MATH1005/MATH6005: Discrete Mathematical Models

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Semester 1, 2021

Section A: The language of mathematics and computer science

Part 2: Sets (continued)

Partitions: A structure for recognising that a classification works well

Motivation

A common task in any discipline (science, mathematics, philosophy, humanities, ...) is that of classifying things of a certain type into various sub-types. Thanks to our development of set theoretic tools, we have a way to formalise what it means for such a classification scheme to work really well.

Q: What properties do you think an excellent classification scheme will have?

An example

Which, if any, of the following classification schemes works well?

- We classify each integer as positive, negative or 0.
- We classify each song on the charts as pop, rock or urban.
- We classify each student enrolled in this course as a mathematician or a computer scientist or a physicist.

Disjoint sets

Sets A, B are called **disjoint** when $A \cap B = \emptyset$.

Given a set of sets $\mathcal{S}$, the sets in $\mathcal{S}$ are said to be **pairwise disjoint** when

$$\forall A, B \in \mathcal{S} \;\; A \neq B \to A \cap B = \emptyset.$$

An example

Let P be the set of prime numbers, let C the set of composite numbers, and let E be the set of even integers. Consider the sets

$$\mathcal{A} = \{\{1\}, P, C\} \text{ and } \mathcal{B} = \{\{1\}, P, E \cap \mathbb{N}\}$$

Since

$$\{1\} \cap P = \emptyset, \{1\} \cap C = \emptyset \text{ and } P \cap C = \emptyset,$$

the sets in $\mathcal{A}$ are pairwise disjoint.

Since $(E \cap \mathbb{N}) \cap P = \{2\}$, the sets in $\mathcal{B}$ are not pairwise disjoint.

A formal interpretation of an ‘excellent’ classification scheme

Let S be a set and $\mathcal{A} \subseteq \mathcal{P}(S)$ (so $\mathcal{A}$ is a set, the elements of which are subsets of S). We say that $\mathcal{A}$ is a **partition** of S when each of the following statements is true:

1. $\emptyset \notin \mathcal{A}$
2. every element of s is an element of some set in $\mathcal{A}$ (that is, $\forall s \in S\ \exists A \in \mathcal{A}\ s \in A$)
3. the sets in $\mathcal{A}$ are pairwise disjoint.

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3. the sets in $\mathcal{A}$ are pairwise disjoint.

Q: Do you agree or disagree that the three properties listed in the definition of a partition are a reasonable interpretation of what it means for a classification scheme (that classifies the elements of S) to be ‘excellent.’

Some examples

- $\mathcal{A} = \{\{1\}, P, C\}$ is a partition of $\mathbb{Z}^+$
- $\mathcal{B} = \{\{1\}, P, E \cap \mathbb{N}\}$ is not a partition of $\mathbb{N}$ because the sets in $\mathcal{B}$ are not pairwise disjoint.
- $\mathcal{A} = \{\{1\}, P, C\}$ is not a partition of $\mathbb{N}$ because $0 \in \mathbb{N}$ but 0 is not in any set in $\mathcal{A}$.
- Let P, C, E, O be as above. Then $\{P \cap C, P \cap E, P \cap O\}$ is not a partition of P, because $P \cap C = \emptyset$.
- Let P, C, E, O be as above. Then $\{P \cap E, P \cap O\}$ is a partition of P.

Section A: The language of mathematics and computer science

Part 3: Relations and functions

Relations

Relations

Let A, B be sets. Any subset of $A \times B$ is called a **relation from** A **to** B. A relation from A to A is called a **relation on** A.

Given a relation R from A to B and an element $(a, b) \in A \times B$, we usually write $a \; R \; b$ instead of $(a, b) \in R$ and we usually write $a \not{R} b$ instead of $(a, b) \notin R$.

An example will help us see why relations are important, and why these choices of notation are made.

An example

Let

$$
\begin{aligned}
B &= \{\text{One Direction}, \text{5SOS}, \text{Cody Simpson}, \text{BTS}\} \\
L &= \{\text{My daughter is a fan}, \text{I've seen them perform live}, \\
&\quad \text{I would slay their tunes on Just Dance}\} \\
\sim &= \{(\text{One Direction}, \text{My daughter is a fan}), \\
&\quad (\text{5SOS}, \text{My daughter is a fan}), \\
&\quad (\text{Cody Simpson}, \text{My daughter is a fan}), \\
&\quad (\text{5SOS}, \text{I've seen them perform live}) \\
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\end{aligned}
$$

Then $\sim$ is a relation from B to L.

Example: From arithmetic to inequalities

We define

$$< \; = \; \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \mid b - a \in \mathbb{Z}^+\}.$$

We have defined a relation $<$ on $\mathbb{Z}$.

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A: We define

$$\geq = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \mid a - b \in \mathbb{N}\}.$$

A: For all integers a and b,

$$(a \geq b) \Leftrightarrow (a - b \in \mathbb{N}).$$

Inverse relation

The **inverse** R^{-1} of a relation $R \subseteq A \times B$ is the relation $R^{-1} \subseteq B \times A$ defined by

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$$\begin{aligned} b\,R^{-1}\,a \quad &\iff \quad a\,R\,b \\ &\iff \quad a \text{ sells to } b \\ &\iff \quad b \text{ buys from } a. \end{aligned}$$

Diagram of a relation

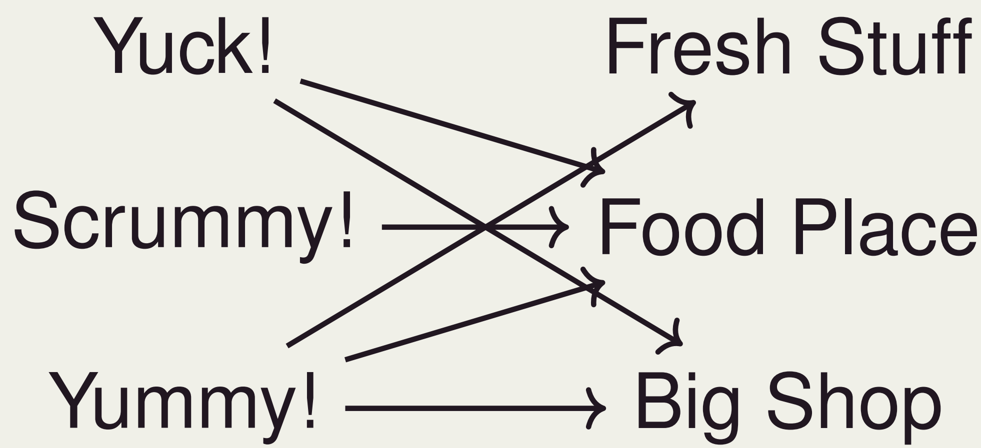
For small sets, relations can be expressed with **arrow diagrams**.

Example: Let

$$
\begin{aligned}
A &= \{\text{Yummy!}, \text{Scrummy!}, \text{Yuck!}\} \\
B &= \{\text{Big Shop}, \text{Food Place}, \text{Fresh Stuff}\} \\
R &= \{(\text{Yummy!}, \text{Big Shop}), (\text{Yummy!}, \text{Food Place}), \\
&\quad (\text{Yummy!}, \text{Fresh Stuff}), (\text{Scrummy!}, \text{Food Place}), \\
&\quad (\text{Yuck!}, \text{Food Place}), (\text{Yuck!}, \text{Big Shop})\}
\end{aligned}
$$

Perhaps aRb means “a sells to b.”
Q: How would you represent R in a diagram?

Directed arrows from elements in set A to elements in set B can be used to show which elements are related.



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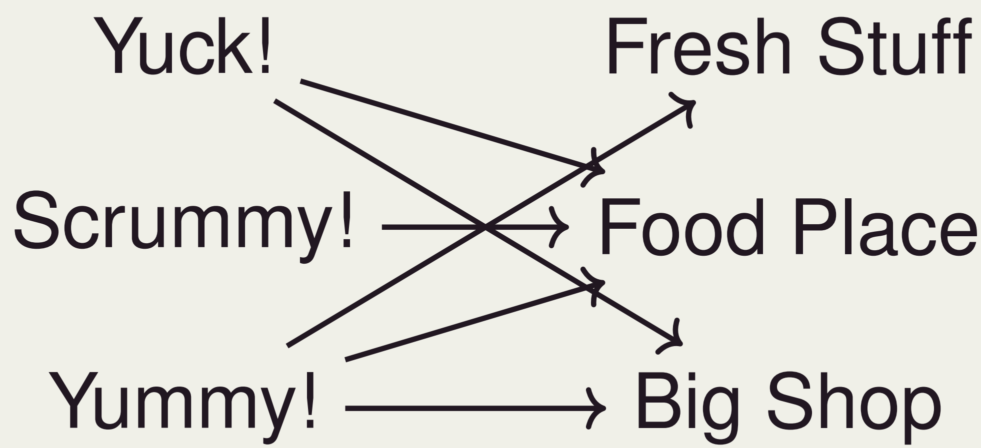

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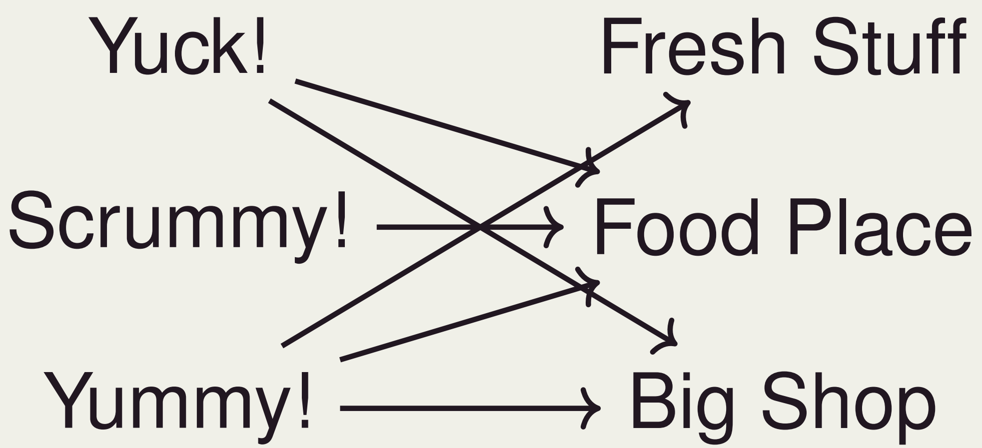


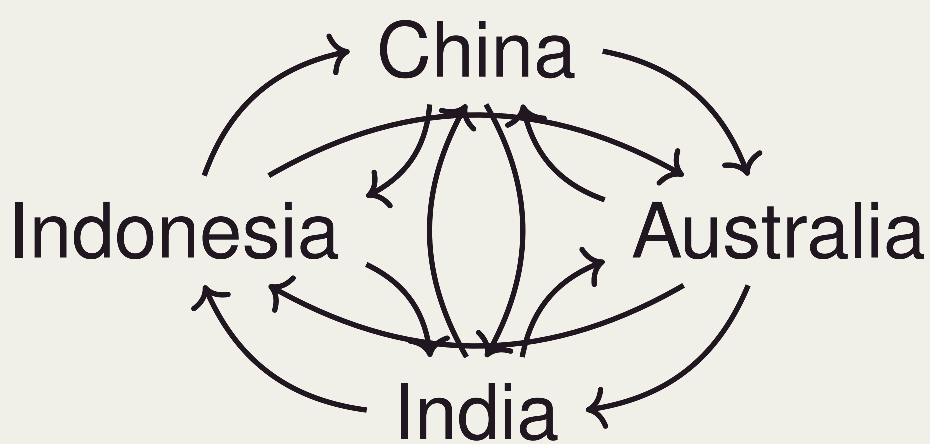
Q: How would you change the diagram above to represent the inverse relation R^{-1}?

A: Change the direction of the arrows.

Directed Graphs

A **directed graph** or **digraph** is a set A of **vertices** together with a subset $R \subseteq A \times A$ of **directed edges**. If $(x, y) \in R$ we say 'there is a directed edge from x to y'. When A is small, a digraph can be drawn with the vertices as points and directed edges as arrows.

Example: A a set of countries. $a \, R \, b$ means a exports to b.



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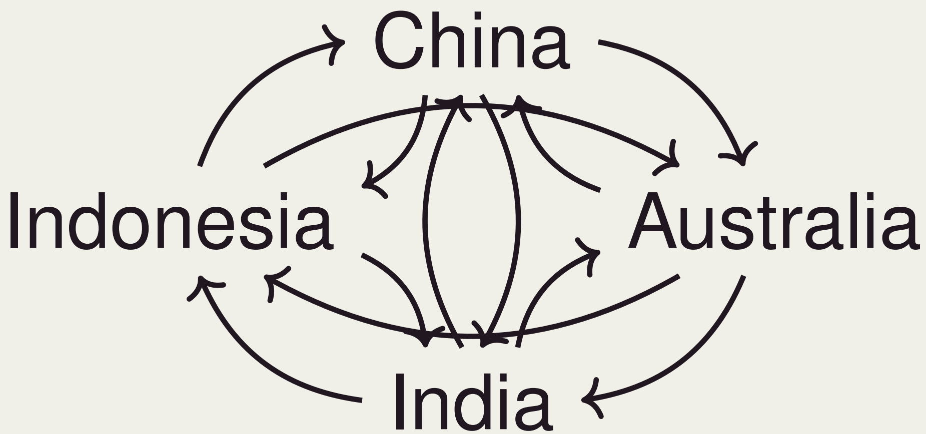

Each of the four countries exports to all the other three.

Example: Directed Graphs

Friends Ami, Bo, Chi and Di took photos of themselves visiting Parliament House.

- Ami took photos of Bo and Chi.
- Bo took photos of each of the others.
- Chi didn't take any photos.
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Draw a digraph for the relation P on $\{$Ami,Bo,Chi,Di$\}$ given by

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$$\begin{tikzcd} \text{Ami} \arrow[r, bend left] \arrow[dr] & \text{Bo} \arrow[l, bend left] \arrow[r] \arrow[d] & \text{Di} \arrow[loop right] \\ & \text{Chi} & \end{tikzcd}$$

(You can position the vertices how you like, of course.)