THESIS DRAFT

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Abstract.

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1. Introduction

1.1. **The theorem.** Let X be a path-connected topological space and choose basepoints $a, b \in X$. Let $\pi_X(a, b)$ be the set of homotopy classes of paths in X from a to b. In particular when $a = b$ we obtain the fundamental group of X. Notice that $\pi_1(X, a)$ acts on $\pi_X(a, b)$ on the right by concatenation of paths, so that the free abelian group $\mathbf{Z}\pi_X(a, b)$ is a right $\mathbf{Z}\pi_1(X)$-module. Let $\mathcal{I}_a \subset \mathbf{Z}\pi_1(X, a)$ be the augmentation ideal.

Consider now the following subspaces of X^n

$$X(n)^a := \{(x_1, \ldots, x_n) \in X^n : x_1 = a \text{ or } x_i = x_{i+1} \text{ for } i \in \{1, \ldots, n-1\}\}$$

$$X(n)_b := \{(x_1, \ldots, x_n) \in X^n : x_n = b \text{ or } x_i = x_{i+1} \text{ for } i \in \{1, \ldots, n-1\}\}$$

and let $X(n)^a_b := X(n)^a \cup X(n)_b$.

Theorem 1.1. *There is a map*

$$\mathbf{Z}\pi_X(a, b)/\mathbf{Z}\pi_X(a, b)\mathcal{I}_a^{n+1} \to H_n(X^n, X(n)^a_b; \mathbf{Z})$$

which is a surjection with kernel $\mathbf{Z}$ *if* $a = b$ *and an isomorphism if* $a \neq b$.

The author first encountered this theorem in a paper of Looijenga ([Loo25], Theorem 1.1), who attributes this theorem (or a dual version with $\mathbf{Q}$-coefficients) to Deligne and Goncharov, ([DG05], Proposition 3.4). In this paper, the authors attribute it to Beilinson.

1.2. **Outline.**

Date: February 22, 2026.

2. Coalgebras

As suggested by the name, a *coalgebra* is the dual notion to that of an algebra. We could leave it at that, but they are much more unfamiliar objects; the maps don't go in the way we are used to. Moreover, the two objects are not dual on the nose: while the dual of every coalgebra is an algebra, the converse is not true without some finiteness assumptions. The reader should feel free to skip this chapter if they are familiar with the theory of (co)associative (co)algebras.

2.1. **First definitions.** Let R be a commutative ring.

Definition 2.1. A *coassociative R-coalgebra* is an R-module C with an R-linear map $\Delta : C \to C \otimes C$, called the *coproduct* (or *diagonalization*), such that the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathrm{id} \otimes \Delta"] \\ C \otimes C \arrow[r, "\Delta \otimes \mathrm{id}"'] & C \otimes C \otimes C \end{tikzcd}$$

One way to think about the coproduct is that it tells one how to decompose a given element in the module. Consider the following example:

Example 2.2 ([JR79], §2)**.** Let $C = R[x]$, the ring of polynomials over R. Define a coproduct on a basis element x^n via

$$\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}.$$

Then...

Given an associative algebra A it is intuitive how to compose the product to obtain a map $A^{\otimes n} \to A$. With a coassociative coalgebra (C, Δ) there is an analogous notion. Define the *iterated coproduct* $\Delta^n : C \to C^{\otimes n+1}$ inductively with $\Delta^0 = \mathrm{id}$, $\Delta^1 = \Delta$, and

$$\Delta^n = \underbrace{\Delta \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{n \text{ operations}} \circ \Delta^{n-1}.$$

Coassociativity tells us that we could have inserted the coproduct anywhere within the above tensor product. A *morphism of coassociative coalgebras* $f : C \to D$ is an R-linear map commuting with the coproduct, i.e.

$$(f \otimes f) \circ \Delta_C = \Delta_D \circ f.$$

Just as a unital associative R-algebra A is one admitting a unital morphism $R \to A$, we say dually that a coassociative R-coalgebra C is *counital* if there is a morphism $\epsilon : C \to R$ such that the following diagram commutes:

$$\begin{tikzcd} & C \arrow[dl, "\cong"'] \arrow[d, "\Delta"] \arrow[dr, "\cong"] & \\ R \otimes C & C \otimes C \arrow[l, "\epsilon \otimes \mathrm{id}"] \arrow[r, "\mathrm{id} \otimes \epsilon"'] & C \otimes R \end{tikzcd}.$$

The simplest example of a counital coassociative R-coalgebra is R itself, with the coproduct given by $1 \mapsto 1 \otimes 1$ and the counit given by $1 \mapsto 1$. We call a morphism $\eta : R \to C$ a *coaugmentation*, and in this case say that C is *coaugmented*. Beccause η is a morphism of coalgebras, it must commute with the counit maps, so we obtain that $\epsilon_C \circ \eta_C = \epsilon_R = \mathrm{id}_R$. It then follows that $C \cong \ker \epsilon \oplus R$, and we denote this kernel by $\overline{C}$ and call it the *reduced coalgebra*. We can think of $\overline{C}$ as either a submodule or a quotient of C. The reduced coalgebra also has a coproduct $\overline{\Delta}$ given by

$$\overline{\Delta}(x) = \Delta(x) - 1 \otimes x - x \otimes 1.$$

2.2. **Conilpotency, cofreeness, and the tensor coalgebra.** Let (C, Δ) be a coaugmented coalgebra. Define the *coradical* (sometimes also called *canonical*) filtration on C as follows:

$$F_0C = R, \quad F_rC = \{x \in \overline{C} : \bar{\Delta}^n(x) = 0 \text{ for } n \geq r\} \text{ for } r \geq 1.$$

Then we say C is *conilpotent* or *connected* if this filtration is exhaustive. Conilpotency is important is in the following definition.

Definition 2.3. The *cofree* coassociative R-coalgebra over a R-module M is a conilpotent coassociative coalgebra $\mathcal{F}^c M$ equipped with an R-linear map $s : \mathcal{F}^c M \to M$ sending 1 to 0 and satisfying the following universal property:

> Given any R-linear map $f : B \to M$ factors through $\mathcal{F}^c M$, i.e. there exists a unique map $\tilde{f} : B \to \mathcal{F}^c M$ such that $s \circ \tilde{f} = f$.

As with other objects defined via universal properties, the cofree coalgebra is unique up to unique isomorphism. In the categorial language we want this functor

$$\mathcal{F}^c : \mathsf{Mod}_R \to \mathsf{conilCoalg}_R$$

to be right adjoint to the forgetful function sending a conilpotent coalgebra to its underlying module. The reason we restrict ourselves to conilpotent coalgebras here is that the cofree objects are familiar, as we will soon see. In the general category of coalgebras they are large and unwieldy.

Definition 2.4. Let M be an R-module. The *tensor coalgebra over M*, denoted $T^c M$, is the coalgebra whose underlying module is

$$T^c M := R \oplus M \oplus M^{\otimes 2} \oplus \cdots$$

and whose coproduct $T^c M \to T^c M \otimes T^c M$ is given by

$$1 \mapsto 1 \otimes 1 \quad \text{and} \quad x_1 \otimes \cdots \otimes x_n \mapsto \sum_{i=0}^{n} (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_n).$$

For example, $x \in M$ gets mapped to $1 \otimes x + x \otimes 1$.

Proposition 2.5. *$T^c M$ is coassociative, counital, and conilpotent.*

Proof. Coassociativity... The counit is given by the map $T^c M \to R$ which is the identity on R and zero on the higher summands. Conilpotency... □

Proposition 2.6. *Let M be an R-module. Any linear map $M \to TM$ extends uniquely to a derivation $TM \to TM$.*

Proof. This follows from the more general Proposition 1.1.8, [LV12]. □

Remark 2.7. For an R-module M, the same tensor module $R \oplus M \oplus M^{\otimes 2} \oplus \cdots$ also has an algebra structure given by tensor multiplication. We denote it by TM rather than $T^c M$ to discern between the two structures. With this product it becomes the free associative R-algebra over M. In fact, the product and coproducts are compatible, making TM into an *bialgebra*, and in fact a *Hopf algebra*.

Proposition 2.8. *$T^c M$ is the cofree conilpotent coalgebra over M.*

Proof. Let $x \in T^c M$. □

2.3. **Differential graded coalgebras.** A *differential graded* (often abbreviated to *dg*) coalgebra is a coassociative coalgebra (C, Δ) with

(1) a grading, i.e. $C = \oplus_{n \in \mathbf{N}} C_n$ such that the product sends C_n to $\oplus_{i+j=n} C_i \otimes C_j$, and
(2) a differential d sending C_i to C_{i-1}, satisfying the dual identity

$$\Delta \circ d = (d \otimes \mathrm{id} + \mathrm{id} \otimes d) \circ \Delta.$$

Similarly this implies that the coproduct is a morphism of chain complexes, and we can think of dga coalgebras as either (1) chain complexes with a compatible coproduct, or (2) coalgebras with compatible differential. For example, the tensor (co)algebra from before is graded over $\mathbf{N}$.

Given a dg R-coalgebra $C = \oplus_{n \in \mathbf{N}} C_n$, we can define its *suspension* sC to be C shifted up a degree, i.e. $(sV)_i = V_{i-1}$, and its *desuspension* $s^{-1}C$ to be C shifted down a degree, i.e. $(s^{-1}C)_j = C_{j+1}$. Then any $x \in C_n$ determines an element $sx \in (sC)_{n+1}$ and $s^{-1}x \in (s^{-1}C)_{n-1}$. Defining these turns out to be helpful for degree bookkeeping reasons.

Example 2.9. Consider the singular chains $(C_\bullet(X; \mathbf{Z}), \partial)$ on a topological space X with basepoint x. We claim that they form a coaugmented dg $\mathbf{Z}$-coalgebra (in fact also a bialgebra). For the coproduct, notice that we always have a diagonal map $x \mapsto (x, x)$, which induces a chain map $C_\bullet(X) \to C_\bullet(X \times X)$. Now is a natural quasi-isomorphism $C_\bullet(X) \otimes C_\bullet(X)$ (for more details see Section 4.4... this ordering is not good). $C_\bullet(X)$ is counital with the map $\epsilon : C_\bullet(X) \to \mathbf{Z}$ given by sending all 0-simplicies to 1 and has a coaugmentation $\eta : \mathbf{Z} \to C_\bullet(X)$ which sends 1 to the 0-simplex at x.

Example 2.10. The tensor (co)algebra

3. Adams' cobar construction

In this chapter we define the cobar construction of a coalgebra and build up to Adams' remarkable 1956 theorem [Ada56], which (roughly) says that given a based topological space (X, x), one can recover the singular chains of the based loop space $\Omega_x X$ via a purely algebraic construction on the singular chains on X. We conclude by highlighting some recent refinements of Adams' theorem, which will be used in the proof of Looijenga's theorem.

3.1. **The cobar construction.** Let R be a commutative unital ring and let $(C = \overline{C} \oplus R, \Delta)$ be a dg coaugmented counital coassociative R-coalgebra. The cobar construction is a functor from this category of coalgebras to the category of dg associative algebras. There are many references for the following definition, including Adams' original paper. We follow the modern presentation given in [Riv22].

Definition 3.1. Let (C, Δ, ∂) be as above. The *cobar construction* on C is the dg R-algebra whose underlying algebra is the tensor algebra over the desuspension of the reduced coalgebra C:

$$\mathsf{Cobar}(C) := T(s^{-1}\overline{C}) = R \oplus s^{-1}\overline{C} \oplus s^{-1}\overline{C}^{\otimes 2} \oplus \cdots$$

and whose differential is given by extending the linear map

$$-s^{-1} \circ \partial \circ s^{+1} + (s^{-1} \otimes s^{-1}) \circ \Delta \circ s^{+1} : s^{-1}\overline{C} \to s^{-1}\overline{C} \oplus s^{-1}\overline{C}^{\otimes 2}$$

to all of $T(s^{-1}\overline{C})$, which yields a linear map of degree -1 from $T(s^{-1}\overline{C})$ to itself.

The definition of the differential here is valid owing to Proposition 2.6, which says that it suffices to define the differential on the component $s^{-1}\overline{C} \subset T(s^{-1}\overline{C})$. Regardless, it will be helpful to be a little more explicit. By the discussion in the previous chapter we have that as a chain complex, $\mathsf{Cobar}(C)$ has terms

$$
\begin{aligned}
\mathsf{Cobar}_k(C) &= T_k(s^{-1}\overline{C}) \\
&= \bigoplus_{q\geq 0} (s^{-1}\overline{C})_k^{\otimes q} \\
&= \bigoplus_{q\geq 0} \bigoplus_{n_1+\cdots+n_q=k} (s^{-1}\overline{C})_{n_1}\otimes\cdots\otimes(s^{-1}\overline{C})_{n_q} \\
&= \bigoplus_{q\geq 0} \bigoplus_{n_1+\cdots+n_q=k} \overline{C}_{n_1+1}\otimes\cdots\otimes\overline{C}_{n_q+1} \\
&= \bigoplus_{q\geq 0} \bigoplus_{m_1+\cdots+m_q=k+q} \overline{C}_{m_1}\otimes\cdots\otimes\overline{C}_{m_q} \\
&= \bigoplus_{p-q=k} \bigoplus_{m_1+\cdots+m_q=p} \overline{C}_{m_1}\otimes\cdots\otimes\overline{C}_{m_q}
\end{aligned} \tag{3.2}
$$

The purpose of introducing the dummy variable p at the end of this tedious indexing exercise was to suggest that we should be able to realize the cobar construction as the totalization of a bicomplex in p and q. Indeed, in the cobar differential, we utilize both the internal differential ∂ of C as well as the coproduct Δ. One of these will turn out to raise q and the other will lower p.

Definition 3.3. Let (C, Δ, ∂) be as above. The *cobar bicomplex* is the bicomplex $\mathsf{Cobar}_\bullet^\bullet(C)$ with

$$\mathsf{Cobar}_p^q = \bigoplus_{m_1+\cdots+m_q=p} \overline{C}_{m_1}\otimes\cdots\otimes\overline{C}_{m_q}$$

and differentials

$$d_H : \mathsf{Cobar}_{p,q} \longrightarrow \mathsf{Cobar}_{p-1}^q, \quad x_1\otimes\cdots x_q \mapsto \sum_{i=1}^q (-1)^{\sigma(x_i)} x_1\otimes\cdots\otimes\partial(x_i)\otimes\cdots\otimes x_q$$

$$d_V : \mathsf{Cobar}_p^q \longrightarrow \mathsf{Cobar}_p^{q+1}, \quad x_1\otimes\cdots\otimes x_q \mapsto \sum_{i=1}^q (-1)^{\sigma(x_i)} x_1\otimes\cdots\otimes\overline{\Delta}(x_i)\otimes\cdots\otimes x_q$$

where the x_i are elements of $(s^{-1}\overline{C})_{n_i}$ and $\sigma(x_i)$ is the Koszul sign $\deg x_1 + \cdots + \deg x_{i-1}$.

Proposition 3.4. *The totalization of* $\mathsf{Cobar}_\bullet^\bullet(C)$ *given by*

$$\mathrm{Tot}_k(\mathsf{Cobar}_\bullet^\bullet) = \bigoplus_{p-q=k} \mathsf{Cobar}_p^q$$

with differential $\partial_{\mathrm{Tot}} = d_H + (-1)^q d_V$ *on* Cobar_p^q*, is isomorphic as a dg algebra to* $\mathsf{Cobar}(C_\bullet)$.

Proof. By the preceeding discussion and derivation, we have an equality of groups $\mathrm{Tot}_k(\mathsf{Cobar}_\bullet^\bullet)$ and $\mathsf{Cobar}_k(C)$. Now by Proposition 2.6, it suffices to check that ∂_{Tot} extends the map $s^{-1}\overline{C} \to s^{-1}\overline{C}\oplus s^{-1}\overline{C}^{\otimes 2}$ given in the definition of the cobar complex. Indeed, we have that on

$$s^{-1}\overline{C} = \bigoplus_{n_1=p\geq 0} (s^{-1}\overline{C})_{n_1} = \bigoplus_{p\geq 0} \mathsf{Cobar}_p^1,$$

the differential is $\partial_{\mathrm{Tot}} = d_H - d_V$ which is exactly the cobar differential. $\square$

What is miraculous is that this purely algebraic construction encodes a good amount of topological information. This is the content of Adams' theorem:

Theorem 3.5 ([Ada56])**.** *Let (X, x) be a simply connected based topological space. Let $\Omega_x X$ be the space of loops in X based at x. Then there is a natural isomorphism*

$$H_\bullet(\mathsf{Cobar}(C_\bullet(X))) \cong H_\bullet(\Omega_x X).$$

Thus we can regard the cobar construction as the algebraic analogue of the loop space functor. Some stuff about classifying spaces, adjointness, etc. Bar construction, etc

3.2. **Refinements.** The isomorphism of Adams' theorem actually comes from a chain-level quasi-isomorphism. Moreover, the assumption that X is simply connected can be removed.

Theorem 3.6 ([Riv22], Theorem 1)**.** *Let (X, x) be a path connected based topological space. Let $\Omega_x X$ be the space of loops in X based at x. Then $\mathsf{Cobar}(C_\bullet(X))$ is quasi-isomorphic to $C_\bullet(\Omega_x X)$ as dg algebras.*

Remark 3.7. We have stated the theorem for singular chains $C_\bullet(X)$ rather than some modified chains $C_\bullet(X, b)$ (not the relative chains!), but these are quasi-isomorphic?

Remark 3.8. Cobar not preserving quasi isomorphisms

Moreover, there is a canonical way in which the cobar construction is related to the augmentation ideal-adic truncations of the fundamental group ring:

Theorem 3.9 ([Gad23], Theorem 1.2(3))**.** *Let A be a PID, and Γ a group. Let $\mathcal{I}$ be the augemntation ideal of the group ring $A[\Gamma]$. Then we have the following isomorphism*

$$(\mathcal{I}^n)^\perp \cong H^0(\mathsf{Bar}_{<n}(C^\bullet(B\Gamma)); A)$$

where the left hand side is the orthogonal complement of $\mathcal{I}^n < A[\Gamma]$, and the left hand side is the zeroth cohomology of the subcoalgebra of the bar construction on the cochains of the classifying space $B\Gamma$, consisting of weight $< n$ tensors. □

Taking $A = \mathbf{Z}$, $\Gamma = \pi_1(X, x)$, and dualizing, we obtain the following:

Corollary 3.10. *There is an isomorphism of groups*

$$\mathbf{Z}\pi_1(X, x)/\mathcal{I}^{n+1} \cong H_0(\mathsf{Cobar}_{\leq n}(C_\bullet(B\pi_1(X, x))))$$

and consequently

$$\mathbf{Z}\pi_1(X, x)/\mathcal{I}^{n+1} \cong H_0(\mathsf{Cobar}_{\leq n}(C_\bullet(X))).$$

Proof. The first statement follows immediately from Gadish's theorem. The second statement follows since the zeroth homology of the cobar construction depends only on the fundamental group of a space. To see this, □

4. (Co)simplicial objects

> *A cosimplicial object of a category $\mathcal{C}$ could be defined simply as a simplicial object of the opposite category $\mathcal{C}^{\mathrm{op}}$. This is not really how the human brain works...*
>
> —Stacks Project, [Sta26, Tag 016I]

In this chapter we define cosimplicial objects, the totalization of a cosimplicial space, and provide some examples. We will state the Dold–Kan correspondence and use it to prove a useful statement for later on. Finally we conclude with an exposition of the Alexander–Whitney map, which couldn't find a home in any other chapter. We assume some familiarity with simplicial objects.

4.1. **Definitions.** The *simplex category*, denoted $\mathbf{\Delta}$, is the category with

(1) objects: finite nonempty totally ordered sets. We write $[n]$ for the set $\{0 < 1 < \cdots < n\}$,
(2) morphisms: order-preserving maps, i.e. if $i \leq j$ then $f(i) \leq f(j)$.

Then a *cosimplicial object* in a category C is a functor $X^\bullet : \mathbf{\Delta} \to C$. We denote the image of $[n]$ under $X^\bullet$ by X^n. A morphism of two simplicial objects $X^\bullet, Y^\bullet : \mathbf{\Delta} \to C$ is a natural transformation of functors, i.e. morphisms $X^n \to Y^n$ for all n that commute with morphisms in $\mathbf{\Delta}$. We will denote the category of of cosimplicial objects in C by $\mathsf{cs}C$. In this thesis we will concern ourselves with the cases where $C = \mathsf{Top}$ or $C = \mathsf{Set}$.

Remark 4.1. Even though simplicial objects are *contravariant* functors, the convention is to denote the simplicies using subscripts, e.g. X_n. Conversely, even though simplicial spaces are *covariant* functors, the convention is to denote the cosimplicies with superscripts. Maybe the presence of the suffix co- explains this.

Recall that the morphisms in the simplex category are generated by two distinguished classes of maps. For $n \geq 1$ and $j \in [n]$, we have injections $\delta_j : [n-1] \to [n]$ where δ_j skips $j \in [n]$. For $n \geq 0$ and $j \in [n]$, we have $n+1$ surjections $\sigma_j : [n+1] \to [n]$ where σ_j sends both $j, j+1$ to $j \in [n]$. For a cosimplicial object $X^\bullet : \mathbf{\Delta} \to C$, we call the images of the δ_j *coface maps*, usually denoted d^j, and the images of the σ_j *codegeneracy maps*, usually denoted s^j. So to specify a cosimplicial object in C it also suffices to list a sequence of objects $X^n \in C$ for $n \geq 0$, as well as coface and codegeneracy maps satisfying the following *cosimplicial identities*:

(1) If $i < j$, then $d^j \circ d^i = d^i \circ d^{j-1}$.
(2) If $i < j$, then $s^j \circ d^i = d^i \circ s_{j-1}$.
(3) $\mathrm{id} = s^j \circ d^j = s^j \circ d^{j+1}$.
(4) If $i > j+1$, then $s^j \circ d^i = d^{i-1} \circ s^j$.
(5) If $i \leq j$, then $s^j \circ s^i = s^i \circ s^{j+1}$.

One should think of a cosimplicial object $X^\bullet : \mathbf{\Delta} \to C$ as a diagram

$$X^0 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; X^1 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; X^2 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \cdots$$

where the rightward pointing arrows are the coface maps and the leftward pointing arrows are the codegeneracy maps. In general it helps to think of the coface maps as "duplicating a coordinate" and the codegeneracy maps as "forgetting a coordinate." We will see this in the following examples.

Example 4.2 (The topological simplicies)**.** Define the functor $\Delta^\bullet : \mathbf{\Delta} \to \mathsf{Top}$ which sends $[n]$ to the topological n-simplex:

$$\Delta^n = \{(x_1, ..., x_n) \in \mathbf{R}^n : 0 \leq x_1 \leq \cdots \leq x_n \leq 1\}.$$

Then the coface maps $\Delta^{n-1} \to \Delta^n$ are the inclusions of faces, where d^j is the inclusion of the face opposite the jth vertex. The codegeneracy map s^j collapses the line joining the jth and jth vertex. In coordinates, we have

$$d^j(x_1, ..., x_n) = (x_1, ..., x_j, x_j, ..., x_{n-1}),$$

$$s^j(x_1, ..., x_n) = (x_1, ..., \widehat{x_j}, ..., x_n)$$

where $\widehat{\ }$ indicates omission.

Example 4.3 (Path spaces)**.** Let X be a topological space with $a, b \in X$. Define the cosimplicial space $P^\bullet_{a,b}X$ whose cosimplicies are

$$P^0_{a,b}X = \{*\}, \quad P^n_{a,b}X = X^n \text{ for } n \geq 1.$$

The coface maps $d^j : P^{n-1}_{a,b}X \to P^n_{a,b}$ are given by

$$d^j(x_1, \ldots, x_{n-1}) = \begin{cases} (a, x_1, \ldots, x_{n-1}) & j = 0 \\ (x_1, \ldots, x_j, x_j, \ldots, x_{n-1}) & j \in \{1, \ldots, n-1\} \\ (x_1, \ldots, x_{n-1}, b) & j = n \end{cases}$$

The codegeneracy maps $s^j : P^{n+1}_{a,b}X \to P^n_{a,b}X$ are given by

$$s^j(x_1, \ldots, x_{n+1}) = (x_1, \ldots, \widehat{x_{j+1}}, \ldots x_{n+1}), \quad j \in \{0, \ldots, n\}$$

where the $\widehat{}$ denotes omission.

We leave it to the reader to verify that the maps in both examples satisfy the cosimplicial identities. Why we refer to Example 4.3 by *path spaces* will become evident in the next section.

4.2. **Totalization.** Cosimplicial spaces provide a useful model for many types of topological spaces, including the based path and loop spaces. This is done via *totalization*, which is dual to the notion of geometric realization of a simplicial set.

Given a cosimplicial space $X^\bullet : \mathbf{\Delta} \to \mathsf{Top}$, define the *totalization* of $X^\bullet$ to be the space of maps from the cosimplicial simplices to $X^\bullet$:

$$\mathrm{Tot}(X^\bullet) := \mathrm{Hom}_{\mathsf{csTop}}(\Delta^\bullet, X^\bullet),$$

i.e. maps $f^n : \Delta^n \to X^n$ for all $n \geq 0$ that commute with the coface and codegeneracy maps. We topologize it as a subspace of $\prod_{n \geq 0} \mathrm{Hom}(\Delta^n, X^n)$ with the compact-open topology. Thus totalization gives us a functor from csTop to Top.

What seems to be happening here is that by iterating the face maps, we are creating finer and finer piecewise subdivisons of paths whose endpoints are at a and b. Indeed,

Proposition 4.4. $\mathrm{Tot}(P^\bullet_{a,b}X)$ *is homeomorphic to the path space $\Omega_{a,b}X$ of paths in X beginning at a and ending at b.*

Proof. A point of $\mathrm{Tot}(P^\bullet_{a,b}X) = \mathrm{hom}_{\mathsf{csTop}}(\Delta^\bullet, P^\bullet_{a,b}X)$ us a sequence of continuous maps

$$f = \{f_i : \Delta^i \to X^i\}_{i \geq 0}$$

commuting with the coface and codegeneracy maps. Fix $n \geq 2$ and $k \in \{1, \ldots, n\}$. Consider the following composition of codegeneracy maps

$$\alpha_{n,k} := \underbrace{s^{n-1} \circ s^{n-2} \circ \cdots \circ s^{k-2} \circ s^k \circ \cdots \circ s^0}_{n-1 \text{ maps}}$$

where we compose all the degeneracies except for s^{k-1}. This gives us a map $\Delta^n \to \Delta^1$ and likewise for $X^n \to X^1$. Then for $f = \{f_0, f_1, \ldots\} \in \mathrm{Tot}(P^\bullet_{a,b}X)$, we have by commutativity that

$$f_1 \circ \alpha_{n,k} = \alpha_{n+1,k} \circ f_n.$$

But now the right hand side is just picking out the kth coordinate of f_n. Hence for $n \geq 2$, f_n is completely determined by f_1, so that the projection $\Phi : \mathrm{Tot}(P^\bullet_{a,b}X) \to \mathrm{Map}(\Delta^1, X)$ given by $\{f_i\}_{i \geq 0} \mapsto f_1$ is injective.

We claim next that Φ is actually a map into $\Omega_{a,b}X \subset \mathrm{Map}(\Delta^1, X)$. The cosimplicial relations imply

$$f_1 \circ d^0 = d^0 \circ f_0 = \mathrm{const}_a, \quad f_1 \circ d^1 = d^1 \circ f_0 = \mathrm{const}_b$$

so $f_1(0) = a$ and $f_1(1) = b$ as desired.

Lastly we define an inverse to $\Phi : \mathrm{Tot}(P^\bullet_{a,b}X) \to \Omega_{a,b}X$. For a given path $\gamma \in \Omega_{a,b}X$ consider the family of maps $\{f_i : \Delta^i \to X^i\}_{i\geq 0}$ given as follows. We let f^0 be the constant map, $f^1 = \gamma$, and for $n \geq 2$ define

$$f_n(x_1, \ldots, x_n) = (\gamma(x_1), \gamma(x_1 + x_2), \ldots, \gamma(x_1 + \cdots + x_n)).$$

Clearly this is an inverse to Φ. We leave it to the reader to check that the family of maps $\{f_i\}$ commutes with the coface and codegeneracies, and that Φ and its inverse are continuous maps. ◻

We will call $P^\bullet_{a,b}X$ a cosimplicial model for the path space of X. When $a = b$ we get a cosimplicial model for the based loop space of X. It turns out that the cosimplicial space $P^\bullet_{x,x}X$ is in some sense the underlying cosimplicial set of the cobar construction of $C_\bullet(X)$, as we will see in the course of this thesis.

4.3. **Dold–Kan.** Recall that the standard Dold–Kan correspondence gives an equivalence of categories between simplicial abelian groups and (non-negative) chain complexes of abelian groups. We'll restate some important results and their duals. Although these are duals, there is once again some asymmetry, as suggested by the quote.

Definition 4.5. Let A be a simplicial abelian group. Its *normalized chain complex* $(NA_\bullet, \partial)$ has

$$NA_n = \bigcap_{i=1}^{n} \ker(d^i : A_n \to A_{n-1}), \quad \partial = d_0 : NA_n \to NA_{n-1}$$

and its *Moore complex* $(MA_\bullet, \partial')$ has

$$MA_n = A_n, \quad \partial' = \sum_{i=0}^{n} (-1)^i d_i : A_n \to A_{n-1}.$$

The following results and their proofs can be found in [GJ09], Chapter III.2:

Theorem 4.6 (Dold–Kan correspondence)**.** *Let A be a simplicial abelian group. Then:*

(1) The functor $N : \mathsf{sAb} \to \mathsf{Ch}_+(\mathsf{Ab})$ is an equivalence of categories between simplicial abelian groups and non-negative chain complexes of abelian groups.
(2) The inclusion of chain complexes $NA \hookrightarrow MA$ is a chain homotopy equivalence, natural in A.
(3) There is a functorial direct sum decomposition $MA = NA \oplus DA$, where DA is the subcomplex generated by the images of all the degeneracy maps. Consequently, DA is acyclic.

Now we dualize. Recall that for simplicial objects, the face maps d_i lower degree. For cosimplicial objects, the face maps d^i raise degree, so we should get an equivalence between cosimplicial abelian groups and cochain complexes of abelian groups. This is indeed what happens.

Definition 4.7. Let $C^\bullet$ be a cosimplicial abelian group. Its normalized cochain complex $(NC^\bullet, \partial)$ has

$$NC^n = \mathrm{coker} \bigoplus_{i=1}^{n} (d^i : C^{n-1} \to C^n), \quad \partial = d^0 : NC^n \to NC^{n+1}$$

and its Moore complex has

$$MC^n = C^n, \quad \partial' = \sum_{i=0}^{n+1} (-1)^i d^i : C^n \to C^{n+1}.$$

Analogously we obtain the following statements:

Theorem 4.8 (Dold–Kan correspondence, dual)**.** *Let C be a cosimplicial abelian group.*

(1) The functor N : $\mathsf{csAb} \to \mathsf{coCh}_+(\mathsf{Ab})$ is an equivalence of categories between cosimplicial abelian groups and non-negative cochain complexes of abelian groups.
(2) The quotient of cochain complexes $MC \to NC$ is a cochain homotopy equivalence, natural in C.
(3) There is a functorial direct sum decomposition $MC = NC \oplus DC$, where DC is the subcomplex generated by the images of all the coface maps. Consequently, DC is acyclic.

Really we could have stated the previous four theorems for simplicial objects in any abelian category. In the following discussion, we replace "abelian group" with "R-module" for some commutative ring R. Consider the free R-module functor $R[-] : \mathsf{Set} \to \mathsf{Mod}_R$. Then given any cosimplicial set $X^\bullet : \mathbf{\Delta} \to \mathsf{Set}$ the composition of functors $R[X^\bullet]$ yields a cosimplicial R-module. We first state a useful lemma. The author first encountered it in a MathOverflow comment of Tom Goodwillie, and here we supply the proof:

Lemma 4.9 (Goodwillie's lemma)**.** *Let $n > 0$ and choose $x, y \in X^n$. If x, y are not in the image of any coface maps $d^j : X^{n-1} \to X^n$, then $d^i x = d^j y$ implies $i = j$ and $x = y$.*

Proof. First suppose $i < j$. Then $d^i x = d^j y$ implies

$$d^i s^{j-1} x = s^j d^i x = s^j d^j y = y$$

so that y is in the image of a coface map, a contradiction. The same argument for $j < i$ leaves only the possibility $i = j$. But $s^i d^i = \mathrm{id}$, implying that d^i is injective, so $x = y$. □

Proposition 4.10. *Under the Dold–Kan correspondence, $C^\bullet := R[X^\bullet]$ has zero cohomology in positive degree.*

Proof. □

4.4. **Alexander–Whitney.** In the proof of Looijenga's theorem there is a step where we roughly need to move from singular chains on a product space to the product of chains. There is a canonical way to do so, called the Alexander–Whitney map. We will present it in generality before specializing to our case.

In some sense, the Alexander–Whitney map is a way of moving between the category of simplicial abelian groups and the category of chain complexes of abelian groups that preserves their respective monoidal structure. Let us recall these structures. The monoidal product on sAb is given levelwise: for $A, B \in \mathsf{sAb}$, we have

$$(A \otimes B)_n = A_n \otimes B_n$$

and the face and degeneracy maps are inherited naturally. The monoidal product on the category of chain complexes is given as follows: for $X, Y \in \mathsf{Ch}_+(\mathsf{Ab})$, define

$$(X \otimes Y)_n = \bigoplus_{p+q=n} X_p \otimes Y_q.$$

and the new differential $(X \otimes Y)_n \to (X \otimes Y)_{n-1}$ by the Koszul sign rule:

$$\partial_{X \otimes Y}(x \otimes y) = \partial_X x \otimes y + (-1)^{\deg x} x \otimes \partial_Y y.$$

Definition 4.11. Let $M : \mathsf{sAb} \to \mathsf{Ch}_+(\mathsf{Ab})$ be the Moore complex functor. The *Alexander–Whitney* map is a natural transformation of the functors $\mathsf{sAb} \times \mathsf{sAb} \to \mathsf{Ch}_+(\mathsf{Ab})$

$$\mathcal{AW} : M(- \otimes -) \implies M(-) \otimes M(-)$$

whose components are given as follows. For $A, B \in \mathsf{sAb}$ and $a \in A_n, b \in B_n$, define the map

$$\mathcal{AW}_{A,B} : M(A \otimes B) \to M(A) \otimes M(B), \quad a \otimes b \mapsto \bigoplus_{p+q=n} \tilde{d}^p(a) \otimes d^q(b)$$

where d^p is the face map induced by $[p] \to [n], i \mapsto i$, and d^q is the face map induced by $[q] \to [n], i \mapsto i + p$.

Remark 4.12. $\mathcal{AW}$ restricts to a natural transformation

$$N(- \otimes -) \implies N(-) \otimes N(-)$$

where N is the normalized chains functor. (We will denote this restriction also by $\mathcal{AW}$.)

Proposition 4.13. *Let $A, B \in \mathsf{sAb}$. Then*

$$\mathcal{AW}_{A,B} : N(A \otimes B) \to N(A) \otimes N(B)$$

is a natural quasi-isomorphism.

Proof. Kerodon [Lur26, Tag 00S0]. □

This is a significant fact in homological algebra; now we employ only a fraction of its power. Let X, Y be a topological space, and $C_\bullet(-)$ denote the singular chains functor from Top to sAb. Then $MC_\bullet(X)$ is the singular chain complex of X in the usual sense.

Corollary 4.14. *There is a natural quasi isomorphism*

$$MC_\bullet(X \times Y) \to MC_\bullet(X) \otimes MC_\bullet(Y).$$

Proof. Recall that $C_\bullet(-)$ is the composition of the following functors:

$$\mathsf{Top} \xrightarrow{X \mapsto \mathrm{Map}(\Delta^\bullet, X)} \mathsf{sSet} \xrightarrow{A_\bullet \mapsto \mathbf{Z}[A_\bullet]} \mathsf{sAb}$$

and under this composition we get

$$X \times Y \ \mapsto \mathrm{Map}(\Delta^\bullet, X \times Y) = \mathrm{Map}(\Delta^\bullet, X) \times \mathrm{Map}(\Delta^\bullet, Y) \mapsto \mathbf{Z}[\mathrm{Map}(\Delta^\bullet, X)] \otimes \mathbf{Z}[\mathrm{Map}(\Delta^\bullet, Y)]$$

because the free abelian group functor sends Cartesian products of sets to tensor products of groups. So the simplicial abelian group $C_\bullet(X \times Y)$ is equal to $C_\bullet(X) \otimes C_\bullet(Y)$. Thus applying Proposition 4.13 and the Dold–Kan correspondence we are done. □

Really what we have defined here is a topological Alexander–Whitney map which is a natural transformation of the functors $\mathsf{Top} \times \mathsf{Top} \to \mathsf{sAb}$:

$$MC_\bullet(- \times -) \to MC_\bullet(-) \otimes MC_\bullet(-)$$

For topological spaces X, Y, denote the constituent map by $\mathcal{AW}_{X,Y}$. From now we will use $C_\bullet(X)$ to mean both the singular chain *complex* and the simplicial abelian group, depending on context.

Corollary 4.15. *For $q \geq 0$ and a topological space X, there is a natural quasi-isomorphism of chain complexes*

$$C_\bullet(X^q) \to C_\bullet(X)^{\otimes q}.$$

Proof. Iterate the topological Alexander–Whitney map

$$C_\bullet(X \times X^{q-1}) \xrightarrow{\mathcal{AW}_{X,X^{q-1}}} \cdots \xrightarrow{\mathrm{id}^{\otimes q-2} \otimes \mathcal{AW}_{X,X}} C_\bullet(X)^{\otimes q}.$$

Since each component is a natural quasi-isomorphism, so is the composition. □

5. The proof

Here we present the proof of Theorem 1.1. Throughout this section let X be a path connected topological space with basepoints a, b. Our strategy will be use the cosimplicial path space model to construct the E^0-page of a spectral sequence which (1) converges to the group $H_n(X^n, X(n)^a_b)$ and (2) is related to the cobar construction, hence the groups $\mathbf{Z}\pi_1(X, a)/\mathbf{Z}\pi_1(X, a)\mathcal{I}_a^{n+1}$.

5.1. **This spectral sequence...** Recall the cosimplicial path space $P^\bullet_{a,b}X$ of Example 4.3. From now we denote it by $P^\bullet$ for simplicity. Notice first that the union of the images of the coface maps $d^i : X^{n-1} \to X^n$ is exactly the subspace $X(n)^a_b$. This is a good sign.

Now consider the bicomplex $\mathscr{C}^\bullet_\bullet$ which is morally obtained by applying singular chains to $P^\bullet$:

$$\mathscr{C}^q_p = C_p(P^q)$$

with "horizontal" differential $d^{\mathscr{C}}_H : C_p(P^q) \to C_{p-1}(P^q)$ given by the singular boundary map, and "vertical" differential $d^V_{\mathscr{C}} : C_p(P^q) \to C_p(P^q)$ given by the differential of the Moore complex of the cosimplicial abelian group $C_p(P^\bullet)$.

Proposition 5.1. *$C^\bullet_\bullet$ is a bicomplex.*

Proof. That $d^{\mathscr{C}}_H$ and $d^V_{\mathscr{C}}$ both square to zero follows from the fact that each row is a chain complex and each row is a cochain complex. It remains to check commutativity of the squares:

$$\begin{tikzcd}
C_{p-1}(X^{q+1}) & C_p(X^{q+1}) \arrow[l] \\
C_{p-1}(X^q) \arrow[u] & C_p(X^q) \arrow[l] \arrow[u]
\end{tikzcd}$$

Starting with a map $f : \Delta^p \to X^q$, if we first go up then left, we get the map

$$\Delta^{p-1} \xrightarrow{([p]\to[p-1])^*} \Delta^p \xrightarrow{f} X^q \xrightarrow{\sum(-1)^i d^i} X^{q+1}$$

and proceeding in the other direction gives the exact same map. (This is really obvious, might not need to include). $\square$

Each column p of this bicomplex forms a cochain complex $\mathscr{C}^\bullet_p$ which is the Moore complex of $C_p(F^\bullet)$. Consider now the bicomplex $\mathscr{N}^\bullet_\bullet$ whose columns are instead the normalized cochain complexes of $C_p(F^\bullet)$. This is a bicomplex for the same reason as above, and the inclusion $\mathscr{N} \hookrightarrow \mathscr{C}$ is a quasi-isomorphism (?) of bicomplexes by Dold–Kan. Let us spell out what the groups in $\mathscr{N}$ actually are. Following the definition, we have

$$\mathscr{N}^q_p = \operatorname{coker} \bigoplus_{i=1}^n ((d^i)_* : C_p(X^{n-1}) \to C^p(X^n)) = C^p(X^n)/\langle \sum_{i=1}^n \operatorname{im}(d^i)_* \rangle = C_p(X^n, X(n)_b).$$

Moreover, the vertical differential $\mathscr{N}^q_p \to \mathscr{N}^{q+1}_p$ is induced by the zeroth coface map d^0.

Now consider the bicomplex obtained from $\mathscr{N}$ by setting all rows above $q = n$ to zero, which we will denote by $(\mathscr{N}^{\le n})^\bullet_\bullet$.

Proposition 5.2. *Filtering $(\mathscr{N}^{\le n})^\bullet_\bullet$ in the horizontal direction i.e. by the lower index, we obtain a spectral sequence collapsing on the second page with*

$$E^2_{n,n} = H_n(X^n, X(n)^a_b), \quad E^2_{0,0} = \begin{cases} \mathbf{Z} & a = b \\ 0 & a \ne b \end{cases}$$

converging to $H_\bullet(\operatorname{Tot}(\mathscr{N}^{\le n})^\bullet_\bullet)$.

Proof. Convergence is given by the fact that our bicomplex is concentrated in the first quadrant. To compute the E^2-page, let us begin in the E^0-page, whose pth column is the following normalized cochain complex (beginning below in degree $q = 0$):

$$C_p(*) \to C_p(X, \{b\}) \to \cdots \to C_p(X^n, X(n)_b) \to 0 \to \cdots$$

of the Moore complex

$$C_p(*) \to C_p(X) \to \cdots \to C_p(X^n) \to 0 \to \cdots \tag{5.3}$$

so they have the same cohomology. Moreover, the E^0-differential is the cochain differential d^0 given our filtration. But now (5.3) is just the cochain complex obtained by taking the free abelian group on the cosimplicial set $[q] \mapsto \mathrm{Map}(\Delta^p, F^q)$, with terms $q > n$ set to zero. Therefore, by Proposition 4.10, we have that $E^1_{p,q} = 0$ for $q \neq 0, n$. For $q = n$ and $q = 0$, we have

$$E^1_{p,n} = C_p(X^n, X(n)_b)/\mathrm{im}(d^0) = C_p(X^n, X(n)^a_b)$$

$$E^1_{p,0} = \ker(d^0 : C_p(*) \to C_p(X, \{b\})) = \begin{cases} \mathbf{Z} & p = 0, a = b \\ 0 & \text{otherwise} \end{cases}.$$

On the E^1-page, the differential is now induced by singular boundary map. Thus on the only two possibly nonzero rows $q = n$ of the E^1-page, we just compute the singular homology of to obtain

$$E^2_{p,n} = H_p(X^n, X(n)^a_b).$$

And on row $q = 0$, we have

$$E^2_{p,0} = \begin{cases} H_p(\mathbf{Z} \leftarrow 0 \leftarrow \cdots) & a = b \\ 0 & a \neq b \end{cases}$$

as desired.

At this point, because the only nonzero rows are $q = n$ and possibly $q = 0$, the differentials on the pages $E^{\geq 2}$ are all zero, except possibly on page E^{n+1}, where we have a differential of bidegree $(-n-1, -n)$ from $E^{n+1}_{n+1,n} = H_{n+1}(X^n, X(n)^a_b)$ to $E^{n+1}_{0,0} = \mathbf{Z}$. □

Corollary 5.4. *We have a surjection*

$$H_0(\mathrm{Tot}(\mathcal{N}^{\leq n})^\bullet_\bullet) \twoheadrightarrow H_n(X^n, X(n)^a_b)$$

whose kernel is $\mathbf{Z}$ *if* $a = b$ *and trivial otherwise.*

Proof. Follows from convergence of the spectral sequence. □

5.2. **... (sort of) computes cobar.** It remains to compute H_0 of this totalization. For this we return to our original bicomplex $\mathscr{C}^\bullet_\bullet$ and instead of normalizing the columns, we first look at the rows $\mathscr{C}^q_\bullet$ for each q, which are the singular chain complexes $C_\bullet(X^q)$. Then we can apply the topological Alexander–Whitney map on each row:

$$\mathscr{C}^q_\bullet = C_\bullet(X^q) \to C_\bullet(X)^{\otimes q}$$

to obtain a new bicomplex, which we will call $\mathscr{D}^\bullet_\bullet$. It follows from Corollary 4.15 that this map $\mathscr{C} \to \mathscr{D}$ of bicomplexes is a quasi-isomorphism. By definition of tensor product of chain complexes, $\mathscr{D}$ has a bigrading given by

$$\mathscr{D}^q_p = (C_\bullet(X)^{\otimes q})_p = \bigoplus_{n_1+\cdots+n_q=p} C_{n_1}(X) \otimes \cdots \otimes C_{n_q}(X). \tag{5.5}$$

Proposition 5.6. *Suppose* $x := a = b$. *The coface maps* $d^i : X^q \to X^{q+1}$ *and codegeneracy maps* $s^i : X^q \to X^{q-1}$ *respectively induce maps* $\mathscr{D}(d^i) : \mathscr{D}^q_\bullet \to \mathscr{D}^{q+1}_\bullet$, $\mathscr{D}(s^i) : \mathscr{D}^q_\bullet \to \mathscr{D}^{q-1}_\bullet$ *given by*

$$\mathscr{D}(d^i) = \begin{cases} \eta(1) \otimes \mathrm{id}^{\otimes q} & i = 0 \\ \mathrm{id}^{\otimes i-1} \otimes \Delta \otimes \mathrm{id}^{\otimes q-i} & i \in \{1, \ldots, q\} \\ \mathrm{id}^{\otimes q} \otimes \eta(1) & i = q \end{cases} \quad \textit{and} \quad \mathscr{D}(s^i) = \mathrm{id}^{\otimes i} \otimes \varepsilon \otimes \mathrm{id}^{q-i-1}$$

where by a slight abuse of notation, $\eta(1)$ *is the constant map at* x*.*

Proof. Naturality of Alexander–Whitney implies that the following diagram commutes:

$$\begin{tikzcd}
C_\bullet(X^q) \arrow[r, "\cong"] \arrow[d, "\mathcal{AW}_q"'] \arrow[rr, bend left, "C_\bullet(d^0)"] & C_\bullet(* \times X^q) \arrow[r, "C_\bullet(\iota_x \times \mathrm{id}_{X^q})"] \arrow[d, "\mathcal{AW}_{*,X^q}"'] & C_\bullet(X^{q+1}) \arrow[d, "\mathcal{AW}_{q+1}"] \\
C_\bullet(X)^{\otimes q} \arrow[r, "\cong"] \arrow[rr, bend right, "\mathcal{D}(d^0)"'] & C_\bullet(*) \otimes C_\bullet(X)^{\otimes q} \arrow[r, "C_\bullet(\iota_x) \otimes \mathrm{id}^{\otimes q}"] & C_\bullet(X)^{\otimes q+1}
\end{tikzcd}$$

where ι_x is the inclusion at the basepoint. But now we make the identifications

$$C_\bullet(*) \otimes C_\bullet(X)^{\otimes q} \cong C_\bullet(X)^{\otimes q}, \text{ and } C_\bullet(\iota_x) = \eta(1)$$

to conclude

$$(\eta(1) \otimes \mathrm{id}^{\otimes q}) \circ \mathcal{AW}_q = \mathcal{AW}_{q+1} \circ C_\bullet(d^0)$$

which implies $\mathcal{D}(d^0) = \eta(1) \otimes \mathrm{id}^{\otimes q}$. The arguments for the other maps are analogous. □

We now want to replace each column of $\mathcal{D}$:

$$\mathcal{D}_p^\bullet : C_p(*) \to C_p(X) \to C_p(X)^{\otimes 2} \to \cdots$$

with its normalized cochain complex, which is the reason we computed these maps. Here we use an alternative formulation (prove) of the normalized cochain complex to obtain

$$\begin{aligned}
N\mathcal{D}_p^q &= \bigcap_{i=0}^{q-1} \ker(s^i : \mathcal{D}_p^q \to \mathcal{D}_p^{q-1}) \\
&= \bigcap_{i=0}^{q-1} \ker(\mathrm{id}^i \otimes \varepsilon \otimes \mathrm{id}^{\otimes q-i-1}) \\
&= (\ker(\varepsilon)^{\otimes q})_p \\
&= \bigoplus_{m_1+\cdots+m_q=p} \overline{C_{n_1}(X)} \otimes \cdots \otimes \overline{C_{n_q}(X)}
\end{aligned} \tag{5.7}$$

Now this is beginning to look very similar to the cobar construction on $C_\bullet(X)$. Because the vertical differential raises q, and the horizontal differential lowers p, when we totalize $N\mathcal{D}_\bullet^\bullet$ we take total degree $k = p - q$ to obtain a chain complex, similar to the construction of the cobar bicomplex.

Theorem 5.8. *Suppose* $x := a = b$*. Then*
(1) There is an isomorphism $\mathrm{Tot}\, N\mathcal{D} \to \mathsf{Cobar}(C_\bullet(X))$.
(2) There is a quasi-isomorphism $\mathsf{Cobar}(C_\bullet(X)) \to \mathrm{Tot}_\bullet(\mathcal{N})$.

Proof. (2) follows from (1) in the following manner. See the following diagram where the solid arrows are quasi-isomorphisms previously constructed in this section. Their composition with (1) gives the desired quasi-isomorphism.

$$\begin{tikzcd} & & \operatorname{Tot}\mathscr{C} \arrow[r, two heads] \arrow[d, shift right, "\mathcal{AW}"'] & \operatorname{Tot}\mathscr{N} \\ & \operatorname{Tot} N\mathscr{D} \arrow[r, hook] & \operatorname{Tot}\mathscr{D} \arrow[u, shift right, "\mathcal{EZ}"'] & \\ \mathsf{Cobar}(C_\bullet(X)) \arrow[ur, dashed, "(1)"] \arrow[urrr, dashed, bend right] & & & \end{tikzcd}$$

To see (1), notice that $N\mathscr{D}$ is exactly the cobar bicomplex of $C_\bullet(X)$ in Definition 3.3, so that Proposition 3.4 yields an isomorphism $\operatorname{Tot} N\mathscr{D} \to \mathsf{Cobar}(C_\bullet(X))$. $\square$

Remark 5.9. We have not yet used Adams' theorem. But what we have proven should suggest that the theorem is true for the following reason. We have exhibited a quasi-isomorphism between $\mathsf{Cobar}(C_\bullet(X))$ and the totalization of $\mathscr{C}$, which is itself the bicomplex constructed by taking singular chains on the cosimplicial loop space! So if we can somehow commute "chains on a totalization" and "totalization of chains" then we would obtain Adams' theorem. See the following diagram:

$$\begin{tikzcd} \mathsf{Cobar}(C_\bullet(X)) \arrow[r, "\simeq"] \arrow[d, dashed, "\text{Adams' theorem}"'] & \operatorname{Tot}(\mathscr{C}) = \operatorname{Tot}(C_\bullet P^\bullet_{x,x} X) \arrow[d, dashed, "?"] \\ C_\bullet(\Omega_x X) \arrow[r, equal] & C_\bullet(\operatorname{Tot}(P^\bullet_{x,x} X)) \end{tikzcd}$$

The arrow on the right turns out to be an isomorphism, but it is beyond the scope of this thesis.

Proof of Theorem 1.1. In the quasi-isomorphism $\mathsf{Cobar}(C_\bullet(X)) \to \operatorname{Tot}_\bullet(\mathscr{N})$ produced in Proposition 5.8, the truncation of $\mathscr{N}$ by q corresponds to the tensor length truncation of Cobar. It follows that the restriction

$$\mathsf{Cobar}_{\leq n}(C_\bullet(X)) \to \operatorname{Tot}_\bullet(\mathscr{N}^{\leq n})$$

is still a quasi-isomorphism. On zeroth homology, we therefore have an isomorphism when

$$H_0 \,\mathsf{Cobar}_{\leq n}(C_\bullet(X)) \to H_0 \operatorname{Tot}_\bullet(\mathscr{N}^{\leq n})$$

which by Corollary 3.10 yields an isomorphism when $x := a = b$:

$$\mathbf{Z}\pi_1(X,x)/\mathcal{I}^{n+1} \to H_0 \operatorname{Tot}_\bullet(\mathscr{N}^{\leq n}).$$

By Corollary 5.4, we get a surjection

$$\mathbf{Z}\pi_1(X,x)/\mathcal{I}^{n+1} \to H_0 \operatorname{Tot}_\bullet(\mathscr{N}^{\leq n}) \twoheadrightarrow H_n(X^n, X(n)^x_x)$$

whose kernel is $\mathbf{Z}$, completing the proof. $\square$

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Appendix A. Group rings

. One important class of algebras are group rings. For a fixed ring R and a group G, the *group ring of G over R*, denoted $R[G]$, has elements finite formal linear combinations of elements in G with coefficients in R, with addition and multiplication given by

$$(\sum_{g\in G} r_g g) + (\sum_{g\in G} s_g g) = \sum_{g\in G}(r_g + s_g)g, \quad (\sum_{g\in G} r_g g)(\sum_{g\in G} s_g g) = \sum_{g\in G}\sum_{g_1g_2=g}(r_{g_1}s_{g_2})g.$$

Then the action of R on $R[G]$ given by multipliying coefficients gives $R[G]$ the structure of an R-algebra. One should think of $R[G]$ as some sort of free module over R with basis G. Group rings abound in the representation theory of groups, where any representation $\rho : G \to \mathrm{GL}(V)$ of a group G over a k-vector space V corresponds to a module over the group ring $k[G]$. In this thesis, however, we do not need to take this perspective (unless we could?)

As an algebra, $R[G]$ comes with a natural augmentation, given by the map

$$\varepsilon : R[G] \to R, \quad \varepsilon(\sum_{g\in G} r_g g) = \sum_{g\in G} r_g.$$

We call its kernel the *augmentation ideal*. Clearly we have a splitting $R[G] \cong \ker\varepsilon \oplus R$. The augmentation ideal is an interesting object of study. For example, we have the following observation:

Proposition A.1. *Let $\mathcal{I}$ be the augmentation ideal of the integral group ring $\mathbf{Z}[G]$. Then $\mathcal{I}/\mathcal{I}^2 \cong G^{\mathrm{ab}}$, the abelianzation of G.*

Proof. The proof relies on the following two facts. First, that $\{g-1 : g \in G\}$ is a generating set of $\mathcal{I}$. Second, that the abelianzation of G is the quotient of G by its commutator subgroup $[G, G]$, which is generated by group elements of the form $g^{-1}h^{-1}gh$. Then we can define an explicit homomorphism

$$\mathcal{I}/\mathcal{I}^2 \to G^{\mathrm{ab}} = G/[G,G], \quad [g-1] \mapsto [g]$$

and extending linearly to all of $\mathcal{I}/\mathcal{I}^2$. Then the inverse map is $[g] \to [g-1]$. $\square$

Given a path connected, based topological space with (X, x), consider its integral fundamental group ring $\mathbf{Z}\pi_1(X, x)$ and corresponding augmentation ideal $\mathcal{I}$. The previous proposition implies that

$$\mathcal{I}/\mathcal{I}^2 \cong \pi_1(X,x)^{\mathrm{ab}} \cong H_1(X;\mathbf{Z}).$$

This is the simplest case of the main theorem we are trying to prove. It hints at the fact that we can gain information about the fundamental group of the space by examining its homology.