Question 1
Given an initial value problem $y' = y^2 \sin(t)$ with $y(0) = -3$.
a. Approximate $y\left(\frac{\pi}{2}\right)$ using Euler's method with $n = 2$ steps.
b. Approximate $y\left(\frac{\pi}{2}\right)$ using Taylor's method of second order with $n = 2$ steps.
c. Approximate $y\left(\frac{\pi}{2}\right)$ using Modified Euler method with $n = 2$ steps.
d. Given the exact solution is $y(t) = \frac{3}{3\cos(t)-4}$. Compare the relative errors for the methods in (a), (b) and (c).

Solution.

(a) Euler's method
With $n = 2$ we have $h = (\pi/2)/2 = \pi/4$, $t_i = \pi \cdot i/4$, $w_0 = y(0) = -3$:

$$w_{i+1} = w_i + h(w_u^2 \sin(t_i)) = w_i + \frac{\pi}{4}(w_i^2 \sin(\pi i/4))$$

For $t_0 = 0$

$$w_0 = y(0) = -3.0000$$

For $t_1 = \frac{\pi}{4}$

$$w_1 = w_0 + \frac{\pi}{4}\left(w_0^2 \sin(0)\right) = -3.0000$$

For $t_2 = \frac{\pi}{2}$

$$w_2 = w_1 + \frac{\pi}{4}\left(w_1^2 \sin(\frac{\pi}{4})\right) = 1.9982$$

Therefore

$$y(\frac{\pi}{2}) = w_2 = 1.9982$$

(b) Taylor's method of second order
With $n = 2$ we have $h = (\pi/2)/2 = \pi/4$, $t_i = \pi \cdot i/4$, $w_0 = y(0) = -3$:
And

$$\begin{aligned} f'(t) &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \\ &= y^2\cos(t) + \left(y^2 \sin(t)\right) \cdot 2y\sin(t) \\ &= y^2\cos(t) + 2y^3\sin^2(t) \end{aligned}$$

Here the iteration function is

$$w_{i+1} = w_i + h\left(f(t_i, w_i) + \frac{h}{2} f'(t_0, w_0)\right)$$

For $t_0 = 0$

$$w_0 = y(0) = -3.0000$$

For $t_1 = \frac{\pi}{4}$

$$\begin{aligned} w_1 &= w_0 + h\left(f\left(t_0, w_0\right) + \frac{h}{2} f'\left(t_0, w_0\right)\right) \\ &= (-3) + \frac{\pi}{4}\left((-3)^2 \sin(0) + \frac{1}{2} \cdot \frac{\pi}{4} \cdot \left((-3)^2 \cos(0) + 2(-3)^3 \sin(0)\right)\right) \\ &= -3 + \frac{9\pi^2}{32} = -0.2242 \end{aligned}$$

For $t_2 = \frac{\pi}{2}$

$$\begin{aligned} w_2 &= (-0.2242) + \frac{\pi}{4}\left((-0.2242)^2 \sin\left(\frac{\pi}{4}\right) + \frac{\pi}{8}\left(2(-0.2242)^3 \sin^2\left(\frac{\pi}{4}\right) + (-0.2242)^2 \cos\left(\frac{\pi}{4}\right)\right)\right) \\ &= -0.1888 \end{aligned}$$

Therefore

$$y(\frac{\pi}{2}) = w_2 = -0.1888$$

(c) Modified Euler method

With $n = 2$ we have $h = (\pi/2)/2 = \pi/4$, $t_i = \pi \cdot i/4$, $w_0 = y(0) = -3$:
Here the iteration function is

$$w_{i+1} = w_i + \frac{h}{2}\left(f\left(t_i, w_i\right) + f\left(t_{i+1}, w_i + hf\left(t_i, w_i\right)\right)\right)$$

For $t_0 = 0$

$$w_0 = y(0) = -3.0000$$

For $t_1 = \frac{\pi}{4}$

$$\begin{aligned} w_1 &= -3 + \frac{\pi}{8}\left((-3)^2 \sin(0) + \left(-3 + \frac{\pi}{4}\left((-3)^2 \sin(0)\right)\right)^2 \sin\left(\frac{\pi}{4}\right)\right) \\ &= \frac{9\sqrt{2}\pi}{16} - 3 = -0.5009 \end{aligned}$$

For $t_2 = \frac{\pi}{2}$

$$\begin{aligned} W_2 &= (-0.5009) + \frac{\pi}{8}\left((-0.5009)^2 \sin\left(\frac{\pi}{4}\right) + \left((-0.5009) + \frac{\pi}{4}\left((-0.5009)^2 \sin\left(\frac{\pi}{4}\right)\right)^2 5\text{in}\left(\frac{\pi}{2}\right)\right)\right) \\ &= -0.3799 \end{aligned}$$

Therefore

$$y(\frac{\pi}{2}) = w_2 = -0.3799$$

(d) Compare relative error

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\begin{aligned}
\varepsilon_{\text{Euler}} &= \left|\frac{-0.75-(1.9982)}{-0.75}\right| = 3.6643 \\
\varepsilon_{\text{taylor}} &= \left|\frac{-0.75-(-0.2242)}{-0.75}\right| = 0.7483 \\
\varepsilon_{\text{Modified}} &= \left|\frac{-0.75-(-0.1888)}{-0.75}\right| = 0.4935
\end{aligned}
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Question 2
Given an initial value problem $y' = -(xy^2 + y)$ with $y(0) = 1$. Approximate $y(0.3)$ using Runge-Kutta method of order four with step length 0.1.

Solution.
With $h = 0.1$ we have $n = 3$, $t_i = 0.1i$, $w_0 = y(0) = 1$:
The Approximation $y(0.1)$ could be given that:

$$\begin{aligned}
w_0 &= 1 \\
k_1 &= hf(t_0, w_0) = 0.1 \cdot f(0,1) = -0.1000 \\
k_2 &= hf\left(t_0 + \frac{h}{2}, w_0 + \frac{1}{2}k_1\right) = 0.1 \cdot f(0.05, 1 - 0.05) = -0.0995 \\
k_3 &= hf\left(t_0 + \frac{h}{2}, w_0 + \frac{1}{2}k_2\right) = 0.1 \cdot f(0.05, 1 - 0.0498) = -0.0995 \\
k_4 &= hf(t_{0+1}, w_0 + k_3) = 0.1 \cdot f(0.1, 1 - 0.0995) = -0.0982 \\
w_1 &= w_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.9006,
\end{aligned}$$

The approximation y(0.2) could be given that:

$$\begin{aligned}
w_1 &= 0.9006 \\
k_1 &= hf(t_1, w_1) = 0.1 \cdot f(0.1, 0.9006) = -0.0982 \\
k_2 &= hf\left(t_1 + \frac{h}{2}, w_1 + \frac{1}{2}k_1\right) = 0.1 \cdot f(0.15, 0.9006 - 0.0491) = -0.0960 \\
k_3 &= hf\left(t_1 + \frac{h}{2}, w_1 + \frac{1}{2}k_2\right) = 0.1 \cdot f(0.15, 0.9006 - 0.0480) = -0.0962 \\
k_4 &= hf(t_2, w_1 + k_3) = 0.1 \cdot f(0.2, 0.9006 - 0.0962) = -0.0934 \\
w_2 &= w_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.8046,
\end{aligned}$$

The approximation y(0.3) could be given that:

$$\begin{aligned}
w_2 &= 0.8046 \\
k_1 &= hf(t_2, w_1) = 0.1 \cdot f(0.2, 0.8046) = -0.0934 \\
k_2 &= hf\left(t_2 + \frac{h}{2}, w_2 + \frac{1}{2}k_1\right) = 0.1 \cdot f(0.25, 0.8046 - 0.0467) = -0.0902 \\
k_3 &= hf\left(t_2 + \frac{h}{2}, w_2 + \frac{1}{2}k_2\right) = 0.1 \cdot f(0.25, 0.8046 - 0.0451) = -0.0904 \\
k_4 &= hf(t_3, w_2 + k_3) = 0.1 \cdot f(0.3, 0.8046 - 0.0904) = -0.0867 \\
w_3 &= w_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.7144,
\end{aligned}$$

Therefore the approximation of y(0.3)=$w_3 = 0.7144$

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Question 3
Consider the linear system

$$\begin{aligned} 4.3x_1 + 6.6x_2 - 5.3x_3 + 6.8x_4 &= 48.81 \\ 2.5x_1 - 1.2x_2 + 6.6x_3 - 2.0x_4 &= -30.50 \\ 5.4x_1 + 2.2x_2 - 2.6x_3 + 3.5x_4 &= 45.69 \\ -7.2x_1 + 5.3x_2 - 1.3x_3 + 4.9x_4 &= -18.15 \end{aligned}$$

a. Solve the system by using the method of Gaussian Elimination. In each arithmetic operation, round to two decimal places.
b. Solve the system by using the method of Gaussian Elimination with partial pivoting. In each arithmetic operation, round to two decimal places.

Solution.

(a) using the method of Gaussian Elimination
Here we can represent the linear system with the augmented matrix as

$$\begin{bmatrix} 4.30 & 6.60 & -5.30 & 6.80 & : & 48.81 \\ 2.50 & -1.20 & 6.60 & -2.00 & : & -30.50 \\ 5.40 & 2.20 & -2.60 & 3.50 & : & 45.69 \\ -7.20 & 5.30 & -1.30 & 4.90 & : & -18.15 \end{bmatrix}$$

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Question 4

Solution.

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