THESIS DRAFT

YIMING SONG

Abstract.

Contents

1. Introduction

2. Towards Adams' theorem

In this chapter we give a brief overview of the important algebraic constructions used throughout the thesis. The reader should feel free to skip this the first two subsections if they are familiar with the theory of (co)associative (co)algebras. The main reference for the background material is from the first two chapters of [LV12]. We then state Adams' remarkable 1956 theorem [Ada56], which (roughly) says that given a based topological space (X, x), one can recover the singular chains of the based loop space $\Omega_x X$ via a purely algebraic construction on the singular chains on X. We conclude by highlighting some recent refinements of Adams' theorem, which will be used in the proof of Looijenga's theorem.

2.1. **Algebras.** In a first course in ring theory, one encounters, given a base ring R, the ring of polynomials $R[x]$. They come with an addition and multiplication determined by the ring operations of R. In other words, $R[x]$ is an *algebra* over R. In general, an *associative R-algebra* is a (possibly

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non-unital) ring A that is also an R-module, such that the ring product is R-bilinear. That is, for all $r \in R$ and $x, y \in A$:

$$r \cdot (xy) = (r \cdot x)y = x(r \cdot y).$$

Remark 2.1. In much literature, the ring product $A \times A \to A$ of an R-algebra A is instead written as a map $A \otimes_R A \to A$. By the universal property of tensor products these presentations are equivalent.

If A is unital, then the algebra A is also said to be *unital*. A *morphism of associative R-algebras* $f : A \to B$ is a map respecting both the R-module structure and ring structure, i.e. for $r \in R$ and $a, b \in A$, we require

$$r \cdot f(ab) = f(r \cdot ab) = r \cdot f(a)f(b).$$

An algebra morphism from R-algebra A to R itself is called an *augmentation*. In this case we say A is *augmented*. The simplest example of an R-algebra is R itself. Other examples include the aforementioned ring of polynomials $R[x]$ or square matrices of size n with entries in R.

2.1.1. *Group rings.* One important class of algebras are group rings. For a fixed ring R and a group G, the *group ring of G over R*, denoted $R[G]$, has elements finite formal linear combinations of elements in G with coefficients in R, with addition and multiplication given by

$$(\sum_{g \in G} r_g g) + (\sum_{g \in G} s_g g) = \sum_{g \in G} (r_g + s_g) g, \quad (\sum_{g \in G} r_g g)(\sum_{g \in G} s_g g) = \sum_{g \in G} \sum_{g_1 g_2 = g} (r_{g_1} s_{g_2}) g.$$

Then the action of R on $R[G]$ given by multipliying coefficients gives $R[G]$ the structure of an R-algebra. One should think of $R[G]$ as some sort of free module over R with basis G. Group rings abound in the representation theory of groups, where any representation $\rho : G \to \mathrm{GL}(V)$ of a group G over a k-vector space V corresponds to a module over the group ring $k[G]$. In this thesis, however, we do not need to take this perspective (unless we could?)

As an algebra, $R[G]$ comes with a natural augmentation, given by the map

$$\varepsilon : R[G] \to R, \quad \varepsilon(\sum_{g \in G} r_g g) = \sum_{g \in G} r_g.$$

We call its kernel the *augmentation ideal*. Clearly we have a splitting $R[G] \cong \ker \varepsilon \oplus R$. The augmentation ideal is an interesting object of study. For example, we have the following observation:

Proposition 2.2. *Let $\mathcal{I}$ be the augmentation ideal of the integral group ring $\mathbf{Z}[G]$. Then $\mathcal{I}/\mathcal{I}^2 \cong G^{\mathrm{ab}}$, the abelianzation of G.*

Proof. The proof relies on the following two facts. First, that $\{g - 1 : g \in G\}$ is a generating set of $\mathcal{I}$. Second, that the abelianzation of G is the quotient of G by its commutator subgroup $[G, G]$, which is generated by group elements of the form $g^{-1}h^{-1}gh$. Then we can define an explicit homomorphism

$$\mathcal{I}/\mathcal{I}^2 \to G^{\mathrm{ab}} = G/[G, G], \quad [g - 1] \mapsto [g]$$

and extending linearly to all of $\mathcal{I}/\mathcal{I}^2$. Then the inverse map is $[g] \to [g - 1]$. □

Given a path connected, based topological space with (X, x), consider its integral fundamental group ring $\mathbf{Z}\pi_1(X, x)$ and corresponding augmentation ideal $\mathcal{I}$. The previous proposition implies that

$$\mathcal{I}/\mathcal{I}^2 \cong \pi_1(X, x)^{\mathrm{ab}} \cong H_1(X; \mathbf{Z}).$$

This is the simplest case of the main theorem we are trying to prove. It hints at the fact that we can gain information about the fundamental group of the space by examining its homology.

2.1.2. *Differential graded algebras.* We begin with an example. Consider the singular chains $C_\bullet(X)$ on a topological space X. They are $\mathbf{N}$-graded, where each $C_n(X)$ is the free abelian group on the set of n-simplicies of X:

$$C_n(X) = \mathbf{Z}\{\text{continuous maps } \Delta^n \to X\}.$$

Another name given to $\mathbf{Z}$-modules is "abelian group," and indeed

$$C_\bullet = \bigoplus_{n\in\mathbf{N}} C_n(X)$$

is an abelian group. Next, we have a product

$$C_p(X) \times C_q(X) \to C_{p+q}(X)$$

given by the decomposition of the geometric $(p+q)$-simplex into a sum of p-simplices and q-simplicies (see []) for a reference. These maps assemble to a product

$$\times : C_\bullet(X) \times C_\bullet(X) \to C_\bullet(X).$$

One can check that this product is compatible with the $\mathbf{Z}$-module structure on $C_\bullet(X)$, making it into a *graded* $\mathbf{Z}$*-algebra*. As icing on the cake, we also have a singular boundary map $\partial : C_n(X) \to C_{n-1}(X)$ satisfying $\partial^2 = 0$ and the graded Leibniz rule: for $\sigma \in C_p(X)$ and $\tau \in C_q(X)$,

$$\partial(\sigma \times \tau) = (\partial\sigma) \times \tau + (-1)^p \sigma \times (\partial\tau).$$

Thus $C_\bullet(X)$ has the structure of a *differential graded* $\mathbf{Z}$*-algebra*. In general, a *differential graded R-algebra*, often abbreviated as a *dg algebra*, is a graded R-algebra $A_\bullet$ with a differential satisfying the graded Leibniz rule. If the differential lowers degree, we say $A_\bullet$ is *homologically graded*; if it raises degree, we say $A_\bullet$ is *cohomologically graded*. We can think of dg algebras as (co)chain complexes with a product, or algebras with a chain structure.

2.1.3. *Freeness and the tensor algebra.* Just as we can construct a free group from a set, we can also construct a free R-algebra given any R-module A. Before we given an explicit construction, we give a description of the *free associative algebra over A*, denoted $\mathcal{F}A$, in terms of its *universal property*:

> *There is an R-linear map $i : A \hookrightarrow \mathcal{F}A$ such that any R-algebra morphism $f : A \to B$ extends into a unique morphism $\tilde{f} : \mathcal{F}A \to B$ with $\tilde{f} \circ i = f$.*

This is entirely analogous to universal property of a free group. Sending an R-module A to the free associative algebra over A gives us a functor from the category of R-modules (supposing R is commutative), Mod_R to the category of associative R-algebras, denoted Alg_R. Conversely, we have a forgetful functor sending a R-algebra to its underlying module. In this language, the universal property tells us that these two functors are *adjoint*, i.e. there is a natural isomorphism

$$\hom_{\mathsf{Alg}_R}(\mathcal{F}A, B) \cong \hom_{\mathsf{Mod}_R}(A, \mathsf{forget}(B)).$$

This universal property guarantees that two manifestations of the free associative algebra are isomorphic via a unique isomorphism. With this in mind, let us now define the *tensor algebra* over an R-module A. Denoted $T(A)$, its underlying R-module is given by

$$T(A) := R \oplus A \oplus A^{\otimes 2} \oplus \cdots$$

and the product $T(A) \otimes T(A) \to T(A)$ is given by concatenation. On homogenous tensors $a \in A^{\otimes p}$ and $b \in A^{\otimes q}$, we have

$$(a_1 \cdots a_p) \otimes (b_1 \cdots b_q) \mapsto (a_1 \cdots a_p b_1 \cdots b_q) \in A^{\otimes p+q}.$$

This is a unital and associative R-algebra, with augmentation given by the identity on $R \subset T(A)$ and zero on higher tensor powers. The augmentation ideal, also called the *reduced tensor algebra* is denoted
$$\overline{A} = A \otimes A^{\otimes 2} \otimes \cdots$$
and is a nonunital associative R-algebra. The grading on $T(A)$ is given by tensor *length*, so a homogenous element $x \in A^{\otimes n}$ is said to have length n. Later in this thesis, we will be working with A a dg algebra, so the notions of grading can get confusing. But that is a problem for later.

Proposition 2.3. *$T(A)$ is the free associative algebra over A.*

Proof. Let $i : A \to T(A)$ be the inclusion into the second factor. Let $f : A \to B$ be an R-algebra morphism. Then $\tilde{f} : T(A) \to B$ can be defined by $\tilde{f}(1) = 1$ on $R \subset T(A)$, $\tilde{f}(x) = x$ for $x \in A \subset T(A)$, and $\tilde{f}(x_1 \cdots x_n) = f(x_1) \cdots f(x_n)$. Then $\tilde{f}$ extends f and is uniquely determined by f. We leave it to the reader to check that $\tilde{f}$ is an R-algebra morphism. □

2.2. **Coalgebras.** As suggested by the name, a *coalgebra* is the dual notion to that of an algebra. We could leave it at that, but they are much more unfamiliar objects; the maps don't go in the way we are used to. Moreover, the two objects are not dual on the nose: while the dual of every coalgebra is an algebra, the converse is not true without some finiteness assumptions.

Let us begin. A *coassociative R-coalgebra* is an R-module C with an R-linear map $\Delta : C \to C \otimes C$, called the *coproduct* which is coassociative, i.e. the following diagram commutes:
$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathrm{id} \otimes \Delta"] \\ C \otimes C \arrow[r, "\Delta \otimes \mathrm{id}"'] & C \otimes C \otimes C \end{tikzcd}$$
A *morphism of coassociative coalgebras* $f : C \to D$ is an R-linear map commuting with the coproduct, i.e.
$$(f \otimes f) \circ \Delta_C = \Delta_D \circ f.$$
Earlier, we could have defined a unital algebra to be one with a morphism from R. Dually, we say a coassociative R-coalgebra C is *counital* if there is a morphism $\epsilon : C \to R$ such that the following diagram commutes:
$$\begin{tikzcd} & C \arrow[dl, "\cong"'] \arrow[d, "\Delta"] \arrow[dr, "\cong"] & \\ R \otimes C & C \otimes C \arrow[l, "\epsilon \otimes \mathrm{id}"] \arrow[r, "\mathrm{id} \otimes \epsilon"'] & C \otimes R \end{tikzcd}.$$
The simplest example of a counital coassociative R-coalgebra is R itself, with the coproduct given by $1 \mapsto 1 \otimes 1$ and the counit given by $1 \mapsto 1$. We call a morphism $u : R \to C$ a *coaugmentation*, and in this case say that C is *coaugmented*. Beccause u is a morphism of coalgebras, it must commute with the counit maps, so we obtain that $\epsilon_C \circ u_C = \epsilon_R = \mathrm{id}_R$. It then follows that $C \cong \ker \epsilon \oplus R$, and we denote this kernel by $\overline{C}$.

- iterated coproducts

2.2.1. *Conilpotency, cofreeness, and the tensor coalgebra.*

2.2.2. *Differential graded coalgebras.*

2.2.3. *Singular chains.* Singular chains $C_\bullet(X)$ on a topological space X also admit the structure of a $\mathbf{Z}$-coalgebra (and enough things are compatible so this makes them into a *bialgebra*, but we don't need that).

2.3. **Adams' theorem.**

2.3.1. *Cobar as a bicomplex.*

2.3.2. *Refinements.*

3. Cosimplicial spaces

> *A cosimplicial object of a category* $\mathcal{C}$ *could be defined simply as a simplicial object of the opposite category* $\mathcal{C}^{\mathrm{op}}$*. This is not really how the human brain works...*
>
> —Stacks Project, 14.5 [Sta26]

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