**Question 1** The forward-difference formula can be expressed as:

$$f'\left(x_0\right)=\frac{1}{h}\left[f\left(x_0+h\right)-f\left(x_0\right)\right]-\frac{h}{2} f''\left(x_0\right)-\frac{h^2}{6} f''\left(x_0\right)+O\left(h^3\right)$$

Use extrapolation to derive an $O\left(h^3\right)$ formula for $f'\left(x_0\right)$.

*Solution.*
Here we have that:

$$f'\left(x_0\right)=\frac{1}{h}\left(f\left(x_0+h\right)-f\left(x_0\right)\right)-\frac{h}{2} f''\left(x_0\right)-\frac{h^2}{6} f'''\left(x_0\right)+O\left(h^3\right) \tag{1}$$

Here we replace $h$ with $2h$

$$f'\left(x_0\right)=\frac{1}{2h}\left(f\left(x_0+2h\right)-f\left(x_0\right)\right)-h f''\left(x_0\right)-\frac{4h^2}{6} f'''\left(x_0\right)+O\left(h^3\right) \tag{2}$$

Then we multiply 1 with 2 and subtract the 2:

$$f'\left(x_0\right)=\frac{2}{h}\left(f\left(x_0+h\right)-f\left(x_0\right)\right)-\frac{1}{2h}\left(f\left(x_0+2h\right)-f\left(x_0\right)\right)-\frac{h^2}{3} f''\left(x_0\right)+\frac{2h^2}{3} f'''\left(x_0\right)+O\left(h^3\right) \tag{3}$$

Next we replace the h with 2h in 3:

$$f'\left(x_0\right)=\frac{1}{h}\left(f\left(x_0+2h\right)-f\left(x_0\right)\right)-\frac{1}{4h}\left(f\left(x_0+4h\right)-f\left(x_0\right)\right)+\frac{4h^2}{3} f^{m}\left(x_0\right)+O\left(h^3\right) \tag{4}$$

Then we multiply the3 with 4 and substract 4:

$$\begin{aligned} 3f'\left(x_0\right)=&\frac{8}{h}\left(f\left(z_0+h\right)-f\left(x_0\right)\right)-\frac{2}{h}\left(f\left(x_0+2h\right)-f\left(x_0\right)\right)+\frac{4h^2}{3} f''\left(x_0\right) \\ &-\frac{1}{h}\left(f\left(x_0+2h\right)-f\left(x_0\right)\right)+\frac{1}{4h}\left(f\left(x_0+4h\right)-f\left(x_0\right)\right)-\frac{4h^2}{3} f''\left(z_0\right)+O\left(h^3\right)\end{aligned} \tag{5}$$

And we can summarize that:

$$f'\left(x_0\right)=\frac{1}{12h}\left(f\left(x_0+4h\right)-12 f(20+2h)+32 f\left(x_0+h\right)-21 f\left(x_0\right)\right)+O\left(h^3\right) \tag{6}$$

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**Question 2** Determine the values of $n$ and $h$ required to approximate $\int_0^2 x^2 \sin(-x)dx$ to within $10^{-6}$ using the Composite Trapezoid Rule, Composite Midpoint Rule and Composite Simpson's Rule respectively.
Hint: You do not need to solve the numerical integration.

*Solution.*

Here we start with the Composite Trapezoid Rule, we recall that:

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f\left(x_j\right) + f(b)\right] - \frac{b-a}{12}h^2 f''(\mu) \tag{7}$$

Here we have the error term as $\frac{b-a}{12}h^2 f''(\mu)$, and then we find that maximum of of the error turn, first find the $|f''_{max}|$

$$f(x) = x^2 \sin(-x) \tag{8}$$

$$f'(x) = 2x\sin(-x) - x^2\cos(-x) \tag{9}$$

$$f''(x) = \left(x^2 - 2\right)\sin(x) - 4x\cos(x) \tag{10}$$

Here we try to find the absolute value of second derivative:

$$\begin{aligned} |f''(x)| &= \left|\left(x^2-2\right)\sin(x) - 4x\cos(x)\right| \\ &\leq \left|\left(x^2-2\right)\sin(x)\right| + |4x\cos(x)| \\ &\leq \left|x^2 - 2\right| + |4x| \leq \left|(2)^2 - 2\right| + |4\cdot 2| = 10 \end{aligned} \tag{11}$$

Therefore we can derive the error term's maximum

$$\frac{b-a}{12}h^2 f''(\mu) \leq \frac{2-0}{12}h^2 \cdot 10 \leq 10^{-6}$$

And it turn out that:

$$h \leq 5.477 \times 10^{-4}$$

And we can have the $n$ that:

$$n \geq \frac{b-a}{h} = 3652$$

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Next we find the Composite Midpoint Rule

$$\int_a^b f(x)dx = 2h\sum_{j=0}^{n/2} + \frac{b-a}{6}h^2 f''(\mu) \tag{12}$$

And we have the second derivative same with the previous one:

$$f''(x) = \left(x^2 - 2\right)\sin(x) - 4x\cos(x) \tag{13}$$

This part is similar with the part in 11, and we can caculate the error term here:

$$\frac{b-a}{b} h^2 f''(\mu) \leq \frac{2-0}{6} h^2 \cdot 10 \leq 10^{-6}$$

$$h \leq 5.477 \times 10^{-4}$$

And we can turn out that:

$$n \geq \frac{b-a}{h} = 3652$$

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Next we find the Composite Simpson Rule

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{(n/2)-1} f\left(x_{2j}\right) + 4\sum_{j=1}^{n/2} f\left(x_{2j-1}\right) + f(b)\right] - \frac{b-a}{180} h^4 f^{(4)}(\mu)$$

Here the error term is $\frac{b-a}{180} h^4 f^{(4)}(\mu)$, then we first find the fourth derivative:

$$\begin{gathered} f''(x) = \left(x^2 - 2\right)\sin(x) - 4x\cos(x) \\ f'''(x) = \left(x^2 - 6\right)\cos(x) + 6x\sin(x) \\ f^{(4)}(x) = 8x\cos(x) - \left(x^2 - 12\right)\sin(x) \end{gathered}$$

Next determine the maximum of the fourth derivative:

$$\begin{aligned} \left|f^{(4)}(x)\right| &= \left|8x\cos(x) - \left(x^2 - 12\right)\sin(x)\right| \\ &\leq |8x\cos(x)| + \left|\ \left(x^2 - 127\sin(x)\ \right|\right. \\ &\leq |8x| + \left|x^2 - 12\right| = 8x + 12 - x^2 \\ &= -(x-4)^2 + 28 \leq 24 \end{aligned}$$

And the error term could be transform into below form:

$$\begin{aligned} \frac{2-0}{180} h^4 \cdot 24 \leq 10^{-6} \\ h \leq 4.401 \times 10^{-2} \end{aligned} \tag{14}$$

And we can turn out that:

$$n \geq \frac{b-a}{h} = 46 \tag{15}$$

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**Question 3**
Given a function, $f(t) = \sqrt{t}$
a) Apply the Romberg Integration to find $R_{3,3}$ for the integral $\int_1^4 f(t)dt$.
b) Apply the Composite Simpson's Rule to approximate $\int_1^4 f(t)dt$ using eight intervals.
c) Comment on your results in (a) and (b).

*Solution.*

(a)

$$\begin{aligned}
R_{1,1} &= \frac{4-1}{2}(\sqrt{1}+\sqrt{4}) = \frac{9}{2} = 4.500 \\
R_{2,1} &= \frac{4-1}{4}\left(\sqrt{1.1}+2\sqrt{\frac{1+4}{2}}+\sqrt{4}\right) = \frac{3}{4}(3+\sqrt{10}) = 4.622 \\
R_{2,2} &= R_{2,1} + \frac{1}{3}\left(R_{2,1}-R_{1,1}\right) = 4.663 \\
R_{3,1} &= \frac{4-1}{8}\left(\sqrt{1}+2\sqrt{\frac{7}{4}}+2\sqrt{\frac{5}{2}}+2\sqrt{\frac{13}{4}}+\sqrt{4}\right) = \frac{3}{8}(3+\sqrt{7}+\sqrt{10}+\sqrt{13}) = 4.655 \\
R_{3,2} &= R_{3,1} + \frac{1}{3}\left(R_{3,1}-R_{2,1}\right) = 4.666 \\
R_{3,3} &= R_{3,2} + \frac{1}{4^2-1}\left(R_{3,2}-R_{2,2}\right) = 4.666
\end{aligned} \tag{16}$$

And we can plot the graph as:

| | | |
|---|---|---|
| 4.500 | | |
| 4.622 | 4.663 | |
| 4.655 | 4.666 | 4.666 |

(b) Here $n = 8$, $h = \frac{4-1}{8}$, $x_j = 1 + \frac{3}{8}j$:

$$\begin{aligned}
\int_1^4 f(t)dt &= \frac{1}{3}\frac{3}{8}\left(f(1) + 2\left(f\left(x_2\right)+f\left(x_4\right)+f\left(x_6\right)\right) + 4\left(f\left(x_1\right)+f\left(x_3\right)+f\left(x_5\right)+f\left(x_7\right)+f(49)\right)\right. \\
&= \frac{1}{8}\left(f(1) + 2\left(f\left(\frac{7}{4}\right)+f\left(\frac{5}{2}\right)+f\left(\frac{13}{4}\right)\right)+\right. \\
&\quad \left. 4\left(f\left(\frac{11}{8}\right)+f\left(\frac{17}{8}\right)+f\left(\frac{23}{8}\right)+f\left(\frac{29}{8}\right)\right)+f(4)\right) \\
&= \frac{1}{8}(3+\sqrt{7}+\sqrt{10}+\sqrt{13}+\sqrt{22}+\sqrt{34}+\sqrt{46}+\sqrt{58}) \\
&= 4.667
\end{aligned} \tag{17}$$

(c)Here we can first calculate the exact value of the solution

$$\int_1^4 f(t)dt = \int_1^4 t^{\frac{1}{2}} dt = \left(\frac{2}{3}t^{\frac{3}{2}}\right)_1^4 = \frac{2}{3}\cdot(8-1) = \frac{14}{3} = 4\frac{2}{3}$$
$$= 4.666\cdots$$

And we can find that both method give a very close answer with the exact value. And Composite Simpson Law seems to be closer to the exact result.However due to the rounding off 4 digits, we can not see what exact is the error for this 2 methods,

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