

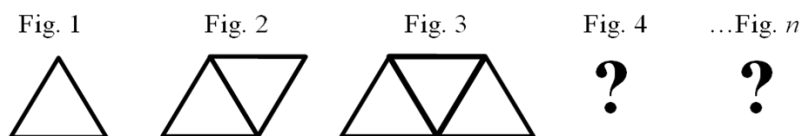
# “I Don’t Know—I’m Just Genius!”: Distinguishing Between the Process and the Product of Student Algebraic Reasoning

Jose Francisco Gutierrez, Graduate School of Education,  
University of California—Berkeley, Berkeley, CA 94720-1670, josefrancisco@berkeley.edu

**Abstract:** This paper reports on a semiotic-cultural analysis of two high-school minority seniors’ participation in a mathematics instructional intervention conducted at a school for academically at-risk students. Focusing on the students’ speech acts, gestures, and artifact production during their successful collaborative engagement in an algebraic pattern-finding task, I adopt and adapt Radford’s (2003) framework to evaluate the quality of each student’s actual learning as implicated by their discursive contributions. By analyzing each student’s utterances as either “to” a semiotic mode (generalization) or just “in” the mode, and then overlaying these coded utterances, I show how one student’s participation, which appears to mark a developing algebraic command, in fact blinds the teachers to underlying discontinuity in the student’s meaning construction. The study illuminates critical tradeoffs in the design and facilitation of collaborative problem solving.

## Introduction: The Story of Be’zhawn and Deana

We join Be’zhawn and Deana, two high school seniors classified by their mathematics teacher as low-achieving students, in the midst of a tutoring session involving a collaborative algebraic problem-solving activity. Be’zhawn and Deana are at the board, working together on a sequence of several geometric figures in an attempt to formulate its pattern algebraically (see Figure 1 below). For twenty-five minutes, the students have been bandying mathematical ideas about: some ideas are taken up and built upon, while others are ignored or proven inefficacious. Throughout this duration, both students sustain a high level of engagement and, to all appearances, contribute equally to the problem-solving process. Eventually, Be’zhawn is the first to verbally express a correct solution. And yet, when asked to explain his reasoning, he replies, “I don’t know—I’m just genius!”



**Figure 1.** The triangle sequence—a paradigmatic reform-based algebra problem. The task objective is to express  $U_n$ , the total number of line segments in the  $n^{\text{th}}$  figure. For example, “Fig. 1” consists of 3 line segments, “Fig. 2” consists of 5, “Fig. 3” consists of 7, etc., so that the  $n^{\text{th}}$  figure consists of  $2n+1$  line segments.

Back in our laboratory, all three teachers—the author included—discussed the students’ behaviors during this activity and the nature of their mathematical contributions. We all agreed that this particular tutoring session was unique, as compared to previous sessions, in terms of the quality of students’ mathematical reasoning. Also, as this was the first time they had collaborated on a pattern-finding problem, we attributed the students’ engagement to the problem type’s unique task structure. Be’zhawn, in particular, had exhibited much difficulty with algebraic concepts in prior sessions, where the problem items had been more routine. The pattern-finding problem was not the only item Be’zhawn solved quicker than his peers; however, this was the first time he had ever indicated that he held a positive perception of himself as a mathematics doer, when he confidently declared that he was “genius.”

This particular tutoring episode was part of a larger mathematics-education research project conducted at a small alternative high school in the San Francisco Bay area. Broadly, we sought to identify and respond to learning challenges faced by academically at-risk mathematics students with respect to both reform and traditional algebra curricula. The pattern-finding episode appeared to demonstrate success and not only challenge, and so I selected this episode in an attempt to articulate principles of design and facilitation that contributed to its apparent success. That said, my enthusiasm was tempered by acknowledging that Be’zhawn’s inability to explain his correct solution might implicate his solution strategies as problematic. What might it mean for a student to solve an algebraic pattern-finding problem yet not be able to explain his solution? What methodological tools could address this puzzle?

## Objectives of the Study

Researchers of mathematics education constantly develop theoretical frameworks for articulating what students struggle with as they attempt to solve algebraic problems. At the same time, though, the national achievement gap is

constantly increasing. This, despite constant calls for equity in mathematics education and, in particular, for increasing the accessibility of algebra content for students from minority groups and economically disadvantaged backgrounds (for a review see DiME, 2007). The current study embarks from a conjecture that some recent theoretical frameworks developed particularly for tackling algebraic reasoning (for a review see Kieran, 2007) harbor the potential to illuminate new directions for broadening diverse participation in mathematics and, in particular, to support general success in this “gatekeeper” content. The objective of this paper is to build upon some delimited empirical data so as to illustrate what we may need to attend to as we pave new paths through the gate.

Central to the theoretical argument of this paper is Luis Radford’s (2003, 2008) *semiotic-cultural approach* to theorizing the process and content of students’ algebraic reasoning. This powerful approach views learning as an evolving process co-constrained by both cognitive and socio-cultural factors. Specifically, mathematics learning is conceptualized as constructing personal meaning for canonical semiotic artifacts (e.g., algebraic symbols such as the variable “ $x$ ”). Through consolidation and iteration of such constructions, students appropriate these notations and, reciprocally, build personal meaning for mathematical content as well as fluency with the disciplinary procedures. That is, individuals who build mathematical signs upon personal meaning develop more than fluency with procedures—they develop substantive and articulated understanding of the targeted content. Radford’s approach takes into account a vast arsenal of personal and interpersonal resources that students bring to bear in solving mathematical situations, including linguistic devices and mathematical tools. Most relevant to our current study, Radford has demonstrated that critical steps within individual learning trajectories can be explained by noting subtle shifts in the subjective function and status of the semiotic artifacts (Abrahamson, 2009; Sfard, 2007).

I here propose Radford’s approach as a powerful tool for untangling the compound challenges faced by students who are attempting to develop a mathematical register even as they are learning new content. Specifically, I propose, the semiotic-cultural approach enables researchers to delve into the cognitive/semiotic plane below external discursive manifestations. That is, students may manifest discursive productions in a proper mathematical register, yet these “correct” utterances may belie the absence of underlying personal meaning. The semiotic-cultural approach, I submit, could be leveraged so as to expose tension between students’ overt mathematical speech contributions and their covert, (dis)connected reasoning.

A potential theoretical contribution of this paper is the qualification I propose for Radford’s approach, as I adapt the approach to the particular needs of at-risk students. These students’ disposition toward the classroom “leading discourse” (Sfard, 2007) is complex, because for them, “talking the mathematical talk” is at once enabling and threatening. We therefore may witness gaps between these students’ overt expression and covert meaning. This tension bore out in the analysis. Namely, Whereas Radford’s approach was pivotal to making initial sense of these students’ utterances, I gradually realized, however, that the approach was not enabling me to unwrap students’ covert meanings. As the paper will demonstrate, I was thus impelled to revisit some of the implicit assumptions of Radford’s semiotic-cultural framework and then make a pivotal amendment. Namely, whereas Radford’s interpretation of collaborative problem solving may imply that all team members are party to critical semiotic transitions inherent to generalization acts, I submit that some students manifest productions *in* milestones of this procedure without having partook in advancing *to* these semiotic modes. Distinguishing the within-vs.-between semiotic nature of students’ discursive productions is important, because, as Radford claims, it is the *transitions* from one semiotic mode to the next—not discursive acts per se—that are crucial to building meaning (Radford, 2003). Equipped with this qualification, the semiotic-cultural approach helps me characterize weaknesses in some students’ learning processes during collaborative problem solving as resulting from an impetus to participate within a particular semiotic mode without necessarily having personally made the careful transition to that mode.

## Theoretical Background

From the semiotic-cultural perspective, mathematics learning is viewed as a mediated, distributed, and dynamically reciprocal process in which students’ emerging presymbolic knowledge is iteratively objectified in progressively sophisticated forms of historically and culturally endowed semiotic systems. One such semiotic system—which is prevalent in the mainstream of school mathematics programs—is algebra. Viewed from the semiotic-cultural perspective, “algebraic thinking is a particular form of reflecting mathematically” (Radford, 2003), that is, algebra is about “using signs to think in a distinctive way” (Radford, 2008, p. 87).

To illuminate the content and process of this distinctive way of thinking characteristic of algebra, Radford elaborates a theoretical framework for describing and analyzing students’ reasoning during collaborative engagement with pattern-generalization algebraic activities such as the triangle sequence (see Figure 1, above). I now use this problem as a context to elaborate on Radford’s framework, which is central to my data analysis, and then outline a potential shortcoming with its extant formulation and finally propose a solution to this problem.

A key construct in Radford's framework is *knowledge objectification*, which is defined as the process of making the objects of knowledge apparent (Radford, 2003). For example, a mathematics learner, in an attempt to convey a certain aspect of a concrete object, such as its shape or size, will tap a variety of semiotic artifacts such as mathematical symbols and inscriptions, words, gestures, calculators, and so forth. In patterning activities, however, some of the objects of knowledge are general and therefore "cannot be fully exhibited in the concrete world" (Radford, 2008, p. 87). More generally, mathematical notions may not be cognitively accessible, because they do not exist in the world for empirical investigation (Duval, 2006), that is, these notions are never apparent to direct perception. Therefore, in order to instantiate (objectify) these ephemeral notions, students must resort instead to personally and culturally available forms such as linguistic, diagrammatic, symbolic, and substantive artifacts as well as the body, which Radford (2003; 2008) collectively terms *semiotic means of objectification*.

In patterning activities, students often resort to using one of two strategies when they attempt to construct generalizations from an initial set of numeric and/or figural cues (Radford, 2008) such as in the triangles sequence.

1. The first strategy, *naïve induction* is characterized by a process of disconnected or minimally principled "guess and check." For example, a student might propose a simple rule such as "the figure-number times two" and check its validity on a few cases. Finding that the proposed rule does not satisfy one or more cases, the student might then modify her rule or propose an entirely different rule, for example, "the figure-number times two, plus one" and check its validity as well, and so on.
2. The second and more sophisticated strategy, *generalizing*, is expressed as an active search for recurrent relationship structures and/or pattern commonalities among constituent elements in the problem space. This latter strategy, in turn, typically produces two types of generalization, *arithmetic* and/or *algebraic*.
  - 2.1. Arithmetic generalizations typically take the form of a *recursive solution* that only indirectly expresses any term in the sequence. In the triangle sequence, the number of line segments required to construct each consecutive figure always increases by a factor of two with respect to the previous figure, an observation expressible as the arithmetic generalization  $U_n = U_{n-1} + 2$ . The commensurate *explicit solution*  $U_n = 2n + 1$ , however, is more powerful, because it determines directly any item along the infinite sequence.
  - 2.2. Making the leap from recursive to explicit solutions involves generalizing a pattern *algebraically*, which requires grasping and objectifying recurrent  $n$ -to- $U_n$  relations along the sequence, then elaborating this objectification into a generalization that would *directly* express any term in the projected sequence.
  - 2.3. Finally, algebraic generalizations are subdivided into three types in accordance with their level of generality: *factual*, *contextual*, and *symbolic* (see Figure 2, below, for a summary as well as examples).

Solution Strategies	Generalizing			
	<u>Naïve Induction</u>			
	<ul style="list-style-type: none"><li>Characterized by “guess and check”</li><li>E.g., “It’s ‘<math>2n+2</math>’... No. Try ‘<math>2n+1</math>’. YES!”</li></ul>	<ul style="list-style-type: none"><li>Active search for recurrent relationships/commonalities among constituent elements in the problem space</li></ul>		
Generalization Types		<u>Algebraic Generalization</u>		
	<u>Arithmetic Generalization</u>	<ul style="list-style-type: none"><li>Explicit formula (<math>U</math> is expressed in terms of the figure’s ordinal position)</li><li>E.g., <math>U_n = 2n + 1</math></li></ul>		
Levels of Generality		<u>Factual</u>	<u>Contextual</u>	<u>Symbolic</u>
		<ul style="list-style-type: none"><li>Objects and operations are bound to concrete level</li><li>E.g., “1 plus 2, 2 plus 3, 3 plus 4...”</li></ul>	<ul style="list-style-type: none"><li>Objects and operations are abstracted and generalized</li><li>E.g., “The figure plus the next figure.”</li></ul>	<ul style="list-style-type: none"><li>Objects and operations are expressed through formal symbolism</li><li>E.g., “<math>n + (n+1)</math>”</li></ul>

Figure 2. Semiotic-cultural taxonomy of algebraic pattern-finding solution processes (adapted, Radford, 2003; 2008)

## A Proposed Qualification

When I first applied the semiotic-cultural framework to the data (see Analysis Techniques, below), it indeed revealed disconnection between the students' utterances and their underlying meaning making. Namely, Deana's

utterances reflected generalizing acts, whereas Be'zhawn's did not. Essentially, applying Radford's framework revealed that one student 'got it' while the other did not 'get it.' However, what the extant framework did not explain was why it appeared to the instructors *in situ* that *both* students 'got it.'

A second analysis of the data was conducted so as to provide an alternative description of the students' behaviors—a description that would explain how the researcher-as-analyst could see what the researcher-as-teacher could not. So doing, I realized that inherent to Radford's framework were what might be called three *semiotic modes* of action, in which both Deana and Be'zhawn were participating. By "mode" I am offering an analytic construct that is linguistic but not psychological; that is, by "mode" I am referring to the surface features of student utterances, temporarily putting aside the question of the student's subjective semiotic grounding. I could now conjecture that students' utterances *within* each of these semiotic modes were similar, thus giving the teachers the impression that both students shared similar learning experiences toward those utterance.

By introducing "modes" as an elaboration to the framework, I differentiate two meanings of "generalization": the *process* of generalizing (i.e., the way that students engage with the problem), and the *product* that results from this process. This distinction is important, because it suggests that researchers and teachers may attribute understanding to students when such attribution is not warranted. In particular, a student's utterance *within* a semiotic mode may be interpreted as marking the production of a semiotic generalization *to* that mode, when in fact the student is only appropriating and elaborating on the product of another student's generalization to the mode.

The proposed process-vs.-product distinction reflects a pedagogical position that the generalizing activity is not merely to create opportunities for students to operate *in* each of the modes, but rather to support them as they progress *along* the desired chain of signification from the factual through to the symbolic mode, thus ensuring that each and every student performs the cognitive work necessary to personally ground the target signs.

Having presented a semiotic-cultural framework and outlined a potential problem with its extant formulation as well as a proposed solution to this problem, I am now in a position to restate my research questions in light of Radford's seminal theory. Namely:

- Can a mathematics learner be *in* a particular semiotic mode without having generalized *to* that mode?
- What are the consequences of such superficial semiotic production for learning?

Next I describe the methods used to address these questions.

## Methods

For this preliminary empirical study, the research team, which included the author as well as two undergraduate apprentice researchers, conducted an instructional intervention—*The Mathematics Workshop*—at a small high-school for academically at-risk students. The intervention, which was designed to prepare a small cohort of senior students ( $N=5$ ) for a state exit exam, provided one-to-one and small-group tutoring sessions with a focus on algebra content. The research team all shared in the design, facilitation, and preliminary study of the workshop. For this paper I focus on two case studies, Be'zhawn (male) and Deana (female; both pseudonyms), and conduct my data analysis from the semiotic-cultural perspective. To address my research questions (see above), I examined students' speech acts, gestures, and artifact production during collaborative engagement with the "triangle sequence." I analyzed each of these students' reasoning processes during the focus episode in terms of its semiotic trajectory and then juxtaposed these two trajectories. This juxtaposition reveals "hidden" dyad dynamics in the joint activity of interpreting the sequence and constructing its algebraic reformulation.

*Data Sources:* The entire data corpus from the intervention includes students' original work, a total of six hours of video footage from the implementation, and a project wiki (online collaborative archive) that we used to coordinate resources, document fieldnotes and meeting minutes, and share ongoing reflections. However, the study reported in this paper focuses on a span of only 28 minutes of video footage, in which the triangle sequence was implemented. I selected this particular teaching episode for deep analysis both because this activity has been previously studied (e.g., Radford, 2003; Rivera, 2008) and because the participants' behaviors as they engaged in the solution of this problem helped me first notice the process-vs.-product of semiotic constructions.

*Materials and procedures:* The teaching episode in question begins with an utterance from the first teacher (hence "T1," the author; "T2" = the second teacher, one of the undergraduate apprentice researchers; "T3" = the third teacher, the other undergraduate apprentice researcher). T1 sketched the first three figural cues on the whiteboard and instructed Deana and Be'zhawn to search for a mathematical pattern and express it as an algebraic equation. Aside from these initial figural cues, a set of guiding questions was also written on the board: "How many lines are there in Fig. 1? Fig. 2? Fig. 3? Fig. 4? Fig. 12?" Furthermore, a key design feature for implementing the triangle sequence was to substitute increasingly larger numbers (e.g., Fig. 50, and Fig. 67) as a way to impress upon students that ultimately the arithmetic/recursive strategy is inefficient, thus motivating the need for more powerful tools and strategies such as algebraic generalizing as well as the use of explicit formulas.

## Analysis Techniques

I produced and analyzed a transcription of the 28-minute teaching episode, which captures all verbal, gestural, inscriptional, and other semiotic actions that were clearly observable in the video. The transcription was parsed into a total of 440 “Lines,” where each Line is defined as a segment of speaker continuous speech. Next, I identified all student utterances involving mathematical propositions (127 out of the 440 Lines). For this study, I focus only on these Lines of the transcription, for which two main analytic questions were asked pertaining to its semiotic nature:

1. Generalization Type (generalization *to* the semiotic mode—i.e., the *products*):
  - 1a. Is the expression the product of naïve induction or generalizing?
  - 1b. [If so,] Is the expression an arithmetic or algebraic generalization?
  - 1c. [If so,] Is the expression a factual, contextual, or symbolic generalization?
2. Semiotic Mode (operating *in* the semiotic mode, and therefore having ostensibly arrived *to* the mode and ostensibly engaging in the *process* of generalizing to the next mode):
  - 2a. Is the student referring to and/or expressing an action upon concrete objects (i.e., operating in the factual mode)?
  - 2b. Is the student referring to and/or expressing an action upon abstract yet contextually situated objects (i.e., operating in the contextual mode)?
  - 2c. Is the student referring to and/or expressing an action upon algebraic symbolical objects (i.e., operating in the symbolic mode)?

Working with both the video footage and the abbreviated transcription, a first pass of the data was done using analytic questions 1a-1c (see above). I initially evaluated whether or not each utterance reflected a generalization *to* a particular mode or merely a reiteration *in* the mode. This evaluation was based on a qualitative microgenetic analysis (Schoenfeld, Smith, & Arcavi, 1993) of students’ behaviors during their conversation. So doing, I determined whether the students engaged in authentic generalizing acts (i.e., grasping and objectifying recurrent  $n$ -to- $U_n$  relations and providing a direct expression for any term along the sequence) or resorted instead to naïve induction and/or appropriating the speech forms of others. Following this qualitative analysis, a second pass through the data was done using analytic questions 2a-2c, whereby all student utterances involving mathematical propositions were coded only for their semiotic mode (factual, contextual, or symbolic). For example, the student utterance “Fig. 3 has seven lines” was coded as an assertion *in* the factual mode, because Fig. 3 was perceptually available when the utterance was verbalized (i.e., Fig. 3 was drawn on the board; it was “present”). And yet, the utterance “Fig. 5 is going to have eleven” was also coded as *in* the factual mode even though Fig. 5 was not perceptually available (i.e., Fig. 5 was not drawn on the board; it was “absent”), because the utterance “Fig. 5” refers to a specific figure along the sequence, which could be instantiated as a concrete entity. In contrast, utterances such as, “We just add a triangle to every one” were coded as assertions *in* the contextual mode because the student expressed an action upon abstract objects, whereby “one” is a pronoun referring to the general term “figure.” Finally, transcript Lines containing students’ inscriptional actions that involve symbolical notation were coded as assertions *in* the symbolic mode. For example, the inscription “ $x^2 + 1$ ,” which a study participant wrote on the board, was coded as *in* the symbolic mode.

## Results and Discussion

The results and the discussion begin by presenting a qualitative analysis of several transcription segments, followed by further quantitative analysis of students’ utterances in terms of their semiotic mode. I present the results in this order—which is atypical of social science research articles—because I wish to provide for the reader a chronological description of how my initial investigation of the data led to the need for theoretical extension. Combined, these analyses reveal that the students’ contributions during their conversation occupied each of the three modes, however, Deana’s but not Be’zhawn’s utterances generalized *to* these semiotic modes.

### Qualitative Results

I here present qualitative data analyses of a series of selected transcript segments. Due to space constraints, these selected segments do not represent all of the students’ mathematical activity during the 28-minute episode; however, these segments are representative of the students’ behaviors during the episode in question. With these segments, I aim to elaborate on findings from the quantitative analysis by illustrating that: (1) both Deana and Be’zhawn’s discursive production were *in* each semiotic mode; and yet by-and-large (2) Deana’s manifestations ultimately generalized *to* each mode, whereas Be’zhawn’s did not.

*Segment 1:* At the onset of the activity, Deana immediately noticed that the figures could be construed as a succession of accruing triangles; she also noticed that the number of line segments for each consecutive figure

increased by an addend of two. Deana continued to operate both *in* the factual and contextual mode. So doing, she constructed an arithmetic generalization *to* the contextual mode in the form of  $U_{n+1} = U_n + 2$ . Deana thus began a process of authentic generalizing *in* the factual and contextual modes and this qualitative analysis suggests that her subsequent semiotic productions could be grounded in concrete elements from the initial problem space. [Each segment of data presented below is labeled by its Line number from the original transcription, the speaker, utterance, and its ostensible semiotic mode: “F” = factual; “C” = contextual; “S” = symbolic]

Line 14	Deana	“[gazes at <i>Figs 1, 2, and 3 drawn on board</i> ] Fig. 4 is gonna have nine.”	F
Line 16	Deana	“We just add a triangle to every one, right?”	C
Line 33	Deana	“[ <i>Fig.</i> ] 4 has nine. It goes up every two lines, so Fig. 5 is going to have eleven...”	C

*Segment 2:* This segment illustrates how Be’zhawn first appropriates Deana’s utterance and builds upon it, such that his speech acts are *in* (but not *to*) the factual and contextual modes. Furthermore, Be’zhawn gave no indication that he had interpreted the original problem-situation and had not noticed the patterns that Deana was verbalizing.

Line 47	Deana	“[addressing no one in particular, thinking aloud, referring to <i>Fig. 12</i> ] Twelve and two... no... twelve and two is twenty-four.”	F
Line 49	Be’zhawn	“Twenty-four! That’s the answer. [...] What’s twelve times two? Eleven times two is twenty-two. Ten times ten is twenty.”	F
Line 57	Be’zhawn	“They all go up by two.”	C
Line 68	Teacher 1	“[after asking Be’zhawn to draw <i>Fig. 4 on the board</i> ] They’re just triangles, right?”	--
Line 69	Be’zhawn	“Oh, they’re just triangles? What the—”	--

*Segment 3:* For the remainder of the activity, Be’zhawn employed naïve induction. For example, he guessed that Fig. 67 is comprised of 33 lines (i.e.,  $U_n \approx n / 2$ ), and then changed his guess to 134 (i.e.,  $U_n \approx n * 2$ ).

Line 191	Be’zhawn	“Thirty-three... That’s how many lines [ <i>Fig.</i> ] 67 has. Nah, that’s too little. [ <i>Fig.</i> ] 12 is twenty-six.”	F
Line 199	Be’zhawn	“But it’s not thirty-three though, cuz only [ <i>Fig.</i> ] 12 is twenty-six.”	F
Line 254	Be’zhawn	“I think it’s one hundred and thirty-four.”	F

*Segment 4:* Interestingly, it was Be’zhawn who first produced a non-symbolic version of the correct solution procedure. Assisted by the teachers, Be’zhawn noticed that his proposed rule,  $U_n = n * 2$ , was off mark by a difference of 1. Using this piece of new information, Be’zhawn cobbled together a correct solution procedure, namely,  $U_n = n * 2 + 1$ , and he used it to articulate the solution for the case of Fig. 67.

Line 308	Teacher 2	[to Be’zhawn] That was the guess, but these are each off a little bit so what do we think this is actually gonna be?	--
Line 309	Be’zhawn	Unless you add one to it.	C
Line 310	Teacher 1	So what do you think the actual—	--
Line 311	Be’zhawn	One-thirty-five!	F

Even though Be’zhawn’s solution is mathematically correct, I argue that his semiotic constructions were disconnected from the original problem space, because they were the result either of naïve induction or of appropriating the speech forms of others, mainly Deana’s. Namely, my analysis reveals that Be’zhawn was guessing and operating on proximal numerical values in isolation from the initial set of figural cues. At least, Be’zhawn could

not reconstruct how he arrived at these constructions. Indeed, when a teacher asked Be'zhawn, "What's the formula?" he responded as follows: "I don't know—I'm just genius!"

*Segment 5:* Deana, on the other hand, was able to build upon her initial arithmetic generalization (see Segment 1 above). So doing, at the end of the activity she was able to connect components of the solution procedures to elements from the original problem situation. Thus, I maintain, her inscription " $x2 + 1$ ," which she wrote on the board, was semioitcally grounded, from the factual through to the symbolic.

Line 366	Deana	"The formula. Okay. You know sixteen, right? Times two, multiply by two, is thirty-two, right? "And you add one more cuz of the extra line, [ <i>gestures the shape of the top of a triangle</i> " $\wedge$ "] cuz it's three lines [ <i>referring to missing bottom line</i> ]. And you just—since it's three lines."	F
Line 406	Deana	"It's gonna work because you got the two and then you add one more line."	C
Line 414	Deana	"[ <i>writes "<math>x2 + 1 = ?</math>" on the board</i> ] $x$ times two plus one." [ <i>Be'zhawn walks to the other side of the board, gazes toward Deana's inscription, then he writes "Figure <math>x2 + 1</math>," where the "<math>x</math>" is read as a multiplication symbol</i> ]	S

The qualitative analyses above illustrate how Deana was able to construct both arithmetic and algebraic generalizations, whereas Be'zhawn was not. We see the subtle shifts in Deana's perception of the function and status of the semiotic artifacts—beginning with her initial grasp of a recurrent relationship between figures, "we just add a triangle to every one" (Line 16), to her arithmetic generalization expressed in terms of the line segments, "it goes up every two lines" (Line 33), to her final algebraic inscription of " $x2 + 1 = \text{Lines}$ " (Line 414). Be'zhawn's semiotic productions, on the other hand, certainly manifested *within* each of the generalization modes, however, his shifts *between* the different modes are weakly connected at best, and non-existent at worst.

## Quantitative Results

Figure 3, below, represents students' semiotic-mode participation sequences—it depicts a synoptic left-to-right view of each student's participation through the lens of the semiotic mode they are *in*. Every node in the graph represents a student utterance coded as articulated in one of the three semiotic modes. By overlaying their respective semiotic sequences, it may appear that both Deana and Be'zhawn are making forward advances between modes, yet a closer analysis of their manifest productions reveals that this is not the case.

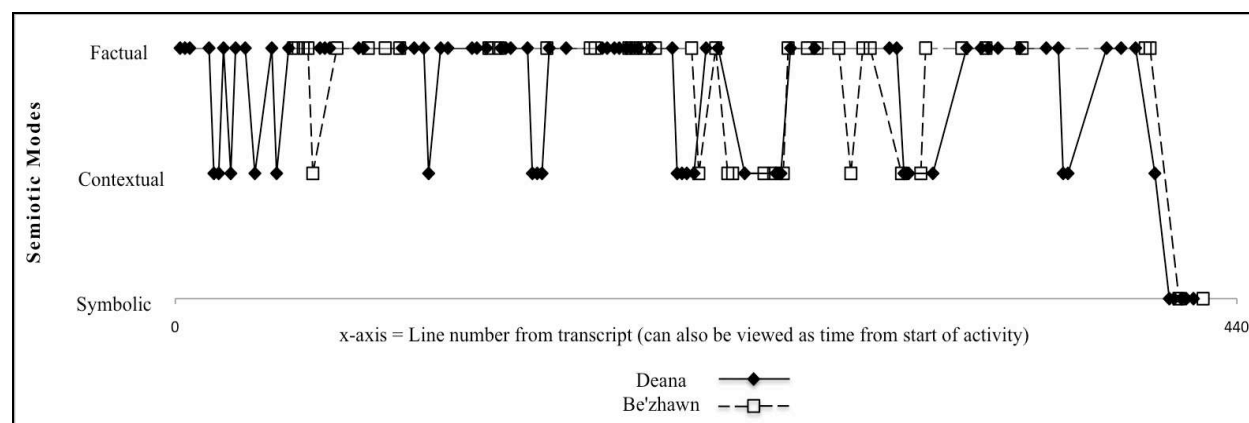


Figure 3. Representation of juxtaposed student participation sequences (ostensible semioitc trajectories)

Figure 3 clearly illustrates that Be'zhawn and Deana's respective semiotic-mode participation sequences were indeed similar, which could account for why the teachers, in situ, interpreted their behaviors as indicating similar and mutually beneficial reasoning processes. And yet, in Figure 3, it is important to notice that Deanna's move from the factual to the symbolic (at the very end) occurs before Be'zhawn's move. We also see that Deanna's sequence passes through the contextual mode en route to the symbolic whereas Be'zhawn's does not, an observation

supporting the hypothesis that his semiotic productions were disconnected, as apparent in his “I don’t know” statement mentioned earlier in my qualitative analysis.

## Conclusions and Implications

When students participate in collaborative problem solving, their discursive productions do not necessarily reflect adequate grounding of mathematical symbols, even in the cases where these contributions are instrumental to the group’s overall success. Namely, while participating in collaborative problem solving, students’ individual discursive productions can occupy each of the three semiotic modes identified by Radford, yet these manifest productions might nevertheless conceal a discontinuous semiotic trajectory, resulting in suboptimal learning. Previous research has challenged the claim that mathematics learning is best done in small groups or dyads (e.g., Barron, 2003), and this paper contributes to this previous research by providing a semiotic-cultural articulation of the variability of meaning construction for individuals participating in a dyad. Notably, the paper certainly does not argue against collaboration as a mode of classroom practice. Rather, it has been my intention to provide analytic tools that problematize any assumptions that educators might harbor with respect to issues of equity that emerge when two or more students are assigned a joint problem-solving task (cf. Esmonde, 2009).

In addition to these issues of design and facilitation, the paper has also made two tentative contributions to mathematics-education research: (1) a theoretical contribution is a proposed refinement to Radford’s semiotic-cultural approach; and (2) a methodological contribution is the initial development of an analytic approach to distinguishing between the process and product of students’ discursive contributions to collaborative (algebraic) problem solving.

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