of the algebraic reference "*n=4,5,6*" at this moment in interaction is also informative in terms of respective limitations of each medium and their mutually constitutive function for communication. Davidcyl's chat message not only provided further specificity to the recently produced diagrams, but also marked or announced the completion of his drawing work. This is revealing in terms of the kinds of *illocutionary acts* (Austin, 1962) that can be achieved by users in this dual-media online environment. In particular, although a drawing and its production process may be available for all members to observe in the shared whiteboard, diagrams by themselves cannot fulfill the same kind of interactional functions achieved by text postings such as "asking a question" or "expressing agreement." In other words, whiteboard objects are made interactionally relevant through chat messages that either (a) project their production as a next action or (b) refer to already produced objects. This can also be seen as members' orientation to a limitation of this online environment as a communication platform; one can act only in one space at a given time, so it is not possible to perform a simultaneous narration of a drawing as one can do in a face-to-face setting. Therefore, each interaction space as a communicative medium seems to enable and/or hinder certain kinds of actions, i.e., they carry specific *communicative affordances* (Hutchby, 2001) for collaborative problem solving online.

The way Davidcyl has put some of the features of the whiteboard —like dragging and copy/paste—into use in the episode described above demonstrates some of its key affordances as a medium for producing shared drawings. In particular, we have observed how copying and pasting is used to avoid additional drawing effort, and how collections of objects are selected, dragged, and positioned to produce specific stages of a geometric pattern. Such possibilities for action are supported by the object-oriented design of the whiteboard. Davidcyl's drawing actions show that, as compared to other physical drawing media such as paper or blackboard, the electronic whiteboard affords unique possibilities for constructing and modifying shared mathematical diagrams in ways that have mathematical, collaborative, semantic and communicative power.

It is analytically significant that Davidcyl changed from building the pattern row-wise in Figure 2 to building it column-wise subsequently and that he "computed" the height of the new column in several different ways in Figures 3, 4 and 5. This indicates that he did not have an explicit solution "in his head"—a mental model that he just had to illustrate in the world with the whiteboard. Rather, he worked out the solution gradually through emergent whiteboard activities and his recognition of what appeared in the whiteboard. Significantly, the other group members could observe the same thing.

An important concern for our group-cognitive approach is to investigate how students make use of the technological features available to them to explore mathematical ideas in an online environment like VMT. Drawing features such as copy/paste, dragging, coloring, etc. are important affordances of the shared whiteboard not simply because of their respective advantages as compared to other drawing media. The mathematical significance of these features relies on the way single actions like copy/paste or dragging are sequentially organized as part of a broader drawing activity that aims towards constructing a shared mathematical artifact. For instance, through such a sequence of drawing actions Davidcyl demonstrated to us (as analysts) and to his peers (a) how to construct a stair-step pattern as a spatially organized assemblage of squares, and (b) how to derive a new stage of the stair-step pattern from a copy of the prior stage by adding a new column of squares to its right. Moreover, Davidcyl's engagement with the squares (rather than with the sticks that make up the squares) displays his explicit orientation to this particular aspect of the shared task (i.e., finding the number of squares at a given stage). Hence, the availability of these drawing actions as a sequence of changes unfolding in the shared visual space allows group members to witness the *reasoning* process embodied in the sequential and spatial organization of those actions. In other words, the sequentially unfolding details of the construction process provide specificity (and hence meaning) to the mathematical artifact that is being constructed.

Besides figuring out ways to connect their own actions across dual-interaction spaces, VMT users also coordinate their actions with the actions of their peers to be able to meaningfully participate in the ongoing discussion. The ways participants produce and deploy mathematical artifacts in the shared space implicate or inform what procedures and methods may be invoked next to produce other mathematical artifacts, or to modify existing ones as the discussion progresses towards a solution to the task at hand. For instance, 137's competent contribution to Davidcyl's sequentially unfolding line of reasoning in Table 1 shows that shared mathematical understanding at the group level is an interactional achievement that requires coordinated *co-construction* of mathematical artifacts. The *co* prefix for the term "co-construction" highlights the *intersubjective* nature of the mathematical artifacts produced during collaborative work; they are not mere mental constructs easily ascribed to certain individuals. As we have just observed in the excerpt above, *intersubjectivity* is evidenced in the ways participants organize their actions to display their relevance to prior actions. 137's anticipation and production of the next relevant step in the joint problem-solving effort serves as strong evidence of mutual understanding between him and Davidcyl. Moreover, the term *construction* signals that mathematical artifacts are not simply passed down by the mathematical culture as ready-made Platonic entities external to the group. Once enacted in group discourse, culturally transmitted artifacts such as "Gaussian Sum" need to be made sense of and appropriated in relation to the task at hand. Hence, our use of the combined term *co-construction* implies an interactional process of sense making by a group of students—even in an excerpt like the present one in which one individual takes an extended turn in the group discourse to develop a complex presentation. The fact that it is a visible construction worked out in collaborative media and designed for reception by others makes it a co-construction from which the speaker is as likely to learn as are the other group members.

When co-construction takes place in an online environment like a chat tool, the construction process must take place through observable interactions within technical media. This requires student groups to invent, adapt or appropriate methods to co-construct mathematical artifacts. It also makes it possible for them to explicitly reflect on the *persistent traces* of their co-constructions by investigating the persistent content provided by the technology. Therefore, the persistent nature of actions provides the necessary infrastructure for joint action, and hence is a key affordance of CSCL environments like VMT, where actors work at a distance in a disembodied environment. In addition, the persistent records of interactions also allow researchers to analyze the co-construction process as it unfolded in real-time, as this paper demonstrates.

Through similar case studies of other VMT sessions, we observed that students make use of additional resources (such as the explicit referencing tool, locational pronouns, color names in chat) to methodically achieve referential relationships between shared diagrams and chat messages (Cakir, Zemel & Stahl, 2009). Chat postings use a broad and sophisticated array of such methods to refer to matters constructed graphically. Due to their recurrent appearance as a practical concern for the participants in this dual-media online environment, we refer to the collection of these methods as *referential practices*. Referential practices are of particular importance to the study of mathematical understanding as a group-cognitive phenomenon, because they are enacted in circumstances where participants explicitly orient to the task of achieving relationships between the textual and graphical contributions that they have been exchanging online—a phenomenon that is given significance in the math education literature as characterizing deep mathematical understanding. Likewise, one can use the term *representational practices* to refer to the spatial and temporal organization of whiteboard actions that produce shared diagrams, which simultaneously give further specificity to the mathematical artifacts that the team has been working on—e.g., Davidcyl's methodical sequencing of copy/paste operations to indicate growth patterns. Through referential and representational practices, participants co-construct mathematical artifacts that reify mathematical understandings. The understanding or meaning is not simply located inside students' individual brains or in the chat/drawing artifacts themselves. The meaning is embodied in the sequentially organized and coordinated actions through which those artifacts were co-constructed. To sum up, group referential and representational practices play a key role in the ways mathematical artifacts are (a) appropriated by active teams from historically developed cultural tools, and (b) emergent from ways of communicating and symbolizing within local collectivities as shared, meaningful resources for mathematical discourse, collaborative learning and group understanding.

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