XIAMEN UNIVERSITY MALAYSIA

Course Code	:	BSC128
Course Name	:	Numerical Method
Lecturer	:	Goh Chien Yong
Academic Session	:	202204
Assessment Title	:	Assignment 3
Submission Due Date	:	2022/06/04

Prepared by :

Student ID	Student Name
PHY2009481	Guo Yiming

Date Received :

Feedback from Lecturer:

Mark:

Own Work Declaration

I/We hereby understand my/our work would be checked for plagiarism or other misconduct, and the softcopy would be saved for future comparison(s).

I/We hereby confirm that all the references or sources of citations have been correctly listed or presented and I/we clearly understand the serious consequence caused by any intentional or unintentional misconduct.

This work is not made on any work of other students (past or present), and it has not been submitted to any other courses or institutions before.

Signature: Guo Yiming

Date: 04/06/2022

DESCRIPTION OF COURSEWORK

Course Code	BSC128/CST303*
Course Name	Numerical Methods
Lecturer	Goh Chien Yong
Academic Session	2022/04
Assessment Title	Assignment 3

A. Introduction/ Situation/ Background Information

Through this assignment, students are exploring to the knowledge of error analysis, numerical differentiation and integration.

B. Course Learning Outcomes (CLO) covered

At the end of this assessment, students are able to:

CLO 1 Apply the concepts and theories in numerical methods.

CLO 2 Calculate the solution of a mathematical problem numerically with an appropriate algorithm.

C. University Policy on Academic Misconduct

1. Academic misconduct is a serious offense in Xiamen University Malaysia. It can be defined as any of the following:
   i. **Plagiarism** is submitting or presenting someone else's work, words, ideas, data or information as your own intentionally or unintentionally. This includes incorporating published and unpublished material, whether in manuscript, printed or electronic form into your work without acknowledging the source (the person and the work).
   ii. **Collusion** is two or more people collaborating on a piece of work (in part or whole) which is intended to be wholly individual and passed it off as own individual work.
   iii. **Cheating** is an act of dishonesty or fraud in order to gain an unfair advantage in an assessment. This includes using or attempting to use, or assisting another to use materials

that are prohibited or inappropriate, commissioning work from a third party, falsifying data, or breaching any examination rules.

2. All the assessment submitted must be the outcome of the student. Any form of academic misconduct is a serious offense which will be penalised by being given a zero mark for the entire assessment in question or part of the assessment in question. If there is more than one guilty party as in the case of collusion, both you and your collusion partner(s) will be subjected to the same penalty.

D. Instruction to Students

1. This is an individual assignment.
2. You are required to complete all the Three questions given.
3. You have to submit the assignment (softcopy) through Moodle before 04th Jun 2022, 5pm. Late submission will be penalised.
4. Handwriting/drawing is allowed in this assignment.
5. Attached the PDF-file upon submission, with naming as **StudentID_CourseCode_A3.pdf**.
6. The given cover page of assignment needs to be attached.
7. Always start a new question in new page.
8. Please do it on your own, any suspected misconduct will be directly reported to the Office of Academic Affair.

E. Evaluation Breakdown

No.	**Component Title**	**Percentage (%)**
1.	Question 1	10
2.	Question 2	40
3.	Question 3	40
4.	Report	10
	TOTAL	**100**

F. Task(s)

1. The forward-difference formula can be expressed as:

$$f'(x_0) = \frac{1}{h}\left[f(x_0+h) - f(x_0) \right] - \frac{h}{2} f''(x_0) - \frac{h^2}{6} f'''(x_0) + O\left(h^3\right)$$

   Use extrapolation to derive an $O\left(h^3\right)$ formula for $f'(x_0)$. [10]

2. Determine the values of *n* and *h* required to approximate $\int_0^2 x^2 \sin(-x)\,dx$ to within 10^{-6} using the Composite Trapezoid Rule, Composite Midpoint Rule and Composite Simpson's Rule respectively. [40]

   Hint: You do not need to solve the numerical integration.

3. Given a function, $f(t) = \sqrt{t}$.

   a) Apply the Romberg Integration to find $R_{3,3}$ for the integral $\int_1^4 f(t)\,dt$. [20]

   b) Apply the Composite Simpson's Rule to approximate $\int_1^4 f(t)\,dt$ using eight intervals. [10]

   c) Comment on your results in (a) and (b). [10]

Question 1 The forward-difference formula can be expressed as:

$$f'\left(x_0\right)=\frac{1}{h}\left[f\left(x_0+h\right)-f\left(x_0\right)\right]-\frac{h}{2} f''\left(x_0\right)-\frac{h^2}{6} f''\left(x_0\right)+O\left(h^3\right)$$

Use extrapolation to derive an $O\left(h^3\right)$ formula for $f'\left(x_0\right)$.

Solution.
Here we have that:

$$f'\left(x_0\right)=\frac{1}{h}\left(f\left(x_0+h\right)-f\left(x_0\right)\right)-\frac{h}{2} f''\left(x_0\right)-\frac{h^2}{6} f'''\left(x_0\right)+O\left(h^3\right) \tag{1}$$

Here we replace h with $2h$

$$f'\left(x_0\right)=\frac{1}{2h}\left(f\left(x_0+2h\right)-f\left(x_0\right)\right)-h f''\left(x_0\right)-\frac{4h^2}{6} f'''\left(x_0\right)+O\left(h^3\right) \tag{2}$$

Then we multiply 1 with 2 and subtract the 2:

$$f'\left(x_0\right)=\frac{2}{h}\left(f\left(x_0+h\right)-f\left(x_0\right)\right)-\frac{1}{2h}\left(f\left(x_0+2h\right)-f\left(x_0\right)\right)-\frac{h^2}{3} f'''\left(x_0\right)+\frac{2h^2}{3} f'''\left(x_0\right)+O\left(h^3\right) \tag{3}$$

Next we replace the h with 2h in 3:

$$f'\left(x_0\right)=\frac{1}{h}\left(f\left(x_0+2h\right)-f\left(x_0\right)\right)-\frac{1}{4h}\left(f\left(x_0+4h\right)-f\left(x_0\right)\right)+\frac{4h^2}{3} f'''\left(x_0\right)+O\left(h^3\right) \tag{4}$$

Then we multiply the 3 with 4 and substract 4:

$$\begin{aligned} 3f'\left(x_0\right)= & \frac{8}{h}\left(f\left(x_0+h\right)-f\left(x_0\right)\right)-\frac{2}{h}\left(f\left(x_0+2h\right)-f\left(x_0\right)\right)+\frac{4h^2}{3} f''\left(x_0\right) \\ & -\frac{1}{h}\left(f\left(x_0+2h\right)-f\left(x_0\right)\right)+\frac{1}{4h}\left(f\left(x_0+4h\right)-f\left(x_0\right)\right)-\frac{4h^2}{3} f''\left(x_0\right)+O\left(h^3\right) \end{aligned} \tag{5}$$

And we can summarize that:

$$f'\left(x_0\right)=\frac{1}{12h}\left(f\left(x_0+4h\right)-12 f(x_0+2h)+32 f\left(x_0+h\right)-21 f\left(x_0\right)\right)+O\left(h^3\right) \tag{6}$$

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Question 2 Determine the values of n and h required to approximate $\int_0^2 x^2 \sin(-x)dx$ to within 10^{-6} using the Composite Trapezoid Rule, Composite Midpoint Rule and Composite Simpson's Rule respectively.
Hint: You do not need to solve the numerical integration.

Solution.

Here we start with the Composite Trapezoid Rule, we recall that:

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{b-a}{12}h^2 f''(\mu) \tag{7}$$

Here we have the error term as $\frac{b-a}{12}h^2 f''(\mu)$, and then we find that maximum of of the error turn, first find the $|f''_{max}|$

$$f(x) = x^2 \sin(-x) \tag{8}$$

$$f'(x) = 2x\sin(-x) - x^2\cos(-x) \tag{9}$$

$$f''(x) = \left(x^2 - 2\right)\sin(x) - 4x\cos(x) \tag{10}$$

Here we try to find the absolute value of second derivative:

$$\begin{aligned} |f''(x)| &= \left|\left(x^2-2\right)\sin(x) - 4x\cos(x)\right| \\ &\le |\left(x^2-2\right)\sin(x)| + |4x\cos(x)| \\ &\le \left|x^2-2\right| + |4x| \le \left|(2)^2 - 2\right| + |4\cdot 2| = 10 \end{aligned} \tag{11}$$

Therefore we can derive the error term's maximum

$$\frac{b-a}{12}h^2 f''(\mu) \le \frac{2-0}{12}h^2 \cdot 10 \le 10^{-6}$$

And it turn out that:

$$h \le 7.7460 \times 10^{-4}$$

And we can have the n that:

$$n \ge \frac{b-a}{h} = 2582$$

Next we find the Composite Midpoint Rule

$$\int_a^b f(x)dx = 2h\sum_{j=0}^{n/2} + \frac{b-a}{6}h^2 f''(\mu) \tag{12}$$

And we have the second derivative same with the previous one:

$$f''(x) = \left(x^2 - 2\right)\sin(x) - 4x\cos(x) \tag{13}$$

This part is similar with the part in 11, and we can caculate the error term here:

$$\frac{b-a}{b}h^2 f''(\mu) \leq \frac{2-0}{6}h^2 \cdot 10 \leq 10^{-6}$$

$$h \leq 5.477 \times 10^{-4}$$

And we can turn out that:

$$n \geq \frac{b-a}{h} = 3652$$

....

Next we find the Composite Simpson Rule

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{(n/2)-1} f\left(x_{2j}\right) + 4\sum_{j=1}^{n/2} f\left(x_{2j-1}\right) + f(b)\right] - \frac{b-a}{180}h^4 f^{(4)}(\mu)$$

Here the error term is $\frac{b-a}{180}h^4 f^{(4)}(\mu)$, then we first find the fourth derivative:

$$\begin{aligned} f''(x) &= \left(x^2-2\right)\sin(x) - 4x\cos(x) \\ f'''(x) &= \left(x^2-6\right)\cos(x) + 6x\sin(x) \\ f^{(4)}(x) &= 8x\cos(x) - \left(x^2-12\right)\sin(x) \end{aligned}$$

Next determine the maximum of the fourth derivative:

$$\begin{aligned} \left|f^{(4)}(x)\right| &= \left|8x\cos(x) - \left(x^2-12\right)\sin(x)\right| \\ &\leq |8x\cos(x)| + \left|\left(x^2-12\right)\sin(x)\right| \\ &\leq |8x| + \left|x^2-12\right| = 8x + 12 - x^2 \\ &= -(x-4)^2 + 28 \leq 24 \end{aligned}$$

And the error term could be transform into below form:

$$\begin{aligned} \frac{2-0}{180}h^4 \cdot 24 \leq 10^{-6} \\ h \leq 4.401 \times 10^{-2} \end{aligned} \tag{14}$$

And we can turn out that:

$$n \geq \frac{b-a}{h} = 46 \tag{15}$$

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Question 3
Given a function, $f(t) = \sqrt{t}$
a) Apply the Romberg Integration to find $R_{3,3}$ for the integral $\int_1^4 f(t)dt$.
b) Apply the Composite Simpson's Rule to approximate $\int_1^4 f(t)dt$ using eight intervals.
c) Comment on your results in (a) and (b).

Solution.

(a)

$$\begin{aligned}
R_{1,1} &= \frac{4-1}{2}(\sqrt{1}+\sqrt{4}) = \frac{9}{2} = 4.500 \\
R_{2,1} &= \frac{4-1}{4}\left(\sqrt{1.1} + 2\sqrt{\frac{1+4}{2}} + \sqrt{4}\right) = \frac{3}{4}(3+\sqrt{10}) = 4.622 \\
R_{2,2} &= R_{2,1} + \frac{1}{3}\left(R_{2,1} - R_{1,1}\right) = 4.663 \\
R_{3,1} &= \frac{4-1}{8}\left(\sqrt{1} + 2\sqrt{\frac{7}{4}} + 2\sqrt{\frac{5}{2}} + 2\sqrt{\frac{13}{4}} + \sqrt{4}\right) = \frac{3}{8}(3+\sqrt{7}+\sqrt{10}+\sqrt{13}) = 4.655 \\
R_{3,2} &= R_{3,1} + \frac{1}{3}\left(R_{3,1} - R_{2,1}\right) = 4.666 \\
R_{3,3} &= R_{3,2} + \frac{1}{4^2-1}\left(R_{3,2} - R_{2,2}\right) = 4.666
\end{aligned} \tag{16}$$

And we can plot the table as:

4.500		
4.622	4.663	
4.655	4.666	4.666

. . . .

(b) Here $n = 8$, $h = \frac{4-1}{8}$, $x_j = 1 + \frac{3}{8}j$:

$$\begin{aligned}
\int_1^4 f(t)dt &= \frac{1}{3}\frac{3}{8}\left(f(1) + 2\left(f\left(x_2\right) + f\left(x_4\right) + f\left(x_6\right)\right) + 4\left(f\left(x_1\right) + f\left(x_3\right) + f\left(x_5\right) + f\left(x_7\right) + f(49)\right)\right. \\
&= \frac{1}{8}\left(f(1) + 2\left(f\left(\frac{7}{4}\right) + f\left(\frac{5}{2}\right) + f\left(\frac{13}{4}\right)\right) + \right. \\
&\quad \left. 4\left(f\left(\frac{11}{8}\right) + f\left(\frac{17}{8}\right) + f\left(\frac{23}{8}\right) + f\left(\frac{29}{8}\right)\right) + f(4)\right) \\
&= \frac{1}{8}(3+\sqrt{7}+\sqrt{10}+\sqrt{13}+\sqrt{22}+\sqrt{34}+\sqrt{46}+\sqrt{58}) \\
&= 4.667
\end{aligned} \tag{17}$$

. . . .

(c)Here we can first calculate the exact value of the solution

$$\int_1^4 f(t)dt = \int_1^4 t^{\frac{1}{2}} dt = \left(\frac{2}{3}t^{\frac{3}{2}}\right)_1^4 = \frac{2}{3} \cdot (8-1) = \frac{14}{3} = 4\frac{2}{3}$$
$$= 4.666\cdots$$

And we can find that both method give a very close answer with the exact value. And Composite Simpson Law seems to be closer to the exact result.However due to the rounding off with 4 digits, we can not determine the accurate error in this situation because the error is relatively too small. We may first verify the result with the computer's result, here list the result by composite Simpson's formula and Romberg integration result respectively (The code is attached in footnote link):

Here first is the result from Composite Simpson's formula:

$$f_{simpson} = 4.666631373794358$$

And next we list the result of Romberg integration result:

4.500000000000000	0	0
4.621708245126285	4.662277660168379	0
4.655092592511359	4.666220708306383	4.666483578182250

And we can calculate and find that for composite Simpson formula: the absolute error is 0.00003529, relative error is 0.00000756. As for Romberg Formula: the absolute error is 0.00018309, relative error is 0.00003923. Which means the the Composite Simpson law is much accurate in this situation.

We are not sure if these two methods have different results in other n intervals.Here we use Matlab to derive the 2^n interval for 2 method and compute their relative error for 2 methods. And we can plot the graph as 1. And we can find that the relative error for Romberg integration is rapidly decrease at first time and later become flat as the increase of interval, comparing with the composite Simpson Rules, it initially start with the relatively small at $n = 2$ and decrease with the interval increase mildly. Additionally, we can calculate and find out that since the interval increase to $2^5 = 32$, the Romberg start to have a better prediction campaign with Composite Simpson.

Therefore we can conclude that in this example we can find that the Composite Simpson Formula can have a better prediction when the interval number is small. But if the interval number increasing, the relative error for Romberg will reduce rapidly and finally the accuracy will be better than the Composite Simpson Formula.

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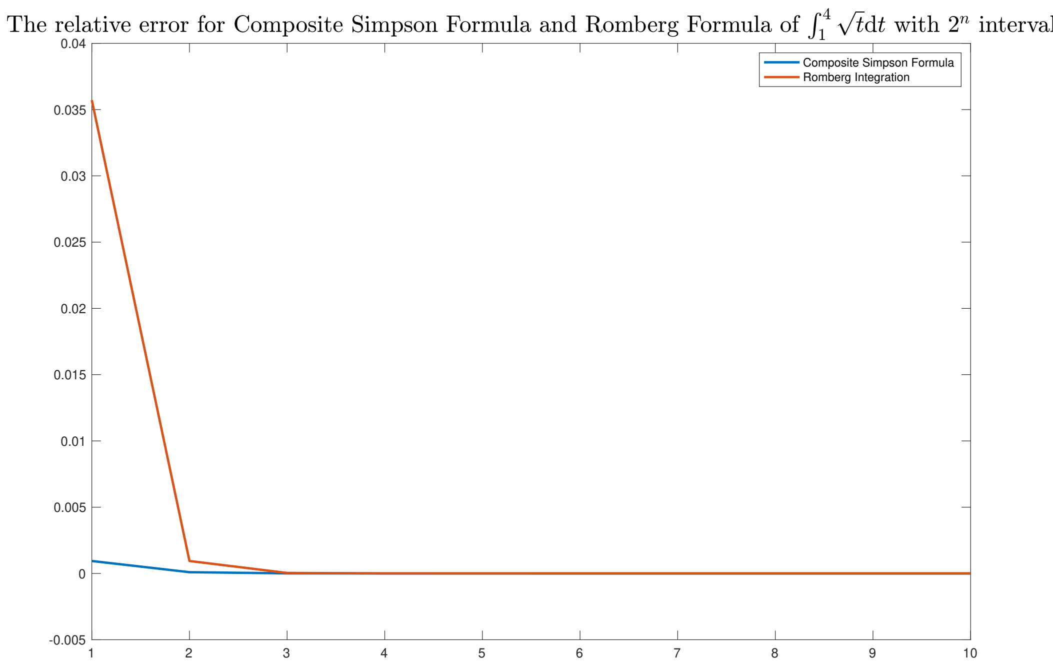

Figure 1: The relative error for Composite Simpson Formula and Romberg Formula of $\int_1^4 \sqrt{t}\mathrm{d}t$ with 2^n interval