A CAROUSEL PROPERTY FOR COMPACT CONVEX SETS

YIMING SONG

Abstract. We prove that if A_0 and A_1 are contained in a convex n-gon G with vertices $g_1, \dots, g_n$, and n is strictly greater than the number of common supporting lines of A_0 and A_1, then there exist $i \in \{0, 1\}$ and $j \in \{0, \dots, n\}$ such that A_i is in the convex hull of A_{1-i} and $(\{g_0, \dots, g_n\} \setminus \{g_j\})$. This recovers and generalizes previous results of Adaricheva–Bolat, and Czédli. We also show that this bound is sharp for even n.

Contents

1. Introduction

1.1. **Background.** In 2019, K. Adaricheva and M. Bolat proved the following result:

Theorem (Theorem 3.1, [AB19])**.** *If A_0 and A_1 are closed disks in $\mathbb{R}^2$ and G is a triangle with vertices g_0, g_1, g_2 such that $A_0, A_1 \subset G$, then there exist $i \in \{0, 1\}$ and $j \in \{0, 1, 2\}$ such that*

$$A_i \subset \mathrm{Conv}(A_{1-i}, \{g_0, g_1, g_2\} \setminus \{g_j\}).$$

where $\mathrm{Conv} : \mathcal{P}(\mathbb{R}^2) \to \mathcal{P}(\mathbb{R}^2)$ denotes the *convex hull* operator, mapping a subset of the plane to the smallest convex set containing it. Their result has implications for the representation theory of convex geometries. A *convex geometry* (not to be confused with the field of study) is a combinatorial set system abstracting the notion of convexity found in point arrangements, posets, and more. For a formal exposition, see Chapter 5 of [AN16] or [EJ85]. An important result is that convex geometries are equivalent to meet-distributive lattices [AGT03].

A major area of research concerns lattice structure-preserving maps of convex geometries into Euclidean space. An *embedding* result asserts that a convex geometry admits a structure-preserving injection into the lattice of convex subsets of a Euclidean point configuration. A seminal result of Kashiwabara–Nakamura–Okamoto [KNO05] states that every finite convex geometry embeds into the lattice of convex subsets of a suitable point configuration in $\mathbb{R}^d$ for sufficiently large d. In [Czé14], G. Czédli shows that using disks

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Department of Mathematics, Columbia University. Email: ys3635@columbia.edu.

rather than points in $\mathbb{R}^2$ yields more flexibility, allowing the embedding of a larger class of convex geometries as arrangements of sets in Euclidean space.

Slightly stronger, a *representation* result is an isomorphism between a convex geometry and a lattice of convex subsets of a Euclidean point configuration. Richter and Rogers show in [RR17] that any convex geometry of convex dimension n (see [EJ85]) can be represented as convex n-gons in $\mathbb{R}^2$.

A natural question to ask is whether such representation results can be strengthened if one uses arbitrary convex shapes rather than points, n-gons, or disks. The result of Adaricheva–Bolat provides a geometric reason for the failure of disks in $\mathbb{R}^2$ to represent convex geometries. Understanding analogous geometric results for other families of convex compact shapes would provide a wealth of (counter)examples in this aspect.

1.2. **Results.** In this paper, we generalize the theorem of Adaricheva–Bolat to arbitrary convex compact shapes. Given a pair $\mathcal{A} = \{A_0, A_1\}$ of convex compact subsets of the plane, and a convex n-gon G containing the elements of $\mathcal{A}$, we say $(\mathcal{A}, G)$ satisfy the *weak carousel rule* if there exist $i \in \{0, 1\}$ and $j \in \{1, \ldots, n\}$ such that

$$A_i \subset \operatorname{Conv}(A_{1-i}, \operatorname{vert}(G) \setminus \{g_j\}).$$

where $\operatorname{vert}(G)$ denotes the vertices of G. In this language, their theorem asserts that if G is a triangle containing both disks in $\mathcal{A} = \{A_0, A_1\}$, then $(\mathcal{A}, G)$ satisfy the weak carousel rule.

For G a triangle, and $\mathcal{A}$ consisting of arbitrary convex compact A_0, A_1, it is not difficult to construct examples where the weak carousel rule fails for $(\mathcal{A}, G)$. Our main result is that the weak carousel rule holds for arbitrary convex compact A_0, A_1 if $n = \#\operatorname{vert}(G)$ is sufficiently large as a function of A_0 and A_1. Precisely, let s be the number of *common supporting lines* of A_0 and A_1 (common tangent lines that contain both A_0 and A_1 on one side). Then our main theorem is the following:

Theorem 1.1. *If $s < n$, then the weak carousel rule holds for* $(\mathcal{A}, G)$.

This recovers and generalizes the result of Adaricheva–Bolat. We also construct examples to show that this bound is sharp for even n. Our proof is geometric in nature, relying at times on classical concepts from convex analysis. One step owes heavily to Czédli–Stachó's notion of *slide-turning*, detailed in [CS16].

1.3. **Organization.** We collect the necessary background on supporting lines, support functions, sectors, and slide-turning in Section 2. In Section 3, we prove the main theorem, exhibit corollaries, prove sharpness of bound, and conclude with a few questions.

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2. Preliminaries

The strategy behind the proof will be to observe the tangent lines to A_0 and A_1, and manipulate them around via a process called "slide-turning," all of which will be made formal in this section.

2.1. **Common supporting lines.** Let $A \subset \mathbb{R}^n$ be a convex set, and let l be a hyperplane in $\mathbb{R}^n$. We say that l *supports* A, or is a *supporting hyperplane* of A, if $l \cap A$ is nonempty and A lies in a closed halfspace bounded by l. Given a collection $\mathcal{A}$ of convex sets in $\mathbb{R}^n$, we say that l is a *common supporting hyperplane* of $\mathcal{A}$ if l supports all A_i in the same halfspace. A classical theorem in convex analysis states:

Theorem 2.1 ([Roc70])**.** *Let $A \subseteq \mathbb{R}^n$ be a closed, convex set. For any $p \in \partial A$, there exists a supporting hyperplane of A containing p.* □

For the remainder of this paper, we will work with $n = 2$, so (common) supporting hyperplanes are just (common) supporting lines. By a slight abuse of notation, we will use $\mathrm{csl}(A)$ or $\mathrm{csl}(\mathcal{A})$ to denote the set of common supporting lines of either a single convex set $A \subset \mathbb{R}^2$, or a collection of convex sets $\mathcal{A}$.

For convex $A \subset \mathbb{R}^2$ it is useful to consider its *support function* $h_A : \mathbb{R}^2 \to \mathbb{R}$, given by

$$h_A(v) = \sup_{a \in A}(v \cdot a)$$

where $\cdot$ denotes the standard inner product. If A is compact and nonempty, then the support function h_A is continuous [Sch13]. Importantly, the support function gives a helpful characterization of supporting lines, and is often how the supporting lines are defined:

Lemma 2.2 ([Tro22], §2.5)**.** *For a fixed unit vector $\eta \in \mathbb{R}^2$, the line*

$$l_\eta := \{x \in \mathbb{R}^2 : x \cdot \eta = h_A(\eta)\}$$

is in $\mathrm{csl}(A)$*, and A is contained in the closed halfplane*

$$\{x \in \mathbb{R}^2 : x \cdot \eta \leq h_A(\eta)\}.$$

Corollary 2.3. *Let $\mathcal{A}$ be a collection of convex sets in the plane. For a fixed unit vector $\eta \in \mathbb{R}^2$ and constant $c \in \mathbb{R}$, the line $\{x \in \mathbb{R}^2 : \eta \cdot x = c\}$ is in* $\mathrm{csl}(\mathcal{A})$ *iff the values of $h_A(\eta)$ are equal for all $A \in \mathcal{A}$.* □

Now let $\mathcal{A} = \{A_i\}_{i \in I}$ be a collection of convex subsets of $\mathbb{R}^2$. For $l \in \mathrm{csl}(\mathcal{A})$, there is a choice of sign for its unit direction vector. We will use v_l^L to denote the direction vector of l such that $\cup\mathcal{A}$ lies on the *left* of the directed line (l, v_l^L). Let $v_l^R := -v_l^L$. For simplicity, we will write l^L and l^R for the directed lines (l, v_l^L) and (l, v_l^R). For the sets of directed supporting lines we will write $\mathrm{csl}^L(\mathcal{A}) := \{l^L : l \in \mathrm{csl}(\mathcal{A})\}$ and $\mathrm{csl}^R(\mathcal{A}) := \{l^R : l \in \mathrm{csl}(\mathcal{A})\}$.

2.2. **Adjacency and sectors.** We now define an order on the common supporting lines, which will turn out to be very important. For the remainder of this section, let $\mathcal{A}$ be a collection of compact convex subsets of $\mathbb{R}^2$. Following [CK19], identifying S^1 with $[0, 2\pi]/(0 \sim 2\pi)$, define

$$\mathrm{dir}^L : \mathrm{csl}(\mathcal{A}) \to S^1, \quad \mathrm{dir}^R : \mathrm{csl}(\mathcal{A}) \to S^1$$

via $l \mapsto v_l^L$ and $l \mapsto v_l^R$ respectively. Now define

$$\mathrm{nor}^L(l) := \mathrm{dir}^L(l) + \pi/2, \quad \mathrm{nor}^R(l) := \mathrm{dir}^R(l) - \pi/2$$

but notice that $\mathrm{nor}^L(l) = \mathrm{nor}^R(l)$, since they are both the outward pointing normal vectors away from the subset that l supports, so we will just write $\mathrm{nor}(l)$.

Lemma 2.4. *For compact convex $A \subset \mathbb{R}^2$, $\mathrm{nor} : \mathrm{csl}(A) \to S^1$ is a bijection.*

Proof. For injectivity, if two distinct supporting lines of A had the same normal vector, then by definition of supporting line, A would have to lie in two disjoint subsets of the plane, a contradiction. Surjectivity is given by Lemma 2.2. $\square$

This lemma implies that we have a total cyclic order on $\mathrm{csl}(\mathcal{A})$ if it is finite. This is because $\mathrm{csl}(\mathcal{A})$ is a subset of $\mathrm{csl}(\mathrm{Conv}(\mathcal{A}))$, and $\mathrm{Conv}(\mathcal{A})$ is convex and compact. Then, $\mathrm{csl}(\mathcal{A})$ inherits a total cyclic order as a subset of $\mathrm{csl}(\mathrm{Conv}(\mathcal{A}))$. We say that $l_a, l_b \in \mathrm{csl}(\mathcal{A})$ are *adjacent* if $\mathrm{nor}(l_a)$ and $\mathrm{nor}(l_b)$ are consecutive in this order. Then, given s common supporting lines we also have s pairs of adjacent lines. See Figure 1 for an illustration. From here we will assume that $\mathrm{csl}(\mathcal{A})$ is indexed accordingly, i.e. l_i and l_{i+1} are adjacent, where indices are understood modulo s.

We now seek to define a sector between two supporting lines, similar to but more general than the definition of a comet in [CK19]. Let A be convex compact, and let l_1, l_2 be distinct elements of $\mathrm{csl}(A)$. In S^1, we have two closed arcs whose endpoints are $\mathrm{nor}(l_1)$ and $\mathrm{nor}(l_2)$. Call them $\Lambda^+(l_1, l_2)$ and $\Lambda^-(l_1, l_2)$. For $\eta \in S^1$, let $H_A(\eta)$ be the closed halfplane

$$H_A(\eta) = \{x \in \mathbb{R}^2 : x \cdot \eta \leq h_A(\eta)\}$$

Finally, define the *sectors of* l_1, l_2 *by* A via:

$$\mathrm{Sect}^{\pm}(l_1, l_2; A) = \bigcap_{\eta \in \Lambda^{\pm}(l_1, l_2)} H_A(\eta).$$

Because compact convex subsets are the intersection of all halfplanes containing them, it is clear that $A \subset \mathrm{Sect}^{\pm}(l_1, l_2; A)$.

We can fix a choice of sign for $\Lambda^{\pm}(l_1, l_2; A)$ as follows. Orient S^1 clockwise. We fix $\Lambda^+(l_1, l_2)$ to be the arc obtained by proceeding from $\mathrm{nor}(l_1)$ to $\mathrm{nor}(l_2)$ in this clockwise direction, and $\Lambda^-(l_1, l_2)$ to be the complementary arc. So the order in which we write l_1, l_2 matters.

Proposition 2.5. *Let* $\mathcal{A} = \{A_0, A_1\}$. *If* $l_1, l_2 \in \mathrm{csl}(\mathcal{A})$ *are adjacent, then at least one of the following inclusions holds:*

(1) $A_0 \subset \mathrm{Sect}^+(l_1, l_2; A_1)$,
(2) $A_1 \subset \mathrm{Sect}^+(l_1, l_2; A_0)$.

Proof. Consider the continuous function $\delta : S^1 \to \mathbb{R}$ given by

$$\theta \mapsto h_{A_0}(\theta) - h_{A_1}(\theta).$$

Since $l_1, l_2 \in \mathrm{csl}(\mathcal{A})$, we have by Corollary 2.3 that $\delta(\mathrm{nor}(l_1)) = \delta(\mathrm{nor}(l_2)) = 0$.

Suppose for contradiction that none of the inclusions hold. Then, the failure of (1) implies that a point of A_0 must lie outside $\mathrm{Sect}^+(l_1, l_2; A_1)$. By convexity and compactness, the subset $A_0 \setminus \mathrm{Sect}^+(l_1, l_2; A_1)$ must contain a point on the boundary of A_0, say a. By Theorem 2.1, there exists a supporting line l_a of A_0, passing through a. Moreover, we have $\mathrm{nor}(l_a) \in \Lambda^+(l_1, l_2)$. Because l_a separates a from $\mathrm{Sect}^+(l_1, l_2; A_1)$, which contains A_1, we have that for all $b \in A_1$ that

$$h_{A_0}(\mathrm{nor}(l_a)) = \mathrm{nor}(l_a) \cdot a > \mathrm{nor}(l_a) \cdot b$$

so $h_{A_0}(\mathrm{nor}(l_a)) > \sup_{b \in A_1} \mathrm{nor}(l_a) \cdot b = h_{A_1}(\mathrm{nor}(l_a))$. Therefore, $\delta(\mathrm{nor}(l_a)) > 0$.

By the same argument, (2) implies that there exists some $l_b \in \mathrm{csl}(A_1)$ such that $\mathrm{nor}(l_b) \in \Lambda^+(l_1, l_2)$ and $\delta(\mathrm{nor}(l_b)) < 0$. By the intermediate value theorem, there is some $\eta \in \Lambda^+(l_1, l_2) \setminus \{\mathrm{nor}(l_1), \mathrm{nor}(l_2)\}$ such that $\delta(\eta) = 0$, which yields another common supporting line of A_0 and A_1, whose normal lies strictly between $\mathrm{nor}(l_1)$ and $\mathrm{nor}(l_2)$ in the clockwise order, contradicting the adjacency of l_1 and l_2. $\square$

2.3. **Slide-turning and expansion.** In [CS16], the authors formalize a notion of "slide-turning" support lines. Intuitively, one can slide support lines around the boundary of a convex shape in a continuous manner. A *pointed supporting line* of A is a pair (a, l) where $a \in \partial A$ and $l \in \operatorname{csl}(A)$ passes through a. They define the set

$$\operatorname{Sli}(A) := \{(a, \operatorname{dir}^L(l)) : (a, l) \text{ is a pointed supporting line of } A\} \subset \mathbb{R}^2 \times S^1 \subset \mathbb{R}^4$$

and prove the following theorem about Sli:

Theorem 2.6. [CS16] *If A is nonempty, compact and convex, then $\operatorname{Sli}(A)$ is a rectifiable, simple, closed curve in $\mathbb{R}^4$.* □

In particular, this implies that we can parametrize $\operatorname{Sli}(A)$ and consequently $\operatorname{csl}(A)$. But there is no guarantee that this parametrization is nice, which we want for our purposes. Instead, we will use the inverse of the bijection $\operatorname{nor} : \operatorname{csl}(A) \to S^1$:

$$P_A(t) := \operatorname{nor}^{-1}(t)$$

where we identify S^1 with $[0, 2\pi]$ modulo endpoints. With this parametrization, increasing t corresponds to rotating the supporting line clockwise along A, and vice versa. Then, slide-turning l_1 along A, clockwise by an angle of α, or counterclockwise by an angle of β corresponds to the maps

$$t \mapsto P_A(\operatorname{nor}(l_1) + \alpha t), \quad t \mapsto P_A(\operatorname{nor}(l_1) - \beta t)$$

for $t \in [0, 1]$. For ease of notation, let us write

$$l_{i,\alpha;A} := P_A(\operatorname{nor}(l_i) + \alpha).$$

Then the following lemma follows directly from definition:

Lemma 2.7. *If $\alpha + \beta \le \operatorname{nor}(l_2) - \operatorname{nor}(l_1) \pmod{2\pi}$, then*

$$\operatorname{Sect}^+(l_1, l_2; A) \subset \operatorname{Sect}^+(l_{1,\alpha;A}, l_{2,-\beta;A}; A).$$

and

$$\operatorname{Sect}^-(l_1, l_2; A) \subset \operatorname{Sect}^-(l_{1,-\alpha;A}, l_{2,\beta;A}; A).$$

Proof. Unraveling the definitions, since $\alpha + \beta \le \operatorname{nor}(l_2) - \operatorname{nor}(l_1) \pmod{2\pi}$, the arcs Λ satisfy

$$\Lambda^\pm(l_{1,\pm\alpha;A}, l_{2,\mp\beta;A}) \subset \Lambda^\pm(l_1, l_2)$$

which immediately implies the desired relations. □

For this reason, we will call $\operatorname{Sect}^+(l_{1,\alpha;A}, l_{2,-\beta;A})$ an *expanded* sector of $\operatorname{Sect}^+(l_1, l_2; A)$. See Figure 2 for a diagram of a sector and its expansion.

3. The weak carousel rule

Throughout this section, let $\mathcal{A} = \{A_0, A_1\}$ be convex, compact, and let G be a convex n-gon containing $\mathcal{A}$. Label the vertices of G clockwise as $g_1, \dots, g_n$, and the edges clockwise $e_1, .., e_n$, where e_i is the edge joining g_i and g_{i+1}. Let s denote the number of common supporting lines of $\mathcal{A}$. Fix one common supporting line as l_1, and in the clockwise cyclic order on $\operatorname{nor}(\operatorname{csl}(\mathcal{A}))$, label each consecutive line $l_2, l_3, \dots, l_s$. Let us restate our main theorem:

Theorem 1.1. *If $s < n$, then the weak carousel rule holds for $(\mathcal{A}, G)$.*

The proof of Theorem 1.1 will consist in showing that out of the s adjacent pairs in $\operatorname{csl} \mathcal{A}$, there exists some adjacent pair of lines (l_i, l_{i+1}) whose sector $\operatorname{Sect}^+(l_i, l_{i+1})$ can be expanded so that it is equal to the convex hull in the definition of the weak carousel rule.

3.1. **Setting up the proof, endpoints.** We first need to consider the points where the supporting lines making up a sector intersect the boundary of G. Define the functions $\Omega^L : \mathrm{csl}(A_i) \to \partial G$ and $\Omega^R : \mathrm{csl}(A_i) \to \partial G$ as follows. For $l \in \mathrm{csl}(A_i)$, fix a point $a \in l \cap A_i$, and let $t^* \in \mathbb{R}$ be the maximum $t \in \mathbb{R}$ such that $a + t\,\mathrm{dir}^L(l)$ is in G. Then set $\Omega^L(l) := a + t^*\,\mathrm{dir}^L(l)$. Similarly, define $\Omega^R(l) := a + t^\dagger\,\mathrm{dir}^R(l)$ where $t^\dagger$ is the maximum $t \in \mathbb{R}$ such that $a + t\,\mathrm{dir}^R(l)$ is in G. These are well-defined maps since G is compact.

Now we state a definition that is geometrically intuitive but technical to state. For an adjacent pair $l_1, l_2 \in \mathrm{csl}(\mathcal{A})$, we have by Proposition 2.5 that

$$\begin{cases} A_0 \subset \mathrm{Sect}^+(l_1, l_2; A_1), \text{ or} \\ A_1 \subset \mathrm{Sect}^+(l_1, l_2; A_0). \end{cases}$$

Without loss of generality, suppose the first inclusion holds. If there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha + \beta \leq \mathrm{nor}(l_2) - \mathrm{nor}(l_1) \pmod{2\pi}, \text{ and } \Omega^L(l_{1,\alpha;A_1}), \Omega^R(l_{2,-\beta;A_1}) \in \mathrm{vert}(G),$$

say $\Omega^L(l_{1,\alpha;A_1}) = g_a$ and $\Omega^R(l_{2,-\beta;A_1}) = g_b$, we define the (*vertices between $l_{1,\alpha;A_1}$ and $l_{2,-\beta;A_1}$*) by

$$\mathrm{vert}_G(l_{1,\alpha;A_1}, l_{2,-\beta;A_1}) := \begin{cases} \{g_b, g_{b+1}, \ldots, g_a\} & \text{if the segment of } \partial G \text{ moving clockwise} \\ & \text{from } g_b \text{ to } g_a \text{ lies in } \mathrm{Sect}^+(l_{1,\alpha;A_1}, l_{2,-\beta;A_1}; A_1) \\ \{g_a, g_{a+1}, \ldots, g_b\} & \text{otherwise.} \end{cases}$$

This definition is perhaps best served by a diagram. See Figure 3.

Proposition 3.1. $G \cap \mathrm{Sect}^+(l_{1,\alpha;A_1}, l_{2,-\beta;A_1}; A_1) \subset \mathrm{Conv}(A_1, \mathrm{vert}_G(l_{1,\alpha;A_1}, l_{2,-\beta;A_1}))$.

Proof. For ease of notation let us write

$$S := G \cap \mathrm{Sect}^+(l_{1,\alpha;A_1}, l_{2,-\beta;A_1}; A_1), C := \mathrm{Conv}(A_1, \mathrm{vert}_G(l_{1,\alpha;A_1}, l_{2,-\beta;A_1}))$$

Since S is the intersection of halfplanes and a convex subset, it is convex. By the Krein–Milman theorem [KM40], it suffices to show that the boundary points, hence extreme points of S are contained in C. The boundary of S consists of

(1) A segment of $l_{1,\alpha;A_1}$ contained in the segment of $l_{1,\alpha;A_1}$ from a point on ∂G to $\Omega^L(l_{1,\alpha;A_1})$. This is contained in C, since ∂G and $\Omega^L(l_{1,\alpha;A_1})$ are in C.
(2) Likewise for the segment of $l_{2,-\beta;A_1}$.
(3) If nonempty, the segment $S \cap \partial G$, which is contained, by definition, in $\mathrm{Conv}(\mathrm{vert}_G(l_{1,\alpha;A_1}, l_{2,-\beta;A_1})))$.

All of these are clearly contained in C. □

Now we can state the idea from the beginning of this section formally. If we can expand the sector of an adjacent pair so that the endpoints of the lines meet the vertices of G, and the vertices between the lines do not make up all the vertices of G, then we are done:

Proposition 3.2. *If there exists an adjacent pair $l_1, l_2 \in \mathrm{csl}(\mathcal{A})$ satisfying*

(1) There exist $\alpha, \beta \in \mathbb{R}$ such that $\Omega^L(l_{1,\alpha;A_1}), \Omega^R(l_{2,-\beta;A_1}) \in \mathrm{vert}(G)$ and $\alpha + \beta \leq \mathrm{nor}(l_2) - \mathrm{nor}(l_1) \pmod{2\pi}$
(2) $\#\,\mathrm{vert}_G(l_{1,\alpha;A_1}, l_{2,-\beta;A_1}) < \#\,\mathrm{vert}(G)$,

then $(\mathcal{A}, G)$ satisfy the weak carousel rule.

Proof. We have the inclusions

$$\begin{aligned} A_0 &\subset G \cap \mathrm{Sect}^+(l_1, l_2; A_1) && \text{by Proposition 2.5} \\ &\subset G \cap \mathrm{Sect}^+(l_{1,\alpha;A_1}, l_{2,-\beta;A_1}; A_1) && \text{by Lemma 2.7} \\ &\subset \mathrm{Conv}(A_1, \mathrm{vert}_G(l_{1,\alpha;A_1}, l_{2,-\beta;A_1})) && \text{by Proposition 3.1} \end{aligned}$$

By assumption, this last convex hull is always contained in $\mathrm{Conv}(A_1, \mathrm{vert}(G) \setminus \{g_j\})$ for some $j \in \{1, \dots, n\}$, which concludes the proof. □

3.2. **Casework and last steps.** It remains to show that if $s < n$, we can always find an adjacent pair satisfying the assumptions of Proposition 3.2. The important fact to notice is that the segments on ∂G moving clockwise from $\Omega^L(l_i)$ to $\Omega^L(l_{i+1})$ for $i \in \{1, \dots, s\}$ partition ∂G. To make this rigorous, first define the *clockwise angle between l_i and l_{i+1}* to be

$$\Delta_i := \mathrm{nor}(l_{i+1}) - \mathrm{nor}(l_i)$$

Then we define the *left sweep of l_i* and *right sweep of l_{i+1} about A* by

$$\mathrm{sweep}^L(l_i) := \{\Omega^L(l_{i,\alpha;A}) : \alpha \in [0, \Delta_i]\},$$

$$\mathrm{sweep}^R(l_{i+1}) := \{\Omega^R(l_{i+1,-\beta;A}) : \beta \in [0, \Delta_i]\}$$

where $A = \mathrm{Conv}(A_0, A_1)$. The reason we take A here is that after completing one sweep, moving to the next common supporting line, the new line and its corresponding sector may satisfy the inclusion in 2.5 for a different element of $\mathcal{A}$. See Figure 4 for an illustration. Using this definition of sweeps we can restate Proposition 3.2:

Proposition 3.3. *If there exists an adjacent pair $l_1, l_2 \in \mathrm{csl}(\mathcal{A})$ such that*

(a) $\mathrm{sweep}^L(l_1) \cap \mathrm{vert}(G) \neq \varnothing \neq \mathrm{sweep}^R(l_2) \cap \mathrm{vert}(G)$*, and*
(b) not $(\mathrm{sweep}^L(l_1) \cap \mathrm{vert}(G) = \{g_i\}, \quad \mathrm{sweep}^R(l_2) \cap \mathrm{vert}(G) = \{g_{i+1}\})$*,*

then $(\mathcal{A}, G)$ satisfy the weak carousel rule.

Proof. (a) implies (1) of Proposition 3.2, and (b) implies (2) of Proposition 3.2. Then apply Proposition 3.2 □

Lemma 3.4. *The sweeps partition the boundary of G, i.e.*

$$\bigcup_{i=1}^{s} \mathrm{sweep}^L(l_i) = \bigcup_{i=1}^{s} \mathrm{sweep}^R(l_i) = \partial G$$

and

$$\mathrm{sweep}^L(l_i) \cap \mathrm{sweep}^L(l_{i+1}) = \Omega^L(l_{i+1}), \quad \mathrm{sweep}^R(l_i) \cap \mathrm{sweep}^R(l_{i-1}) = \Omega^R(l_{i-1}).$$

Proof. We prove the statement for the left sweep. By definition, $\mathrm{sweep}^L(l_i)$ is a segment of ∂G beginning at $\Omega^L(l_i)$ and ending at $\Omega^L(l_{i,\Delta_i;A})$. But this latter point is by definition of Δ_i equal to $\Omega^L(l_{i+1})$. Hence the sweep is a contiguous segment proceeding clockwise from $\Omega^L(l_i)$ to $\Omega^L(l_{i+1})$. Iterating the argument for $i \in \{1, \dots, s\}$ completes the proof. □

We are now ready to prove the main theorem. For notation, let lowercase $\omega^L(l)$ and $\omega^R(l)$ denote the edge of G that $\Omega^L(l)$ and $\Omega^R(l)$ lie on. If $\Omega^L(l)$ is a vertex of G, let $\omega^L(l)$ be the edge lying clockwise from it, and if $\Omega^R(l)$ is a vertex of G, let $\omega^R(l)$ be the edge lying counterclockwise from it. This convention ensures that ω^L and ω^R are well-defined, and makes the proof easier to work with.

Proof of Theorem 1.1. By Lemma 3.4 and the pigeonhole principle, there exists some $i \in \{1, \dots, s\}$ such that $\mathrm{sweep}^L(l_i)$ contains more than one vertex of G. Without loss of generality, let $i = 1$. If $\mathrm{sweep}^R(l_2)$ contains at least one vertex of G, we have satisfied the assumptions of Proposition 3.3, implying the weak carousel rule for $(\mathcal{A}, G)$.

Otherwise, suppose $\mathrm{sweep}^R(l_2)$ sweeps no vertices. Then by Lemma 3.4, $\Omega^R(l_2)$ lies clockwise of $\Omega^R(l_1)$ on the same edge. Without loss of generality suppose $\omega^L(l_1) = e_1$. Since $\mathrm{sweep}^L(l_1)$ contains more than one vertex of G, $\omega^L(l_2)$=e_j for $j \in \{3, \dots, n, 1\}$. Let $\omega^R(l_2) = \omega^R(l_1) = e_k$. In the following, for $a, b, c \in \{1, \dots, n\}$, by $a < b < c$ we mean that b lies in the interval $a + 1, a + 2, \dots, c - 1$ working modulo n. We have the following cases. For a visual of each case, see Figure 6.

(1) $j < k < 1$. Let $\mathcal{H}_1$ be the halfplane whose boundary is l_1, in the direction containing $\mathcal{A}$. Then

$$\begin{aligned} A_0, A_1 &\subset \mathcal{H}_1 \cap G \\ &= \mathrm{Conv}(\Omega^L(l_1), \Omega^R(l_1), \mathrm{vert}(G) \cap \mathcal{H}_1) \\ &\subset \mathrm{Conv}(g_1, g_n, \dots, g_3) \end{aligned}$$

Hence the weak carousel rule holds for either A_0 or A_1.

(2) $1 < k < j$. Then we have the following order of the given points, proceeding clockwise around ∂G:

$$\Omega^L(l_1), \Omega^R(l_1), \Omega^R(l_2), \Omega^L(l_2).$$

Then l_1 and l_2 divide G into three disjoint regions. By the fact that l_1, l_2 are common supporting lines, $\mathcal{A}$ must lie in two of these regions, a contradiction.

(3) $k = j$. By our definition of ω, this implies that e_j is on the line l_2, so $\Omega^R(l_2)$ is g_j, contained in $\mathrm{sweep}^R(l_2)$. And $\mathrm{sweep}^L(l_1)$ contains at least two elements g_1, g_2. We conclude by applying Proposition 3.3.

(4) $k = 1$. Then $\omega^R(l_1) = \omega^R(l_2) = e_1 = \omega^L(l_1)$. By our definition of ω, this implies that e_1 is on the line l_1, so $\Omega^L(l_1)$ is a vertex contained in $\mathrm{sweep}^L(l_1)$, and $\mathrm{sweep}^R(l_2) = \mathrm{vert}(G)$. We conclude by applying Proposition 3.3.

This enumerates through all possibilities for $j, k \in \{1, \dots, n\}$, concluding the proof. □

With this theorem we can immediately deduce the previous result of Adaricheva–Bolat, as well as similar results for ellipses. Because the boundaries of disks and ellipses are smooth algebraic plane curves, their common tangent lines correspond to common intersection points of their dual curves.

Corollary 3.5. *Let $\mathcal{A} = \{A_0, A_1\}$ be subsets of $\mathbb{R}^2$. If ∂A_0 and ∂A_1 are smooth plane curves of degree d_1, d_2, and G is a convex n-gon with*

$$n > d_1(d_1 - 1)d_2(d_2 - 1),$$

then $(\mathcal{A}, G)$ satisfy the weak carousel rule.

Proof. If a curve has degree d, then its dual has degree at most $d(d - 1)$ by Plücker's first formula. By Bézout's theorem [Sha13], distinct plane curves of degree x and y intersect at most xy points. Hence the maximum number of intersections of the duals of ∂A_0 and ∂A_1 is $d_1(d_1 - 1)d_2(d_2 - 1)$. Hence $\#\,\mathrm{csl}(\mathcal{A}) \le d_1(d_1 - 1)d_2(d_2 - 1)$. □

Corollary 3.6. *Two ellipses contained in a pentagon satisfy the weak carousel rule.*

Proof. Apply the previous statement with $d_1 = d_2 = 2$. □

Corollary 3.7 (Theorem 3.1, [AB19] and Theorem 1.1, [CK19])**.** *Two disks contained in a triangle satisfy the weak carousel rule.*

Proof. Out of four possible common tangent lines to two circles, two lines are internal tangents, which are not common supporting lines. Hence two disks have at most two common supporting lines. ◻

Lastly, we leave it to the reader to use this theorem to prove Theorem 1.1 of [CK19], which states that the weak carousel rule holds when G is a triangle and A_0, A_1 are compact convex sets which are positive homotheties or translations of one another. Using properties of support functions, one can show that A_0, A_1 have at most two common supporting lines.

3.3. **The bound is sharp for even n.** We now construct a family of examples to show that the bound in Theorem 1.1 is sharp for even n. That is, given $n > 2$, we construct a convex n-gon G and two convex shapes $\mathcal{A} = \{A_0, A_1\}$ contained in G with $\#\operatorname{csl}(\mathcal{A}) = n$ such that $(\mathcal{A}, G)$ fails to satisfy the weak carousel rule.

Identifying $\mathbb{R}^2$ with the complex plane, let G be the n-gon whose vertices are the nth roots of unity $\{\exp 2k\pi i/n : k = 0, 1, \ldots, n-1\}$. The common supporting lines will be the lines connecting the midpoints of adjacent edges of G, so that $\cup \operatorname{csl}(\mathcal{A}) \cap G$ forms the boundary of a similar n-gon, inscribed inside G. Denote these edges of this smaller n-gon by $l_1, \ldots, l_n$, with the convention that l_i and l_{i+1} are adjacent. Trisect each line l_k, and name the points of trisection p_k^a, p_k^b, proceeding clockwise. Let A_0 be the polygon whose vertices are $p_1^a, p_2^b, p_3^a, \ldots, p_n^b$, and A_1 the polygon whose vertices are those remaining points.

Proposition 3.8. *$(\mathcal{A}, G)$ do not satisfy the weak carousel rule.*

Proof. The proof will proceed by coordinate bash. By symmetry, it suffices to check that the inclusion

$$A_j \subset \operatorname{Conv}(A_{1-j}, \operatorname{vert} G \setminus \{1\})$$

does not hold for $j = 0, 1$. We will switch between $\mathbb{C}$ and $\mathbb{R}^2$ whenever convenient. Let l_1 be the common supporting line joining the midpoints of the two edges of G joining 1 to $\exp(2\pi i/n)$ and $\exp(2\pi i(n-1)/n) = \exp(-2\pi i/n)$. In real coordinates, l_1 intersects the boundary of G at the points

$$\left(\frac{1+\cos 2\pi/n}{2}, \frac{\sin 2\pi/n}{2}\right), \left(\frac{1+\cos 2\pi/n}{2}, -\frac{\sin 2\pi/n}{2}\right),$$

so then

$$p_1^a = \left(\frac{1+\cos 2\pi/n}{2}, \frac{\sin 2\pi/n}{6}\right), \quad p_1^b = \left(\frac{1+\cos 2\pi/n}{2}, -\frac{\sin 2\pi/n}{6}\right)$$

Checking first for $j = 1$,

$$C := \operatorname{Conv}(A_0, \operatorname{vert} G \setminus \{1\}) = \operatorname{Conv}(p_1^a, \operatorname{vert} G \setminus \{1\})$$

The lower rightmost edge of C is the line connecting p_1^a to $\exp(-2\pi i/n)$, which has equation

$$y - \frac{\sin 2\pi/n}{6} = \frac{7}{3}(\cot \pi/n)\left(x - \frac{1+\cos 2\pi/n}{2}\right)$$

so any point in C must satisfy the inequality

$$y - \frac{\sin 2\pi/n}{6} \geq \frac{7}{3}(\cot \pi/n)\left(x - \frac{1+\cos 2\pi/n}{2}\right)$$

But plugging in $p_1^b = (\frac{1+\cos 2\pi/n}{2}, -\frac{\sin 2\pi/n}{6})$ for (x, y), we obtain

$$\text{L.H.S.} = -\frac{\sin 2\pi/n}{3}$$

$$\text{R.H.S.} = \frac{7}{3}(\cot \pi/n)(\frac{1+\cos 2\pi/n}{2} - \frac{1+\cos 2\pi/n}{2}) = 0$$

and since $\sin 2\pi/n$ is strictly positive, we have that $p_1^b \notin C$, hence

$$A_1 \not\subset \operatorname{Conv}(A_0, \operatorname{vert} G \setminus \{1\})$$

and the statement for $j = 0$ is identical, reflected across the y-axis. □

For a figure of this construction when $n = 4, 6$, see Figure 7.

3.4. **Questions.** We conclude with a few questions. The first follows from the sharpness example in the previous section. We have not been able to construct such an example for odd n in a similar manner. In fact, all examples we have constructed with odd $s = n$ still satisfy the weak carousel rule. Hence:

Question 3.1. Is the bound in Theorem 1.1 sharp for odd n? If not, is it $s \geq n$?

Next, in Theorem 1.1, [Czé17a], Czédli shows that this property characterizes disks in $\mathbb{R}^2$. That is, suppose $\mathcal{A} = \{A_0, A_1\}$, and A_1 is obtained from A_0 by isometry. Then if $(\mathcal{A}, G)$ satisfies the weak carousel rule for all triangles G, then A_0 must be a disk. One natural question in this vein, following Corollary 3.6, is whether this property uniquely characterizes ellipses in $\mathbb{R}^2$. That is,

Question 3.2. Does there exist a non-ellipse compact convex subset $A \subseteq \mathbb{R}^2$ such that given any distinct isometric copy A' of A, we have $\# \operatorname{csl}(A, A') \leq 4$?

Returning to the subject of convex geometries, the authors in [AB19] show a slightly stronger result than stated in the introduction. By their Theorem 5.1, the weak carousel rule in fact holds for A_0, A_1 disks, and $G = \operatorname{Conv}(X_1, X_2, X_3)$ where X_1, X_2, X_3 are also disks. That is, there exist $i \in \{0, 1\}$ and $j \in \{1, 2, 3\}$ such that

$$A_i \subset \operatorname{Conv}(A_{1-i}, \{X_1, X_2, X_3\} \setminus \{X_j\})$$

whenever $A_0, A_1 \subset G$. This result allows them to deduce a statement about the non-representability of convex geometries as disks. Unfortunately, it is difficult to extend this result for $X_1, X_2, \ldots$ arbitrary convex shapes. For example, it is possible to construct two ellipses $\mathcal{A} = \{A_0, A_1\}$ with $\# \operatorname{csl}(\mathcal{A}) = 2$, and ellipses X_1, X_2, X_3 such that $A_0, A_1 \subset G := \operatorname{Conv}(X_1, X_2, X_3)$ yet the weak carousel rule fails for $(\mathcal{A}, G)$, as in Figure 5. We ask:

Question 3.3. Does there exist a suitable generalization of Theorem 1.1 where the n-gon G is replaced by the convex hull of n convex compact shapes? For example, does the weak carousel rule hold for two ellipses contained in the convex hull of five other ellipses, i.e. can we replace the pentagon in Corollary 3.6 by the convex hull of five ellipses?

Lastly, we ask whether some reformulation of the weak carousel property holds in higher dimensions. As shown by Czédli in Example 4.1, [Czé17b], the most simple generalization, for an arrangement of two spheres in a tetrahedron in $\mathbb{R}^3$, fails. The problem remains open, and seems unamenable using the proof techniques from this paper, seeing as even in the most simple case, two spheres in $\mathbb{R}^3$ in general position can have infinitely many common supporting planes.

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Figures

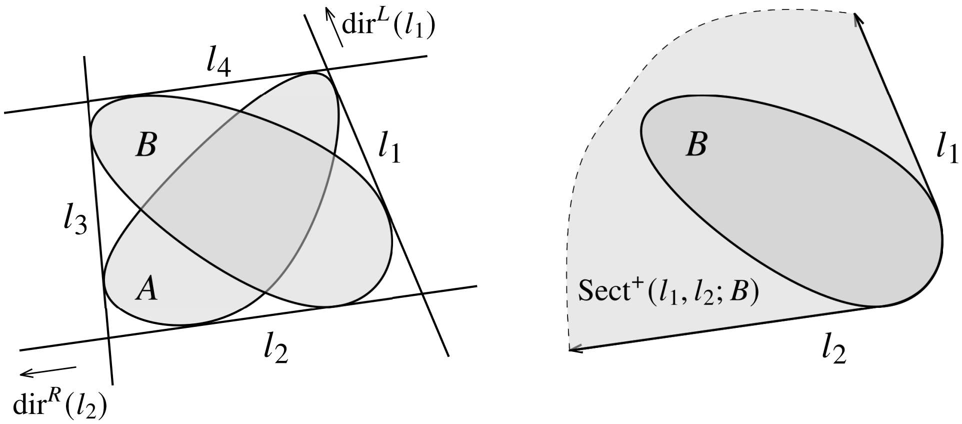

Figure 1. On the left, l_i and l_{i+1} are adjacent, for $i = 1, 2, 3, 4$, indices $(\mathrm{mod}\, 4)$. On the right, the sector $\mathrm{Sect}^+(l_1, l_2; B)$ continues along $\mathrm{dir}^L(l_1)$ and $\mathrm{dir}^L(l_2)$.

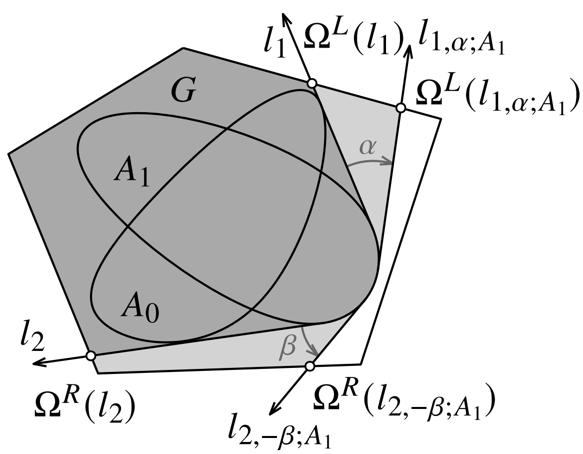

Figure 2. $\mathrm{Sect}^+(l_1, l_2; A_1) \cap G$ in dark gray, and the expanded $\mathrm{Sect}^+(l_{1,\alpha;A_1}, l_{2,-\beta;A_1}; A_1) \cap G$ in light gray.

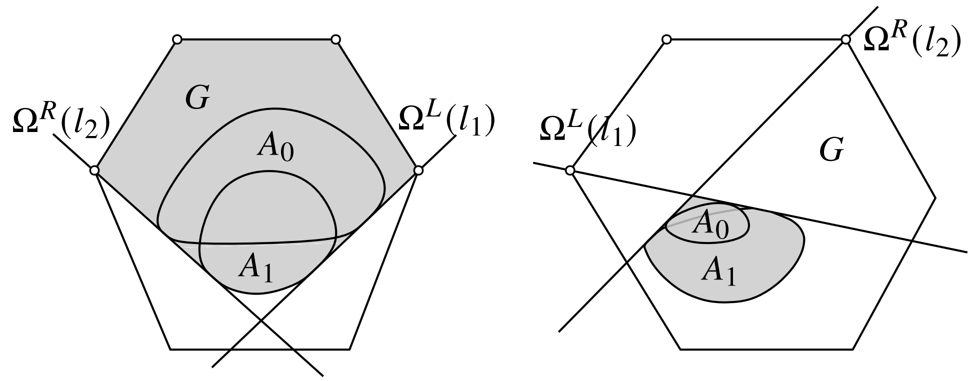

FIGURE 3. The vertices between l_1 and l_2 are dotted, and the sectors $\mathrm{Sect}^+(l_1, l_2; A_1)$ are shaded. Note the difference in order depending on whether the sector intersects G.

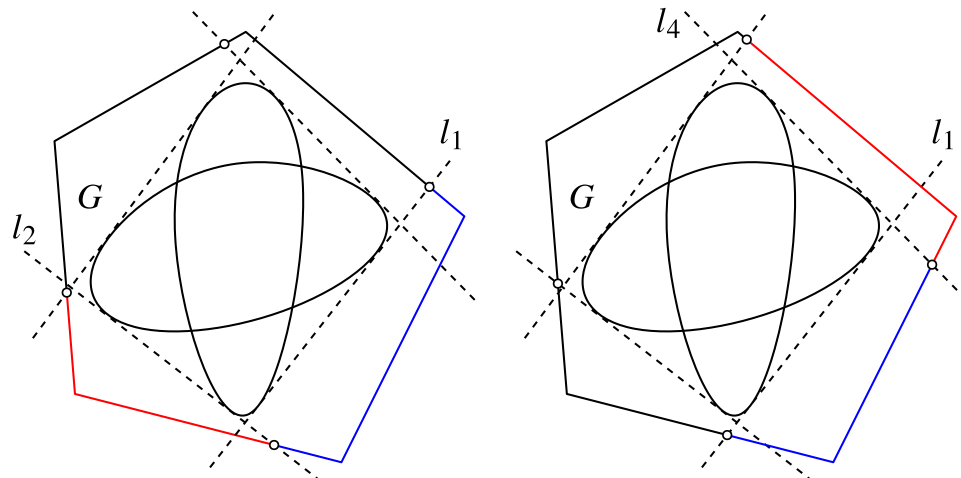

FIGURE 4. On the left, $\mathrm{sweep}^L(l_1)$ in blue and $\mathrm{sweep}^L(l_2)$ in red. On the right, $\mathrm{sweep}^R(l_1)$ in blue and $\mathrm{sweep}^R(l_4)$ in red.

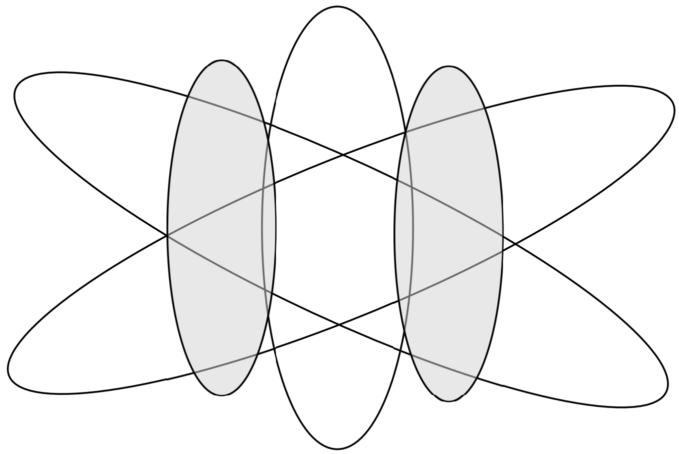

FIGURE 5. The ellipses A_0, A_1 shaded, the ellipses X_1, X_2, X_3 unshaded. The weak carousel rule fails for $\mathcal{A} = \{A_0, A_1\}$ and $G = \mathrm{Conv}(X_1, X_2, X_3)$.

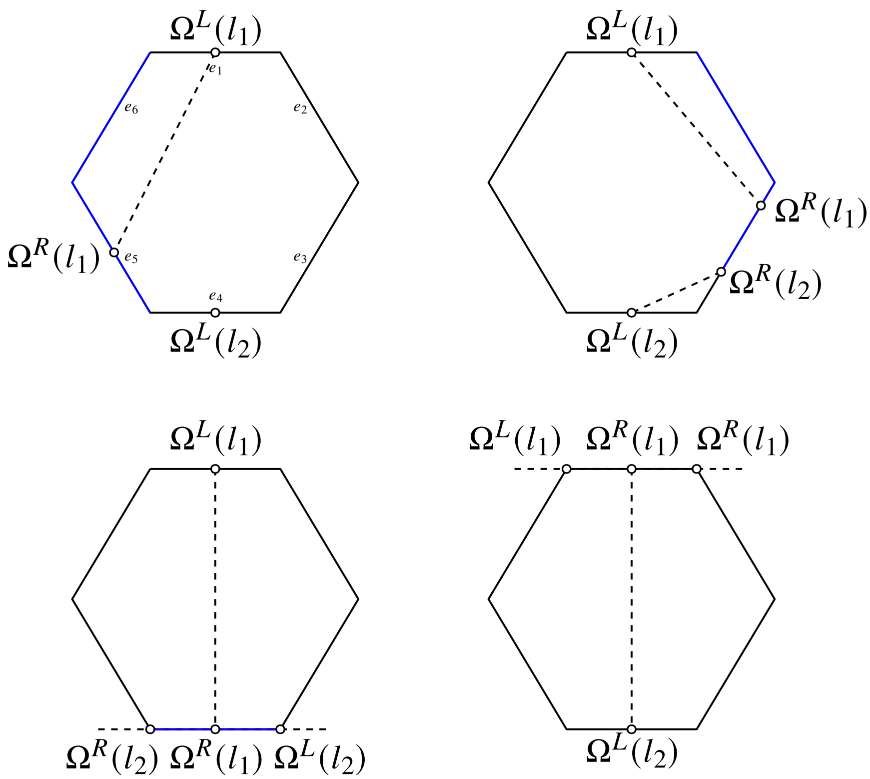

FIGURE 6. Illustrations for various cases in the proof of Theorem 1.1. Top to bottom, left to right are cases (1), (2), (3), and (4). Here we fix $j = 4$. The possible locations for $\Omega^R(l_1)$ are in blue.

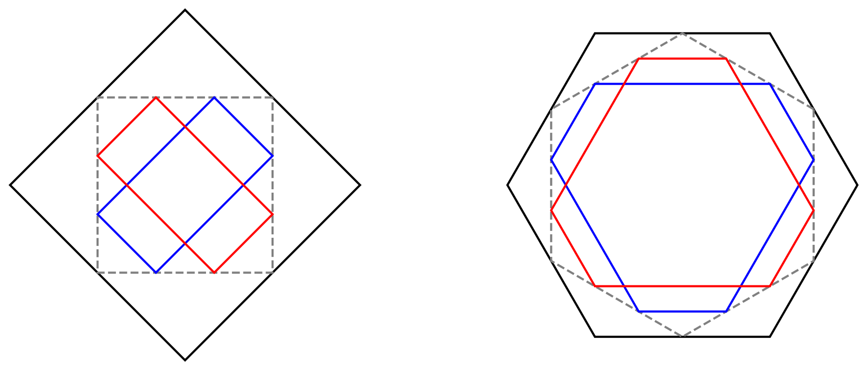

Figure 7. Examples to demonstrate sharpness for $n = 4, 6$. In red and blue are A_0, A_1. The common supporting lines are dotted, and G is the black polygon.