Supporting Students' Reasoning with Inscriptions

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Abstract: In this paper we look at how students can be supported to reason with a mathematical inscription system. We do so by analyzing several episodes from a classroom design experiment that centered on supporting middle school students' understanding of proportional relations in the context of measurement. We illustrate how the teacher orchestrated whole-class conversations that built on students' diverse ways of reasoning and how she used an inscription system to revoice their contributions. We explain that such efforts made it possible for students to start communicating and reasoning mathematically with the inscriptions.

The contention that mathematical inscriptions play a key role in the development of students' mathematical reasoning is widely accepted. Both Piagetian (e.g., Kaput, 1994; P. W. Thompson, 1992) and socio-cultural (e. g., Dörfler, 1993; van Oers, 1996) accounts of mathematical learning recognize inscriptions as paramount. In this paper, we contribute to the understanding of how students can be supported to reason with mathematical inscription-systems in a classroom setting. We do so by analyzing several episodes from a classroom design experiment that centered on supporting middle school students' understanding of proportional relations in the context of measurement. We illustrate how an inscription system became constituted—with the proactive support of the teacher in whole-class conversations—as a pivotal tool for mathematical communication and reasoning within the classroom community.

Our interest in symbols derives from our work as classroom researchers who are concerned with issues related to instructional design. The perspective we adopt towards *symbol use* (Nemirovsky, 1994) is consistent with the basic Vygotskian insight that students' use of symbols profoundly influences both the process of their mathematical development and its products, increasingly sophisticated mathematical ways of knowing (Dörfler, 1993; Kaput, 1994; Meira, 1995; van Oers, 1996). This perspective contrasts with instructional design approaches that view inscriptions as external representations which students internalize as they develop mathematical understandings. Instead, in the view we adopt, symbol use is considered an integral aspect of a mathematical practice (Cobb, 2002), one that is constituted in the social interactions of a classroom community. Previous analyses have shown how a focus on symbol use like the one we adopt can serve to trace the evolution of mathematical meaning within a classroom across extended periods of time (cf. Cobb, 2002; Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997). The purpose of this analysis is to further the agenda by focusing on how an inscription system can become constituted within a classroom community as a tool for mathematical communication and reasoning.

In this paper, we start by briefly describing the classroom design experiment and then clarify the methodological and instructional orientations of the study. Next we present an analysis of several classroom episodes that took place during the first instructional session. These episodes illustrate how the use of an inscription system became constituted in the classroom as the teacher orchestrated whole-class conversations. We conclude by explaining how the bottom-up instructional approach followed by the teacher supported the students in coming to view the inscription system as a collective resource for mathematical communication and reasoning.

Setting and Data

The classroom design experiment involved 10 teaching sessions conducted with a group of 7th grade students at an urban public school in the southeast United States. The school served an ethnically diverse population. In the case of the 10 participating students, five were Caucasian, four African Americans, and one

Indian American. The teaching sessions took place twice a week during the students' activity period (i.e. the last period of the school day). A member of the research team assumed the role of the classroom teacher and was assisted on numerous occasions by two additional researchers who also participated in planning the instruction. During the classroom design experiment, the research team tested and revised a conjectured learning trajectory (Simon, 1995) aimed at supporting middle-school students' understanding of proportional relations in the context of measurement (cf. Freudenthal, 1983; Simon & Blume, 1996; A. G. Thompson & Thompson, 1996). Central to this conjectured trajectory was the use of an inscription system (Cobb, in press; McClain, 2002; Stephan, Cobb, Gravemeijer, & Estes, 2001)based on a double-scaled number line (Gravemeijer, 1998; Streefland,1984).

As we began our development efforts, we drew on our previous research to frame the design of the inscription system. We were guided by two requirements that are not easy to reconciliate (Cobb et al., 1997). On the one hand, the inscriptions needed to be experienced by the students as providing a means of faithfully representing their ways of thinking. When this is not the case, it becomes difficult for students to appropriate them in building new understandings. On the other hand, the inscriptions needed to afford the possibility of representing rather sophisticated mathematical ideas, or else the teacher would not be able to use them to address key aspects of her mathematical agenda. For this reason, the tools were designed as means of support for the teacher, not carriers of meaning (Kaput, 1994; Meira, 1995, 1998; van Oers, 1996).

The data for our analysis consist of videotape recordings from two cameras, one focused on the teacher and the other on the students. In addition, a set of field notes and copies of all student work are part of the data corpus.

Analytical Framework

Our analysis is guided by the emergent perspective (Cobb & Yackel, 1996). The emergent perspective involves coordinating constructivist analyses of individual students' activities and meanings with an analysis of the communal mathematical practices in which they occur. This framework was developed out of attempts to coordinate individual students' mathematical development with the evolving social processes in which they participate in order to account for learning in the social context of the classroom. It therefore places the students' and teacher's activity in social context by explicitly coordinating sociological and psychological perspectives.

Method

The approach followed for analyzing the whole class data generated during the design experiment involves a method described by Cobb and Whitenack (1996) for analyzing sets of classroom data. This method is an adaptation of Glaser and Strauss' (1967) constant comparative method. The initial orientation for a retrospective analysis is provided by the tentative and eminently revisable conjectures that are developed both prior to and while actually conducting the classroom design experiment. The method involves continually testing and revising conjectures while working through the data chronologically. This constant comparison of conjectures with data results in the formulation of claims or assertions that span the data set but yet remain empirically grounded in the details of specific episodes (cf. Cobb, McClain, & Gravemeijer, 2003). Though in this paper we only make reference to events that took place during the first instructional session, we look at them with hindsight, taking into account the retrospective analysis of the entire data corpus.

Our interest in proportional reasoning was motivated by previous experiences with middle grades students in which they had difficulty making reasonable mathematical interpretations of data represented by ratios (e.g., "price per ounce," "income per capita," etc.). Given the importance of phenomena that are commonly organized into ratios in science and social studies, we considered developing the instructional approach an important aspect of middle-school mathematics.

In designing and conducting the instructional sessions, we followed a bottom-up instructional approach (Cobb et al., 1997) that centered on the orchestration of whole-class conversations (McClain, 2002). The main goal when following this approach is to capitalize on the diversity of students' reasoning, as it is reflected in their contributions to whole class discussions, so that significant mathematical ideas can become—in students' eyes—sensible topics of conversation. This approach requires careful planning that involves anticipating students' diverse ways of reasoning, and developing conjectures about how to capitalize upon them as they become public during whole class conversations. The image that results is that of a teacher adjusting her interventions as she pursues an instructional agenda, based on an ongoing interpretation of classroom events.

Results of Analysis

In this section, we analyze several episodes that took place during the first instructional session. Our purpose is to illustrate how the teacher made use of an inscription system as she introduced a problem to the class, revoiced students' mathematical contributions, and addressed salient mathematical issues during the whole class conversation. We also describe how students started to appropriate the inscription system introduced by the teacher as they reasoned about the problem at hand and made contributions to the ongoing conversation.

Supporting Productive Interpretation of the Mathematical Problem

The teacher's instructional goal for the first session was to engage students in thinking about the proportional accumulation of two quantities (A. G. Thompson & Thompson, 1996). At the beginning of the first session, the teacher introduced a problem that focused on the proportional relationship between miles driven and gallons of gas consumed. She explained that normally she would drive 204 miles a week using 9 gallons of gasoline, but because of a new schedule starting the next week she would be driving about 300 miles. She wanted to know how much gasoline her car would consume weekly on this new schedule. While explaining the problem, the teacher drew the double-scaled number line on the board (Gravemeijer, 1998; Streefland, 1984) as shown in figure 1.

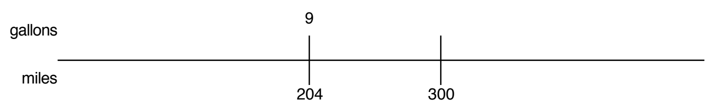

Figure 1. The double-scaled number line

The teacher introduced the double-scaled number line to represent the proportional relations between distance and amount of gasoline. Her intent was that the line would become constituted as a record (of a narrative) of the event that was taking place for the students (i.e., a car covering distance). The vertical lines were intended to represent the endpoints of the two accumulations that were simultaneously occurring as the car was driven (i.e., driving distance and gasoline consumption). The teacher's expectation was that the students would come to view these outcomes as being proportionally equivalent. If this occurred, they would take it as self-evident that were the car to travel twice the distance as it did before, it would consume twice as much gasoline. In this sense, the distance between the vertical marks was proportional to the values they represented. For example, the mark 300 was about 3/2 as far from the origin of the line as the mark 204 because it represented a value that was about 3/2 as big (see Figure 1).

At the beginning of the discussion, there was no indication that the students considered the graph to be related to the problem at hand. One of the students, Megan, for example, proposed to use cross multiplication to solve the problem.

Megan: I think you can put it in fraction, say 9 over 204 equals…and then compared it to X over 300?
Teacher: What would that tell me?
Megan: Huh…huh…If you make it into a fraction,
Other students: Proportion…
Megan: Oh, proportion, proportion! If you make it into a proportion... If you make 9 over 204 and equals…or… I don't know, let's say equals, X over 300. You simplify the bottom and cross multiply…

Megan's approach was constituted as a reasonable strategy by the rest of the class as none of the other students voiced a different opinion and some of them tried to further elaborate on the method of carrying out a cross multiplication. This indicated to the teacher that the students did not view the double-scaled number line as a useful resource for reasoning about the problem at hand. Instead, they seemed to interpret the mathematical task to be that of using a known procedure (i.e., cross multiplication) to figure out the numerical result. Although this was a mathematically adequate way of dealing with the problem, it seemed that solving word problems was, for the students, a process of identifying and applying a previously taught algorithmic method. The teacher's and students' interpretations of the situation were therefore in conflict. From the teacher's perspective, the students' procedural interpretation of the activity did not support the development of reasoning about the proportional covariation of two quantities that was central to her mathematical agenda (i.e., the distance and the gasoline consumption). Thus, the teacher was faced with the challenge of making the mathematical issues that she wanted the students to reason about an explicit focus of discussion.

The double-scaled number line (see Figure 1) became an important resource for the teacher as she addressed this challenge. In an effort to support the students to reinterpret the problem situation, the teacher told them that she was interested in knowing what the approximate amount of gas was needed for a 300 mile drive before she conducted any precise calculations. She then guided a whole-class conversation that focused on generating this estimate, in the course of which she made frequent use of the double-scaled number line.

> *Teacher: I want to get a sense of what you think the answer is gonna be. How much gas you are going to use? We know when you drive that distance 204 miles* [motioning along the double-scaled number line to indicate a distance of 204 miles], *you watch the gas gauge you use 9 gallons and you then have to stop at the gas station and put 9 gallons in, right* [pointing to the numbers]*? Instead of driving that distance 204, you will be driving 300 miles. Do you think you will be using twice as much gas as you used before?*

The teacher explained the problem again with the intent to redefine the nature of the activity. While doing so, she developed a narrative by referring to a gas gauge moving and a gas refill. Her intent in doling so was to make the problem situation experientially real for the students and, therefore, to enable them to view the numbers involved as measured quantities. The teacher used the inscription to support the development of the narrative by purposefully tracing with her hand the distance from the origin of the line to the mark 9/204 (see Figure 1).

Many students indicated "no" to the teacher's question about whether she would need twice as much gas to drive a 300-mile distance, and Alice concluded by saying "It won't be twice as much, because you are not going twice the distance." The teacher then asked if the students thought the car would use the same amount of gasoline for both the distance of 204 miles and 300 miles, and another student responded:

> *Amanda: I think it will be more than 9 gallons because if it is 9 gallons for 204 miles, 300 miles is more than 204 miles, so it will probably be more than 9 gallons* [pointing to the graph when she mentioned 204 miles, 300 mils and 9 gallons].
>
> *Teacher: So you are going further. See I need some more gas in there* [pointing to the distance between the 204 and 300]*, right? Yet it is not twice as far so you are not going to use twice as much gas* [motioning with her arm back and forth along the number line]. *So you are going to figure out how much more gas you are going to use. Does that make sense? We don't need the exact answer. We can just estimate.*

Amanda's contribution was the first in the whole class discussion in which a student referred to the graph as she made a contribution. The teacher, by gesturing to the double-scaled number line as she recast Amanda's argument, legitimized Amanda's way of thinking and oriented the group to use the double-scaled number line as a means of communication. In doing so, the teacher attempted to help the students build an image of the two quantities (i.e. distance and gasoline) simultaneously accumulating in a proportional way based on the double-scaled number line. It was at this point in the episode that most of the students first seemed to view the double-scaled number line as signifying an actual event, and started to view the inscribed numbers as representing quantities that were relevant for them to solve the task as hand.

The episodes we have presented illustrate how the purpose of the activity was redefined within the group. It seemed that students started to view the activity as that of figuring out relations between real quantities rather than manipulating numbers by applying algorithms. To make this shift possible, the teacher continually used the double-scaled number line to develop the narrative of the actual event (i.e. a car covering distance) as she intervened in the ongoing whole-class conversation and revoiced students' contributions. As a result, students' interpretations of the double-scaled number line seemed to evolve, from seeing it as a graphical record of the numbers involved in a calculational problem, to a representation that could be used to think about quantitative relations involved in the situation at hand. As we explain below, during the whole-class conversation that followed, the students started to focus on the proportional relations between the accumulations of two quantities and did so by *reasoning with* the graph.

Reasoning with the Inscription System

The teacher's efforts to support the redefinition of the problem situation proved to be effective. As the session continued, the following exchange took place that indicates a shift in at least some of the students' ways of thinking about the covariation in the accumulations of two quantities.

> *Teacher: Ok, so we want to use this information* [pointing to the "9/204" mark] *to try to get this estimate* [pointing to the "300" mark]. *Right? Is that what you are saying. Ok. So does anybody want to try that? Ok, Megan.*
>
> *Megan: Ok,* [pointing from her desk at different parts of the graph] *since 300 is about 100 more than 204, we need to find out how many gallons of gas you would use for say 100 miles. So if you divide 204 by, in half, it's about 100. So you would divide 9 into half, which would be 4 and a half.*
>
> *Teacher: Ok*
>
> *Megan: And so I guess to get… it would be about 9 plus 4 and a half*

Megan's approach was to find correspondences based on the proportional relations between the driving distance and the gas consumption. Her approach can be considered a continuation of the type of reasoning supported by the activity of estimating. As she explained her thinking to the group, Megan publicly used the double-scaled number line as a resource to figure out relationships between different quantities. She seemed to reason that since half the distance required half the amount of gasoline, one could find out how much gasoline was needed for the additional 100 miles by halving the amount of gas needed for 200 miles.

Megan's argument was important to the teacher's mathematical agenda since it focused on the proportional relations of two quantities (i.e. distance and gas consumption) as they accumulated simultaneously. She indicated that she particularly valued Megan's reasoning by inscribing her approach on the double-scaled number line, adding the inscriptions "9 + 4.5 = 13.5" and "4.5/100" (see Figure 2). She then asked the class if they could explain "Megan's way."

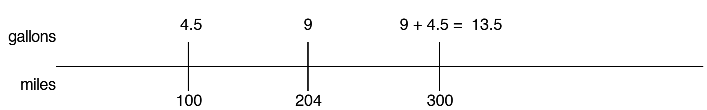

Figure 2. The evolving double-scaled number line

> *Nicolas: Ok, first she had to figure out how many miles she was using per100 miles. Right?*
>
> *Teacher: Ok I got a question for both of you guys* [Nicolas and Megan] *or anybody else. Why are you trying to find out about 100 miles* [pointing to the "100" in the "4.5/100" mark; see figure 2]*?*
>
> *Nicolas: Because it is about 100… 300 is close to 100 miles more, a little less than 100 miles, than 204 miles. So, ah, in order to find bout how much gas you'll be using with 300 miles, you have to find out what 100 miles more is.*
>
> *Teacher: Ok. So you are taking half of the 200* [indicating the distance between the "100" and the "204"] *and then you are just finding out what half the gas is* [indicating the distance between the "4.5" and the "9"]. *Because that's like an extra half* [indicating the distance between the "204" and the "300"]. *So if that's 200, an extra half would be 300* [indicating again the distance between "204" and the "300"]*, so that's got to be an extra half there* [indicating the distance between the "9" and the "13.5"]. *Is that kind of how you are thinking about it?*

By having students give explanations and by referring to the double-scaled number line as she recast aspects of students' contributions that were salient to her mathematical agenda (i.e., half the distance, half the gasoline), the teacher seemed to help the rest of the class make sense of both Megan's solution and of the definition of the problem situation on which it was based. This became evident as the session progressed. It should be clear from the episode that even though it was the teacher who was constantly developing the double-scaled number line to include more complicated inscriptions, these inscriptions signified students' contributions and were integral to the ongoing whole-class conversation.

The discussion continued as the teacher asked the students to estimate how much gasoline her car would consume if she drove 500 miles. While doing so, she marked "500" on the double-scaled number line. Rashied proposed to add the 9 gallons that corresponded to 204 miles, to the 13.5 gallons that corresponded to 300 miles. In this way, he said, one could find out how much gas the car would consume for 500 miles (i.e. 204 + 300 = 500 miles [approximately]; hence, 9 + 13.5 = 22.5 gallons). Rashied's approach resembled Megan's in that it was based on putting together "pieces" of known proportional correspondences (i.e. 9 gallons to 204 miles and 13.5 gallons to 300 miles) to figure out the unknown one (i.e. the gas needed for 500 miles). It is worth noting that the *known* proportional correspondences (such as 13.5 gallons and 300 miles) that were pivotal in Rashied's reasoning were those that had been marked on the double-scaled number line during the previous whole-class conversations.

After all the students indicated they had understood Rashied's way, Alice contributed to the conversation by proposing a different approach. Alice argued that because 500 miles was 5 times 100 miles, they could multiply 4.5 gallons by five to find out how much gasoline the car would consume for 500 miles (i.e. since 5 times 100 miles equals 500 miles, the car would consume 5 times 4.5 gallons of gas). Alice's approach was significant to the teacher's long-term mathematical agenda because it used a proportional correspondence as a multiplicand to find a larger correspondence. Similar to how she built on Megan's approach, the teacher asked other students to explain Alice's way and meanwhile, she added to the graph 5x100=500 and 5x4.5=22.5 in an effort to clarify this approach. In later sessions of the design experiment, the teacher capitalized on *normalized-ratio-like* contributions of this type (e.g., using a construct like "gallons per 100 miles" to find bigger correspondences) to address issues related to using a ratio such as *consumption per household* to measure phenomena that are dependent on the proportional relation between two quantities (e.g., the relative consumption of a commodity). In achieving her goals, it seemed to be critical that the students had come to view the double-scaled number line as a resource for reasoning and communicating mathematically about the situation at hand.

In this episode, the double-scaled number line seemed to become a taken-for-granted resource for the students to solve the mathematical tasks. It was used by the students to mathematize the problem not only to produce correct numerical results but more so, to contextualize the calculations and give rise to new ways of reasoning. As the students reasoned about the relationship between different quantities, they *reasoned with* the double-scaled number line. In other words, the double-scaled number line constituted a central aspect of the mathematical reality in which the students reasoned and acted. The emerging relationship between the students and the double-scaled number line might seem miraculous if one did not take into consideration the proactive efforts of the teacher throughout the session. As we may recall, by using and elaborating the double-scaled number line as she developed a narrative of an event with the students, the teacher helped her students (Megan for example) start to reason about quantities that were inscribed on the double-scaled number line. Additionally, the teacher used the graph to make different students' reasoning an explicit topic of the whole-class conversation and by doing so, she seemed to help students (Rashied for example) come to think about the problem in terms of proportional relations. Moreover, by documenting the students' diverse ways of reasoning, the teacher developed the double-scaled number line as a resource that they could use to build on each other's contribution. For instance, the teacher's inquiry about the "100 mile correspondence" and the inscription she made of it on the double-scaled number line seemed to lead Alice to develop her approach.

Conclusion

In this paper we have illustrated how students can be supported to use an inscription system as a resource for reasoning mathematically. The episodes we analyzed illustrate how, with the proactive support of the teacher, the double-scaled number line became constituted in the classroom as a resource for reasoning about the proportional covariation of two accumulations. The teacher's effectiveness involved supporting the redefinition of the instructional activity as a meaningful situation that involved dealing with quantities rather than merely doing something with numbers. It was crucial to this process that students came to view the inscriptions produced by the teacher as adequately representing the contributions they made.

Key to this success was the bottom-up instructional approach followed by the teacher. She engaged in instruction not with a fixed plan, but with an instructional agenda that allowed her to make adjustments based on her continual assessment of how students were interpreting the ongoing classroom events. From the students' perspective, it seemed to be their contributions that became topics of whole-class conversations, and their thinking that was inscribed on the double-scaled number line. This perception oriented them to use the inscription system as they reasoned about new situations.

Previous investigations have recognized the use of tools and symbols as integral aspects of mathematical reasoning (e.g. Dörfler, 1993; Kaput, 1994; van Oers, 1996). These works have shown how the use of symbols in a classroom can serve to trace the evolution of mathematical meaning. In this paper, we have illustrated how students can come to view an inscription system as a resource for reasoning mathematically. This can be possible when a skillful teacher pursues a mathematical agenda by using the system to meaningfully represent students' contributions to whole-class discussions about—what in their eyes are—significant issues.

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