A CAROUSEL PROPERTY FOR COMPACT CONVEX SETS

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Carleton University
Algorithms Seminar

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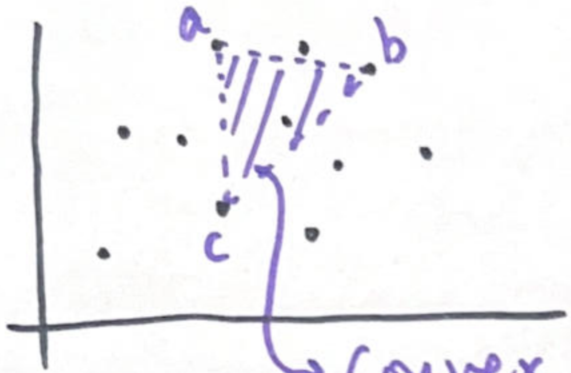
① CONVEXITY :



Convex hull of $\{a, b, c\}$

$\left(\text{points in } \mathbb{R}^2 \ , \ \text{convex hull} \right)$

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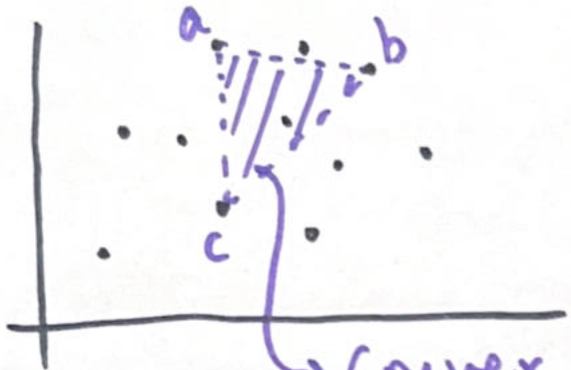

(points in $\mathbb{R}^2$, convex hull)

Convex hull of $\{a, b, c\}$

More generally, a <u>CONVEX GEOMETRY</u> is a set X and a closure operator $\phi : 2^X \to 2^X$ satisfying:

CLOSURE PROPERTIES:

$$A \subset \phi(A)$$

$$A \subset B \Rightarrow \phi(A) \subset \phi(B)$$

$$\phi(A) = \phi(\phi(A))$$

ANTI-EXCHANGE RULE: For convex A, $x, y \notin A$, $x \in \phi(A \cup \{y\})$ $\Rightarrow y \notin \phi(A \cup \{x\})$

Q. Does every convex geometry look like

$$\left\{ \begin{matrix} \text{points in} \\ \mathbb{R}^n \end{matrix} , \text{convex hull} \right\} ?$$

$$\Big\updownarrow \cong ?$$

$$\left\{ \begin{matrix} \text{a set} \\ X \end{matrix} , \begin{matrix} \text{closure operator} \\ \phi : 2^X \longrightarrow 2^X \end{matrix} \right\}$$

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$$\left\{ \begin{matrix} \text{points in} \\ \mathbb{R}^n \end{matrix} , \text{convex hull} \right\} \; ?$$

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$$\left(\begin{matrix} \text{a set} \\ X \end{matrix} , \begin{matrix} \text{closure operator} \\ \phi : 2^X \longrightarrow 2^X \end{matrix} \right)$$

A. NO.

$$\left(\{a, b\}, \phi \right) \xrightarrow{\not\cong} \left(\begin{matrix} \text{two points} \\ \text{in } \mathbb{R}^n \end{matrix} , \begin{matrix} \text{convex} \\ \text{hull} \end{matrix} \right)$$

$$\boxed{\phi(\{a\}) = \{a, b\}}$$

$$\phi(\{b\}) = \{b\}$$

$$\phi(\{a, b\}) = \{a, b\}$$

$$\phi(\{\}) = \{\}$$

No representation with points in $\mathbb{R}^n$ exists.

A. NO... but possible with CIRCLES

$$\left(\{a, b\}, \phi\right) \longrightarrow \left(\text{circles in } \mathbb{R}^2, \text{ convex hull}\right)$$

$a \longmapsto$ 

$b \longmapsto$

$\phi(\{a\}) = \{a, b\}$

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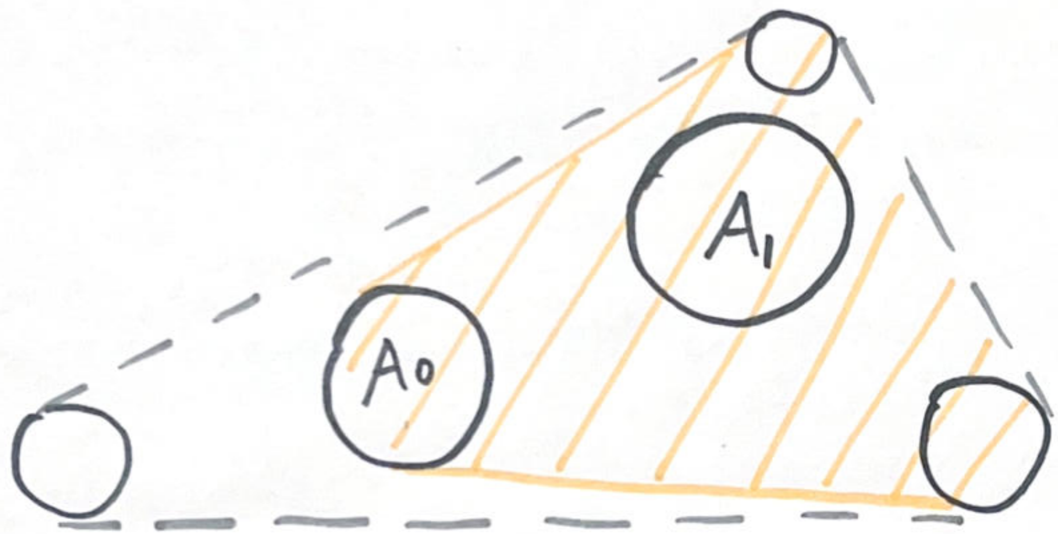
More geometry $\Longrightarrow$ More representations

Restricting to $\mathbb{R}^2$, are circles enough?

NO. (ADARICHEVA-BOLAT, 2019)

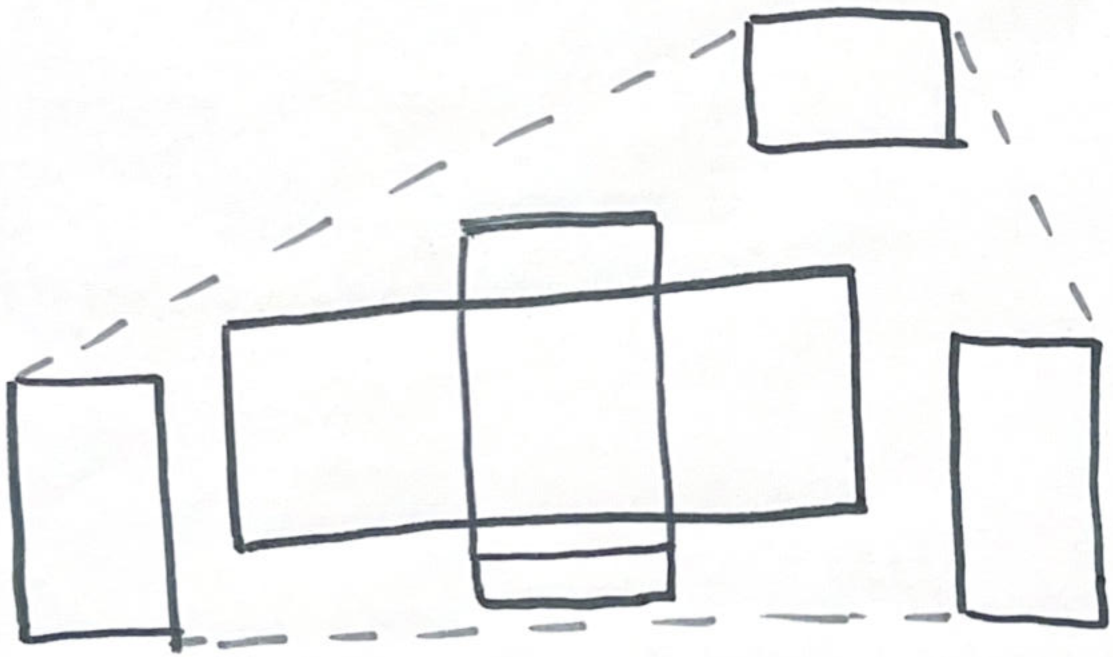


<u>Thm.</u> If A_0, A_1 are disks in $\mathbb{R}^2$ and G is the convex hull of three disks g_1, g_2, g_3, where $A_0, A_1 \subset \mathrm{Conv}(g_1, g_2, g_3)$, then there exist $i \in \{0,1\}$, $j \in \{1,2,3\}$ such that

$$A_i \subset \mathrm{Conv}(A_{1-i}, \{g_1, g_2, g_3\} \setminus \{g_j\})$$

More complex polygons don't have this issue.

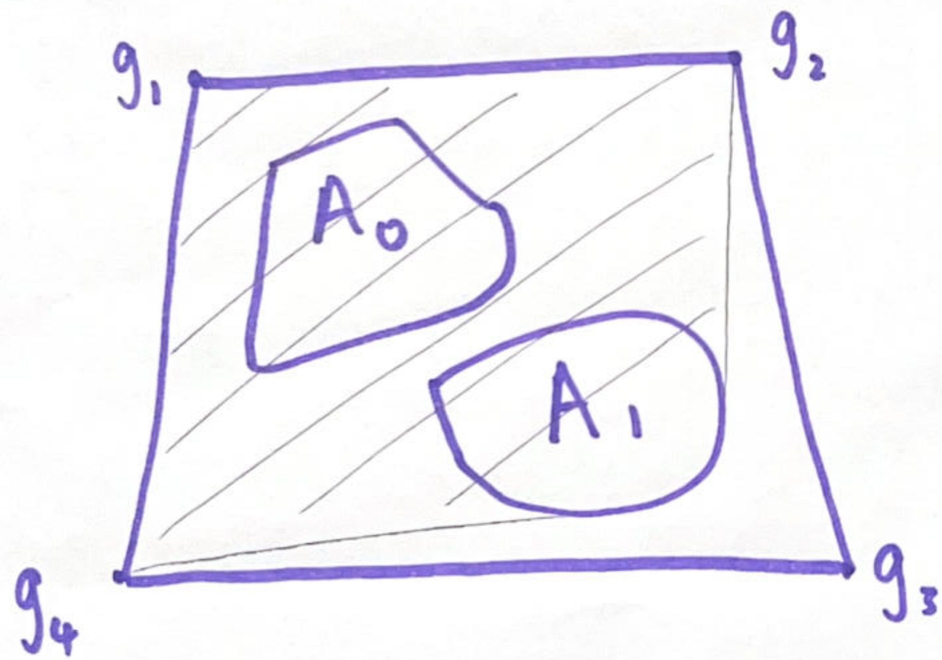
<u>Thm</u> (Richter-Rogers, 2017). Any convex geometry can ~~by~~ be represented by n-gons in $\mathbb{R}^2$ for sufficiently large n.



The **WEAK CAROUSEL RULE** : Given $\mathcal{A} = \{A_0, A_1\}$ convex compact subsets of the plane, and $G = \mathrm{Conv}(g_1, \ldots, g_n)$ a convex n-gon containing $\mathcal{A}$, we say $(\mathcal{A}, G)$ satisfy the WCR if :

$$\exists\, i \in \{0,1\},\ j \in \{1, \ldots, n\} \text{ such that}$$

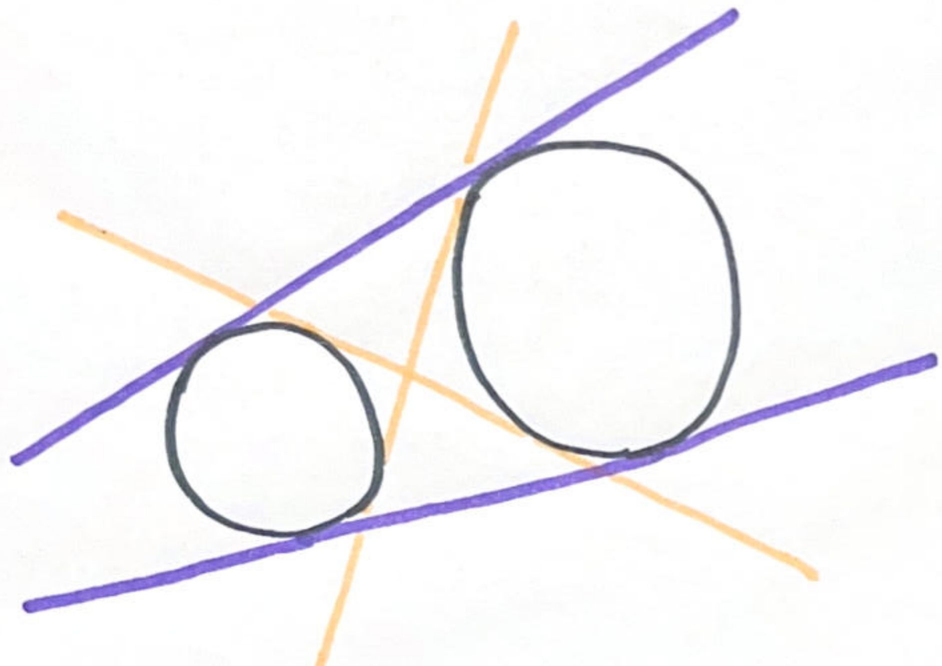
$$A_i \subset \mathrm{Conv}\left(A_{1-i}, \{g_1, \ldots, g_n\} \setminus \{g_j\}\right).$$



<u>Thm</u>. (S., 2025). Suppose

$$\#\left\{\begin{matrix}\text{common supporting} \\ \text{lines of } A_0, A_1\end{matrix}\right\} < n. \text{ Then}$$

the weak carousel rule holds.

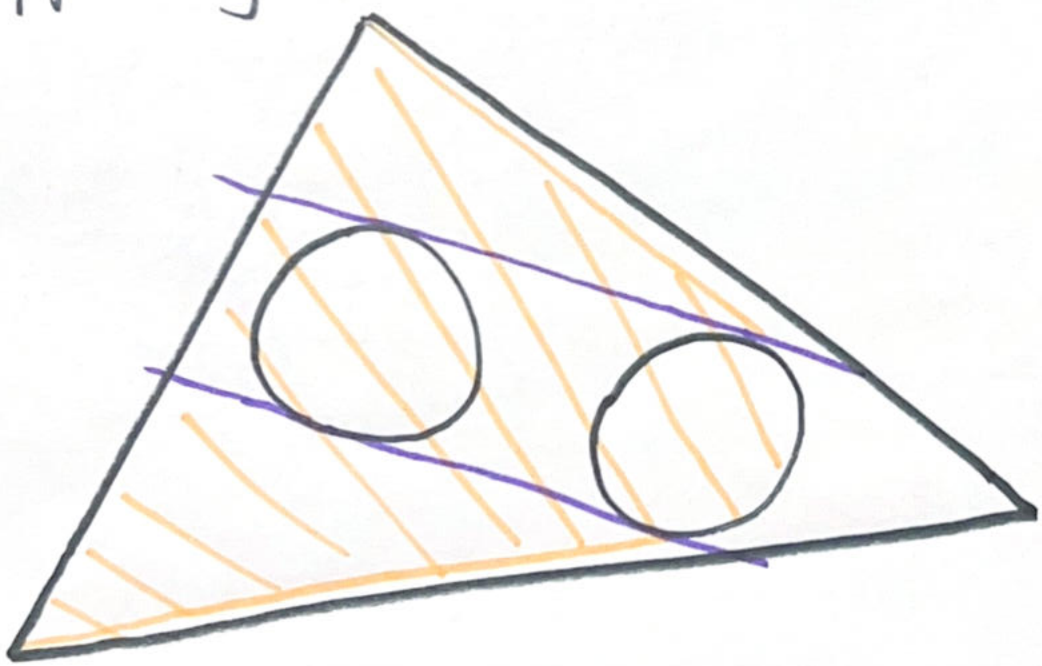


common supporting lines

not

Cor. (Adaricheva – Bolat, 2019). Two disks in a triangle satisfy the weak carousel rule.

Pf. Two Disks have at most two common supporting lines.



Thm. If $A_0, A_1 \subset \mathbb{R}^2$ are compact convex sets contained in a convex n-gon $G = \mathrm{Conv}(g_1, \dots, g_n)$, $\exists\, i \in \{0,1\},\ j \in \{1, \dots, n\}$ such that

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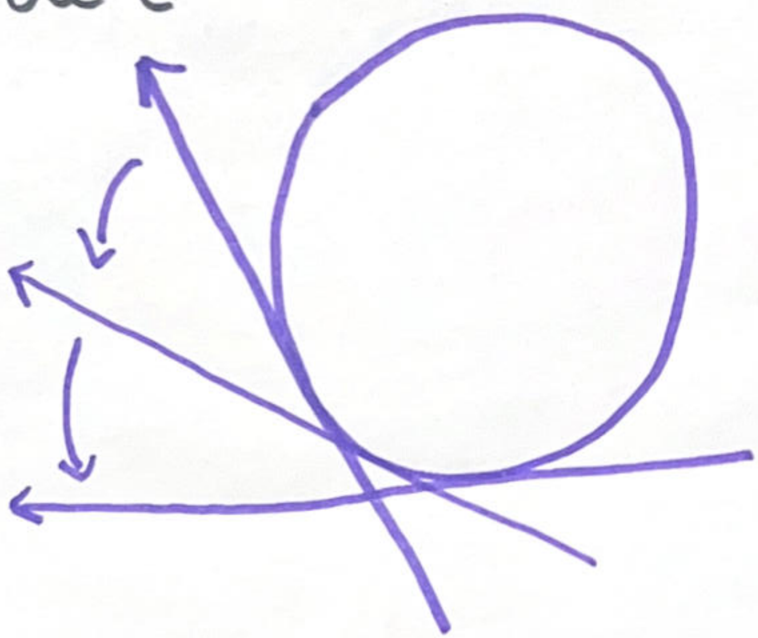
<u>Pf</u> idea.



"slide-turning"

(Czédli – Stachó, 2016)

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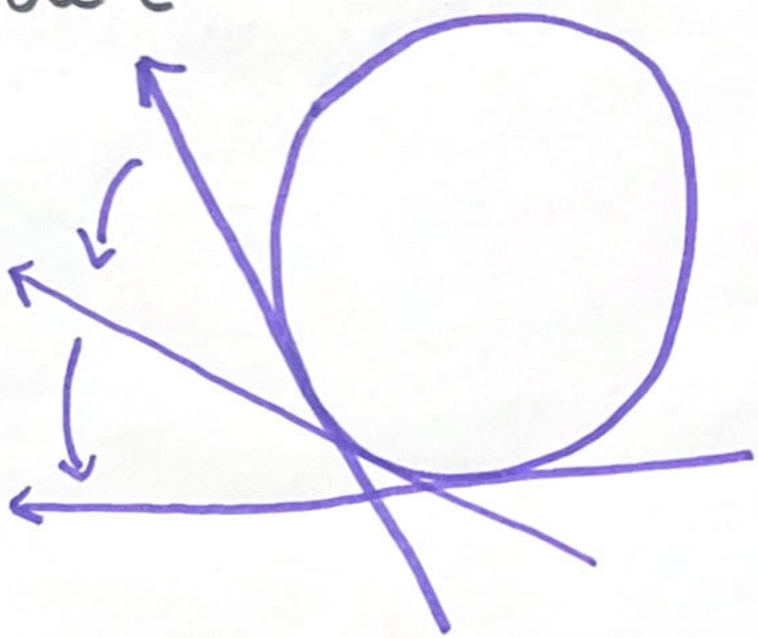

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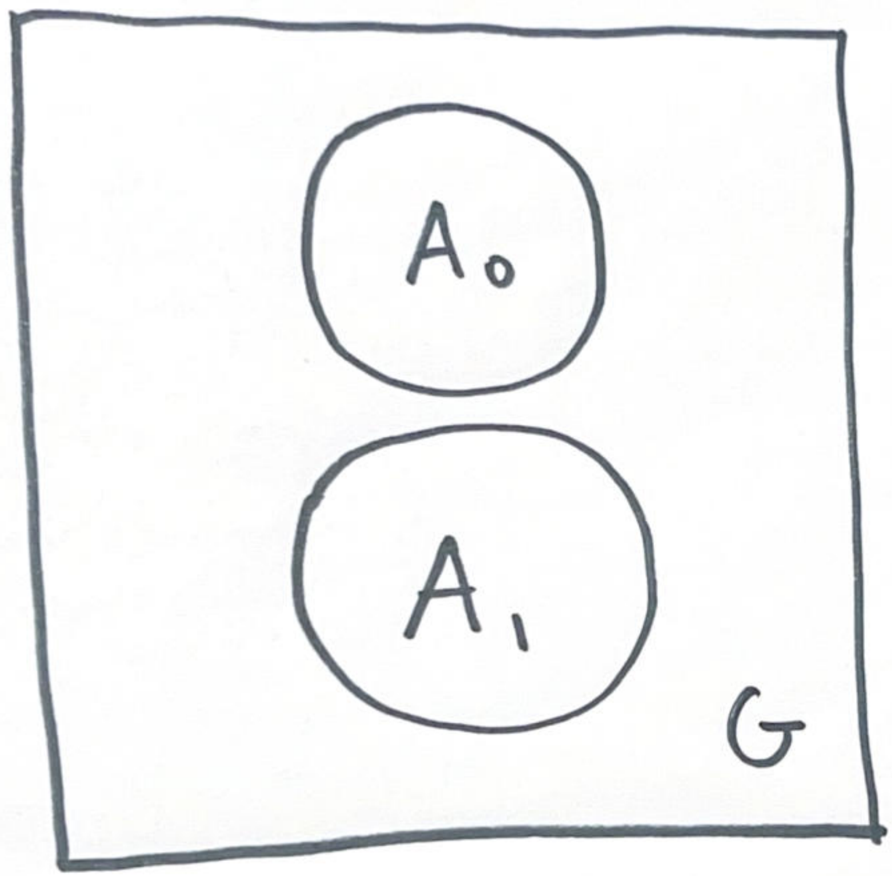
Slide-turning



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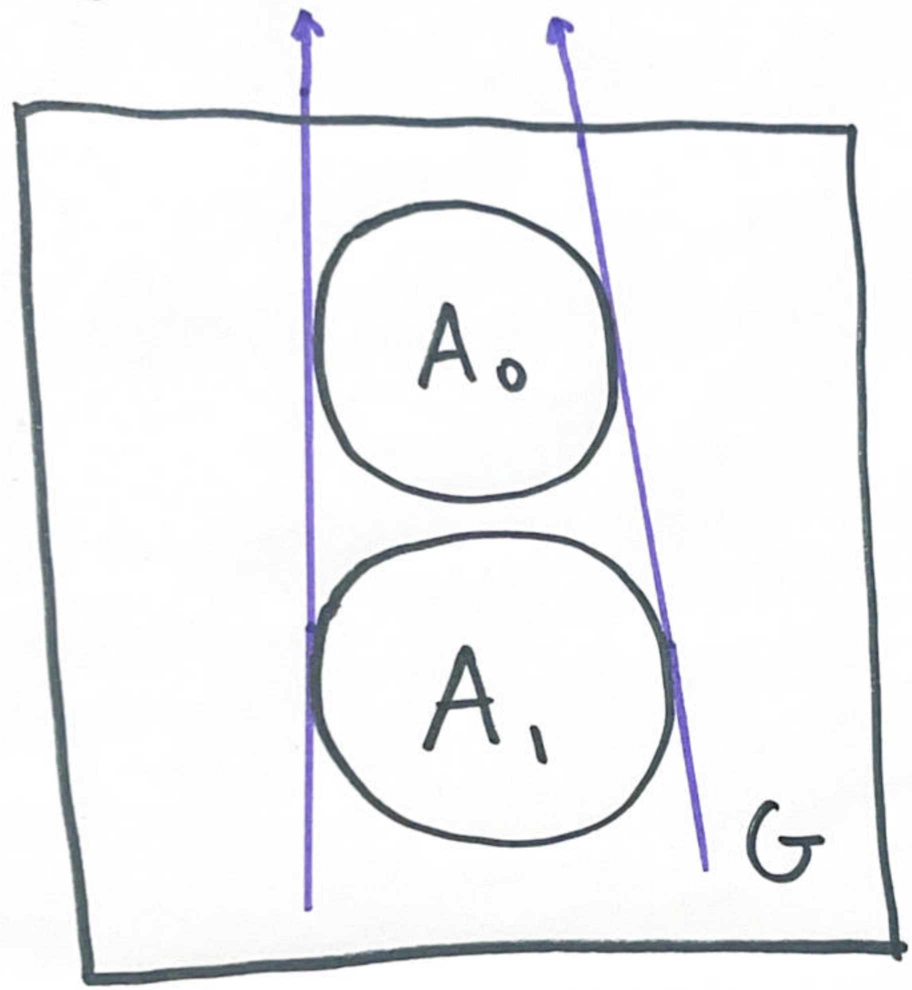

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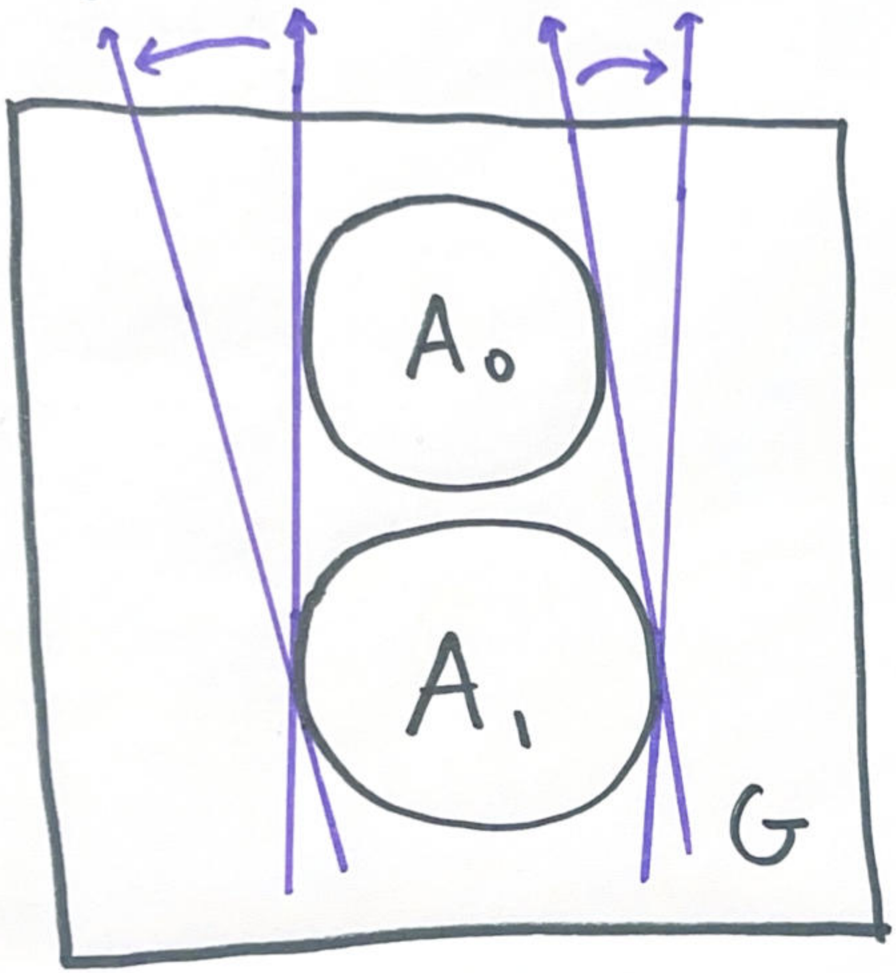

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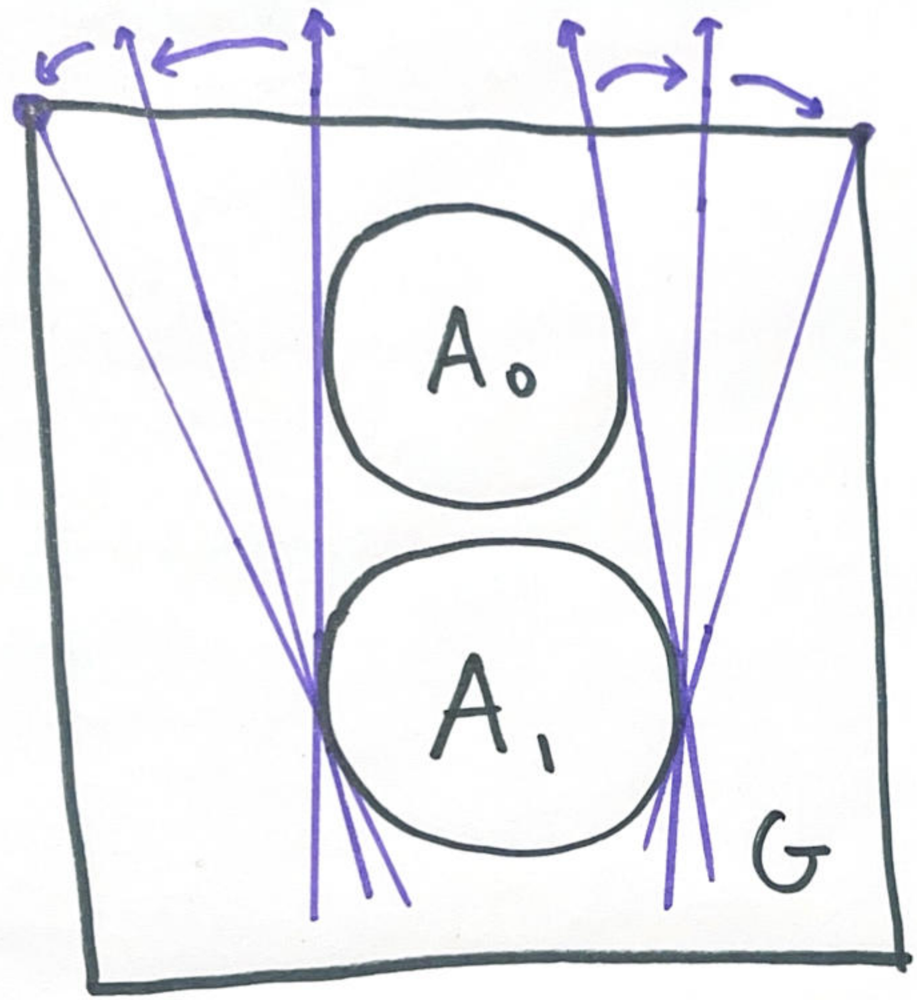

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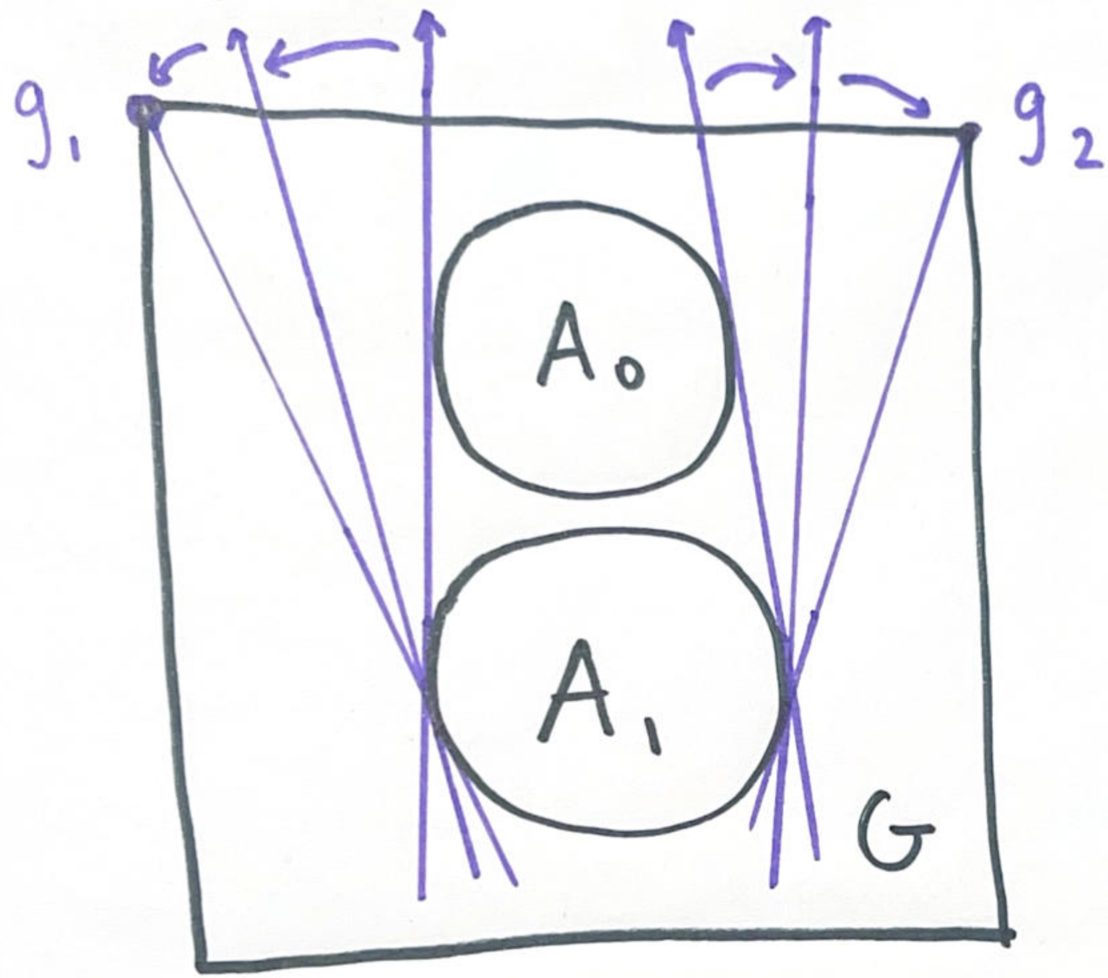
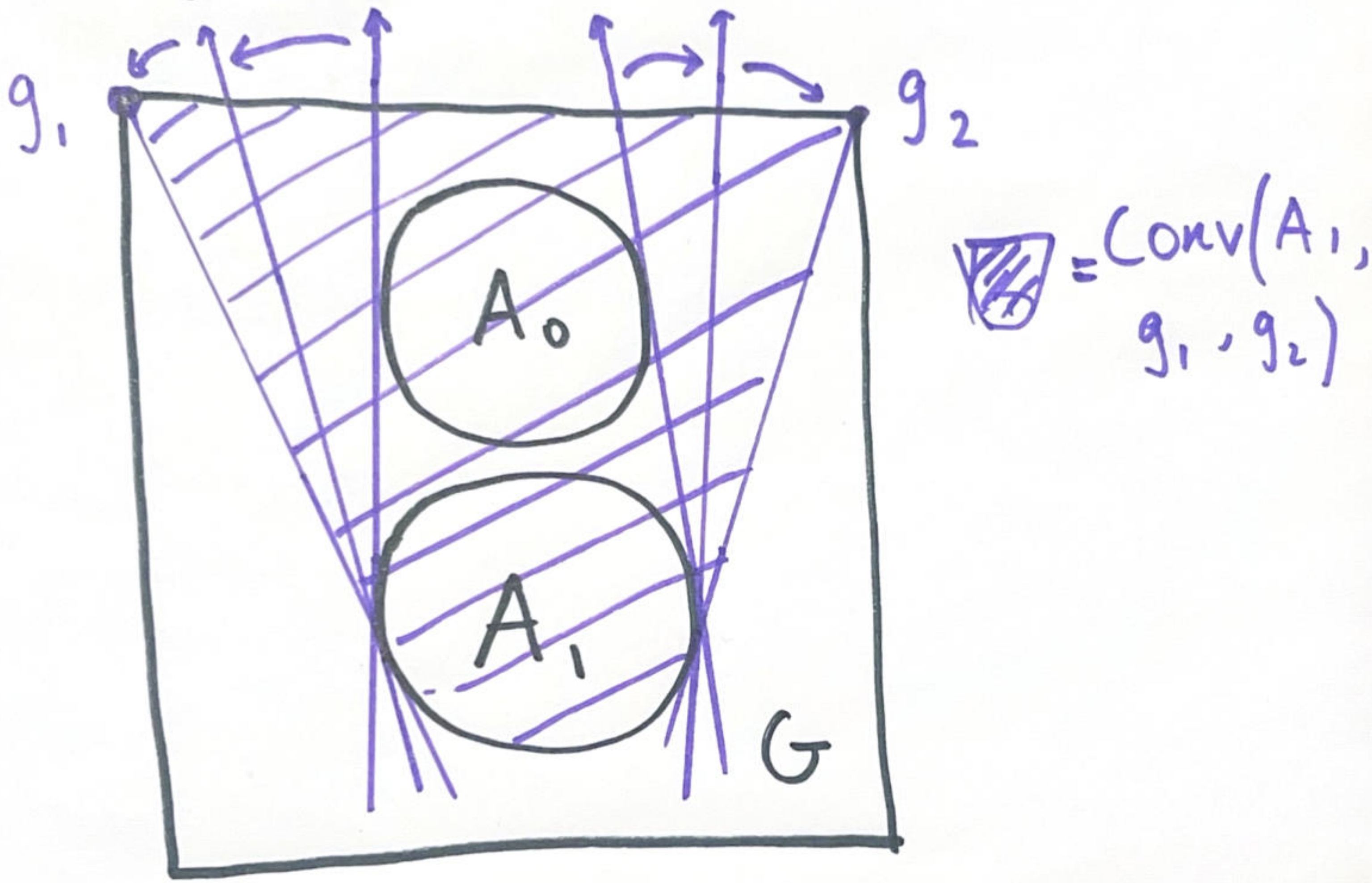


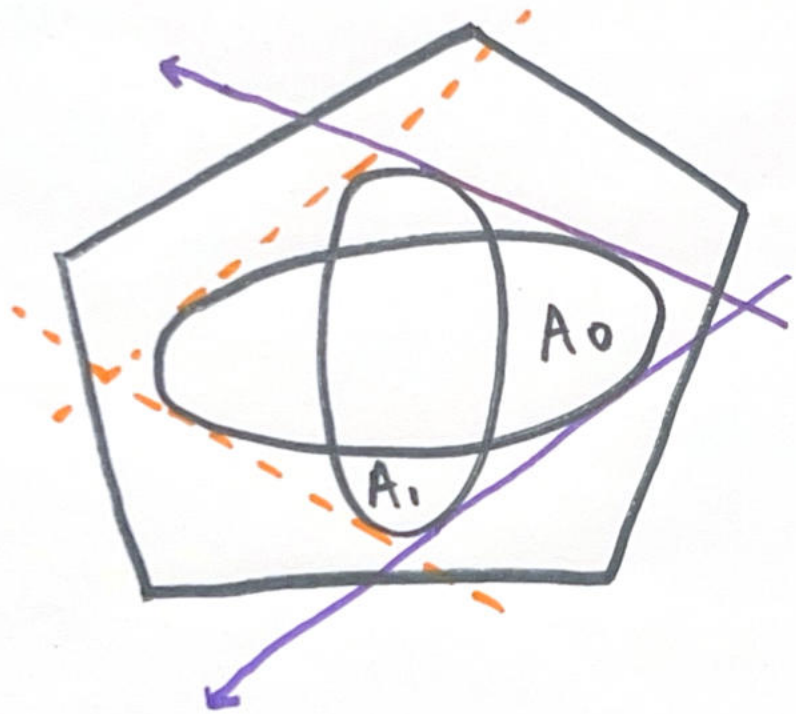
If endpoints of supporting lines intersect vertices of G, we are in luck!

Slide-turning



If endpoints of supporting lines intersect vertices of G, we are in luck!

New goal : Find an <u>adjacent</u> pair of supporting lines that each sweep to a vertex of G.



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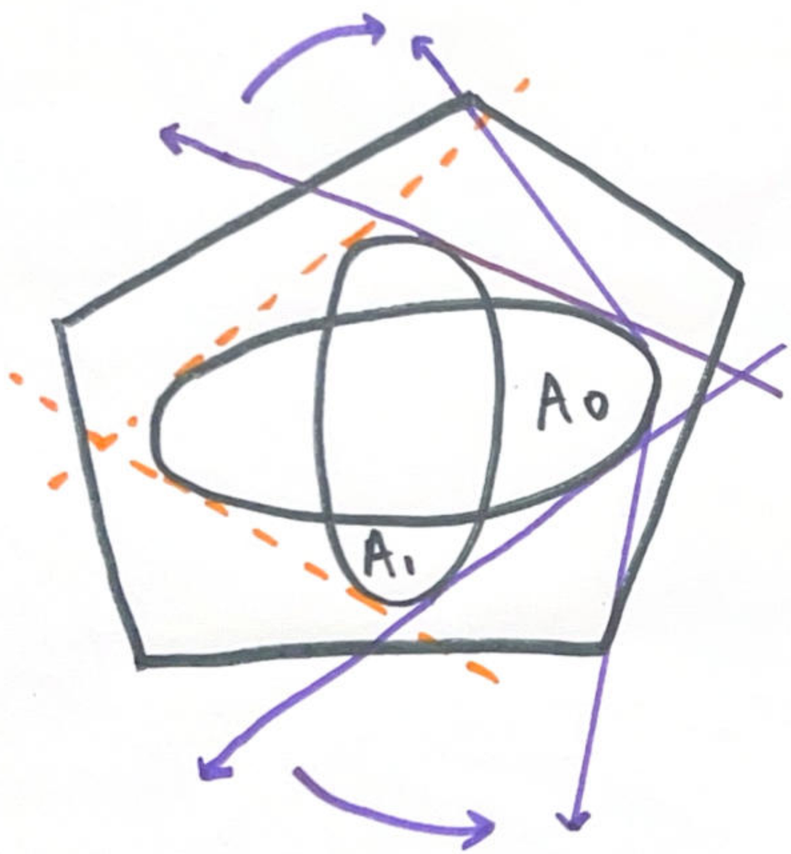

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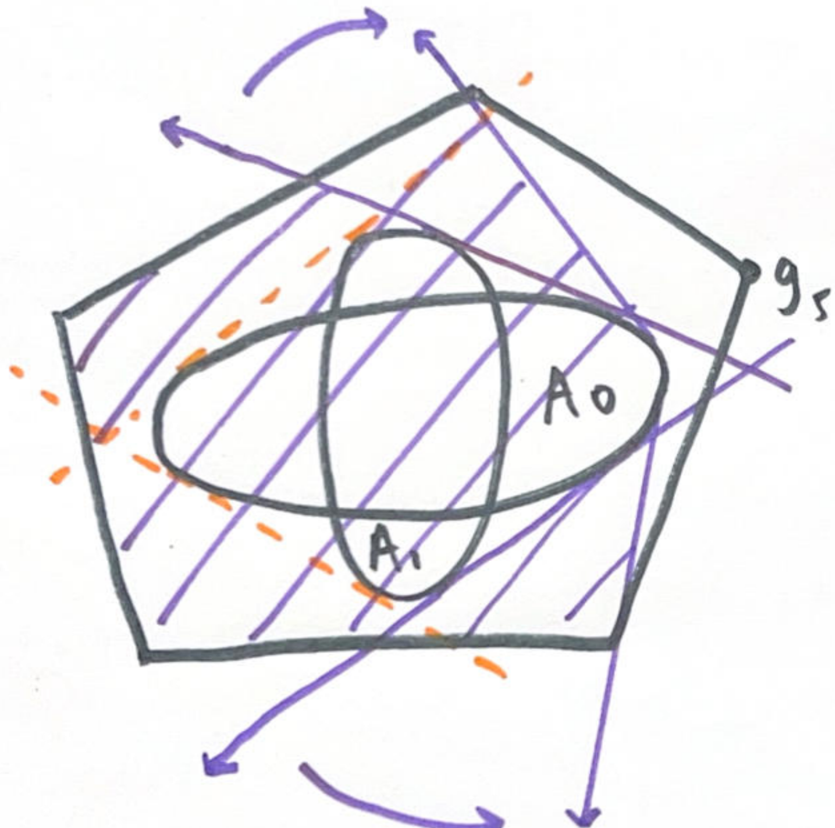

/// : $\mathrm{Conv}\left(A_0, \{g_1, \ldots, g_5\} \setminus \{g_5\}\right)$

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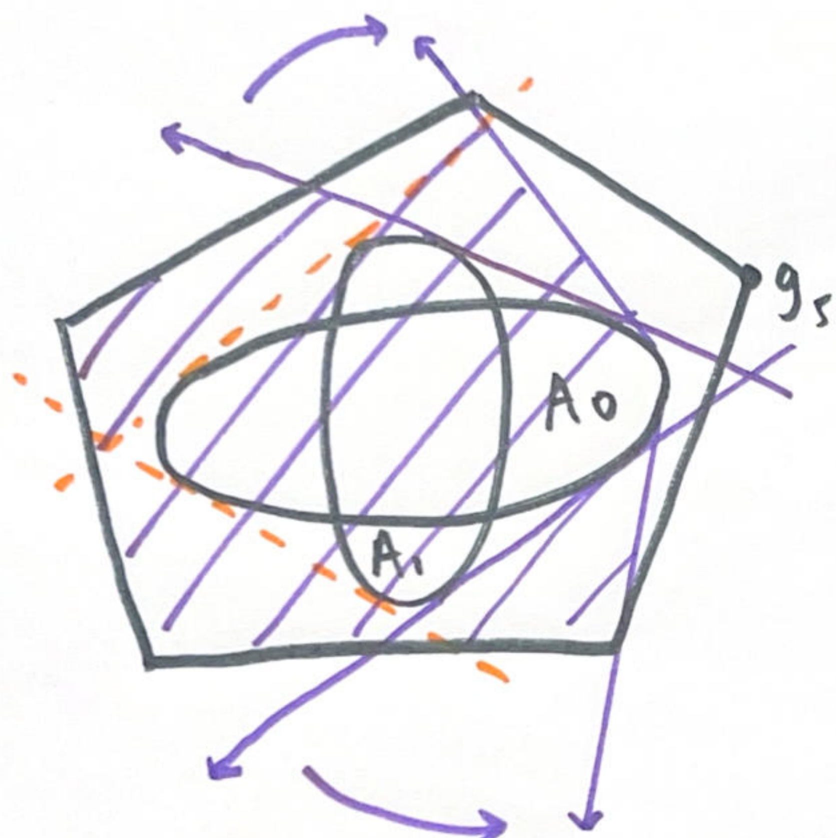

$/// : \mathrm{Conv}\left(A_0, \{g_1, \dots, g_5\} \setminus \{g_5\}\right)$

If $\#\left\{\begin{matrix}\text{common supporting} \\ \text{lines}\end{matrix}\right\} < \#\{\text{vertices of } G\}$, these always exist.

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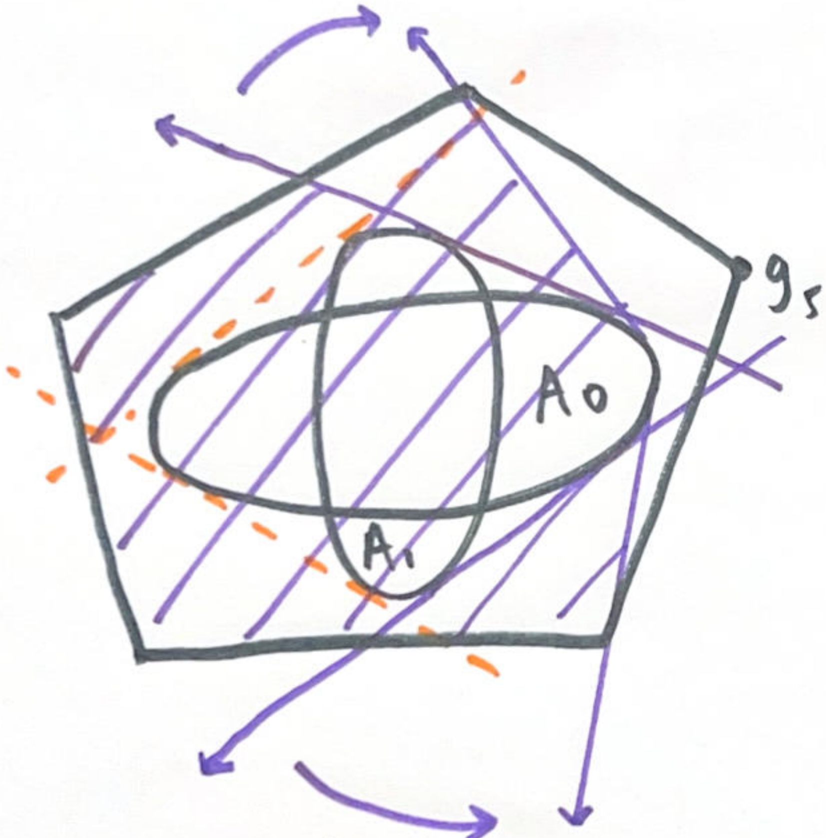

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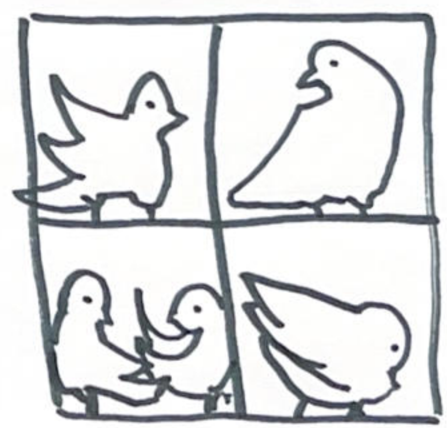

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<u>Cor</u>. $\mathcal{A} = \{A_0, A_1\}$ compact subsets of $\mathbb{R}^2$.

If ∂A_0 and ∂A_1 are smooth plane curves of degree d_1, d_2, and G is a convex n-gon with

$$n > d_1(d_1 - 1)(d_2 - 1)d_2,$$

then $(\mathcal{A}, G)$ satisfy the weak carousel rule.

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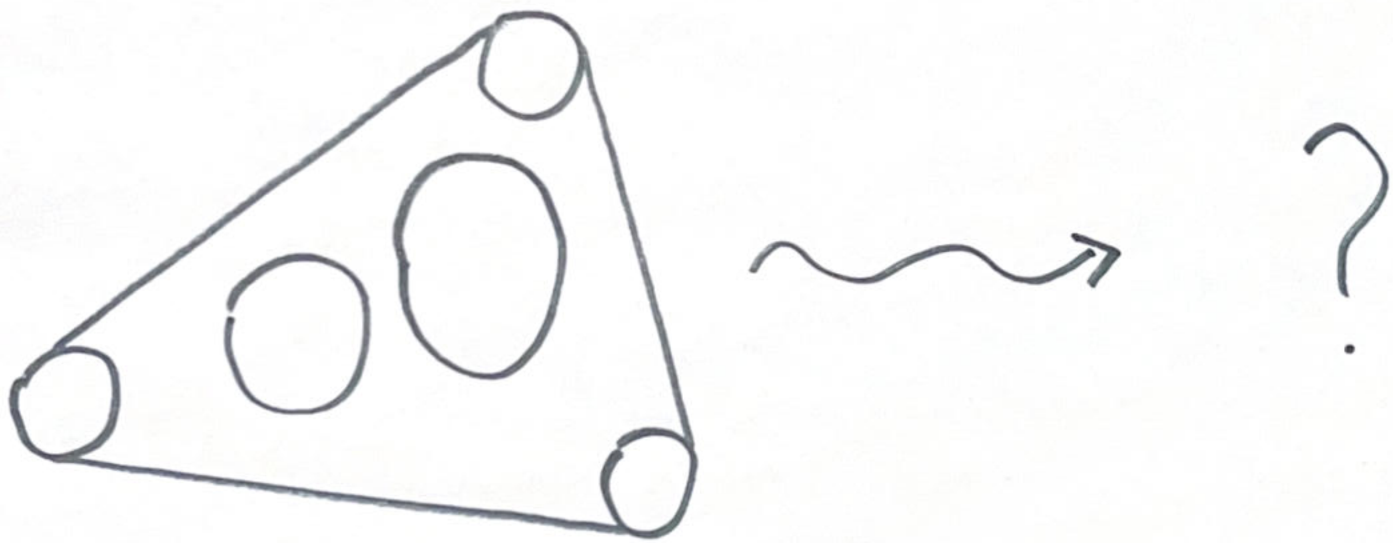
then $(\mathcal{A}, G)$ satisfy the weak carousel rule.

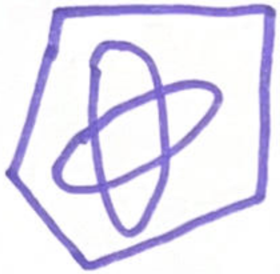
Cor. (Two disks, triangle)

(Two ellipses, pentagon)

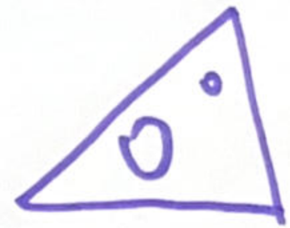
Returning to convex geometries.

Instead of $G = \mathrm{Conv}(\text{points})$, want a theorem for $G = \mathrm{Conv}\left(\begin{array}{c}\text{other convex} \\ \text{compact shapes}\end{array}\right)$





Thank you!



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