

# MCDUFF-SEGAL’S ARGUMENT FOR GROUP COMPLETION

YIMING SONG

ABSTRACT. We present the details in the proof of the group completion theorem by McDuff–Segal [MS76].

## CONTENTS

1. Introduction	1
2. Various constructions and definitions	2
2.1. Homotopy quotient by a monoid	2
2.2. Homotopy fibers and homology fibrations	2
2.3. Localization and right fractions	3
3. Proposition 2 implies group completion	4
3.1. The case where $\pi := \pi_0(M)$ is the monoid $\mathbf{N}$ .	4
3.2. Reducing to $\pi$ finitely generated	5
4. Proof of Proposition 2	6
4.1. Homology fibration v.s. homology-fibration	6
4.2. Cleaning up and last steps	8
Acknowledgements	10
References	10

## 1. INTRODUCTION

Given a topological monoid  $M$ , recall that we can construct its classifying space  $BM$  by regarding  $M$  as a one-object category and taking the geometric realization of its nerve. When  $M$  is grouplike, i.e.  $\pi := \pi_0(M)$  is a group, the functor  $B$  can be regarded as some sort of “delooping” of  $M$ . That is, the map  $M \rightarrow \Omega BM$  sending  $m$  to the loop corresponding to the 1-simplex of  $BM$  labeled by  $m$  is a homotopy equivalence. Consequently, the induced map  $H_*(M) \rightarrow H_*(\Omega BM)$  is an isomorphism. However, if  $M$  is just a monoid, then  $H_*(M)$  and  $H_*(\Omega BM)$  are not isomorphic via this map.

It turns out that the only obstruction to the isomorphism is that  $\pi := \pi_0(M)$  is not a group. But it remains a multiplicatively closed subset, thanks to the monoid operation. So perhaps we should localize at its image in  $H_0(M)$ . This is exactly what we missed:

**Theorem 1.1** (Group completion). *If  $\pi$  is in the center of  $H_*(M)$ , then*

$$H_*(M)[\pi^{-1}] \rightarrow H_*(\Omega BM)$$

*is an isomorphism.*

This theorem has many applications across geometric topology and  $K$ -theory. In this note, we will expound on the nice proof given in the paper [MS76].

## 2. VARIOUS CONSTRUCTIONS AND DEFINITIONS

Before moving on to the proof, we need to define a few essential pieces of machinery.

**2.1. Homotopy quotient by a monoid.** Let  $M$  be a topological monoid acting on a space  $X$ . If  $M$  was a topological group, then we could have formed the homotopy quotient  $X_M$  by taking the universal bundle  $EM \rightarrow BM$  and setting  $X_M := (X \times EM)/M$ . Then  $X_M$  has a natural map to  $BM$ , given by projection onto the second factor, with fiber  $X$ . Even when  $M$  is not a group, it is still possible to construct a space with these properties. First consider the category  $\mathcal{X}$  with objects  $X$  and morphisms  $M \times X$  where a pair  $(m, x)$  is a morphism from  $x$  to  $mx$ . The nerve of  $\mathcal{X}$  is:

$$\begin{aligned} N_0\mathcal{X} &= X \\ N_1\mathcal{X} &= M \times X \\ N_2\mathcal{X} &= M \times M \times X \\ &\dots \end{aligned}$$

where the  $k$ th simplex is given by a string of morphisms

$$x \rightarrow m_1x \rightarrow m_2m_1x \cdots \rightarrow m_km_{k-1} \cdots m_1x.$$

We then define  $X_M := |N_\bullet\mathcal{X}| = \sqcup_{p \geq 0} (\Delta^p \times X \times M^p) / \sim$  to be the geometric realization of this nerve. The projection ignoring the middle  $X$  coordinate still gives a map to  $BM$  with fiber  $X$ , so we have done our job. (It is worth noting that this map is *not* usually a fibration.)

**2.2. Homotopy fibers and homology fibrations.** If we have an arbitrary map of spaces  $p : E \rightarrow B$ , we can turn it into a fibration as follows. Fix a basepoint  $b \in B$ . Consider the space

$$E_p := \{(e, \gamma) : e \in E, \gamma : I \rightarrow B, \gamma(0) = p(e)\}$$

considered as a subspace of  $E \times \text{Map}(I, B)$ . Then:

**Proposition 2.1.**  $E_p \rightarrow B, (e, \gamma) \mapsto \gamma(1)$  is a fibration.

*Proof.* Suppose we have a homotopy  $g_t : X \rightarrow B$  and a lift  $\tilde{g}_0 : X \rightarrow E_p$ . We can write  $\tilde{g}_0(x) = (h(x), \gamma_x) \in E_p$ , for some  $h : X \rightarrow E$  and  $\gamma_x : I \rightarrow B$ . Then we get a lift  $\tilde{g}_t(x) := (h(x), \gamma_x * g_{[0,t]}(x))$ , where  $g_{[0,t]}$  is the homotopy  $g$  ran until time  $t$ .  $\square$

Now fix a basepoint  $b \in B$ . Then  $E$  includes into the space  $E_p$  via  $e \mapsto (e, c_{p(e)})$  where  $c_{p(e)}$  is the constant map at  $p(e)$ . Moreover,  $E_p$  deformation retracts onto  $E$  just by retracting the given path  $\gamma$  until it is the constant map at  $b$ . Therefore, under this identification  $E \simeq E_p$ , we have a fibration  $E \rightarrow B$  whose fiber over a point  $b$  we will call the *homotopy fiber of  $p : E \rightarrow B$* , denoted  $\text{hofib}(p)_b$ . Like many homotopical constructions, these are only unique up to homotopy equivalence. Let us follow with some useful facts about homotopy fibers, the first of which follows from this discussion.

**Lemma 2.2.** The homotopy fiber of a map  $p : E \rightarrow B$  is the pullback of  $p$  along the the pathspace fibration  $f : PB \rightarrow B$ .  $\square$

**Lemma 2.3.** If  $E_p$  is contractible, then  $\text{hofib}(p)_b \simeq \Omega B_b$ .

*Proof.* Consider first the map  $i : \{*\} \hookrightarrow B$  with  $i(*) = b$ . Following the construction, we have  $E_i = \{(*, \gamma) : \gamma : I \rightarrow B, \gamma(0) = b\}$  and this is clearly homeomorphic to  $PB_b = \{\gamma : I \rightarrow B : \gamma(0) = b\}$ . The map  $E_i \cong PB_b \rightarrow B$  is given by evaluating  $\gamma \mapsto \gamma(1)$ . The fiber of this map over  $b$  consists of loops in  $B$  based at  $b$ , so  $\text{hofib}(i)_b \cong \Omega B$ .

$PB_b$  is contractible by shrinking each path to the constant path based at  $b$ , so we have some homotopy equivalence  $PB_b \rightarrow E_p$ , since  $E_p \simeq E$  is contractible. We then get a commutative diagram

$$\begin{array}{ccccc} \mathrm{hofib}(i)_b \cong \Omega B_b & \hookrightarrow & E_i \cong PB_b & \xrightarrow{\gamma \mapsto \gamma(1)} & B \\ \downarrow & & \downarrow \simeq & & \downarrow \simeq \\ \mathrm{hofib}(p)_b & \hookrightarrow & E_p & \longrightarrow & B \end{array}$$

By taking the long exact sequence on homotopy groups, and applying the five lemma, we see that the dotted arrow is a homotopy equivalence.  $\square$

**Lemma 2.4.** *If  $f : E \rightarrow B$  is a fibration, then  $f^{-1}(b) \simeq \mathrm{hofib}(f)_b$ .*

*Proof.* Let  $p : E_f \rightarrow B$  be the fibration constructed previously. Since both  $f$  and  $p$  are fibrations, we get two fiber sequences assembling into a commutative diagram:

$$\begin{array}{ccccc} \mathrm{hofib}(f)_b & \hookrightarrow & E_p & \xrightarrow{p} & B \\ \downarrow & & \downarrow \simeq & & \downarrow \simeq \\ F & \hookrightarrow & E & \xrightarrow{f} & B \end{array}$$

and the rest of the argument is the same as in the previous lemma.  $\square$

In the cases where  $p$  is not as nice as a fibration, we can still define:

**Definition 2.5.** We say a map  $p : E \rightarrow B$  is a *homology fibration* if the map  $p^{-1}(b) \rightarrow \mathrm{hofib}(p)_b$  is a homology equivalence for all basepoints  $b \in B$ .

By 2.4, all fibrations are homology fibrations.

**2.3. Localization and right fractions.** Now to do some honest algebra. Let  $R$  be a ring and let  $S$  be a multiplicative subset of  $R$ , i.e. contains 1, closed under multiplication. Then we can construct the localization  $S^{-1}R$ , being mindful of the fact that  $R$  need not be commutative, so things are a little weird.

**Definition 2.6.**  $S^{-1}R$  can be constructed by *right fractions* if every element of it can be written as  $rs^{-1}$ , and  $r_1/s_1 = r_2/s_2$  iff there exists elements  $t_1, t_2$  in  $S$  such that  $r_1t_1 = r_2t_2$  and  $s_1t_1 = s_2t_2$ .

This is a somewhat technical condition, slightly weaker than  $S$  being contained in the center of  $R$ . It is actually necessary to stipulate the first phrase, that the element can be written as  $rs^{-1}$ , as  $R$  is not necessarily commutative.

**Lemma 2.7.** *If  $S$  is contained in the center of  $R$ , then  $S^{-1}R$  can be constructed by right fractions.*

*Proof.* Since  $S$  is in the center, we do not need to worry about distinguishing between  $s^{-1}r$  and  $rs^{-1}$ . First suppose  $r_1/s_1 = r_2/s_2$ . Then there exists  $q \in S$  such that  $qr_1s_2 = qr_2s_1$ . Let  $t_1 := qs_2$  and  $t_2 := qs_1$ . Then  $r_1t_1 = r_1qs_2 = qr_1s_2 = qr_2s_1 = qs_1r_2 = r_2t_2$ , and  $s_1t_1 = s_1qs_2 = s_2qs_1 = s_2t_2$ .

Conversely, suppose such  $t_1, t_2$  exist. Then let  $q = t_1t_2$ , from which we obtain  $qr_1s_2 = r_1t_1s_2t_2 = r_2t_2s_1t_1 = qr_2s_1$  as desired.  $\square$

This somewhat contrived definition of “right fractions” is needed for the following proposition:

**Lemma 2.8.** *If  $R[s^{-1}]$  can be constructed by right fractions, then*

$$\operatorname{colim}(R \xrightarrow{\cdot s} R \xrightarrow{\cdot s} \cdots) \cong R[s^{-1}]$$

*Proof.* We can represent elements of the colimit as pairs  $(r, k)$  where  $r$  is the element in the  $k$ th copy of  $R$ . Thus we have the relation  $(r, k) \sim (rs, k+1)$ . Consider the map from the colimit to  $R[s^{-1}]$  given by  $(r, k) \mapsto r/s^k$ , which is clearly surjective. If  $(r, k) \mapsto 0$ , then  $r/s^k \sim 0/s^0$ , and because  $R[s^{-1}]$  can be constructed by right fractions, there exist  $s^n, s^m$  such that

$$rs^n = 0s^m = 0, \quad s^k s^n = s^0 s^m.$$

From this and the relation we obtain  $(r, k) \sim (rs^n, k+n) = (0, m) \sim (0s^m, 0) = (0, 0)$ , so the map is injective too.  $\square$

### 3. PROPOSITION 2 IMPLIES GROUP COMPLETION

Let's see how the proof usually goes if  $M$  is a group, in which case we call it  $G$ . It follows very quickly from the commutative diagram of the form

$$\begin{array}{ccccc} G & \longrightarrow & EG & \longrightarrow & BG \\ \downarrow & & \downarrow \simeq & & \downarrow \simeq \\ \Omega BG & \longrightarrow & PBG & \longrightarrow & BG \end{array}$$

where the top row is the principal  $G$ -bundle and the bottom row is the path space fibration. We have a homotopy equivalence in the middle arrow because path spaces are contractible. Then everything commutes, and the five lemma finishes us off. The issue when  $M$  is not a group is that the top row of this diagram doesn't exist, at least not with homotopy fiber  $M$ , so we need to try something else. Let us begin with a proposition.

**Proposition 3.1** (Proposition 2. in [MS76]). *If  $X$  is a space with the action of a topological monoid  $M$ , and for all  $m \in M$ ,  $x \rightarrow m \cdot x$  is homology equivalence, then  $X_M \rightarrow BM$  is a homology fibration.*

The proof is slightly tricky, so we will save that for later and first see how group completion follows from this. We will first prove the group completion theorem for  $\pi_0(M) \cong \mathbf{N}$ , then generalize to the case where  $\pi_0(M)$  is an arbitrary monoid.

**3.1. The case where  $\pi := \pi_0(M)$  is the monoid  $\mathbf{N}$ .** In nature, the prototypical monoid is  $\mathbf{N}$ , so it perhaps makes sense to start here. Choose  $m$  in the component of the generator  $\mathbf{1} \in \mathbf{N}$ , and let  $X$  in 3.1 be  $M_\infty = \operatorname{colim}(M \xrightarrow{\cdot m} M \xrightarrow{\cdot m} \cdots)$ . Since homology preserves filtered colimits, we have

$$H_*(X) \cong \operatorname{colim}(H_*(M) \xrightarrow{\cdot m_*} H_*(M) \xrightarrow{\cdot m_*} \cdots) \cong H_*(M)[m_*^{-1}]$$

where the last isomorphism is due to 2.8. Now since we've picked  $m \in \mathbf{1} \in \pi$ , the induced map  $\cdot m_*$  in homology corresponds to  $\mathbf{1} \in \pi = \langle \mathbf{1} \rangle$ , hence

$$H_*(X) \cong H_*(M)[m_*^{-1}] \cong H_*(M)[\mathbf{1}^{-1}] \cong H_*(M)[\pi^{-1}]$$

Finally, consider the left action of  $M$  on  $X$ . For  $n \in M$ , the action is by multiplication, and this induces the following commutative diagram on homology:

$$\begin{array}{ccccc} H_*(M) & \xrightarrow{\cdot m_*} & H_*(M) & \xrightarrow{\cdot m_*} & \cdots \\ \downarrow n_* & & \downarrow n_* & & \\ H_*(M) & \xrightarrow{\cdot m_*} & H_*(M) & \xrightarrow{\cdot m_*} & \cdots \end{array}$$

but since  $n_* \in \pi$ , multiplication by  $n_*$  is invertible in  $H_*(M_\infty) = H_*(M)[\pi^{-1}]$ , so the action of  $M$  on  $X$  is a homology equivalence. So by 3.1, we conclude that  $p : X_M = (M_\infty)_M \rightarrow BM$  is a homology fibration, i.e.  $p^{-1}(b) \rightarrow \text{hofib}(p)_b$  is a homology equivalence.

It remains to figure out what  $p^{-1}(b)$  and  $\text{hofib}(p)_b$  are. By the previous section, the homotopy quotient  $X_M$  is the realization of the nerve  $N_*X$  whose  $k$ -simplex is

$$N_k X = M^k \times M_\infty = M^k \times \text{colim}(M \rightarrow M \rightarrow \dots) = \text{colim}(M^{k+1} \rightarrow M^{k+1} \rightarrow \dots)$$

where the last equality is again because filtered colimits commute with products. Then, because geometric realization commutes with colimits, we have

$$\begin{aligned} X_M = (M_\infty)_M &= \left| N_0 X \rightrightarrows N_1 X \rightrightarrows N_2 X \rightrightarrows \dots \right| \\ &= \left| \text{colim}(M \rightarrow \dots) \rightrightarrows \text{colim}(M^2 \rightarrow \dots) \rightrightarrows \dots \right| \\ &= \left| \text{colim} \left( M \rightrightarrows M^2 \rightrightarrows \dots \right) \right| \\ &= \text{colim} \left| M \rightrightarrows M^2 \rightrightarrows \dots \right| \\ &= \text{colim}(EM) \end{aligned}$$

The filtered colimit of a sequence of contractible spaces is contractible, hence  $X_M$  is too. Therefore by 2.3,  $p : X_M \rightarrow BM$  has homotopy fiber  $\Omega BM$ . The regular fiber of the map  $(M_\infty)_M \cong \text{colim}(EM) \rightarrow BM$  is just  $M_\infty$ , since on the level of nerves, we have projections

$$N_k X = M^k \times M_\infty \rightarrow M^k$$

whose fibers are  $M_\infty$ , and taking the realization of the constant simplicial space at  $M_\infty$  gives  $M_\infty$ . Looking back to the homology fibration definition, we thus obtain a homology isomorphism

$$H_*(M)[\pi^{-1}] \cong H_*(X) \cong H_*(p^{-1}(b)) \rightarrow H_*(\text{hofib}(p)_b) \cong H_*(\Omega BM)$$

as desired.

**3.2. Reducing to  $\pi$  finitely generated.** Suppose instead now that  $\pi$  is generated by some  $s_1, \dots, s_k$ . Then setting  $s := s_1 s_2 \dots s_k$ , and letting  $X = \text{colim}(M \xrightarrow{m} M \xrightarrow{m} \dots)$  for some  $m \in s$ , by the same argument in the previous section we have

$$H_*(X) \cong H_*(M)[m_*^{-1}] \cong H_*(M)[s^{-1}] \cong H_*(M)[\pi^{-1}]$$

The action of  $M$  on  $X$  is a homology isomorphism by the same reason as above, and the rest of the argument proceeds identically. Unfortunately, not every monoid is finitely generated. Fortunately,

**Theorem 3.2.** *Every monoid  $(M, \cdot)$  is the quotient of a free monoid by a set of relations. Moreover,  $M$  is isomorphic to a filtered colimit of finitely generated monoids.*

*Proof.* Let  $A$  be the generating set of  $M$  (it may be that  $A = M$ ). Let  $FA$  be the free monoid on  $A$ . Let  $R$  consist of all equations satisfied by the elements of  $A$ . Then we have a map  $FA \rightarrow M$  given by sending each letter of  $A$  to the corresponding element in  $M$ , and concatenation of letters to monoid multiplication. This map is clearly surjective, and it factors through  $R$  by definition, so  $FA/R \cong M$ .

Now consider the category  $P$  whose objects are pairs  $(A_0, R_0)$  where  $A_0 \subseteq A$  and  $R_0 \subseteq R$  are finite subsets such that all equations in  $R_0$  consist only of elements in  $A_0$ . The morphisms are given by inclusion. This category is filtered; it is clearly nonempty, and for any pair of elements  $(A_0, R_0), (A_1, R_1) \in P$  includes into  $(A_0 \cup A_1, R_0 \cup R_1)$ . We then have a

functor  $\mathcal{F} : P \rightarrow \text{Monoid}$  sending  $(A_0, R_0)$  to the monoid  $A_0/R_0$ , and morphisms in  $P$  to the inclusion map in Monoid.

We claim that  $M \cong \text{colim}(\mathcal{F})$ . For any class  $[x] \in \mathcal{F}(A_0, R_0) = A_0/R_0$ , there is a map sending  $[x]$  to the element  $x \in M$ . So we have maps from each element of the diagram  $\mathcal{F}$  to  $M$ , so by the universal property of colimits this gives some map  $\Phi : \text{colim}(\mathcal{F}) \rightarrow M$ . Surjectivity is obvious. For injectivity, suppose  $[x] \in \text{colim}(\mathcal{F})$  maps to zero. The class  $[x]$  is represented by some  $x \in FA_0/R_0$  for some  $(A_0, R_0) \in P$ , so then  $\Phi([x]) = 0$  implies that  $x = 0$  is a relation of  $M$ . Then the class of  $x$  is zero in  $(A_0, R_0 \cup \{x = 0\}) \in P$ , hence zero in the colimit.  $\square$

Now we can assume  $\pi = \text{colim}(\mathcal{F})$  for some functor  $\mathcal{F}$  mapping into Monoid. Then, for each object  $(A_i, R_i)$  in the category  $P$ , we have that  $\mathcal{F}(A_i, R_i) =: \pi_{(i)}$  is a finitely generated monoid, so we can repeat the same trick in the finitely generated case above to obtain  $m_i$  in the component  $s_i$  which generates  $\pi_{(i)}$ . Then we get a space  $X_i = \text{colim}(M \xrightarrow{m_i} \dots)$  such that

$$H_*(X_i) \cong H_*(M)[(m_i)_*^{-1}] \cong H_*(M)[(s_i)_*^{-1}] \cong H_*(M)[\pi_{(i)}^{-1}]$$

and taking colimits, we have

$$H_*(X) \cong \text{colim}(H_*(X_i)) \cong \text{colim} H_*(M)[\pi_{(i)}^{-1}] \cong H_*(M)[\pi^{-1}]$$

as desired. Then  $M$  acts on  $X$  by homology equivalences since it acts on each  $X_i$  by homology equivalence. And  $X_M$  remains contractible since homotopy quotients commute with filtered colimits. This completes the proof of group completion.

#### 4. PROOF OF PROPOSITION 2

To prove Proposition 2, the authors end up adopting a alternate definition of homology fibration, which they call a *homology-fibration*. This is different because there is a hyphen. But they turn out to be not so different, as the names suggest. In this section we first explain the difference, then complete the proof of Proposition 2.

##### 4.1. Homology fibration v.s. homology-fibration.

**Definition 4.1.** A map  $p : E \rightarrow B$  is a *homology-fibration* if any  $b \in B$  has arbitrarily small contractible neighborhoods  $U$  such that  $p^{-1}(b') \hookrightarrow p^{-1}(U)$  is a homology equivalence for all  $b' \in U$ .

Recall that earlier in the proof all we really needed was for the map  $X_M \rightarrow BM$  to be a homology fibration. We know that  $BM$  can be modeled as the geometric realization of a simplicial set, which means that  $BM$  can always be given a CW structure. In particular, CW complexes are always paracompact and locally contractible. Under these assumptions, we can forget the hyphen:

**Proposition 4.2** (Proposition 5, [MS76]). *If  $B$  is a paracompact, locally contractible space, and  $p : E \rightarrow B$  is homology-fibration, then it is a homology fibration.*

To prove this, we will make use of the following proposition, a mix of statements from §4 and §5 in [Seg68].

**Proposition 4.3.** *Let  $\mathcal{B}$  be a basis for a paracompact, locally contractible, and contractible space  $B$ . Let  $p : E \rightarrow B$  be a homology-fibration. Then:*

- (1) *If  $C$  is the category whose elements are finite subsets of  $\mathcal{B}$ , and morphisms are inclusions, then  $|\mathcal{B}| := |NC|$  is homotopy equivalent to  $B$ .*

- (2) There exists a space  $E_{\mathcal{B}}$  homotopy equivalent to  $E$ , and a map  $E_{\mathcal{B}} \rightarrow |\mathcal{B}|$  giving rise to a spectral sequence  $E_{*,*}^*$  with

$$E_{p,q}^2 = H_p(|\mathcal{B}|, \mathcal{H}_q) \implies H_{p+q}(E_{\mathcal{B}}) = H_{p+q}(E)$$

where  $\mathcal{H}$  is the local coefficient system  $U \mapsto p^{-1}(U)$ .  $\square$

In the remaining discussion, we will abbreviate “basis consisting of contractible sets” by “contractible basis.”

**Corollary 4.4** (Proposition 6, [MS76]). *If  $s : X \rightarrow Y$  is a homology-fibration, and  $Y$  is paracompact, locally contractible, and contractible, then  $s^{-1}(y) \rightarrow X$  is a homology equivalence for each  $y \in Y$ .*

*Proof.* Since  $s$  is a homology-fibration, given a contractible basis  $\mathcal{Y}$  of  $Y$  we know that  $s^{-1}(y) \rightarrow s^{-1}(U)$  is a homology equivalence for all  $U \in \mathcal{Y}$ . Observing the  $E_{0,q}^2$  term of the spectral sequence from the previous proposition, we obtain a chain of isomorphisms

$$H_q(s^{-1}(y)) \cong H_q(s^{-1}(U)) \cong \mathcal{H}_q(U) \cong H_0(|\mathcal{Y}|; \mathcal{H}_q) \cong H_q(X)$$

where the penultimate isomorphism is because  $Y \simeq |\mathcal{Y}|$  is contractible, and the final isomorphism is via the spectral sequence.  $\square$

*Proof of 4.2.* Let  $f : PB_b \rightarrow B$  be the pathspace fibration. By 2.2,  $\text{hofib}(p)_b \simeq f^*E$ , we have the pullback square below, where we have also added the fibers on top for  $\gamma \in PB_b$  and  $b \in B$  satisfying  $f(\gamma) = b$ :

$$\begin{array}{ccc} (f^*p)^{-1}(\gamma) & \xrightarrow{\simeq} & p^{-1}(b) \\ \downarrow & & \downarrow \\ f^*E = \text{hofib}(p)_b & \longrightarrow & E \\ f^*p \downarrow & & \downarrow p \\ PB_b & \xrightarrow{f} & B \end{array}$$

Importantly, the fibers  $(f^*p)^{-1}(\gamma)$  and  $p^{-1}(b)$  are equivalent, since we have a pullback.

We claim first that  $f^*p$  is a homology-fibration, i.e. for any  $\gamma \in PB_b$ , there exists a contractible neighborhood  $V \ni \gamma$  such that  $(f^*p)^{-1}(\gamma) \rightarrow (f^*p)^{-1}(V)$  is a homology equivalence. To see this, choose a contractible basis  $\mathcal{B}$  of  $B$ . Since  $p$  is a homology-fibration, the map  $p^{-1}(b) \rightarrow p^{-1}(U)$  is a homology equivalence for all  $U \in \mathcal{B}$ . By some point-set technicalities, we can find a contractible basis  $\mathcal{B}^*$  of  $PB_b$  such that for all  $V \in \mathcal{B}^*$ ,  $f(V) \in \mathcal{B}$  and  $V \rightarrow f(V)$  is a fibration. Now we can restrict the pullback square above to obtain

$$\begin{array}{ccc} f^*p^{-1}(f(V)) & \xrightarrow{\pi} & p^{-1}(f(V)) \\ f^*p \downarrow & & \downarrow p \\ V & \xrightarrow{f} & f(V) \end{array}$$

where the bottom arrow is a fibration and homotopy equivalence (since both spaces are contractible). Then this makes the top arrow  $\pi$  a fibration since pullbacks of fibrations are also fibrations. But  $\pi$  is the projection  $(v, e) \mapsto e$ . The fiber over a point  $e \in p^{-1}(f(V))$

with  $p(e) = b$  is

$$\begin{aligned}
\pi^{-1}(e) &= \{(v, e) \in f^* p^{-1}(f(V)) : \pi(v, e) = e\} \\
&= \{(v, e) \in V \times p^{-1}(f(V)) : f(v) = p(e) = b\} \\
&\cong \{v \in V \mid f(v) = b\} \\
&= f^{-1}(b)
\end{aligned}$$

Since  $f$  is a homotopy equivalence and a fibration, we have by the long exact sequence in homotopy groups that the fibers  $f^{-1}(b) = \pi^{-1}(e)$  are contractible. Hence  $\pi$  is a weak homotopy equivalence, hence a homology equivalence. Restricting once again to the fibers:

$$\begin{array}{ccc}
(f^* p)^{-1}(\gamma) & \xrightarrow{\pi} & p^{-1}(b) \\
f^* p \downarrow & & \downarrow p \\
V & \xrightarrow{f} & f(V)
\end{array}$$

we see that the top, right, and bottom arrows are all homology equivalences, and thus so is the left arrow. Hence  $f^* p$  is a homology-fibration.

Then, applying 4.4, taking  $s = f^* p$ ,  $X = f^* E$ , and  $Y = PB_b$ , we have that  $(f^* p)^{-1}(\gamma) \rightarrow f^* E$  is a homology equivalence. But by earlier, we have that  $(f^* p)^{-1} \simeq p^{-1}(p)$ , so actually  $p^{-1}(p) \rightarrow f^* E = \text{hofib}(p)_b$  is a homology equivalence for all  $b \in B$ . Hence  $p$  is a homology equivalence (without hyphen).  $\square$

**4.2. Cleaning up and last steps.** Now back to the proof of 3.1, whose conclusion is that a certain map is a homology fibration. We now know that homology-fibrations are homology fibrations under certain circumstances, so it suffices to prove the conclusion that the map is a homology-fibration.

**Definition 4.5.** Let  $X, Y, Z$  be topological spaces. The *double mapping cylinder* of maps  $f : X \rightarrow Y, g : X \rightarrow Z$  is the quotient

$$\text{Cyl}(Z \leftarrow X \rightarrow Y) := (X \times [0, 1] \sqcup Y \sqcup Z) / (x, 0) \sim f(x), (x, 1) \sim g(x)$$

When one of  $Y$  or  $Z$  is the one point space, then we recover the mapping cone. When one of the maps is the identity, we recover the mapping cylinder of the other map. Notice that this construction is functorial: levelwise maps from  $Z_1 \leftarrow X_1 \rightarrow Y_1$  to  $Z_2 \leftarrow X_2 \rightarrow Y_2$  induce a map of double mapping cylinders. The advantage of the modified definition, the authors write, is that it “makes the following proposition obvious.” Let’s see:

**Proposition 4.6** (Proposition 3, [MS76]). *Suppose we have a commutative diagram*

$$\begin{array}{ccccc}
E_1 & \longleftarrow & E_0 & \longrightarrow & E_2 \\
\downarrow p_1 & & \downarrow p_0 & & \downarrow p_2 \\
B_1 & \xleftarrow{f_1} & B_0 & \xrightarrow{f_2} & B_2
\end{array}$$

where the vertical arrows are homology-fibrations, and  $p_0^{-1}(b) \rightarrow p_i^{-1}(f_i(b))$  is a homology-equivalence for all  $b \in B_0$ , then the induced map

$$p : \text{Cyl}(E_1 \leftarrow E_0 \rightarrow E_2) \rightarrow \text{Cyl}(B_1 \leftarrow B_0 \rightarrow B_2)$$

is a homology-fibration.

*Proof.* Let  $c \in \text{Cyl}(B_1 \leftarrow B_0 \rightarrow B_2)$ . Then there exist arbitrarily small neighborhoods  $U$  of  $c$  that are equal to mapping cylinders of the form  $\text{Cyl}(U_0 \rightarrow U_i)$  for  $i = 0, 1, 2$ , where  $U_i \subset B_i$  is open, and this will imply the proposition. To see this, choose a point  $x \in \text{Cyl}(B_1 \leftarrow B_0 \rightarrow B_2)$ . We have a few cases:

- (1)  $x$  lies entirely within the component of  $B_i, i = 1, 2$  of the cylinder. Then we can find a contractible open set  $U_i \subset B_i$ . Now let  $U = f_i^{-1}(U_i)$ . Then  $x \in \text{Cyl}(f : U \rightarrow U_i)$ . Then  $p^{-1}(x) \rightarrow p^{-1}(\text{Cyl}(f : U \rightarrow U_i))$  is a homology equivalence since  $\text{Cyl}(f : U \rightarrow U_i) \simeq U_i$  and this is just the map  $p_i^{-1}(x) \rightarrow p_i^{-1}(U_i)$ , which is a homology equivalence by assumption.
- (2)  $x = (b, t)$  lies in the component  $B_0 \times (0, 1)$ . Then we can find a neighborhood  $U \ni b$  in  $B_0$ , and some small  $\varepsilon > 0$  such that  $x \in U \times (t - \varepsilon, t + \varepsilon)$ . This is homotopy equivalent to the identity mapping cylinder  $\text{Cyl}(U \rightarrow U)$ . Just like before, the map  $p^{-1}(x) \rightarrow p^{-1}(\text{Cyl}(U \rightarrow U))$  is the map  $p_0^{-1}(x) \rightarrow p_0^{-1}(\text{Cyl}(U \rightarrow U)) \simeq p_0^{-1}(U)$ , which is a homology equivalence by assumption.
- (3)  $x$  lies in the gluing points, i.e.  $x = (b, 0) \sim f_1(b)$  or  $x = (b, 1) \sim f_2(b)$  for  $b \in B_0$ . In the former case, we can take some open  $U_1 \ni f_1(b)$  in  $B_1$ , and let  $U = f_1^{-1}(U_1)$ . Then consider  $U_1 \cup_{f_1} U \times [0, \varepsilon) \simeq \text{Cyl}(U \rightarrow U_1)$ . Then  $p^{-1}(x) \rightarrow p^{-1}(\text{Cyl}(U \rightarrow U_1)) \simeq$  is the map

$$(p_0^{-1}(b), 0) \xrightarrow{\text{homology eq.}} p_1^{-1}(f_1(b)) \rightarrow p_1^{-1}(U_1)$$

which is a homology equivalence.

This completes the proof.  $\square$

In the next proposition,  $\| - \|$  denotes the *fat* geometric realization functor, which is the same as the regular realization except that we do not quotient out by the degeneracy maps. The fat realization of a simplicial space  $E$  is homotopy equivalent to the usual realization, as long as the space is *good*, that is if  $E_k \hookrightarrow E_{k+1}$  is a closed cofibration, which is the case for all CW complexes.

We will use  $\|E\|^{(n)}$  to denote the image of  $\sqcup_{k \leq n} \Delta^k \times E_k$  in  $\|E\|$ . We will call  $\|E\|^{(n)}$  the “ $n$ -skeleton” of  $\|E\|$ . This allows to induct on  $n$  to make arguments about  $\|E\|$ , which is defined as the colimit of  $(\|E\|^{(0)} \rightarrow \|E\|^{(1)} \rightarrow \dots)$ , as we will see below. The reference for this material is §1 and the appendix in [Seg74].

**Proposition 4.7** (Proposition 4, [MS76]). *Suppose  $p : E \rightarrow B$  is a map of simplicial spaces such that  $E_k \rightarrow B_k$  is a homology-fibration for all  $k \geq 0$ . Suppose also that for any simplicial map  $\theta : [k] \rightarrow [l]$ , and any  $b \in B_l$ , the map  $p_l^{-1}(b) \rightarrow p_k^{-1}(\theta^*b)$  is a homology equivalence. Then the induced map  $p_* : \|E\| \rightarrow \|B\|$  is a homology-fibration.*

*Proof.* We will prove this by inducting on the skeletons. For  $k = 0$ , the homology-fibration  $E_0 \rightarrow B_0$  directly yields the homology-fibration  $\|E\|^{(0)} \rightarrow \|B\|^{(0)}$ .

Now suppose that the restricted induced map  $p_* : \|E\|^{(n-1)} \rightarrow \|B\|^{(n-1)}$  is a homology-fibration. The  $n$ -skeleton of  $\|E\|$  is given by the pushout

$$\begin{array}{ccc} \partial \Delta^k \times E_k & \hookrightarrow & \Delta^k \times E_k \\ \downarrow & & \downarrow \\ \|E\|^{(k-1)} & \hookrightarrow & \|E\|^{(k)} \end{array}$$

but this is just the double mapping cylinder

$$\text{Cyl}(\|E\|^{(k-1)} \leftarrow \partial \Delta^k \times E_k \rightarrow \Delta^k \times E_k)$$

and likewise for  $B$ , so we can assemble these maps into a commutative diagram

$$\begin{array}{ccccc} ||E||^{(k-1)} & \longleftarrow & \partial\Delta^k \times E_k & \xhookrightarrow{i_E} & \Delta^k \times E_k \\ & & \downarrow p_* & & \downarrow \text{id} \times p_k \\ & & \partial\Delta^k \times B_k & \xhookrightarrow{i_B} & \Delta^k \times B_k \end{array}$$

satisfying the assumptions of the previous proposition:

- (1) The vertical arrows are homology-fibrations. This is by the inductive hypothesis and the assumption that  $E_k \rightarrow B_k$  is a homology-fibration for all  $k$ .
- (2) For  $x \in \partial\Delta^k \times B_k$ , the top right arrow is a homeomorphism  $(\text{id} \times p_k)^{-1}(x) \xrightarrow{i_E} (\text{id} \times p_k)^{-1}(i_B(x))$ , hence a homology equivalence.
- (3) The top left arrow is a homology equivalence by the inductive hypothesis.

By the previous proposition, we therefore have a homology-fibration  $||E||^{(k)} \rightarrow ||B||^{(k)}$  for all  $k \geq 0$ , so  $||E|| \rightarrow ||B||$  is a homology-fibration.  $\square$

Finally, we come to the proof of Proposition 2 (3.1):

*Proof.* We will first use 4.7 with  $||E|| \simeq |E| = X_M$ , and  $||B|| \simeq |B| = BM$ . Recall that in this case  $E$  and  $B$  are the simplicial sets with  $E_k = X \times M^k$  and  $B_k = M^k$ . The map  $E_k \rightarrow B_k$  is just projection, hence a homology-fibration. For the second assumption, it suffices to check for  $\theta$  a coface/codegeneracy map. So let  $\theta : [k] \rightarrow [k+1]$  be a coface map. Then we have the diagram

$$\begin{array}{ccc} [k+1] & \longrightarrow & X \times M^{k+1} \xrightarrow{p_{k+1}} M^{k+1} \\ \uparrow \theta & & \downarrow \theta^* \\ [k] & \longrightarrow & X \times M^k \xrightarrow{p_k} M^k \end{array} \quad \begin{array}{c} \ni m = (m_1, \dots, m_{k+1}) \\ \downarrow \\ \theta^*(m) = (m_1, \dots, m_j m_{j+1}, \dots, m_{k+1}) \end{array}$$

but now  $\theta^* : p_{k+1}^{-1}(m) \rightarrow p_k^{-1}(\theta^*(m))$  is a map  $X \times \{m\} \rightarrow X \times \{\theta^*(m)\}$  induced by the monoid multiplication, hence a homology equivalence by the assumptions of Proposition 2. If  $\theta$  is a codegeneracy map the proof proceeds identically. Hence 4.7 implies that  $||E|| = X_M \rightarrow ||B|| = BM$  is a homology-fibration. By 4.2, since  $BM$  is a CW complex, it is paracompact and locally contractible, so  $X_M \rightarrow BM$  is a homology fibration, no hyphen.  $\square$

#### ACKNOWLEDGEMENTS

Thanks to the organizers of the Columbia DRP, and to Azélie Picot for being a wonderful, patient, and thoughtful mentor.

#### REFERENCES

- [MS76] D McDuff and G Segal. Homology fibrations and the “group-completion” theorem. *Invent. Math.*, 31(3):279–284, October 1976.
- [Seg68] Graeme Segal. Classifying spaces and spectral sequences. *Publ. Math. Inst. Hautes Etudes Sci.*, 34(1):105–112, January 1968.
- [Seg74] Graeme Segal. Categories and cohomology theories. *Topology*, 13(3):293–312, 1974.