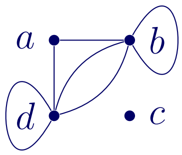
1. *This question is is mostly just about definitions; almost no calculations are required.* A graph G has $V(G) = \{a, b, c, d\}$ and $E(G) = \{\{a, b\}, \{a, d\}, \{b, b\}, \{b, d\}, \{d, b\}, \{d, d\}\}$.

(a) Draw the graph.

Answer:



Note that $E(G)$ is a multiset, so even though $\{b, d\} = \{d, b\}$, we still get two edges.

(b) What is the order of the graph?

Answer: 4. *(There are 4 vertices.)*

(c) Write out a table of edges for G.

Answer:

Edge:	e_1	e_2	e_3	e_4	e_5	e_6
End points:	a,b	a,d	b,b	b,d	b,d	d,d

(d) Write out a vertex adjacency listing for G.

Answer:

Vertex:	a	b	c	d
Adjacent to:	b,d	a,b,d,d	–	a,b,b,d

(Listings without the repeat vertices for b and d would also be acceptable.)

(e) What extra information is included in the vertex adjacency listing that is not in the table of edges?

Answer: *There is an isolated vertex c.*

(f) Write down the adjacency matrix for G.

Answer: $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

(g) Does G contain loops? How can you tell from the adjacency matrix?

Answer: *Yes. There are non-zero entries on the diagonal.*

(h) Does G contain parallel edges? How can you tell from the adjacency matrix?

Answer: *Yes. There are entries greater than one.*

(i) Is G a simple graph? Why?

Answer: *No. It contains loops and parallel edges.*
[Having just one of either of these is enough to render the graph non-simple.]

(j) Name all the isolated vertices in G. How can you tell from the adjacency matrix?

Answer: *Vertex c only. The third row is all zeros. No other such rows.*

(k) Is the adjacency matrix symmetric? Why?

Answer: *Yes. This is always so for a graph. On the other hand, digraphs usually have non-symmetric adjacency matrices.*

(l) How many paths of length 2 are there from a to b? Is it possible to get this information from the adjacency matrix? If so, How?

Answer: 3. *(e_1e_3, e_2e_4, e_2e_5) This information is not available from A directly. The value 3 is the entry in row 1, column 2 of A^2.*

(m) Is G connected? Is it possible to tell this from the adjacency matrix? If so, How?

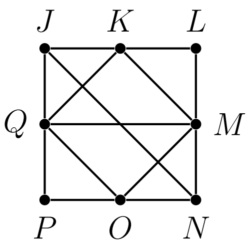
Answer: *No. The row of zeros implies this (see (j)).*
However, in general it is not obvious from the adjacency matrix.

(n) What are the degrees of a, b, c and d?

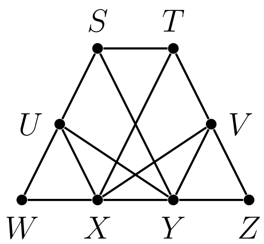
Answer:

Vertex:	a	b	c	d
Degree:	2	5	0	5

2. Although it may seem implausible, the graphs G and H below are isomorphic. Find an isomorphism $b : V(G) \to V(H)$ that verifies this.
Hint: For every vertex v of G, the degree of $b(v)$ must be the same as the degree of v.



Graph G

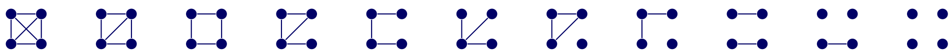


Graph H

edges								
v	J	K	L	M	N	O	P	Q
degree	3	4	2	5	3	4	2	5
$b(v)$	S	U	W	X	T	V	Z	Y
OR:	T	V	Z	Y	S	U	W	X
edges								

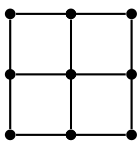
3. How many non-isomorphic simple graphs have exactly four vertices? Draw one of each.

Answer: 11 :

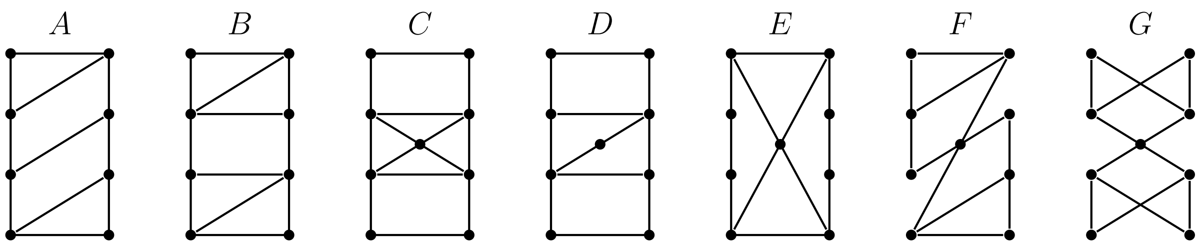


4.

Γ :



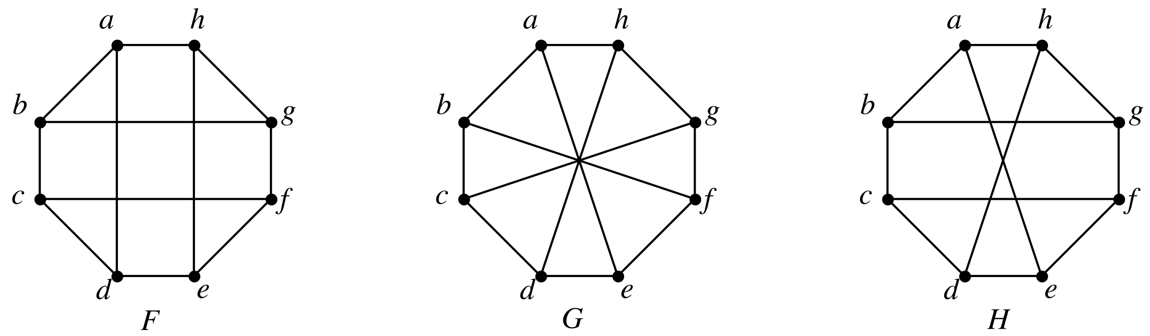
The graph Γ at left is not isomorphic to any of the graphs A - G below. In each case state an invariant property of Γ that is not possessed by the other graph. (An 'invariant' property is one that that does not depend on any particular labelling of the vertices.)



Answer:
A : Γ has 9 vertices. (A has 8.) *B : Γ has 9 vertices. (B has 8.)*
C : Γ has 12 vertices. (C has 14.) *D : Γ has a vertex of degree 4. (D does not.)*
E : No vertex of degree 2 in Γ is adjacent to another vertex of degree 2. (Not true for E.)
F : Every vertex of degree 3 is adjacent to a vertex of degree 4. (Not true for F.)
G : Γ has exactly 4 circuits of length 4. (G has 6.)

Note: There are many other valid answers.

5. [*Challenge*] For each pair of the graphs F, G, H below, decide whether or not the two graphs are isomorphic. For two graphs that are isomorphic, give an isomorphism between them, as well as a convincing argument showing why your function is an isomorphism. For two graphs that are not isomorphic, state an invariant property possessed by one but not by the other.



Answer:

F and G are not isomorphic because G contains a 5-cycle (circuit of length 5) [e.g. abcdea] whereas F does not.

F and H are not isomorphic for a similar reason.

*G and H **are** isomorphic, but an isomorphism is not easy to find, since all vertices of both graphs have the same degree (3).*

One clue is that the 'outer' edges on the G diagram each belong to exactly one 4-cylcle, wheras the 'diagonals' each belong to two. Moving across to H, we find that the edges belonging to two 4-cycles are ah, bg, cf, de. So these must correspond in some way to the G-edges ae, bf, cg, dh.

Since the diagram of G is rotationally symmetrical, we may as well choose $\phi(a) = a$ where $\phi : V(G) \to V(H)$ is the isomorphism we seek. Since in both graphs the edge ab belongs to exactly one 4-cycle, we can also choose $\phi(b) = b$. Then, following round these 4-cycles, we are forced to set $\phi(f) = g$ and $\phi(e) = h$.

In G the third vertex adjacent to a is h and in H it is e, so we must set $\phi(h) = e$. In similar manner we find we must set $\phi(c) = c$, $\phi(g) = f$ and $\phi(d) = d$.

So the complete isomorphism is

$x:$	a	b	c	d	e	f	g	h
$\phi(x):$	a	b	c	d	h	g	f	e

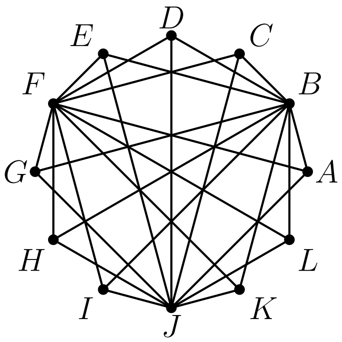
You can now check that for any $x, y \in \{a, b, c, d, e, f, g, h\}$:

$$xy \in E(G) \text{ if and only if } \phi(x)\phi(y) \in E(H).$$

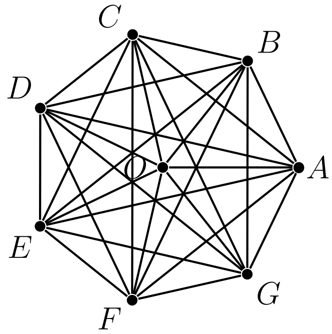
Visually, the isomorphism we have found indicates that the diagram for G can be transformed into the diagram for H by swapping the positions of e and h and at the same time swapping the positions of f and g, and leaving all connections the same by shrinking or stretching edges as appropriate.

Note: Our ϕ is only one of a total of 16 isomorphisms $G \to H$.

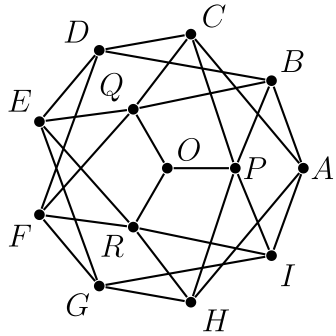
6. For each of the graphs G_1, G_2, G_3 below decide whether it is bipartite and/or complete and/or complete bipartite, or none of these. In the case of bipartite graphs, specify the two partite sets of vertices. If appropriate, give the standard K-names for the graphs.



Graph G_1



Graph G_2



Graph G_3

Answer:

G_1 is the complete bipartite graph $K_{3,9}$.

The two sets are $\{B, J, F\}$ and $\{A, C, D, E, G, H, I, K, L\}$.

G_2 is the complete graph K_8.

G_3 is bipartite with sets $\{O, B, C, E, F, H, I\}$ and $\{A, D, G, P, Q, R\}$.

It is not complete bipartite; e.g. OA is not an edge.

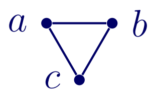
7. Recall that a graph H is a subgraph of a graph G if, and only if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Don't forget that every set has itself and the empty set as subsets.

(a) The complete graph K_2 with $V(K_2) = \{a, b\}$ and $E(K_2) = \{\{a, b\}\}$ has 5 subgraphs. Draw them.

Answer:

(b) The complete graph K_3 has 18 subgraphs. Provide a counting argument to justify this. Ensure that your counting method does not miss any subgraphs nor count any more than once.

Answer: *Let S be a subgraph of K_3:*



- *For $|V(S)| = 3$ there are $2^3 = 8$ possibilities for S, since each of the three edges may or may not be in $E(S)$.*
- *For $|V(S)| = 2$ there are $\binom{3}{2} = 3$ choices for the vertices and for each of these 2 possibilities; an edge or no edge.*
- *For $|V(S)| = 1$ there is just the $\binom{3}{1} = 3$ choices of vertex.*
- *For $|V(S)| = 0$ there is only the empty subgraph.*

Hence the number of subgraphss is $8 + 3\times 2 + 3 + 1 = 18$.

(c) How many subgraphs has the complete graph K_4? (No argument required.)

Answer: *First note that K_4 has 6 edges. Then, by a similar counting method to that used in (b) we get*

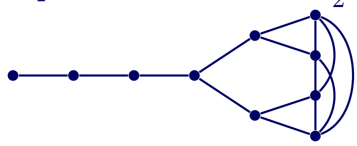
$2^6 + \binom{4}{3}\times 2^3 + \binom{4}{2}\times 2 + \binom{4}{1} + 1 = 64 + 32 + 12 + 4 + 1 = 113.$

8. Prove or disprove that there exists a simple graph with:

(a) One vertex of degree 1, two vertices of degree 2, three vertices of degree 3 and four vertices of degree 4.

Answer: *Total degree* $= 1 + 2\times2 + 3\times3 + 4\times4 = 30$. *This is even so we can attempt to make such a graph. It must have* $\frac{30}{2} = 15$ *edges.*

Here is one of many examples:



(b) One vertex of degree 1, two vertices of degree 2, three vertices of degree 3, four vertices of degree 4 and five vertices of degree 5.

Answer: *Total degree* $= 1 + 2\times2 + 3\times3 + 4\times4 + 5\times5 = 55$.
This is odd, so no such graph exists.

9. For the 'boxer' graph below give an example of each of the following:
Note: *There are many other examples besides those given below.*

(a) A simple path.
Answer: abc

(b) A non-simple path.
Answer: $bjkabc$

(c) A walk that is not a path.
Answer: aba

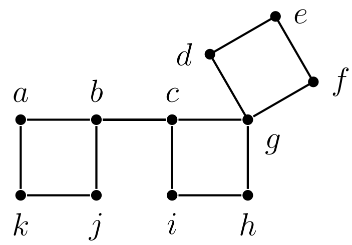
(d) A simple circuit.
Answer: $abjka$

(e) A non-simple circuit.
Answer: $cgdefghic$

(f) A closed walk that is not a circuit.
Answer: aba

(g) A bridge.
Answer: bc

(h) A cut vertex that is not part of a bridge.
Answer: g



10. For the 'boxer' graph of Q9, find those of the following walks that exist. Justify any claim that a particular walk does not exist.

(a) An Euler path **Answer:** $bjkabcgdefghic$

(b) A Hamilton path **Answer:** $akjbcihgdef$

(c) An Euler circuit **Answer:** *Does not exist.*
There are vertices of odd degree (b and c)

(d) A Hamilton circuit **Answer:** *Does not exist.*
No graph containing a bridge can have a Hamilton circuit since the bridge would have to be crossed in both directions, repeating that edge.

11. Which, if any, of the graphs A - G of Q4 have an Euler circuit? Justify your answer, but there is no need to actually specify any circuits.

Answer: *Only* C*, since all its vertices have even degree, whereas all the others have at least one vertex of odd degree.*

12. [*Challenge*] For each of the graphs G_1 - G_3 of Q6 prove or disprove that the graph has a Hamilton circuit.

Answer:

G_1 is $K_{3,9}$, but $K_{m,n}$ only has a Hamilton circuit if $m = n$. This is because edges on the circuit must have endpoints in different partites and so there must be an equal number of vertices in each partite if all vertices are to be visited. So G_1 does not have a Hamilton circuit.

G_2 is K_8 and like all complete graphs with at least three vertices does have a Hamilton circuit. One example is $OABCDEFGO$.

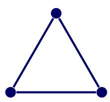
G_3 is bipartite with unequal partites (6 and 7 vertices). So it does not have a Hamilton circuit for a similar reason to that for G_1.

13. How many different walks of length 6 on the complete graph K_3:

(a) return to their starting point; (b) do not return to their starting point.

Hint: Use powers of the adjacency matrix. Take into account all the different start and end vertices of the walk.

Answer:



K_3

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix}$$

$$A^6 = (A^3)^2 = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 22 & 21 & 21 \\ 21 & 22 & 21 \\ 21 & 21 & 22 \end{bmatrix}.$$

(a) $22 + 22 + 22 = 66$.

(b) $21 + 21 + 21 + 21 + 21 + 21 = 126$.

14. Find the connected components of the graph whose adjacency matrix A is shown at right.

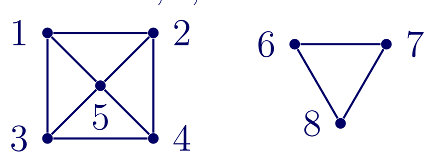
$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Answer:

Observe that $A = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & C \end{array}\right]$,

where B is 5×5 and C is 3×3.

So vertices $1, 2, 3, 4, 5$ are not connected to vertices $6, 7, 8$. However the two groups are each internally connected:



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Hence these two subgraphs are the required connected components.

15. A graph G has 100 vertices labelled $0, 1, \ldots, 98, 99$ and a total of 179 edges comprising two sets

$$E_1 = \{\{i, i{+}10\} : i = 0, \ldots, 89\}$$
$$E_2 = \{\{i, i{+}11\} : i = 0, \ldots, 88\}$$

Prove that G is connected.
[Hint: As an example, a path from 35 to 0 is 35 - 45 - 55 - 44 - 33 - 22 - 11 - 0.]

Answer:

We first establish that for every vertex labelled with non-zero n there is a path from vertex 0 to vertex n.

Suppose n has two digit decimal representation tu, where t might be 0, but t and u can't both be 0. Then $n = 10\mathrm{t}{+}\mathrm{u}$ and we consider three cases:

Case 1; t=u: *Then $n = 11\mathrm{t}$ and so vertex n can be reached from vertex 0 using t E_2-edges. E.g. 0 - 11 - 22 - 33.*

Case 2; t > u: *Then $n = 10(\mathrm{t}-\mathrm{u}) + 11\mathrm{u}$. Vertex n can be reached from vertex 0 using $\mathrm{t}-\mathrm{u}$ E_1-edges followed by u E_2-edges. E.g. 0 - 10 - 21 - 32.*

Case 3; t < u: *Then $n = 11\mathrm{u}-10(\mathrm{u}-\mathrm{t})$. Vertex n can be reached from vertex 0 using u E_2-edges followed by $\mathrm{u}-\mathrm{t}$ E_1-edges. E.g. 0 - 11 - 22 - 12.*

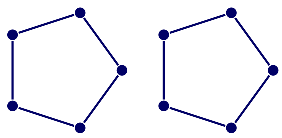
Since any path in G can be traversed in either direction, a path from vertex m to vertex n can be found by reversing the path from 0 to m and adjoining the path from 0 to n.

16. A simple graph G has ten vertices and ten edges. Prove or disprove:

(a) G must be connected. (b) G must contain a circuit.

Answer:

(a) This is false. Counterexample:



(b) This is true. Since G has more than $n{-}1$ edges it is not a tree. Then:

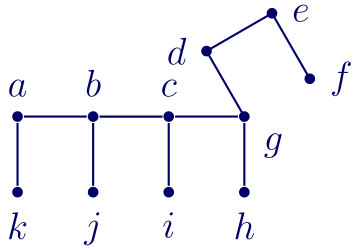
- *If G is connected it contains a circuit by definition of 'tree'.*
- *If G is not connected then consider its connected components. At least one of these must have at least as many edges as vertices, otherwise the total number of edges (10) would be less than the total number of vertices (also 10). Such a component is connected but not a tree, and so, as above, contains a circuit.*

17. For the 'boxer' graph of Q9:

(a) Draw a spanning tree. (b) How many spanning trees does the graph have?

Answer:

(a) One example:



(b) Boxer has $n = 11$ *vertices and* 13 *edges, so we need to remove* 3 *edges to reach the required* $n - 1$ *edges for a tree. To do this and at the same time eliminate all circuits, we need to remove one edge from each of the three 'boxes'. There are* 4 *choices for each box and all sets of choices leave the graph connected and hence a tree. Thus the total number of spanning trees is* $4^3 = \boxed{64}$.

18. [*Challenge*] A tree T has 30 vertices, of which 10 have degree 3. What is the greatest number of vertices of degree 2 that T could have?

Answer:

As T *is a tree, it has* $30 - 1 = 29$ *edges and so has total degree* $2 \times 29 = 58$. *The* 10 *vertices of degree* 3 *account for* 30 *of this total, leaving* 28 *for the remaining* 20 *vertices. Since* T *is connected, each of these* 20 *vertices has degree at least* 1, *leaving only* 8 *spare to distribute amongst vertices of degree* 2 *or more.*

So the most degree-2 vertices possible is $\boxed{8}$ *(when there are no vertices of degree more than* 3.*)*