QUILLEN'S FRAMEWORK FOR HOMOLOGICAL STABILITY

Abstract. We'll see how the proof of homological stability for symmetric groups generalizes to broader sequences of groups, following Quillen's framework. ⋆

Contents

1. Previously...

Previously in this seminar, Azélie sketched the proof of Nakaoka's theorem:

Theorem 1.1. *The sequence $\Sigma_0 \to \Sigma_1 \to \cdots$ exhibits homological stability. That is, the induced map on the homology of the classifying spaces*

$$H_\bullet(B\Sigma_n; \mathbf{Z}) \to H_\bullet(B\Sigma_{n+1}; \mathbf{Z})$$

is surjective for $\bullet \leq n/2$ and an isomorphism for $\bullet \leq (n-1)/2$.

This was first proven in [Nak60] then improved. The following concepts, constructions, facts, etc. were important for the proof:

(1) *Classifying spaces*. For a group G, its classifying space BG is the geometric realization of the nerve of G regarded as a one-object category.
(2) *Homological connectivity*. A space X is homologically d-connected if $H_\bullet(X; \mathbf{Z}) = 0$ for $\bullet < d$. A map $X \to Y$ is homologically d-connected if it is an isomorphism on $H_{<d}$ and a surjection on H_d.
(3) *Homotopy quotients*. For a given G-space X, we can construct the homotopy quotient $X /\!\!/ G$ via the Borel construction: take a contractible space EG with free G-action, and define $X /\!\!/ G = (X \times EG)/G$. We have the following properties:
   (a) G-equivariant maps of G-spaces $X \to Y$ induce maps $X /\!\!/ G \to Y /\!\!/ G$. Moreover, homological connectivity gets preserved.
   (b) Homotopy quotients commute with geometric realization.
   (c) For a G-set S (regarded as a discrete topological space) with transitive G-action, $S /\!\!/ G \simeq B\,\mathrm{Stab}_G(s)$, the subgroup of G which stabilizes any $s \in S$.
(4) *Semi-simplicial sets*. The category $\mathbf{\Delta}_{\mathrm{inj}}$ has nonempty finite totally ordered sets as objects and injective order-preserving functions as morphisms. A semi-simplicial set is a contravariant functor from $\mathbf{\Delta}_{\mathrm{inj}}$ to **Set**. It comes with an analogous notion of geometric realization.

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The proof proceeds roughly as follows. First replace $B\Sigma_n \simeq * /\!\!/ \Sigma_n$ by some other homotopy quotient $X /\!\!/ \Sigma_n$ where X is homologically close to a point, i.e. homologically highly-connected in the range we care about. The best choice for X turns out to be the geometric realization of a semi-simplicial set $X_\bullet$ where each X_k is a transitive Σ_n-set. Then by looking at the geometric realization spectral sequence, we can relate $H_\bullet(B\Sigma_n)$ to the stabilizer subgroups $\mathrm{Stab}_{\Sigma_n}(X_k)$.

Today we will wrap up the proof of Nakaoka's theorem by looking at the final spectral sequence step. For completeness we'll also include the preliminaries and the earlier parts of the proof in these notes. Then we will see that none of this depended too much on symmetric groups, and present a strategy of Quillen that works for more general sequences of groups. All of this material was taken from Kupers' minicourse notes [Kup21] and Wahl's survey [Wah23].

2. Proving Nakaoka

The semi-simplicial set that we replace the point by in the previous argument is the *injective words*, which for simplicity we will denote by $W^n_\bullet$. Let $[a]$ be the set $[0, \dots, a]$ and $\underline{b}$ be the set $[1, \dots, b]$. Let **FI** be the category of finite sets with injective maps. Then $W^n_\bullet$ has simplicies

$$W^n_k = \hom_{\mathbf{FI}}([k], \underline{n}).$$

We can write a k-simplex of $W^n_\bullet$ as a $k+1$-letter word with letters in $\underline{n}$. For example, W^2 has

- 0-simplices W^2_0 the one-letter words (1) and (2).
- 1-simplices W^2_1 the words (12) and (21).
- no higher simplices.

So W^2 geometrically realizes to a circle. The Σ_n-action on W^n is given by post composition. Remember that the goal is to replace the point with $||W^n_\bullet||$, such that each simplex is a transitive Σ_n set. Indeed:

Proposition 2.1. *$||W^n_\bullet||$ is homologically $((n-1)/2)$-connected.*

Proof. Mayer–Vietoris spectral sequence and induction. See [RW]. □

Proposition 2.2. *W^n_k is a transitive Σ_n-set. For $x \in W^n_k$, $\mathrm{Stab}_{\Sigma_n}(x)$ is the group of permutations of $\underline{n} \setminus \mathrm{im}(x)$.*

Proof. Clearly Σ_n acts transitively on any injective map $[k] \to \underline{n}$ by postcomposition. And a map x is stabilized by a permutation σ iff σ doesn't touch the elements in $\mathrm{im}\, x$. □

Armed with these facts, we continue to the *geometric realization spectral sequence*. The geometric realization of a (semi-)simplicial space $X_\bullet$ has a filtration by skeleta

$$F_r||X_\bullet|| = \Big(\bigsqcup_{0 \le p \le r} \Delta^p \times X_p\Big)/\sim$$

and this gives rise to a spectral sequence:

Theorem 2.3. *There is a strongly convergent first quadrant spectral sequence*

$$E^1_{p,q} = H_q(X_p; \mathbf{Z}) \implies H_{p+q}(||X_\bullet||; \mathbf{Z})$$

with differentials $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$. Moreover, d^1 is given by the alternating sum of face maps $\sum_{i=0}^p (-1)^i (d_i)_*$. The edge homomorphism $E^1_{0,q} \to E^\infty_{0,q} \to H_q(||X_\bullet||)$ is induced by the inclusion $X_0 \hookrightarrow X_\bullet$.

Let's now see the proof of Nakaoka stability. This is a proof by strong induction on n.

2.1. **First step: substitution.** By 2.1, $||W_\bullet^{n+1}||$ is homologically $n/2$-connected, so the map from it to the point is homologically $(n/2)+1$-connected. Then, taking homotopy quotient by Σ_{n+1}, we have that

$$||W_\bullet^{n+1}|| /\!\!/ \Sigma_{n+1} \to * /\!\!/ \Sigma_{n+1} = B\Sigma_{n+1}$$

is $((n/2)+1)$-connected and likewise for Σ_n. So it suffices to show that the map

$$B\Sigma_n \to ||W_\bullet^{n+1}|| /\!\!/ \Sigma_{n+1}$$

is an isomorphism/surjection in the desired range. And because homotopy quotients commute with geometric realization, this reduces to showing that

$$B\Sigma_n \to ||W_\bullet^{n+1} /\!\!/ \Sigma_{n+1}||$$

has the desired properties in the desired range.

2.2. **Second step: setting up the spectral sequence.** We now want to examine the homology of $||W_\bullet^{n+1} /\!\!/ \Sigma_{n+1}||$. Fortunately we have the spectral sequence from the previous section:

$$E^1_{p,q} = H_q(W_p^{n+1} /\!\!/ \Sigma_{n+1}) \implies H_{p+q}(||W_\bullet^{n+1} /\!\!/ \Sigma_{n+1}||)$$

Since W_p^{n+1} is a transitive Σ_{n+1}-set, for an element $x \in W_p^{n+1}$ (i.e. a p-letter word with letters in $\underline{n+1}$) we have a homotopy equivalence

$$* /\!\!/ \mathrm{Stab}_{\Sigma_{n+1}}(x) \to W_p^{n+1} /\!\!/ \Sigma_{n+1}$$

sending $*$ to x, because $\mathrm{Stab}_{\Sigma_{n+1}}(x)$ is the permutation group of $\underline{n+1} \setminus \mathrm{im}(x)$. If we take x to be the inclusion ι_p into the last elements on $\underline{n}$, i.e.

$$0 \mapsto n-p+1, \cdots, p-1 \mapsto n, p \mapsto n+1$$

then the map

$$\Sigma_{n-p} \cong \mathrm{Stab}_{\Sigma_{n+1}}(x) \hookrightarrow \Sigma_{n+1}$$

is just the inclusion of permutations that only act on the first $n-p$ elements. So we can make the identification

$$E^1_{p,q} \cong H_q(B\Sigma_{n-p}).$$

The next step is to figure out what the differential $d^1 : H_q(B\Sigma_{n-p}) \to H_q(B\Sigma_{n-(p-1)})$ is actually doing. The theorem tells us that it is induced by the alternating sums of the semi-simplicial face maps (which are themselves given by precomposition with the face maps from the simplex category), but after making our identification between the homotopy quotient of the semi-simplicial set and our symmetric groups, things become a little murky. By even and oddness, and the theorem identifying the d^1 differential, we have

$$d^1 = \begin{cases} \sum_{i=0}^{p}(-1)^i \sigma_* & p > 0, p \text{ even} \\ 0 & \text{else} \end{cases}$$

where $\sigma : \Sigma_j \to \Sigma_{j+1}$ is the canonical inclusion into the first j elements. This is really good, because now the edge homomorphism

$$E^1_{0,q} = H_q(B\Sigma_n) \to H_q(||W_\bullet^{n+1} /\!\!/ \Sigma_{n+1}||) \simeq H_q(B\Sigma_{n+1})$$

is the map we are analyzing, and the inductive hypothesis tells us that the lower left chunk of the E^2 page is zero, since σ_* is an isomorphism there.

2.3. **Third step: the edge homomorphism.** To complete the proof, we will do casework on the parity of n. Precisely, the inductive hypothesis tells us that for $m \leq n$, the maps

$$\sigma_* : H_q(B\Sigma_m) \to H_q(B\Sigma_{m+1})$$

are isomorphisms for $q \leq (n-1)/2$. So in column q, the rows which have alternating differentials 0 and σ_* are exact for q in some small range. To be precise, we need to consider the parity of n.

(1) If n is odd, so $n = 2k+1$, then the isomorphism range is

$$q \leq (2k+1-1)/2 = k.$$

Restricting ourselves to these rows, the inductive hypothesis gives that σ_* is an isomorphism for

$$q \leq (n-p-1)/2 = (2k+1-p-1)/2 = k - p/2.$$

So upon moving to the E^2 page, this bottom left chunk becomes zero, and for $q \leq k$ there are no more contributions in the differential, implying that

$$H_q(B\Sigma_n) = E^2_{0,q} \to E^\infty_{0,q} = H_q(B\Sigma_{n+1})$$

is an isomorphism for $q \leq k$. And the surjection comes from the fact that the top term $q = k+1$ has only one differential going into it, so that term quotiented by the image of that differential gives the E^∞ term.

(2) We'll omit the case where n is even. It is basically the same.

This completes the proof! See [Kup21] for a diagram of the spectral sequence and examples when $n = 8, 9$.

3. Quillen's framework

In the previous last section we really didn't use much about symmetric groups. The strategy was to find some nice semi-simplicial objects W_n that G_n act on transitively, and argue using those instead. Here we will show how that works, and how it recovers Nakaoka stability.

3.1. **Braided monoidal groupoids.** A *groupoid* is a category where all morphisms are isomorphisms. A *monoidal* category is a category C with a functor (called the monoidal product)

$$\otimes : C \times C \to C$$

which is associative and unital. We say $(C, \otimes)$ is *braided* if we also have isomorphisms

$$\sigma_{X,Y} : X \otimes Y \to Y \otimes X$$

for $X, Y \in \mathrm{ob}(C)$ making the following diagrams commute:

$$\begin{tikzcd}
X\otimes Y\otimes Z \arrow[r, equal] \arrow[d, "\mathrm{id}\otimes\sigma_{Y,Z}"'] & X\otimes Y\otimes Z \arrow[d, "\sigma_{X,Y}\otimes\mathrm{id}"] \\
X\otimes Z\otimes Y \arrow[d, "\sigma_{X,Z}\otimes\mathrm{id}"'] & Y\otimes X\otimes Z \arrow[d, "\mathrm{id}\otimes\sigma_{X,Z}"] \\
Z\otimes X\otimes Y \arrow[d, "\mathrm{id}\otimes\sigma_{X,Y}"'] & Y\otimes Z\otimes X \arrow[d, "\sigma_{Y,Z}\otimes\mathrm{id}"] \\
Z\otimes Y\otimes X \arrow[r, equal] & Z\otimes Y\otimes X
\end{tikzcd}
\qquad
\begin{tikzcd}
X\otimes Y \arrow[r, "\sigma_{X,Y}"] \arrow[d, "f\otimes g"'] & Y\otimes X \arrow[d, "g\otimes f"] \\
Z\otimes W \arrow[r, "\sigma_{W,Z}"'] & W\otimes Z
\end{tikzcd}$$

Figure 1. Braided monoidal category identities.

So a symmetric monoidal category is a braided monoidal category in which $\sigma_{X,Y}\circ\sigma_{Y,X}$ is the identity. Some examples of braided monoidal *groupoids* are:

(1) The category of sets with bijections. The product is given by disjoint union.
(2) The category of R-modules with isomorphisms. The product is given by direct sum.
(3) The category of n-manifolds with diffeomorphisms. The product is connected sum.

Everything here is just a subcategory of a symmetric monoidal category where we take only the isomorphisms. What's important is that these braided monoidal categories give rise to sequences of groups as follows. Let $(\mathcal{G},\otimes)$ be a monoidal groupoid. Pick two objects $A, X\in\mathcal{G}$. Then we get a sequence of groups $G_1\to G_2\to\cdots$ with

$$G_n = \mathrm{Aut}_{\mathcal{G}}(A\otimes X^{\otimes n})$$

and maps

$$\sigma_n : G_n \to G_{n+1}$$

given by $f\mapsto f\otimes\mathrm{id}$. Pictorially this makes me think of the groups $\mathrm{Diff}_\partial(S_{n,1})$. Indeed, under this framework, for suitable choice of groupoid $\mathcal{G}$ we can recover most examples previously discussed:

(1) The groupoid of finite sets and isomorphisms with monoidal product given by disjoint union. Then for $A=\varnothing, X=\{*\}$ we have

$$G_n = \mathrm{Aut}_{\mathsf{Set}}(\{1,\dots,n\}) = \Sigma_n$$

which recovers the symmetric groups and embeddings $\Sigma_i\to\Sigma_{i+1}$.

(2) For a ring R, the groupoid of R-modules and isomorphisms, with a symmetric monoidal product given by direct sum. Then for $A=0, X=R$, we have

$$G_n = \mathrm{Aut}_{\mathsf{Mod_R}}(R^{\oplus n}) = \mathrm{GL}_n(R)$$

where the map $G_n\to G_{n+1}$ is given by adding a zero to the bottom right corner of the matrix.

To recover the other examples, and to state the general theorem, we need the concept of modules over a braided monoidal groupoid. A *right module over a monoidal groupoid $\mathcal{G}$* is a category C with a functor (the right action)

$$\otimes : C\times\mathcal{G}\to C$$

that is unital and associative. Then taking $A\in C$ and $X\in\mathcal{G}$, we can define a sequence

$$G_n = s\,\mathrm{Aut}_C(A\otimes X^{\otimes n})$$

where $G_n \to G_{n+1}$ is similarly given by $f \mapsto f \otimes \mathrm{id}$. This is a more general version of the sequence we constructed earlier, which is the scenario where $\mathcal{G}$ is taken to be a right module over itself. Here we'll stick with that (unless...)

3.2. **The space of destabilizations.** Now that we have our sequence of groups $\{G_n\}$, we need some nice enough simplicial objects W_n on which the groups act transitively. By 'nice enough' we mean that the tricks from the proof of Nakaoka stability (connectivity, stabilizer being a smaller G_n etc.) should transfer analogously. In the following, we will define a semi-simplicial set $W_\bullet$ given a sequence of groups $G_n = \mathrm{Aut}_{\mathcal{C}}(A \otimes X^{\otimes n})$, $A \in \mathcal{C}, X \in \mathcal{G}$, coming from the action of a braided monoidal groupoid $\mathcal{G}$ on a module $\mathcal{C}$ over it.

Definition 3.1. Fix $\mathcal{G}, \mathcal{C}, A, X$ as above. The *space of destabilizations* $W_n(A, X)_\bullet$ is the semi-simplicial set with simplices

$$W_n(A, X)_p = \{(B, f) | B \in \mathrm{ob}(\mathcal{C}), f : B \otimes X^{\otimes p+1} \to A \otimes X^{\otimes n}\}/\sim$$

where $(B, f) \sim (B', f')$ if there is an isomorphism $g : B \to B'$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd} B \otimes X^{\otimes p+1} \arrow[rr, "g \otimes \mathrm{id}^{p+1}"] \arrow[dr, "f"'] & & B \otimes X^{\otimes p+1} \arrow[dl, "f'"] \\ & A \otimes X^{\otimes n} & \end{tikzcd}$$

The face map $d_i : W^n(A, X)_p \to W^n(A, X)_{p-1}$ sends $[B, f]$ to $[B \otimes X, d_i f]$ where $d_i f$ is the composition

$$d_i f : B \otimes X \otimes X^{\otimes p-1+1} \xrightarrow{\mathrm{id} \otimes \tau_{X^{\otimes i}, X} \otimes \mathrm{id}} B \otimes X^{\otimes i} \otimes X \otimes X^{\otimes p-i} \xrightarrow{f} A \otimes X^{\otimes n}$$

where τ is the inverse of the braiding σ.

Just like Σ_n acts on the injective words by postcomposition, $G_n = \mathrm{Aut}_{\mathbf{C}}(A \otimes X^{\otimes n})$ also acts on $W_n(A, X)_\bullet$ by postcomposition. Now we'll list the properties that we wanted these objects to have. We unfortunately need mild assumptions.

(1) If $Y \otimes A^{\otimes p+1} \cong A \otimes X^{\otimes n}$ can be cancelled to obtain $Y \cong X^{\otimes n-p-1}$, then G_n acts transitively on $W_n(A, X)_p$.
(2) If the map $G_{n-p-1} \to G_n$ is injective, then there is an isomorphism $G_{n-p-1} \cong \mathrm{Stab}_{G_n}(x)$ for any simplex $x \in W_n(A, X)_p$.

Combining these two statements, we get that under these assumptions,

$$W_n(A, X)_p \cong G_n / \mathrm{Stab}_{G_n}(x) \cong G_n / G_{n-p-1}.$$

Continuing with the example with $\mathcal{G} = \mathcal{C} = \mathsf{finSet}$, $A = \emptyset, X = \{*\}$, let's see how this space of destabilizations recovers the injective words. We have

$$W_n(A, X)_p = \{(B, f) | B \in \mathsf{finSet}, f : B \sqcup \{0, ..., p\} \to \{1, ..., n\}\}/\sim$$

Because $\mathcal{C} = \mathcal{G}$ is a groupoid, f is forced to be an isomorphism, so (B, f) and $(B, f') \in W_n(A, X)_p$ represent the same class iff B and B' have the same cardinality. That is, f is determined entirely by where it maps the set $\{0, ..., p\}$. Hence

$$W_n(A, X)_p = \{\text{injective maps } [p] \to \underline{n}\}$$

as we had before. One can check that the face maps induced by the braiding yields what we had previously, and that both properties (1) and (2) above are satisfied.

3.3. **The argument and the theorem.** Now we can mimic the spectral sequence argument from before prove our general homological stability result.

Theorem 3.2. *Let $\mathcal{C}$ be a module over a braided monoidal groupoid $\mathcal{G}$. Fix $A \in \mathcal{C}$, $X \in \mathcal{G}$. Suppose $(\mathcal{G}, \mathcal{C}, A, X)$ satisfy the assumptions (1) and (2). If for all $n \geq 0$ there exists $k \geq 2$ such that $||W_n(A, X)_\bullet||$ is $(n-2)/k$-connected, then*

$$H_i(BG_n) \to H_i(BG_{n+1})$$

is an isomorphism for $i \leq (n-1)/k$ and surjective for $i \leq n/k$.

Proof sketch. Let $W_n = W_n(A, X)$ and $G_n = \mathrm{Aut}_{\mathcal{C}}(A \otimes X^{\otimes n})$. We'll induct on n. As before, we are interested in the homology of BG_{n+1}, which by the assumptions is homologically equivalent to the homotopy quotient

$$W_n /\!\!/ G_{n+1}$$

in the desired ranges.

We start by taking a free $\mathbf{Z}G_n$-resolution of $\mathbf{Z}$, call this $E_\bullet G_n$. For example, by taking augmented singular chains on the space EG we get a free resolution

$$\cdots \to C_2(EG) \to C_1(EG) \to C_0(EG) \to \mathbf{Z}.$$

Next, let $\tilde{C}_\bullet(W_n)$ be the augmented cellular chains of W_n. Then the tensor product

$$E_\bullet G_n \otimes \tilde{C}_\bullet(W_n)$$

gives a bicomplex, hence the E^0 page of a spectral sequence. Because we've assumed that W_n is highly connected, filtering according to the singular chain direction tells us that the E^1 page has zeroes in the lower left corner range. Filtering in the direction of the free resolution, we have

$$E^1_{p,q} = H_q(G_n; C_p(W_n))$$

but now $C_p(W_n)$ is the cellular p-chains on W_n, i.e. the free abelian group on the p-cells of W_n, which we know G_n acts transitively on. So

$$C_p(W_n) = \mathbf{Z}\{p\text{-cells of } W_n\} = \mathbf{Z}\{G_n/\mathrm{Stab}_{G_n}(x)\}$$

for any p-cell x. By the second assumption

$$G_n/\mathrm{Stab}_{G_n}(x) \cong G_n/G_{n-p-1}$$

so we have

$$E^1_{p,q} = H_p(G_n; \mathbf{Z}\{G_n/G_{n-p-1}\}) \cong H_p(G_{n-p-1})$$

by some homological algebra. Then we make the observation that the E^1 differential is again zero if p is odd, and σ_* if p is even, and conclude in the same way. □

3.4. **Conclusion.** We've seen that gives us homological stability for symmetric groups. A few other sequences of groups which exhibit homological stability are:

(1) General linear groups over a ring R. Here we need some assumptions on R (finite stable rank?) to make sure the conditions (1) and (2) are satisfied.
(2) Automorphism groups of free groups. Somebody asked last time if it was true that all these slopes were linear. Hatcher initially gives a bound of

$$i^2/4 + 2i - 1 < n$$

through some more complicated proof. But Galatius improves the bound to the same one as Nakaoka.
(3) Diffeomorphism groups or mapping class groups of surfaces.

(4) and more!

Some groups not exhibiting stability: The *pure braid groups*, which are the kernel of the "forgetful" map $B_n \to \Sigma_n$, do not exhibit homological stability. What the reason is, I'm not too sure. Another example of a sequence of groups arising from a braided monoidal groupoid but *not* satisfying homological stability is the following. Consider the braided monoidal groupoid of finite sets with Cartesian instead of disjoint product. Then choosing $A = \{0\}$, $X = \{0, 1\}$ yields $G_n = \Sigma_{2^n}$. In this case, the first stabilization map is zero which breaks everything, and our proof method fails to apply because the corresponding space of destabilizations is disconnected.

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