THESIS DRAFT

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ABSTRACT.

CONTENTS

1. INTRODUCTION

1.1. **The theorem.** Let X be a path-connected topological space and choose basepoints $a, b \in X$. Let $\pi_X(a, b)$ be the set of homotopy classes of paths in X from a to b. In particular when $a = b$ we obtain the fundamental group of X. Notice that $\pi_1(X, a)$ acts on $\pi_X(a, b)$ on the right by concatenation of paths, so that the free abelian group $\mathbf{Z}\pi_X(a, b)$ is a right $\mathbf{Z}\pi_1(X)$-module. Let $\mathcal{I}_a \subset \mathbf{Z}\pi_1(X, a)$ be the augmentation ideal.

Consider now the following subspaces of X^n

$$X(n)^a := \{(x_1, \dots, x_n) \in X^n : x_1 = a \text{ or } x_i = x_{i+1} \text{ for } i \in \{1, \dots, n-1\}\}$$

$$X(n)_b := \{(x_1, \dots, x_n) \in X^n : x_n = b \text{ or } x_i = x_{i+1} \text{ for } i \in \{1, \dots, n-1\}\}$$

and let $X(n)_b^a := X(n)^a \cup X(n)_b$.

Theorem 1.1. *There is a map*

$$\mathbf{Z}\pi_X(a, b)/\mathbf{Z}\pi_X(a, b)\mathcal{I}_a^{n+1} \to H_n(X^n, X(n)_b^a; \mathbf{Z})$$

which is a surjection with kernel $\mathbf{Z}$ *if* $a = b$ *and an isomorphism if* $a \neq b$.

The author first encountered this theorem in a paper of Looijenga ([Loo25], Theorem 1.1), who attributes this theorem (or a dual version with $\mathbf{Q}$-coefficients) to Deligne and Goncharov, ([DG05], Proposition 3.4). In this paper, the authors attribute it to Beilinson.

1.2. **Outline.**

Date: March 1, 2026.

2. Coalgebras

As suggested by the name, a *coalgebra* is the dual notion to that of an algebra. We could leave it at that, but they are much more unfamiliar objects; the maps don't go in the way we are used to. Moreover, the two objects are not dual on the nose: while the dual of every coalgebra is an algebra, the converse is not true without some finiteness assumptions. The reader should feel free to skip this chapter if they are familiar with the theory of (co)associative (co)algebras.

2.1. First definitions.

Let R be a commutative ring.

Definition 2.1. A *coassociative R-coalgebra* is an R-module C with an R-linear map $\Delta : C \to C \otimes C$, called the *coproduct* (or *diagonalization*), such that the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathrm{id} \otimes \Delta"] \\ C \otimes C \arrow[r, "\Delta \otimes \mathrm{id}"'] & C \otimes C \otimes C \end{tikzcd}$$

One way to think about the coproduct is that it tells one how to decompose a given element in the module. Consider the following example:

Example 2.2 ([JR79], §2)**.** Let $C = R[x]$, the ring of polynomials over R. Define a coproduct on a basis element x^n via

$$\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}.$$

Then...

Given an associative algebra A it is intuitive how to compose the product to obtain a map $A^{\otimes n} \to A$. With a coassociative coalgebra (C, Δ) there is an analogous notion. Define the *iterated coproduct* $\Delta^n : C \to C^{\otimes n+1}$ inductively with $\Delta^0 = \mathrm{id}$, $\Delta^1 = \Delta$, and

$$\Delta^n = \underbrace{\Delta \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{n \text{ operations}} \circ \Delta^{n-1}.$$

Coassociativity tells us that we could have inserted the coproduct anywhere within the above tensor product. A *morphism of coassociative coalgebras* $f : C \to D$ is an R-linear map commuting with the coproduct, i.e.

$$(f \otimes f) \circ \Delta_C = \Delta_D \circ f.$$

Just as a unital associative R-algebra A is one admitting a unital morphism $R \to A$, we say dually that a coassociative R-coalgebra C is *counital* if there is a morphism $\epsilon : C \to R$ such that the following diagram commutes:

$$\begin{tikzcd} & C \arrow[dl, "\cong"'] \arrow[d, "\Delta"] \arrow[dr, "\cong"] & \\ R \otimes C & C \otimes C \arrow[l, "\epsilon \otimes \mathrm{id}"] \arrow[r, "\mathrm{id} \otimes \epsilon"'] & C \otimes R \end{tikzcd}.$$

The simplest example of a counital coassociative R-coalgebra is R itself, with the coproduct given by $1 \mapsto 1 \otimes 1$ and the counit given by $1 \mapsto 1$. We call a morphism $\eta : R \to C$ a *coaugmentation*, and in this case say that C is *coaugmented*. Beccause η is a morphism of coalgebras, it must commute with the counit maps, so we obtain that $\epsilon_C \circ \eta_C = \epsilon_R = \mathrm{id}_R$. It then follows that $C \cong \ker \epsilon \oplus R$, and we denote this kernel by $\overline{C}$ and call it the *reduced coalgebra*. We can think of $\overline{C}$ as either a submodule or a quotient of C. The reduced coalgebra also has a coproduct $\overline{\Delta}$ given by

$$\overline{\Delta}(x) = \Delta(x) - 1 \otimes x - x \otimes 1.$$

2.2. **Conilpotency, cofreeness, and the tensor coalgebra.** Let (C, Δ) be a coaugmented coalgebra. Define the *coradical* (sometimes also called *canonical*) filtration on C as follows:

$$F_0 C = R, \quad F_r C = \{x \in \overline{C} : \bar{\Delta}^n(x) = 0 \text{ for } n \geq r\} \text{ for } r \geq 1.$$

Then we say C is *conilpotent* or *connected* if this filtration is exhaustive. Conilpotency is important is in the following definition.

Definition 2.3. The *cofree* coassociative R-coalgebra over a R-module M is a conilpotent coassociative coalgebra $\mathcal{F}^c M$ equipped with an R-linear map $s : \mathcal{F}^c M \to M$ sending 1 to 0 and satisfying the following universal property:

> Given any R-linear map $f : B \to M$ factors through $\mathcal{F}^c M$, i.e. there exists a unique map $\tilde{f} : B \to \mathcal{F}^c M$ such that $s \circ \tilde{f} = f$.

As with other objects defined via universal properties, the cofree coalgebra is unique up to unique isomorphism. In the categorial language we want this functor

$$\mathcal{F}^c : \mathsf{Mod}_R \to \mathsf{conilCoalg}_R$$

to be right adjoint to the forgetful function sending a conilpotent coalgebra to its underlying module. The reason we restrict ourselves to conilpotent coalgebras here is that the cofree objects are familiar, as we will soon see. In the general category of coalgebras they are large and unwieldy.

Definition 2.4. Let M be an R-module. The *tensor coalgebra over M*, denoted $T^c M$, is the coalgebra whose underlying module is

$$T^c M := R \oplus M \oplus M^{\otimes 2} \oplus \cdots$$

and whose coproduct $T^c M \to T^c M \otimes T^c M$ is given by

$$1 \mapsto 1 \otimes 1 \quad \text{and} \quad x_1 \otimes \cdots \otimes x_n \mapsto \sum_{i=0}^{n} (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_n).$$

For example, $x \in M$ gets mapped to $1 \otimes x + x \otimes 1$.

Proposition 2.5. *$T^c M$ is coassociative, counital, and conilpotent.*

Proof. Coassociativity... The counit is given by the map $T^c M \to R$ which is the identity on R and zero on the higher summands. Conilpotency... □

Proposition 2.6. *Let M be an R-module. Any linear map $M \to TM$ extends uniquely to a derivation $TM \to TM$.*

Proof. This follows from the more general Proposition 1.1.8, [LV12]. □

Remark 2.7. For an R-module M, the same tensor module $R \oplus M \oplus M^{\otimes 2} \oplus \cdots$ also has an algebra structure given by tensor multiplication. We denote it by TM rather than $T^c M$ to discern between the two structures. With this product it becomes the free associative R-algebra over M. In fact, the product and coproducts are compatible, making TM into an *bialgebra*, and in fact a *Hopf algebra*.

Proposition 2.8. *$T^c M$ is the cofree conilpotent coalgebra over M.*

Proof. Let $x \in T^c M$. □

2.3. **Differential graded coalgebras.** A *differential graded* (often abbreviated to *dg*) coalgebra is a coassociative coalgebra (C, Δ) with

(1) a grading, i.e. $C = \oplus_{n\in\mathbf{N}} C_n$ such that the product sends C_n to $\oplus_{i+j=n} C_i \otimes C_j$, and
(2) a differential d sending C_i to C_{i-1}, satisfying the dual identity

$$\Delta \circ d = (d \otimes \mathrm{id} + \mathrm{id} \otimes d) \circ \Delta.$$

Similarly this implies that the coproduct is a morphism of chain complexes, and we can think of dga coalgebras as either (1) chain complexes with a compatible coproduct, or (2) coalgebras with compatible differential. For example, the tensor (co)algebra from before is graded over $\mathbf{N}$.

Given a dg R-coalgebra $C = \oplus_{n\in\mathbf{N}} C_n$, we can define its *suspension* sC to be C shifted up a degree, i.e. $(sV)_i = V_{i-1}$, and its *desuspension* $s^{-1}C$ to be C shifted down a degree, i.e. $(s^{-1}C)_j = C_{j+1}$. Then any $x \in C_n$ determines an element $sx \in (sC)_{n+1}$ and $s^{-1}x \in (s^{-1}C)_{n-1}$. Defining these turns out to be helpful for degree bookkeeping reasons.

Example 2.9. The tensor (co)algebra

Example 2.10. Consider the singular chains $(C_\bullet(X; \mathbf{Z}), \partial)$ on a topological space X with basepoint x. We claim that they form a coaugmented dg $\mathbf{Z}$-coalgebra (in fact also a bialgebra). For the coproduct, notice that we always have a diagonal map $x \mapsto (x, x)$, which induces a chain map $C_\bullet(X) \to C_\bullet(X \times X)$. Now there a natural quasi-isomorphism $C_\bullet(X) \otimes C_\bullet(X)$ (for more details see the following section). $C_\bullet(X)$ is counital with the map $\epsilon : C_\bullet(X) \to \mathbf{Z}$ given by sending all 0-simplicies to 1 and has a coaugmentation $\eta : \mathbf{Z} \to C_\bullet(X)$ which sends 1 to the 0-simplex at x.

2.4. **Comodules.** Continuing with our previous example of singular chains on a space X, the situation changes if we want to talk about two distinct basepoints $a, b \in X$. In this case there is no preferred choice of coaugmentation for the coalgebra $C_\bullet(X)$. For this we need the notion of a comodule.

Definition 2.11. A *left comodule* over a unital R-coalgebra (C, Δ, ϵ) is an R-module N with a left action

$$\Delta^L : N \to C \otimes N$$

compatible with the coproduct and counit in the sense that the following diagrams commute:

$$\begin{tikzcd} N \arrow[r, "\Delta^L"] \arrow[d, "\Delta^L"'] & C \otimes N \arrow[d, "\mathrm{id}\otimes\Delta^L"] \\ C \otimes N \arrow[r, "\Delta\otimes\mathrm{id}"'] & C \otimes C \otimes N \end{tikzcd} \qquad \begin{tikzcd} N \arrow[d, "\Delta^L"'] \arrow[dr, "\cong"] & \\ C \otimes N \arrow[r, "\epsilon\otimes\mathrm{id}"'] & R \otimes N \end{tikzcd}$$

We can similarly define a *right* comodule over C as an R-module M with the analogous right action

$$\Delta^R : M \to M \otimes C.$$

If a left and right comodule M also satsifies the following compatibility condition:

$$\begin{tikzcd} M \arrow[r, "\Delta^L"] \arrow[d, "\Delta^R"'] & C \otimes M \arrow[d, "\mathrm{id}\otimes\Delta^R"] \\ M \otimes C \arrow[r, "\Delta^L\otimes\mathrm{id}"'] & M \otimes C \otimes M \end{tikzcd}$$

then we call M a *co-bimodule*. The simplest example of a co-bimodule over a coalgebra C is C itself, where the left right actions are just given by the coproduct. Returning to the example of singular chains, let $C = C_\bullet(X)$. Then the chains on the pathspace

3. (Co)simplicial objects

A cosimplicial object of a category $\mathcal{C}$ could be defined simply as a simplicial object of the opposite category $\mathcal{C}^{\mathrm{op}}$. This is not really how the human brain works...

—Stacks Project, [Sta26, Tag 016I]

In this chapter we define cosimplicial objects, the totalization of a cosimplicial space, and provide some examples. We will state the Dold–Kan correspondence and use it to prove a useful statement for later on. Finally we conclude with an exposition of the Alexander–Whitney map, which couldn't find a home in any other chapter. We assume some familiarity with simplicial objects.

3.1. **Definitions.** The *simplex category*, denoted $\mathbf{\Delta}$, is the category with

(1) objects: finite nonempty totally ordered sets. We write $[n]$ for the set $\{0 < 1 < \cdots < n\}$,
(2) morphisms: order-preserving maps, i.e. if $i \leq j$ then $f(i) \leq f(j)$.

Then a *cosimplicial object* in a category $\mathcal{C}$ is a functor $X^\bullet : \mathbf{\Delta} \to \mathcal{C}$. We denote the image of $[n]$ under $X^\bullet$ by X^n. A morphism of two simplicial objects $X^\bullet, Y^\bullet : \mathbf{\Delta} \to \mathcal{C}$ is a natural transformation of functors, i.e. morphisms $X^n \to Y^n$ for all n that commute with morphisms in $\mathbf{\Delta}$. We will denote the category of of cosimplicial objects in $\mathcal{C}$ by $\mathsf{cs}\mathcal{C}$. In this thesis we will concern ourselves with the cases where $\mathcal{C} = \mathsf{Top}$ or $\mathcal{C} = \mathsf{Set}$.

Remark 3.1. Even though simplicial objects are *contravariant* functors, the convention is to denote the simplicies using subscripts, e.g. X_n. Conversely, even though simplicial spaces are *covariant* functors, the convention is to denote the cosimplicies with superscripts. Maybe the presence of the suffix co- explains this.

Recall that the morphisms in the simplex category are generated by two distinguished classes of maps. For $n \geq 1$ and $j \in [n]$, we have injections $\delta_j : [n-1] \to [n]$ where δ_j skips $j \in [n]$. For $n \geq 0$ and $j \in [n]$, we have $n+1$ surjections $\sigma_j : [n+1] \to [n]$ where σ_j sends both $j, j+1$ to $j \in [n]$. For a cosimplicial object $X^\bullet : \mathbf{\Delta} \to \mathcal{C}$, we call the images of the δ_j *coface maps*, usually denoted d^j, and the images of the σ_j *codegeneracy maps*, usually denoted s^j. So to specify a cosimplicial object in $\mathcal{C}$ it also suffices to list a sequence of objects $X^n \in \mathcal{C}$ for $n \geq 0$, as well as coface and codegeneracy maps satisfying the following *cosimplicial identities*:

(1) If $i < j$, then $d^j \circ d^i = d^i \circ d^{j-1}$.
(2) If $i < j$, then $s^j \circ d^i = d^i \circ s_{j-1}$.
(3) $\mathrm{id} = s^j \circ d^j = s^j \circ d^{j+1}$.
(4) If $i > j+1$, then $s^j \circ d^i = d^{i-1} \circ s^j$.
(5) If $i \leq j$, then $s^j \circ s^i = s^i \circ s^{j+1}$.

One should think of a cosimplicial object $X^\bullet : \mathbf{\Delta} \to \mathcal{C}$ as a diagram

$$X^0 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; X^1 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; X^2 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \cdots$$

where the rightward pointing arrows are the coface maps and the leftward pointing arrows are the codegeneracy maps. In general it helps to think of the coface maps as "duplicating a coordinate" and the codegeneracy maps as "forgetting a coordinate." We will see this in the following examples.

Example 3.2 (The topological simplicies)**.** Define the functor $\Delta^\bullet : \mathbf{\Delta} \to \mathsf{Top}$ which sends $[n]$ to the topological n-simplex:

$$\Delta^n = \{(x_1, \ldots, x_n) \in \mathbf{R}^n : 0 \leq x_1 \leq \cdots \leq x_n \leq 1\}.$$

Then the coface maps $\Delta^{n-1} \to \Delta^n$ are the inclusions of faces, where d^j is the inclusion of the face opposite the jth vertex. The codegeneracy map s^j collapses the line joining the jth and jth vertex. In coordinates, we have

$$d^j(x_1, ..., x_n) = (x_1, ..., x_j, x_j, ..., x_{n-1}),$$

$$s^j(x_1, ..., x_n) = (x_1, ..., \widehat{x_j}, ..., x_n)$$

where $\widehat{\ }$ indicates omission.

Example 3.3 (Path spaces)**.** Let X be a topological space with $a, b \in X$. Define the cosimplicial space $P^\bullet_{a,b}X$ whose cosimplicies are

$$P^0_{a,b}X = \{*\}, \quad P^n_{a,b}X = X^n \text{ for } n \geq 1.$$

The coface maps $d^j : P^{n-1}_{a,b}X \to P^n_{a,b}$ are given by

$$d^j(x_1, ..., x_{n-1}) = \begin{cases} (a, x_1, ..., x_{n-1}) & j = 0 \\ (x_1, ..., x_j, x_j, ..., x_{n-1}) & j \in \{1, ..., n-1\} \\ (x_1, ..., x_{n-1}, b) & j = n \end{cases}$$

The codegeneracy maps $s^j : P^{n+1}_{a,b}X \to P^n_{a,b}X$ are given by

$$s^j(x_1, ..., x_{n+1}) = (x_1, ..., \widehat{x_{j+1}}, ...x_{n+1}), \quad j \in \{0, ..., n\}$$

where the $\widehat{\ }$ denotes omission.

We leave it to the reader to verify that the maps in both examples satisfy the cosimplicial identities. Why we refer to Example 3.3 by *path spaces* will become evident in the next section.

3.2. **Totalization.** Cosimplicial spaces provide a useful model for many types of topological spaces, including the based path and loop spaces. This is done via *totalization*, which is dual to the notion of geometric realization of a simplicial set.

Given a cosimplicial space $X^\bullet : \mathbf{\Delta} \to \mathsf{Top}$, define the *totalization* of $X^\bullet$ to be the space of maps from the cosimplicial simplices to $X^\bullet$:

$$\mathrm{Tot}(X^\bullet) := \mathrm{Hom}_{\mathsf{csTop}}(\Delta^\bullet, X^\bullet),$$

i.e. maps $f^n : \Delta^n \to X^n$ for all $n \geq 0$ that commute with the coface and codegeneracy maps. We topologize it as a subspace of $\prod_{n\geq 0} \mathrm{Hom}(\Delta^n, X^n)$ with the compact-open topology. Thus totalization gives us a functor from csTop to Top.

What seems to be happening here is that by iterating the face maps, we are creating finer and finer piecewise subdivisons of paths whose endpoints are at a and b. Indeed,

Proposition 3.4. $\mathrm{Tot}(P^\bullet_{a,b}X)$ *is homeomorphic to the path space* $\Omega_{a,b}X$ *of paths in* X *beginning at* a *and ending at* b.

Proof. A point of $\mathrm{Tot}(P^\bullet_{a,b}X) = \mathrm{hom}_{\mathsf{csTop}}(\Delta^\bullet, P^\bullet_{a,b}X)$ us a sequence of continuous maps

$$f = \{f_i : \Delta^i \to X^i\}_{i\geq 0}$$

commuting with the coface and codegeneracy maps. Fix $n \geq 2$ and $k \in \{1, ..., n\}$. Consider the following composition of codegeneracy maps

$$\alpha_{n,k} := \underbrace{s^{n-1} \circ s^{n-2} \circ \cdots \circ s^{k-2} \circ s^k \circ \cdots \circ s^0}_{n-1 \text{ maps}}$$

where we compose all the degeneracies except for s^{k-1}. This gives us a map $\Delta^n \to \Delta^1$ and likewise for $X^n \to X^1$. Then for $f = \{f_0, f_1, ...\} \in \mathrm{Tot}(P^\bullet_{a,b}X)$, we have by commutativity that

$$f_1 \circ \alpha_{n,k} = \alpha_{n+1,k} \circ f_n.$$

But now the right hand side is just picking out the kth coordinate of f_n. Hence for $n \geq 2$, f_n is completely determined by f_1, so that the projection $\Phi : \mathrm{Tot}(P^\bullet_{a,b}X) \to \mathrm{Map}(\Delta^1, X)$ given by $\{f_i\}_{i\geq 0} \mapsto f_1$ is injective.

We claim next that Φ is actually a map into $\Omega_{a,b}X \subset \mathrm{Map}(\Delta^1, X)$. The cosimplicial relations imply

$$f_1 \circ d^0 = d^0 \circ f_0 = \mathrm{const}_a, \quad f_1 \circ d^1 = d^1 \circ f_0 = \mathrm{const}_b$$

so $f_1(0) = a$ and $f_1(1) = b$ as desired.

Lastly we define an inverse to $\Phi : \mathrm{Tot}(P^\bullet_{a,b}X) \to \Omega_{a,b}X$. For a given path $\gamma \in \Omega_{a,b}X$ consider the family of maps $\{f_i : \Delta^i \to X^i\}_{i\geq 0}$ given as follows. We let f^0 be the constant map, $f^1 = \gamma$, and for $n \geq 2$ define

$$f_n(x_1, ..., x_n) = (\gamma(x_1), \gamma(x_1 + x_2), ..., \gamma(x_1 + \cdots + x_n)).$$

Clearly this is an inverse to Φ. We leave it to the reader to check that the family of maps $\{f_i\}$ commutes with the coface and codegeneracies, and that Φ and its inverse are continuous maps. □

We will call $P^\bullet_{a,b}X$ a cosimplicial model for the path space of X. When $a = b$ we get a cosimplicial model for the based loop space of X. It turns out that the cosimplicial space $P^\bullet_{x,x}X$ is in some sense the underlying cosimplicial set of the cobar construction of $C_\bullet(X)$, as we will see in the course of this thesis.

3.3. **Dold–Kan.** Recall that the standard Dold–Kan correspondence gives an equivalence of categories between simplicial abelian groups and (non-negative) chain complexes of abelian groups. We'll restate some important results and their duals. Although these are duals, there is once again some asymmetry, as suggested by the quote.

Definition 3.5. Let A be a simplicial abelian group. Its *normalized chain complex* $(NA_\bullet, \partial)$ has

$$NA_n = \bigcap_{i=1}^{n} \ker(d^i : A_n \to A_{n-1}), \quad \partial = d_0 : NA_n \to NA_{n-1}$$

and its *Moore complex* $(MA_\bullet, \partial')$ has

$$MA_n = A_n, \quad \partial' = \sum_{i=0}^{n} (-1)^i d_i : A_n \to A_{n-1}.$$

The following results and their proofs can be found in [GJ09], Chapter III.2:

Theorem 3.6 (Dold–Kan correspondence)**.** *Let A be a simplicial abelian group. Then:*

(1) The functor $N : \mathsf{sAb} \to \mathsf{Ch}_+(\mathsf{Ab})$ is an equivalence of categories between simplicial abelian groups and non-negative chain complexes of abelian groups.
(2) The inclusion of chain complexes $NA \hookrightarrow MA$ is a chain homotopy equivalence, natural in A.
(3) There is a functorial direct sum decomposition $MA = NA \oplus DA$, where DA is the subcomplex generated by the images of all the degeneracy maps. Consequently, DA is acyclic.

Now we dualize. Recall that for simplicial objects, the face maps d_i lower degree. For cosimplicial objects, the face maps d^i raise degree, so we should get an equivalence between cosimplicial abelian groups and cochain complexes of abelian groups. This is indeed what happens.

Definition 3.7. Let $C^\bullet$ be a cosimplicial abelian group. Its normalized cochain complex $(NC^\bullet, \partial)$ has

$$NC^n = \operatorname{coker} \bigoplus_{i=1}^{n} (d^i : C^{n-1} \to C^n), \quad \partial = d^0 : NC^n \to NC^{n+1}$$

and its Moore complex has

$$MC^n = C^n, \quad \partial' = \sum_{i=0}^{n+1} (-1)^i d^i : C^n \to C^{n+1}.$$

Analogously we obtain the following statements:

Theorem 3.8 (Dold–Kan correspondence, dual)**.** *Let C be a cosimplicial abelian group.*

(1) The functor $N : \mathsf{csAb} \to \mathsf{coCh}_+(\mathsf{Ab})$ is an equivalence of categories between cosimplicial abelian groups and non-negative cochain complexes of abelian groups.
(2) The quotient of cochain complexes $MC \to NC$ is a cochain homotopy equivalence, natural in C.
(3) There is a functorial direct sum decomposition $MC = NC \oplus DC$, where DC is the subcomplex generated by the images of all the coface maps. Consequently, DC is acyclic.

Really we could have stated the previous four theorems for simplicial objects in any abelian category. In the following discussion, we replace "abelian group" with "R-module" for some commutative ring R. Consider the free R-module functor $R[-] : \mathsf{Set} \to \mathsf{Mod}_R$. Then given any cosimplicial set $X^\bullet : \mathbf{\Delta} \to \mathsf{Set}$ the composition of functors $R[X^\bullet]$ yields a cosimplicial R-module. We first state a useful lemma, which was taken by a comment of Tom Goodwillie on a MathOverflow thread [hg] related to the forthcoming Proposition 3.10.

Lemma 3.9 (Goodwillie's lemma)**.** *Let $n > 0$ and choose $x, y \in X^n$. If x, y are not in the image of any coface maps $d^j : X^{n-1} \to X^n$, then $d^i x = d^j y$ implies $i = j$ and $x = y$.*

Proof. First suppose $i < j$. Then $d^i x = d^j y$ implies

$$d^i s^{j-1} x = s^j d^i x = s^j d^j y = y$$

so that y is in the image of a coface map, a contradiction. The same argument for $j < i$ leaves only the possibility $i = j$. But $s^i d^i = \mathrm{id}$, implying that d^i is injective, so $x = y$. □

Proposition 3.10. *Under the Dold–Kan correspondence, the normalized cochain complex of $R[X^\bullet]$ has zero cohomology in positive degree.*

Proof. Let $NR[X^\bullet]$ be the normalized cochain complex of $R[X^\bullet]$. In each degree, we have

$$NR[X^n] = R[X^n]/D^n$$

where D^n is the free module generated by $\cup_{i=1}^n \operatorname{im} d^i$. It follows that $NR[X^n]$ is the free module on the set $U^n := X^n \setminus \cup_{i=1}^n \operatorname{im} d^i$. Let A_n be the subset of X^n consisting of elements not in the image of any coface map, and define

$$B^n = \{d^0 x : x \in A^{n-1}, d^0 x \notin \cup_{i=1}^n d^i\}.$$

We make the following two claims:

(1) For $n \geq 1$, $U^n = A^n \sqcup B^n$

Proof. Disjointness and the inclusion $A^n \sqcup B^n \subset U^n$ are clear. Conversely, let $x \in U^n$. If $x \notin A^n$, then $x = d^k(y)$ for some $y \in X^{n-1}$, in fact $k = 0$ by definition of U^n. It remains to show $y \in A^{n-1}$. If $y \notin A^{n-1}$ then $y = d^l(z)$ for some $z \in A^{n-2}$ (here if $n = 1$ this is impossible). So

$x = d^0 d^l(z)$, and applying a cosimplicial identity we have $x = d^{l+1}d^0(z)$, contradicting the fact that $x \in U^n$. This completes the proof of the claim. □

(2) For $n \geq 2$, $B^n = \operatorname{im}(d^0 : X^{n-1} \to X^n)$.

Proof. It suffices to show that if $x \in \operatorname{im} d^0$, then $x \notin \cup_{i=1}^n \operatorname{im} d^i$. Suppose for contradiction that

$$x = d^0 a = d^j b$$

for $a \in A^{n-1}$, $b \in X^{n-1}$, and $j \geq 1$. Applying s^0 to both sides we obtain

$$a = s^0 d^0 a = s^0 d^j b = \begin{cases} b & j = 1 \\ d^{j-1} s^0 b & j > 1 \end{cases}.$$

In the first case we have $d^0 a = d^1 b$, contradicting Goodwillie's lemma since $a = b \in A^{n-1}$. In the second case we see that a is in the image of a coface map d^{j-1}, a contradiction. This completes the proof. □

Now we compute the coface map

$$d^0 : A^n \sqcup B^n \to A^{n+1} \sqcup B^{n+1}.$$

By the two claims, d^0 is a bijection $A^n \to B^{n+1}$. On B^n, we have $d^0 x = d^0 d^0 y = d^1 d^0 y$ for some $y \in X^{n-1}$ by cosimplicial identities, so this is zero in NC^n. It follows that

$$H^n(R[X^\bullet]) = \frac{\ker(d^0 : NC[X^n] \to NC[X^{n+1}])}{\operatorname{im}(d^0 : NC[X^{n-1}] \to NC[X^n])} = \frac{B^n}{B^n} = 0$$

completing the proof. □

Remark 3.11. The analogue of Proposition 3.10 for simplicial sets is completely false! For example, gien a simplicial set $K_\bullet$, the chain complex $NR[K_\bullet]$ computes the homology of K. If these groups were all zero we would be in trouble.

3.4. **Alexander–Whitney and Eilenberg–Zilber.** In the proof of Looijenga's theorem there is a step where we roughly need to move from singular chains on a product space to the product of chains. There is a canonical way to do so, called the Alexander–Whitney map, with a natural inverse called the Eilenberg–Zilber map.

The Alexander–Whitney map is a way of moving between the category of simplicial abelian groups and the category of chain complexes of abelian groups that preserves their respective monoidal structure. Let us recall these structures. The monoidal product on sAb is given levelwise: for $A, B \in \mathsf{sAb}$, we have

$$(A \otimes B)_n = A_n \otimes B_n$$

and the face and degeneracy maps are inherited naturally. The monoidal product on the category of chain complexes is given as follows: for $X, Y \in \mathsf{Ch}_+(\mathsf{Ab})$, define

$$(X \otimes Y)_n = \bigoplus_{p+q=n} X_p \otimes Y_q.$$

and the new differential $(X \otimes Y)_n \to (X \otimes Y)_{n-1}$ by the Koszul sign rule:

$$\partial_{X \otimes Y}(x \otimes y) = \partial_X x \otimes y + (-1)^{\deg x} x \otimes \partial_Y y.$$

Definition 3.12. Let $M : \mathsf{sAb} \to \mathsf{Ch}_+(\mathsf{Ab})$ be the Moore complex functor. The *Alexander–Whitney* map is a natural transformation of the functors $\mathsf{sAb} \times \mathsf{sAb} \to \mathsf{Ch}_+(\mathsf{Ab})$

$$\mathcal{AW} : M(- \otimes -) \implies M(-) \otimes M(-)$$

whose components are given as follows. For $A, B \in \mathsf{sAb}$ and $a \in A_n, b \in B_n$, define the map

$$\mathcal{AW}_{A,B} : M(A \otimes B) \to M(A) \otimes M(B), \quad a \otimes b \mapsto \bigoplus_{p+q=n} \tilde{d}^p(a) \otimes d^q(b)$$

where d^p is the face map induced by $[p] \to [n], i \mapsto i$, and d^q is the face map induced by $[q] \to [n], i \mapsto i + p$.

Remark 3.13. $\mathcal{AW}$ restricts to a natural transformation

$$N(- \otimes -) \implies N(-) \otimes N(-)$$

where N is the normalized chains functor. (We will denote this restriction also by $\mathcal{AW}$.)

Proposition 3.14. *Let $A, B \in \mathsf{sAb}$. Then*

$$\mathcal{AW}_{A,B} : N(A \otimes B) \to N(A) \otimes N(B)$$

is a natural quasi-isomorphism.

Proof. Kerodon [Lur26, Tag 00S0]. □

This is a significant fact in homological algebra; now we employ only a fraction of its power. Let X, Y be a topological space, and $C_\bullet(-)$ denote the singular chains functor from Top to sAb. Then $MC_\bullet(X)$ is the singular chain complex of X in the usual sense.

Corollary 3.15. *There is a natural quasi isomorphism*

$$MC_\bullet(X \times Y) \to MC_\bullet(X) \otimes MC_\bullet(Y).$$

Proof. Recall that $C_\bullet(-)$ is the composition of the following functors:

$$\mathsf{Top} \xrightarrow{X \mapsto \mathrm{Map}(\Delta^\bullet, X)} \mathsf{sSet} \xrightarrow{A_\bullet \mapsto \mathbf{Z}[A_\bullet]} \mathsf{sAb}$$

and under this composition we get

$$X \times Y \ \mapsto \mathrm{Map}(\Delta^\bullet, X \times Y) = \mathrm{Map}(\Delta^\bullet, X) \times \mathrm{Map}(\Delta^\bullet, Y) \mapsto \mathbf{Z}[\mathrm{Map}(\Delta^\bullet, X)] \otimes \mathbf{Z}[\mathrm{Map}(\Delta^\bullet, Y)]$$

because the free abelian group functor sends Cartesian products of sets to tensor products of groups. So the simplicial abelian group $C_\bullet(X \times Y)$ is equal to $C_\bullet(X) \otimes C_\bullet(Y)$. Thus applying Proposition 3.14 and the Dold–Kan correspondence we are done. □

Really what we have defined here is a topological Alexander–Whitney map which is a natural transformation of the functors $\mathsf{Top} \times \mathsf{Top} \to \mathsf{sAb}$:

$$MC_\bullet(- \times -) \to MC_\bullet(-) \otimes MC_\bullet(-)$$

For topological spaces X, Y, denote the constituent map by $\mathcal{AW}_{X,Y}$. From now we will use $C_\bullet(X)$ to mean both the singular chain complex and the simplicial abelian group, depending on context.

Corollary 3.16. *For $q \geq 0$ and a topological space X, there is a natural quasi-isomorphism of chain complexes*

$$\mathcal{AW}_q : C_\bullet(X^q) \to C_\bullet(X)^{\otimes q}.$$

Proof. Iterate the topological Alexander–Whitney map

$$C_\bullet(X \times X^{q-1}) \xrightarrow{\mathcal{AW}_{X,X^{q-1}}} \cdots \xrightarrow{\mathrm{id}^{\otimes q-2} \otimes \mathcal{AW}_{X,X}} C_\bullet(X)^{\otimes q}.$$

Since each component is a natural quasi-isomorphism, so is the composition. □

As promised, there is also a map going the other way, called the *Eilenberg–Zilber* map, a natural transformation of functors

$$\mathcal{EZ} : M(-) \otimes M(-) \implies M(- \otimes -).$$

We are content without having an explicit formula for this map; it is slightly more finnicky than the one for Alexander–Whitney, and all we need to know is the following theorem:

Theorem 3.17 (Eilenberg–Zilber theorem, [EZ53])**.** *For simplicial abelian groups A, B, we have chain homotopies*

$$\mathcal{AW} \circ \mathcal{EZ} \simeq \mathrm{id}_{M(A) \otimes M(B)}, \quad \mathcal{EZ} \circ \mathcal{AW} \simeq \mathrm{id}_{M(A \otimes B)}.$$

Consequently, $\mathcal{EZ}$ is also a quasi-isomorphism, and these maps are inverses. Specializing to our topological setting, we obtain

Corollary 3.18. *For $q > 0$ and a topological space X, there is a natural quasi-isomorphim of chain complexes*

$$\mathcal{EZ}_q : C_\bullet(X)^{\otimes q} \to C_\bullet(X^q).$$

Moreover, $\mathcal{EZ}_q \circ \mathcal{AW}_q$ is chain homotopic to the identitty on $C_\bullet(X^q)$ and $\mathcal{AW}_q \circ \mathcal{EZ}_q$ is chain homotopic to the identity on $C_\bullet(X)^{\otimes q}$. □

Remark 3.19. The proof of the Eilenberg–Zilber theorem appearing in [EZ53] did not contain any explicit formulas. Rather the proof made use of acyclic models. Explicit formulas for the maps involved can be found in [EML53], stated in terms of (p, q)-*shuffles*, which correspond to the way simplicies can be multiplied and divided.

4. Adams' cobar construction

In this chapter we define the cobar construction of a coalgebra and introduce Adams' remarkable 1956 theorem [Ada56], which (roughly) says that given a based topological space (X, x), one can recover the singular chains of the based loop space $\Omega_x X$ via a purely algebraic construction on the singular chains on X. We then extend the cobar construction to account for two basepoints by using comodules over a coalgebra, and conclude by elaborating on the relationship between the cobar construction and the fundamental group.

4.1. The cobar construction.

Let R be a commutative unital ring and let $(C = \overline{C} \oplus R, \Delta)$ be a dg coaugmented counital coassociative R-coalgebra. The cobar construction is a functor from this category of coalgebras to the category of dg associative algebras. There are many references for the following definition, including Adams' original paper [Ada56] and [Che73]. We follow the presentation given in [Riv22].

Definition 4.1. Let (C, Δ, ∂) be as above. The *cobar construction* on C is the dg R-algebra whose underlying algebra is the tensor algebra over the desuspension of the reduced coalgebra C:

$$\mathsf{Cobar}(C) := T(s^{-1}\overline{C}) = R \oplus s^{-1}\overline{C} \oplus s^{-1}\overline{C}^{\otimes 2} \oplus \cdots$$

and whose differential is given by extending the linear map

$$-s^{-1} \circ \partial \circ s^{+1} + (s^{-1} \otimes s^{-1}) \circ \Delta \circ s^{+1} : s^{-1}\overline{C} \to s^{-1}\overline{C} \oplus s^{-1}\overline{C}^{\otimes 2}$$

to all of $T(s^{-1}\overline{C})$, which yields a linear map of degree -1 from $T(s^{-1}\overline{C})$ to itself.

The definition of the differential here is valid owing to Proposition 2.6, which says that it suffices to define the differential on the component $s^{-1}\overline{C} \subset T(s^{-1}\overline{C})$. Regardless, it will be helpful to be a little more explicit. By the discussion in the previous chapter we have that as a chain complex, $\mathsf{Cobar}(C)$ has terms

$$\begin{aligned}
\mathsf{Cobar}_k(C) &= T_k(s^{-1}\overline{C}) \\
&= \bigoplus_{q\geq 0}(s^{-1}\overline{C})_k^{\otimes q} \\
&= \bigoplus_{q\geq 0}\bigoplus_{n_1+\cdots+n_q=k}(s^{-1}\overline{C})_{n_1}\otimes\cdots\otimes(s^{-1}\overline{C})_{n_q} \\
&= \bigoplus_{q\geq 0}\bigoplus_{n_1+\cdots+n_q=k}\overline{C}_{n_1+1}\otimes\cdots\otimes\overline{C}_{n_q+1} \\
&= \bigoplus_{q\geq 0}\bigoplus_{m_1+\cdots+m_q=k+q}\overline{C}_{m_1}\otimes\cdots\otimes\overline{C}_{m_q} \\
&= \bigoplus_{p-q=k}\bigoplus_{m_1+\cdots+m_q=p}\overline{C}_{m_1}\otimes\cdots\otimes\overline{C}_{m_q}
\end{aligned} \tag{4.2}$$

The purpose of introducing the dummy variable p at the end of this tedious indexing exercise was to suggest that we should be able to realize the cobar construction as the totalization of a bicomplex in p and q. Indeed, in the cobar differential, we utilize both the internal differential ∂ of C as well as the coproduct Δ. One of these will turn out to raise q and the other will lower p.

Definition 4.3. Let (C, Δ, ∂) be as above. The *cobar bicomplex* is the bicomplex $\mathsf{Cobar}_\bullet^\bullet(C)$ with

$$\mathsf{Cobar}_p^q := \bigoplus_{m_1+\cdots+m_q=p}\overline{C}_{m_1}\otimes\cdots\otimes\overline{C}_{m_q}$$

and differentials

$$d_H : \mathsf{Cobar}_{p,q} \to \mathsf{Cobar}_{p-1}^q, \quad x_1\otimes\cdots x_q \mapsto \sum_{i=1}^q(-1)^{\sigma(x_i)}x_1\otimes\cdots\otimes\partial(x_i)\otimes\cdots\otimes x_q$$

$$d_V : \mathsf{Cobar}_p^q \to \mathsf{Cobar}_p^{q+1}, \quad x_1\otimes\cdots\otimes x_q \mapsto \sum_{i=1}^q(-1)^{\sigma(x_i)}x_1\otimes\cdots\otimes\overline{\Delta}(x_i)\otimes\cdots\otimes x_q$$

where the x_i are elements of $(s^{-1}\overline{C})_{n_i}$ and $\sigma(x_i)$ is the Koszul sign $\deg x_1 + \cdots + \deg x_{i-1}$.

Proposition 4.4. *The totalization of* $\mathsf{Cobar}_\bullet^\bullet(C)$ *given by*

$$\mathrm{Tot}_k(\mathsf{Cobar}_\bullet^\bullet) = \bigoplus_{p-q=k}\mathsf{Cobar}_p^q$$

with differential $\partial_{\mathrm{Tot}} = d_H + (-1)^q d_V$ *on* Cobar_p^q*, is isomorphic as a dg algebra to* $\mathsf{Cobar}(C_\bullet)$.

Proof. By the preceeding discussion, we have an equality of groups $\mathrm{Tot}_k(\mathsf{Cobar}_\bullet^\bullet)$ and $\mathsf{Cobar}_k(C)$. Now by Proposition 2.6, it suffices to check that ∂_{Tot} extends the map $s^{-1}\overline{C} \to s^{-1}\overline{C}\oplus s^{-1}\overline{C}^{\otimes 2}$ given in the definition of the cobar complex. Indeed, we have that on

$$s^{-1}\overline{C} = \bigoplus_{n_1=p\geq 0}(s^{-1}\overline{C})_{n_1} = \bigoplus_{p\geq 0}\mathsf{Cobar}_p^1,$$

the differential is $\partial_{\mathrm{Tot}} = d_H - d_V$ which is exactly the cobar differential. □

What is miraculous is that this purely algebraic construction encodes a good amount of topological information. This is the content of Adams' theorem:

Theorem 4.5 ([Ada56])**.** *Let (X, x) be a simply connected based topological space. Let $\Omega_x X$ be the space of loops in X based at x. Then there is a natural isomorphism*

$$H_\bullet(\mathsf{Cobar}(C_\bullet(X))) \cong H_\bullet(\Omega_x X).$$

Thus we can regard the cobar construction as the algebraic analogue of the based loop space functor $\Omega_x(-)$. As the prefix *co-* suggests, the cobar construction is adjoint to a functor

$$\mathsf{Bar} : \mathsf{dgAlg} \to \mathsf{dgConilCoalg}$$

called the *bar construction*, which is (roughly) the algebraic version of the delooping of a space. See Chapter 2 of [LV12] for precise statements and a comparison of the two constructions.

Remark 4.6. The name *bar* construction comes from typesetting. Pre-LaTeX, a tensor $x_1 \otimes \cdots \otimes x_n$ would be written $x_1|\cdots|x_n$. Adams attributes the name *cobar* to H. Cartan (see footnote 3 of [Ada56]).

4.2. **Two-sided version.** The cobar construction of a dg coalgebra is a special case of a more general *two-sided* cobar construction. An analogous two-sided bar construction is well documented (for example [May72], [Zha19], [MZZ25]). Here we give an exposition of the dual construction, which seems to not appear in the literature.

Definition 4.7. Let $C = \overline{C} \oplus R$ be a coaugmented R-coalgebra and let N, M be left and right comodules over C, respectively. The *two-sided cobar construction on* (N, C, M) is the cosimplicial R-module

$$\mathsf{Cobar}(N, C, M)^q = M \otimes C^{\otimes q} \otimes N$$

with cofaces $d^i : \mathsf{Cobar}^{q-1} \to \mathsf{Cobar}^q$ given by

$$d^i = \begin{cases} \Delta^R \otimes \mathrm{id}^{\otimes q} & i = 0 \\ \mathrm{id}^{\otimes i} \otimes \Delta \otimes \mathrm{id}^{q-i} & i \in \{1, \dots, q-1\} \\ \mathrm{id}^{\otimes q} \otimes \Delta^L & i = q \end{cases}$$

and codegeneracies $s^i : \mathsf{Cobar}^{q+1} \to \mathsf{Cobar}^q$ given by

$$s^i = \mathrm{id}^{\otimes i} \otimes \epsilon \otimes \mathrm{id}^{\otimes q-i-1}$$

If C, N, M have additional dg structure (i.e. C is a dg coalgebra and N, M are dg comodules), then we also get an internal differential on $\mathsf{Cobar}(N, C, M)$ in addition to the differential coming from the coalgebra and comodule maps. Taking these new differentials into account, we obtain the following notion, generalizing the previous discussion of the cobar bicomplex:

Definition 4.8. Let (C, ∂_C), (N, ∂_N), and (M, ∂_M) be as above. The *two-sided cobar bicomplex* has underlying R-module

$$\mathsf{Cobar}(N, C, M)^\bullet_\bullet = \bigoplus_{q \geq 0} M \otimes C^{\otimes q} \otimes N$$

and bigrading

$$\mathsf{Cobar}(N, C, M)^q_p = \bigoplus_{a + b_1 + \cdots + b_q + c = p} M_a \otimes C_{b_1} \otimes \cdots \otimes C_{b_q} \otimes N_c$$

with vertical differential given by the alternating sums

$$d^V : \mathsf{Cobar}(N, C, M)^{q-1}_\bullet \to \mathsf{Cobar}(N, C, M)^q_\bullet, \quad d_V = \sum_{i=0}^{q} d^i$$

and horizontal differential coming from the comodule and coalgebra differentials:

$$d_H : \mathsf{Cobar}(N,C,M)^q_p \to \mathsf{Cobar}(N,C,M)^q_{p-1}$$

$$\begin{aligned} d_H(m\otimes c_1\otimes\cdots\otimes c_q\otimes n) = \;& \partial_M(m)\otimes c_1\otimes\cdots\otimes c_q\otimes n \\ &+\sum_{i=1}^{q}(-1)^{|m|+\sigma(c_i)} m\otimes c_1\otimes\cdots\otimes\partial_C(c_i)\otimes\cdots\otimes c_q\otimes n \\ &+(-1)^{|m|+\sigma(c_q)} m\otimes c_1\otimes\cdots\otimes c_q\otimes\partial_N(n) \end{aligned} \tag{4.9}$$

where $\sigma(x_k) = |x_1| + \cdots |x_{k-1}|$ is the Koszul sign. Then totalizing with total degree $p-q$ and differential $d_H + (-1)^q d^V$ on $\mathsf{Cobar}(N,C,M)^q_\bullet$ yields a dg algebra, the *two-sided cobar construction* of (N,C,M).

This two-sided construction reduces to the one in the previous subsection as follows. Let C be a dg coalgebra with coaugmentation $\eta : R \to C$. Set $N = M = R$, regarded as a dg R-module with R concentrated in degree zero. Define the left action $\Delta^L : N \to C\otimes N$ via

$$\begin{tikzcd} R \arrow[d, "\eta"'] \arrow[dr, dashed, "\Delta^L"] & \\ C \arrow[r, "\cong"'] & C\otimes R \end{tikzcd}$$

and likewise for the right action. This is the correct reduction:

Proposition 4.10. $\operatorname{Tot}\mathsf{Cobar}(R,C,R)$ *as above is quasi-isomorphic to the single cobar construction from Definition 3.1.*

For the proof we need a fact from homological algebra.

Lemma 4.11. *Let $f : D_{\bullet,\bullet} \to E_{\bullet,\bullet}$ be a map of first quadrant bicomplexes of abelian groups. If f is a row-wise or column-wise quasi-isomorphism, then the induced map on totalizations $f_* : \operatorname{Tot} D \to \operatorname{Tot} E$ is a quasi-isomorphism.*

Proof. We do the proof where f is a row-wise quasi-isomorphism, i.e. for fixed q we have a quasi-isomorphism

$$f : D_{\bullet,q} \to E_{\bullet,q}.$$

Let us first consider the truncated bicomplexes $D^{\le n}_{\bullet,\bullet}, E^{\le n}_{\bullet,\bullet}$ where the first n rows are nonzero. Then f_* restricts to a map $D^{\le n}_{\bullet,\bullet} \to E^{\le n}_{\bullet,\bullet}$. We will induct on n. If $n = 0$, there is nothing to prove. Suppose now that $n > 1$ and $f : D^{\le n-1}_{\bullet,\bullet} \to E^{\le n-1}_{\bullet,\bullet}$ induces a quasi-isomorphism of totalizations.

We totalize by taking $\operatorname{Tot}(D)_k = \bigoplus_{p+q=k} D_{p,q}$. So we have an exact sequence of chain complexes

$$\operatorname{Tot}(D^{\le n-1}) \hookrightarrow \operatorname{Tot}(D^{\le n}) \twoheadrightarrow D_{\bullet,n}[-n]$$

where $D_{\bullet,n}[-n]$ is the chain complex beginning below in degree 0,

$$D_{\bullet,n}[-n] : 0 \leftarrow \cdots \leftarrow D^{0,n} \leftarrow D^{1,n} \leftarrow \cdots$$

having first nonzero term $D^{0,n}$ in degree n. Repeating the same construction for E, we get a commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Tot}(D^{\le n-1}) \arrow[r, hook] \arrow[d, "f_*"'] & \operatorname{Tot}(D^{\le n}) \arrow[r, two heads] \arrow[d, dashed] & D_{\bullet,n}[-n] \arrow[r] \arrow[d, "f_*"] & 0 \\ 0 \arrow[r] & \operatorname{Tot}(E^{\le n-1}) \arrow[r, hook] & \operatorname{Tot}(E^{\le n}) \arrow[r, two heads] & E_{\bullet,n}[-n] \arrow[r] & 0 \end{tikzcd}$$

in which the two outer arrows are quasi-isomorphisms. The five lemma concludes the induction. Hence $f_* : \mathrm{Tot}(D^{\leq n}) \to \mathrm{Tot}(E^{\leq n})$ is a quasi-isomorphism for all n.

We can now write D and E as the direct limits (which are colimits) of their truncated subcomplexes

$$D = \varinjlim_n D^{\leq n}, \quad E = \varinjlim_n E^{\leq n}.$$

Now totalization is a colimit, and colimits commute with colimits, so f_* assembles into a map of direct limits

$$f_* : \varinjlim_n \mathrm{Tot}(D^{\leq n}) \to \varinjlim_n \mathrm{Tot}(E^{\leq n}).$$

Since homology commutes with filtered colimits, we a map of colimits

$$\varinjlim_n H_* \,\mathrm{Tot}(D^{\leq n}) \to \varinjlim_n H_* \,\mathrm{Tot}(E^{\leq n})$$

where each map $H_* \,\mathrm{Tot}(D^{\leq n}) \to H_* \,\mathrm{Tot}(E^{\leq n})$ is an isomorphism, implying that

$$f_* : H_* \,\mathrm{Tot}(D) \to H_* \,\mathrm{Tot}(E)$$

is too. The proof supposing f is a column-wise quasi-isomorphism proceeds identically, by truncating column-wise. □

Because we will use this notion many more times times in this thesis, we make the following definition:

Definition 4.12. A map $f : D \to E$ of first quadrant spectral sequences is a *quasi-isomorphism* if it is a row- or column-wise quasi-isomorphism.

Then Lemma 4.11 says that a quasi-isomorphism of bicomplexes induces a quasi-isomorphism of totalizations.

Proof of Proposition 4.10. We take the normalized cochain complex on each column of $\mathsf{Cobar}^\bullet_p$ to obtain:

$$\begin{aligned} N\,\mathsf{Cobar}^q_p &= \bigcap_{i=0}^{q-1} \ker(s^i : \mathsf{Cobar}^q_p \to \mathsf{Cobar}^{q-1}_p) \\ &= \bigcap_{i=0}^{q-1} \ker(\mathrm{id}^i \otimes \epsilon \otimes \mathrm{id}^{\otimes q-i-1})_p \\ &= (\ker(\epsilon)^{\otimes q})_p \\ &= \bigoplus_{m_1+\cdots+m_q=p} \overline{C_{n_1}} \otimes \cdots \otimes \overline{C_{n_q}} \\ &= oldcobar. \end{aligned} \tag{4.13}$$

Upon checking differentials, we find that the normalized bicomplex $N\,\mathsf{Cobar}$ is equal to the reduced cobar construction on C. Since $N\,\mathsf{Cobar} \hookrightarrow \mathsf{Cobar}$ is a quasi-isomorphism, we conclude by Proposition 4.10. □

Finally, we should check that this generalization of the cobar construction extends Adams' theorem.

Theorem 4.14. *Let X be a space with basepoints a, b. Regard $C_\bullet(\{a\})$ and $C_\bullet(\{b\})$ as ...*

There is a natural quasi-isomorphism from the totalization of the two-sided cobar construction

$$\mathsf{Tot}\,\mathsf{Cobar}(C_\bullet(\{b\}), C_\bullet(X), C_\bullet(\{a\}))$$

to the singular chains on the path space $C_\bullet(\Omega_{a,b}X)$.

Proof. ◻

4.3. **Homology and the fundamental group.** Adams' theorem allows one to use homology and the cobar construction to approximate the fundamental group:

Theorem 4.15 ([Sta75], Theorem 5.4)**.** *Let R be a ring and (X, x) a path-connected space. Let $\mathcal{I}$ be the augmentation ideal of the group ring $R[\pi_1(X, x)]$. Let E be the spectral sequence associated to the cobar construction on $C_\bullet(X)$ with basepoint x. Then there is an isomorphism*

$$\mathcal{I}^p/\mathcal{I}^{p+1} \cong E^\infty_{-p,p}.$$

A dual version of this theorem with cohomology and the bar construction can be found in [Gad23], Theorem 1.2(3). The goal of this subsection will be to prove an integral version of this theorem for the two-sided cobar construction.

Theorem 4.16. *Let X be a path-connected space with basepoints a, b. Let $\mathcal{I}$ be the augmentation ideal of the group ring $\mathbf{Z}\pi_1(X, a)$. Let E be the spectral sequence associated to the two-sided cobar construction*

$$\mathsf{Cobar}(NC_\bullet(\{b\}), NC_\bullet(X), NC_\bullet(\{a\}))$$

Then there is an isomorphism

$$\mathcal{I}^p/\mathcal{I}^{p+1} \to E^\infty_{-p,p}.$$

Proof. It will again to turn out to be easier to work with the normalized cobar, a la the proof of Proposition 4.10. So abbreviate $N\,\mathsf{Cobar}(\cdots)$ by Cobar. By Theorem 4.14, we have an isomorphism

$$H_0(\mathrm{Tot}(\mathsf{Cobar})) \to H_0(\Omega_{a,b}X) \cong \mathbf{Z}\pi_0(\Omega_{a,b}X).$$

Recall that the bigrading on the two-sided cobar construction is given by

$$\mathsf{Cobar}(N, C, M)^q_p = \bigoplus_{a+b_1+\cdots+b_q+c=p} M_a \otimes C_{b_1} \otimes \cdots \otimes C_{b_q} \otimes N_c$$

so that the degree zero totalization is given exactly by terms with $p = -q$. Since $NC_\bullet(\{a\})$ and $NC_\bullet(\{b\})$ have only nonzero terms concentrated in degree zero, while in the reduced complex we have $\overline{NC_0(X)} = 0$, we are forced to have

$$\mathsf{Cobar}^p_{-p} = NC_\bullet(\{a\}) \otimes \overline{NC_1(X)}^{\otimes p} \otimes NC_\bullet(\{b\}).$$

We can then define a map from $\mathrm{Tot}_0(\mathsf{Cobar})$ to a rank one free module over $\mathbf{Z}\pi_1(X, a)$ as follows. Fix a spanning tree of X with basepoint at a, i.e. a collection of paths for each x in a

$$\{s : [0, 1] \to X : s_x(0) = a, s_x(1) = x\}.$$

with the requirement that s_a is the constant path at a. Then define $\tilde{\phi} : \mathrm{Tot}_0(\mathsf{Cobar}) \to \mathbf{Z}\pi_1(X, a)[s_b]$ via

$$\tilde{\phi}(1_a \otimes \sigma_1 \otimes \cdots \sigma_p \otimes 1_b) = (1_a - \tilde{\sigma}_1) \cdots (1_a - \tilde{\sigma}_p)s_b$$

where

$$\tilde{\sigma}_i = s_{\sigma_i(0)} * \sigma_i * s_{\sigma_i(1)}.$$

Clearly the image of $\tilde{\phi}$ is all of $\mathcal{J}^p$. We claim that $\tilde{\phi}$ descends to $H_0\,\mathrm{Tot}(\mathsf{Cobar})$. ◻

5. Proof of main theorem

Here we present the proof of Theorem 1.1. Throughout this section let X be a path connected topological space with basepoints a, b. Our strategy will be use the cosimplicial path space model to construct the E^0-page of a spectral sequence which (1) converges to the group $H_n(X^n, X(n)^a_b)$ and (2) is related to the cobar construction, hence the groups $\mathbf{Z}\pi_1(X,a)/\mathbf{Z}\pi_1(X,a)\mathcal{I}_a^{n+1}$.

5.1. **Building the spectral sequence.** Recall the cosimplicial path space $P^\bullet_{a,b}X$ of Example 3.3. From now we denote it by $P^\bullet$ for simplicity. Notice first that the union of the images of the coface maps $d^i : X^{n-1} \to X^n$ is exactly the subspace $X(n)^a_b$. This is a good sign.

Now consider the bicomplex $\mathscr{C}^\bullet_\bullet$ which is obtained by applying singular chains to $P^\bullet$:

$$\mathscr{C}^q_p = C_p(P^q)$$

with "horizontal" differential $d^{\mathscr{C}}_H : C_p(P^q) \to C_{p-1}(P^q)$ given by the singular boundary map, and "vertical" differential $d^V_{\mathscr{C}} : C_p(P^q) \to C_p(P^q)$ given by the differential of the Moore complex of the cosimplicial abelian group $C_p(P^\bullet)$.

Proposition 5.1. *$C^\bullet_\bullet$ is a bicomplex.*

Proof. That $d^{\mathscr{C}}_H$ and $d^V_{\mathscr{C}}$ both square to zero follows from the fact that each row is a chain complex and each row is a cochain complex. It remains to check commutativity of the squares:

$$\begin{array}{ccc} C_{p-1}(X^{q+1}) & \longleftarrow & C_p(X^{q+1}) \\ \uparrow & & \uparrow \\ C_{p-1}(X^q) & \longleftarrow & C_p(X^q) \end{array}$$

Starting with a map $f : \Delta^p \to X^q$, if we first go up then left, we get the map

$$\Delta^{p-1} \xrightarrow{([p]\to[p-1])^*} \Delta^p \xrightarrow{f} X^q \xrightarrow{\sum(-1)^i d^i} X^{q+1}$$

and proceeding in the other direction gives the exact same map. (This is really obvious, might not need to include). ◻

Each column p of this bicomplex forms a cochain complex $\mathscr{C}^\bullet_p$ which is the Moore complex of $C_p(F^\bullet)$. Consider now the bicomplex $\mathscr{N}^\bullet_\bullet$ whose columns are instead the normalized cochain complexes of $C_p(F^\bullet)$. This is a bicomplex for the same reason as above, and the inclusion $\mathscr{N} \hookrightarrow \mathscr{C}$ is a quasi-isomorphism of bicomplexes by Dold–Kan. Let us spell out what the groups in $\mathscr{N}$ actually are. Following the definition, we have

$$\mathscr{N}^q_p = \operatorname{coker} \bigoplus_{i=1}^{n} ((d^i)_* : C_p(X^{n-1}) \to C^p(X^n)) = C^p(X^n)/\langle \sum_{i=1}^{n} \operatorname{im}(d^i)_* \rangle = C_p(X^n, X(n)_b).$$

Moreover, the vertical differential $\mathscr{N}^q_p \to \mathscr{N}^{q+1}_p$ is induced by the zeroth coface map d^0.

Now consider the bicomplex obtained from $\mathscr{N}$ by setting all rows above $q = n$ to zero, which we will denote by $(\mathscr{N}^{\le n})^\bullet_\bullet$.

Proposition 5.2. *Filtering $(\mathscr{N}^{\le n})^\bullet_\bullet$ in the horizontal direction i.e. by the lower index, we obtain a spectral sequence collapsing on the second page with*

$$E^2_{n,n} = H_n(X^n, X(n)^a_b), \quad E^2_{0,0} = \begin{cases} \mathbf{Z} & a = b \\ 0 & a \neq b \end{cases}$$

converging to $H_\bullet(\mathrm{Tot}(\mathscr{N}^{\le n})^\bullet_\bullet)$.

Proof. Convergence is given by the fact that our bicomplex is concentrated in the first quadrant. To compute the E^2-page, let us begin in the E^0-page, whose pth column is the following normalized cochain complex (beginning below in degree $q = 0$):

$$C_p(*) \to C_p(X, \{b\}) \to \cdots \to C_p(X^n, X(n)_b) \to 0 \to \cdots$$

of the Moore complex

$$C_p(*) \to C_p(X) \to \cdots \to C_p(X^n) \to 0 \to \cdots \tag{5.3}$$

so they have the same cohomology. Moreover, the E^0-differential is the cochain differential d^0 given our filtration. But now (5.3) is just the cochain complex obtained by taking the free abelian group on the cosimplicial set $[q] \mapsto \mathrm{Map}(\Delta^p, F^q)$, with terms $q > n$ set to zero. Therefore, by Proposition 3.10, we have that $E^1_{p,q} = 0$ for $q \neq 0, n$. For $q = n$ and $q = 0$, we have

$$E^1_{p,n} = C_p(X^n, X(n)_b)/\mathrm{im}(d^0) = C_p(X^n, X(n)^a_b)$$

$$E^1_{p,0} = \ker(d^0 : C_p(*) \to C_p(X, \{b\})) = \begin{cases} \mathbf{Z} & p = 0, a = b \\ 0 & \text{otherwise} \end{cases}.$$

On the E^1-page, the differential is now induced by singular boundary map. Thus on the only two possibly nonzero rows $q = n$ of the E^1-page, we just compute the singular homology of to obtain

$$E^2_{p,n} = H_p(X^n, X(n)^a_b).$$

And on row $q = 0$, we have

$$E^2_{p,0} = \begin{cases} H_p(\mathbf{Z} \leftarrow 0 \leftarrow \cdots) & a = b \\ 0 & a \neq b \end{cases}$$

as desired.

At this point, because the only nonzero rows are $q = n$ and possibly $q = 0$, the differentials on the pages $E^{\geq 2}$ are all zero, except possibly on page E^{n+1}, where we have a differential d_{n+1} of bidegree $(-n-1, -n)$ from $E^{n+1}_{n+1,n} = H_{n+1}(X^n, X(n)^a_b)$ to $E^{n+1}_{0,0} = \mathbf{Z}$. To see that $d_{n+1} = 0$, consider the map of pairs $f : (X, x) \to (*, *)$. Repeating the constructions in this section, we get a spectral sequence $D_{\bullet,\bullet}$ computing $H_n(*^n, *(n)^*_*) = 0$. By functoriality, f induces a morphism of spectral sequences. Specializing to the $n+1$th page, we have a commutative square

$$\begin{tikzcd} H_{n+1}(X^n, X(n)^x_x) \arrow[r, equal] & E^{n+1}_{n+1,n} \arrow[r, "d^E_{n+1}"] \arrow[d, "f_*"'] & E^{n+1}_{0,0} \arrow[r, equal] \arrow[d, "f_*"] & \mathbf{Z} \\ H_{n+1}(*, *) = 0 \arrow[r, equal] & D^{n+1}_{n+1,n} \arrow[r, "d^D_{n+1}"'] & D^{n+1}_{0,0} \arrow[r, equal] & \mathbf{Z} \end{tikzcd}$$

But now the $f_* : \mathbf{Z} \to \mathbf{Z}$ is the identity since it is the induced map $H_0(X, x) \to H_0(*, *)$, forcing $d^E_{n+1} = 0$, so $E^2 = E^\infty$ as desired. □

Corollary 5.4. *We have a surjection*

$$H_0(\mathrm{Tot}(\mathcal{N}^{\leq n})^\bullet_\bullet) \twoheadrightarrow H_n(X^n, X(n)^a_b)$$

whose kernel is $\mathbf{Z}$ *if* $a = b$ *and trivial otherwise.*

Proof. Follows from convergence of the spectral sequence. □

5.2. **Functoriality.** It remains to compute H_0 of this totalization. For this we return to our original bicomplex $\mathscr{C}_\bullet^\bullet$ and instead of normalizing the columns, we first look at the rows $\mathscr{C}_\bullet^q$ for each q, which are the singular chain complexes $C_\bullet(X^q)$. Then we can apply the topological Alexander–Whitney map on each row:

$$\mathscr{C}_\bullet^q = C_\bullet(X^q) \to C_\bullet(X)^{\otimes q}$$

to obtain a new bicomplex, which we will call $\mathscr{D}_\bullet^\bullet$.

Proposition 5.5. *The map $\mathscr{C} \to \mathscr{D}$ induced by Alexander–Whitney is a quasi-isomorphism of bicomplexes, with inverse quasi-isomorphism induced by the Eilenberg–Zilber map.*

Proof. Combine Corollary 3.16, Theorem 3.17, and Proposition 4.11, □

By definition of tensor product of chain complexes, $\mathscr{D}$ has a bigrading given by

$$\mathscr{D}_p^q = (C_\bullet(X)^{\otimes q})_p = \bigoplus_{n_1+\cdots+n_q=p} C_{n_1}(X) \otimes \cdots \otimes C_{n_q}(X). \tag{5.6}$$

Theorem 5.7. *Let $NC_\bullet(-)$ be the normalized singular chains functor. Regard $NC_\bullet(\{a\})$ as a right $NC_\bullet(X)$-comodule via the right comodule action*

$$\Delta^R : NC_\bullet(\{a\}) \xrightarrow{\mathcal{AW}} NC_\bullet(\{a\}) \otimes NC_\bullet(\{a\}) \xrightarrow{\mathrm{id}\otimes \iota_a} NC_\bullet(\{a\}) \otimes NC_\bullet(X)$$

and likewise for $C_\bullet(\{b\})$ as a left $C_\bullet(X)$-comodule. Then there is a natural quasi-isomorphism

$$\mathsf{Cobar}(NC_\bullet(\{b\}), NC_\bullet(X), NC_\bullet(\{a\})) \to \mathscr{D}.$$

Proof. Denote $\mathsf{Cobar}(\cdots)$ by Cobar. We will exhibit an isomorphism from Cobar into a bicomplex that is quasi-isomorphic to $\mathscr{D}$.

Note that $NC_\bullet(\{a\})$ and $NC_\bullet(\{b\})$ are just the chain complexes with a $\mathbf{Z}$ term concentrated in degree zero, i.e. the identity object in the symmetric monoidal structure on $\mathsf{Ch}_+(\mathsf{Ab})$. Hence we have natural isomorphisms

$$\psi_a : NC_\bullet(\{a\}) \otimes NC_\bullet(X) \to NC_\bullet(X)$$

$$\psi_b : NC_\bullet(X) \otimes NC_\bullet(\{b\}) \to NC_\bullet(X)$$

yielding a natural isomorphism

$$\begin{aligned} \phi : \mathsf{Cobar}_p^q = & \bigoplus_{n+b_1+\cdots+b_q+m=p} NC_n(\{a\}) \otimes NC_{b_1}(X) \otimes \cdots \otimes NC_{b_q}(X) \otimes NC_m(\{b\}) \\ & \to \bigoplus_{b_1+\cdots+b_q=p} NC_{b_1}(X) \otimes \cdots \otimes NC_{b_q}(X) =: N\mathscr{D}_p^q. \end{aligned} \tag{5.8}$$

where $N\mathscr{D}$ bicomplex obtained by taking normalized chains on each row of $\mathscr{C}_\bullet^\bullet$ then applying the Alexander–Whitney map, as in the diagram below:

$$\begin{array}{ccc} C_\bullet(X^q) & \xrightarrow{\mathcal{AW}_q} & C_\bullet(X)^{\otimes q} = \mathscr{D}_\bullet^q \\ \uparrow & & \uparrow \\ NC_\bullet(X^q) & \xrightarrow[\mathcal{AW}_q]{} & NC_\bullet(X)^{\otimes q} =: N\mathscr{D}_\bullet^q \end{array}$$

By Dold–Kan, the inclusion $N\mathscr{D} \hookrightarrow \mathscr{D}$ is a bicomplex quasi-isomorphism.

We now check that the differentials on Cobar and $N\mathscr{D}$ are compatible with ϕ. Because $NC_\bullet(\{a\})$ and $NC_\bullet(\{b\})$ have all differentials zero, the horizontal cobar differential formula 4.9 gives

$$\begin{aligned}
\phi(d_H^{\mathsf{Cobar}}(m\otimes c_1\otimes\cdots\otimes c_q\otimes n)) &= \phi(\sum_{i=1}^{q}(-1)^{\sigma(c_i)}m\otimes c_1\otimes\cdots\otimes\partial_{NC}(c_i)\otimes\cdots\otimes c_q\otimes n)\\
&= \sum_{i=1}^{q}(-1)^{\sigma(c_i)}\psi_a(m\otimes c_1)\otimes\cdots\otimes\partial_{NC}(c_i)\otimes\cdots\otimes\psi_b(c_q\otimes n)\\
&= d_{N\mathscr{D}}(\phi(m\otimes c_1\otimes\cdots\otimes c_q\otimes n)).
\end{aligned}$$

The vertical differential in both cases is given by the alternating sum of coface maps. We will show that $\phi\circ d^i_{\mathsf{Cobar}} = d^i_{N\mathscr{D}}\circ\phi$ for all i. For $i=0$, the map $d^i_{N\mathscr{D}} : N\mathscr{D}^q_\bullet \to N\mathscr{D}^{q+1}_\bullet$ is induced by the following inclusion under the topological Alexander–Whitney map:

$$(x_1,\dots,x_q)\to(a,x_1,\dots,x_q).$$

Naturality of Alexander–Whitney implies that the following diagram commutes:

$$\begin{tikzcd}
NC_\bullet(X^q) \arrow[r, "\cong"] \arrow[d, "\mathcal{AW}_q"'] \arrow[rr, bend left, "NC_\bullet(d^0)"] & NC_\bullet(\{a\}\times X^q) \arrow[r, "NC_\bullet(\iota_a\times\mathrm{id}_{X^q})"] \arrow[d, "\mathcal{AW}_{\{a\},X^q}"'] & NC_\bullet(X^{q+1}) \arrow[d, "\mathcal{AW}_{q+1}"] \\
NC_\bullet(X)^{\otimes q} \arrow[r, "\cong"] \arrow[rr, bend right, "d^0_{\mathscr{D}}"'] & NC_\bullet(\{a\})\otimes NC_\bullet(X)^{\otimes q} \arrow[r, "NC_\bullet(\iota_a)\otimes\mathrm{id}^{\otimes q}"] & NC_\bullet(X)^{\otimes q+1}
\end{tikzcd}$$

where ι_a is the inclusion $\{a\}\hookrightarrow X$. But now we make the identifications

$$NC_\bullet(\{a\})\otimes NC_\bullet(X)^{\otimes q}\cong NC_\bullet(X)^{\otimes q},\ \text{and } NC_\bullet(\iota_a)=\eta_a$$

to conclude

$$(\eta_a\otimes\mathrm{id}^{\otimes q})\circ\mathcal{AW}_q = \mathcal{AW}_{q+1}\circ NC_\bullet(d^0)$$

which implies $d^0_{N\mathscr{D}}=\eta_a\otimes\mathrm{id}^{\otimes q}$. (By a slight abuse of notation we use η_a to denote both the constant map at η_a as well as the constant zero simplex at a). Then by naturality of the Alexander–Whitney map,

$$\begin{aligned}
\phi(d^0_{\mathsf{Cobar}}(m\otimes c_1\otimes\cdots\otimes c_q\otimes n)) &= \phi(\Delta^R(m)\otimes c_1\otimes\cdots\otimes c_q\otimes n)\\
&= \phi\big(((\mathrm{id}\otimes\iota_a)\circ\mathcal{AW}(m))\otimes c_1\otimes\cdots\otimes c_q\otimes n\big)\\
&= \phi(m\otimes\eta_a\otimes c_1\otimes\cdots\otimes c_q\otimes n)\\
&= \psi_a(m\otimes\eta_a)\otimes c_1\otimes\cdots\otimes\psi_b(c_q\otimes n)\\
&= \eta_a\otimes\psi_a(m\otimes c_1)\otimes\cdots\otimes\psi_b(c_q\otimes n)\\
&= (\eta_a\otimes\mathrm{id}^{\otimes q})\big(\psi_a(m\otimes c_1)\otimes\cdots\otimes\psi_b(c_q\otimes n)\big)\\
&= d^0_{N\mathscr{D}}(\phi(m\otimes c_1\otimes\cdots\otimes c_q\otimes n))
\end{aligned}$$

The proof for $d^1,\dots,d^q$ are analogous. Hence ϕ is a bicomplex quasi-isomorphism $\mathsf{Cobar}\to N\mathscr{D}$, and postcomposing with the inclusion $N\mathscr{D}\hookrightarrow\mathscr{D}$ yields the desired quasi-isomorphism. □

Theorem 5.9. *There is a quasi-isomorphism of bicomplexes*

$$\mathsf{Cobar}(NC_\bullet(\{b\}),NC_\bullet(X),NC_\bullet(\{a\}))\to\mathscr{N}.$$

Proof. Composing the quasi-isomorphisms and isomorphisms obtained so far, we have

$$\mathsf{Cobar}(NC_\bullet(\{b\}), NC_\bullet(X), NC_\bullet(\{a\})) \xrightarrow{5.7} \mathscr{D} \underset{\mathcal{EZ}}{\overset{\mathcal{AW}}{\rightleftarrows}} \mathscr{C} \xrightarrow{(\S 5.1)} \mathscr{N}$$

which is a quasi-isomorphism. □

Proof of Theorem 1.1. In the quasi-isomorphism $\mathsf{Cobar} \to \mathscr{N}$ produced in Theorem 5.9, the truncation of $\mathscr{N}$ by q corresponds to the tensor length truncation of Cobar. It follows by Lemma 4.11 that the induced map on totalizations of the truncated bicomplexes

$$\mathrm{Tot}\,\mathsf{Cobar}_{\leq n} \to \mathrm{Tot}_\bullet(\mathscr{N}^{\leq n})$$

is still a quasi-isomorphism. On zeroth homology, we therefore have an isomorphism

$$H_0\,\mathrm{Tot}\,\mathsf{Cobar}_{\leq n} \to H_0\,\mathrm{Tot}_\bullet(\mathscr{N}^{\leq n})$$

LAST STEP: which by (???) yields an isomorphism

$$\mathbf{Z}\pi_X(a,b)/\mathbf{Z}\pi_X(a,b)\mathcal{I}^{n+1} \to H_0\,\mathrm{Tot}_\bullet(\mathscr{N}^{\leq n}).$$

By Corollary 5.4, we get a surjection

$$\mathbf{Z}\pi_X(a,b)/\mathbf{Z}\pi_X(a,b)\mathcal{I}^{n+1} \to H_0\,\mathrm{Tot}_\bullet(\mathscr{N}^{\leq n}) \twoheadrightarrow H_n(X^n, X(n)^x_x)$$

whose kernel is $\mathbf{Z}$ is $a = b$ and trivial otherwise, completing the proof. □

6. To do

(1) Stallings' paper to two basepoints. Rewrite 'homology and the fundamental group' subsection. Complete proof of 4.16. Probably don't need 4.14 in proof of 4.16
(2) In def of two sided cobar maybe begin by normalizing everything?
(3) Present the alternative cochain normalization in 4.10
(4) Notation/convention issues:
   (a) Spectral sequence indexing
   (b) Twosided/one sided cobar notation
   (c) Variants of AW and EZ

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Appendix A. Group rings

One important class of algebras are group rings. For a fixed ring R and a group G, the *group ring of G over R*, denoted $R[G]$, has elements finite formal linear combinations of elements in G with coefficients in R, with addition and multiplication given by

$$(\sum_{g\in G} r_g g) + (\sum_{g\in G} s_g g) = \sum_{g\in G}(r_g + s_g)g, \quad (\sum_{g\in G} r_g g)(\sum_{g\in G} s_g g) = \sum_{g\in G}\sum_{g_1 g_2 = g}(r_{g_1}s_{g_2})g.$$

Then the action of R on $R[G]$ given by multipliying coefficients gives $R[G]$ the structure of an R-algebra. One should think of $R[G]$ as some sort of free module over R with basis G. Group rings abound in the representation theory of groups, where any representation $\rho : G \to \mathrm{GL}(V)$ of a group G over a k-vector space V corresponds to a module over the group ring $k[G]$. In this thesis, however, we do not need to take this perspective (unless we could?)

As an algebra, $R[G]$ comes with a natural augmentation, given by the map

$$\varepsilon : R[G] \to R, \quad \varepsilon(\sum_{g\in G} r_g g) = \sum_{g\in G} r_g.$$

We call its kernel the *augmentation ideal*. Clearly we have a splitting $R[G] \cong \ker\varepsilon \oplus R$. The augmentation ideal is an interesting object of study. For example, we have the following observation:

Proposition A.1. *Let $\mathcal{I}$ be the augmentation ideal of the integral group ring $\mathbf{Z}[G]$. Then $\mathcal{I}/\mathcal{I}^2 \cong G^{\mathrm{ab}}$, the abelianzation of G.*

Proof. The proof relies on the following two facts. First, that $\{g-1 : g \in G\}$ is a generating set of $\mathcal{I}$. Second, that the abelianzation of G is the quotient of G by its commutator subgroup $[G,G]$, which is generated by group elements of the form $g^{-1}h^{-1}gh$. Then we can define an explicit homomorphism

$$\mathcal{I}/\mathcal{I}^2 \to G^{\mathrm{ab}} = G/[G,G], \quad [g-1] \mapsto [g]$$

and extending linearly to all of $\mathcal{I}/\mathcal{I}^2$. Then the inverse map is $[g] \to [g-1]$. □

Given a path connected, based topological space with (X,x), consider its integral fundamental group ring $\mathbf{Z}\pi_1(X,x)$ and corresponding augmentation ideal $\mathcal{I}$. The previous proposition implies that

$$\mathcal{I}/\mathcal{I}^2 \cong \pi_1(X,x)^{\mathrm{ab}} \cong H_1(X;\mathbf{Z}).$$

This is the simplest case of the main theorem we are trying to prove. It hints at the fact that we can gain information about the fundamental group of the space by examining its homology.