MATH1005/MATH6005: Discrete Mathematical Models

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Section A: The language of mathematics and computer science

Part 3: Relations and functions (cont,)

Functions

Relations (recap)

Let A, B be sets. Any subset of $A \times B$ is called a **relation from A to B**. A relation from A to A is called a **relation on A**.

Given a relation R from A to B and an element $(a, b) \in A \times B$, we usually write $a \; R \; b$ instead of $(a, b) \in R$ and we usually write $a \not{R} b$ instead of $(a, b) \notin R$.

Functions

Let A, B be sets. A relation f from A to B is called a **function from A to B** when

$$\forall a \in A\ \exists! b \in B\ (a, b) \in f$$

The set A is called the **domain** of the function; the set B is called the **codomain** of the function. The **range** of f is the set

$$\{b \in B \mid \exists a \in A\ (a, b) \in f\}.$$

We write $f : A \to B$ to say that f is a function from A to B. Even though a function is a relation, we usually write $f(a) = b$ instead of $(a, b) \in f$ or afb.

Functions as machines that take inputs and produce outputs

You may like to think of a function as an abstract machine (this is an example of a ‘model’). It takes inputs. When given an input, the machine runs and produces an output according to some process. Using this model, we may interpret the function $f : A \to B$ as follows:

- the domain A is the set of allowed inputs to the function f
- the codomain B is the set from which outputs are selected by f
- the range is the subset of the codomain comprising the elements that are actually selected as outputs of the function.

About functions

We use the language of inputs and outputs to observe that:

- every input is assigned exactly one output—this is exactly the criteria under which a relation is a function;
- in general, the same output may be assigned to any number of input
- in general, some elements of the codomain may not be selected as the output for any input—that is, the range and the codomain may be different.

Example: Consider the function $s : \mathbb{Z} \to \mathbb{Z}$ that squares each input. Then $s(-3) = 9 = s(3)$, and -2 is in the codomain but not the range.

How to specify a function

To describe a function f you must:

- Describe the domain A
- Describe the codomain B
- Specify enough about "how the machine works" that for all $a \in A$, the predicate

  $$p_a(b) : f(a) = b$$

  is true for exactly one $b \in B$. It does not have to be easy to describe how to get $f(a)$ from a (in particular, the description may be very complicated), but the description must effectively select one output for each a.

Equal functions (a technical point)

Function f and g are **equal** if they have the same domain, the same codomain, and $f(x) = g(x)$ for all x in the domain.

Q: Are the following functions equal?
$f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(z) = z^2$
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A: No, because they have different codomains.

Another example

Q: Are the following functions equal?

$f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(z) = \dfrac{z^2 + 2z + 1}{(z+1)}$

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A: Yes, because $\frac{z^2+2z+1}{(z+1)} = \frac{(z+1)^2}{(z+1)} = z + 1$.

Examples: Different notation to define a function

Here are several ways to define the same function:

- Let f be the function from $\mathbb{Z}$ to $\mathbb{Z}$ that squares its input.
- Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be defined by the rule $f(z) = z^2$.
- Let $f = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid b = a^2\}$.
- Consider the function $\mathbb{Z} \to \mathbb{Z}$ given by $z \overset{f}{\mapsto} z^2$
- Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be $z \mapsto z^2$

In the above, the arrow $\to$ is read 'to' and the arrow $\mapsto$ is read 'maps to'.

Another example: multiple inputs

Here is a function that takes an input from a Cartesian product. This is a sneaky way to take two inputs, yet still fit the definition of a function.

Let $A = \{\text{cat}, \text{dog}, \text{chicken}\}$ and $B = \{+, -\}$ and $C = \{1, 2, 3\}$ and let $R : A \times B \to C$ be defined by the rule:

$$R((\text{pet}, \text{test})) = \begin{cases} 1, \text{if test is } + \\ 2, \text{if test is } - \text{ and pet is not cat} \\ 3, \text{if test is } - \text{ and pet is cat.} \end{cases}$$

In a mild abuse of notation, for such functions we tend to write $R(\text{cat}, +)$ instead of $R((\text{cat}, +))$.

Functions can be well-defined, but slow to compute

Let $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ be defined by the rule

$$f(z) = \begin{cases} 1, \text{if } z = 1 \\ \text{the smallest prime that divides } z, \text{if } z \neq 1 \end{cases}$$

We do not know ‘fast’ ways to compute $f(z)$ when z is large.

This example illustrates that the ‘rule of the function’ (a description of the process by which the output is determined by the input) does not have to be easily expressible in a single simple formula. As long as each element of the domain is related to exactly one element of the codomain, a function has been defined.

Functions can be well-defined, but difficult to compute

Let $\cos : \mathbb{R} \to \mathbb{R}$ be the function such that θ maps to the x-coordinate of the point reached by starting at the point $(1, 0)$ in the Euclidean plane and travelling counter-clockwise θ units around the unit circle centred at $(0, 0)$.

Let $\sin : \mathbb{R} \to \mathbb{R}$ be the function such that θ maps to the y-coordinate of the point reached by starting at the point $(1, 0)$ in the Euclidean plane and travelling counter-clockwise θ units around the unit circle centred at $(0, 0)$.

A very difficult question (without a computer):
Evaluate $\cos(2.3)$ and $\sin(-1.45)$.

Properties of functions: injective

Let $f : A \to B$ be a function. We say that f is **one-to-one** or f is **injective** or f is an **injection** when

$$\forall a_1, a_2 \in A \ (a_1 \neq a_2) \to (f(a_1) \neq f(a_2))$$

So f is injective when different inputs must give different outputs.

Properties of functions: surjective

Let $f : A \to B$ be a function. We say that f is **onto** or f **maps onto** B or f is **surjective** or f is a **surjection** when

$$\forall b \in B \; \exists a \in A \; f(a) = b.$$

So f is surjective when its codomain and range are equal; f is surjective when every element of the codomain is an actual output of the function).

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To determine whether or not a function in surjective, the codomain must be explicitly understood.

Example: Consider the function f that takes any integer as input, and outputs the absolute value of the input. Then f is surjective if the codomain is $\mathbb{N}$, but not surjective if the codomain is $\mathbb{Z}$.

Properties of functions: bijective

Let $f : A \to B$ be a function. We say that f is **bijective** or f is a **bijection** when f is injective and surjective.

We say that A and B are in one-to-one correspondence when there exists a bijection $f : A \to B$.

Composition

Let A, B and C be subsets of a universe U. If $f : A \to B$ and $g : B \to C$ are functions, then the rule $a \mapsto g(f(a))$ defines a function called the **composition of g and f** and denoted $g \circ f$. Note

$$(g \circ f)(a) = g(f(a)).$$

The order of composition matters...

Order of composition matters

Let $A = \{\text{cat}, \text{dog}, \text{chicken}\}$ and let $f : A \to \mathbb{N}$ be defined by the rule

$$\begin{aligned} &f \\ \text{cat} &\mapsto 70 \\ \text{dog} &\mapsto 90 \\ \text{chicken} &\mapsto 50, \end{aligned}$$

and let $g : \mathbb{N} \to \mathbb{N}$ be defined by the rule

$$g(z) = 3z.$$

Then $g \circ f$ is defined, but $f \circ g$ is undefined.

Inverse function

Let A, B be subsets of a universe U. Recall that if R is a relation from A to B, then the inverse relation R^{-1} from B to A is determined by the rule

$$aRb \Leftrightarrow bR^{-1}a.$$

Theorem: Let A, B be subsets of a universe U, and let $f : A \to B$ be a function. The inverse relation f^{-1} is a function from B to A (called the **inverse of** f) if only if f is a bijection.

If f and f^{-1} are both functions, we call them **inverse functions** and we say that f is **invertible**.

Inverse functions and identity functions

For any set A, the identity function on A is the function $i_A : A \to A$ defined by the rule $a \mapsto a$.

If $f : A \to B$ is a bijection, then $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

An example

Let's unpack something written using the vocabulary we have defined above.

Let $B = \{0, 1\}$ and $n \in \mathbb{Z}^+$. We define a set

$$B_n = \underbrace{B \times B \times \cdots \times B}_{n \text{ times}},$$

and a function $H_n : B_n \times B_n \to \mathbb{N}$ by the rule

$$H_n(s,t) = \text{ the number of coordinate (bit) positions where } s \text{ and } t \text{ differ.}$$

Q: Explain, in your own words, what has been defined by the above.

Example continued

An infinite number of sets $B_1, B_2, B_3, \ldots$ and functions $H_1, H_2, H_3, \ldots$ (called the **Hamming functions**) have been defined. For each positive integer n, B_n is the set of n-tuples of binary digits (bits). So, for example, $(0, 1, 1, 0, 0, 1) \in B_6$.

The function H_n takes as input two n-tuples of bits (in the form of an ordered pair), compares them to see in which places they agree, and outputs the number of coordinates in which they disgree. For example,

$$H_5((0, 0, 0, 1, 1), (1, 0, 1, 0, 1)) = 3$$

because the n-tuples given disagree in the first, third and fourth coordinates.

Challenge

Describe, if possible, a way to arrange the elements of B_n around a circle so that $H_n(s,t) = 1$ whenever s and t are adjacent in your arrangement.