# Practice Midsemester Exam

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**Problem 1 (5 marks)** (a) For each of the following terms, state a formal definition of the term and give an example: statement, predicate, argument.

(b) Write a sentence which does not use the word 'not' and is logically equivalent to the negation of the following statement:

For all integers $a$ and $b$, if $a < b$, then there exists a unique solution to the equation $E(a, b)$.

(c) Represent the form of the following argument using one of the notations introduced in class, and then establish or refute the validity of the argument.

"There is no water on Planet X. If there is life on Planet X, then Planet X contains water. Planet X will be invited to join the Confederation of Lively Planets only if there is life on Planet X. Therefore, Planet X will not be invited to join the Confederation of Lively Planets."

(d) Is the set $\{\neg, \vee\}$ a functionally complete set of logical connectives? Justify your answer.

**Problem 2 (5 marks)** (a) Describe an example of each of the following: a set $A$ containing 6 elements defined using set-roster notation; a set $B$ containing 27 elements defined using set-builder notation; a partition of the integers that contains exactly four sets.

(b) Let $A$ denote the lower-case English alphabet $\{a, b, c, \ldots, z\}$ and let $X = A \times \mathbb{Z} \times A \times (\mathbb{Q} \setminus \mathbb{Z})$. Use set-roster notation to give an example of a subset of $X$ that has exactly three elements.

(c) Recall that $\mathbb{N}$ denotes the set of positive integers. We define a sequence by $(a_n)_{n \in \mathbb{N}}$ by

$$\begin{cases} a_1 = 2 \\ \forall n \in \mathbb{N}\ a_{n+1} = a_n + (n+1)2^{n+1} \end{cases}$$

Prove that the following is an explicit definition of the same sequence

$$\forall n \in \mathbb{N}\ \ a_n = (n-1)2^{n+1} + 2.$$

(d) Use the logical equivalences below and the definitions of set operations to prove, or provide a counterexample to disprove, the following set identity:

$$(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$$

The following is taken from the optional text. The text uses $\sim$ instead of $\neg$ for negation. You should use $\neg$.

**Theorem 2.1.1 Logical Equivalences**

Given any statement variables $p$, $q$, and $r$, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

| | | |
|---|---|---|
| 1. *Commutative laws:* | $p \wedge q \equiv q \wedge p$ | $p \vee q \equiv q \vee p$ |
| 2. *Associative laws:* | $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ | $(p \vee q) \vee r \equiv p \vee (q \vee r)$ |
| 3. *Distributive laws:* | $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ | $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ |
| 4. *Identity laws:* | $p \wedge \mathbf{t} \equiv p$ | $p \vee \mathbf{c} \equiv p$ |
| 5. *Negation laws:* | $p \vee \sim p \equiv \mathbf{t}$ | $p \wedge \sim p \equiv \mathbf{c}$ |
| 6. *Double negative law:* | $\sim(\sim p) \equiv p$ | |
| 7. *Idempotent laws:* | $p \wedge p \equiv p$ | $p \vee p \equiv p$ |
| 8. *Universal bound laws:* | $p \vee \mathbf{t} \equiv \mathbf{t}$ | $p \wedge \mathbf{c} \equiv \mathbf{c}$ |
| 9. *De Morgan's laws:* | $\sim(p \wedge q) \equiv \sim p \vee \sim q$ | $\sim(p \vee q) \equiv \sim p \wedge \sim q$ |
| 10. *Absorption laws:* | $p \vee (p \wedge q) \equiv p$ | $p \wedge (p \vee q) \equiv p$ |
| 11. *Negations of* **t** *and* **c**: | $\sim \mathbf{t} \equiv \mathbf{c}$ | $\sim \mathbf{c} \equiv \mathbf{t}$ |

**Problem 3 (5 marks)** (a) Let $A$ and $B$ be sets. State the definition of each of the following: a relation from $A$ to $B$; a relation on $A$; the inverse relation $R^{-1}$ of a relation $R$ from $A$ to $B$.

(b) A relation $R$ is defned on the set $\{1, 2, 3, 4, 5\}$ by the rule

$$xRy \Leftrightarrow x + 2y < 10.$$

Draw a digraph representing $R$.

(c) Let $A$ and $B$ be sets. Suppose that $R$ is a relation from $A$ to $B$. What must be true about $R$ for it to be an injective function? State you answer in English using quantification and a predicate.

(d) Is the function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ given by $f(z) = (z - 3)^2 + 1$ surjective? Justify your answer.

(e) Prove or disprove the following: Let $A$, $B$ and $C$ be sets, and let $f : A \to B$ and $g : B \to C$ be functions. If $f$ and $g$ are injective, then $g \circ f$ is an injective function from $A \to C$.

**Problem 4 (5 marks)** (a) Write the hexadecimal and decimal representations of the integer: $(1101001101)_2$

(b) Find all of the integers $z$ such that $z \text{ mod } 11 = 3$ and $z \text{ div } 11 = -4$

(c) Suppose that we agree to represent a non-zero rational number $q = (-1)^2 \times m \times 2^n$ by 12 bits using: 1 bit to store $s$; followed by 3 bits to store a non-negative integer representing $n + 3$; followed by 8 bits to store the bits that appear after the binary point in the mantissa $m$. The hexadecimal string $B3A$ represents a rational number $q$ in this scheme. Find the decimal representation of $q$.

(d) The list $(X_i)_{1..8} = (D, I, S, C, R, E, T, E)$ is to be sorted using Selection sort. The sorting is to be achieved by progressively modifying an index function $\pi$, rather than by shuffling members of the list itself. So initially $(X_i)_{1..8} = (X_{\pi(i)})_{1..8}$ where $\pi(i) = i$ for $i = 1, ..., 8$, and when sorting is complete $\pi$ is sufficiently changed so that $(X_{\pi(i)})_{1..8}$ is in alphabetical order. Find the state of the index function $\pi$ after the least element algorithm has been called three times by Selection Sort.