THESIS DRAFT

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Abstract.

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1. Introduction

2. Coalgebras

As suggested by the name, a *coalgebra* is the dual notion to that of an algebra. We could leave it at that, but they are much more unfamiliar objects; the maps don't go in the way we are used to. Moreover, the two objects are not dual on the nose: while the dual of every coalgebra is an algebra, the converse is not true without some finiteness assumptions. The reader should feel free to skip this chapter if they are familiar with the theory of (co)associative (co)algebras.

2.1. First definitions.

Let R be a commutative ring.

Definition 2.1. A *coassociative R-coalgebra* is an R-module C with an R-linear map $\Delta : C \to C \otimes C$, called the *coproduct* (or *diagonalization*), such that the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathrm{id} \otimes \Delta"] \\ C \otimes C \arrow[r, "\Delta \otimes \mathrm{id}"'] & C \otimes C \otimes C \end{tikzcd}$$

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One way to think about the coproduct is that it tells one how to decompose a given element in the module. Consider the following example:

Example 2.2 ([JR79], §2)**.** Let $C = R[x]$, the ring of polynomials over R. Define a coproduct on a basis element x^n via
$$\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}.$$
Then...

Given an associative algebra A it is intuitive how to compose the product to obtain a map $A^{\otimes n} \to A$. With a coassociative coalgebra (C, Δ) there is an analogous notion. Define the *iterated coproduct* $\Delta^n : C \to C^{\otimes n+1}$ inductively with $\Delta^0 = \mathrm{id}$, $\Delta^1 = \Delta$, and
$$\Delta^n = \underbrace{\Delta \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{n \text{ operations}} \circ \Delta^{n-1}.$$
Coassociativity tells us that we could have inserted the coproduct anywhere within the above tensor product. A *morphism of coassociative coalgebras* $f : C \to D$ is an R-linear map commuting with the coproduct, i.e.
$$(f \otimes f) \circ \Delta_C = \Delta_D \circ f.$$
Just as a unital associative R-algebra A is one admitting a unital morphism $R \to A$, we say dually that a coassociative R-coalgebra C is *counital* if there is a morphism $\epsilon : C \to R$ such that the following diagram commutes:
$$\begin{tikzcd}
 & C \arrow[dl, "\cong"'] \arrow[d, "\Delta"] \arrow[dr, "\cong"] & \\
R \otimes C & C \otimes C \arrow[l, "\epsilon \otimes \mathrm{id}"] \arrow[r, "\mathrm{id} \otimes \epsilon"'] & C \otimes R
\end{tikzcd}.$$
The simplest example of a counital coassociative R-coalgebra is R itself, with the coproduct given by $1 \mapsto 1 \otimes 1$ and the counit given by $1 \mapsto 1$. We call a morphism $u : R \to C$ a *coaugmentation*, and in this case say that C is *coaugmented*. Beccause u is a morphism of coalgebras, it must commute with the counit maps, so we obtain that $\epsilon_C \circ u_C = \epsilon_R = \mathrm{id}_R$. It then follows that $C \cong \ker \epsilon \oplus R$, and we denote this kernel by $\overline{C}$ and call it the *reduced coalgebra*. We can think of $\overline{C}$ as either a submodule or a quotient of C.

2.2. **Conilpotency, cofreeness, and the tensor coalgebra.** Let (C, Δ) be a coaugmented coalgebra. Define the *coradical* (sometimes also called *canonical*) filtration on C as follows:
$$F_0 C = R, \quad F_r C = \{x \in \overline{C} : \bar{\Delta}^n(x) = 0 \text{ for } n \geq r\} \text{ for } r \geq 1.$$
Then we say C is *conilpotent* or *connected* if this filtration is exhaustive. Conilpotency is important is in the following definition.

Definition 2.3. The *cofree* coassociative R-coalgebra over a R-module M is a conilpotent coassociative coalgebra $\mathcal{F}^c M$ equipped with an R-linear map $s : \mathcal{F}^c M \to M$ sending 1 to 0 and satisfying the following universal property:

> Given any R-linear map $f : B \to M$ factors through $\mathcal{F}^c M$, i.e. there exists a unique map $\tilde{f} : B \to \mathcal{F}^c M$ such that $s \circ \tilde{f} = f$.

As with other objects defined via universal properties, the cofree coalgebra is unique up to unique isomorphism. In the categorial language we want this functor
$$\mathcal{F}^c : \mathsf{Mod}_R \to \mathsf{conilCoalg}_R$$

to be right adjoint to the forgetful function sending a conilpotent coalgebra to its underlying module. The reason we restrict ourselves to conilpotent coalgebras here is that the cofree objects are familiar, as we will soon see. In the general category of coalgebras they are large and unwieldy.

Definition 2.4. Let M be an R-module. The *tensor coalgebra over* M, denoted $T^c M$, is the coalgebra whose underlying module is

$$T^c M := R \oplus M \oplus M^{\otimes 2} \oplus \cdots$$

and whose coproduct $T^c M \to T^c M \otimes T^c M$ is given by

$$1 \mapsto 1 \otimes 1 \quad \text{and} \quad x_1 \otimes \cdots \otimes x_n \mapsto \sum_{i=0}^{n} (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_n).$$

For example, $x \in M$ gets mapped to $1 \otimes x + x \otimes 1$.

Proposition 2.5. *$T^c M$ is coassociative, counital, and conilpotent.*

Proof. Coassociativity... The counit is given by the map $T^c M \to R$ which is the identity on R and zero on the higher summands. Conilpotency... □

Remark 2.6. For an R-module M, the same tensor module $R \oplus M \oplus M^{\otimes 2} \oplus \cdots$ also has an algebra structure given by tensor multiplication. We denote it by TM rather than $T^c M$ to discern between the two structures. With this product it becomes the free associative R-algebra over M. In fact, the product and coproducts are compatible, making TM into an *bialgebra*, and in fact a *Hopf algebra*.

Proposition 2.7. *$T^c M$ is the cofree conilpotent coalgebra over M.*

Proof. Let $x \in T^c M$. □

2.3. **Differential graded coalgebras.**

Example 2.8. Singular chains on a space and the Alexander–Whitney product

3. Adams' cobar construction

In this chapter we define the cobar construction of a coalgebra and build up to Adams' remarkable 1956 theorem [Ada56], which (roughly) says that given a based topological space (X, x), one can recover the singular chains of the based loop space $\Omega_x X$ via a purely algebraic construction on the singular chains on X. We conclude by highlighting some recent refinements of Adams' theorem, which will be used in the proof of Looijenga's theorem.

3.1. **The statement.**

3.2. **Cobar as a bicomplex.**

3.3. **Refinements.**

4. Cosimplicial objects

A cosimplicial object of a category C could be defined simply as a simplicial object of the opposite category C^{op}. This is not really how the human brain works...

—Stacks Project, 14.5 [Sta26]

In this chapter we define cosimplicial objects, the totalization of a cosimplicial space, and provide some examples. We assume some familiarity with simplicial objects, for they are ubiquitous throughout algebraic topology.

4.1. **Definitions.** The *simplex category*, denoted $\mathbf{\Delta}$, is the category with

(1) objects: finite nonempty totally ordered sets. We write $[n]$ for the set $\{0 < 1 < \cdots < n\}$,
(2) morphisms: order-preserving maps, i.e. if $i \leq j$ then $f(i) \leq f(j)$.

Then a *cosimplicial object* in a category C is a functor $X^\bullet : \mathbf{\Delta} \to C$. We denote the image of $[n]$ under $X^\bullet$ by X^n. A morphism of two simplicial objects $X^\bullet, Y^\bullet : \mathbf{\Delta} \to C$ is a natural transformation of functors, i.e. morphisms $X^n \to Y^n$ for all n that commute with morphisms in $\mathbf{\Delta}$. We will denote the category of of cosimplicial objects in C by $\mathsf{cs}C$. In this thesis we will concern ourselves with the cases where $C = \mathsf{Top}$ or $C = \mathsf{Set}$.

Remark 4.1. Even though simplicial objects are *contravariant* functors, the convention is to denote the simplicies using subscripts, e.g. X_n. Conversely, even though simplicial spaces are *covariant* functors, the convention is to denote the cosimplicies with superscripts. Maybe the presence of the suffix co- explains this.

Recall that the morphisms in the simplex category are generated by two distinguished classes of maps. For $n \geq 1$ and $j \in [n]$, we have injections $\delta_j : [n-1] \to [n]$ where δ_j skips $j \in [n]$. For $n \geq 0$ and $j \in [n]$, we have $n+1$ surjections $\sigma_j : [n+1] \to [n]$ where σ_j sends both $j, j+1$ to $j \in [n]$. For a cosimplicial object $X^\bullet : \mathbf{\Delta} \to C$, we call the images of the δ_j *coface maps*, usually denoted d^j, and the images of the σ_j *codegeneracy maps*, usually denoted s^j. So to specify a cosimplicial object in C it also suffices to list a sequence of objects $X^n \in C$ for $n \geq 0$, as well as coface and codegeneracy maps satisfying the following *cosimplicial identities*:

(1) If $i < j$, then $d^j \circ d^i = d^i \circ d^{j-1}$.
(2) If $i < j$, then $s^j \circ d^i = d^i \circ s_{j-1}$.
(3) $\mathrm{id} = s^j \circ d^j = s^j \circ d^{j+1}$.
(4) If $i > j+1$, then $s^j \circ d^i = d^{i-1} \circ s^j$.
(5) If $i \leq j$, then $s^j \circ s^i = s^i \circ s^{j+1}$.

One should think of a cosimplicial object $X^\bullet : \mathbf{\Delta} \to C$ as a diagram

$$X^0 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; X^1 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; X^2 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \cdots$$

where the rightward pointing arrows are the coface maps and the leftward pointing arrows are the codegeneracy maps. In general it helps to think of the coface maps as "duplicating a coordinate" and the codegeneracy maps as "forgetting a coordinate." We will see this in the following examples.

Example 4.2 (The topological simplicies)**.** Define the functor $\Delta^\bullet : \mathbf{\Delta} \to \mathsf{Top}$ which sends $[n]$ to the topological n-simplex:

$$\Delta^n = \{(x_1, ..., x_n) \in \mathbf{R}^n : 0 \leq x_1 \leq \cdots \leq x_n \leq 1\}.$$

Then the coface maps $\Delta^{n-1} \to \Delta^n$ are the inclusions of faces, where d^j is the inclusion of the face opposite the jth vertex. The codegeneracy map s^j collapses the line joining the jth and jth vertex. In coordinates, we have

$$d^j(x_1, ..., x_n) = (x_1, ..., x_j, x_j, ..., x_{n-1}),$$

$$s^j(x_1, ..., x_n) = (x_1, ..., \widehat{x_j}, ..., x_n)$$

where $\widehat{\ }$ indicates omission.

Example 4.3 (Path spaces)**.** Let X be a topological space with $a, b \in X$. Define the cosimplicial space $P^\bullet_{a,b}X$ whose cosimplicies are

$$P^0_{a,b}X = \{*\}, \quad P^n_{a,b}X = X^n \text{ for } n \geq 1.$$

The coface maps $d^j : P^{n-1}_{a,b}X \to P^n_{a,b}$ are given by

$$d^j(x_1, \dots, x_{n-1}) = \begin{cases} (a, x_1, \dots, x_{n-1}) & j = 0 \\ (x_1, \dots, x_j, x_j, \dots, x_{n-1}) & j \in \{1, \dots, n-1\} \\ (x_1, \dots, x_{n-1}, b) & j = n \end{cases}$$

The codegeneracy maps $s^j : P^{n+1}_{a,b}X \to P^n_{a,b}X$ are given by

$$s^j(x_1, \dots, x_{n+1}) = (x_1, \dots, \widehat{x_{j+1}}, \dots x_{n+1}), \quad j \in \{0, \dots, n\}$$

where the $\widehat{}$ denotes omission.

We leave it to the reader to verify that the maps in both examples satisfy the cosimplicial identities. Why we refer to Example 4.3 by *path spaces* will become evident in the next section.

4.2. **Totalization.** Cosimplicial spaces provide a useful model for many types of topological spaces, including the based path and loop spaces. This is done via *totalization*, which is dual to the notion of geometric realization of a simplicial set.

Given a cosimplicial space $X^\bullet : \mathbf{\Delta} \to \mathsf{Top}$, define the *totalization* of $X^\bullet$ to be the space of maps from the cosimplicial simplices to $X^\bullet$:

$$\mathrm{Tot}(X^\bullet) := \mathrm{Hom}_{\mathsf{csTop}}(\Delta^\bullet, X^\bullet),$$

i.e. maps $f^n : \Delta^n \to X^n$ for all $n \geq 0$ that commute with the coface and codegeneracy maps. We topologize it as a subspace of $\prod_{n \geq 0} \mathrm{Hom}(\Delta^n, X^n)$ with the compact-open topology. Thus totalization gives us a functor from csTop to Top.

What seems to be happening here is that by iterating the face maps, we are creating finer and finer piecewise subdivisons of paths whose endpoints are at a and b. Indeed,

Proposition 4.4. $\mathrm{Tot}(P^\bullet_{a,b}X)$ *is homeomorphic to the path space* $\Omega_{a,b}X$ *of paths in* X *beginning at* a *and ending at* b.

Proof. A point of $\mathrm{Tot}(P^\bullet_{a,b}X) = \mathrm{hom}_{\mathsf{csTop}}(\Delta^\bullet, P^\bullet_{a,b}X)$ us a sequence of continuous maps

$$f = \{f_i : \Delta^i \to X^i\}_{i \geq 0}$$

commuting with the coface and codegeneracy maps. Fix $n \geq 2$ and $k \in \{1, \dots, n\}$. Consider the following composition of codegeneracy maps

$$\alpha_{n,k} := \underbrace{s^{n-1} \circ s^{n-2} \circ \cdots \circ s^{k-2} \circ s^k \circ \cdots \circ s^0}_{n-1 \text{ maps}}$$

where we compose all the degeneracies except for s^{k-1}. This gives us a map $\Delta^n \to \Delta^1$ and likewise for $X^n \to X^1$. Then for $f = \{f_0, f_1, \dots\} \in \mathrm{Tot}(P^\bullet_{a,b}X)$, we have by commutativity that

$$f_1 \circ \alpha_{n,k} = \alpha_{n+1,k} \circ f_n.$$

But now the right hand side is just picking out the kth coordinate of f_n. Hence for $n \geq 2$, f_n is completely determined by f_1, so that the projection $\Phi : \mathrm{Tot}(P^\bullet_{a,b}X) \to \mathrm{Map}(\Delta^1, X)$ given by $\{f_i\}_{i \geq 0} \mapsto f_1$ is injective.

We claim next that Φ is actually a map into $\Omega_{a,b}X \subset \mathrm{Map}(\Delta^1, X)$. The cosimplicial relations imply

$$f_1 \circ d^0 = d^0 \circ f_0 = \mathrm{const}_a, \quad f_1 \circ d^1 = d^1 \circ f_0 = \mathrm{const}_b$$

so $f_1(0) = a$ and $f_1(1) = b$ as desired.

Lastly we define an inverse to $\Phi : \mathrm{Tot}(P^\bullet_{a,b}X) \to \Omega_{a,b}X$. For a given path $\gamma \in \Omega_{a,b}X$ consider the family of maps $\{f_i : \Delta^i \to X^i\}_{i\geq 0}$ given as follows. We let f^0 be the constant map, $f^1 = \gamma$, and for $n \geq 2$ define

$$f_n(x_1, ..., x_n) = (\gamma(x_1), \gamma(x_1 + x_2), ..., \gamma(x_1 + \cdots + x_n)).$$

Clearly this is an inverse to Φ. We leave it to the reader to check that the family of maps $\{f_i\}$ commutes with the coface and codegeneracies, and that Φ and its inverse are continuous maps. □

We will call $P^\bullet_{a,b}X$ a cosimplicial model for the path space of X. When $a = b$ we get a cosimplicial model for the based loop space of X. It turns out that the cosimplicial space $P^\bullet_{x,x}X$ is in some sense the underlying cosimplicial set of the cobar construction of $C_\bullet(X)$, as we will see in the course of this thesis.

4.3. **Dold–Kan and homotopy.** "... there is no interesting homotopy theory of cosimplicial sets!" concludes a MathOverflow answer [hg] of Tom Goodwillie. In this subchapter we will try to understand this comment. Recall that the standard Dold–Kan correspondence gives an equivalence of categories between simplicial abelian groups and (non-negative) chain complexes of abelian groups. We'll restate some important results and their duals. Although these are duals, there is once again some asymmetry, as suggested by the quote.

Definition 4.5. Let A be a simplicial abelian group. Its *normalized chain complex* $(NA_\bullet, \partial)$ has

$$NA_n = \bigcap_{i=1}^{n} \ker(d^i : A_n \to A_{n-1}), \quad \partial = d_0 : NA_n \to NA_{n-1}$$

and its *Moore complex* $(MA_\bullet, \partial')$ has

$$MA_n = A_n, \quad \partial' = \sum_{i=0}^{n} (-1)^i d_i : A_n \to A_{n-1}.$$

The following two results and their proofs can be found in [GJ09], Chapter III.2:

Theorem 4.6 (Dold–Kan correspondence)**.** *The functor $N : \mathsf{sAb} \to \mathsf{Ch}_+(\mathsf{Ab})$ is an equivalence of categories between simplicial abelian groups and non-negative chain complexes of abelian groups.*

Theorem 4.7. *The inclusion of chain complexes $NA \hookrightarrow MA$ is a chain homotopy equivalence, natural in A.*

Now we dualize. Recall that for simplicial objects, the face maps d_i lower degree. For cosimplicial objects, the face maps d^i raise degree, so we should get an equivalence between cosimplicial abelian groups and cochain complexes of abelian groups. This is indeed what happens.

Definition 4.8. Let $C^\bullet$ be a cosimplicial abelian group. Its normalized cochain complex $(NC^\bullet, \partial)$ has

$$NC^n = \mathrm{coker} \bigoplus_{i=1}^{n} (d^i : C^{n-1} \to C^n), \quad \partial = d^0 : NC^n \to NC^{n+1}$$

and its Moore complex has

$$MC^n = C^n, \quad \partial' = \sum_{i=0}^{n+1} (-1)^i d^i : C^n \to C^{n+1}.$$

And analogously we obtain the following two theorems:

Theorem 4.9 (Dold–Kan correspondence, dual)**.** *The functor $N : \mathsf{csAb} \to \mathsf{coCh}_+(\mathsf{Ab})$ is an equivalence of categories between cosimplicial abelian groups and non-negative cochain complexes of abelian groups.*

Theorem 4.10. *The quotient of cochain complexes $MC \to NC$ is a cochain homotopy equivalence, natural in C.*

Really we could have stated the previous four theorems for simplicial objects in any abelian category. In the following discussion, we replace "abelian group" with "R-module" for some commutative ring R. Consider the free R-module functor $R[-] : \mathsf{Set} \to \mathsf{Mod}_R$. Then given any cosimplicial set $X^\bullet : \mathbf{\Delta} \to \mathsf{Set}$ the composition of functors $R[X^\bullet]$ yields a cosimplicial R-module.

Proposition 4.11. *Under the Dold–Kan correspondence, $R[X^\bullet]$ has zero cohomology in positive degree.*

Proof. Is this true??? □

Proposition 4.12. *Let $P^\bullet_{a,b}X$ be the cosimplicial path space. For fixed $p \geq 0$, define the cosimplicial abelian group $C_p(F^\bullet_{a,b}X)$ with*

$$C_p(F^q_{a,b}X) := \mathbf{Z}[\mathrm{Map}(\Delta^p, P^q_{a,b}X)]$$

i.e. take singular p-chains on the qth cosimplex of $P^\bullet_{a,b}X$. Under the cosimplicial Dold–Kan correspondence, the associated cochain complex has cohomology

$$H^* = \begin{cases} \mathbf{Z} & * = 0, a = b \\ 0 & \textit{otherwise} \end{cases}$$

Proof. We have the following order of operations. We begin with our cosimplicial space $P^\bullet_{a,b}X$. Then we build the cosimplicial set

$$[q] \mapsto \mathrm{Map}(\Delta^p, P^q_{a,b}X) = \mathrm{Map}(\Delta^p, X^q)$$

Then passing to the free abelian group and applying the Moore complex functor, we obtain the cochain complex (which we choose to write here in terms of the free abelian groups rather than using the reduced notation $C_p(X^q)$):

$$\mathbf{Z}\,\mathrm{Map}(\Delta^p, *) \xrightarrow{\partial_0} \mathbf{Z}\,\mathrm{Map}(\Delta^p, X) \xrightarrow{\partial_1} \mathbf{Z}\,\mathrm{Map}(\Delta^p, X^2) \to \cdots$$

where ∂_i is the map defined on the basis via

$$\mathbf{Z}\,\mathrm{Map}(\Delta^p, X^q) \to \mathbf{Z}\,\mathrm{Map}(\Delta^p, X^{q+1}), \quad f \mapsto \sum_{i=0}^{q} (-1)^i d^i \circ f.$$

Consider the following family of maps:

$$h_q : \mathbf{Z}\,\mathrm{Map}(\Delta^p, X^q) \to \mathbf{Z}\,\mathrm{Map}(\Delta^p, X^{q-1}), \quad f \mapsto \begin{cases} s^0 \circ f \end{cases}$$

□

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