CSCI 8715
Class Project – Final Report
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Mathematical Measures to Estimate Partisan Gerrymandering

Abstract

Gerrymandering is one of the most critical issues in the political lexicon. It refers to candidates and their parties manipulating district maps as loopholes in elections to maximize their interests. In the United States, the presidential election occurs every four years, and senators from each state will act as representatives of each region. Senators will be the ones to submit final party seats based on the votes cast by voters in that region. However, it is recognized that different electoral divisions may create different bias levels. This paper will describe six mathematical approaches for estimating partisan gerrymandering. We analyze the behavior of measures for an exemplary case and apply it to a realistic dataset with multiple proposed plans. We explore the rationality for each proposed redistricting plan based on its cumulative score of partisan bias. Finally, we will benchmark and rank the six mathematical measures we described for estimating partisan bias and show the most effective approaches. Through our exploration, the efficiency gap, seats to votes ratio, distance to mean, and ranked-marginal deviation are suitable mathematical measures to estimate partisan gerrymandering. Our measures might effectively quantify partisan gerrymandering bias for future elections and have a more comprehensive application other than voting.

1. Introduction

The word "Gerrymandering" came from a political comic in the state of Massachusetts. At the time, governor Elbridge Gerry redrew the state's electoral district to ensure that his party could win the next election. Gerry adopted the strategies that concentrated the voters for his party in one place and spread the voters against his party across the board instead of the rule of the electoral district, which depends on geographic location. Gerry's strategy ensured the voters who supported his party won in most of the districts so that they won the election. No one can deny that his strategy pointed out holes in democracy. When people realized the seriousness of gerrymandering, solving it became a puzzle for many scientists.

To estimate partisan gerrymandering problems, we firstly studied Wikipedia examples to illustrate the measures defined in the paper (see section 4). Our strategy to measure gerrymandering bias is to simulate the simplified cases and perform similar computations in complex real world cases. Our approach adopts two different types of datasets, one is the spatial dataset, which contains information on location, coordinate system, perimeter, and area in the different districts. The other dataset is regarding the general vote, which provides the population and distribution of the parties. In this paper, we mainly observe the 2020 Minnesota election datasets and the last census data in Minnesota. We implemented six different measurements and analyzed the reasons and results for the partisan gerrymandering in the election (see section 4).

The challenges of measuring/calculating the partisan bias are the following: difficult to obtain a district spatial dataset (*e.g.,* perimeter and polygon area in the partition areas, area border information), limited information related to alternative district plans, and that, there are no common explanations/approaches to interpreting the results of gerrymandering. Although mathematicians and researchers in public policy institutions are devoted to defining the concepts and matrices to measure the gerrymandering related issues, summarizing and concluding the mathematical approaches regarding gerrymandering issues are the key challenges nowadays. So, it is crucial to define the approaches and interpretations by combining several mathematical concepts.

One of the main causes of the gerrymandering problem is redistricting maps after every census. The person who formally serves as the electoral college provides the proposed district plan before the election. Besides, investigating the population in the census is to obtain population mobility in order to reduce the partisan bias due to immigration issues in the next election. Mathematical tests are widely used in districts partitioning in today's elections, yet the critical challenge is to create effective methods to determine the district partitioning and reduce bias in drawing district boundaries.

In this paper, we adopted six different measurements to investigate the Minnesota dataset. The measurements include two types of data: spatial dataset and general vote dataset. We computed "Perimeter to area ratio" and "Area Ratio Convexity Measure" in seven of the proposed plans for election 2020. In other words, there are 7 x 2 entries in the matrix, which relate to the geospatial dataset. The rest of the matrix entries correspond to general vote datasets, including "Distance to mean," "Efficiency gap," "Ranked-marginal deviation," and "Seats to votes ratio." Each mathematical concept will be defined in detail in sections 3 and 4. As a result, there are 7 x 6 matrix entries in our approach. For the validation of our approach, we searched for similar results in the official reports, benchmarked our measurements' results, and calculated the error rate to verify the correctness of our approach.

Contribution: In this article, we will study the basic example of the gerrymandering problem in Wikipedia and then we will formally define several measurements to estimate partisan gerrymandering bias. We then perform similar calculations for Minnesota's presidential election in 2020, comparing and contrasting the results of each measurement. Finally, we will quantify the partisan gerrymandering bias by mathematical approaches and validate the effectiveness and accuracy of our proposed measurements in Minnesota's presidential election 2020.

Outline: the article is organized as follows: Section 2 provides the problem definition of our study and introduces the key concepts, i.e., spatial bias and computational definitions and formal problem definition. Section 3 introduces our experiment methodology. Section 4 presents the results and discusses/benchmarks each measurement in the article. Section 5 states the literature reviews and motivation of our study. Section 6 will conclude the article and discuss further possible future work. Supplementary information corresponding to all calculation details in the Minnesota election 2020 dataset and the simplified model of the Wikipedia page are provided at the end.

2. Problem Definition

Recently, with many released Census redistricting and demographic datasets on publicly accessible open data portals, decision-makers use various reports and algorithms to provide interpretation and decisions or general guides to the public. However, advocates, reporters, and others have voiced concerns about the bias of algorithms used to guide public decisions and the data that empower them.

Despite significant progress in spatial data science, relatively little research has been done on a standardized approach for assessing spatial bias in open data. Therefore, it is essential to provide methods and tools to quantify spatial data to fill this gap. The initial input is a very simplified model for an election, an imaginary state with uniform population distribution, where the Yellow party dominates the left side of the state, and the Blue party dominates the right side of the state. We calculated the following measures for the gerrymandering example on the Wikipedia page: i) Perimeter to area ratio; ii) Ranked-marginal deviation; iii) Distance to mean; iv) Seats to votes ratio; v) Area Ratio Convexity Measure, and vi) Efficiency gap. These measures are calculated for each partition (*i.e.*, district) and summed to get a total value representing the map. Based on the ground truth of this example, we estimate which measures might better indicate spatial bias. The output is the measure(s) with the highest accuracy in estimating gerrymandering across various plans. The objective is to use the gerrymandering example to illustrate spatial bias and quantify it using different criteria. The constraint is that we cannot change the partitions. One possible setback is that limited quantitative data exist that measure the extent of gerrymandering. We conduct a statewide analysis to evaluate our method to observe the cumulative impact of gerrymandering in Minnesota using the Census dataset and 2020 presidential election dataset.

2.1 Definition and Computation of Measures

2.1.1 Perimeter to area ratio

Detailed census data and election data have made it easier to construct electorates with an all-but-guaranteed political leaning. For example, we can usually spot a gerrymandered district by its wacky boundaries. Or we can use the perimeter to area ratio, $\frac{P_i}{A_i}$, where P_i is the district's total perimeter, and A_i is the district's area, and n denotes the number of congressional districts within a map:

$$\frac{P}{A} = \sum_{i=1}^{n} \frac{P_i}{A_i}$$

This formula gives lower gerrymandering scores to districts with a low perimeter to area ratio. Higher scores go to irregularly shaped districts. One should note that the formula doesn't take into account how a district got its odd contours [1].

2.1.2 Ranked-marginal deviation

Herschlag et al. [2] proposed ranked marginal deviation as a novel metric to study a redistricting plan within the underlying space of possible plans. They considered the baseline provided by the ordered marginal medians. Based on this method, for each districting plan, we quantify distances by considering the standard Euclidean norm (*i.e.*, L2 norm) between the ordered marginal means and the observed vote fractions of the ordered districts. Thus this metric could examine deviations in the ordered marginal vote fraction structure. For a given map, we take Party A's ordered vote fractions of plan, $\vec{d}_i$, and also, the ordered marginal means of Party A's vote fractions within all maps, $\vec{m}$, and consider the differences between the mean and the observed Party A's percentage. This provides a set of numbers to compute the two-norm, a distance metric:

$$\left\| \vec{m} - \vec{d}_i \right\|_2 = \sqrt{\sum_{j=1}^{n} (m_j - d_{ij})^2}$$

The ranked marginal deviation is minor when all the ordered Party A's vote percentages are the mean values precisely. However, this is an implausible scenario, as the percentages in the different districts are highly correlated. To understand the range of and interpret possible values, we could also plot the PDF of the ranked marginal deviation of our ensemble of randomly generated reasonable redistricting plans.

2.1.3. Distance to mean

Despite all algorithmically conducted geospatial analysis and cartographic visualizations to discuss which various practices of redistricting is optimal, the most spatial analysis and cartographic visualization of gerrymandering to date have relied almost exclusively on Euclidean, absolute representations of space. The distance to mean measure introduced in this paper indicates these spaces. This measure uses the average number of seats won in all districts, $m_{Seats} = \frac{\sum_{i=1}^{r} S_i}{r}$, where r denotes the number of redistricting plans, to calculate the absolute distance between the share of winning votes in a given redistricting plan, S_i, from this average value:

$$d_i = \left| S_i - m_{Seats} \right|$$

Distance to mean differentiates disproportionate versus proportionate outcomes. Lower distance to mean measure seems to be descriptive of the proportional outcome, whereas large measures indicate extreme outcomes. However, one can demonstrate how strictly Manhattan or Euclidean perspectives may fail to account for the quotidian experiences of space, that is, how individuals move through space in their everyday lives, which such approaches negligibly ignore. A set of complex mathematical techniques has been suggested in the new and improved computational tools to implement these determining parameters [3].

2.1.4 Seats to votes ratio

During the stable periods, which Burnham (1970) noted to be approximately every thirty-two years, where a critical issue influences the election that rearranges the existing party coalitions and cues in a new era of stable party coalitions, the normal baseline vote tends to favor one of the two political parties, manifesting itself in one party's dominance of presidential and congressional elections. However, the level of party voting fluctuates (or swings) around the regular vote, and the lesser party can enjoy temporary electoral success in what is known as a deviating election. There have been significant regional and urban-rural components to the party coalitions throughout the political development of the United States. The distribution of coalition groups into and within states affects the seats to votes ratio by way of partitioning and redistricting.

Congressional districts within states must be theoretically equally populated because each state receives at least one congressional district. However, districts are slightly malapportioned across states favoring smaller populations, predominantly rural states. In addition, districts must be contiguous and compact, which means that most districts will not stretch to combine heavily rural and urban areas. Recent polls find that urban areas tend to favor one party and rural areas favor the opposition party. The rules for allocating and redrawing districts would therefore slightly favor the dominance of the opposition party [4].

For a given electoral system, the votes to seat ratio can be described by plotting the share of votes against the share of seats a party won. It is common in such analyses to calculate vote and seat ratios for the major party vote or seats after removing minor party and write-in candidates from the calculation. This is because the two major political parties have dominated recent US elections and minor party candidates tend to win insignificant votes and rarely win seats. As a result, the seats to votes curves become more robust by omitting minor parties and showing how the electoral system works for the two major political parties. The resulting curves tend to follow an *S-curve* shape, which Kendall and Stuart (1950) asserted followed a law of cubic proportions. Further analysis (Tufte 1973) noted that this was not an ironclad law. The curve must not necessarily pass through the *fifty percent vote/fifty percent seat* point, and the slope was variable among electoral systems. Furthermore, for US elections, we are most interested in the vote range between 40-50 percent, where most recent elections occur and not in the tail areas. Therefore, it is common to characterize the seats to votes curve as a linear function. *Responsiveness* (the line slope) can be estimated as the coefficient on the share of votes between seats and votes in a bivariate regression. *Bias* is the shift from fifty percent where a party will win fifty percent of the seats. Bias represents the coefficient on the regression intercept and the estimated responsiveness [5]. Here, $\frac{S_i}{V_i}$, is calculated for each district and summed, where S_i denoted seats per district and is 1 if won in that district (gained more than fifty percent of casted voted) and 0 otherwise; V_i is the vote percentage in that particular district. Therefore:

$$\frac{S}{V} = \sum_{i=1}^{n} \frac{S_i}{V_i}$$

2.1.5 Area Ratio Convexity Measure

Mathematically speaking, the area ratio convexity measure indicates how closely a simple closed polygon P is superimposed onto its corresponding convex. Here, the ratio of the area of the largest convex set (or *endogon*), A_i, in P to that of the convex hull, C_i, of P, *i.e.*, $\frac{A_i}{C_i}$, is calculated and summed:

$$ARCM = \sum_{i=1}^{n} \frac{A_i}{C_i}$$

Many algorithms are available to compute the convex hull and also the endogon. We define this ratio to investigate whether such polygons are nearly convex and use the result to decide when legislative districts are nicely shaped. A legislative district is *nicely shaped* if it is equally or almost convex (*i.e.*, the ratio is nearly equal to one). Otherwise, it is considered *poorly shaped* (*i.e.*, the ratio is far less than one). There are reasons to believe that this method for measuring the shape of legislative districts as a signature of gerrymandering is as good as if not better than the other techniques proposed in other studies [5].

2.1.6 Efficiency gap

A gerrymandering party does not need to eliminate all of its inefficient votes. It only needs to end up with fewer wasted votes than the opposition party by winning its seats by smaller margins on average. The opposition party is left winning a small number of seats by large margins and losing a large number of seats where it claims many votes but still falls short of victory. The strategies that produce these results include *cracking* (breaking a cluster of voters and spreading them among multiple districts to dilute their vote rather than allowing them to exert enormous influence in fewer or even one single district) and *packing* (cramming voters into one district or as few districts as possible, leaving their numbers in the other districts too scant to win elections) [6]. Though the nuances vary, cracking and packing is how gerrymandering is constructed.

The efficiency gap is enlightened and calculated based on the idea that gerrymandering is always carried out by cracking or packing [7]. Cracking and packing both produce wasted votes, *i.e.*, votes with no contribution to a candidate's election. Wasted votes include both *lost* votes (those cast for a losing candidate) and *surplus* votes (those cast for a winning candidate but above what they needed to prevail). In the case of cracking, all votes for the loser candidate are wasted; in the case of packing, votes for the winning candidate above the 50% (plus one) threshold (which is needed for victory) are considered wasted. Each party's wasted votes are summed, one sum is subtracted from the other, and then, to be comparable across systems, this difference is divided by the total number of votes. This measure captures how district lines crack and pack one party's voters more than that of the other [8]:

$$Efficiency\ gap\ =\ \frac{\left|Wasted\ votes_{Party\ A} - Wasted\ votes_{Party\ B}\right|}{Total\ votes}$$

3. Methods

3.1 Wikipedia gerrymandering example illustration

As previously mentioned, we first studied the simplified Wikipedia gerrymandering example (*Figure 1*). This simplified example shows the four different district plans and explains the consequences of the gerrymandering phenomenon, which is caused by different partitioning. We assume that each district has the same electoral vote, but the results are different.

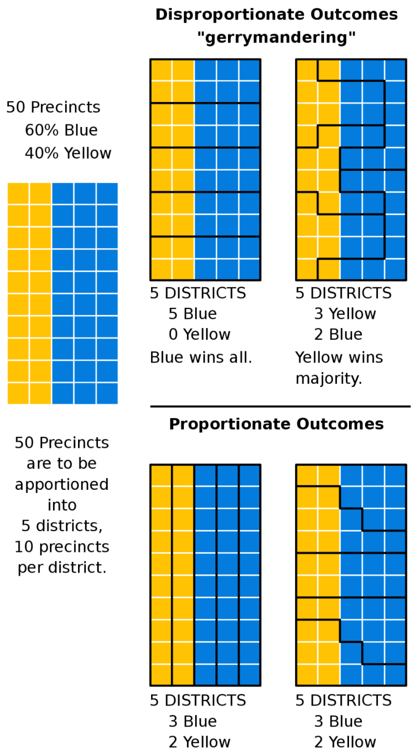

Figure 1. A simplified model to illustrate gerrymandering based on Wikipedia

According to these four maps, we implemented six measurements to explain which measurements can be used to identify disproportionate outcomes. Besides, we are devoted to proposing or identifying the measurements that can be used to identify disproportionate outcomes. The computation results have been summarized in *Table 1*, which demonstrates the gerrymandering measurements. The top right and top left partitioning maps would figure out disproportionate outcomes in this example. This is because different partitions contradict the assumption. As a result, it causes gerrymandering. To perform similar computation in the 2020 presidential election in Minnesota, we used the observation results from this example to discuss the gerrymandering phenomenon.

Our methodology makes the hypothesis that distance to mean differentiates disproportionate versus proportionate outcomes. Lower distance to mean measure seems to be descriptive of the proportional outcome, whereas large measures indicate disproportionate outcomes. In general, we consider picking a certain value as the threshold for the efficiency gap above which a district plan would be presumptively unconstitutional. The assistant professor of law at the University of Chicago, Nicholas Stephanopoulos, suggests the threshold of 8% or greater in the efficiency gap. As a result, the efficiency gap can identify disproportionate outcomes since the efficiency gaps of upper images are comparable yet far above the 20% threshold, whereas the bottom images fall below the 20% threshold in terms of efficiency gap. The supplementary information of the Wikipedia example can be found at the end of this paper.

Table 1. Quantifying gerrymandering for different exemplary maps based on frequently used estimators

	Perimeter to area ratio	Ranked-marginal deviation	Distance to mean	Seats to votes ratio	Area Ratio Convexity Measure	Efficiency gap
Top left image	7.0	0.89	1.75	8.33	5.0	30%
Top right image	9.8	1.05	1.25	2.22	3.32	30%
Bottom left image	11.0	1.41	0.25	3.0	5.0	10%
Bottom right image	7.8	0.96	0.25	4.17	4.33	10%

3.2 Overall short outline of the Minnesota dataset

In this project, we used the precinct shapefiles with election results collected and processed by the MGGG group, which are available in a Minnesota Election Shapefile dataset [9]. The data has been created by the Minnesota Secretary of State's Election Division and is a compilation of multiple partial data sourced from variable larger projects titled Minnesota Geospatial Commons [10], Census API [11], and IGER/Line Shapefiles [12]. These shapefiles came with election data already joined and precincts assigned to districts corresponding to our report's Existing map (Map1). Minnesota has six formally proposed alternate maps other than the existing map (*Figure 2*):

- The *Murphy* map was submitted by the DFL caucus in the state House of Representatives (Map2).
- The *Torkelso*n map was submitted by the GOP caucus in the state House (Map3).

- The *Wattson* map modifies the current districts as little as possible while meeting other criteria (Map4).
- The *Sachs* map represents the interests of Minnesota Democrats (Map5).
- The *Corrie* map aims to maximize the interests of communities of color (Map6).
- The *Anderson* map represents the interests of Minnesota Republicans (Map7).

Two maps (*i.e.*, Map2 and Map3) were produced by political party caucuses in the state Legislature, which were not formally submitted to the court. The other four different redistricting proposals (*i.e,* Map4, Map5, Map6, and Map7) were submitted by other groups of citizens who intervened in the legal process. The judges are likely to draw Minnesota's final map based on these plans. The corresponding shapefiles for all six alternative plans have been accessed and downloaded from the Minnesota Legislature website [13].

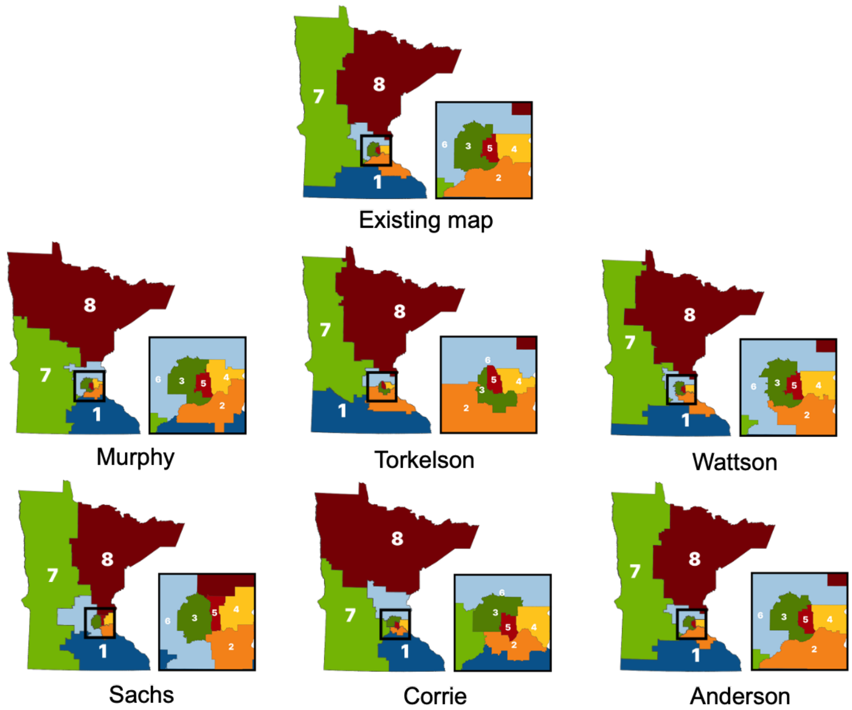

Figure 2. Proposed Minnesota redistricting plans [14]

3.3 Library import

We used the "GeoPandas" [15], an open-source library facilitating working with geospatial data in python, to compute geometries. GeoPandas is an extension of pandas' data types to allow spatial operations on geometric types. Geopandas further depends on the "shapely" library to perform geometric operations, the "fiona" library for file access, and the "matplotlib" library for plotting.

4. Results and Discussion

4.1 mathematical measurements in estimating partisan gerrymandering

We will talk about how mathematical measurements behave in estimating partisan gerrymandering. As mentioned earlier, we implemented six different measurements to estimate partisan bias. The following measurements are *efficiency gap*, *seats to votes ratio, distance to mean, rank-marginal deviation, perimeter to area ratio, and area ratio convexity measure*. *Figure 3* shows the overall spatial bias we compute in the six measurements, and then we accumulate the computation results together to quantify the partisan gerrymandering bias.

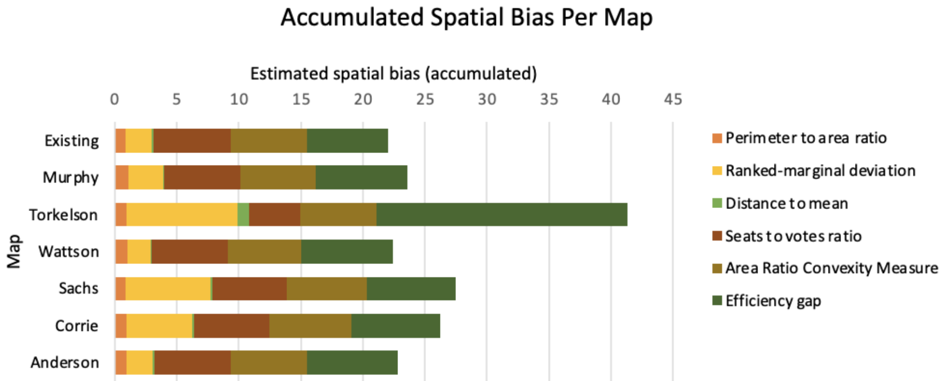

Figure 3. Overall estimation of gerrymandering bias in 2020 US presidential election in MN

4.1.1 Efficiency gap

Wasted votes are the basis of the efficiency gap measure of gerrymandering, where voters are grouped into electoral districts in such a way as to increase the wasted votes of one political fraction while decreasing the wasted votes of the other. We assumed at least 51% of the votes must be gained to determine the party's win. Except for the *Torkelson* map, in the rest of the six proposed maps, both the Democratic Party and Republican Party won half of the districts in Minnesota. Our calculations are based on the democratic party, if the democrat's support rate is greater and equal to 51%, we think the democratic party wins the district, then the rest of the votes beyond the 51% can be treated as wasted votes. On the other hand, all the votes in the lost districts are treated as wasted votes. By the formula, the efficiency gap is the difference between the wasted votes of the two parties divided by the total number of votes so the efficiency gap for the existing map is 6.6%. By a similar calculation, the efficiency gap of the rest of the maps is as follows: 7.4%, 20.2%, 7.4%, 7.2%, 7.2%, and 7.3% (*Figure 4*). We tend to rationalize that the smaller the value of the efficiency gap the smaller the gerrymandering in a given redistricting map. By looking at the formula of efficiency gap, the denominator (the number of votes) is constant, whereas the numerator varies as it is the difference between wasted votes of two parties. If the numerator limits to 0 (which means closer wasted votes for the two parties), the resulting value of the measure would be close to 0. We interpret that fewer wasted votes for the two parties infer less bias in the partitioning map. In general, we might consider picking a certain value as the threshold for the efficiency gap above which a district plan would be presumptively unconstitutional. According to the University of Chicago's assistant professor Nicholas Stephanopoulos, a threshold of 8% or greater in the efficiency gap could be viable. We intend to take a step further to suggest a threshold of 15%. With this new threshold, the result of the efficiency gap in the *Torkelson* map shows explicit gerrymandering bias because *Torkelson* cuts out suburban areas in favor of more rural territory thus leaning towards Republicans. Overall, we conclude that in the ideal redistrict plan, the difference in wasted votes between the two parties is close to zero, and so is the value of the efficiency gap. Besides, the efficiency gap can be a good mathematical measure to quantize the gerrymandering bias.

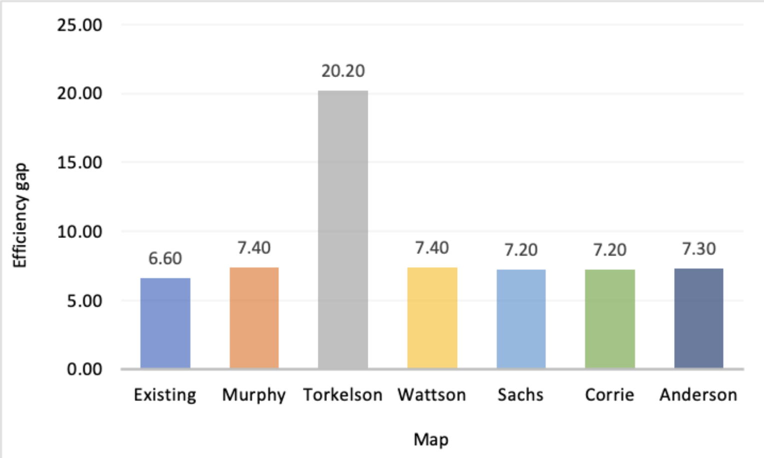

Figure 4. Efficiency gap in seven districting maps in MN

4.1.2 Seats to votes ratio

The shape of the seats-votes curve provides information on the partisan characteristics of a particular state, such as how election outcomes are affected by changes in voting and whether elections are fair to both parties. *Figure 5* shows the seats to votes ratio in seven redistricting plans in our experiment. This bar chart determines partisan bias. For example, if the party won 60% of the seats at 50% of the vote, we can conclude that the partisan bias is 10%. In the *Torkelson* map, the democrat party won 5 out of 8 election seats and gained 72.3% of the total votes on average; the ratio of seats to votes is 4.149 in the districts and the sum of seats to vote is 4.19 for the democrat party. The most ideal situation would be for the parties that won equally four districts, and each of the districts is very competitive. In other words, the total votes in each of the eight districts in both parties are very close. Then, we can say the party used 50% to obtain 50% of the total eight districts. In this case, the sum of seats to vote is 8 in the fairest plan (4*1/0.5). Similar calculations for the other proposed maps yield 6.21, 6.11, 6.15, 6.04, 6.09, and 6.16. As we mentioned earlier, the denominator of the six proposed plans is the same. In other words, the denominator is 50%, which indicates half of the district was won by a single party. In this case, the worst scenario is the opposite: nearly 100% of supporters won 4/8 seats (50%). So the sum of seats to votes is 4. Therefore, the value of the seat to vote ratio is closer to the optimal (fairest) plan indicating the plan has the best performance. In this case, the *Sachs* map has the best performance in seats to votes ratio. In this district, Democrats use 67% of the votes to win 50% of the seats. The seats to votes ratio in the *Torkelson* map is the outlier because Democrats win only in 3 out of 8 districts. Therefore, we cannot compare the *Torkelson* map to the same standard as other maps. In conclusion, to estimate bias by using the seat to vote approach we need to first carefully determine the best case and the worst-case scenarios, and find the range between them. Then, calculate the difference between the best/worst cases to identify the partisan bias.

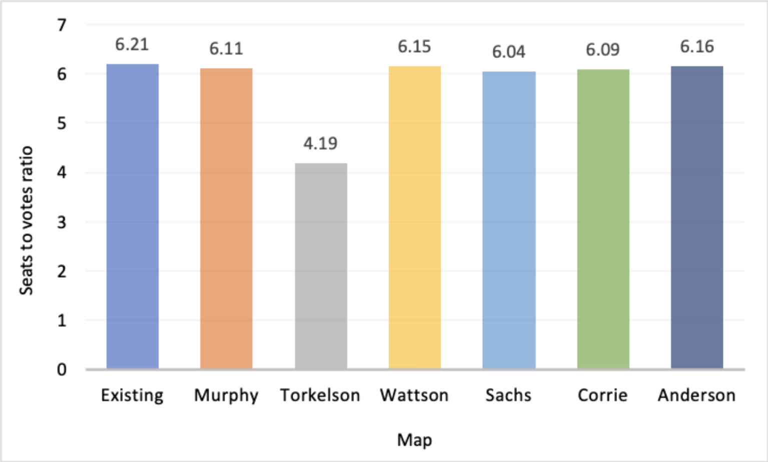

Figure 5. Seats to votes ratio in seven districting maps in MN

4.1.3 Ranked-marginal deviation

The purpose of the ranked-marginal deviation approach is to compare the plans within the dataset. It is used to calculate the distance of deviation in average for the collection of redistricting plans. Upon our calculations, the average supporter percentage in eight districts are 45.17, 54.69, 61.23, 68.8, 81.49, 39.81, 34.51, 43.06, respectively. Therefore, we will be using the average supporter percentage calculated as a standard. This has further helped us calculate the differences between the standard versus the other proposed maps. From the calculations, the ranked-marginal deviations of the seven proposed plans are 2.08, 2.81, 8.95, 1.82, 6.81, 5.32, and 2.17, respectively (*Figure 6*).

Our calculations demonstrate that it is easy to rank the performance of the existing redistricting plans in the dataset. By trying to use this approach to evaluate the new maps, we must apply the new average in the collection of redistricting plans. In other words, this approach only applies to comparing and finding the local optimal plans. By the formula of the ranked-marginal deviation, the difference in the average winning percentage in each partition area in the *Torkelson* is higher than in other partition cases, which causes the deviation to be higher. However, if the proportion of party A and party B under special circumstances is much closer than the *Ground Truth*, the deviation should be much closer to zero. So, we conclude that better partitioning has a smaller deviation (closer to 0). To sum up, using a ranked-marginal deviation to estimate the partisan bias has the restriction that must re-update the average percentage in the collection of redistricting plans. It cannot independently score the maps like efficiency gap and seats to votes ratio. Besides, the divergence between standardization and the proposed maps is smaller and means less partisan bias.

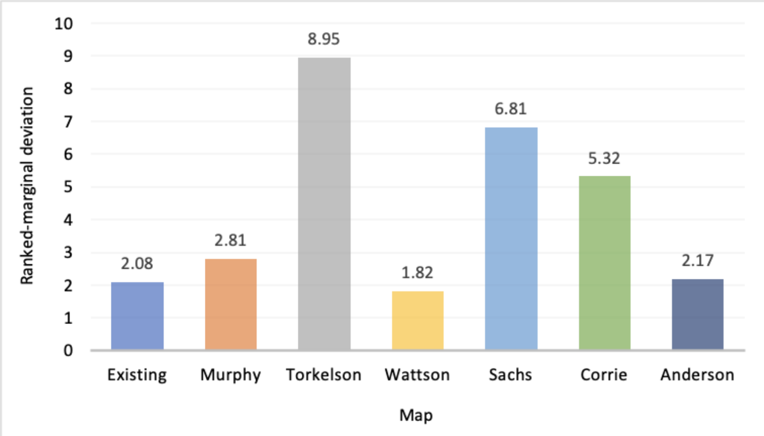

Figure 6. Ranked-marginal deviation in seven districting maps in MN

4.1.4 Distance to mean

The strategy of distance to mean is similar to the rank-marginal deviation. First, it is necessary to calculate the average number of seats the party won in the collection of the plans. In the Minnesota dataset, we noted that except for *Torkelson*, where Democrats won only 3 out of 8 seats, both parties won half of the seats each in any of the other plans. Therefore, the Democrats won 3.86 districts on average (*Figure 7*). We further calculated the distance to mean in each proposed map, and we found the *Torkelson* map has unusual results. We conclude that *Torkelson* has the largest partisan bias in the collection of the maps. In addition, using distance to mean to estimate the bias has a limitation that is only evaluated within the collection. In conclusion, compared with the average number of winning seats in the party within the collection, the smaller distance to average seats won means less partisan bias.

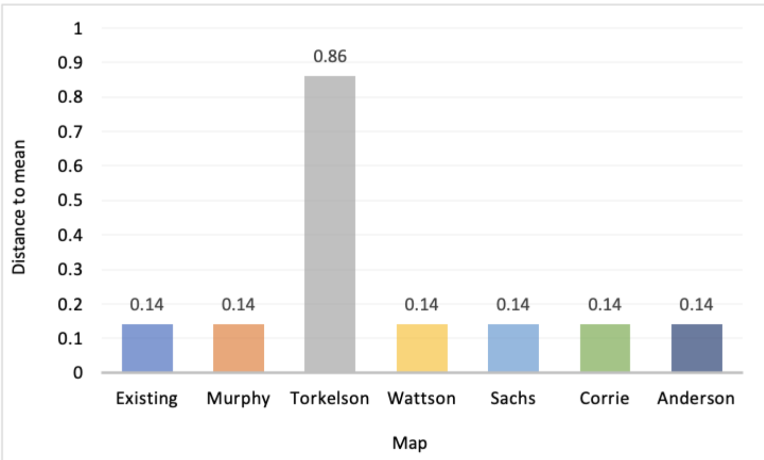

Figure 7. Distance to mean in seven districting maps in MN

4.1.5 Perimeter to area ratio

We calculated the perimeter and areas of the districts in Minnesota by its geospatial data. The ratio of perimeter to areas from Map 1 to Map 7 are 0.88, 1.06, 0.98, 1.04, 0.88, 0.93, 0.91, respectively (*Figure 8*). By observing these results, there is no significant evidence to show the measure is hugely influenced by partisan bias. We think it is least robust in estimating partisan bias due to its results only depending on the geometries and not highly related to election seats. What may influence the votes heavily depends on human factors such as migration, income, education level, and urbanization rather than mathematical approaches. For example, one of the principles to redistrict maps is to share similar social, cultural, and economic places together and we need to take these factors into consideration. However, given we assumed all the maps from the dataset are appropriate, we cannot conclude this measurement is effective in estimating the partisan bias in this experiment.

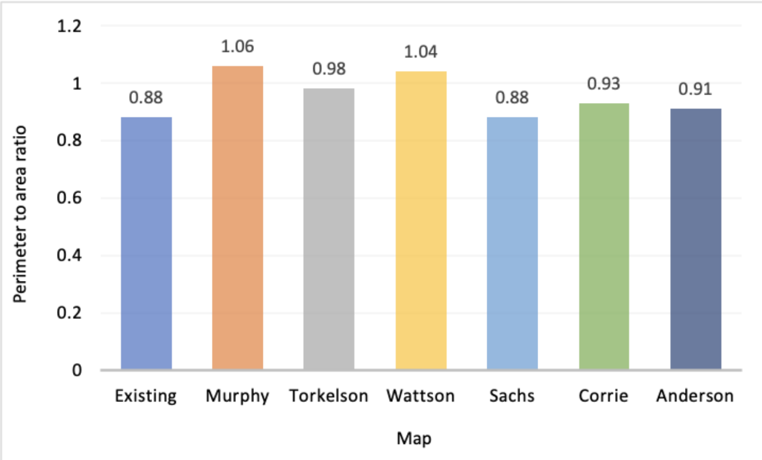

Figure 8. Perimeter to area ratio in seven districting maps in MN

4.1.6 Area Ratio Convexity Measure

Calculating the convex polygon area is different from the computing area. The convex hull is the intersection of all convex sets containing a given subset of Euclidean space or equivalently as the set of all convex combinations of points in the subset. The convex hull may be imagined as the shape enclosed by a rubber band stretched around the subset for a bounded subset of the plane [16]. The calculated results of the area ratio convexity measures include 6.15, 6.10, 6.15, 5.87, 6.42, 6.58, and 6.11 for Map1 through Map7, respectively (*Figure 9*). We found each of these results to be very close to one another. That means the measurement does not have sufficient robustness to estimate partisan bias. The approaches in 4.1.5 and 4.1.6 have similar input, that is area. However, the convex polygon area has a different geometrical representation compared with the perimeter. In this measurement, we only discuss the geometric representation of different maps in the dataset. In the report "What you need to know about redistricting in Minnesota", the author Montgomery mentions that *Torkelson* trims the suburbs to support more rural areas, thus tilting them toward the Republican party. We used the data sources in this published report to infer the calculations of the perimeter to area ratio and area ratio convexity measure. Our result does not show any anomalies in the latter case and hence, we cannot conclude a correlation between ARCM and partisan bias. All in all, there is no significant evidence to show the geometrical representation in different maps to be great indicators of partisan bias. It is necessary to include human factor features before re-estimating/re-evaluating the maps.

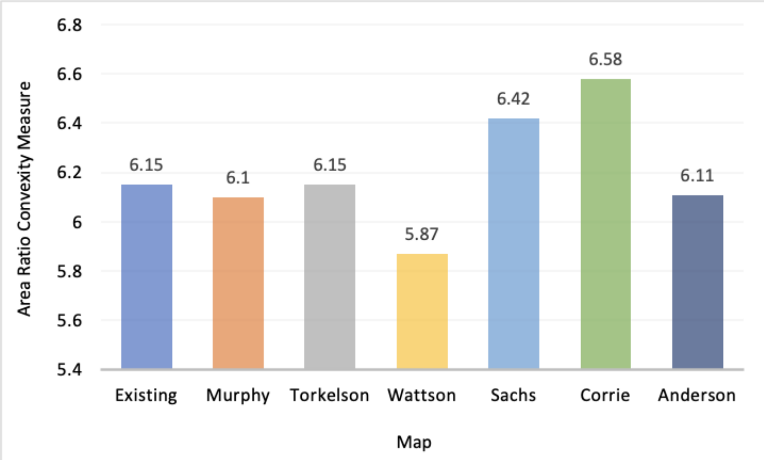

Figure 9. Area Ratio Convexity Measure in seven districting maps in MN

4.2 Suitable mathematical measures to estimate partisan gerrymandering

In this section, we will be talking about the suitable mathematical measures to estimate partisan gerrymandering and compare the strengths and weaknesses of each measure. The effectiveness of our six measurements can be divided into three tiers. Tier 1 contains the efficiency gap and seats to votes ratio; Tier 2 includes distance to mean and rank-marginal deviation, and Tier 3 involves perimeter to area ratio and area ratio convexity measure.

4.2.1 Tier 1

Tier 1 includes the "most powerful" measures to estimate partisan bias, because, according to the results, the efficiency gap and seats to votes ratio can directly quantify partisan bias. In other words, these two measurements have a range between the worst and the best cases. For example, the efficiency gap has a threshold of 8%, if the result is greater than the threshold it indicates the plan has serious partisan bias.

4.2.2 Tier 2

Estimating partisan bias in Tier 2 is stricter than in Tier 1 because the ranked-marginal deviation and the distance to mean lack the independence to conduct such an estimate. In other words, these two approaches depend on the average level of the collected data. Using Tier 2, it is easier to get the partisan bias rank inside the collection but hard to score/evaluate/rank a newly generated map without comparison.

4.2.3 Tier 3

Tier 3 is the least robust in estimating partisan bias as it requires more information to fulfill the estimate. There is no direct evidence that perimeter to area ratio and area ratio convexity measure approaches correlate with partisan bias. As mentioned earlier, we have assumed that the seven maps have appropriate district partitions, and there is no huge change/development in the state of Minnesota. For example, logging, mining, and tourism are located in northeastern Minnesota and the farming economy is located northwest Minnesota. The location of these industries will not presumably change for dozens of years. However, in the real situation, changing of location would occur very commonly. Once the economy, education, and income level develop, that would influence the party supporter's immigration as a result of changing population and redistricting the maps. We might think about the relationship between geometric/spatial definitions to take more complex information into consideration. This is also the limitation of our experiment, so we cannot conclude how the approaches in Tier 3 affect the partisan bias. This is why we believe these two methods are the least robust in estimating partisan bias.

4.3 Spatial bias

In addition to recognizing data bias as an umbrella term, we also need to watch out for spatial bias that negatively influences our decision-making process. We can identify four types of spatial bias: *map area*, *projection*, *disputed territory*, and *exclusion* [17].

4.3.1 Map area bias

Map areas bias refers to focusing primarily on larger regions on a map and less on smaller ones. A classic example is choropleth maps of US presidential elections, which draw our attention to the geometries of US states rather than their population size or the number of electoral votes. Conventional maps tend to exaggerate the political influence of rural states with larger geographic areas (like spacious Wyoming with a population of <600,000) and undermine the role of urban states with small areas (like tiny Rhode Island with a population of +1,000,000). For example, with a size 80 times larger than Rhode Island, Wyoming casts only three electoral votes in US presidential races, whereas Rhode Island has four. However, most map readers cannot easily distinguish because larger areas draw our eyes, not populations.

4.3.2 Projection bias

Projection bias is how maps portray geographic areas. Different projection systems have been developed to display a three-dimensional globe on a two-dimensional surface. As one of the most common projection systems, Mercator inflates the size of many countries within Europe and North America and diminishes the relative size (and importance) of those in Central Africa and Central America that lie closer to the equator. In recent years, the growing popularity of Google Maps and similar online services has made their default projection system, known as Web Mercator [18], ubiquitous, further framing distorted geography in our minds.

4.3.3 Disputed territory bias

Disputed territory bias refers to how web world views might display differently, depending on where you access them. For example, Russia flared a geopolitical dispute when it forcibly seized the Crimean Peninsula away from Ukraine in 2014. Google created two versions of its border with Ukraine on its platform to continue making profits in Russia. When viewed from a Russian IP address, Google Maps shows a solid-line border to signify Russia controls the territory. When viewed from anywhere else globally, Google Maps draws a dotted-line border representing a disputed territory. Despite claims to remain neutral on geopolitical disputes, the corporation took a side by displaying a solid border for Russian viewers [19].

4.3.4 Map exclusion bias

Map exclusion bias refers to ways that we fail to represent people or land through the act of omission. An example is the District of Columbia (DC): The nation's capital is not counted as a state, nor does it have a voting representative in the US Congress. Yet, with +700,000 residents (more than Wyoming or Vermont), it is granted electoral votes as if it were a state (though it can never have more than the least populous state). Similarly, US maps represent Puerto Rico, a territory with over 3 million US citizens, or other US territories such as American Samoa, Guam, the Northern Mariana Islands, and the US Virgin Islands do have no votes in Congress or for the Presidency. It is crucial that when data exists for these places, the maps make them visible, not make them vanish. If the latter, one might need to consider if the act of omission is also a type of underlying spatial bias, given that most residents in DC and other above territories are Black, Latino, and Pacific Islanders. These examples clearly show how crucial it is to recognize and systematically measure spatial bias in the data to improve decision-making. When considering gerrymandering and how it changes the outcome of political competitions, it mostly correlates with the map exclusion sort of bias.

5. Related Work

The 2010 Census led to multiple allegations of gerrymandering. Studies suggest that the 2010 Census redistricting maps mark the most gerrymandered in the history of the US. Such results have enticed geospatial scientists to create better methods to measure gerrymandering [20].

Mathematical tests have been applied to districts to determine whether they have been gerrymandered. For instance, in the *mean-median difference*, districts with a disproportionate representation of given demographics that are prone to vote for one party or those excessively diluted across multiple districts are candidates to determine bias in district boundaries. Other tests, called *partisan bias*, also include looking at the spread of votes across districts to see if districts have a more significant bias toward one party. In North Carolina, the non-partisan group PlanScore [21] used three tests, including the two mentioned, and a third test to show that North Carolina demonstrated evident gerrymandering in all three measures. The third test used in the North Carolina case was the efficiency gap. The wasted votes and votes for the losing party are tallied, and the method measures if districts are too packed, that is, too many voters for one party in one area, or cracked, that is, voters spread too thin for one party. A high percentage of wasted votes in such districts having little effect on an outcome denotes a bias for the district in representing the population.

Computer simulations to create alternative maps also apply different tests to see if districts are too packed or too cracked than dispersed relative to other possible mappings. Researchers think these ensemble approaches can measure bias in the next round of redistricting [22].

Other methods have been developed, including those that depend on the spatial shapes of districts to detect potential gerrymandering. Shape compactness metrics help determine if given profiles of districts are peculiarly shaped. Metrics for roundness (such as Polsby-Popper [23a], Schwartzberg [23b], and Reock [23c]), and context of other districts can create a threshold that a given district is spatially very different from others and potentially gerrymandered [24].

Another method uses a combination of geospatial boundaries and characteristics of the given districts, such as population and infrastructure, to generate less biased redistricting maps. A *capacitated double p-median problem with preference (CDPMP-P)* approach is used in this case. For spatial aspects, the method looks at boundaries and how they align with each other. Other assessment factors include population balance, preferences for potential facilities, and possible allocation of multiple polling places. The method uses spatial properties, how well balanced the representative population is, and if the district provisions for voting. Districts with a high score in these categories are considered less gerrymandered, while those with biases and a lack of supreme voting conditions are more likely to be gerrymandered [25].

Academic researchers have demonstrated that gerrymandering is a problem in the US and many representative democracies, mainly as the population shifts often, and a fair representation demands some form of redistricting. The critical challenge is to create effective methods to determine what less biased district maps might be alike. Still, these methods also need to be acceptable to the judiciary. For example, in an earlier Supreme Court case, the justices ruled that professionally created maps are not necessarily bias-free. The challenge, therefore, is to develop approaches that alleviate different stakeholders and be as objective as possible in drawing district boundaries.

6. Conclusion and Future Work

In conclusion, we have shown that:

- The measurement of the perimeter to area ratio and Area Ratio Convexity Measure has the least significant power in estimating partisan bias
- *Torkelson* map (Map 3) shows the largest bias
- Approved map (Map 1) shows the best performance
- Efficiency gap, seats to votes ratio, distance to mean, and ranked-marginal deviation are suitable mathematical measures to estimate partisan gerrymandering

In addition to the work that we have already done, there are some methods we could explore to improve spatial bias detection accuracy. For example, we could add further mathematical measures to the study of partisan gerrymandering or add weights to each measure defining partisan bias to create an integrated approach to estimating partisan gerrymandering. Furthermore, we could incorporate more political aspects into the computation. Another possible avenue for exploration would be to perform a similar case study for another state other than Minnesota with known gerrymandered maps. This is beneficial as to how generalizable the estimators are for new cases. For instance, as a brief foray in this direction, one could look into Kansas. A Kansas district court judge struck down a new congressional map, and instead, the judge ordered the state legislators to draft a new map for the first time. Further analyses are under study.

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Supplementary Information

Wikipedia gerrymandering example calculation

Calculations for the real-world dataset (Minnesota 2020 US house)