Question 1
Given an initial value problem $y' = y^2 \sin(t)$ with $y(0) = -3$.
a. Approximate $y\left(\frac{\pi}{2}\right)$ using Euler's method with $n = 2$ steps.
b. Approximate $y\left(\frac{\pi}{2}\right)$ using Taylor's method of second order with $n = 2$ steps.
c. Approximate $y\left(\frac{\pi}{2}\right)$ using Modified Euler method with $n = 2$ steps.
d. Given the exact solution is $y(t) = \frac{3}{3\cos(t)-4}$. Compare the relative errors for the methods in (a), (b) and (c).

Solution.

(a) Euler's method
With $n = 2$ we have $h = (\pi/2)/2 = \pi/4$, $t_i = \pi \cdot i/4$, $w_0 = y(0) = -3$:

$$w_{i+1} = w_i + h(w_u^2 \sin(t_i)) = w_i + \frac{\pi}{4}(w_i^2 \sin(\pi i/4))$$

For $t_0 = 0$

$$w_0 = y(0) = -3.0000$$

For $t_1 = \frac{\pi}{4}$

$$w_1 = w_0 + \frac{\pi}{4}\left(w_0^2 \sin(0)\right) = -3.0000$$

For $t_2 = \frac{\pi}{2}$

$$w_2 = w_1 + \frac{\pi}{4}\left(w_1^2 \sin(\frac{\pi}{4})\right) = 1.9982$$

Therefore

$$y(\frac{\pi}{2}) = w_2 = 1.9982$$

. . . .

(b) Taylor's method of second order
With $n = 2$ we have $h = (\pi/2)/2 = \pi/4$, $t_i = \pi \cdot i/4$, $w_0 = y(0) = -3$:
And

$$\begin{aligned} f'(t) &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \\ &= y^2\cos(t) + \left(y^2\sin(t)\right)\cdot 2y\sin(t) \\ &= y^2\cos(t) + 2y^3\sin^2(t) \end{aligned}$$

Here the iteration function is

$$w_{i+1} = w_i + h\left(f(t_i, w_i) + \frac{h}{2}f'(t_0, w_0)\right)$$

For $t_0 = 0$

$$w_0 = y(0) = -3.0000$$

For $t_1 = \frac{\pi}{4}$

$$\begin{aligned}
w_1 &= w_0 + h\left(f\left(t_0, w_0\right) + \frac{h}{2} f'\left(t_0, w_0\right)\right) \\
&= (-3) + \frac{\pi}{4}\left((-3)^2 \sin(0) + \frac{1}{2} \cdot \frac{\pi}{4} \cdot \left((-3)^2 \cos(0) + 2(-3)^3 \sin(0)\right)\right) \\
&= -3 + \frac{9\pi^2}{32} = -0.2242
\end{aligned}$$

For $t_2 = \frac{\pi}{2}$

$$\begin{aligned}
w_2 &= (-0.2242) + \frac{\pi}{4}\left((-0.2242)^2 \sin\left(\frac{\pi}{4}\right) + \frac{\pi}{8}\left(2(-0.2242)^3 \sin^2\left(\frac{\pi}{4}\right) + (-0.2242)^2 \cos\left(\frac{\pi}{4}\right)\right)\right) \\
&= -0.1888
\end{aligned}$$

Therefore

$$y(\frac{\pi}{2}) = w_2 = -0.1888$$

(c) Modified Euler method

With $n = 2$ we have $h = (\pi/2)/2 = \pi/4$, $t_i = \pi \cdot i/4$, $w_0 = y(0) = -3$:
Here the iteration function is

$$w_{i+1} = w_i + \frac{h}{2}\left(f\left(t_i, w_i\right) + f\left(t_{i+1}, w_i + hf\left(t_i, w_i\right)\right)\right)$$

For $t_0 = 0$

$$w_0 = y(0) = -3.0000$$

For $t_1 = \frac{\pi}{4}$

$$\begin{aligned}
w_1 &= -3 + \frac{\pi}{8}\left((-3)^2 \sin(0) + \left(-3 + \frac{\pi}{4}\left((-3)^2 \sin(0)\right)\right)^2 \sin\left(\frac{\pi}{4}\right)\right) \\
&= \frac{9\sqrt{2}\pi}{16} - 3 = -0.5009
\end{aligned}$$

For $t_2 = \frac{\pi}{2}$

$$\begin{aligned}
W_2 &= (-0.5009) + \frac{\pi}{8}\left((-0.5009)^2 \sin\left(\frac{\pi}{4}\right) + \left((-0.5009) + \frac{\pi}{4}\left((-0.5009)^2 \sin\left(\frac{\pi}{4}\right)\right)^2 5\operatorname{in}\left(\frac{\pi}{2}\right)\right)\right) \\
&= -0.3799
\end{aligned}$$

Therefore

$$y(\frac{\pi}{2}) = w_2 = -0.3799$$

(d) Compare relative error

$$\varepsilon_{\text{Euler}} = \left|\frac{-0.75-(1.9982)}{-0.75}\right| = 3.6643$$

$$\varepsilon_{\text{taylor}} = \left|\frac{-0.75-(-0.2242)}{-0.75}\right| = 0.7483$$

$$\varepsilon_{\text{Modified}} = \left|\frac{-0.75-(-0.1888)}{-0.75}\right| = 0.4935$$

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Question 2
Given an initial value problem $y' = -\left(xy^2 + y\right)$ with $y(0) = 1$. Approximate $y(0.3)$ using Runge-Kutta method of order four with step length 0.1.

Solution.
With $h = 0.1$ we have $n = 3$, $t_i = 0.1i$, $w_0 = y(0) = 1$:
The Approximation $y(0.1)$ could be given that:

$$\begin{aligned}
w_0 &= 1 \\
k_1 &= hf\left(t_0, w_0\right) = 0.1 \cdot f(0,1) = -0.1000 \\
k_2 &= hf\left(t_0 + \frac{h}{2}, w_0 + \frac{1}{2}k_1\right) = 0.1 \cdot f(0.05, 1 - 0.05) = -0.0995 \\
k_3 &= hf\left(t_0 + \frac{h}{2}, w_0 + \frac{1}{2}k_2\right) = 0.1 \cdot f(0.05, 1 - 0.0498) = -0.0995 \\
k_4 &= hf\left(t_{0+1}, w_0 + k_3\right) = 0.1 \cdot f(0.1, 1 - 0.0995) = -0.0982 \\
w_1 &= w_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 0.9006,
\end{aligned}$$

The approximation y(0.2) could be given that:

$$\begin{aligned}
w_1 &= 0.9006 \\
k_1 &= hf\left(t_1, w_1\right) = 0.1 \cdot f(0.1, 0.9006) = -0.0982 \\
k_2 &= hf\left(t_1 + \frac{h}{2}, w_1 + \frac{1}{2}k_1\right) = 0.1 \cdot f(0.15, 0.9006 - 0.0491) = -0.0960 \\
k_3 &= hf\left(t_1 + \frac{h}{2}, w_1 + \frac{1}{2}k_2\right) = 0.1 \cdot f(0.15, 0.9006 - 0.0480) = -0.0962 \\
k_4 &= hf\left(t_2, w_1 + k_3\right) = 0.1 \cdot f(0.2, 0.9006 - 0.0962) = -0.0934 \\
w_2 &= w_1 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 0.8046,
\end{aligned}$$

The approximation y(0.3) could be given that:

$$\begin{aligned}
w_2 &= 0.8046 \\
k_1 &= hf\left(t_2, w_1\right) = 0.1 \cdot f(0.2, 0.8046) = -0.0934 \\
k_2 &= hf\left(t_2 + \frac{h}{2}, w_2 + \frac{1}{2}k_1\right) = 0.1 \cdot f(0.25, 0.8046 - 0.0467) = -0.0902 \\
k_3 &= hf\left(t_2 + \frac{h}{2}, w_2 + \frac{1}{2}k_2\right) = 0.1 \cdot f(0.25, 0.8046 - 0.0451) = -0.0904 \\
k_4 &= hf\left(t_3, w_2 + k_3\right) = 0.1 \cdot f(0.3, 0.8046 - 0.0904) = -0.0867 \\
w_3 &= w_2 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 0.7144,
\end{aligned}$$

Therefore the approximation of $y(0.3) = w_3 = 0.7144$

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Question 3
Consider the linear system

$$\begin{aligned}
4.3x_1 + 6.6x_2 - 5.3x_3 + 6.8x_4 &= 48.81 \\
2.5x_1 - 1.2x_2 + 6.6x_3 - 2.0x_4 &= -30.50 \\
5.4x_1 + 2.2x_2 - 2.6x_3 + 3.5x_4 &= 45.69 \\
-7.2x_1 + 5.3x_2 - 1.3x_3 + 4.9x_4 &= -18.15
\end{aligned}$$

a. Solve the system by using the method of Gaussian Elimination. In each arithmetic operation, round to two decimal places.
b. Solve the system by using the method of Gaussian Elimination with partial pivoting. In each arithmetic operation, round to two decimal places.

Solution.

(a) Using the method of Gaussian Elimination
Here we can represent the linear system with the augmented matrix as

$$\tilde{A} = \tilde{A}^{(1)} = \begin{bmatrix} 4.30 & 6.60 & -5.30 & 6.80 & : & 48.81 \\ 2.50 & -1.20 & 6.60 & -2.00 & : & -30.50 \\ 5.40 & 2.20 & -2.60 & 3.50 & : & 45.69 \\ -7.20 & 5.30 & -1.30 & 4.90 & : & -18.15 \end{bmatrix}$$

Perform the operation $(E_2-(0.58)E_1) \rightarrow (E_2)$, $(E_3-(1.26)E_1) \rightarrow (E_3)$, $(E_4-(-1.67)E_1) \rightarrow (E_4)$,

$$\tilde{A}^{(2)} = \begin{bmatrix} 4.30 & 6.60 & -5.30 & 6.80 & : & 48.81 \\ 0.01 & -5.03 & 9.67 & -5.94 & : & -58.81 \\ -0.02 & -6.12 & 4.08 & -5.07 & : & -15.81 \\ -0.02 & 16.32 & -10.15 & 16.26 & : & 63.36 \end{bmatrix}$$

Perform the operation $(E_3 - (1.22)E_2) \rightarrow (E_3)$, $(E_4 - (-3.24)E_2) \rightarrow (E_4)$,

$$\tilde{A}^{(3)} = \begin{bmatrix} 4.30 & 6.60 & -5.30 & 6.80 & : & 48.81 \\ 0.01 & -5.03 & 9.67 & -5.94 & : & -58.81 \\ -0.03 & 0.02 & -7.72 & 2.18 & : & 55.94 \\ 0.01 & 0.02 & 21.18 & -2.99 & : & -127.18 \end{bmatrix}$$

Perform the operation $(E_4 - (-2.74)E_3) \rightarrow (E_4)$,

$$\tilde{A}^{(4)} = \begin{bmatrix} 4.30 & 6.60 & -5.30 & 6.80 & : & 48.81 \\ 0.01 & -5.03 & 9.67 & -5.94 & : & -58.81 \\ -0.03 & 0.02 & -7.72 & 2.18 & : & 55.94 \\ -0.07 & 0.07 & 0.03 & 2.98 & : & 26.10 \end{bmatrix}$$

Finally, the matrix is converted back into a linear system that has a solution equivalent to the solution of the original system and the backward substitution is applied:

$$x_4 = 9.09$$

$$x_3 = \left[a_{3,5} - \sum_{j=4}^{4} a_{3j}x_j\right] / a_{33} = -4.71$$

$$x_2 = \left[a_{2,5} - \sum_{j=3}^{4} a_{2j}x_j\right] / a_{22} = -8.09$$

$$x_1 = \left[a_{1,5} - \sum_{j=2}^{4} a_{1j}x_j\right] / a_{11} = 3.59$$

(b) Using the method of Gaussian Elimination with partial pivoting

Here we can represent the linear system with the augmented matrix as

$$\tilde{A}^{(1)} = \begin{bmatrix} 4.30 & 6.60 & -5.30 & 6.80 & : & 48.81 \\ 2.50 & -1.20 & 6.60 & -2.00 & : & -30.50 \\ 5.40 & 2.20 & -2.60 & 3.50 & : & 45.69 \\ -7.20 & 5.30 & -1.30 & 4.90 & : & -18.15 \end{bmatrix}$$

Here find tha maximum on E_4. Switch the row $(E_4) \leftrightarrow (E_1)$

$$\begin{bmatrix} -7.20 & 5.30 & -1.30 & 4.90 & : & -18.15 \\ 2.50 & -1.20 & 6.60 & -2.00 & : & -30.50 \\ 5.40 & 2.20 & -2.60 & 3.50 & : & 45.69 \\ 4.30 & 6.60 & -5.30 & 6.80 & : & 48.81 \end{bmatrix}$$

Perform the operation $(E_2 - (-0.35)E_4) \rightarrow (E_2)$, $(E_3 - (-0.75)E_4) \rightarrow (E_3)$, $(E_1 - (-0.60)E_4) \rightarrow (E_1)$,

$$\tilde{A}^{(2)} = \begin{bmatrix} -7.20 & 5.30 & -1.30 & 4.90 & : & -18.15 \\ -0.02 & 0.66 & 6.15 & -0.28 & : & -36.85 \\ 0.00 & 6.18 & -3.58 & 7.18 & : & 32.08 \\ -0.02 & 9.78 & -6.08 & 9.74 & : & 37.92 \end{bmatrix}$$

Here find tha maximum on E_4. Switch the row $(E_4) \leftrightarrow (E_2)$

$$\begin{bmatrix} -7.20 & 5.30 & -1.30 & 4.90 & : & -18.15 \\ -0.02 & 9.78 & -6.08 & 9.74 & : & 37.92 \\ 0.00 & 6.18 & -3.58 & 7.18 & : & 32.08 \\ -0.02 & 0.66 & 6.15 & -0.28 & : & -36.85 \end{bmatrix}$$

Perform the operation $(E_3 - (0.63)E_1) \rightarrow (E_3)$, $(E_2 - (0.07)E_1) \rightarrow (E_2)$,

$$\tilde{A}^{(3)} = \begin{bmatrix} -7.20 & 5.30 & -1.30 & 4.90 & : & -18.15 \\ -0.02 & 9.78 & -6.08 & 9.74 & : & 37.92 \\ 0.01 & 0.02 & 0.25 & 1.04 & : & 8.19 \\ -0.02 & -0.02 & 6.58 & -0.96 & : & -39.50 \end{bmatrix}$$

Here find tha maximum on E_4. Switch the row $(E_4) \leftrightarrow (E_3)$

$$\begin{bmatrix} -7.20 & 5.30 & -1.30 & 4.90 & : & -18.15 \\ -0.02 & 9.78 & -6.08 & 9.74 & : & 37.92 \\ -0.02 & -0.02 & 6.58 & -0.96 & : & -39.50 \\ 0.01 & 0.02 & 0.25 & 1.04 & : & 8.19 \end{bmatrix}$$

Perform the operation $(E_3 - (0.04)E_2) \rightarrow (E_3)$,

$$\tilde{A}^{(4)} = \begin{bmatrix} -7.20 & 5.30 & -1.30 & 4.90 & : & -18.15 \\ -0.02 & 9.78 & -6.08 & 9.74 & : & 37.92 \\ -0.02 & -0.02 & 6.58 & -0.96 & : & -39.50 \\ 0.01 & 0.02 & -0.01 & 1.08 & : & 9.77 \end{bmatrix}$$

Finally, the matrix is converted back into a linear system that has a solution equivalent to the solution of the original system and the backward substitution is applied:

$$x_4 = 9.05$$

$$x_3 = \left[a_{3,5} - \sum_{j=4}^{4} a_{3j} x_j \right] / a_{33} = -4.70$$

$$x_2 = \left[a_{2,5} - \sum_{j=3}^{4} a_{2j} x_j \right] / a_{22} = -8.05$$

$$x_1 = \left[a_{1,5} - \sum_{j=2}^{4} a_{1j} x_j \right] / a_{11} = 3.60$$

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Question 4
Use the LU factorization to solve the following system.

$$\begin{aligned} x - y + 2z &= -1 \\ -x + 2y - 4z &= 4 \\ 2x - 4y + 9z &= -9 \end{aligned}$$

Solution.
Here we can transform the equation system into A and $\mathbf{b}$ and written the linear system as $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1.00 & -1.00 & 2.00 \\ -1.00 & 2.00 & -4.00 \\ 2.00 & -4.00 & 9.00 \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} -1.00 \\ 4.00 \\ -9.00 \end{bmatrix}$$

Here we first determine the LU factorization:
Perform the operation $(E_2 - (-1.00)E_1) \to (E_2)$, $(E_3 - (2.00)E_1) \to (E_3)$, $(E_3 - (-2.00)E_2) \to (E_3)$, And turn out the $\mathbf{U}$ upper triangle matrix.

$$\mathbf{U} = \begin{bmatrix} 1.00 & -1.00 & 2.00 \\ 0.00 & 1.00 & -2.00 \\ 0.00 & 0.00 & 1.00 \end{bmatrix}$$

As the multipliers m_{ij} could construct $\mathbf{L}$ the lower triangle matrix

$$\mathbf{L} = \begin{bmatrix} 1.00 & 0.00 & 0.00 \\ -1.00 & 1.00 & 0.00 \\ 2.00 & -2.00 & 1.00 \end{bmatrix}$$

Then we can produce the factorization:

$$\mathbf{A} = \mathbf{LU} = \begin{bmatrix} 1.00 & 0.00 & 0.00 \\ -1.00 & 1.00 & 0.00 \\ 2.00 & -2.00 & 1.00 \end{bmatrix} \begin{bmatrix} 1.00 & -1.00 & 2.00 \\ 0.00 & 1.00 & -2.00 \\ 0.00 & 0.00 & 1.00 \end{bmatrix}$$

Next we solve:

$$\mathbf{LUx} = \begin{bmatrix} 1.00 & 0.00 & 0.00 \\ -1.00 & 1.00 & 0.00 \\ 2.00 & -2.00 & 1.00 \end{bmatrix} \begin{bmatrix} 1.00 & -1.00 & 2.00 \\ 0.00 & 1.00 & -2.00 \\ 0.00 & 0.00 & 1.00 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1.00 \\ 4.00 \\ -9.00 \end{bmatrix}$$

we first introduce the substitution $\mathbf{y} = U\mathbf{x}$. Then $\mathbf{b} = L(U\mathbf{x}) = L\mathbf{y}$. That is,

$$\begin{bmatrix} 1.00 & 0.00 & 0.00 \\ -1.00 & 1.00 & 0.00 \\ 2.00 & -2.00 & 1.00 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1.00 \\ 4.00 \\ -9.00 \end{bmatrix}$$

That we can determine by forward-substitution process:

$$\begin{aligned} y_1 & & &= -1 \\ -y_1 + y_2 & & &= 4 \\ 2y_1 - 2y_2 & &+y_3 &= -9 \end{aligned}$$

And

$$\begin{cases} y_1 = -1 \\ y_2 = 3 \\ y_3 = -1 \end{cases}$$

Then solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$, the solution of the original system; that is,

$$\begin{bmatrix} 1.00 & -1.00 & 2.00 \\ 0.00 & 1.00 & -2.00 \\ 0.00 & 0.00 & 1.00 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$$

Using backward substitution we obtain $x_3 = -1$,$x_2 = 1$,$x_1 = 2$.

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