

Designing for Playful Math Engagement Across Learning Environments

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Abstract: This symposium offers considerations for designing playful mathematics learning environments by synthesizing empirical research across four contexts that differ in location, formality, timescale, materials, facilitation, and participation structures. Despite the importance of play across the lifespan, mathematics classrooms are often spaces students feel a void of agency and enjoyment. This has strong implications for students' sense of competence and their identities as individuals who participate in and enjoy mathematics. Thus, designing playful environments where children enjoy and participate in mathematics as a form of meaning making is a critical issue for equity in education. By putting these four studies in conversation with one another, we hope to develop a refined understanding of how particular design decisions influence learner engagement in mathematics, with an eye towards making mathematical play more accessible both in and out of school.

Overview of symposium

The focus of this symposium is to empirically examine how aspects of in-school and out-of school-learning ecologies designed to afford joyful, child-driven engagement can support and expand children's participation in the domain of mathematics. In the ongoing climate of accountability and budget cuts, learners' experiences with mathematics are increasingly narrowed with an emphasis on ends over means that fails to affirm learner dignity (Espinoza & Vossoughi, 2014). In this symposium, we ask: *How do different aspects of learning ecologies (i.e., tasks and activities structures, participation structures, roles of adults) designed to afford playful engagement shift children's participation in mathematics?* The papers in this symposium work in concert to examine designs for learning that promote *playful engagement with mathematics* — a description of engagement we use to highlight the emphasis on eliciting and leveraging learners' curiosity, imagination, and enjoyment.

We draw from research on open engagement and the importance of enjoyment during learning to define playful engagement with mathematics. Learning environments that allow for open engagement tend to foster *disciplinary enjoyment* in ways that more heavily guided environments do not (Sengupta-Irving & Enyedy, 2015). Because the contexts in each of the studies are designed for open engagement and participants *appear* to be playfully engaged, we take *enjoyment* to be a common experience of participation in these contexts. Further, because of the playful nature of engagement, many participants in the featured contexts may not view their activity as mathematics even if their activity is mathematically rich. Indeed, in many ways, such playful contexts are inherently interdisciplinary as they involve children doing something typically not categorized as mathematics in service of mathematics learning (Fisher, Hirsh-Pasek, Golinkoff, Singer, & Berk, 2011; Sarama & Clements, 2009). Thus, we define playful engagement with mathematics as learners' exploration of structures and systems that are driven by curiosity, imagination, and enjoyment in activities that are not typically viewed as mathematics learning.

Mathematics is inherently creative and imaginative. Yet we rarely promote this image of what it means to do mathematics and be engaged in mathematics in classrooms. Collectively, our papers study children's playful mathematics activity across various contexts and timescales to a) address how different aspects of learning ecologies (i.e., tasks and activities structures, participation structures, roles of adults) designed to afford playful engagement shift children's participation in mathematics and b) to broaden what we count as mathematics. Paper 1 attends to how first and second grade teachers guide role play by pretending to be sea creatures (kelp, sea urchin, sea otter, plankton, etc.) living in an underwater kelp forest ecosystem. Analysis of the five-month long curriculum revealed how playful activities became meaningful sensemaking resources for students as they shifted their use of mathematical concepts to represent their understanding of the kelp forest ecosystem. Paper 2 examines a fifteen-day unit in a first-grade classroom that used children's imaginary resources of playing with rules and boundaries to participate in mathematical defining and construction. Moments where children's imaginary resources were positioned as central to the classroom's defining practice led to a repositioning of children's contributions as mathematically productive and a broadening of children's images of and construction of 3-D shapes beyond

prototypical examples. Paper 3 describes how middle school youth design and create woven artifacts in a school setting across 6 sessions. In these sessions, a facilitator supports youth as they are prompted by weaving activities to play, experiment, and explore with mathematical sequences and structures, supporting the playful reimagining of what math looks like. Paper 4 attends to parent-child interactions in an out-of-school space for family math-play. The data corpus of 345 children supports the development of a grounded theory of how adult family members can shape the nature of children's mathematics in free-play with materials designed to afford increasingly complex mathematical engagement.

We offer implications not only for design towards empowered, enjoyable mathematics learning but also for critiquing and extending visions of what mathematics is and could be across both formal and informal learning environments. Together, these papers attend to how adults shaped open learning environments in different contexts (in and out of schools), activities, and timescales, revealing the importance of playfulness as a design principle for engaging learners in broader and meaningful math learning. To facilitate a discussion that brings together the multiple ways these papers understand play and math participation, we bring in Dr. Andee Rubin whose expertise designing and studying mathematics education in both formal and informal contexts through an equity-seeking lens will offer unique insights and inspirations for future work.

Paper 1: Play as a sensemaking resource for engaging students in mathematics

Christine Lee

Research has shown open learning environments can elicit joyful sensemaking experiences that lead to a rich understanding of STEM concepts (Sengupta-Irving & Enyedy, 2015). If we want to make room for these types of learning environments, it is important that we design and create spaces that value learners' agency and affect (Horn, 2017; Jaber & Hammer, 2016; Keifert et al., 2017). Play is one such learning environment where a diversity of experiences and understandings can come together for students to engage as authentic participants in sensemaking experiences (Elkind, 2008; Hirsh-Pasek et al., 2009; Kernan, 2007). This study provides insight on how teacher-guided play, where teachers improvised and made room for student-led inquiry, led to a shift in students' use of mathematical concepts.

As a shared goal to understand how play can support agency, enjoyment, and learning, the three participating teachers co-designed and collaborated with the researchers to implement play-based learning into their classrooms. The study took place at a progressive elementary school in two mixed-age and dual-language classrooms (English and Spanish) of first and second grade students ($n = 51$ students). The larger data corpus of this study including video data, student work, and pre and post drawing assessments were collected across five months as students learned about the kelp forest. Students and teachers in the study pretended to play as various marine animals, algae, and plants (kelp, sea urchin, sea otter, plankton, etc.) and explored how each sea creature compared in size and shape from one another.

We draw from both Vygotsky (1978) and Fisher and colleagues (2011) to define play as when students dramatically explore and take on roles within an imaginary context. However, since these activities took place in school, teachers in our study often guided students' sensemaking opportunities by improvising and following student-led ideas as they spontaneously unfolded by 1) co-playing with students, 2) listening to students' ideas and sharing it with the class, giving students opportunities to regularly propose and test ideas, and 3) accepted various forms of participation as students pretended to be sea creatures, which lead to multiple interactions and investigations to simultaneously unfold.

In this paper, I ask how playful learning environments can become places where students build and draw on mathematical resources. I first analyzed students' pre (Figure 1a) and post drawings (Figure 1b) and found math learning gains in students' representations of a healthy ocean ecosystem. Most notably, students used spatial relations to organize their drawings in ways that reflected their participation in play. Using interaction analysis (Jordan & Henderson, 1995), I then analyzed 147 minutes of video data from two weeks of teacher-guided play that was captured by two cameras. I marked all moments where teachers improvised or playfully followed students' ideas and questions. Within these improvised and playful episodes, I analyzed teacher-student interactions as students organized and navigated play to make sense of their roles in relation to other characters in the kelp forest.

While playing in the underwater kelp forest ecosystem, students playing as kelp discovered the importance of size in relation to other sea creatures. Not only does kelp need to reach the top of the ocean water to be close to the sun (kelp harvests energy through photosynthesis), but it needs to be tall enough for sea creatures who are hiding from predators. We first saw this understanding of how and why kelp needed to be the tallest character in relation to other sea creatures unfold during play. These experiences and discoveries were later

reflected in students' drawings as they clearly communicated and organized the details of the size of kelp in relation to the water level, other sea creatures, and the sun (Figure 1b).

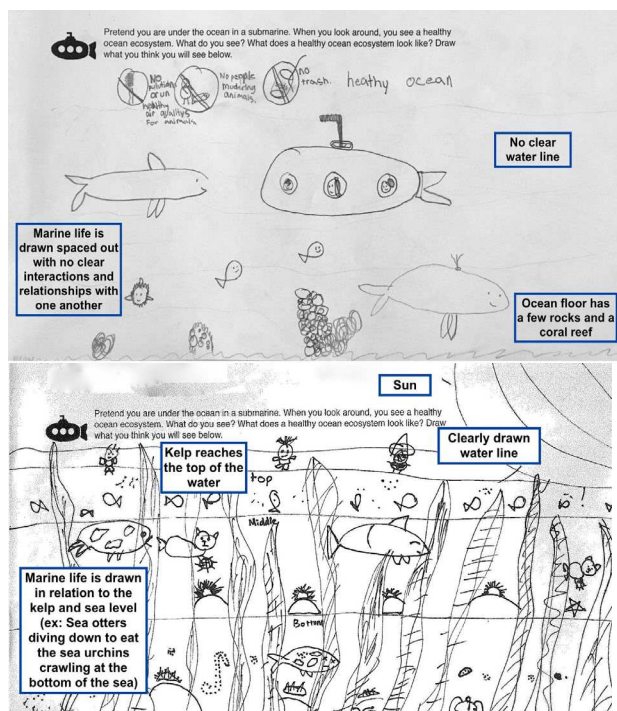


Figure 1. Student's pre-drawing (a) and post-drawing (b) of a healthy marine ecosystem.

Findings show that play created a memorable and meaningful experience for students as they imagined themselves as sea creatures varying in shape, size, movement, and roles. Becoming kelp meant being tall enough to be closer to the sun, while also providing shelter for other smaller animals. By playing and taking on the role of kelp, students had to coordinate, participate, and deeply think about how their size and movement would impact others. This playful approach of dramatically pretending to be sea creatures provided students with opportunities to deeply embrace the character they were playing as and to imagine and experience how their characters moved and behaved in space in relation to others in the kelp forest ecosystem. The importance of spatially orienting kelp as the tallest character in play then became a meaningful and memorable resource, one that students consistently felt the need to communicate and illustrate in their post drawings. In conclusion, designing playful experiences can welcome diverse and meaningful resources for students as they engage and make sense of math concepts, bringing attention to the need to expand on what we count as participation in mathematics and make room for play and enjoyment in math education.

Paper 2: When boundary testing and rule breaking leads to mathematical defining

Megan Wongkamalasai

Play has the potential to give young children opportunities to test roles and rules in imaginary situations (Vygotsky, 1967). In these play activities, children can engage with objects and tools, and attach their own meaning in ways that align with their imaginary scenarios (Vygotsky, 1978). This coordination of attaching actions and objects to meaning and goals is essential to what it means to learn and participate in practice, as opposed to learning how to follow the rules (Rouse, 2015). In order to engage in new and inclusive math practices, it is important to understand how to utilize play's unique ability to engage children in boundary testing as a meaning making activity that is central to participating in mathematical practices. In this paper, I explore how playful boundary testing or rule negotiation in a first-grade classroom can lead to new and productive practices of mathematical defining. Specifically, I analyze how one child's boundary testing, which often results in him being disciplined, get repositioned as a central role and change the perceived "challenging" behavior as mathematically productive. In my findings I address how the repositioning of children's playful boundary testing aligned with the

goals of mathematical defining occurred, and in what ways this repositioning led to developments in the classroom's collective defining practice.

This case-study comes from a longer design-study in a first-grade classroom where children are introduced to two new forms of mathematical practices, *defining* and *construction*, through 3-D construction play. During the fifteen days of instruction, children in this classroom built and analyzed 3-D structures using Magformers®. Instruction started with children's open construction of structures. Students then looked at their constructions and thought about different ways to group or classify their structures—open vs. closed, pointy vs. box-like. From this initial sorting activity, we further explored what makes a structure pointy and box-like and generated a mathematical definition for pyramids and prisms. Children engaged in analyzing a range of pyramids and prisms to generate a classroom definition for these two forms of polyhedra. Students then used their imagination to construct examples of each polyhedra that they had never seen before but that followed their definitions. These activities introduced a new form of mathematical construction that involves exploring variability within a system of constraints and led to the refinement of the classroom's definitions of pyramids and prisms.

Based on classroom observations and field notes, I identified the focal child, Tony, for further case analysis. Tony was routinely positioned in the classroom as being disruptive. His desk was separated from the other children, throughout the day he routinely got into power struggles with the teacher, and he was frequently sent out of the classroom. However, during the design-study, Tony's boundary testing during whole group discussions often led to developments in the classroom's definitions. This mathematical development was only after other children in the classroom laughed at Tony's contributions as silly or off topic. To understand how Tony's contributions got repositioned from silly to mathematically productive, I identified moments in classroom whole group discussions where Tony's contributions were met with laughter or strong disagreement. Next, I characterized what type of contribution Tony was making based on Kobiela and Lehrer's (2015) framework for aspects of defining. I then used interaction analysis (Jordan & Henderson, 1995) to understand how and when Tony's contributions were repositioned. In particular, these repositionings were dependent on a teacher taking up his contribution, which led me to focus on the different ways the teacher either took up or ignored his contributions and how the subsequent turns at talk resulted in a development in the classroom's definition practice.

Tony's contributions included analyzing and making relations between properties of structures, proposing of examples and non-examples of polyhedral, and bids to revise the classroom definitions for pyramids and prisms. To demonstrate how his contributions got positioned as central to the classroom's defining practice, I present one example of Tony's analysis of properties and describe how this contribution was repositioned by the teacher. As a first step towards defining prisms, the class was describing how two structures, a hexagonal and pentagonal prism, were similar. Tony takes one of the structures from the teacher's hands and proceeds to roll the structure across the floor and exclaims that they are the same because they both can roll. This creates a quiet hush in the classroom as the classroom teacher starts calling Tony out for interjecting, but the researcher steps in and asks students to describe what about the structures make both structures roll-able. From this taking-up of Tony's contribution, the class proceeds to describe how both structures have two matching (congruent) bases connected by squares (rectangles)—two defining properties of prisms. Here Tony's testing of the boundaries of what is allowable in the classroom, rolling math manipulative across the floor, not only leads to describing defining properties of prisms, but he also opens up new considerations about how motion can help describe geometric properties. In order for this repositioning to take place; however, the researcher had to have a way to recognize the mathematical fruitfulness of his contribution, but also know how to revoice his contribution as aligned with the emergent goals of defining.

Children in early elementary grades are often enculturated into a school of mathematics that is ritualistic and rule-governed. Rarely are young children involved in mathematics as a meaning making endeavor that requires understanding how objects and actions or procedures serve and derive their meaning from the goals of mathematical practice. This paper demonstrates how children's boundary testing can serve as a resource for developing mathematical practices. However, this requires activities and teaching that privilege and can recognize boundary testing as a central part of the doing of mathematics and not as a need for discipline.

Paper 3: Math engagement through playful weaving

Naomi Thompson

The constructionist and feminist lens of epistemological pluralism (Turkle & Papert, 1990), or validating diverse ways of thinking and knowing, can help educators and researchers consider more carefully how playful and diverse mathematical practices can be recognized and valued. In mathematics contexts, furthering an agenda of epistemological pluralism involves giving learners space to solve problems through playful and unconventional

strategies and tools, or even to question the nature of math as it is commonly understood (e.g., Gutiérrez, 2012). This inclusive approach has potential to reaffirm youth practices as valid, allowing diverse experiences to be valued in STEM spaces. Weaving has been shown to be highly mathematical in certain cultural contexts (e.g., Saxe & Gearhart, 1990), yet work has not explored the mathematical nature of weaving in schools.

Rules, an important factor in play spaces, feature prominently in playing with weaving designs as they provide boundaries for learners to both act within and push outward. These productive constraints (e.g., Thompson, Peppler, & Danish, 2017) create “double-binds” (e.g., Engeström, 1987), or mismatches between a current and a desired state, requiring learners to solve problems, make choices, and interact with the contexts in innovative ways. Providing space for learners to question rules may also work toward affirming student dignity in learning spaces (Espinoza & Vossoughi, 2014) as it positions student input as valuable and valid. The specific link between play, weaving, and emergent diverse mathematical practices is not yet known. I ask, how does playing through weaving support middle school youth in engaging with mathematical practices and principles?

I implemented and video/audio recorded a 6-session weaving unit with middle school youth in a project-focused school in the Midwestern United States. Thirteen youth signed up to do weaving activities with me during their “design studio” time. I prompted the youth to consider the mathematics in their weaving, but math content was not explicitly taught. I also periodically checked in with individual youth asking prompting questions such as “What are you working on? How did you solve this issue?” Additionally, I took photos of the woven and drawn artifacts produced in every session. I used techniques from mediated discourse analysis (Wohlwend, 2014) and artifact analysis (Pahl & Rowsell, 2010) to attend to math in youths’ actions, talk, and artifacts.

Findings show that youth became more adventurous with their designs and artifacts as the playful nature of the activities allowed them to explore the possibilities of the materials and structures. As youth progressed from their first to last projects, 30% either planned or implemented different “over, under” sequences, playing with the outcomes of different combinations. This is evidence of mathematical engagement as Common Core Math Practice Standard 8 (2010) calls for learners to “Look for and express regularity in repeated reasoning.” An example can be seen in Figure 2a where the youth adopted an “over two, under three” sequence that resulted in a horizontal chevron pattern. Additionally, 76% of youth played with shape, particularly pushing against the inherently gridded structure of weaving by building in curved lines and shapes. This is evidence of mathematical engagement as Common Core Math Practice Standard 7 (2010) calls for learners to “Look for and make use of structure.” An example of this can be seen in Figure 2b where the youth tried different methods to create a feeling of moving water. In fact, in both of these examples, the youth move above and beyond the math curriculum by inventing and experimenting, or playing, with the concepts and discovering new results.



Figure 2. Chevron pattern (a); Moving water (b); HRC design (c); HRC artifact (d).

In one case, a student named Marg showcased the way playing with the rules and structure of the materials led to mathematical practices. For Marg’s final project, she decided to recreate the Human Rights Campaign flag: a royal blue background with an embedded yellow equal sign in the center. She planned her design on grid paper, taking care to ensure the equal sign was centered and symmetrical. She also colored in every other square for the first two rows to indicate her plan to use an over one, under one pattern (Figure 2c). Yellow yarn was not available on Day 4. Marg worked on her blue rows up until the point she wanted to add the yellow equal sign and stopped so she could come back to it when the yellow thread was available. The yarn was still not available on Day 5, and Marg initially planned to read during the session. I encouraged her to think about another way she could continue with her project for the time being. Through discussion with me, Marg decided to continue weaving with the blue, skipping the area where the yellow would be and bringing the blue thread behind the strings in this area. This also helped solve Marg’s concern that there would be a gap in between the yellow and blue portions of the artifact. This strategy required Marg to work around the imagined shape of the equal sign, using mathematical resources such as parallel counting and her design plan as a guide. On Day 6, the yellow yarn was available, and Marg was able to complete the first half of her equal sign design (Figure 2d).

The playful and experimental approach to mathematics that was made possible through weaving allowed Marg and her classmates to follow their passions, experiment with mathematics principles and practices in

meaningful and interest-driven ways, and experience a new way of participating in mathematics. Through a lens of epistemological pluralism, we can recognize knowing and doing in mathematics in broader and more equitable ways. When we understand that math may look different and more playful for learners from different backgrounds, experiences, and epistemologies, we can begin to adjust our frameworks to better value youth's intellectual work.

Paper 4: Schoolitizing versus mathematizing: Family play and children's mathematical sensemaking

Lara Jasien

Research in mathematics education has well established that learner agency is paramount for sustaining engagement in disciplinary sensemaking (Boaler & Greeno, 2000; Engle & Conant, 2002; Gresalfi, Martin, Hand, & Greeno, 2009; Horn, 2017). Yet still today, students rarely have opportunities to exercise their agency in math class (Jacobs et al., 2006; Litke, 2015), let alone opportunities to engage in mathematical play. Given that much of children's waking hours are spent outside of formal educational contexts, it is time to study children's play — one of the highest agency activities (Elkind, 2008) — in out-of-school spaces designed to afford mathematical engagement. This research is timely as such high-agency, out-of-school mathematics spaces are becoming increasingly widespread and accessible (e.g., Danielson et al., n.d.; Hirsh Pasek et al., 2015).

This paper examines children's and families' mathematics in one such space — Math On-A-Stick (MOAS), an out-of-school context (at the USA's Minnesota State Fair) that centers children's need for play above the mathematics itself. In the year of data collection (2016), MOAS consisted of nine tables that families could move between freely. Each table contained unique materials designed for both aesthetic appeal (to invite exploration) and multiple affordances for mathematical engagement. For example, one table — the *Tiles and Patterns* exhibit — consisted of rectangular tiles composed of two Truchet Tiles (a canonical mathematical object) of black and colored triangles on one side and a solid paint on the other side. The variations in color afforded increasingly complex patterns as families sustained engagement over multiple iterations of patterning, thus attuning them to affordances of the tiles that were often unrealized in initial play. Another table, the *Pentagons* exhibit, contained wooden pentagons that made concrete the spiraling pentagons of the mathematical paper from Gailiunas (2000) and the problem of *What if there is another kind of tiling pentagon?* as in the mathematical paper from Lord (2016).

The data corpus of 345 children (aged 4–16) includes head-mounted GoPro videos of activity, surveys to collect both demographic information and Likert scale self-reports of school mathematics experiences, as well as semi-structured interviews about experiences at MOAS. Typical stay time at MOAS was between 20 and 30 minutes, and learner and family engagement at particular exhibits often lasted five minutes or less.

Notably, the timescale of playful engagement at MOAS was relatively short in comparison to how long we might expect children engage with a task in school, yet the children and families visiting this math space in their leisure time engaged meaningfully with the materials on their own terms. This study thus provides insight into how families might engage in mathematics playfully without expert guidance (such as at home) — an important undertaking for fostering increased equity in mathematics education (Banks et al., 2007). Even more, learning more about families' out-of-school mathematics is important not only for school-based instruction but also for the sake of supporting out-of-school learning itself (Ito et al., 2013). Indeed, learning outside of school is quite important for equity: A retrospective survey of STEM professionals found that 94 percent of those surveyed reported out-of-school experiences as influencing their decisions to pursue STEM careers (Jones, Taylor, & Forrester, 2011). Such experiences must be better understood and made accessible to all learners. This study contributes by carefully looking at how families interact at MOAS, attending to the nature of their mathematical engagement and playfulness and also providing design conjectures for playful mathematics learning environments.

Through purposeful selection of cases of adult-child mathematical engagement and subsequent iterative open coding of video (Strauss & Corbin, 1990), categories for the *types of mathematical practices* (e.g., patterning, defining, deconstructing to understand structure, etc.) and *who* was engaging in them (e.g., parent, child, multiple children, parent-child dyad, whole family, etc.) were developed and then condensed into two categories of *schoolitizing* and *mathematizing*. Schoolitizing happened when adults or children attempted to direct play towards content and skills heavily valued in school math (such as arithmetic); mathematizing happened when adults or children noticed and wondered about patterns and possibilities within children's use of play objects. In this way, mathematizing is conceptualized as a form of play as it involves “playing along” with the mathematical affordances of the materials as they emerge in children's activity. Schoolitizing tended to be accompanied by heavy adult directives while mathematizing tended to extend children's engagement and support their exploration.

By describing a spectrum of family engagement practices, the paper outlines (1) how spontaneous, well-meaning adult interventions result degrees of “schoolitizing” and “mathematizing,” and (2) how these two forms of intervention have differential outcomes for children’s mathematics in play. The paper concludes by offering principles for facilitating mathematical play and makes a call for extending typical notions inquiry to include an element of playfulness in order to support learner enjoyment-in and ownership-of mathematics.

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