Electrodynamics Homework II

1. Maths behind magnetic dipole moment (20 marks)

   Consider infinitesimal elements $\vec{dl}'$ of a closed path are located at position vectors $\vec{r}'$. The observation point, where we would like to know the magnetic potential, is at point $\vec{r}$. Prove the following identity

   $$\oint (\hat{r} \cdot \vec{r}')\vec{dl}' = -\hat{r} \times \int \vec{da},$$

   connecting the integral over the closed loop with the area spanned by the loop.

2. Another uniqueness theorem (30 marks)

   Prove that electric field is uniquely defined within a volume if we know charge density in this volume as well as potential V and its normal derivative $\frac{\partial V}{\partial n}$ on the boundary.

3. Something relaxing (20 marks)

   Two positive point charges q_A and q_B (masses m_A and m_B) are at rest, held together by a massless string of length a. Now the string is cut, and the particles fly off in opposite directions. How fast is each one going when they are far apart?

4. Computational problem: electric field inside an atom (30 marks)

   Recall quantum description of the hydrogen atom. Stationary proton is located at the origin and the electron is described by a wave function $\psi_{nlm}(r, \theta, \phi)$. Assume that the atom is in the ground state $n = 1, l = 0, m = 0$ and its charge density has two components: positive point charge e at the origin (proton) and negative charge distribution $\rho_e(r, \theta, \phi) = -e|\psi_{100}(r, \theta, \phi)|^2$ describing the electron. Explain qualitatively the electric field of this configuration. Then prepare precise plot of electric field as a function of radial distance.

A few comments:

- Please send your answers in a single file, e.g. photos of calculations or a report.
- Deadline: 17 June 2022.
- Plagiarism: I highly encourage you to discuss these topics, by writing down the answers / calculations should be done individually

Guo Yiming PHY2009481

Question 1. Show that. Griffth 1.108

$$\oint (\hat{r}\cdot\vec{r}')\,d\vec{l}' = -\hat{r}\times\int d\vec{a}$$

Here we first prove a lemma.

lemma 1

$$\oint (\vec{c}\cdot\vec{r})\,d\vec{l} = \vec{a}\times\vec{c} \quad (1)$$

Here we first focus on LHS, let $T=\vec{c}\cdot\vec{r}$ (2), next determine $\vec{\nabla}T$

$$\vec{\nabla}T = \vec{\nabla}(\vec{c}\cdot\vec{r}) = \vec{c}\times(\vec{\nabla}\times\vec{r}) + (\vec{c}\cdot\vec{\nabla})\vec{r}$$

Here we known that $\vec{\nabla}\times\vec{r}=0$, $(\vec{c}\cdot\vec{\nabla})\vec{r} = \left((c_x\hat{x}+c_y\hat{y}+c_z\hat{z})\cdot(\hat{x}\partial_x+\hat{y}\partial_y+\hat{z}\partial_z)\right)(x\hat{x}+y\hat{y}+z\hat{z})$

$$= \left(c_x\frac{\partial}{\partial x}+c_y\frac{\partial}{\partial y}+c_z\frac{\partial}{\partial z}\right)(x\hat{x}+y\hat{y}+z\hat{z})$$

$$= c_x\hat{x}+c_y\hat{y}+c_z\hat{z} = \vec{c}$$

$$\therefore \vec{\nabla}T = \vec{c} \quad (3)$$

We substitute (2)(3) into (1), by theorem of curl.

$$\oint T\,d\vec{l} = -\int(\nabla T)\times d\vec{a}.$$

$$\oint(\vec{c}\cdot\vec{r})\,d\vec{l} = -\int\vec{c}\times d\vec{a}$$

$$= -\vec{c}\times\int\vec{a} \quad (4)$$

Now we input $\hat{r}=\vec{c}$, $\vec{r}'=\vec{r}$ $d\vec{l}'=d\vec{l}$

$$\therefore \oint(\hat{r}\cdot\vec{r}')\,d\vec{l}' = -\hat{r}\times\int d\vec{a}$$

厦門大學
XIAMEN UNIVERSITY
ADD:FUJIAN XIAMEN CABLE:0633 P.C:361005

Question 2

Show that if ρ, Q in volume, V or $\frac{\partial V}{\partial n}$ on boundary known. Griffth 3.5 Problem.

The Electric field uniquely determined.

Here we apply similar methed with 2th uniqueness theorem, we assume there exist 2 set of $\vec{E}$ field satisfied, $\vec{E}_1$, $\vec{E}_2$.

Since we are clear on charge density, therefore, we have gauss law in $\vec{\nabla}$ form set

$$\vec{\nabla}\cdot\vec{E}_1 = \vec{\nabla}\cdot\vec{E}_2 = \frac{\rho}{\varepsilon_0} \quad (1)$$

And also Gauss law in integral form also set, both inner boundarg and outer boundary.

inner $\oint \vec{E}_1\cdot d\vec{a} = \oint \vec{E}_2\, d\vec{a} = \frac{Q_i}{\varepsilon_0}$ (2) outer (total) $\oint \vec{E}_1 \cdot d\vec{a} = \oint \vec{E}_2\cdot d\vec{a} = \frac{Q_{tot}}{\varepsilon_0}$ (3)

Next we let $\vec{E_3} = \vec{E_1} - \vec{E_2}$ as difference of 2 set of E field

From (1) $\vec{\nabla}\cdot\vec{E_3} = \vec{\nabla}\cdot(\vec{E}_1 - \vec{E}_2) = \frac{\rho}{\varepsilon_0} - \frac{\rho}{\varepsilon_0} = 0 \iff \vec{\nabla}\cdot\vec{E}_3 = 0$

From (2)(3) $\oint \vec{E}_3\cdot d\vec{a} = \oint(\vec{E}_1 - \vec{E}_2)\cdot d\vec{a} = \frac{Q-Q}{\varepsilon_0} = 0 \iff \oint \vec{E}_3\cdot d\vec{a} = 0$

Next we apply the same trick as we do in 2th uniquenss theorem, that make connection V_3 and $\vec{E_3}$ (Page 123).

$$\vec{\nabla}\cdot(V_3\vec{E}_3) = V_3(\vec{\nabla}\cdot\vec{E}_3) + \vec{E}_3\cdot(\vec{\nabla}V_3) = V_3(0) - (\vec{E}_3)\cdot(\vec{E}_3) = -|E_3|^2 \quad (4)$$

厦 門 大 學
XIAMEN UNIVERSITY
ADD:FUJIAN XIAMEN CABLE:0633 P.C:361005

Next we Integrat in whole volume with (4)

$$\int_{\mathcal{V}} \vec{\nabla} \cdot (V_3 \vec{E_3})\, d\tau = \oint_S V_3 \vec{E_3}\, d\vec{a} = -\int_{\mathcal{V}} (E_3)^2 d\tau. \quad (5) \ \star$$

divergence theorem

Next, we discuss the 2 condition seperately

1° V set, that $V_1 = V_2$, that $V_3 = V_1 - V_2 = 0$

$$\therefore \oint_S V_3 \vec{E_3} \cdot d\vec{a} = 0 = -\int_{\mathcal{V}} (E_3)^2 d\tau \Leftrightarrow E_3 = 0,$$ uniqueness exist

2° $\frac{\partial V}{\partial n}$ set. that $\frac{\partial V_1}{\partial n} = \frac{\partial V_2}{\partial n}$ that $\frac{\partial V_3}{\partial n} = \frac{\partial V_1 - V_2}{\partial n} = 0$

that. $\vec{E_\perp} = 0$ that $\vec{E} \cdot \vec{a} = 0$.

$$\therefore \oint_S V_3 \vec{E_3} \cdot d\vec{a} = 0 \quad -\int_{\mathcal{V}} (E_3)^2 d\tau \Leftrightarrow E_3 = 0$$ uniqueness exist

3. 2 point charge. q_A, q_B, mass m_A, m_B, length a, rod cut final speed.

The initial potential $V_0 = \frac{1}{4\pi\varepsilon_0}\frac{q_A q_B}{a}$, the final potential $V_\infty \to 0$.

$$\begin{cases} \Delta V = V_0 = \frac{1}{4\pi\varepsilon_0}\frac{q_A q_B}{a} = \frac{1}{2} m_A V_A^2 + \frac{1}{2} m_B V_B^2 \\ 0 = m_A V_A + m_B V_B \end{cases}$$

Here conseration of momentum, A, B move on the same line, we let V_A as positive.

$$\boxed{\begin{aligned} V_A &= \frac{1}{m_A}\sqrt{\frac{2 m_A m_B}{m_A + m_B}\frac{1}{4\pi\varepsilon_0}\frac{q_A q_B}{a}} \\ V_B &= -\frac{1}{m_B}\sqrt{\frac{2 m_A m_B}{m_A + m_B}\frac{1}{4\pi\varepsilon_0}\frac{q_A q_B}{a}} \end{aligned}}$$

厦門大學
XIAMEN UNIVERSITY
ADD:FUJIAN XIAMEN CABLE:0633 P.C:361005

Question 4

$$\psi_{100} = \frac{1}{\sqrt{4\pi}} \cdot \frac{1}{a_0^{3/2}} \cdot 2\, e^{-r/a_0}$$

$$\rho = -e|\psi|^2 = -e \frac{1}{4\pi} \frac{1}{a_0^3} \cdot 4\, e^{-2r/a_0}$$

$$\oint \vec{E} \cdot d\vec{a} = \left(e + \int \rho \,(r^2 \sin\theta \, dr \, d\theta \, d\phi) \right) / \varepsilon_0$$

$$= \frac{e}{\varepsilon_0} \left(1 - \cancel{\frac{1}{4\pi}} \frac{1}{a_0^3} \cdot 4 \int e^{-2r/a_0} r^2 dr \cdot \cancel{2 \cdot 2\pi} \right)$$

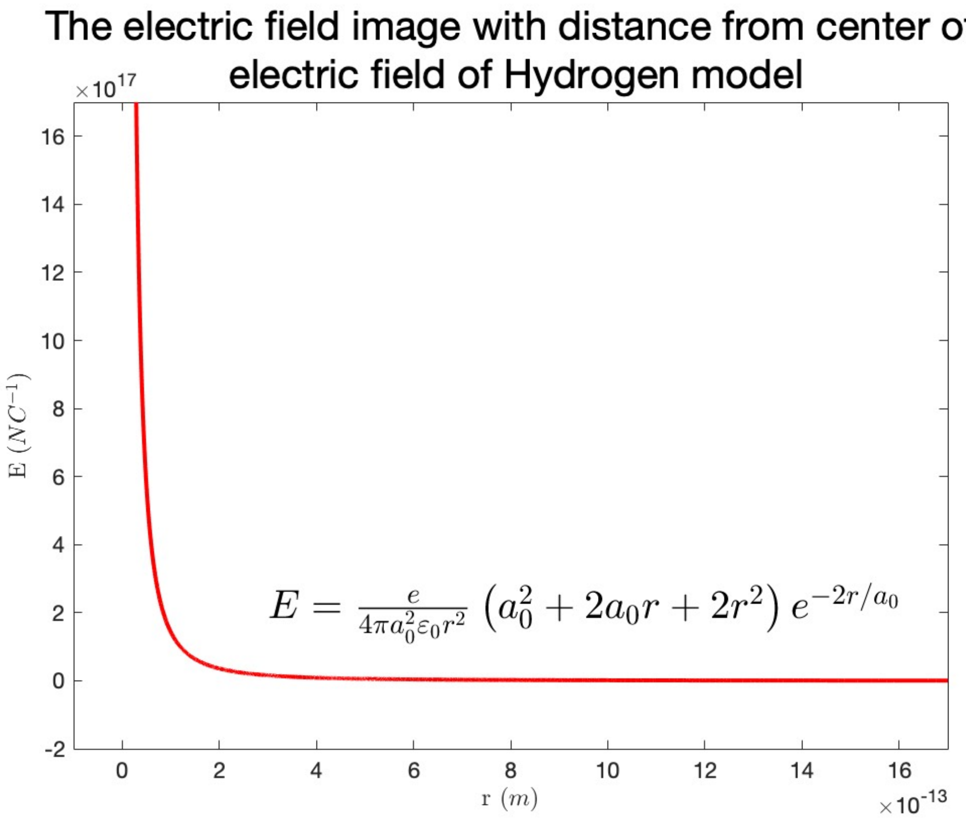
$$= \frac{e}{\varepsilon_0} \left(1 - \frac{4}{a_0^3} \int_0^r e^{-2r/a_0} r^2 dr \right)$$

$$= \frac{e}{\varepsilon_0} \left(1 - \frac{4}{a_0^3} \cdot \frac{1}{4} \left(a_0^3 - a_0 e^{-2r/a_0} (a_0^2 + 2a_0 r + 2r^2) \right) \right)$$

$$= \frac{e}{\varepsilon_0} \left(1 - 1 + \frac{1}{a_0^2} (a_0^2 + 2a_0 r + 2r^2) e^{-2r/a_0} \right)$$

$$E \cdot 4\pi r^2 = \frac{e}{a_0^2 \varepsilon_0} (a_0^2 + 2a_0 r + 2r^2) e^{-2r/a_0}$$

$$\boxed{E = \frac{e}{4\pi a_0^2 \varepsilon_0 r^2} (a_0^2 + 2a_0 r + 2r^2) e^{-2r/a_0}}$$



The electric go to positive infinite closing to the proton point. And rapidly decrease to the 0 as the radius increase, because hydrogen atom is neutral , therefore when distance go to the infinity the total e field tend to 0.

The code and figure is attached in github. https://github.com/yimingio/Lecture_Electrodynamics/tree/main/Homework_2