Linear Algebra

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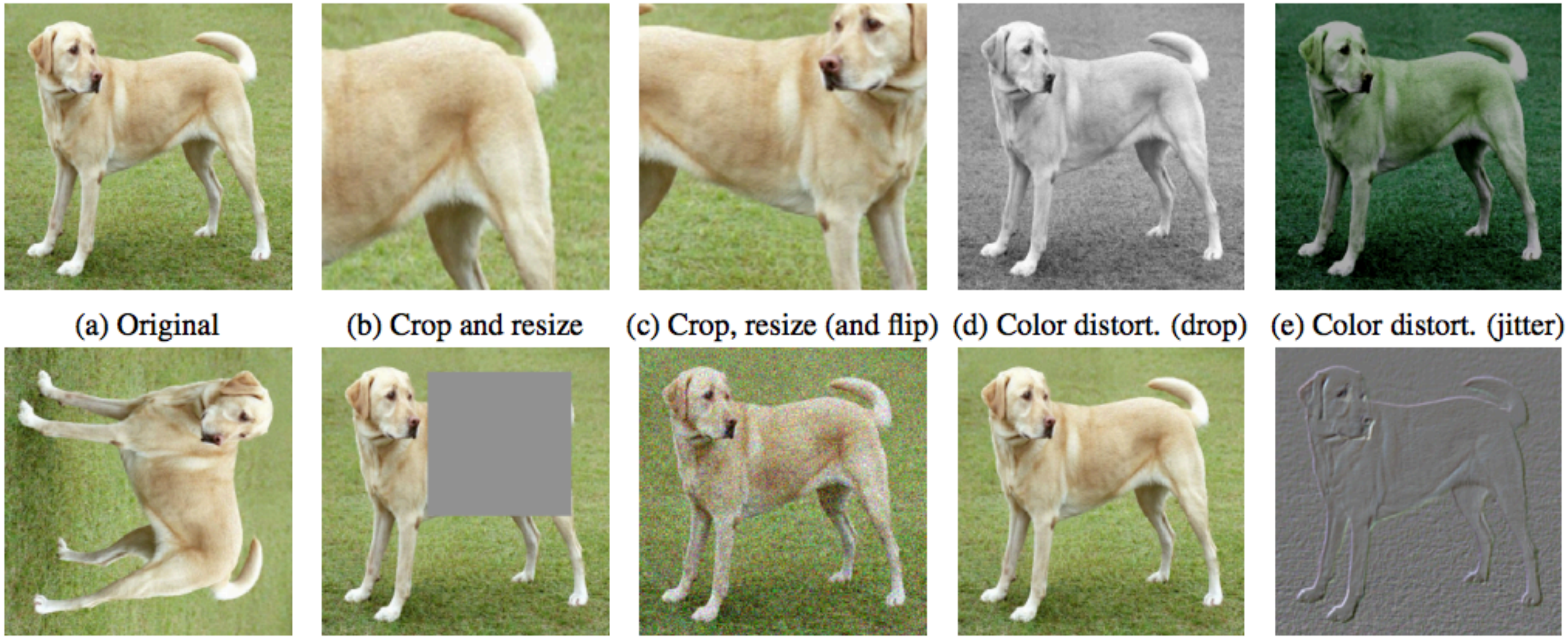
Self-supervised learning T. Chen et al., ICML 2020



(a) Original (b) Crop and resize (c) Crop, resize (and flip) (d) Color distort. (drop) (e) Color distort. (jitter)

(f) Rotate $\{90°, 180°, 270°\}$ (g) Cutout (h) Gaussian noise (i) Gaussian blur (j) Sobel filtering

We apply various transformations to the original image. The resulting images should share the same label.

2.4.1 Groups

⊗为任意运算符

- Consider a set $\mathcal{G}$ and an operation $\otimes: \mathcal{G} \otimes \mathcal{G} \to \mathcal{G}$ defined on $\mathcal{G}$. Then $G := (\mathcal{G}, \otimes)$ is called a group if the following holds
  - Closure of $\mathcal{G}$ under $\otimes$: $\forall x, y \in \mathcal{G}: x \otimes y \in \mathcal{G}$ 运算结果仍在G中
  - Associativity: $\forall x, y, z \in \mathcal{G}: (x \otimes y) \otimes z = x \otimes (y \otimes z)$ 满足结合律
  - Neutral element: $\exists e \in \mathcal{G}\ \forall x \in \mathcal{G}: x \otimes e = x$ and $e \otimes x = x$
  - Inverse element: $\forall x \in \mathcal{G}\ \exists y \in \mathcal{G}: x \otimes y = e$ and $y \otimes x = e$. We often write x^{-1} to denote the inverse element of x

- Additionally, If $\forall x, y \in \mathcal{G}, x \otimes y = y \otimes x$ (commutative), then $G := (\mathcal{G}, \otimes)$ is an Abelian group. 阿贝尔群：满足交换律

- Examples
- $(\mathbb{Z}, +)$ is a group and an Abelian group
  - …,-5, -4, -3, -2, -1, 0, 1, 2, 3,4, …

Closure: √
Associativity: $(x + y) + z = x + (y + z)$ √
Neutral element: 0 √
Inverse element: $\forall x \in \mathbb{Z}, y = -x \in \mathbb{Z}$ √

- $(\mathbb{Z}, -)$ is not a group: it does not satisfy associativity, has no neutral element or inverse element

~~Associativity~~: $(x - y) - z \neq x - (y - z)$

- Examples
- $(\mathbb{R}^{m\times n}, +)$, the set of $m\times n$-matrices is Abelian (component-wise addition).
按位相加

  - Closure: addition of any two matrices in $\mathbb{R}^{m\times n}$ is a matrix in $\mathbb{R}^{m\times n}$
  - Associativity: $\forall \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}^{m\times n}$, $(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C} = \boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})$
  - Neutral element: $\mathbf{0}$
  - Inverse element: $\forall \boldsymbol{A} \in \mathbb{R}^{m\times n}$, there exists its inverse element $-\boldsymbol{A}$
  - Commutative: $\forall \boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m\times n}$, $\boldsymbol{A}+\boldsymbol{B} = \boldsymbol{B}+\boldsymbol{A}$

2.4.2 Vector spaces

- Definition 由一个set和两个operation组成
- A real-valued vector space $V = (\mathcal{V}, +, \cdot)$ is a set $\mathcal{V}$ with two operations

$$+ : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$$

$$\cdot : \mathbb{R} \otimes \mathcal{V} \to \mathcal{V}$$

R：实数集，即：任意系数（scaler）与V相乘值域仍为V
·：标量乘法，以下条件都与该运算有关

- where
  - $(\mathcal{V}, +)$ is an Abelian group
  - Distributivity:

$$\forall \lambda \in \mathbb{R}, \boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}: \qquad \lambda \cdot (\boldsymbol{x} + \boldsymbol{y}) = \lambda \cdot \boldsymbol{x} + \lambda \cdot \boldsymbol{y}$$

$$\forall \lambda, \varphi \in \mathbb{R}, \boldsymbol{x} \in \mathcal{V}: \qquad (\lambda + \varphi) \cdot \boldsymbol{x} = \lambda \cdot \boldsymbol{x} + \varphi \cdot \boldsymbol{x}$$

  - Associativity (outer operation ·):

$$\forall \lambda, \varphi \in \mathbb{R}, \boldsymbol{x} \in \mathcal{V}: \qquad \lambda \cdot (\varphi \cdot \boldsymbol{x}) = (\lambda\varphi) \cdot \boldsymbol{x}$$

  - Neutral element (w.r.t to outer operation ·):

$$\forall \boldsymbol{x} \in \mathcal{V}: \qquad 1 \cdot \boldsymbol{x} = \boldsymbol{x}$$

2.4.2 Vector spaces

- Elements $\boldsymbol{x} \in \mathcal{V}$ are called vectors
- The neutral element of $(\mathcal{V}, +)$ is the zero vector $\mathbf{0} = [0, \cdots, 0]^{\mathrm{T}}$
- $+$ is called vector addition
- Elements $\lambda \in \mathbb{R}$ are called scalars
- Outer operation $\cdot$ is a multiplication by scalars

- Example
- $\mathcal{V} = \mathbb{R}^n, n \in \mathbb{N}$ is a vector space. Its operations are defined as
    - Addition: $\boldsymbol{x} + \boldsymbol{y} = (x_1, \cdots, x_n)^{\mathrm{T}} + (y_1, \cdots, y_n)^{\mathrm{T}} = (x_1 + y_1, \cdots, x_n + y_n)^{\mathrm{T}}$, for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$
    - Multiplication by scalars: $\lambda \boldsymbol{x} = \lambda (x_1, \cdots, x_n)^{\mathrm{T}} = (\lambda x_1, \cdots, \lambda x_n)^{\mathrm{T}}$, for $\boldsymbol{x} \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$
- Custom
- We usually write $\boldsymbol{x} \in \mathbb{R}^n$ in a column vector $\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Vector spaces - example

- $\mathcal{V} = \mathbb{R}^n, n \in \mathbb{N}$ is a vector space. Its operations are defined as

  - Addition: for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

  $\boldsymbol{x} + \boldsymbol{y} = (x_1, \cdots, x_n)^{\mathrm{T}} + (y_1, \cdots, y_n)^{\mathrm{T}} = (x_1 + y_1, \cdots, x_n + y_n)^{\mathrm{T}}$, for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

  - Multiplication by scalars: for $\boldsymbol{x} \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$

  $\lambda\boldsymbol{x} = \lambda(x_1, \cdots, x_n)^{\mathrm{T}} = (\lambda x_1, \cdots, \lambda x_n)^{\mathrm{T}}$

- We usually write $\boldsymbol{x} \in \mathbb{R}^n$ in a column vector $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

2.4.3 Vector Subspaces

- Sets contained in the original vector space 为原向量空间的子集
- "closed" 在向量子空间中做运算时值不会超出该子空间
- When we perform vector space operations on elements within this subspace, we will never leave it

- $U = (\mathcal{U}, +, \cdot)$ is called vector subspace of $V = (\mathcal{V}, +, \cdot)$, if
- $\mathcal{U} \subseteq \mathcal{V}$,
- $\mathcal{U} \neq \emptyset$, in particular $\mathbf{0} \in \mathcal{U}$

实际上只需U $\neq$ $\emptyset$
由于封闭性原则，对于元素x，scaler为-1时，-1*x = -x必须也在U中，而-x+x = 0必须在U中。

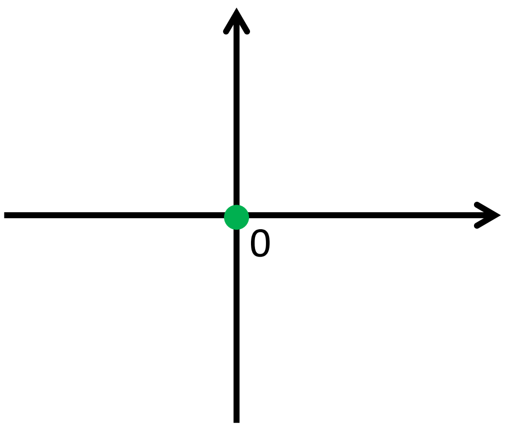
- Closure of U
  - $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{U}, \boldsymbol{x} + \boldsymbol{y} \in \mathcal{U}$
  - $\forall \boldsymbol{x} \in \mathcal{U}, \lambda \in \mathbb{R}, \lambda \boldsymbol{x} \in \mathcal{U}$

2.4.3 Vector Subspaces

- Examples
- For every vector space V, the trivial subspaces are V itself and $\{\mathbf{0}\}$

平凡子空间　　0为零向量，非标量

- Is it a subspace of $\mathbb{R}^2$?



Is it a subset of $\mathbb{R}^2$? Yes

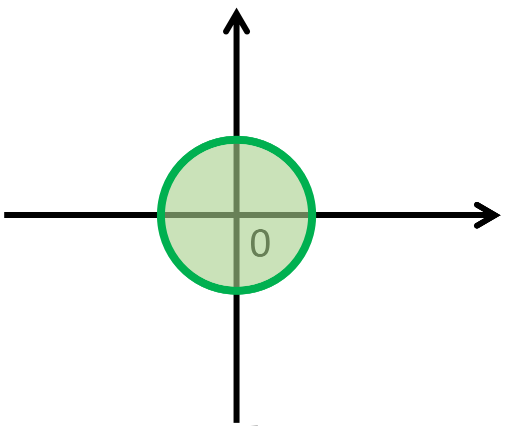
Does it satisfy $\mathcal{U} \neq \emptyset$, in particular $\mathbf{0} \in \mathcal{U}$ Yes

Does it satisfy closure? Yes

$$\boxed{\begin{array}{l} x + y \in \{0\} \\ \lambda x \in \{0\} \end{array}}$$

2.4.3 Vector Subspaces

- Examples
- Is it a subspace of $\mathbb{R}^2$?



Is it a subset of $\mathbb{R}^2$? Yes

Does it satisfy $\mathcal{U} \neq \emptyset$, in particular $\mathbf{0} \in \mathcal{U}$ Yes

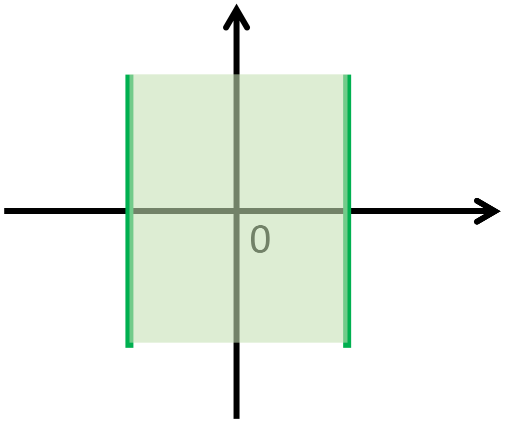
Does it satisfy closure? No

- Type equation here.

$$(0.8, 0) + (0.9, 0) = (1.7,0) \notin \mathcal{U}$$

2.4.3 Vector Subspaces

- Examples
- Is it a subspace of $\mathbb{R}^2$?



Is it a subset of $\mathbb{R}^2$? Yes

Does it satisfy $\mathcal{U} \neq \emptyset$, in particular $\mathbf{0} \in \mathcal{U}$ Yes

Does it satisfy closure? No

2.4.3 Vector Subspaces

- Examples

- The solution set of a homogeneous system of linear equations $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ with n unknowns $\boldsymbol{x} = [x_1, \cdots, x_n]^{\mathrm{T}}$. Is it a subspace of $\mathbb{R}^n$?

Is it a subset of $\mathbb{R}^n$? Yes

Does it satisfy $\mathcal{U} \neq \emptyset$, in particular $\mathbf{0} \in \mathcal{U}$ Yes

Does it satisfy closure? Yes

$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{U}$, we have $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}, \boldsymbol{A}\boldsymbol{y} = \mathbf{0}$

1) We investigate whether $\boldsymbol{x} + \boldsymbol{y} \in \mathcal{U}$.

Because $\boldsymbol{A}(\boldsymbol{x} + \boldsymbol{y}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{A}\boldsymbol{y} = \mathbf{0}$,

We know $\boldsymbol{x} + \boldsymbol{y}$ is a solution, thus belonging to $\mathcal{U}$

2) We investigate whether $\lambda\boldsymbol{x} \in \mathcal{U}$.

Because $\boldsymbol{A}(\lambda\boldsymbol{x}) = \lambda(\boldsymbol{A}\boldsymbol{x}) = \mathbf{0}$,

We know $\lambda\boldsymbol{x}$ is a solution, thus belonging to $\mathcal{U}$

2.4.3 Vector Subspaces

- Examples

- The solution set of an inhomogeneous system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}, \boldsymbol{b} \neq \boldsymbol{0}$. Is it a subspace of $\mathbb{R}^n$?

Is it a subset of $\mathbb{R}^2$? Yes

Does it satisfy $\mathcal{U} \neq \emptyset$, in particular $\boldsymbol{0} \in \mathcal{U}$ No

Does it satisfy closure? No

Linear combination

- Consider a vector space V and k vectors $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k \in V$. For $\lambda_1, \cdots, \lambda_k \in \mathbb{R}$, $\boldsymbol{v} \in V$ is called a linear combination of vectors $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k$, if

$$\boldsymbol{v} = \lambda_1 \boldsymbol{x}_1 + \cdots + \lambda_k \boldsymbol{x}_k = \sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i \in V$$

2.5 Linear Independence

- Consider a system of linear functions $\lambda_1 \boldsymbol{x}_1 + \cdots + \lambda_k \boldsymbol{x}_k = \boldsymbol{0}$

  非平凡解，即非0解
- If there is a non-trivial solution, $\lambda_1, \ldots, \lambda_k$, with at least one $\lambda_i \neq 0$, the vectors $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k$ are linearly dependent

- If only the trivial solution exists, i.e., $\lambda_1 = \cdots = \lambda_k = 0$, then vectors $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k$ are linearly independent

- Intuitively, a set of linearly independent vectors consists of vectors that have no redundancy, *i.e.*, if we remove any of those vectors from the set, we will lose something.

  从非线性相关的数据中移除任意数据都会导致信息丢失。

How to determine linear (in)dependence

- Write all vectors $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k$ as columns of a matrix $\boldsymbol{A}$
- Perform Gaussian elimination until the matrix is in row echelon form
- The pivot columns correspond to independent vectors

$$\begin{array}{ccc} & & \\ \multicolumn{3}{c}{\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}} \\ x_1 & x_2 & x_3 \end{array}$$

$$\boxed{x_2 = 2x_1}$$

pivots, linearly independent

- All column vectors are linearly independent if and only if all columns are pivot columns.
- If there is at least one non-pivot column, the columns are linearly dependent.

Determine linear (in)dependence

- Consider three vectors in $\mathbb{R}^3$

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \boldsymbol{x}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \boldsymbol{x}_3 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\text{R1+R2->R2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\text{Swap R2 and R3}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\xrightarrow{\text{R3-2R2->R3}} \begin{array}{ccc} \multicolumn{3}{c}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}} \\ x_1 & x_2 & \underline{x_3} \end{array}$$

$$\underline{x_3 = x_1 + 2x_2}$$

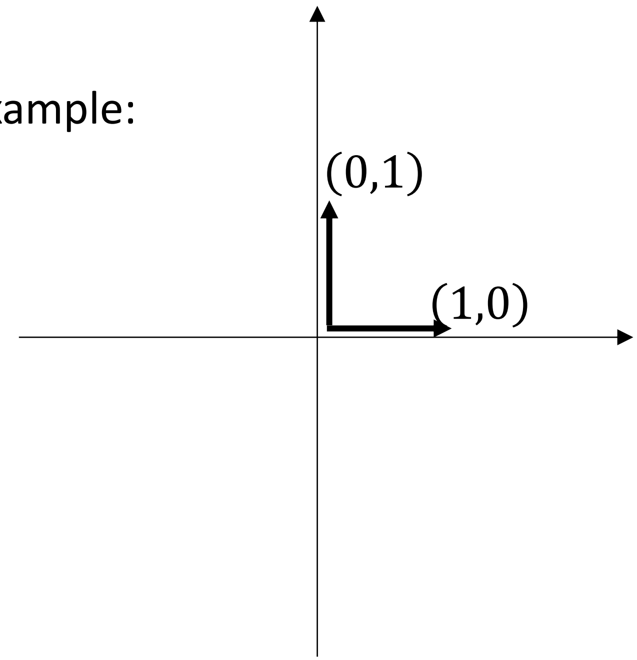
The Basis of a vector space

- A set of vectors $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k\}$ is said to form a basis for a vector space if

(1) The vectors $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k\}$ span the vector space: every vector in this space can be represented by a linear combination of $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k\}$

(2) The vectors $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k\}$ are linearly independent.

Example:



- Example
- In $\mathbb{R}^3$, the canonical/standard basis is

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- Different bases in $\mathbb{R}^3$ are $\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

First, this REF has three pivots, so the three bases are linearly independent.

Second, do the three bases span $\mathbb{R}^3$?

Specifically, $\forall\ [a, b, c]^T \in \mathbb{R}^3$, we examine whether it can be obtained by a linear combination by the three bases.

$$\lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

We can obtain the solution

$$\begin{cases} \lambda_3 = c \\ \lambda_2 = b - c \\ \lambda_1 = a - b \end{cases}$$

- Another different basis in $\mathbb{R}^3$ is

$$\mathcal{B}_2 = \left\{ \begin{bmatrix} 0.5 \\ 0.8 \\ 0.4 \end{bmatrix}, \begin{bmatrix} 1.8 \\ 0.3 \\ 0.3 \end{bmatrix}, \begin{bmatrix} -2.2 \\ -1.3 \\ 3.5 \end{bmatrix} \right\}$$

- Another example

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -4 \end{bmatrix} \right\}$$

is linearly independent, but not a basis of $\mathbb{R}^4$: For instance, the $[1,0,0,0]^\top$ cannot be obtained by a linear combination of elements in $\mathcal{A}$.

So, a couple of things about basis

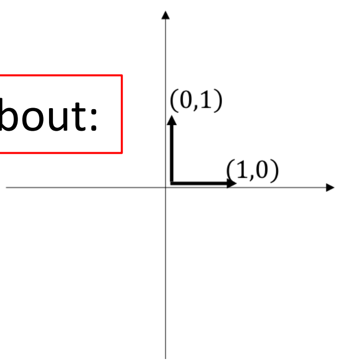
- Let $V = (\mathcal{V}, +, \cdot)$ be a vector space and $\mathcal{B} \subseteq \mathcal{V}, \mathcal{B} \neq \emptyset$ be a basis of V.
- $\mathcal{B}$ is a maximal linearly independent set of vectors in V, i.e., adding any other vector to this set will make it linearly dependent.

  Removing any vector will make it cannot represent any vector in the space V

- Every vector $\boldsymbol{x} \in V$ is a linear combination of vectors from $\mathcal{B}$, and every linear combination is unique, i.e., with

$$x = \sum_{i=1}^{k} \lambda_i \boldsymbol{b}_i = \sum_{i=1}^{k} \psi_i \boldsymbol{b}_i$$

Think about:



and $\lambda_i, \psi_i \in \mathbb{R}, \boldsymbol{b}_i \in B$ it follows that $\boldsymbol{\lambda}_i = \psi_i, i = 1, \cdots, k$.

must have, or at least have

- Every vector space V possesses a basis $\mathcal{B}$.
- There can be many bases of a vector space.
- All bases possess the same number of elements, called the basis vectors

$\boxed{\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \text{ then } \dim(\mathcal{B}) = 3}$

- Dimension of (V): number of basis vectors of V. We write $\dim(V)$
- If $U \subseteq V$ is a subspace of V , then $\dim(U) \leq \dim(V)$
- $\dim(U) = \dim(V)$ if and only if $U = V$

充要条件

Determining a Basis

- Write the spanning vectors as columns of a matrix $\boldsymbol{A}$
- Determine the row-echelon form of $\boldsymbol{A}$.
- The spanning vectors associated with the pivot columns are a basis of U.

- Example
- For a vector subspace $U \subseteq \mathbb{R}^5$, spanned by the vectors

已知span，不必再验证

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \\ -2 \end{bmatrix}, \quad \boldsymbol{x}_3 = \begin{bmatrix} 3 \\ -4 \\ 3 \\ 5 \\ -3 \end{bmatrix}, \quad \boldsymbol{x}_4 = \begin{bmatrix} -1 \\ 8 \\ -5 \\ -6 \\ 1 \end{bmatrix} \in \mathbb{R}^5$$

Determining a Basis - Example

- Which vectors of $\boldsymbol{x}_1, \dots, \boldsymbol{x}_4$ are a basis for U?
- Check whether $\boldsymbol{x}_1, \dots, \boldsymbol{x}_4$ are linearly independent.

$$\boxed{\sum_{i=1}^{4} \lambda_i \boldsymbol{x}_i = \boldsymbol{0}}$$

- A homogeneous system of equations with matrix

$$[\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4] = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ -1 & 1 & 3 & -5 \\ -1 & 2 & 5 & -6 \\ -1 & -2 & -3 & 1 \end{bmatrix}$$

- Through Gaussian Elimination, we obtain the row-echelon form

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ -1 & 1 & 3 & -5 \\ -1 & 2 & 5 & -6 \\ -1 & -2 & -3 & 1 \end{bmatrix} \rightsquigarrow \cdots \rightsquigarrow \begin{array}{c} \begin{array}{cccc} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 & \boldsymbol{x}_4 \end{array} \\ \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_4$ are linearly independent. Therefore, $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_4\}$ is a basis of U

2.6.2 Rank 秩

- The number of linearly independent columns of a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ is called the rank of $\boldsymbol{A}$, denoted by $\mathrm{rk}(\boldsymbol{A})$

- $\mathrm{rk}(\boldsymbol{A})$ also equals the number of linearly independent rows

- Rank gives us an idea of how much information a matrix contains

Important properties

- $\mathrm{rk}(\boldsymbol{A}) = \mathrm{rk}(\mathbf{A}^{\mathrm{T}})$
- Columns and rows of $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ can both span subspaces of the same dimension $\mathrm{rk}(\boldsymbol{A})$

  无论横竖，矩阵包含的信息是相同的，因而秩相同，pivot列数也相同——只是看待矩阵的方向不同。

- The basis of the subspace spanned by columns (rows) can be found by Gaussian elimination to $\boldsymbol{A}$ $(\mathbf{A}^{\mathrm{T}})$ to identify the pivot columns.

- For all $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ it holds that $\boldsymbol{A}$ is regular (invertible) if and only if $\mathrm{rk}(\boldsymbol{A}) = n$.

$$\begin{bmatrix} * & & & & \\ & * & & & \\ & & * & & \\ & & & \ddots & \\ & & & & * \end{bmatrix}_{n \times n}$$

- Example

- We use Gaussian elimination to determine the rank

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix} \rightsquigarrow \cdots \rightsquigarrow \begin{array}{c} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \end{array} \end{array}$$

- 2 pivot columns. So $\mathrm{rk}(\boldsymbol{A}) = 2$

More properties

- For all $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and all $\boldsymbol{b} \in \mathbb{R}^{m}$ it holds that the linear equation system $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ can be solved if and only if $\mathrm{rk}(\boldsymbol{A}) = \mathrm{rk}(\boldsymbol{A}|\boldsymbol{b})$, where $\boldsymbol{A}|\boldsymbol{b}$ denotes the augmented matrix

- For $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ the subspace of solutions for $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{0}$ possesses dimension $n - \mathrm{rk}(\boldsymbol{A})$.

Let's look at a simpler case where $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and $\mathrm{rk}(\boldsymbol{A}) = n$.
In this scenario, the dimension of the solution space is $n - \mathrm{rk}(\boldsymbol{A}) = \boldsymbol{0}$.
The only solution is $\boldsymbol{x} = \boldsymbol{0}$.

More properties

- A matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ has full rank if its rank equals the largest possible rank for a matrix of the same dimensions.

- The rank of a full-rank matrix is the lesser of the number of rows and columns, i.e., $\mathrm{rk}(\boldsymbol{A}) = \min(m, n)$.

For example, for $\boldsymbol{A} \in \mathbb{R}^{5 \times 3}$, $\mathrm{rk}(\boldsymbol{A})$ does not exceed 3.

- A matrix is said to be rank deficient if it does not have full rank.

2.7 Linear Mappings

- For vector spaces V, W, a mapping $\Phi: V \to W$ is called a linear mapping if

$$\forall \boldsymbol{x}, \boldsymbol{y} \in V, \forall \lambda, \psi \in \mathbb{R} : \Phi(\lambda \boldsymbol{x} + \psi \boldsymbol{y}) = \lambda \Phi(\boldsymbol{x}) + \psi \Phi(\boldsymbol{y})$$

- It implies the following

$$\Phi(\boldsymbol{x} + \boldsymbol{y}) = \Phi(\boldsymbol{x}) + \Phi(\boldsymbol{y}) \qquad \Phi(\lambda \boldsymbol{x}) = \lambda\, \Phi(\boldsymbol{x})$$

Example

线性映射的形式↓

- The mapping $\Phi: \mathbb{R}^2 \to \mathbb{C}, \Phi(\boldsymbol{x}) = x_1 + ix_2$, is a linear mapping:

$$\Phi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = (x_1 + y_1) + i(x_2 + y_2) = x_1 + ix_2 + y_1 + iy_2$$

$$= \Phi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + \Phi\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right)$$

$$\Phi\left(\lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \lambda x_1 + \lambda i x_2 = \lambda(x_1 + ix_2) = \lambda\Phi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

2.7 Linear Mappings

- For linear mappings $\Phi: V \rightarrow W$ and $\Psi: W \rightarrow X$, the mapping $\Phi \circ \Psi: V \rightarrow X$ is also linear 传递性

- If $\Phi: V \rightarrow W$ and $\Psi: V \rightarrow W$ are both linear mappings, then $\Phi + \Psi$ and $\lambda\Phi, \lambda \in \mathbb{R}$ are also linear.

Coordinates of a vector

- Consider a vector space V and an ordered basis $B = (\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$ of V. For any $\boldsymbol{x} \in V$ we obtain a unique representation

$$\boldsymbol{x} = a_1 \boldsymbol{b}_1 + \ldots + a_n \boldsymbol{b}_n$$

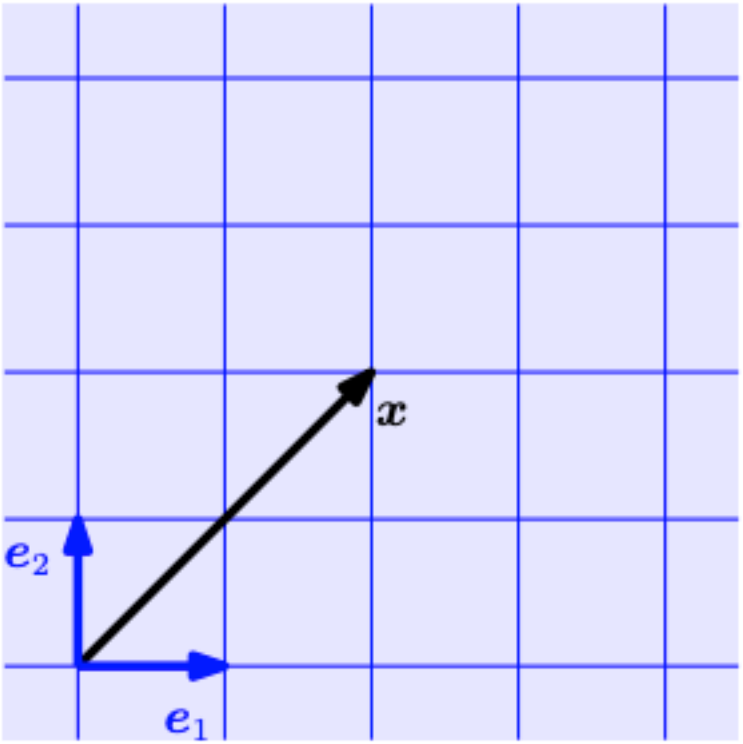
of $\boldsymbol{x}$ with respect to B. Then $\alpha_1, \cdots, \alpha_n$ are the coordinates of $\boldsymbol{x}$ with respect to B, and the vector

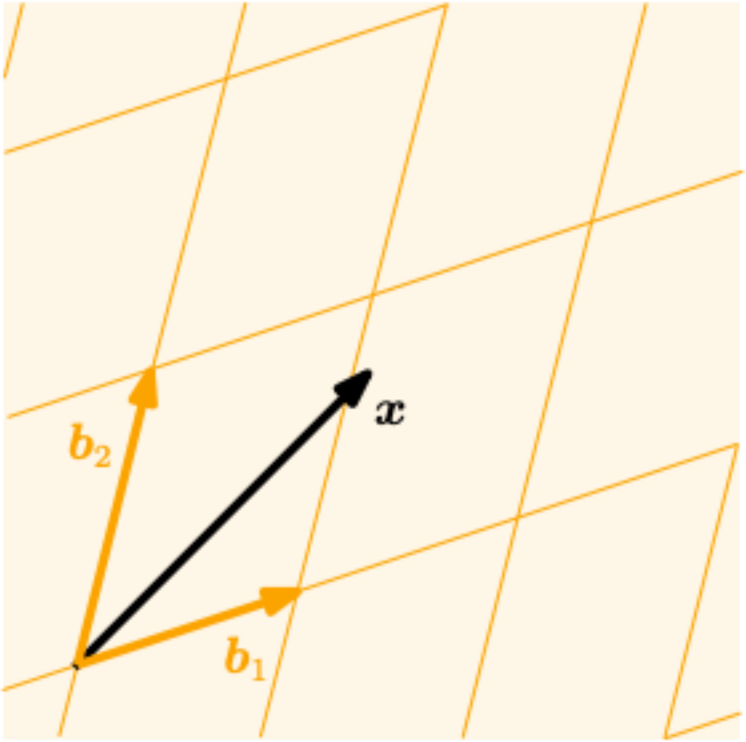
$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{R}^n$$

is the coordinate vector/coordinate representation of $\boldsymbol{x}$ with respect to the ordered basis B.

Coordinates of a vector

- [Left] A Cartesian coordinate system in two dimensions, which is spanned by the canonical basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2$.

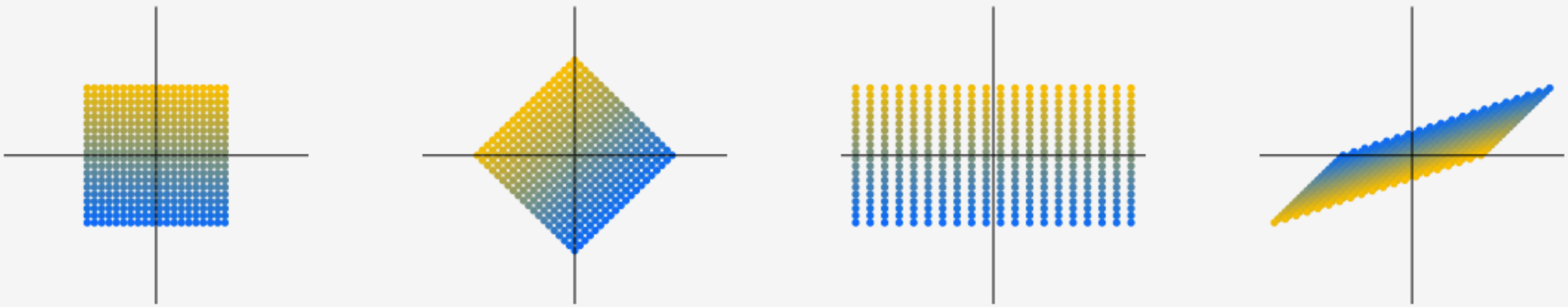




- The same vector $\boldsymbol{x}$ may have different coordinates under different basis.

2.7.1 Matrix Representation of Linear Mappings

- Example - Linear Transformations of Vectors



(a) Original data. (b) Rotation by 45°. (c) Stretch along the horizontal axis. (d) General linear mapping.

同6528中的线性变换。

- The following three linear transformations are used

$$A_1 = \begin{bmatrix} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix} \quad A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad A_3 = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix}$$

- Consider vector spaces V, W with corresponding bases $B = (\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$ and $C = (\boldsymbol{c}_1, \cdots, \boldsymbol{c}_m)$. We consider a linear mapping $\Phi\colon V \to W$. For $j \in \{1, \cdots, n\}$,

$$\Phi(\boldsymbol{b}_j) = \alpha_{1j}\boldsymbol{c}_1 + \cdots + \alpha_{mj}\boldsymbol{c}_m = \sum_{i=1}^{m} \alpha_{ij}\boldsymbol{c}_i$$

is the unique representation of $\Phi(\boldsymbol{b}_j)$ with respect to C. Then, we call the $m \times n$-matrix $\boldsymbol{A}_\Phi$ the transformation matrix of Φ, whose elements are given by

$$A_\Phi(i, j) = \alpha_{ij}$$

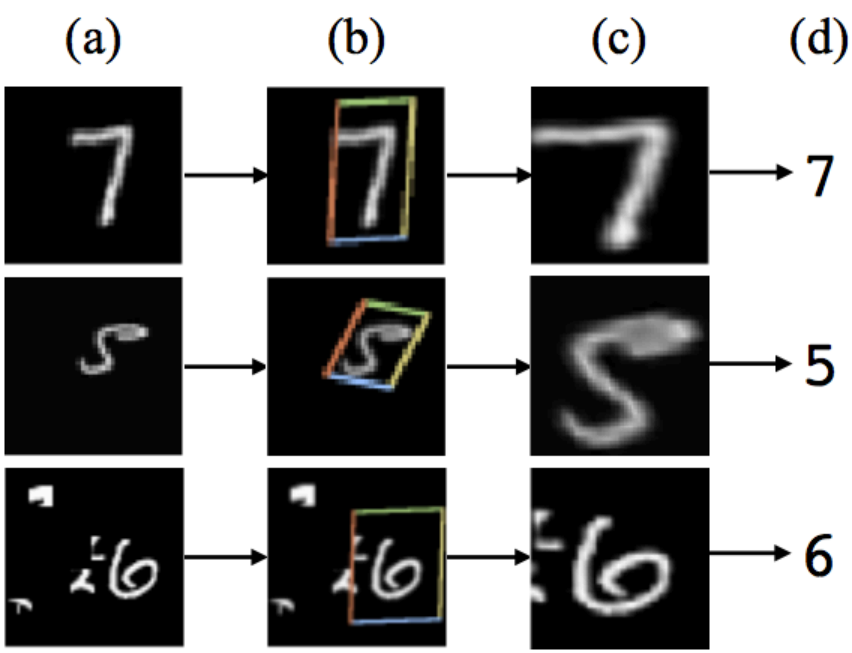
- If $\hat{\boldsymbol{x}}$ is the coordinate vector of $\boldsymbol{x} \in V$ with respect to B, and $\hat{\boldsymbol{y}}$ the coordinate vector of $\boldsymbol{y} = \Phi(\boldsymbol{x}) \in W$ with respect to C, then

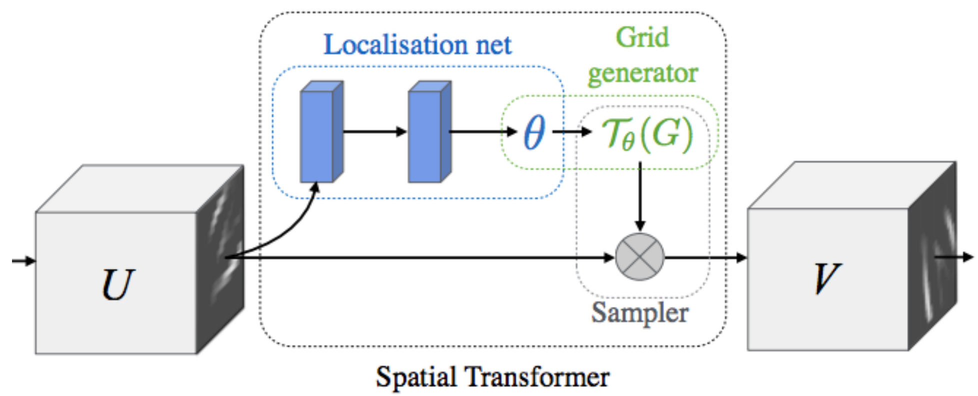
$$\hat{\boldsymbol{y}} = \boldsymbol{A}_\Phi\, \hat{\boldsymbol{x}}$$

Spatial Transformer Networks (Jaderberg et al., NIPS 2015)

$$\begin{pmatrix} x_i^s \\ y_i^s \end{pmatrix} = \mathcal{T}_\theta(G_i) = \mathtt{A}_\theta \begin{pmatrix} x_i^t \\ y_i^t \\ 1 \end{pmatrix} = \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{21} & \theta_{22} & \theta_{23} \end{bmatrix} \begin{pmatrix} x_i^t \\ y_i^t \\ 1 \end{pmatrix}$$

$\boxed{\text{Affine transformation}}$





Check your understanding

- Which of the following statements are correct?

(A) In a vector space, any vector can be represented as a linear combination of a certain set of vectors in this space Yes.For basis, it can represent any

(B) The dimension of a vector equals the dimension of the space itis in. equal or less than

(C) U is a vector subspace of V. Then vectors in U have lower dimension than vectors in V No

(D) The set $\left\{\begin{bmatrix}2\\0\\0\end{bmatrix},\begin{bmatrix}1\\2\\0\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}6\\2\\-2\end{bmatrix}\right\}$ forms a basis for $\mathbb{R}^3$

(E) $U=\{(x,y): x=y, x\in\mathbb{R}, y\in\mathbb{R}\}$ is a subspace of $\mathbb{R}^2$

(F) The vector $\mathbf{0}$ is linearly dependent with any vector in the same vector space