is the ratio of polynomials containing only even powers of s. The interested reader is referred to the subject of *spectral factorization*.

When the driving noise is white and Gaussian, the process specified in eqn. (4.54) is an example of a scalar *Gauss-Markov* process. Higher order *vector Gauss-Markov* processes, such as the system of eqn. (4.55), result when the correlation function is more complex. Additional examples of simple Gauss-Markov process that are often useful in navigation system error model are discussed in Section 4.6.3.

4.6 State Models for Stochastic Processes

As motivation for the study of stochastic systems, consider the error dynamics for a single channel of a tangent plane INS, as depicted in Figure 4.4 and the following equation

$$\begin{bmatrix} \dot{x} \\ \dot{v} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -g \\ 0 & \frac{1}{R} & 0 \end{bmatrix} \begin{bmatrix} x \\ v \\ \phi \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_a \\ \varepsilon_g \end{bmatrix} \tag{4.56}$$

where $[x, v, \phi]$ represents error in the navigation state. The error model is driven by two sources of error: accelerometer error ϵ_a and gyro error ϵ_g. These errors originate in the sensors, but appear in the error model as process noise. The error at each source may take several forms: white noise, correlated (colored) noise, bias error, scale factor error, etc. After any deterministic error sources have been compensated or modeled, the remaining instrument errors will be modeled as random inputs. The use of random variables to model the error sources is reasonable, since although the analyst may not be able to specify the value of the various error sources at some future time, it is usually possible to specify the dynamic nature and statistics of the error sources as a function of time. It is also reasonable to restrict our attention to linear stochastic systems, since the primary focus of the stochastic analysis in navigation systems will be on linearized error dynamics.

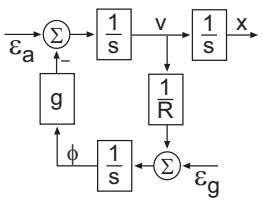

Figure 4.4: Block diagram for single channel INS error model.

The primary objectives for this section are to present: (1) how to account for stochastic sources of error in state space dynamic system modeling; and, (2) how the mean and covariance of system errors propagate in linear stochastic systems. This section presents various results for linear, state space systems with random inputs that will be used throughout the remainder of the book.

4.6.1 Standard Model

The model for a finite-dimensional linear continuous-time system with *stochastic inputs* can be represented as

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{w}(t) \qquad \mathbf{y}(t) = \mathbf{H}(t)\mathbf{x}(t) + \mathbf{v}(t) \tag{4.57}$$

where $\mathbf{w}(t)$ and $\mathbf{v}(t)$ are random variables. The random variable $\mathbf{w}$ is called the *process noise*. The random variable $\mathbf{v}$ is called the *measurement noise*. At each time, $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are also random variables. The designation random variable implies that although the value of the random variable at any time is not completely predictable, the statistics of the random variable may be known. Assuming that the model is constructed so that $\mathbf{v}(t)$ and $\mathbf{w}(t)$ are white, the mean and covariance of the random variables $\mathbf{w}(t)$ and $\mathbf{v}(t)$ will be denoted

$$\begin{aligned} \mu_{\mathbf{w}}(t) &= E\langle \mathbf{w}(t) \rangle = \mathbf{0} \quad \text{and} && (4.58)\\ cov\left(\mathbf{w}(t), \mathbf{w}(\tau)\right) &= \mathbf{Q}(t)\delta(t-\tau), && (4.59)\\ \mu_{\mathbf{v}}(t) &= E\langle \mathbf{v}(t) \rangle = \mathbf{0} \quad \text{and} && (4.60)\\ cov\left(\mathbf{v}(t), \mathbf{v}(\tau)\right) &= \mathbf{R}(t)\delta(t-\tau). && (4.61) \end{aligned}$$

In the analysis that follows, it will often be accurate (and convenient) to assume that the process and measurement noise are independent of the current and previous state

$$\begin{aligned} cov\left(\mathbf{w}(t), \mathbf{x}(\tau)\right) &= \mathbf{0} \text{ for } t \geq \tau && (4.62)\\ cov\left(\mathbf{v}(t), \mathbf{x}(\tau)\right) &= \mathbf{0} \text{ for } t \geq \tau && (4.63) \end{aligned}$$

and independent of each other

$$cov\left(\mathbf{w}(t), \mathbf{v}(\tau)\right) = \mathbf{0} \text{ for all } t, \tau \geq 0. \tag{4.64}$$

In most applications, the random processes $\mathbf{w}(t)$ and $\mathbf{v}(t)$ will be assumed to be Gaussian.

The linear discrete-time model is

$$\mathbf{x}_{k+1} = \mathbf{\Phi}_k \mathbf{x}_k + \mathbf{w}_k \qquad \mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k. \tag{4.65}$$

The mean and covariance of the random variables $\mathbf{w}_k$ and $\mathbf{v}_k$ will be denoted

$$\begin{aligned} \mu_{\mathbf{w}_k} &= E\langle \mathbf{w}_k \rangle = \mathbf{0} \quad \text{and} && (4.66) \\ cov\,(\mathbf{w}_k, \mathbf{w}_l) &= \mathbf{Q}\mathbf{d}_k \delta(k-l), && (4.67) \\ \mu_{\mathbf{v}_k} &= E\langle \mathbf{v}_k \rangle = \mathbf{0} \quad \text{and} && (4.68) \\ cov\,(\mathbf{v}_k, \mathbf{v}_l) &= \mathbf{R}_k \delta(k-l). && (4.69) \end{aligned}$$

Assumptions corresponding to eqns. (4.62–4.64) also apply to the discrete-time model.

In the analysis to follow, the time argument of the signals will typically be dropped, to simplify the notation.

4.6.2 Stochastic Systems and State Augmentation

Navigation system error analysis will often result in equations of the form

$$\begin{aligned} \dot{\mathbf{x}}_n &= \mathbf{F}_n(t)\mathbf{x}_n + \mathbf{G}_n(t)\epsilon && (4.70) \\ \mathbf{y} &= \mathbf{H}_n(t)\mathbf{x}_n + \boldsymbol{\alpha} && (4.71) \end{aligned}$$

where $\boldsymbol{\epsilon}$ and $\boldsymbol{\alpha}$ represent instrumentation error signals and $\mathbf{x}_n$ represents the error in the nominal navigation state. This and subsequent sections of this chapter will present a set of techniques which will allow the error signals $\boldsymbol{\epsilon}$ and $\boldsymbol{\alpha}$ to be modeled as the outputs of linear dynamic systems

$$\begin{aligned} \dot{\mathbf{x}}_\epsilon &= \mathbf{F}_\epsilon \mathbf{x}_\epsilon + \mathbf{G}_\epsilon \boldsymbol{\omega}_\epsilon && (4.72) \\ \boldsymbol{\epsilon} &= \mathbf{H}_\epsilon \mathbf{x}_\epsilon + \boldsymbol{\nu}_\epsilon && (4.73) \end{aligned}$$

and

$$\begin{aligned} \dot{\mathbf{x}}_\alpha &= \mathbf{F}_\alpha \mathbf{x}_\alpha + \mathbf{G}_\alpha \boldsymbol{\omega}_\alpha && (4.74) \\ \boldsymbol{\alpha} &= \mathbf{H}_\alpha \mathbf{x}_\alpha + \boldsymbol{\nu}_\alpha && (4.75) \end{aligned}$$

with the noise processes $\boldsymbol{\omega}_\epsilon(t), \boldsymbol{\nu}_\epsilon(t), \boldsymbol{\omega}_\alpha(t)$ and $\boldsymbol{\nu}_\alpha(t)$ being accurately modeled as white noise processes. By the process of *state augmentation*, equations (4.70-4.75) can be combined into the state space error model

$$\begin{aligned} \dot{\mathbf{x}}_a &= \mathbf{F}_a(t)\mathbf{x}_a(t) + \mathbf{G}_a\boldsymbol{\omega}_a(t) && (4.76) \\ &= \begin{bmatrix} \mathbf{F}_n(t) & \mathbf{G}_n(t)\mathbf{H}_\epsilon & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_\epsilon & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{F}_\alpha \end{bmatrix} \mathbf{x}_a \\ &\quad + \begin{bmatrix} \mathbf{G}_n(t) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_\epsilon & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}_\alpha \end{bmatrix} \begin{bmatrix} \boldsymbol{\nu}_\epsilon \\ \boldsymbol{\omega}_\epsilon \\ \boldsymbol{\omega}_\alpha \end{bmatrix} \end{aligned}$$

which is in the form of eqn. (4.57) and driven only by white noise processes. In these equations, the augmented state is defined as $\mathbf{x}_a = [\mathbf{x}_n^\top, \mathbf{x}_\epsilon^\top, \mathbf{x}_\alpha^\top]^\top$. The measurements of the augmented system are modeled as

$$\begin{aligned} \mathbf{y} &= \mathbf{H}_a(t)\mathbf{x}_a + \boldsymbol{\nu}_\alpha & (4.77) \\ &= \begin{bmatrix} \mathbf{H}_n(t) & \mathbf{0} & \mathbf{H}_\alpha \end{bmatrix} \mathbf{x}_a + \boldsymbol{\nu}_\alpha & (4.78) \end{aligned}$$

which are corrupted only by additive white noise.

For the state augmented model to be an accurate characterization of the actual system, the state space parameters $(\mathbf{F}_a, \mathbf{G}_a, \mathbf{H}_a)$ corresponding to the appended error models and the statistics of the driving noise processes must be accurately specified. Detailed examples of augmented state models are presented in Sections 4.6.3 and 4.9. Section 4.6.3 also discusses several basic building blocks of the state augmentation process.

4.6.3 Gauss-Markov Processes

For the finite dimensional state space system

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{w}(t) \qquad \mathbf{y}(t) = \mathbf{H}(t)\mathbf{x}(t) + \mathbf{v}(t) \tag{4.79}$$

where $\mathbf{w}(t)$ and $\mathbf{v}(t)$ are stochastic processes, if both $\mathbf{w}$ and $\mathbf{v}$ are Gaussian random processes, then the system is an example of a *Gauss-Markov process*. Since any linear operation performed on a Gaussian random variable results in a Gaussian random variable, the state $\mathbf{x}(t)$ and system output $\mathbf{y}(t)$ will be Gaussian random variables. This is a very beneficial property as the Normal distribution is completely described by two parameters (i.e., the mean and covariance) which are straightforward to propagate through time, as is described in Section 4.6.4 and 4.6.5. The purpose of this section is to present several specific types of Gauss-Markov processes that are useful for error modeling.

4.6.3.1 Random Constants

Some portions of instrumentation error (e.g., scale factor) can be accurately represented as constant (but unknown) random variables. If some portion of the constant error is known, then it can be compensated for and the remaining error can be modeled as an unknown constant. An unknown constant is modeled as

$$\dot{x} = 0, \text{ with } P_x(0) = var(x(0), x(0)). \tag{4.80}$$

This model states that the variable x is not changing and has a known initial variance $P_x(0)$.

Example 4.16 *An acceleration measurement $\tilde{a}$ is assumed to be corrupted by an unknown constant bias b_a:*

$$\tilde{a} = a + b_a.$$

The bias is specified to be zero mean with an initial error variance of $var(b_a) = \sigma_b^2$.

1. *What is an appropriate state space model for the accelerometer output error $\epsilon_a(t)$?*
2. *What is the appropriate state space model when this accelerometer error model is augmented to the tangent plane single channel error model of eqn. (3.91)? Assume that the constant bias is the only form of accelerometer error and that $\epsilon_g = 0$.*

Since the bias is assumed to be constant, an appropriate model is

$$\dot{b}_a(t) = 0 \tag{4.81}$$

with initial condition $b_a(t) = 0$ and $P_{b_a} = \sigma_b^2$. Note that there is no process noise in the bias dynamic equation. Since no other forms of error are being modeled, $\epsilon_a = b_a$. Therefore, the augmented single channel error model becomes

$$\begin{bmatrix} \dot{x} \\ \dot{v} \\ \dot{\phi} \\ \dot{b}_a \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -g & 1 \\ 0 & \frac{1}{R} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \\ \phi \\ b_a \end{bmatrix}. \tag{4.82}$$

Note that for eqn. (4.82), if either a position or velocity measurement were available, an observability analysis will show that the system is not observable. The tilt and accelerometer bias states cannot be distinguished. See Exercise 4.15. △

4.6.3.2 Brownian Motion (Random Walk) Processes

The Gauss-Markov process $x(t)$ defined by

$$x(t) = \int_0^t \omega(q)dq \tag{4.83}$$

with $P_x(0) = 0$ and $R_\omega(t) = \sigma_\omega^2 \delta(t)$ is called a Brownian motion or random walk process. The mean of x can be calculated as

$$\begin{aligned} \mu_x(t) &= E\langle x(t) \rangle = E\left\langle \int_0^t \omega(q)dq \right\rangle \\ &= \int_0^t E\langle w(q) \rangle dq = 0. \end{aligned}$$

By the definition of variance in eqn. (4.22), the covariance function for x can be calculated as

$$\begin{aligned} cov_x(t,\tau) &= E\langle x(t)x(\tau)\rangle = E\left\langle \int_0^t \omega(\lambda)d\lambda \int_0^\tau \omega(\zeta)d\zeta \right\rangle \\ &= \int_0^t \int_0^\tau E\langle \omega(\lambda)\omega(\zeta)\rangle d\lambda d\zeta = \int_0^\tau \int_0^t \sigma_\omega^2 \delta(\lambda-\zeta)d\zeta d\lambda \\ &= \int_0^\tau \sigma_\omega^2 d\lambda, \text{ for } \lambda < t \\ &= \sigma_\omega^2 \min(t,\tau), \end{aligned} \tag{4.84}$$

which yields

$$var_x(t) = \sigma_\omega^2 t. \tag{4.85}$$

In navigation modeling, it is common to integrate the output of a sensor to determine particular navigation quantities. Examples are integrating an accelerometer output to determine velocity or integrating an angular rate to determine an angle, as in Example 4.2. If it is accurate to consider the sensor error as white random noise, then the resulting error equations will result in a random walk model.

As with white noise, the units of the random walk process often cause confusion. Note that if x is measured in degrees, then $var_x(t)$ has units of deg^2. Therefore, the units of σ_ω are $\frac{\text{deg}}{\sqrt{s}} = \frac{\text{deg}/s}{\sqrt{Hz}}$ which makes sense given that σ_ω is the PSD of a white angular rate noise process.

One often quoted measure of sensor accuracy is the random walk parameter. For example, a certain gyro might list its random walk parameter to be $4.0\,\text{deg}/\sqrt{hr}$. By eqn. (4.84), the random walk parameter for a sensor quantifies the rate of growth of the integrated (properly compensated) sensor output as a function of time. The $\sqrt{hr}$ in the denominator of the specification reflects that the standard deviation and variance of the random walk variable grow with the square root of time and linearly with time, respectively. When the specification states that the angle random walk parameter is $N\frac{\text{deg}}{\sqrt{hr}}$, then $\sigma_\omega = \frac{N}{60}\frac{\text{deg}/s}{\sqrt{Hz}}$.

For a random walk process, the state space model is

$$\dot{x} = \nu \tag{4.86}$$

where $E\langle x(0)\rangle = 0$, $var(x(0)) = P_x(0) = 0$, and $S_\nu(j\omega) = \sigma_\nu^2$. The transfer function corresponding to eqns. (4.83) and (4.86) is $H(s) = \frac{1}{s}$, so by eqn. (4.49), the PSD of x is

$$S_x(j\omega) = \frac{\sigma_\nu^2}{\omega^2}.$$

Example 4.17 *An accelerometer output is known to be in error by an unknown slowly time-varying bias b_a. Assume that the 'turn-on' bias is*

accurately known and accounted for in the accelerometer calibration; and, that it is accurate to model the residual time-varying bias error as a random walk:

$$\begin{aligned} \delta b_a(t) &= \int_0^t \omega_b(\tau) d\tau && (4.87) \\ var(b(0)) = P_b(0) &= 0 && (4.88) \\ var(\omega_b(t), \omega_b(\tau)) &= Q_\omega \delta(t-\tau) && (4.89) \end{aligned}$$

where ω_b represents Gaussian white noise and Q_ω is the PSD of ω_b. The parameter Q_ω is specified by the manufacturer.

Appending the random walk bias model to a single channel of the inertial frame INS error model:

$$\begin{bmatrix} \delta\dot{p} \\ \delta\dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta p \\ \delta v \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta b_a$$

yields the following augmented state model

$$\begin{bmatrix} \delta\dot{p} \\ \delta\dot{v} \\ \delta\dot{b}_a \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta p \\ \delta v \\ \delta b_a \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega_b(t)$$

$$\mathbf{P_x}(0) = \begin{bmatrix} \sigma_{pp}^2 & \sigma_{pv}^2 & 0 \\ \sigma_{vp}^2 & \sigma_{vv}^2 & 0 \\ 0 & 0 & P_b(0) \end{bmatrix}.$$

This example has made the unrealistic assumption that the turn-on bias is exactly known so that a proper random walk model could be used. In a realistic situation where the initial bias is not perfectly known, a constant plus random walk error model would be identical to the equations shown above with $P_b(0)$ specified to account for the variance in the initial bias estimate. △

4.6.3.3 Scalar Gauss-Markov Process

The scalar Gauss-Markov process refers to the special case of eqn. (4.79) where the state, input, and output are each scalar variables:

$$\left.\begin{aligned} \dot{x} &= -\tfrac{1}{\tau} x + Gw \\ y &= Hx. \end{aligned}\right\} \quad (4.90)$$

where w is Gaussian white noise with PSD denoted by σ^2. In eqn. (4.90), the parameter τ is the *correlation time*. This process has already been used in Examples 4.14 and 4.15.

Eqn. (4.90) is written in its most general form. The PSD of y is

$$S_y(j\omega) = \frac{HG\sigma^2}{\omega^2 + \left(\frac{1}{\tau}\right)^2}$$

which shows that once the correlation time τ is determined, the value of $S_y(j\omega)|_{\omega=0}$ determines the product of the parameters H, G, and σ. Often, one or two of these three parameters is arbitrarily set to one with the remaining parameter values selected to achieve the desired value at $\omega = 0$.

4.6.3.4 Compound Augmented States

The examples of the previous sections each contained a single type of error. In more realistic situations, several forms of error may be present. This section presents an example involving compound error models augmented to simplified INS error equations.

Example 4.18 *The ideal dynamics of a one dimensional INS implemented in inertial space are*

$$\dot{p} = v \tag{4.91}$$
$$\dot{v} = a. \tag{4.92}$$

Two sensors are available. The first sensor measures position. The second sensor measures acceleration.

The position measurement model is

$$\tilde{y}(t) = p(t) + \zeta(t) + \mu(t) \tag{4.93}$$

where $\mu(t)$ represents Gaussian white noise and $\zeta(t)$ is the scalar Gauss-Markov process

$$\dot{\zeta} = -\beta\zeta + \omega_\zeta.$$

The implemented INS is described as

$$\dot{\hat{p}} = \hat{v} \tag{4.94}$$
$$\dot{\hat{v}} = \tilde{a}, \tag{4.95}$$

where the accelerometer model is

$$\tilde{a}(t) = (1 - \delta k)a(t) - \delta b_a(t) - \nu_a(t)$$

where δk is a Gaussian random-constant scale-factor error, δb_a is a constant plus random walk Gaussian bias error, and ν_a is Gaussian white noise. The predicted output at any time is calculated as

$$\hat{y}(t) = \hat{p}(t). \tag{4.96}$$

The differential equations for the error variables are found by subtracting eqns. (4.94-4.95) from eqns. (4.91-4.92)

$$\begin{aligned} \delta\dot{p} &= \delta v \\ \delta\dot{v} &= a(t)\delta k + \delta b_a(t) + \nu_a(t) \end{aligned}$$

with each error term defined as $\delta x = x - \hat{x}$. The model for the residual measurement $\delta y = \tilde{y} - \hat{y}$ is defined by subtracting eqn. (4.96) from eqn. (4.93)

$$\delta y(t) = \delta p(t) + \zeta(t) + \mu(t).$$

The augmented error state equations are then defined to be

$$\begin{bmatrix} \delta\dot{p} \\ \delta\dot{v} \\ \delta\dot{b}_a \\ \delta\dot{k} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & a & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta \end{bmatrix} \begin{bmatrix} \delta p \\ \delta v \\ \delta b_a \\ \delta k \\ \zeta \end{bmatrix} + \begin{bmatrix} 0 \\ \nu_a \\ \omega_b \\ 0 \\ \omega_\zeta \end{bmatrix}$$

$$\delta y = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \end{bmatrix} \delta\mathbf{x} + \mu.$$

These error equations have the form of eqn. (4.79) with Gaussian white noise inputs. This model structure is important for the optimal state estimation methods of Chapter 5.

This example has considered a hypothetical one dimensional INS with only two sensors. The state of the original INS was two and three error states were appended. In realistic navigation systems in a three dimensional world with many more sensed quantities, it should be clear that the dimension of the state of the error model can become quite large. △

When designing a system that will be implemented in a real-time application, there is usually a tradeoff required between reasonable cost and computation time and the desire for accurate modeling. Even in the above single axis example, the dimension of the augmented error state vector $\delta\mathbf{x}$ is large (i.e., 5) relative to the dimension of the original state $\mathbf{x} = [p, v]^\top$ of the INS (i.e., 2), and several more error states could be included in the quest for modeling accuracy. In realistic applications, the dimension of the augmented state vector is potentially quite large, often too large for a real-time system. Therefore, the design may result in two models. The most complex model that accounts for all error states considered to be significant is referred to as the *truth model*. A simplified *Design model* may be constructed from the truth model by eliminating or combining certain state variables. The art is to develop a design model small enough to allow its use in practical realtime implementations without paying a significant performance penalty in terms of state estimation accuracy. The methodology for analyzing the performance tradeoffs is discussed in Chapter 6.

4.6.4 Time-propagation of the Mean

From eqn. (4.65), if the mean of the state vector is known at some time k_0 and $E\langle \mathbf{w}_k \rangle = 0$, then the mean can be propagated forward according to

$$\begin{aligned} E\langle \mathbf{x}_{k+1} \rangle &= E\langle \mathbf{\Phi}_k \mathbf{x}_k \rangle + E\langle \mathbf{w}_k \rangle \\ &= \mathbf{\Phi}_k E\langle \mathbf{x}_k \rangle + E\langle \mathbf{w}_k \rangle \\ E\langle \mathbf{x}_{k+1} \rangle &= \mathbf{\Phi}_k E\langle \mathbf{x}_k \rangle. \end{aligned} \tag{4.97}$$

Intuitively, this formula states that since nothing is known a priori about the specific realization of the process noise for a given experiment, the mean of the state is propagated according to the state model:

$$\boldsymbol{\mu}_{\mathbf{x}_{k+1}} = \mathbf{\Phi}_k \boldsymbol{\mu}_{\mathbf{x}_k} \qquad \boldsymbol{\mu}_{\mathbf{y}_k} = \mathbf{H}_k \boldsymbol{\mu}_{\mathbf{x}_k}. \tag{4.98}$$

The equation for $\boldsymbol{\mu}_{\mathbf{y}}$ is found by analysis similar to that for $\boldsymbol{\mu}_{\mathbf{x}_k}$.

4.6.5 Time-propagation of the Variance

The discrete-time error state covariance matrix is defined as

$$\mathbf{P}_k = E\langle (\mathbf{x}_k - \boldsymbol{\mu}_k)(\mathbf{x}_k - \boldsymbol{\mu}_k)^\top \rangle,$$

where $\boldsymbol{\mu}_k = E\langle \mathbf{x}_k \rangle$. The state covariance at time $k+1$

$$\mathbf{P}_{k+1} = E\left\langle (\mathbf{x}_{k+1} - \boldsymbol{\mu}_{k+1})(\mathbf{x}_{k+1} - \boldsymbol{\mu}_{k+1})^\top \right\rangle$$

can be simplified using eqns. (4.65) and (4.98) to determine eqn. (4.99) for propagating the state error covariance through time:

$$\begin{aligned} \mathbf{P}_{k+1} &= E\left\langle \left(\mathbf{\Phi}_k(\mathbf{x}_k - \boldsymbol{\mu}_k) + \mathbf{w}_k\right)\left((\mathbf{x}_k - \boldsymbol{\mu}_k)^\top \mathbf{\Phi}_k^\top + \mathbf{w}_k^\top\right)\right\rangle \\ &= E\left\langle \mathbf{\Phi}_k(\mathbf{x}_k - \boldsymbol{\mu}_k)(\mathbf{x}_k - \boldsymbol{\mu}_k)^\top \mathbf{\Phi}_k^\top + \mathbf{w}_k \mathbf{w}_k^\top \right. \\ &\quad \left. + \mathbf{w}_k(\mathbf{x}_k - \boldsymbol{\mu}_k)^\top \mathbf{\Phi}_k^\top + \mathbf{\Phi}_k(\mathbf{x}_k - \boldsymbol{\mu}_k)\mathbf{w}_k^\top \right\rangle \\ &= \mathbf{\Phi}_k E\left\langle (\mathbf{x}_k - \boldsymbol{\mu}_k)(\mathbf{x}_k - \boldsymbol{\mu}_k)^\top \right\rangle \mathbf{\Phi}_k^\top + E\left\langle \mathbf{w}_k \mathbf{w}_k^\top \right\rangle \\ \mathbf{P}_{k+1} &= \mathbf{\Phi}_k \mathbf{P}_k \mathbf{\Phi}_k^\top + \mathbf{Q}\mathbf{d}_k. \end{aligned} \tag{4.99}$$

Note that eqns. (4.97) and (4.99) are valid for any zero mean noise process satisfying the assumption of eqn. (4.62). No other assumptions were used in the derivations. If $\mathbf{w}_k$ and $\mathbf{v}_k$ happen to be zero mean Gaussian processes, then $\mathbf{x}_k$ is also a Gaussian stochastic process with mean and variance given by eqns. (4.97) and (4.99).

By limiting arguments (see Chapter 4 in [58]), the covariance propagation for the continuous-time system

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}\boldsymbol{\omega}$$

is described by

$$\dot{\mathbf{P}} = \mathbf{F}\mathbf{P} + \mathbf{P}\mathbf{F}^\top + \mathbf{G}\mathbf{Q}\mathbf{G}^\top. \tag{4.100}$$

Example 4.19 *Consider the scalar Gauss-Markov process with*

$$\dot{x}(t) = -\beta x(t) + \omega(t)$$

with $\beta > 0$ where the PSD of the white noise process ω is $Q > 0$. Letting $var(x) = P$,

$$\begin{aligned} \dot{P} &= -\beta P - P\beta + Q \\ &= -2\beta P + Q. \end{aligned} \tag{4.101}$$

Eqn. (4.101) has the solution

$$P(t) = \frac{Q}{2\beta} + \left(P(0) - \frac{Q}{2\beta} \right) e^{-2\beta t} \tag{4.102}$$

which can be verified by direct substitution. In steady state, $P(\infty) = \frac{Q}{2\beta}$, which is useful in applications involving scalar Gauss-Markov processes when the steady-state covariance $P(\infty)$ and the correlation time $\frac{1}{\beta}$ are known and Q is to be determined. △

4.7 Discrete-time Equivalent Models

When the dynamics of the system of interest evolve in continuous time, but analysis and implementation are more convenient in discrete-time, we will require a means for determining a discrete-time model in the form of eqn. (4.65) which is equivalent to eqn. (4.57) at the discrete-time instants $t_k = kT$. Specification of the equivalent discrete-time model requires computation of the discrete-time state transition matrix Φ_k for eqn. (4.65) and the process noise covariance matrix $\mathbf{Qd}$ for eqn. (4.67). These computations are discussed in the following two subsections for time invariant systems. A method to compute $\mathbf{\Phi}$ and $\mathbf{Qd}$ over longer periods of time for which $\mathbf{F}$ or $\mathbf{Q}$ may not be constant is discussed in Section 7.2.5.2.

4.7.1 Calculation of Φ_k from $\mathbf{F}(t)$

For equivalence at the sampling instants when $\mathbf{F}$ is a constant matrix, $\mathbf{\Phi}$ can be determined as in eqn. (3.61):

$$\begin{aligned} \mathbf{\Phi}(t) = \mathbf{\Phi}(t,0) &= e^{\mathbf{F}\ t}, \\ \mathbf{\Phi}(\tau, t) &= e^{\mathbf{F}(\tau - t)}, \text{ and} \\ \mathbf{\Phi}_k &= e^{\mathbf{F}T} \end{aligned} \tag{4.103}$$

where $T = t_k - t_{k-1}$ is the sample period. Methods for computing the matrix exponential are discussed in Section B.12.

Example 4.20 *Assume that for a system of interest,*

$$\mathbf{F} = \begin{bmatrix} \mathbf{0} & \mathbf{F}_{12} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{F}_{23} \\ \mathbf{0} & \mathbf{0} & \mathbf{F}_{33} \end{bmatrix} \tag{4.104}$$

and the submatrices denoted by $\mathbf{F}_{12}, \mathbf{F}_{23}$, *and* $\mathbf{F}_{33}$ *are constant over the interval* $t \in [t_1, t_2]$. *Then,* $\mathbf{\Phi}(t_2, t_1) = e^{\mathbf{F}T}$ *where* $T = t_2 - t_1$. *Expanding the Taylor series of*

$$e^{\mathbf{F}t} = \mathbf{I} + \mathbf{F}t + \frac{1}{2}(\mathbf{F}t)^2 \ldots$$

is straightforward, but tedious. The result is

$$\mathbf{\Phi}(t_2, t_1) = \begin{bmatrix} \mathbf{I} & \mathbf{F}_{12}T_2 & \mathbf{F}_{12}\mathbf{F}_{23}\int_{t_1}^{t_2}\int_{t_1}^{t} e^{\mathbf{F}_{33}s}dsdt \\ \mathbf{0} & \mathbf{I} & \mathbf{F}_{23}\int_{t_1}^{t_2} e^{\mathbf{F}_{33}s}ds \\ \mathbf{0} & \mathbf{0} & e^{\mathbf{F}_{33}T_2} \end{bmatrix} \tag{4.105}$$

which is the closed form solution. When $\mathbf{F}_{33}$ *can be approximated as zero, the following reduction results*

$$\mathbf{\Phi}(t_2, t_1) = \begin{bmatrix} \mathbf{I} & \mathbf{F}_{12}T_2 & \frac{1}{2}\mathbf{F}_{12}\mathbf{F}_{23}T_2^2 \\ \mathbf{0} & \mathbf{I} & \mathbf{F}_{23}T_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}. \tag{4.106}$$

Eqn. (4.104) corresponds to the $\mathbf{F}$ *matrix for certain INS error models after simplification.* △

If the state transition matrix is required for a time interval $[T_{m-1}, T_m]$ of duration long enough that the $\mathbf{F}$ matrix cannot be considered constant, then it may be possible to proceed by subdividing the interval. When the interval can be decomposed into subintervals $T_{m-1} < t_1 < t_2 < \ldots < T_m$, where $\tau = max(t_n - t_{n-1})$ and the $\mathbf{F}$ matrix can be considered constant over intervals of duration less than τ, then by the properties of state transition matrices,

$$\mathbf{\Phi}(t_n, T_{m-1}) = \mathbf{\Phi}(t_n, t_{n-1})\mathbf{\Phi}(t_{n-1}, T_{m-1}) \tag{4.107}$$

where $\mathbf{\Phi}(t_n, t_{n-1})$ is defined as in eqn. (4.103) with $\mathbf{F}$ considered as constant for $t \in [t_n, t_{n-1})$. The transition matrix $\mathbf{\Phi}(t_{n-1}, T_{m-1})$ is defined from previous iterations of eqn. (4.107) where the iteration is initialized at $t = T_{m-1}$ with $\mathbf{\Phi}(T_{m-1}, T_{m-1}) = \mathbf{I}$. The iteration continues for the interval of time propagation to yield $\mathbf{\Phi}(T_m, T_{m-1})$.

4.7.2 Calculation of $\mathbf{Qd}_k$ from $\mathbf{Q}(t)$

For equivalence at the sampling instants, the matrix $\mathbf{Qd}_k$ must account for the integrated effect of $\mathbf{w}(t)$ by the system dynamics over each sampling

period. Therefore, by integration of eqn. (4.57) and comparison with eqn. (4.65), $\mathbf{w}_k$ must satisfy

$$\mathbf{x}(t_{k+1}) = e^{\mathbf{F}(t_{k+1}-t_k)}\mathbf{x}(t_k) + \int_{t_k}^{t_{k+1}} e^{\mathbf{F}(t_{k+1}-\lambda)}\mathbf{G}(\lambda)\mathbf{w}(\lambda)d\lambda. \tag{4.108}$$

Comparison with eqn. (4.65) leads to the definition:

$$\mathbf{w}_k = \int_{t_k}^{t_{k+1}} e^{\mathbf{F}(t_{k+1}-\lambda)}\mathbf{G}(\lambda)\mathbf{w}(\lambda)d\lambda. \tag{4.109}$$

Then, with the assumption that $\mathbf{w}(t)$ is a white noise process, we can compute $\mathbf{Qd}_k = cov(\mathbf{w_k})$ as follows:

$$\begin{aligned} &E\left\langle \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \mathbf{\Phi}(t_{k+1},s)\mathbf{G}(s)\mathbf{w}(s)\mathbf{w}^\top(\tau)\mathbf{G}^\top(\tau)\mathbf{\Phi}(t_{k+1},\tau)^\top d\tau ds \right\rangle \\ &= \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \mathbf{\Phi}(t_{k+1},s)\mathbf{G}(s)E\left\langle \mathbf{w}(s)\mathbf{w}^\top(\tau)\right\rangle \mathbf{G}^\top(\tau)\mathbf{\Phi}(t_{k+1},\tau)^\top d\tau ds \\ &= \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \mathbf{\Phi}(t_{k+1},s)\mathbf{G}(s)\mathbf{Q}(s)\delta(s-\tau)\mathbf{G}^\top(\tau)\mathbf{\Phi}(t_{k+1},\tau)^\top d\tau ds. \end{aligned}$$

Therefore, the solution is

$$\mathbf{Qd}_k = \int_{t_k}^{t_{k+1}} \mathbf{\Phi}(t_{k+1},s)\mathbf{G}(s)\mathbf{Q}(s)\mathbf{G}^\top(s)\mathbf{\Phi}(t_{k+1},s)^\top ds. \tag{4.110}$$

If $\mathbf{F}$ and $\mathbf{Q}$ are both time invariant, then $\mathbf{Qd}$ is also time invariant.

A common approximate solution to eqn. (4.110) is

$$\mathbf{Qd} \approx \mathbf{GQG}^\top T \tag{4.111}$$

which is accurate only when the eigenvalues of $\mathbf{F}$ are very small relative to the sampling period T (i.e., $\|\mathbf{F}T\| \ll 1$). This approximation does not account for any of the correlations between the components of the driving noise $\mathbf{w}_k$ that develop over the course of a sampling period due to the integration of the continuous-time driving noise through the state dynamics. The reader should compare the results from Examples 4.21, 4.22, and 4.23.

Example 4.21 *For the double integrator system with*

$$\mathbf{F} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{GQG}^\top = \begin{bmatrix} 0.00 & 0.00 \\ 0.00 & 0.01 \end{bmatrix},$$

compute $\mathbf{\Phi}$ *and* $\mathbf{Qd}$ *using eqns. (4.103) and (4.111) for* $T = 1$.

Using the Matlab function 'expm' we obtain $\mathbf{\Phi} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. *Using eqs. (4.111) we obtain*

$$\mathbf{Qd} \approx \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \times 10^{-2}. \tag{4.112}$$

△

Given the state of computing power available, there is no reason why more accurate approximations for $\mathbf{Qd}$ are not used. A few methods for calculating the solution to eqn. (4.110) are described in the following subsections. Each of the following methods assumes that $\mathbf{F}$ and $\mathbf{Q}$ are both time invariant. If $\mathbf{F}$ and $\mathbf{Q}$ are slowly time varying, then these methods could be used to determine approximate solutions by recalculating $\mathbf{Q}_d$ over each sampling interval.

4.7.2.1 Solution by Matrix Exponentials

It is shown in [133] that the exponential of the $2n \times 2n$ matrix

$$\mathbf{\Xi} = \begin{bmatrix} -\mathbf{F} & \mathbf{G}\mathbf{Q}\mathbf{G}^\top \\ 0 & \mathbf{F}^\top \end{bmatrix} T \tag{4.113}$$

is

$$\mathbf{\Upsilon} = e^{\mathbf{\Xi}} = \begin{bmatrix} -\mathbf{D} & \mathbf{\Phi}^{-1}\mathbf{Qd}_w \\ 0 & \mathbf{\Phi}^\top \end{bmatrix} \tag{4.114}$$

where $\mathbf{D}$ is a dummy variable representing a portion of the answer that will not be used. Based on the expressions in the second column of eqn. (4.114), $\mathbf{\Phi}$ and $\mathbf{Qd}_w$ are calculated as

$$\mathbf{\Phi} = \mathbf{\Upsilon}[(n+1:2n),(n+1):2n]^\top \tag{4.115}$$

$$\mathbf{Qd} = \mathbf{\Phi}\mathbf{\Upsilon}[(1:n),(n+1):2n] \tag{4.116}$$

where $\mathbf{\Upsilon}[(i:j),(k:l)]$ denotes the the sub-matrix of $\mathbf{\Upsilon}$ composed of the i through j-th rows and k through l-th columns of matrix $\mathbf{\Upsilon}$.

Example 4.22 *For the simple system defined in Example 4.21, after constructing $\mathbf{\Xi}$ and computing its matrix exponential, we have*

$$\mathbf{\Upsilon} = \begin{bmatrix} 1 & -1 & -\frac{1}{6}\times 10^{-2} & -\frac{1}{2}\times 10^{-2} \\ 0 & 1 & \frac{1}{2}\times 10^{-2} & 1.0\times 10^{-2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \tag{4.117}$$

Therefore, using the lower right 2×2 block, we have that $\mathbf{\Phi} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Using the upper right 2×2 block we have that

$$\mathbf{Qd} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{6} & -\frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \times 10^{-2} = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \times 10^{-2}$$

which is significantly different from the result in eqn. (4.112) that was obtained from the approximate solution in eqn. (4.111). △

4.7.2.2 Solution by Taylor Series

If the Taylor series approximation for $\mathbf{\Phi}$

$$\mathbf{\Phi} = e^{\mathbf{F}T} = I + \mathbf{F}T + \frac{1}{2}(\mathbf{F}T)^2 + \frac{1}{3!}(\mathbf{F}T)^3 + \dots \tag{4.118}$$

is substituted into eqn. (4.110), using $k = 0$ to simplify the notation, the result accurate to third order in $\mathbf{F}$ and 4th order in T is

$$\begin{aligned}\mathbf{Qd} \approx\ & \mathbf{Q}T + \left(\mathbf{FQ} + \mathbf{QF}^\top\right)\frac{T^2}{2} + \left(\mathbf{F}^2\mathbf{Q} + 2\mathbf{FQF}^\top + \mathbf{Q}\left(\mathbf{F}^\top\right)^2\right)\frac{T^3}{6} \\ & + \left(\mathbf{F}^3\mathbf{Q} + 3\mathbf{F}^2\mathbf{QF}^\top + 3\mathbf{FQ}\left(\mathbf{F}^\top\right)^2 + \mathbf{Q}\left(\mathbf{F}^\top\right)^3\right)\frac{T^4}{24}.\end{aligned} \tag{4.119}$$

Although the Taylor series approach is approximate for some implementations, eqn. (4.119) is sometimes a convenient means to identify a closed form solution for either $\mathbf{\Phi}$ or $\mathbf{Qd}$. Even an approximate solution in closed form such as eqn. (4.119) is useful in situations where $\mathbf{F}$ is time-dependent and eqns. (4.114–4.116) cannot be solved on-line.

Example 4.23 *For the simple system defined in Example 4.21,* $\mathbf{F}^n = \mathbf{0}$ *for* $n \geq 2$*; therefore, using eqn. (4.118)*

$$\mathbf{\Phi} = \mathbf{I} + \mathbf{F}T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Simplifying eqn. (4.119) for this specific example gives

$$\mathbf{Qd} = \mathbf{Q}T + \left(\mathbf{FQ} + \mathbf{QF}^\top\right)\frac{T^2}{2} + 2\mathbf{FQF}^\top\frac{T^3}{6}$$

which provides the same result as in Example 4.22. △

4.8 Linear State Estimation

For deterministic systems, Section 3.6 discussed the problem of state estimation. This section considers state estimation for stochastic, linear, discrete-time state space systems

$$\begin{aligned}\mathbf{x}_{k+1} &= \mathbf{\Phi}_k\mathbf{x}_k + \mathbf{\Gamma}_k\mathbf{u}_k + \boldsymbol{\omega}_k \\ \mathbf{y}_k &= \mathbf{H}_k\mathbf{x}_k + \boldsymbol{\nu}_k\end{aligned}$$

where $\mathbf{u}_k$ is a known signal. The standard notation and assumptions stated in Section 4.6.1 apply.