A CAROUSEL PROPERTY FOR COMPACT CONVEX SETS

YIMING SONG †

Abstract. A theorem of K. Adaricheva and M. Bolat states that if A_0 and A_1 are circles in a triangle with vertices g_0, g_1, g_2, then there exist $i \in \{0, 1\}$ and $j \in \{0, 1, 2\}$ such that A_i is in the convex hull of $A_{1-i} \cup (\{g_0, g_1, g_2\} \backslash g_j)$. G. Czedli generalizes this result in the case where A_0 and A_1 are compact sets that are positive homotheties or translations of one another. Here we show that this property holds in the case where A_0, A_1 are arbitrary compact subsets of the plane, when the triangle is replaced with an n-gon, where n is strictly greater than the number of common supporting lines of A_0 and A_1. This implies previous results, and we show additionally that this bound is tight for even n.

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1. Introduction

1.1. **Background.** Let Conv : $\mathcal{P}(\mathbf{R}^2) \to \mathcal{P}(R^2)$ denote the *convex hull* operator, which maps a subset of the plane to the smallest convex set containing it. In 2019, K. Adaricheva and M. Bolat proved the following result about disks:

Theorem (Theorem 3.1, [AB19])**.** *If A_0 and A_1 are closed disks in the $\mathbf{R}^2$ and G is a triangle with vertices g_0, g_1, g_2 such that $A_0, A_1 \subset G$, then there exist $i \in \{0, 1\}$ and $j \in \{0, 1, 2\}$ such that*

$$A_i \subset \mathrm{Conv}(A_{1-i}, \{g_0, g_1, g_2\} \setminus \{g_j\}).$$

Their result has implications for the representation theory of convex geometries. A *convex geometry* (not to be confused with the field of study) is a combinatorial set system abstracting the notion of convexity found in point arrangements, posets, and more. For a formal exposition, see Chapter 5 of [AN] or [EJ85]. A seminal result of Kashiwabara et. al. in [KNO05] states that all convex geometries can be embedded into a particular type of point set in $\mathbf{R}^d$ for sufficiently large d, in a lattice structure-preserving way. This is a *representation* of a convex geometry.

A natural question to ask is whether this number d can be lowered if one works with convex shapes rather than points. In [Czé14], G. Czédli shows that using disks rather than

Date: December 11, 2025.
2020 *Mathematics Subject Classification.* 52C07.
Key words and phrases. convex geometry, discrete geometry, computational geometry, antimatroids.
†Department of Mathematics, Columbia University. Email: ys3635@columbia.edu.

points in $\mathbf{R}^2$ yields more flexibility, allowing the representation of a larger class of convex geometries. Following this result, Richter and Rogers show in [RR17] that any convex geometry can be embedded as convex n-gons in $\mathbf{R}^2$ for sufficiently large n. The result of Adaricheva–Bolat provides a geometric reason for the failure of disks in $\mathbf{R}^2$ to represent convex geometries. Understanding analogous geometric results for other families of convex compact shapes would provide a wealth of (counter)examples in this area.

1.2. **Results.** In this paper, we generalize the theorem of Adaricheva–Bolat to arbitrary convex compact shapes. Given a pair $\mathcal{A} = \{A_0, A_1\}$ of convex compact subsets of the plane, and a convex n-gon G containing the elements of $\mathcal{A}$, we say $(\mathcal{A}, G)$ satisfy the *weak carousel rule* if there exist $i \in \{0, 1\}$ and $j \in \{0, 1, 2\}$ such that

$$A_i \subset \operatorname{Conv}(A_{1-i}, \operatorname{vert}(G) \setminus \{g_j\}).$$

where $\operatorname{vert}(G)$ denotes the vertices of G. In this language, their theorem says that if G is a triangle containing both disks in $\mathcal{A} = \{A_0, A_1\}$, then $(\mathcal{A}, G)$ satisfy the weak carousel rule.

For G a triangle, and $\mathcal{A}$ consisting of arbitrary convex compact A_0, A_1, it is not difficult to construct examples where the weak carousel rule fails for $(\mathcal{A}, G)$. Our main result is that the weak carousel rule holds for arbitrary convex compact A_0, A_1 if $n = \#\operatorname{vert}(G)$ is sufficiently large as a function of A_0 and A_1. Precisely, let s be the number of *common supporting lines* of A_0 and A_1 (common tangent lines of containing both A_0 and A_1 on one side). Then our main theorem is the following:

Theorem 1.1. *If $s < n$, then the weak carousel rule holds for $(\mathcal{A}, G)$.*

This recovers and generalizes the result of Adaricheva–Bolat. We also construct examples to show that this bound is sharp for even n. Our proof is geometric in nature, relying at times on classical concepts from convex analysis. One step owes heavily to Czedli–Stachó's notion of *slide-turning*, detailed in [CS16].

1.3. **Organization.** We collect the necessary background on supporting lines, support functions, sectors, and slide-turning in Section 2. In Section 3, we prove the main theorem and sharpness of bound, and conclude with a few questions.

Acknowledgments. Part of this research was conducted during the 2025 Baruch College Discrete Mathematics NSF Site REU, supported by NSF award DMS-2349366, and by Jane Street. I am greatly indebted to my mentor Kira Adaricheva for her patient and thought-provoking guidance, Adam Sheffer for his tireless organization, and my fellow students for the conversations and laughter.

2. Preliminaries

The strategy behind the proof will be to observe the tangent lines to A_0 and A_1, and manipulate them around via a process called "slide-turning," all of which will be made formal in this section.

2.1. **Common supporting lines.** Let $A \subset \mathbf{R}^n$ be a convex set, and let l be a hyperplane in $\mathbf{R}^n$. We say that l *supports* A, or is a *supporting hyperplane* of A, if $l \cap A$ is nonempty and A lies strictly inside one of the two halfspaces bounded by l. Given a collection $\mathcal{A}$ of convex sets in $\mathbf{R}^n$, we say that l is a *common supporting hyperplane* of $\mathcal{A}$ if l supports all A_i in the same halfspace. A classical theorem in convex analysis states:

Theorem 2.1 ([Roc70])**.** *Let $A \subseteq \mathbf{R}^n$ be a convex set. For any $p \in \partial A$, there exists a supporting hyperplane of A containing p.* □

For the remainder of this paper will we will work with $n = 2$, so (common) supporting hyperplanes are just (common) supporting lines. By a slight abuse of notation, we will use $\mathrm{csl}(A)$ or $\mathrm{csl}(\mathcal{A})$ to denote the set of common supporting lines of either a single convex set $A \subset \mathbf{R}^2$, or a collection of convex sets $\mathcal{A}$.

For convex $A \subset \mathbf{R}^2$ it is useful to consider its *support function* $h_A : \mathbf{R}^2 \to \mathbf{R}$, given by

$$h_A(v) = \sup_{a \in A}(v \cdot a)$$

where $\cdot$ denotes the standard inner product. If A is compact and nonempty, then the support function h_A is continuous [Sch13]. Importantly, the support function gives a helpful characterization of supporting lines, and is often how the supporting lines are defined:

Lemma 2.2. *For a fixed unit vector $\eta \in \mathbf{R}^2$, the line*

$$l_\eta := \{x \in \mathbf{R}^2 : x \cdot \eta = h_A(\eta)\}$$

is in $\mathrm{csl}(A)$*, and A is contained in the closed halfplane*

$$\{x \in \mathbf{R}^2 : x \cdot \eta \leq h_A(\eta)\}.$$

Proof. See [Sch13], ... □

Corollary 2.3. *Let $\mathcal{A}$ be a collection of convex sets in the plane. For a fixed unit vector $\eta \in \mathbf{R}^2$ and $c \in \mathbf{R}$, the line $\{x \in \mathbf{R}^2 : \eta \cdot x = c\}$ is in* $\mathrm{csl}(\mathcal{A})$ *iff $h_A(\eta)$ are equal over all $A \in \mathcal{A}$.* □

Now let $\mathcal{A} = \{A_i\}_{i \in I}$ be a collection of convex subsets of $\mathbf{R}^2$. For $l \in \mathrm{csl}(\mathcal{A})$, there is a choice of sign for its unit direction vector. We will use v_l^L to denote the direction vector of l such that $\cup\mathcal{A}$ lies on the *left* of the directed line (l, v_l^R). Let $v_l^R := -v_l^L$. For simplicity, we will write l^L and l^R for the directed lines (l, v_l^L) and (l, v_l^R). For the sets of directed supporting lines we will write $\mathrm{csl}^L(\mathcal{A}) := \{l^L : l \in \mathrm{csl}(\mathcal{A})\}$ and $\mathrm{csl}^R(\mathcal{A}) := \{l^R : l \in \mathrm{csl}(\mathcal{A})\}$.

2.2. **Adjacency and sectors.** We now define an order on the common supporting lines, which will turn out to be very important. For the remainder of this section, let $\mathcal{A}$ be a colleciton of compact convex subsets of $\mathbf{R}^2$. Following [CK19], identifying S^1 with $[0, 2\pi]/(0 \sim 2\pi)$, define

$$\mathrm{dir}^L : \mathrm{csl}(\mathcal{A}) \to S^1, \quad \mathrm{dir}^R : \mathrm{csl}(\mathcal{A}) \to S^1$$

via $l \mapsto v_l^L$ and $l \mapsto v_l^R$ respectively. Now define

$$\mathrm{nor}^L(l) := \mathrm{dir}^L(l) + \pi/2, \quad \mathrm{nor}^R(l) := \mathrm{dir}^R(l) - \pi/2$$

but notice that $\mathrm{nor}^L(l) = \mathrm{nor}^R(l)$, since they are both the outward pointing normal vectors away from the subset that l supports, so we will just write $\mathrm{nor}(l)$.

Lemma 2.4. *For compact convex $A \subset \mathbf{R}^2$,* $\mathrm{nor} : \mathrm{csl}(A) \to S^1$ *is a bijection.*

Proof. For injectivity, if two distinct supporting lines of A had the same normal vector, then by definition of supporting line, A would have to lie in two disjoint subsets of the plane, a contradiction. Surjectivity is given by Lemma 2.2. □

This lemma implies that we have a total cyclic order on $\mathrm{csl}(\mathcal{A})$ if it is finite. This is because $\mathrm{csl}(\mathcal{A})$ is a subset of $\mathrm{csl}(\mathrm{Conv}(\mathcal{A}))$, and $\mathrm{Conv}(\mathcal{A})$ is convex and compact. Thus $\mathrm{csl}(\mathcal{A})$ inherits the total cyclic order on $\mathrm{csl}(\mathrm{Conv}(\mathcal{A}))$. We say that $l_a, l_b \in \mathrm{csl}(\mathcal{A})$ are *adjacent* if $\mathrm{nor}(l_a)$ and $\mathrm{nor}(l_b)$ are consecutive in this order. Then, given s common supporting lines we also have s pairs of adjacent lines. See 1 for an illustration. From here we will assume that $\mathrm{csh}(\mathcal{A})$ is indexed accordingly, i.e. l_i and l_{i+1} are adjacent, where indices are understood modulo s.

We now seek to define a sector between two supporting lines, similar to but more general than the definition of a comet in [CK19]. Let A be convex compact, and let l_1, l_2 be distinct elements of $\mathrm{csl}(A)$. In S^1, we have two closed arcs whose endpoints are $\mathrm{nor}(l_1)$ and $\mathrm{nor}(l_2)$. Call them $\Lambda^+(l_1, l_2)$ and $\Lambda^-(l_1, l_2)$. For $\eta \in S^1$, let H_η be the closed halfplane

$$H_A(\eta) = \{x \in \mathbf{R}^2 : x \cdot \eta \leq h_A(\eta)\}$$

Finally, define the *sectors of* l_1, l_2 *by* A via:

$$\mathrm{Sect}^\pm(l_1, l_2; A) = \bigcap_{\eta \in \Lambda^\pm(l_1, l_2)} H_A(\eta).$$

Because compact convex subsets are the intersection of all halfplanes containing them, it is clear that $A \subset \mathrm{Sect}^\pm(l_1, l_2; A)$.

We can fix a choice of sign for $\Lambda^\pm(l_1, l_2; A)$ as follows. Orient S^1 clockwise. We fix $\Lambda^+(l_1, l_2)$ to be the arc obtained by proceeding from $\mathrm{nor}(l_1)$ to $\mathrm{nor}(l_2)$ in this clockwise direction, and $\Lambda^-(l_1, l_2)$ to be the complementary arc. So the order in which we write l_1, l_2 matters.

Proposition 2.5. *Let* $\mathcal{A} = \{A_0, A_1\}$. *If* $l_1, l_2 \in \mathrm{csl}(\mathcal{A})$ *are adjacent, then at least one of the following inclusions holds:*

(1) $A_0 \subset \mathrm{Sect}^+(l_1, l_2; A_1)$,
(2) $A_1 \subset \mathrm{Sect}^+(l_1, l_2; A_0)$.

Proof. Consider the continuous function $\delta : S^1 \to \mathbf{R}$ given by

$$\theta \mapsto h_{A_0}(\theta) - h_{A_1}(\theta).$$

Since $l_1, l_2 \in \mathrm{csl}(\mathcal{A})$, we have by Corollary 2.3 that $\delta(\mathrm{nor}(l_1)) = \delta(\mathrm{nor}(l_2)) = 0$.

Suppose for contradiction that none of the inclusions hold. Then, (1) implies that a point of A_0 must lie outside $\mathrm{Sect}^+(l_1, l_2; A_1)$. By convexity and compactness, the subset $A_0 \setminus \mathrm{Sect}^+(l_1, l_2; A_1)$ must contain a point on the boundary of A_0, say a. By Theorem 2.1, there exists a supporting line of l_a of A_0, passing through A. Moreover, we have $\mathrm{nor}(l_a) \in \Lambda^+(l_1, l_2)$. Because l_a separates a from $\mathrm{Sect}^+(l_1, l_2; A_1)$, which contains A_1, we have that for all $b \in A_1$ that

$$h_{A_0}(\mathrm{nor}(l_a)) = \mathrm{nor}(l_a) \cdot a > \mathrm{nor}(l_a) \cdot b$$

so $h_{A_0}(\mathrm{nor}(l_a)) > \sup_{b \in A_1} \mathrm{nor}(l_a) \cdot b = h_{A_1}(\mathrm{nor}(l_a))$. Therefore, $\delta(\mathrm{nor}(l_a)) > 0$.

By the same argument, (2) implies that there exists some $l_b \in \mathrm{csl}\, A_1$ such that $\mathrm{nor}(l_b) \in \Lambda^+(l_1, l_2)$ and $\delta(\mathrm{nor}(l_b)) < 0$. By the intermediate value theorem, there is some $\eta \in \Lambda^+(l_1, l_2) \setminus \{\mathrm{nor}(l_1, l_2)\}$ such that $\delta(\eta) = 0$, which yields another common supporting line of A_0 and A_1, whose normal lies strictly between $\mathrm{nor}(l_1)$ and $\mathrm{nor}(l_2)$ in the clockwise order, contradicting the adjacency of l_1 and l_2. $\square$

2.3. **Slide-turning and expansion.** In [CS16], the authors formalize a notion of "slide-turning" support lines. Intuitively, one can slide tangent lines around the boundary of a convex shape in a continuous manner. A *pointed supporting line* of A is a pair (a, l) where $a \in \partial A$ and $l \in \mathrm{csl}(A)$ passes through a. They define the set

$$\mathrm{Sli}(A) := \{(a, \mathrm{dir}^L(l)) : (a, l) \text{ is a pointed supporting line of } A\} \subset \mathbf{R}^2 \times S^1 \subset \mathbf{R}^4$$

and prove the following theorem about Sli:

Theorem 2.6. [CS16] *If A is nonempty, compact and convex, then* $\mathrm{Sli}(A)$ *is a rectifiable, simple, closed curve in* $\mathbf{R}^4$. ◻

In particular, this implies that we can parametrize $\mathrm{Sli}(A)$ and consequently $\mathrm{csl}(A)$. But there is no guarantee that this parametrization is nice, which we want for our purposes. Instead, we will use the inverse of the bijection $\mathrm{nor} : \mathrm{csl}(A) \to S^1$:

$$P_A(t) := \mathrm{nor}^{-1}(t)$$

where we identify S^1 with $[0, 2\pi]$ modulo endpoints. With this parametrization, increasing t corresponds to rotating the supporting line clockwise along A, and vice versa. Then, slide-turning l_1 along A, clockwise by an angle of α, or counterclockwise by an angle of β corresponds to the maps

$$t \mapsto P_A(\mathrm{nor}(l_1) + \alpha t), \quad t \mapsto P_A(\mathrm{nor}(l_1) - \beta t)$$

for $t \in [0, 1]$. For ease of notation, let us write

$$l_{i,\alpha;A} := P_A(t_i + \alpha).$$

Then the following lemma follows directly from definition:

Lemma 2.7. *If* $\alpha + \beta \leq t_2 - t_1 \pmod{2\pi}$*, then*

$$\mathrm{Sect}^+(l_1, l_2; A) \subset \mathrm{Sect}^+(l_{1,\alpha;A}, l_{2,-\beta;A}; A).$$

and

$$\mathrm{Sect}^-(l_1, l_2; A) \subset \mathrm{Sect}^-(l_{1,-\alpha;A}, l_{2,\beta;A}; A).$$

Proof. Unraveling the definitions, since $\alpha + \beta \leq t_2 - t_1 \pmod{2\pi}$, the arcs Λ satisfy

$$\Lambda^\pm(l_{1,\pm\alpha;A}, l_{2,\mp\beta;A}) \subset \Lambda^\pm(l_1, l_2)$$

which immediately implies the desired relations. ◻

For this reason, we will call $\mathrm{Sect}^+(l_{1,\alpha;A}, l_{2,-\beta;A})$ an *expanded* sector of $\mathrm{Sect}^+(l_1, l_2; A)$. See Figure 2 for a diagram of a sector and its expansion.

3. The weak carousel rule

Throughout this section, let $\mathcal{A} = \{A_0, A_1\}$ be convex, compact, and let G be a convex n-gon containing $\mathcal{A}$. Label the vertices of G clockwise as $g_1, \ldots, g_n$, and the edges clockwise $e_1, .., e_n$, where e_i is the edge joining g_i and g_{i+1}. Let s denote the number of common supporting lines of $\mathcal{A}$. Fix one common supporting line as l_1, and in the clockwise cyclic order on $\mathrm{nor}(\mathrm{csl}(\mathcal{A}))$, label each consecutive line $l_2, l_3, \ldots, l_s$. Let us restate our main theorem:

Theorem 1.1. *If $s < n$, then the weak carousel rule holds for* $(\mathcal{A}, G)$.

The proof of Theorem 1.1 will consist in showing that out of the s adjacent pairs in $\mathrm{csl}\,\mathcal{A}$, there exists some adjacent pair of lines (l_i, l_{i+1}) whose sector $\mathrm{Sect}^+(l_i, l_{i+1})$ can be expanded so that it is equal to the convex hull in the statement of the theorem.

3.1. **Setting up the proof, endpoints.** We first need to consider the points where the supporting lines making up a sector intersect the boundary of G. Define the functions $\Omega^L : \operatorname{csl}(A_i) \to \partial G$ and $\Omega^R : \operatorname{csl}(A_i) \to \partial G$ as follows. For $l \in \operatorname{csl}(A_i)$, fix a point $a \in l \cap A_i$, and let $t^* \in \mathbf{R}$ be the maximum $t \in \mathbf{R}$ such that $a + t \operatorname{dir}^L(l)$ is in G. Then set $\Omega^L(l) := a + t^* \operatorname{dir}^L(l)$. Similarly, define $\Omega^R(l) := a + t^\dagger \operatorname{dir}^R(l)$ where $t^\dagger$ is the maximum $t \in \mathbf{R}$ such that $a + t \operatorname{dir}^R(l)$ is in G. These are well-defined maps since G is compact.

Lemma 3.1. *With respect to the Hausdorff topology on* $\operatorname{csh}(A_i)$*, the maps* Ω^L *and* Ω^R *are continuous.*

Proof. For simplicity we omit some subscripts and superscripts. Let l_n be a sequence in $\operatorname{csh}(A_1)$ converging to l. That is, $l_n \to l$ in the Hausdorff metric, and $\operatorname{dir}(l_n) \to \operatorname{dir}(l)$. Then

$$\Omega(l_n) = a_n + t_n \operatorname{dir}(l_n), \quad \Omega(l) = a + t \operatorname{dir}(l)$$

for $a_n, a \in A_1$ and $t_n, t \in \mathbf{R}$. We first show that $t_n \to t$. The sequence $\{t_n\}$ is bounded since G is compact. By the Bolzano–Weierstrass theorem there exists a convergent subsequence $t_{n_k} \to t_0$. By assumption, $a_{n_k} \to a$ and $\operatorname{dir}(l_{n_k}) \to \operatorname{dir}(l)$, so

$$a_{n_k} + t_{n_k} \operatorname{dir}(l_{n_k}) \to a + t_0 \operatorname{dir}(l)$$

Since G is closed, $a + t_0 \operatorname{dir}(l) \in G$. But since $a + t \operatorname{dir}(l)$ is the last point still in G, along $\operatorname{dir}(l)$, we must have $t_0 \le t$.

If $t_0 < t$, then let $\varepsilon = (t - t_0)/2$ and let $q^* = a + (t_0 + \varepsilon) \operatorname{dir}(l)$. By convexity, $q^* \in G^\circ$. Now consider the sequence $q_{n_k} = a_{n_k} + (t_0 + \varepsilon) \operatorname{dir}(l_{n_k})$. We have that

$$||q_{n_k} - q^*|| = a - a_{n_k} + (t + \varepsilon)(\operatorname{dir}(l) - \operatorname{dir}(l_{n_k}))$$

and since $a_{n_k} \to a$ and $\operatorname{dir}(l_{n_k}) \to \operatorname{dir}(l)$, for large enough k we have that $||q_{n_k} - q^*|| < \delta$. Since $q^* \in G^\circ$, we can take δ to be some value such that $B_\delta(q^*) \subseteq G^\circ$. But then for large k we have that

$$q_{n_k} = a_{n_k} + (t_0 + \varepsilon) \operatorname{dir}(l_{n_k}) \in G^\circ$$

which is a contradiction since $t_{n_k} < t_0 + \varepsilon$. Hence $t_0 = t$.

Finally, we have that if $l_n \to l$, then

$$\begin{aligned} ||\Omega(l_n) - \Omega(l)|| &\le ||a_n - a|| + ||t_n \operatorname{dir}(l_n) - t \operatorname{dir}(l)|| \\ &\le ||a_n - a|| + ||t_n \operatorname{dir}(l_n) - t_n \operatorname{dir}(l) + t_n \operatorname{dir}(l) - t \operatorname{dir}(l)|| \\ &\le ||a_n - a|| + t_n || \operatorname{dir}(l_n) - \operatorname{dir}(l)|| + |t_n - t| || \operatorname{dir}(l)|| \end{aligned}$$

which limits to zero. □

Now we need to state a definition that is geometrically intuitive but technical. It will help save many words later on. For an adjacent pair $l_1, l_2 \in \operatorname{csh}(\mathcal{A})$, we have by Proposition 2.5 that

$$\begin{cases} A_0 \subset \operatorname{Sect}^+(l_1, l_2; A_1), \text{ or} \\ A_1 \subset \operatorname{Sect}^+(l_1, l_2; A_0). \end{cases}$$

W.L.O.G. suppose the first inclusion holds. If there exist $\alpha, \beta \in \mathbf{R}$ such that

$$\alpha + \beta \le \operatorname{nor}(l_2) - \operatorname{nor}(l_1) \pmod{2\pi}, \text{ and } \Omega^L(l_{1,\alpha,A_1}), \Omega^R(l_{2,-\beta,A_1}) \in \operatorname{vert}(G),$$

say $\Omega^L(l_{1,\alpha,A_1}) = g_a$ and $\Omega^R(l_{2,-\beta,A_1}) = g_b$. Then define the (*vertices between* $l_{1,\alpha;A_1}$ *and* $l_{2,-\beta;A_1}$) by

$$\mathrm{vert}_G(l_{1,\alpha,A_1}, l_{2,-\beta,A_1}) := \begin{cases} \{g_b, g_{b+1}, \ldots, g_a\} & \text{if the segment of } \partial G \text{ moving clockwise} \\ & \text{from } g_b \text{ to } g_a \text{ lies in } \mathrm{Sect}^+(l_{1,\alpha;A_1}, l_{2,-\beta;A_1}; A_1) \\ \{g_a, g_{a+1}, \ldots, g_b\} & \text{otherwise.} \end{cases}$$

This definition is perhaps best served by a diagram. See Figure 3.

Proposition 3.2. $G \cap \mathrm{Sect}^+(l_{1,\alpha,A_1}, l_{2,-\beta,A_1}; A_1) \subset \mathrm{Conv}(A_1, \mathrm{vert}_G(l_{1,\alpha,A_1}, l_{2,-\beta,A_1}))$.

Proof. For ease of notation let us write

$$\mathrm{Sect} := G \cap \mathrm{Sect}^+(l_{1,\alpha,A_1}, l_{2,-\beta,A_1}; A_1), \mathrm{Conv} := \mathrm{Conv}(A_1, \mathrm{vert}_G(l_{1,\alpha,A_1}, l_{2,-\beta,A_1}))$$

Since Sect is the intersection of halfplanes and a convex subset, it is convex. By the Krein–Milman theorem [KM40], it suffices to show that the boundary points, hence extreme points of Sect are contained in Conv. The boundary of Sect consists of

(1) A segment of ∂G, contained in Conv.
(2) A segment of l_{1,α,A_1} contained in the segment of l_{1,α,A_1} from a point on ∂G to $\Omega^L(l_{1,\alpha,A_1})$. This is contained in Conv. since ∂G and $\Omega^L(l_{1,\alpha,A_1})$ are in Conv.
(3) Likewise for the segment of $l_{2,-\beta,A_1}$.
(4) A segment of ∂G, in the first case in the definition of $\mathrm{vert}(-)_G$.

All of these are clearly contained in Conv. □

Now we can state the idea from the beginning of this section formally. If we can expand the sector of an adjacent pair so that the endpoints of the lines meet the vertices of G, and the vertices between the lines do not make up all the vertices of G, then we are done:

Proposition 3.3. *If there exists an adjacent pair* $l_1, l_2 \in \mathrm{csh}(\mathcal{A})$ *satisfying*

(1) There exist $\alpha, \beta \in \mathbf{R}$ *such that* $\Omega^L(l_{1,\alpha,A_1}), \Omega^R(l_{2,-\beta,A_1}) \in \mathrm{vert}(G)$ *and* $\alpha + \beta \leq \mathrm{nor}(l_2) - \mathrm{nor}(l_1) \pmod{2\pi}$
(2) $\#\,\mathrm{vert}_G(l_{1,\alpha,A_1}, l_{2,-\beta,A_1}) < \#\,\mathrm{vert}(G)$,

then $(\mathcal{A}, G)$ *satisfy the weak carousel rule.*

Proof. We have the inclusions

$$\begin{aligned} A_0 &\subset G \cap \mathrm{Sect}^+(l_1, l_2; A_1) && \text{by Proposition 2.5} \\ &\subset G \cap \mathrm{Sect}^+(l_{1,\alpha,A_1}, l_{2,-\beta,A_1}; A_1) && \text{by Lemma 2.7} \\ &\subset \mathrm{Conv}(A_1, \mathrm{vert}_G(l_{1,\alpha,A_1}, l_{2,-\beta,A_1})) && \text{by Proposition 3.2} \end{aligned}$$

by assumption, this last convex hull is always contained in $\mathrm{Conv}(A_1, \mathrm{vert}(G) \backslash g_j)$ for some $j \in \{1, ..., n\}$, which concludes the proof. □

3.2. **Casework and last steps.** It remains to show that if $s < n$, we can always find an adjacent pair satisfying the assumptions of Proposition 3.3. The important fact to notice is that the segments on ∂G moving clockwise from $\Omega^L(l_i)$ to $\Omega^L(l_{i+1})$ for $i \in \{1, ..., s\}$ partition ∂G. To make this rigorous, first define the *clockwise angle between* l_i *and* l_{i+1} to be

$$\Delta_i := \mathrm{nor}(l_{i+1}) - \mathrm{nor}(l_i)$$

Then we define the *left sweep of* l_i and *right sweep of* l_{i+1} *about* A by

$$\mathrm{sweep}^L(l_i) := \{\Omega^L(l_{i,\alpha;A}) : \alpha \in [0, \Delta_i]\},$$
$$\mathrm{sweep}^R(l_{i+1}) := \{\Omega^R(l_{i+1,-\beta;A}) : \beta \in [0, \Delta_i]\}$$

where $A = \mathrm{Conv}(A_0, A_1)$. The reason we take A here is that after completing one sweep, moving to the next common supporting line, the new line and its corresponding sector may satisfy the inclusion in 2.5 for a different element of $\mathcal{A}$. See Figure 4 for an illustration. Using this definition of sweeps we can restate Proposition 3.3:

Proposition 3.4. *If there exist and adjacent pair $l_1, l_2 \in \mathrm{csh}(\mathcal{A})$ such that*

(a) $\mathrm{sweep}^L(l_1) \cap \mathrm{vert}(G) \neq \varnothing \neq \mathrm{sweep}^R(l_2) \cap \mathrm{vert}(G)$*, and*
(b) not $(\mathrm{sweep}^L(l_1) \cap \mathrm{vert}(G) = \{g_i\}, \quad \mathrm{sweep}^R(l_2) \cap \mathrm{vert}(G) = \{g_{i+1}\})$*,*

then $(\mathcal{A}, G)$ satisfy the weak carousel rule.

Proof. (1) of Proposition 3.3 implies (a) and (2) of Proposition 3.3 implies (b). □

Lemma 3.5. *The sweeps partition the boundary of G, i.e.*

$$\bigcup_{i=1}^{s} \mathrm{sweep}^L(l_i) = \bigcup_{i=1}^{s} \mathrm{sweep}^R(l_i) = \partial G$$

and

$$\mathrm{sweep}^L(l_i) \cap \mathrm{sweep}^L(l_{i+1}) = \Omega^L(l_{i+1}), \quad \mathrm{sweep}^R(l_i) \cap \mathrm{sweep}^R(l_{i-1}) = \Omega^R(l_{i-1}).$$

Proof. We prove the statement for the left sweep. By Lemma 3.1 and definition, $\mathrm{sweep}^L(l_i)$ is a segment of ∂G beginning at $\Omega^L(l_i)$ and ending at $\Omega^L(l_{i,\Delta_i;A})$. But this latter point is by definition of Δ_i equal to $\Omega^L(l_{i+1})$. Hence the sweep is a contiguous segment proceeding clockwise from $\Omega^L(l_i)$ to $\Omega^L(l_{i+1})$. Iterating the argument for $i \in \{1, \dots, s\}$ completes the proof. □

We are now ready to prove the main theorem. For notation, let lowercase $\omega^L(l)$ and $\omega^R(l)$ denote the edge of G that $\Omega^L(l)$ and $\Omega^R(l)$ lie on. If $\Omega^L(l)$ is a vertex of G, let $\omega^L(l)$ be the edge lying clockwise from it, and if $\Omega^R(l)$ is a vertex of G, let $\omega^R(l)$ be the edge lying counterclockwise from it. This convention ensures that ω^L and ω^R are well-defined, and makes the proof easier to work with.

Proof of Theorem 1.1. By Lemma 3.5 and the pigeonhole principle, there exists some $i \in \{1, \dots, n\}$ such that $\mathrm{sweep}^L(l_i)$ contains more than one vertex of G. Without loss of generality let $i = 1$. If $\mathrm{sweep}^R(l_2)$ contains at least one vertex of G, we have satisfied the assumptions of Proposition 3.4, implying the weak carousel rule for $(\mathcal{A}, G)$.

Otherwise, suppose $\mathrm{sweep}^R(l_2)$ sweeps no vertices. Then by Lemma 3.5, $\Omega^R(l_2)$ lies clockwise of $\Omega(l_1)$ on the same edge. Without loss of generality suppose $\omega^L(l_1) = e_1$. Since $\mathrm{sweep}^L(l_1)$ contains more than one vertex of G, $\omega^L(l_2)$=e_j for $j \in \{3, \dots, n, 1\}$. Let $\omega^R(l_2) = \omega^R(l_1) = e_k$. In the following, for $a, b, c \in \{1, \dots, n\}$, by $a < b < c$ we mean that b lies in the interval $a+1, a+2, \dots, c-1$ working modulo n. We have the following cases. For a visual of each case, see Figure 6.

(1) $j < k < 1$. Let $\mathcal{H}_1$ be the halfplane whose boundary is l_1, in the direction containing $\mathcal{A}$. Then

$$\begin{aligned} A_0, A_1 &\subset \mathcal{H}_1 \cap G \\ &= \mathrm{Conv}(\Omega^L(l_1), \Omega^R(l_1), \mathrm{vert}(G) \cap \mathcal{H}_1) \\ &\subset \mathrm{Conv}(g_1, g_n, \dots, g_3) \end{aligned}$$

Hence the weak carousel rule holds for either A_0 or A_1.

(2) $1 < k < j$. Then we have the following order of the given points, proceeding clockwise around ∂G:

$$\Omega^L(l_1), \Omega^R(l_1), \Omega^R(l_2), \Omega^L(l_2).$$

By convexity of G, this implies that $l_1 \cap G$ and $l_2 \cap G$ are disjoint segments. Consequently, $\mathcal{A}$ lies in two disjoint subsets of G, a contradiction.

(3) $k = j$. By our definition of ω, this implies that e_j and l_2 coincide, so $\Omega^R(l_2)$ is g_j, contained in $\text{sweep}^R(l_2)$. And $\text{sweep}^L(l_1)$ contains at least two elements g_1, g_2. We conclude by applying Proposition 3.4.

(4) $k = 1$. Then $\omega^R(l_1) = \omega^R(l_2) = e_1 = \omega^L(l_1)$. By our definition of ω, this implies that e_1 and l_1 coincide, so $\Omega^L(l_1)$, is a vertex contained in $\text{sweep}^L(l_1)$, and $\text{sweep}^R(l_2) = \text{vert}(G)$. We conclude by applying Proposition 3.4.

This enumerates through all possibilities for $j, k \in \{1, ..., n\}$, concluding the proof. □

With this theorem we can immediately deduce the previous result of Adaricheva–Bolat, as well as similar results for ellipses.

Corollary 3.6 (Theorem 3.1, [AB19] and Theorem 1.1, [CK19])**.** *Two disks contained in a triangle satisfy the weak carousel rule.*

Proof. Three distinct points determine a circle, so two distinct disks have at most two common supporting lines. □

Corollary 3.7. *Two ellipses contained in a pentagon satisfy the weak carousel rule.*

Proof. Any five points determine a conic, so two unique ellipses can have at most four common supporting lines. □

In general, as long as we have an understanding of the common tangents of a convex compact shape, Theorem 1.1 allows us to conclude an analogous result.

3.3. **The bound is sharp for even n.** We now construct a family of examples to show that the bound in Theorem 1.1 is sharp for even n. That is, given $n > 2$, we construct a convex n-gon G and two convex shapes $\mathcal{A} = \{A_0, A_1\}$ contained in G with $\# \operatorname{csh}(\mathcal{A}) = n$ such that $(\mathcal{A}, G)$ fails to satisfy the weak carousel rule.

Identifying $\mathbf{R}^2$ with the complex plane, let G be the n-gon whose vertices are the nth roots of unity $\{\exp 2k\pi i/n\}$ for $k \in \{0, 1, ..., n-1\}$. The common supporting lines will be the lines connecting the midpoints of adjacent edges of G, so that $\cup \operatorname{csh}(\mathcal{A}) \cap G$ forms the boundary of a similar n-gon, inscribed inside G. Denote these edges of this smaller n-gon by $l_1, ..., l_n$, with the convention that l_i and l_{i+1} are adjacent. Trisect each line l_k, and name the points of trisection p_k^a, p_k^b, proceeding clockwise. Let A_0 be the polygon whose vertices are $p_1^a, p_2^b, p_3^a, ..., p_n^b$, and A_1 the polygon whose vertices are those remaining points.

Proposition 3.8. *$(\mathcal{A}, G)$ do not satisfy the weak carousel rule.*

Proof. The proof will proceed by coordinate bash. By symmetry, it suffices to check that the inclusion

$$A_j \subset \operatorname{Conv}(A_{1-j}, \operatorname{vert} G \backslash \{1\})$$

does not hold for $j = 0, 1$. We will switch between $\mathbf{C}$ and $\mathbf{R}^2$ whenever convenient. Let l_1 be the common supporting line joining the midpoints of the two edges of G joining 1 to

$\exp(2\pi i/n)$ and $\exp(2\pi i(n-1)/n) = \exp(-2\pi i/n)$. In real coordinates, l_1 intersects the boundary of G at the points

$$(\frac{1+\cos 2\pi/n}{2}, \frac{\sin 2\pi/n}{2}), (\frac{1+\cos 2\pi/n}{2}, -\frac{\sin 2\pi/n}{2}),$$

so then

$$p_1^a = (\frac{1+\cos 2\pi/n}{2}, \frac{\sin 2\pi/n}{6}), \quad p_1^b = (\frac{1+\cos 2\pi/n}{2}, -\frac{\sin 2\pi/n}{6})$$

Checking first for $j = 1$,

$$C := \mathrm{Conv}(A_0, \mathrm{vert}\, G \backslash \{1\}) = \mathrm{Conv}(p_1^a, \mathrm{vert}\, G \backslash \{1\})$$

The lower rightmost edge of C is the line connecting p_1^a to $\exp(-2\pi i/n)$, which has equation

$$y - \frac{\sin 2\pi/n}{6} = \frac{7}{3}(\cot \pi/n)(x - \frac{1+\cos 2\pi/n}{2})$$

so any point in C must satisfy the inequality

$$y - \frac{\sin 2\pi/n}{6} \geq \frac{7}{3}(\cot \pi/n)(x - \frac{1+\cos 2\pi/n}{2})$$

But plugging in $p_1^b = (\frac{1+\cos 2\pi/n}{2}, -\frac{\sin 2\pi/n}{6})$ for (x, y), we obtain

$$\text{L.H.S.} = -\frac{\sin 2\pi/n}{3}$$

$$\text{R.H.S.} = \frac{7}{3}(\cot \pi/n)(\frac{1+\cos 2\pi/n}{2} - \frac{1+\cos 2\pi/n}{2}) = 0$$

and since $\sin 2\pi/n$ is strictly positive, we have that $p_1^b \notin C$, hence

$$A_1 \not\subset \mathrm{Conv}(A_0, \mathrm{vert}\, G \backslash \{1\})$$

and the statement for $j = 0$ is identical, reflected across the y-axis. □

3.4. **Questions.** We conclude with a few questions. The first follows from the sharpness example in the previous section. We have not been able to construct such an example for odd n in a similar manner. In fact, all examples we have constructed with $s = n$ still satisfy the weak carousel rule. Hence:

Question 3.1. Is the bound in Theorem 1.1 sharp for odd n? If not, is it $s \geq n$?

Next, in Theorem 1.1, [Czé17a], Czédli shows that this property characterizes disks in $\mathbf{R}^2$. That is, suppose $\mathcal{A} = \{A_0, A_1\}$, and A_1 is obtained from A_0 by isometry. Then if $(\mathcal{A}, G)$ satisfies the weak carousel rule for all triangles G, then A_0 must be a disk. One natural question in this vein, following Corollary 3.7, is whether this property uniquely characterizes ellipses in $\mathbf{R}^2$. That is,

Question 3.2. Does there exist a non-ellipse compact convex subset $A \subseteq \mathbf{R}^2$ such that given any distinct isometric copy A' of A, we have $\#\,\mathrm{csl}(A, A') \leq 4$?

Returning to the subject of convex geometries, the authors in [AB19] show a slightly stronger result than stated in the introduction. By their Theorem 5.1, the weak carousel rule in fact holds for A_0, A_1 disks, and $G = \mathrm{Conv}(X_1, X_2, X_3)$ where X_1, X_2, X_3 are also disks. That is, there exist $i \in \{0, 1\}$ and $j \in \{1, 2, 3\}$ such that

$$A_i \subset \mathrm{Conv}(A_{1-i}, \{X_1, X_2, X_3\} \setminus \{X_j\}).$$

whenever $A_0, A_1 \subset G$. This result allows them to deduce a statement about the non-representability of convex geometries as disks. Unfortunately, it is difficult to extend this result for $X_1, X_2, \ldots$ arbitrary convex shapes. For example, it is possible to construct two ellipses $\mathcal{A} = \{A_0, A_1\}$ with $\#\,\mathrm{csl}(\mathcal{A}) = 2$, and ellipses $X_1, \ldots, X_3$ such that $A_0, A_1 \subset G := \mathrm{Conv}(X_1, \ldots, X_3)$ yet the weak carousel rule fails for $(\mathcal{A}, G)$, as in Figure 5. We ask:

Question 3.3. Does there exist a suitable generalization of Theorem 1.1 where the n-gon G is replaced by the convex hull of n convex compact shapes? For example, does the weak carousel rule hold for two ellipses contained the convex hull of five other ellipses, i.e. can we replace the pentagon in Corollary 3.7 by the convex hull of five ellipses?

Lastly, we ask whether some reformulation of weak carousel property holds in higher dimensions. As shown by Czédli in Example 4.1, [Czé17b], the most simple generalization, for an arrangement of two spheres in a tetrahedron in $\mathbf{R}^3$, fails. The problem remains open, and seems unamenable using the proof techniques from this paper, seeing as even in the most simple case, two spheres in $\mathbf{R}^3$ in general position can have infinitely many common supporting planes.

References

[AB19] Kira Adaricheva and Madina Bolat. Representation of convex geometries by circles on the plane. *Discrete Mathematics*, 342(3):726–746, 2019.

[AN] K. Adaricheva and J. B. Nation. Convex geometries. In George Grätzer and Friedrich Wehrung, editors, *Lattice Theory: Special Topics and Applications*, pages 153–179. Springer International Publishing.

[CK19] Gábor Czédli and Árpád Kurusa. A convex combinatorial property of compact sets in the plane and its roots in lattice theory. *Categories and General Algebraic Structures with Applications*, 11:57–92, 2019.

[CS16] Gábor Czédli and László L. Stachó. A note and a short survey on supporting lines of compact convex sets in the plane. *Acta Univ. M. Belii Ser. Math.*, 24:3–14, 2016.

[Czé14] Gábor Czédli. Finite convex geometries of circles. *Discrete Mathematics*, 330:61–75, 2014.

[Czé17a] Gábor Czédli. Characterizing circles by a convex combinatorial property. *Acta Sci. Math. (Szeged)*, 83(3-4):683–701, December 2017.

[Czé17b] Gábor Czédli. An easy way to a theorem of kira adaricheva and madina bolat on convexity and circles. *Acta Scientiarum Mathematicarum*, 83(3):703–712, 2017.

[EJ85] Paul H. Edelman and Robert E. Jamison. The theory of convex geometries. *Geometriae Dedicata*, 19(3):247–270, 1985.

[KM40] M. Krein and D. Milman. On extreme points of regular convex sets. *Studia Mathematica*, 9(1):133–138, 1940.

[KNO05] Kenji Kashiwabara, Masataka Nakamura, and Yoshio Okamoto. The affine representation theorem for abstract convex geometries. *Computational Geometry*, 30(2):129–144, 2005. Special Issue on the 19th European Workshop on Computational Geometry.

[Roc70] Ralph Tyrell Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, 1970.

[RR17] Michael Richter and Luke G. Rogers. Embedding convex geometries and a bound on convex dimension. *Discrete Mathematics*, 340(5):1059–1063, 2017.

[Sch13] Rolf Schneider. *Convex Bodies: The Brunn–Minkowski Theory*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2 edition, 2013.

Figures

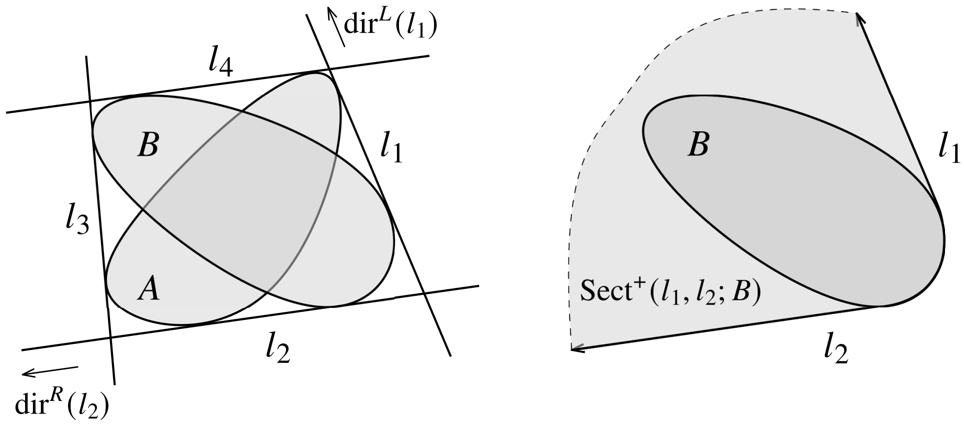

Figure 1. On the left, l_i and l_{i+1} are adjacent, for $i = 1, 2, 3, 4, \pmod 4$. On the right, the sector $\mathrm{Sect}^+(l_1, l_2; B)$ continues along $\mathrm{dir}^L(l_1)$ and $\mathrm{dir}^L(l_2)$.

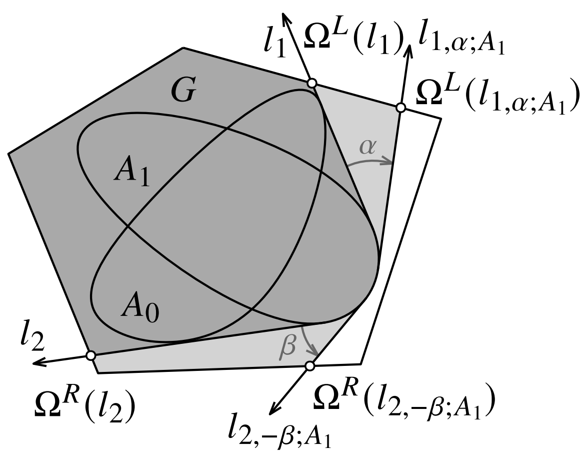

Figure 2. $\mathrm{Sect}^+(l_1, l_2; A_1) \cap G$ in dark gray, and the expanded $\mathrm{Sect}^+(l_{1,\alpha;A_1}, l_{2,-\beta;A_1}; A_1) \cap G$ in light gray.

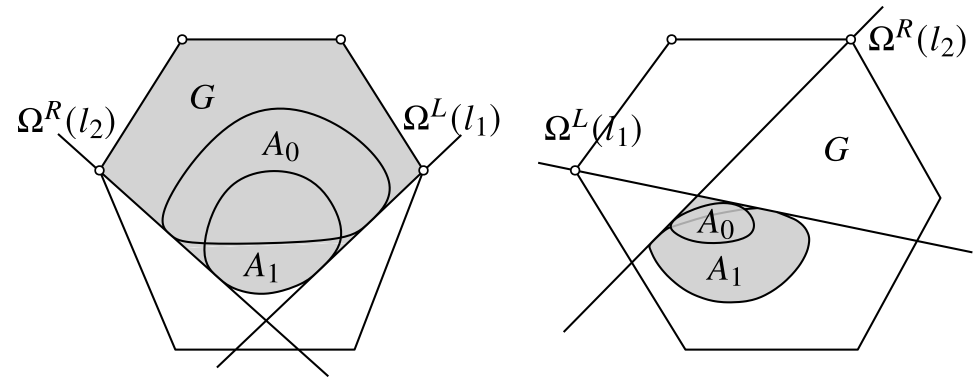

FIGURE 3. The vertices between l_1 and l_2 are dotted, and the sectors $\mathrm{Sect}^+(l_1, l_2; A_1)$ are shaded. Note the difference in order depending on whether the sector intersects G.

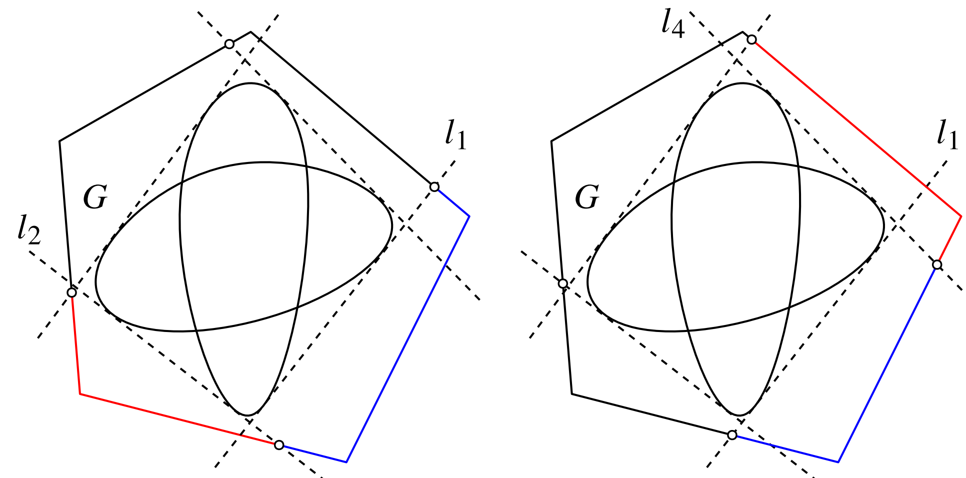

FIGURE 4. On the left, $\mathrm{sweep}^L(l_1)$ in blue and $\mathrm{sweep}^L(l_2)$ in red. On the right, $\mathrm{sweep}^R(l_1)$ in blue and $\mathrm{sweep}^R(l_4)$ in red.

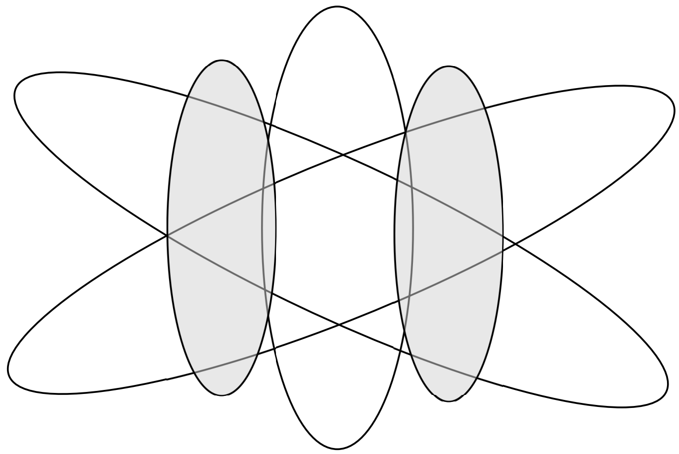

FIGURE 5. The ellipses A_0, A_1 shaded, the ellipses X_1, X_2, X_3 unshaded. The weak carousel rule fails for $\mathcal{A} = \{A_0, A_1\}$ and $G = \mathrm{Conv}(X_1, X_2, X_3)$.

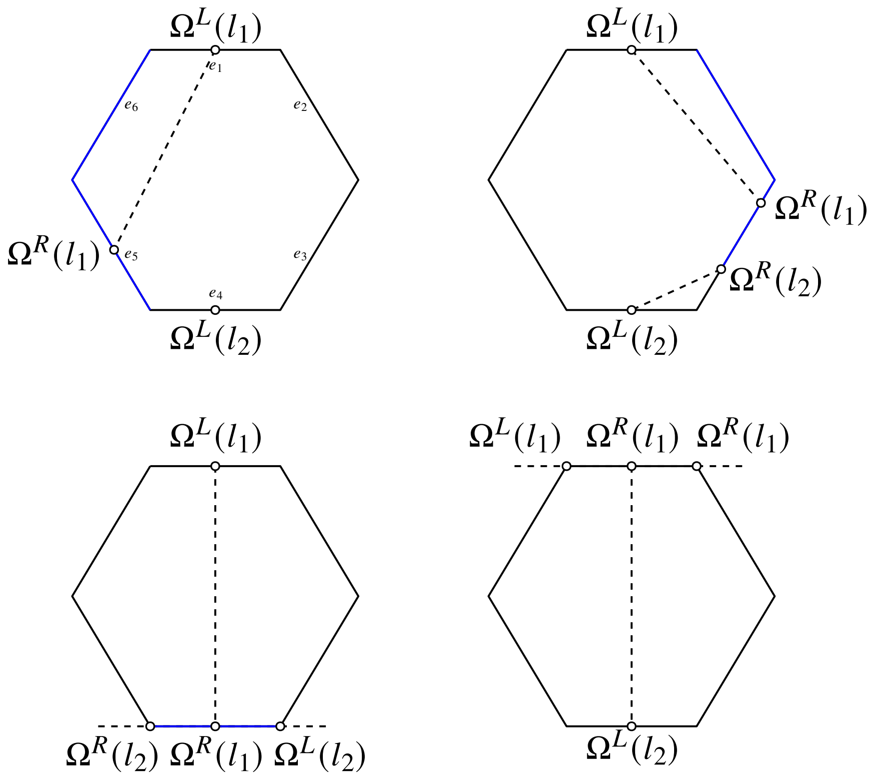

FIGURE 6. Illustrations for various cases in the proof of Theorem 1.1. Top to bottom, left to right are cases (1), (2), (3), and (4). Here we fix $j = 4$. The possible locations for $\Omega^R(l_1)$ are in blue.