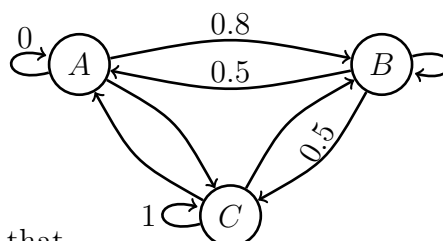


1. A Markov process has two states A and B with transition graph at right.



- Write in the missing probabilities.
- Suppose the system is initially in state B .
Use a tree diagram to find the probability that the system will be in state A after three steps.
- Write out the transition matrix T for this process.
- Use T to recalculate your answer to (b).

2. A Markov process has three states A , B and C with transition graph at right.



- Write in all the missing probabilities.
- Suppose the system is initially in state A .
Use a tree diagram to find the probability that the system will be in state C after two steps.
To simplify your diagram, leave out branches that have zero probability.
- Extend your diagram for (b) to cover four steps. What is the probability that the system will be in state C after four steps?
- Find the probability that the system will be in state C after ten steps starting from A . Do not use a diagram. Generalise from (c) and use complementary probability.
- As for (d), but starting from B .
- Guess the long-term probability that the system will be in state C , no matter what state the system starts in.
- Write out the transition matrix T for this process.
- Calculate T^2 and $T^4 = (T^2)^2$ and use them to confirm your answers to (b) and (c).
- Convert your answer to (f) to a steady state vector S and confirm that answer by verifying that $T'S = S$.

3. Ari is an innovative mathematics teacher. Once a week he sets up his classroom as four activity 'stations' labelled 1, 2, 3 and 4. Students spend 15 minutes at each station. In order to mix up the students, at change-over time Ari randomly divides the groups at stations 1 - 3 into two subgroups, as equally-sized as possible, and randomly sends one subgroup to the next station ($i \rightarrow i+1$) and the other subgroup to the station beyond ($i \rightarrow i+2$ except $3 \rightarrow 1$). Owing to the nature of the activity at station 2, Ari needs to limit the numbers at that station, so he starts with a smaller group there and at changeover time all students at station 4 move only to station 1.

- (a) Compile a transition T matrix representing this (Markov) process. The states are the stations and entry t_{ij} of T specifies, for a student at station i , the probability that, at change-over, the student will move to station j .
- (b) Ari starts with six students at each of stations 1, 3 and 4, and five at station 2. The class lasts an hour. How many students will there be at each station when the class ends? [There are several possible answers here, since odd-sized groups cannot be equally subdivided. Flip a coin to decide to which station each larger subgroup goes.]
- (c) Verify that the steady state is eight students at station 1, four at station 2, six at station 3 and five at station 4.

4. A certain Markov Process has transition matrix T at right. Using a computer some powers of T were calculated and are shown below to three decimal places.

$$T = \begin{bmatrix} .6 & .2 & 0 & .2 \\ .3 & .3 & .2 & .2 \\ .2 & 0 & .2 & .6 \\ .1 & .2 & .1 & .6 \end{bmatrix}$$

$$T^2 = \begin{bmatrix} .44 & .22 & .06 & .28 \\ .33 & .19 & .12 & .36 \\ .22 & .16 & .10 & .52 \\ .20 & .20 & .12 & .48 \end{bmatrix} \quad T^4 = \begin{bmatrix} .335 & .204 & .092 & .368 \\ .306 & .200 & .099 & .396 \\ .276 & .199 & .105 & .421 \\ .276 & .197 & .106 & .421 \end{bmatrix} \quad T^8 = \begin{bmatrix} .302 & .200 & .100 & .398 \\ .300 & .200 & .100 & .400 \\ .299 & .200 & .100 & .401 \\ .299 & .200 & .100 & .401 \end{bmatrix}.$$

Use the powers of T to guess a steady state vector for the process, and then prove your guess is correct.

5. A certain Markov Process has transition matrix T at right. Use a matrix calculation tool such as <https://matrixcalc.org/en/> to calculate T^{16} to three decimal places.

$$T = \begin{bmatrix} .4 & .1 & 0 & .5 \\ .4 & .1 & .2 & .3 \\ 0 & .3 & .4 & .3 \\ .7 & .2 & .1 & 0 \end{bmatrix}$$

Use T^{16} to guess a steady state vector for the process, and then prove your guess is correct.

(If your matrix tool doesn't have a powering function but does have a multiplication function you could first calculate $T \times T = T^2$ then $T^2 \times T^2 = T^4$ and so on. Depending on the tool, with lots of cut-and-paste the only matrix you may need to enter is T .)

6. Calculate the steady state vector S for the Markov process of Question 1. Do this by hand, using the matrix inverse method with short cut to solve $T'S = S$.

7. Recalculate the steady state vector for the Markov process of Question 5 by using the 'Gauss-Jordan Elimination' function in the 'Matrix Reshish'¹ computer application <https://matrix.reshish.com>. Specify your input to, and output from, Reshish.

8. Let T be an $n \times n$ stochastic matrix (rows are probability vectors) and \mathbf{v} a column probability n -vector. Prove that $T'\mathbf{v}$ is always also a probability vector. Try this first for $n = 2$ and then for $n = 3$. Do it for general n if your algebra is up to it.

9. *Hardy-Weinburg Equilibrium*: Consider a gene that has two forms, or *alleles*, A and a . Each individual has two of these genes and so has *genotype* AA , Aa or aa .

Assume that an individual's genotype consists of a random selection of one each of its parents' alleles. So, for example, the offspring of parents who are both Aa has a 50% chance of also being Aa and a 25% chance each of being AA and aa .

Assume further that mating partners are chosen at random.

Let π_{AA} , π_{Aa} and π_{aa} be the proportions of each genotype in a breeding colony.

(a) Explain why $p = \pi_{AA} + \pi_{Aa}/2$ and $q = \pi_{aa} + \pi_{Aa}/2$ are the probabilities that a random allele chosen from a random individual is A or a respectively.

(b) Explain why the parent-to-offspring transition matrix is given by $T = \begin{matrix} & \begin{matrix} AA & Aa & aa \end{matrix} \\ \begin{matrix} AA \\ Aa \\ aa \end{matrix} & \begin{bmatrix} p & q & 0 \\ p/2 & 1/2 & q/2 \\ 0 & p & q \end{bmatrix} \end{matrix}$.

(c) Show that steady state vector is $S = \begin{bmatrix} p^2 \\ 2pq \\ q^2 \end{bmatrix}$.

(d) Show that S is always achieved in just one transition step.

¹When entering decimal values, Reshish requires a digit before the decimal point. *e.g.* enter '0.4', not '.4'.