THESIS DRAFT

YIMING SONG

Abstract.

Contents

1. Introduction

2. Towards Adams' theorem

In this chapter we give a brief overview of the important algebraic constructions used throughout the thesis. The reader should feel free to skip this the first two subsections if they are familiar with the theory of (co)associative (co)algebras. The main reference for the background material is from the first two chapters of [LV12]. We then state Adams' remarkable 1956 theorem [Ada56], which (roughly) says that given a based topological space (X, x), one can recover the singular chains of the based loop space $\Omega_x X$ via a purely algebraic construction on the singular chains on X. We conclude by highlighting some recent refinements of Adams' theorem, which will be used in the proof of Looijenga's theorem.

2.1. **Algebras.** In a first course in ring theory, one encounters, given a base ring R, the ring of polynomials $R[x]$. They come with an addition and multiplication determined by the ring operations of R. In other words, $R[x]$ is an *algebra* over R. In general, an *associative R-algebra* is a (possibly non-unital) ring A that is also an R-module, such that the ring product is R-bilinear. That is, for all $r \in R$ and $x, y \in A$:

$$r \cdot (xy) = (r \cdot x)y = x(r \cdot y).$$

Remark 2.1. In much literature, the ring product $A \times A \to A$ of an R-algebra A is instead written as a map $A \otimes_R A \to A$. By the universal property of tensor products these presentations are equivalent.

Date: February 17, 2026.

If A is unital, then the algebra A is also said to be *unital*. A *morphism of associative R-algebras* $f : A \to B$ is a map respecting both the R-module structure and ring structure, i.e. for $r \in R$ and $a, b \in A$, we require

$$r \cdot f(ab) = f(r \cdot ab) = r \cdot f(a)f(b).$$

An algebra morphism from R-algebra A to R itself is called an *augmentation*. In this case we say A is *augmented*. The simplest example of an R-algebra is R itself. Other examples include the aforementioned ring of polynomials $R[x]$ or square matrices of size n with entries in R.

2.1.1. *Group rings.* One important class of algebras are group rings. For a fixed ring R and a group G, the *group ring of G over R*, denoted $R[G]$, has elements finite formal linear combinations of elements in G with coefficients in R, with addition and multiplication given by

$$(\sum_{g\in G} r_g g) + (\sum_{g\in G} s_g g) = \sum_{g\in G}(r_g + s_g)g, \quad (\sum_{g\in G} r_g g)(\sum_{g\in G} s_g g) = \sum_{g\in G}\sum_{g_1 g_2 = g}(r_{g_1} s_{g_2})g.$$

Then the action of R on $R[G]$ given by multipliying coefficients gives $R[G]$ the structure of an R-algebra. One should think of $R[G]$ as some sort of free module over R with basis G. Group rings abound in the representation theory of groups, where any representation $\rho : G \to \mathrm{GL}(V)$ of a group G over a k-vector space V corresponds to a module over the group ring $k[G]$. In this thesis, however, we do not need to take this perspective (unless we could?)

As an algebra, $R[G]$ comes with a natural augmentation, given by the map

$$\varepsilon : R[G] \to R, \quad \varepsilon(\sum_{g\in G} r_g g) = \sum_{g\in G} r_g.$$

We call its kernel the *augmentation ideal*. Clearly we have a splitting $R[G] \cong \ker\varepsilon \oplus R$. The augmentation ideal is an interesting object of study. For example, we have the following observation:

Proposition 2.2. *Let $\mathcal{I}$ be the augmentation ideal of the integral group ring $\mathbf{Z}[G]$. Then $\mathcal{I}/\mathcal{I}^2 \cong G^{\mathrm{ab}}$, the abelianzation of G.*

Proof. The proof relies on the following two facts. First, that $\{g - 1 : g \in G\}$ is a generating set of $\mathcal{I}$. Second, that the abelianzation of G is the quotient of G by its commutator subgroup $[G, G]$, which is generated by group elements of the form $g^{-1}h^{-1}gh$. Then we can define an explicit homomorphism

$$\mathcal{I}/\mathcal{I}^2 \to G^{\mathrm{ab}} = G/[G, G], \quad [g - 1] \mapsto [g]$$

and extending linearly to all of $\mathcal{I}/\mathcal{I}^2$. Then the inverse map is $[g] \to [g - 1]$. □

Given a path connected, based topological space with (X, x), consider its integral fundamental group ring $\mathbf{Z}\pi_1(X, x)$ and corresponding augmentation ideal $\mathcal{I}$. The previous proposition implies that

$$\mathcal{I}/\mathcal{I}^2 \cong \pi_1(X, x)^{\mathrm{ab}} \cong H_1(X; \mathbf{Z}).$$

This is the simplest case of the main theorem we are trying to prove. It hints at the fact that we can gain information about the fundamental group of the space by examining its homology.

2.1.2. *Differential graded algebras.* We begin with an example. Consider the singular chains $C_\bullet(X)$ on a topological space X. They are $\mathbf{N}$-graded, where each $C_n(X)$ is the free abelian group on the set of n-simplicies of X:

$$C_n(X) = \mathbf{Z}\{\text{continuous maps } \Delta^n \to X\}.$$

Another name given to $\mathbf{Z}$-modules is "abelian group," and indeed

$$C_\bullet = \bigoplus_{n\in\mathbf{N}} C_n(X)$$

is an abelian group. Next, we have a product

$$C_p(X) \times C_q(X) \to C_{p+q}(X)$$

given by the decomposition of the geometric $(p+q)$-simplex into a sum of p-simplices and q-simplicies (see []) for a reference. These maps assemble to a product

$$\times : C_\bullet(X) \times C_\bullet(X) \to C_\bullet(X).$$

One can check that this product is compatible with the $\mathbf{Z}$-module structure on $C_\bullet(X)$, making it into a *graded* $\mathbf{Z}$*-algebra*. As icing on the cake, we also have a singular boundary map $\partial : C_n(X) \to C_{n-1}(X)$ satisfying $\partial^2 = 0$ and the graded Leibniz rule: for $\sigma \in C_p(X)$ and $\tau \in C_q(X)$,

$$\partial(\sigma \times \tau) = (\partial\sigma) \times \tau + (-1)^p \sigma \times (\partial\tau).$$

Thus $C_\bullet(X)$ has the structure of a *differential graded* $\mathbf{Z}$*-algebra*. In general, a *differential graded* R*-algebra*, often abbreviated as a *dg algebra*, is a graded R-algebra $A_\bullet$ with a differential satisfying the graded Leibniz rule. If the differential lowers degree, we say $A_\bullet$ is *homologically graded*; if it raises degree, we say $A_\bullet$ is *cohomologically graded*. We can think of dg algebras as (co)chain complexes with a product, or algebras with a chain structure.

2.1.3. *Freeness and the tensor algebra.* Just as we can construct a free group from a set, we can also construct a free R-algebra given any R-module A. Before we given an explicit construction, we give a description of the *free associative algebra over* A, denoted $\mathcal{F}A$, in terms of its *universal property*:

> *There is an* R*-linear map* $i : A \hookrightarrow \mathcal{F}A$ *such that any* R*-algebra morphism* $f : A \to B$ *extends into a unique morphism* $\tilde{f} : \mathcal{F}A \to B$ *with* $\tilde{f} \circ i = f$.

This is entirely analogous to universal property of a free group. Sending an R-module A to the free associative algebra over A gives us a functor from the category of R-modules (supposing R is commutative), Mod_R to the category of associative R-algebras, denoted Alg_R. Conversely, we have a forgetful functor sending a R-algebra to its underlying module. In this language, the universal property tells us that these two functors are *adjoint*, i.e. there is a natural isomorphism

$$\hom_{\mathsf{Alg}_R}(\mathcal{F}A, B) \cong \hom_{\mathsf{Mod}_R}(A, \mathsf{forget}(B)).$$

This universal property guarantees that two manifestations of the free associative algebra are isomorphic via a unique isomorphism. With this in mind, let us now define the *tensor algebra* over an R-module A. Denoted $T(A)$, its underlying R-module is given by

$$T(A) := R \oplus A \oplus A^{\otimes 2} \oplus \cdots$$

and the product $T(A) \otimes T(A) \to T(A)$ is given by concatenation. On homogenous tensors $a \in A^{\otimes p}$ and $b \in A^{\otimes q}$, we have

$$(a_1 \cdots a_p) \otimes (b_1 \cdots b_q) \mapsto (a_1 \cdots a_p b_1 \cdots b_q) \in A^{\otimes p+q}.$$

This is a unital and associative R-algebra, with augmentation given by the identity on $R \subset T(A)$ and zero on higher tensor powers. The augmentation ideal, also called the *reduced tensor algebra* is denoted

$$\overline{A} = A \otimes A^{\otimes 2} \otimes \cdots$$

and is a nonunital associative R-algebra. The grading on $T(A)$ is given by tensor *length*, so a homogenous element $x \in A^{\otimes n}$ is said to have length n. Later in this thesis, we will be working with A a dg algebra, so the notions of grading can get confusing. But that is a problem for later.

Proposition 2.3. $T(A)$ *is the free associative algebra over* A.

Proof. Let $i : A \to T(A)$ be the inclusion into the second factor. Let $f : A \to B$ be an R-algebra morphism. Then $\tilde{f} : T(A) \to B$ can be defined by $\tilde{f}(1) = 1$ on $R \subset T(A)$, $\tilde{f}(x) = x$ for $x \in A \subset T(A)$, and $\tilde{f}(x_1 \cdots x_n) = f(x_1) \cdots f(x_n)$. Then $\tilde{f}$ extends f and is uniquely determined by f. We leave it to the reader to check that $\tilde{f}$ is an R-algebra morphism. ◻

2.2. **Coalgebras.** As suggested by the name, a *coalgebra* is the dual notion to that of an algebra. We could leave it at that, but they are much more unfamiliar objects; the maps don't go in the way we are used to. Moreover, the two objects are not dual on the nose: while the dual of every coalgebra is an algebra, the converse is not true without some finiteness assumptions.

Let us begin. A *coassociative R-coalgebra* is an R-module C with an R-linear map $\Delta : C \to C \otimes C$, called the *coproduct* which is coassociative, i.e. the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathrm{id} \otimes \Delta"] \\ C \otimes C \arrow[r, "\Delta \otimes \mathrm{id}"'] & C \otimes C \otimes C \end{tikzcd}$$

A *morphism of coassociative coalgebras* $f : C \to D$ is an R-linear map commuting with the coproduct, i.e.

$$(f \otimes f) \circ \Delta_C = \Delta_D \circ f.$$

Earlier, we could have defined a unital algebra to be one with a morphism from R. Dually, we say a coassociative R-coalgebra C is *counital* if there is a morphism $\epsilon : C \to R$ such that the following diagram commutes:

$$\begin{tikzcd} & C \arrow[dl, "\cong"'] \arrow[d, "\Delta"] \arrow[dr, "\cong"] & \\ R \otimes C & C \otimes C \arrow[l, "\epsilon \otimes \mathrm{id}"] \arrow[r, "\mathrm{id} \otimes \epsilon"'] & C \otimes R \end{tikzcd}.$$

The simplest example of a counital coassociative R-coalgebra is R itself, with the coproduct given by $1 \mapsto 1 \otimes 1$ and the counit given by $1 \mapsto 1$. We call a morphism $u : R \to C$ a *coaugmentation*, and in this case say that C is *coaugmented*. Beccause u is a morphism of coalgebras, it must commute with the counit maps, so we obtain that $\epsilon_C \circ u_C = \epsilon_R = \mathrm{id}_R$. It then follows that $C \cong \ker \epsilon \oplus R$, and we denote this kernel by $\overline{C}$.

- iterated coproducts

2.2.1. *Conilpotency, cofreeness, and the tensor coalgebra.*

2.2.2. *Differential graded coalgebras.*

2.2.3. *Singular chains.* Singular chains $C_\bullet(X)$ on a topological space X also admit the structure of a $\mathbf{Z}$-coalgebra (and enough things are compatible so this makes them into a *bialgebra*, but we don't need that).

2.3. **Adams' theorem.**

2.3.1. *Cobar as a bicomplex.*

2.3.2. *Refinements.*

3. Cosimplicial objects

A cosimplicial object of a category C could be defined simply as a simplicial object of the opposite category C^{op}. This is not really how the human brain works...

—Stacks Project, 14.5 [Sta26]

In this chapter we define cosimplicial objects, the totalization of a cosimplicial space, and provide some examples. We assume some familiarity with simplicial objects, for they are ubiquitous throughout algebraic topology.

3.1. **Definitions.** The *simplex category*, denoted $\mathbf{\Delta}$, is the category with

(1) objects: finite nonempty totally ordered sets. We write $[n]$ for the set $\{0 < 1 < \cdots < n\}$,
(2) morphisms: order-preserving maps, i.e. if $i \le j$ then $f(i) \le f(j)$.

Then a *cosimplicial object* in a category C is a functor $X^\bullet : \mathbf{\Delta} \to C$. We denote the image of $[n]$ under $X^\bullet$ by X^n. A morphism of two simplicial objects $X^\bullet, Y^\bullet : \mathbf{\Delta} \to C$ is a natural transformation of functors, i.e. morphisms $X^n \to Y^n$ for all n that commute with morphisms in $\mathbf{\Delta}$. We will denote the category of of cosimplicial objects in C by $\mathsf{cs}C$. In this thesis we will concern ourselves with the cases where $C = \mathsf{Top}$ or $C = \mathsf{Set}$.

Remark 3.1. Even though simplicial objects are *contravariant* functors, the convention is to denote the simplicies using subscripts, e.g. X_n. Conversely, even though simplicial spaces are *covariant* functors, the convention is to denote the cosimplicies with superscripts. Maybe the presence of the suffix co- explains this.

Recall that the morphisms in the simplex category are generated by two distinguished classes of maps. For $n \ge 1$ and $j \in [n]$, we have injections $\delta_j : [n-1] \to [n]$ where δ_j skips $j \in [n]$. For $n \ge 0$ and $j \in [n]$, we have $n+1$ surjections $\sigma_j : [n+1] \to [n]$ where σ_j sends both $j, j+1$ to $j \in [n]$. For a cosimplicial object $X^\bullet : \mathbf{\Delta} \to C$, we call the images of the δ_j *coface maps*, usually denoted d^j, and the images of the σ_j *codegeneracy maps*, usually denoted s^j. So to specify a cosimplicial object in C it also suffices to list a sequence of objects $X^n \in C$ for $n \ge 0$, as well as coface and codegeneracy maps satisfying the following *cosimplicial identities*:

(1) If $i < j$, then $d^j \circ d^i = d^i \circ d^{j-1}$.
(2) If $i < j$, then $s^j \circ d^i = d^i \circ s_{j-1}$.
(3) $\mathrm{id} = s^j \circ d^j = s^j \circ d^{j+1}$.
(4) If $i > j+1$, then $s^j \circ d^i = d^{i-1} \circ s^j$.
(5) If $i \le j$, then $s^j \circ s^i = s^i \circ s^{j+1}$.

One should think of a cosimplicial object $X^\bullet : \mathbf{\Delta} \to C$ as a diagram

$$X^0 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X^1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X^2 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \cdots$$

where the rightward pointing arrows are the coface maps and the leftward pointing arrows are the codegeneracy maps.

Our first example of a cosimplicial space will be the topological simplicies, $\Delta^\bullet : \mathbf{\Delta} \to \mathsf{Top}$. For each n, let Δ^n be the topological n-simplex

$$\Delta^n := \{(x_1, ..., x_n) \in \mathbf{R}^n : 0 \le x_1 \le \cdots \le x_n \le 1\}.$$

Then the coface maps $\Delta^{n-1} \to \Delta^n$ are the inclusions of faces, where d^j is the inclusion of the face opposite the jth vertex. The codegeneracy map s^j collapses the line joining the jth and jth vertex.

In coordinates, we have

$$d^j(x_1, ..., x_n) = (x_1, ..., x_j, x_j, ..., x_{n-1}),$$
$$s^j(x_1, ..., x_n) = (x_1, ..., \widehat{x_j}, ..., x_n)$$

where $\widehat{\ }$ indicates omission. This example is illustrative; in general it helps to think of the coface maps as "duplicating a coordinate" and the codegeneracy maps as "forgetting a coordinate." We will see this again in this following subchapter.

3.2. **Totalization and path spaces.** Cosimplicial spaces provide a useful model for many types of topological spaces, including the based path and loop spaces. This is done via *totalization*, which is dual to the notion of geometric realization of a simplicial set.

Given a cosimplicial space $X^\bullet : \mathbf{\Delta} \to \mathsf{Top}$, define the *totalization* of $X^\bullet$ to be the space of maps from the cosimplicial simplices to $X^\bullet$:

$$\mathrm{Tot}(X^\bullet) := \mathrm{Hom}_{\mathsf{csTop}}(\Delta^\bullet, X^\bullet),$$

i.e. maps $f^n : \Delta^n \to X^n$ for all $n \geq 0$ that commute with the coface and codegeneracy maps. We topologize it as a subspace of $\prod_{n\geq 0} \mathrm{Hom}(\Delta^n, X^n)$ with the compact-open topology. Thus totalization gives us a functor from csTop to Top.

Now for an important example which will feature later. Let X be a topological space with $a, b \in X$. Define the cosimplicial space $F^\bullet_{a,b}X$ whose cosimplicies are

$$F^0_{a,b}X = \{*\}, \quad F^n_{a,b}X = X^n \text{ for } n \geq 1.$$

The coface maps $d^j : F^{n-1}_{a,b}X \to F^n_{a,b}$ are given by

$$d^j(x_1, ..., x_{n-1}) = \begin{cases} (a, x_1, ..., x_{n-1}) & j = 0 \\ (x_1, ..., x_j, x_j, ..., x_{n-1}) & j \in \{1, ..., n-1\} \\ (x_1, ..., x_{n-1}, b) & j = n \end{cases}$$

The codegeneracy maps $s^j : F^{n+1}_{a,b}X \to F^n_{a,b}X$ are given by

$$s^j(x_1, ..., x_{n+1}) = (x_1, ..., \widehat{x_{j+1}}, ...x_{n+1}), \quad j \in \{0, ..., n\}$$

where the $\widehat{\ }$ denotes omission. What seems to be happening here is that by iterating the face maps, we are creating finer and finer piecewise subdivisons of paths whose endpoints are at a and b. Indeed,

Proposition 3.2. $\mathrm{Tot}(P^\bullet_{a,b}X)$ *is homeomorphic to the path space* $\Omega_{a,b}X$ *of paths in* X *beginning at* a *and ending at* b.

Proof. A point of $\mathrm{Tot}(P^\bullet_{a,b}X) = \mathrm{hom}_{\mathsf{csTop}}(\Delta^\bullet, P^\bullet_{a,b}X)$ us a sequence of continuous maps

$$f = \{f_i : \Delta^i \to X^i\}_{i\geq 0}$$

commuting with the coface and codegeneracy maps. Fix $n \geq 2$ and $k \in \{1, ..., n\}$. Consider the following composition of codegeneracy maps

$$\alpha_{n,k} = \underbrace{s^{n-1} \circ s^{n-2} \circ \cdots \circ s^{k-2} \circ s^k \circ \cdots \circ s^0}_{n-1 \text{ maps}}$$

where we compose all the degeneracies except for s^{k-1}. This gives us a map $\Delta^n \to \Delta^1$ and likewise for $X^n \to X^1$. Then for $f = \{f_0, f_1, ...\} \in \mathrm{Tot}(P^\bullet_{a,b}X)$, we have by commutativity that

$$f_1 \circ \alpha_{n,k} = \alpha_{n+1,k} \circ f_n.$$

But now the right hand side is just picking out the kth coordinate of f_n. So we have shown that f_n for $n \geq 2$ is completely determined by f_1, so that the projection $\Phi : \mathrm{Tot}(P^\bullet_{a,b}X) \to \mathrm{Map}(\Delta^1, X)$ given by $\{f_i\}_{i\geq 0} \mapsto f_1$ is injective.

We claim next that Φ is actually a map into $\Omega_{a,b}X \subset \mathrm{Map}(\Delta^1, X)$. To see this, the cosimplicial relations imply

$$f_1 \circ d^0 = d^0 \circ f_0 = \mathrm{const}_a, \quad f_1 \circ d^1 = d^1 \circ f_0 = \mathrm{const}_b$$

so $f_1(0) = a$ and $f_1(1) = b$ as desired.

Lastly we define an inverse to $\Phi : \mathrm{Tot}(P^\bullet_{a,b}X) \to \Omega_{a,b}X$. For a given path $\gamma \in \Omega_{a,b}X$ consider the family of maps $\{f_i : \Delta^i \to X^i\}_{i\geq 0}$ given as follows. We let f^0 be the constant map, $f^1 = \gamma$, and for $n \geq 2$ define

$$f_n(x_1, ..., x_n) = (\gamma(x_1), \gamma(x_1 + x_2), ..., \gamma(x_1 + \cdots + x_n)).$$

Clearly this is an inverse to Φ. We leave it to the reader to check that the family of maps $\{f_i\}$ commutes with the coface and codegeneracies, and that Φ and its inverse are continuous maps. □

We will call $P^\bullet_{a,b}X$ a cosimplicial model for the path space of X. When $a = b$ we get a cosimplicial model for the based loop space of X.

3.3. **Dold–Kan and homotopy.** "... there is no interesting homotopy theory of cosimplicial sets!" concludes a MathOverflow answer [hg] of Tom Goodwillie. This subchapter will be devoted to understanding this comment.

References

[Ada56] J. F. Adams. On the cobar construction. *Proceedings of the National Academy of Sciences of the United States of America*, 42(7):409–412, 1956.

[hg] Tom Goodwillie (https://mathoverflow.net/users/6666/tom goodwillie). What is the cohomology of this complex? MathOverflow. URL:https://mathoverflow.net/q/66045 (version: 2011-05-26).

[LV12] Jean-Louis Loday and Bruno Vallette. *Algebraic Operads*. Springer Berlin Heidelberg, 2012.

[Sta26] The Stacks project authors. The stacks project. https://stacks.math.columbia.edu, 2026.