The Affordable Touchy Feely Classroom: Textbooks Embedded With Manipulable Vectors and Lesson Plans Augment Imagination, Extend Teaching-Learning Practices

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Abstract: Computational media for mathematics and science education have been around for forty years, but teaching and learning practices are still centered surrounding the textbooks in the developing world, mainly due to their affordability. Teachers thus organise their thinking, workflow and classroom interactions using textbooks. However, text media limit students' ability to imagine the dynamics embedded in mathematics and science formalisms, such as vectors. We outline an affordable design that augments textbooks, to: allow all students to imagine such dynamics; help teachers reorganize their thinking and workflow; make all classrooms highly collaborative. The design consists of a 'touchy-feely vector' system (smartphone-based manipulation of vectors), a 'virtual lesson plan' (manipulable simulations that extend textbook figures), and a 'mixed-media' textbook (QR codes pasted next to figures, linked to manipulable simulations). We discuss how this integrated design changed learning and teaching practices, based on an ethnographic study of experimental and control classrooms.

Introduction

Interactive media for making concepts and formal structures in mathematics and science easier to understand (Balacheff & Kaput, 1996) have been around for close to forty years (Papert, 1980, 1994). But these media have not changed classroom practices significantly, as teaching and learning processes and teacher training are still centred around textbooks, particularly in the developing world. This is because textbooks are cheap and widely available, and they thus play an artifactual role in teaching practice, helping organise teachers' thinking, workflow and classroom interaction. The textbook is thus the dominant media in education, but text and figures are limited in conveying the dynamic nature of mathematics and science content, particularly formal structures such as vectors, which are critical for developing model-based reasoning, a key skill needed for practising science (Hestenes, 2010; Magnani, Nersessian, & Thagard, 2012; Nersessian, 1999). Teachers struggle to enact or 'act out' in the classroom the dynamic nature of such formal structures, and students have a hard time imagining the dynamic nature based on the teacher's gestures, words and enaction and dropout physically or cognitively from the classroom learning.

Our design TFV-2

To address this problem for teaching/learning vectors (Barniol & Zavala, 2014; Dorier & Sierpinska, 2002; Hillel, 2000), we developed a multi-pronged design (including cognitive, practice and access factors) to make the dynamic nature of formal structures (geometrical ones in vectors) available to all students, particularly in the developing world.

Cognitive: A simulation system was developed, which allows by touching and manipulating them on any smartphone (Touchy Feely Vectors-2: TFV-2: bit.ly/tfv-2) using *embodied* interactions, for creating (by simple double-consecutive taps), adding (drag one vector over other and double-consecutive taps), and resolving (pinching away gesture) (Fig.1). This makes the formal entities tangible, dynamics visible, and the modes coherent. An earlier version of TFV-2 was tested to be promising in helping students understand and imagine the dynamics better (Karnam, Agrawal, Mishra, & Chandrasekharan, 2016; Karnam, Agrawal, & Chandrasekharan, 2018).

Practice: Teacher adoption and usage of digital resources in the classroom has not been smooth (Bingimlas, 2009). In our experience too, teachers were hesitant to use it in their teaching. To address the integration with teachers' current textbook-based practice, particularly their thinking about vectors and the sequencing of the lessons, we decided to design tasks that could be planned for organised classroom activities. As an exercise, a 'virtual lesson plan' (VLP) was created, in collaboration with teachers, where this manipulable system can be used to *extend* their existing teaching lesson plans.

Access: These VLPs were *embedded* in the textbooks through QR codes (Uluyol & Agca, 2012) augmenting the figures and tasks in the textbook, as well as teacher thinking and practices based on them. This makes the introduction of new artefacts into the classrooms smooth and less abrupt as it builds on the already existing artefacts (textbooks), without replacing them.
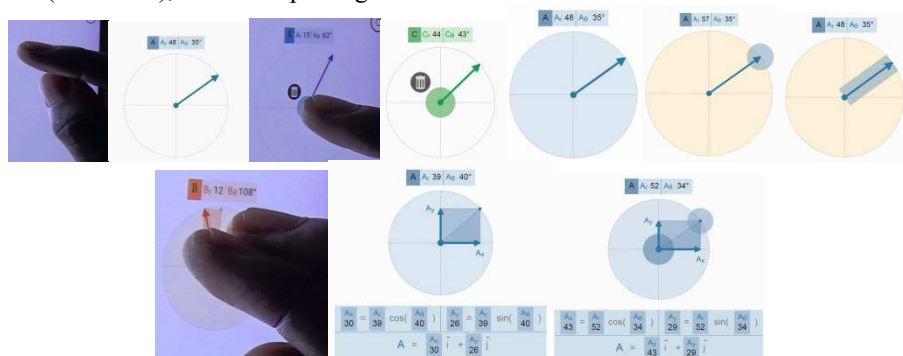

Figure 1. Gesture for creating vector, and for deleting a vector; An active vector, changing magnitude, direction (row-1); Pinching away gesture for resolution and gesture for reversing the resolution of vector (row-2).

We report a study involving classroom teaching using TFV-2 and the changes in teaching and learning practices, based on data from an ethnographic analysis of experimental and control classrooms.

The classroom study

To test whether this design actually meets the design requirements, we did a classroom study (grade 11), with three experimental classrooms where the integrated system was deployed, and three control classrooms where the standard textbook teaching method was used. We worked with 5 physics teachers (T1-T5), in 3 junior colleges (C1-C3) (pre-university level high schools) in Mumbai, India. All the teachers were trained postgraduates in Physics and had about 20 years of experience teaching at this level. Each school had multiple classrooms (divisions/sections) of grade 11. We observed two grade 11 classrooms (Control group - CG, and Experimental group - EG) in each college (a total of 6 classrooms). In CG, the teachers taught using the conventional method. In EG, the teacher used tablets with the TFV-2 deployed.

Table 1: Colleges and classrooms, and corresponding teacher codes. NCERT - national, MH - a provincial (Maharashtra) textbook. * T3 & T5 taught in C2 and C3 due to inter-school transfers (could not be controlled)

College (type, Curriculum)	Classroom	Teacher	No. of students
C1 (Private, MH)	EG1	T1	73
	CG1	T2	70-75
C2 (Public, MH)	EG2	T3	~49
	CG2	T3 & T5*	~ 41
C3 (Public, NCERT)	EG3	T4	45
	CG3	T5 & T3*	45

Methodology and data collection

See figure-2. A brief introduction to the research study and the TFV-2 system was given to the respective college managements and the teachers to get their consent for the study. To get some initial familiarity with the system, the teachers were requested to do some exploration of a few tasks (not mandatory) on an earlier (mouse-control) version of the vector system. All the teachers except T5 attended a Lesson Planning Workshop (LPW) for ~3 hours. In the workshop, the teachers based on their already existing lesson plans, together discussed and arrived at a virtual lesson plan with tasks on TFV-2, to be used in their teaching. We video recorded all the classroom sessions (both in CG and EG for 5-8 teaching sessions of 35/40 minutes each in different colleges). In CG classrooms, the teacher was given no inputs by the researchers. In EG classrooms, as per the VLP, a 1-2 page worksheet with pre-determined tasks (developed during the LPW) was prepared, depending on the topics the teacher planned to teach in the class (one worksheet per student). About 25-30 6" Android tablets (SWIPE)

with the necessary files and QR reading applications pre-installed were taken to the classrooms. The classroom was split into groups of 2-3 students (depending on the total strength) and each group was given one tablet. The tasks are available over the web and can be accessed directly. But the colleges didn't have reliable internet connectivity, so the necessary files were copied to each tablet, and they were accessed locally using QR codes generated for the file paths. In the EG session, the teacher led the class, discussing the day's topics. The students in the groups interacted with the tablet-based system, as directed by the teachers and facilitated by the researchers (fig.3). They also completed related worksheets. The researchers' roles were primarily logistics related to the tablets, technical support (for the teacher as well as the students), facilitating activity based on TFV, and recording the classroom proceedings. After about 1-3 weeks, students in all the 6 classrooms were given a questionnaire, which included questions that sought to captured students conceptual understanding and reasoning approaches in the context of vectors.
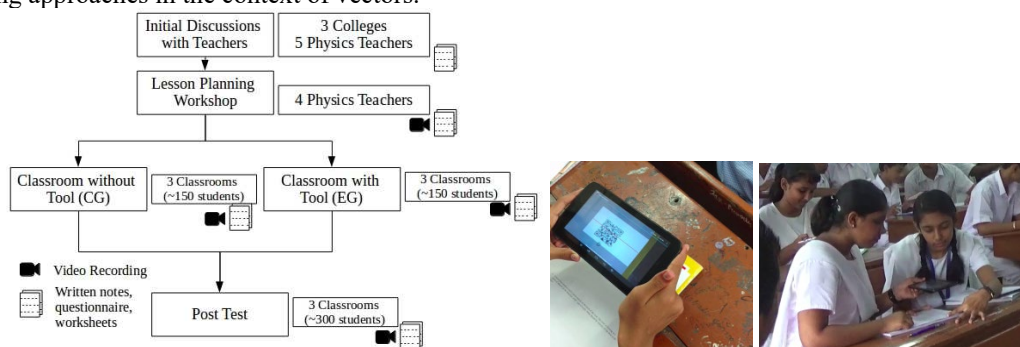

Figure 2. The flow of the study and the sample. Figure 3. Students scanning the QR code in an EG Classroom.

Analysis and findings

Given the space constraints, we confine to reporting here the analysis of qualitative changes, both in students' learning (particularly imagination) and teachers' teaching practices in the classroom setting. The main data sources for the current analysis were video recordings, which were analysed to understand the differences in: classroom dynamics (CG and EG), interactions between teachers and the students, and interactions with various artefacts (textbook, blackboard, notebooks and worksheets, the tablet). For this, the video recordings were analyzed using two broad lenses – differences in learning, particularly imagination, and differences in teacher/teaching related practices. This is not a rigid and complete framework (like situated frames of instrumental genesis (Artigue, 2002) or socio-mathematical norms (Yackel & Cobb, 1996) etc) yet, but, we expect that iteratively useful holistic frames could emerge to analyse the new mixed media based classroom artefacts perhaps enhancing the existing frames incorporating embodied indicators of imagination like gestures.

Learning practices

The main differences in the learning process, particularly related to imagination, between the CG and EG classrooms were in terms of the use of gestures, trajectories, collaboration and flow of learning.

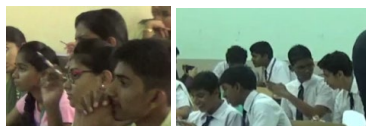#### Gestures: Active and meaningful engagement with content

Figure 4. Student in CG using gestures to remotely access the content on the board (left). Student in EG using gestures to explain to the peer in the context of an activity on TFV (right).

As one of our key design objectives was to help students imagine the vector in geometric terms, we examined the gestures students made in the two classrooms, as these indicate geometric thinking (Kim, Roth, & Thom, 2011). In the CG classroom, students were passive spectators most of the time, except for a few episodes where they took note of what was on the blackboard. Activity, when rarely present, was limited to nodding in agreement or tracing pens over the diagrams seen on the board (fig. 4 left). Gestures of this kind suggest a very limited engagement with the geometric content of vectors. The teacher's enaction of the textbook figures, which is the sole source for triggering the imagination of dynamic operations, is not resonated by the students. In the EG classrooms, more active and meaningful engagement with the content was observed. This was reflected in

more sustained gestures, along with discussions between students, which indicate conceptually deeper engagement with the geometric operations (fig. 4 right). These gestures were usually situated in conversation, where one student was actively trying to explain his imagination to another. The gestures indicate the possibility of students resonating with the teacher's writings (equations) or drawings (diagrams) and meaningfully engagement with content.

Multiple trajectories for learning

In the CG classroom, the learning opportunities were only when the teachers enacted a concept, and most of the activity in the classroom involved taking notes, which followed the teacher's description of the topic and solutions to problems. In the EG classroom, the exploratory and open-ended nature of tasks led to a surprising number of problem-solving trajectories. The diversity of approaches indicate the high imagination and learning potential provided by the TFV-2. To illustrate, in one of the tasks, students had to create two vectors, whose resultant is a given vector with magnitude 60 and a direction of 40°. The following approaches were seen.

- Trial and error (the most common approach, where students manipulate the magnitude and direction of vectors randomly). Some students found some patterns of change as they kept interacting.
- Estimating the rectangular components along x and y-axes and creating them as the two vectors. However, these students struggled when asked to create another set of vectors. Here the students explicitly applied the interconnections between resolution into rectangular components and addition.
- Creating two vectors with magnitude 30 at 20° from the x-axis and then adding to find resultant of magnitude 60 at 20° instead of at 40°. This combination provides the opportunity for students to realise that magnitudes and directions (represented as angles) cannot be algebraically added directly.
- Creating the vector of magnitude 60 at 40° from the x-axis, and arguing that the other vector is a zero vector. This particular group of students tried out this task for a long time using trial and error method, and eventually applied the idea of zero vector quiet intelligently.

Collaboration

In the CG classroom, there is very limited scope for students to engage in discussions with peers. The only conversations recorded between students involved asking for an extra pen or pencil, or to show some part of his/her notes for clarification, or giggles with playing (disinterested in the content of the lesson).

In the EG classroom, the very nature of the lesson plan involved making groups of 2 or 3 students, who worked together on the tablet. This inherently made the class collaborative. In addition, there were many episodes (similar to fig.5) where students naturally started interacting across groups. The nature of collaboration was grounded in the tasks they were performing on the tablets. Collaboration is not necessarily an indicator of an effective classroom, as there could be a lot of social and cognitive loafing (O'Donnell & O'Kelly, 1994), which could have happened in the EG classroom as well. However, as the students were involved in the collaborative activity using the TFV-2, the teacher could monitor student progress, to start discussions.

Figure 5. Students in EGs collaborating within their designated groups as well as beyond their groups.

Control on the flow of learning

In CG classrooms, most of the time was spent on the teacher delivering content in the lecturing mode. The teacher thus had full control over the flow of knowledge and learning in the classroom. There was very little scope for students to intervene in the flow and build their own knowledge. Further, the teacher decided when students could take notes, when to listen to him/her, which book was to be taken, who should respond to questions etc. This authority stance is central to the conventional classroom in developing countries, and this is required to some extent for smooth classroom management. However, this power structure often makes students intimidated, and they thus seldom interact with the teacher. Students tend to overcome this fear using chorus responses. The teacher wielded further control by appreciating (saying words like very good, interesting) or dismissing (ignoring or not attending to) student responses.

EG students actively participated in the enaction process along with the teacher, and this transferred agency to the students, who had more power in directing the flow of learning than in a normal classroom. This was reflected in the free interactions among students, the change in the nature of teacher-student interactions,

and also students' emotional connections with the tasks. The students were active participants with the teacher, and their interaction with the content was active and meaningful (see fig.6) as can be seen in the nature of gestures used. Based on these interactions, along with questions and discussions with the teacher, they controlled the flow of learning, and hence constructed their own knowledge. The teacher was no longer the sole controller of the flow of knowledge. Students generated knowledge during their interactions with their peers mediated by artfeacts like TFV-2, and knowledge flowed horizontally through collaborative work. Moreover, as suggested by the teachers, if tasks could be designed for students to do at home (thus flipping the classroom), the flow of learning could become even more student-centred.
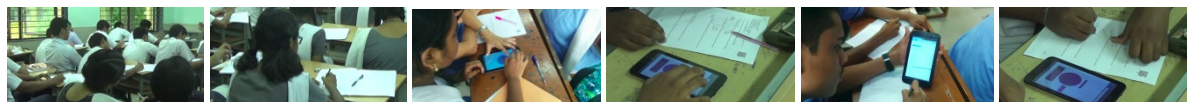

Figure 6. Students in a CG taking notes in their notebooks (1,2) Students in EG interacting with the tablet (3,4) and taking note of it in the worksheet (5,6).

Other observations

The CG classrooms were very calm, and there were never any emotionally charged moments. In contrast, in the EG classroom had a charged feel, and there were numerous moments of excitement and disappointment. There were multiple instances where EG students asked for similar systems for other topics in the curriculum. Students were very excited to see the tablets on the first day. By day 3 and 4, they started engaging with the TFV-2 in a more serious way and were not just excited due to the presence of the tablet.

Teacher/Teaching practices

The introduction of the TFV-2 into the classrooms was not fully and easily embraced by the teachers, though there were changes that indicated possible integration eventually. Here we discuss some of the central differences (and lack of changes) captured in the video data. We start with a discussion of practice elements resistant to change.

Resistant teacher practice

Three out of 5 EG teachers (T1,T2,T5) used the same diagram for teaching parallelogram law of vector addition, where the angle is less acute, even though the vectors used in triangle law, shown simultaneously on the board, were quite different (See fig.7). This drawing closely follows the textbook one and indicates the deep and subtle influence of the textbooks, where the teachers are conditioned by the textbook representation to such an extent that the diagrams seep into their practice without explicit awareness. While the TFV-2 has changed the way teachers present the content (as discussed in later subsections), there were such subtle aspects that were still driven by the textbook representations. We present this result first to indicate how deeply the textbook is integrated into teaching practice. This conditioning by textbooks extended to the way content was presented. In the EG classroom, where teachers had the possibility of widening the scope of discussions (like emphasizing and discussing the patterns of changes in the algebraic expressions through geometric manipulations, or the nature of the x and y components being interconnected by the circle). Such topics were not initiated by the teacher, and the scope of the class was still within the limits of the topics presented in the textbook. This reaffirms the conditioning role played by textbooks in setting classroom practices.
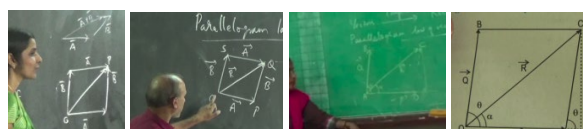

Figure 7. Teachers making parallelograms very similar to the one given in their textbook (right) with almost 90°angle while teaching parallelogram law of vector addition.

In the CG classroom, the teaching narrative usually took the form of a lecture, where the teacher introduced the definition or the statement of a law, explained it using some examples, and then drew diagrams. The teacher used some gestures to enact the diagram. The students took written notes based on these practice elements. When some extra time was available before the end of the session, the teachers solved some numerical problems. The flow of the lesson was very linear and monotonic. In the EG classroom, the narrative was similar in the beginning (introduction to the definitions or statements, making diagrams). It then changed to describe the tasks the students needed to do using the system. But once students started interacting with TFV-2, this was

altered. Interestingly, there was a tendency for both CG and EG teachers to follow a set question and answer template for discussion (especially when stating the definitions or the laws of additions), as this helps students in standard assessment. In later discussions, T1, who has taught in an EG class, noted that despite the teachers being given the flexibility to use new modes of teaching, there is a lot of emphasis on examination results and time limitations. This emphasis comes from the way the education system in India is structured around exams, which restricts the possible narratives in the classroom.

Practice elements that changed

A central change was the nature of teacher talk in EG classrooms, from mere description of static diagrams (like one arrow and another arrow forming two sides of a triangle, and the third side being the resultant, in triangle law of vector addition), to a more dynamic enaction, where teachers described the addition process. For instance, the teacher used the gestures in the TFV-2 to explain the process of adding two vectors saying "*now you know how to take vectors, just bring this vector (2nd vector) and attach here (head of 1st vector), then you will have to press your thumb (at the tail of 1st vector) and other finger here (head of 2nd vector), you will get the resultant*". Teachers in both CG and EG tended to use many gestures and bodily actions when explaining content using diagrams. However, there were differences in the gestures used by CG and EG teachers. In CG, teachers tended to use iconic gestures (like drawing an arrow in the air or using an opened palm to represent a vector) which are linked to diagrams on the board. Some were metaphorical (Roth, 2001) gestures (such as hand movements to show the splitting of vectors during resolution). In the EG classroom, teachers used similar gestures as a CG teacher during lecturing. However, they also incorporated gestures used in the TFV-2. For example, T4 and T3 used gestures influenced by the system when explaining triangle law of vector addition (see quote above). These gestures signify a dynamatisation of the vector operations, based on interactions in TFV-2 .

Figure 8. (Left) Teacher in CG using iconic gestures to show a vector, which is still a diagram. (Right) Teacher in EG using a picking action, reflecting that the vector is a touchable entity.

In the CG classroom, the teacher stayed near the blackboard (as seen in many figures). As the TFV-2 allowed all EG students to enact vector operations, the teacher did not need to demonstrate the enaction in the class, and she thus had a lot of time available to move around in the class and engage in discussions with the students. The role of the TFV-2 in generating this behaviour is clear from the contrasting behaviour of T3 (only teacher present in both EG and CG) in the CG and EG classrooms. The discussions involved clarifying conceptual doubts and questions. This allowed the teacher to also get real-time feedback about students' understanding of vector concepts, and initiate discussion with individual groups or the entire classroom.

Figure 9. (Left and Center) Teachers engaging in discussion with student groups in EG. (Right) Teacher showing a group's work on the tablet to the entire classroom (a potential space for initiating a discussion).
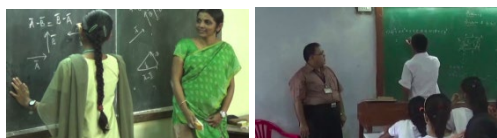

Figure 10. Teachers in CG calling students to the board to answer or solve some questions.

Usually, teachers ask questions to check for understanding (Rosenshine, 1983), or when students are not attentive. In the CG classroom, the monologue of the teacher was often interspersed by questions, where the teacher explained a concept or a problem and asked one of the students to stand or come to the board and solve or answer a similar problem (Fig.10). There were some such episodes in the CG classroom, where the teacher called a student to perform subtraction of two vectors, or derivation of expression of parallelogram law by simple substitution, or to solve a numerical problem similar to the one the teacher had just solved. Irrespective

of the purpose of asking the question, they were close-ended questions with a unique answer.

In contrast, in the EG classroom, the teacher spoke to the entire classroom only in the beginning. The rest of the time was spent answering questions from students. The nature of the questions was open-ended (with no single answer) and exploratory, following the exploratory nature of TFV tasks. Besides the above changes, a few other episodes indicate significant changes in teacher thinking and practice, induced by the TFV-2.

As teachers started discussing possible lesson plans during LPW, they immediately suggested that we could design tasks for students to do at home (as they get more time to engage with the TFV-2) and then discuss their findings and responses in the classroom next day, which was aking to flip-classrooms. After one of the lessons, T3 said - "besides helping the students in visualising and understanding the concepts, it (TVF-2) has clarified and made imagination easier for topics for me too". This shows how systems like TFV-2 could also help teachers understand complex concepts better, and also help develop pedagogical content knowledge (Shulman, 1986), particularly given the acute shortage of good teacher trainers and programs in the developing world. The teachers highly appreciated the TFV-2 when it was first shown to them. As discussed, some of the teachers were not comfortable using it in their teaching. However, after a few classes based on teaching using the system, the teachers became more comfortable and confident. Teachers have repeatedly requested extensions to the TFV-2, particularly to include 3D vectors, and also products of vectors, as well as waves. This is because it is difficult to draw and explain these using the blackboard. A physics teacher who was not part of the study sat through an entire classroom session, to see how the system was being used by students. Also, T5, who could not be part of the LPW, and who taught only in the CG classroom, repeatedly requested deployment of the system in the CG classroom. These indicate a shift to a motivation to use technology from an initial hesitation in teachers.

Discussion

The TFV-2 provides a new model to address the problem of augmenting science and mathematics teaching and learning practices, particularly in the developing world. The focus on smoothly extending textbooks and related practices, rather than replacing them, makes our approach different from other similar approaches to integrating multiple external representations (Bodemer, Ploetzner, Feuerlein, & Spada, 2004), such as GeoGebra. Also, the high scaling possibilities provided by the teacher-driven QR-based augmentation of textbooks offer a new approach to address key resource limitations in developing world classrooms, in terms of both technology access and training costs. The QR-based approach also provides a way for educational research institutions to curate good learning resources (such as videos and problem banks), and include these periodically as annotations to current textbooks, without waiting for the lengthy and bureaucratic process of textbook revision.

We could see a transformation of the classroom space for the students and situations and practices for their learning emerged in them smoothly. Among teachers, we see they could adapt the technological intervention smoothly, but the changes in their practices were far subtler, as we see indicators of them still anchored to textbooks. It is here, a key insight emerges about the central role textbooks play in directing teacher practice, and how building on the textbook could lead to rapid changes in teaching practices, particularly in terms of the way teachers direct/control the flow of knowledge in the classroom. The augmentation of textbooks could also help in swiftly upgrading teacher thinking and pedagogical knowledge of science and mathematics concepts, which is a critical issue in the developing world. Such an upgrade of pedagogical content knowledge (Van Driel, Verloop, & de Vos, 1998) could be better facilitated by QR augmentations that support flipping the classroom, as this feature would raise the number of potential questions from students, and in turn, motivate teachers to come prepared to answer these questions.

The narrow scope of our vector application, which seeks to just seed imagination of geometry, is different from most other interactive systems for learning vectors (such as PHeT), which are scoped more widely. This scoping makes such systems unsuitable while trying to extend teachers' current textbook-based practices, which tend to be very modular. The touchable and manipulable vector also potentially creates a new and embodied way of understanding vectors and related concepts for learners, as has been argued in the case of touchable numbers (Sinclair & de Freitas, 2014). The embodied interaction with vectors also makes our approach different from augmented reality approaches, which focus more on the visual component.

More broadly, our mixed media design stance -- based on an integration of cognitive, practice, and access factors -- treats textbooks as a very useful resource for advancing technology-based solutions for education, particularly in the developing world. This approach seeks to create a bridge from current practice to future practice, where digital platforms such as NetLogo could support the learning of cognitive skills not supported by textbooks, such as computational thinking (Dickes, Sengupta, Farris, & Basu, 2016).

However, we take a more careful design approach towards this shift, analysing the positive role textbooks could play, through their ability to promote working memory based operations such as mental rotation and modelling, particularly by *limiting* dynamic interaction. We believe this reflective stance, and the specific

design instantiation of this stance outlined here provides a way to reevaluate key assumptions underlying educational technologies for learning science and mathematics.

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