# THESIS DRAFT

YIMING SONG

Abstract.

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## 1. Introduction

## 2. (Co)algebras

In this chapter we

2.1. **Algebras.** In a first course in ring theory, one encounters, given a base ring $R$, the polynomials $R[x]$. They come with an addition and multiplication that are essentially determined by the ring operations of $R$. In other words, $R[x]$ is an *algebra* over $R$. In general, an *associative $R$-algebra* is a ring $A$ that is also an $R$-module, in such a way that the module action is compatible with ring multiplication of $A$. That is, for all $r \in R$ and $x, y \in A$:

$$r \cdot (xy) = (r \cdot x)y = x(r \cdot y).$$

One important class of algebras are group rings. For a fixed ring $R$ and a group $G$, the *group ring of $G$ over $R$*, denoted $R[G]$, has elements finite formal linear combinations of elements in $G$ with coefficients in $R$, with addition and multiplication given by

$$(\sum_{g \in G} r_g g) + (\sum_{g \in G} s_g g) = \sum_{g \in G} (r_g + s_g) g, \quad (\sum_{g \in G} r_g g)(\sum_{g \in G} s_g g) = \sum_{g \in G} \sum_{g_1 g_2 = g} (r_{g_1} s_{g_2}) g.$$

Then the action of $R$ on $R[G]$ given by multipliying coefficients gives $R[G]$ the structure of an $R$-algebra. One should think of $R[G]$ as a free module over $R$ with basis $G$.

Group rings abound in the representation theory of groups, where any representation $\rho : G \to \mathrm{GL}(V)$ of a group $G$ over a $k$-vector space $V$ corresponds to a module over the group ring $k[G]$.

Occuring far less in nature are coalgebras.

free tensor coalgebra cdga

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## References

[Loo25] Eduard Looijenga. On the motivic description of truncated fundamental group rings. *Journal of Topology and Analysis*, page 1–13, May 2025.