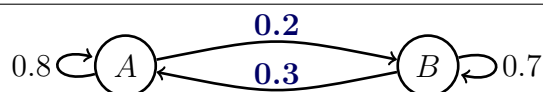


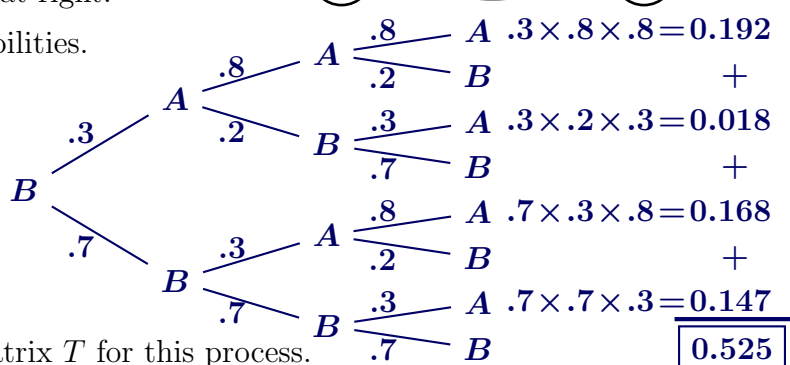
1. A Markov process has two states  $A$  and  $B$  with transition graph at right.



- (a) Write in the missing probabilities.

- (b) Suppose the system is initially in state  $B$ .

Use a tree diagram to find the probability that the system will be in state  $A$  after three steps.



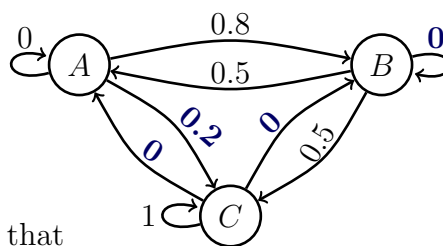
- (c) Write out the transition matrix  $T$  for this process.

$$T = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}, \quad T' = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

- (d) Use  $T$  to recalculate your answer to (b).

$$T'T'T' \begin{bmatrix} 0 \\ 1 \end{bmatrix} = T'T' \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = T' \begin{bmatrix} 0.45 \\ 0.55 \end{bmatrix} = \begin{bmatrix} 0.525 \\ 0.475 \end{bmatrix}, \text{ so probability system in state } A \text{ is } 0.525.$$

2. A Markov process has three states  $A$ ,  $B$  and  $C$  with transition graph at right.

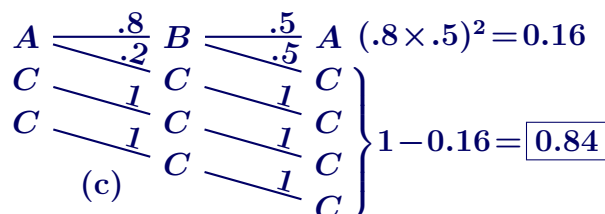
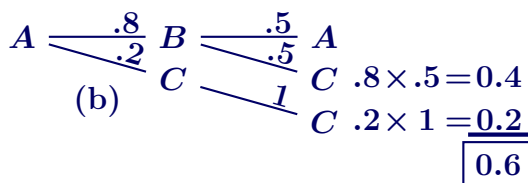


- (a) Write in all the missing probabilities.

- (b) Suppose the system is initially in state  $A$ .

Use a tree diagram to find the probability that the system will be in state  $C$  after two steps.

To simplify your diagram, leave out branches that have zero probability.



- (c) Extend your diagram for (b) to cover four steps. What is the probability that the system will be in state  $C$  after four steps?

- (d) Find the probability that the system will be in state  $C$  after ten steps starting from  $A$ . Do not use a diagram. Generalise from (c) and use complementary probability.

- (e) As for (d), but starting from  $B$ .

- (f) Guess the long-term probability that the system will be in state  $C$ , no matter what state the system starts in.

- (g) Write out the transition matrix  $T$  for this process.

$$T = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.5 & 0.5 & 0 \\ 0.2 & 0 & 1 \end{bmatrix}$$

- (h) Calculate  $T^2$  and  $T^4 = (T^2)^2$  and use them to confirm your answers to (b) and (c).

$$T^2 = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0.4 & 0.6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T^4 = \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0.4 & 0.6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0.4 & 0.6 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.84 & 0.16 & 0 \\ 0.16 & 0.84 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- (i) Convert your answer to (f) to a steady state vector  $S$  and confirm that answer by verifying that  $T'S = S$ .  $S = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 0 & 0.5 & 0 \\ 0.8 & 0 & 0 \\ 0.2 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ .

3. Ari is an innovative mathematics teacher. Once a week he sets up his classroom as four activity 'stations' labelled 1, 2, 3 and 4. Students spend 15 minutes at each station. In order to mix up the students, at change-over time Ari randomly divides the groups at stations 1 - 3 into two subgroups, as equally-sized as possible, and randomly sends one subgroup to the next station ( $i \rightarrow i+1$ ) and the other subgroup to the station beyond ( $i \rightarrow i+2$  except  $3 \rightarrow 1$ ). Owing to the nature of the activity at station 2, Ari needs to limit the numbers at that station, so he starts with a smaller group there and at changeover time all students at station 4 move only to station 1.

- (a) Compile a transition  $T$  matrix representing this (Markov) process. The states are the stations and entry  $t_{ij}$  of  $T$  specifies, for a student at station  $i$ , the probability that, at change-over, the student will move to station  $j$ .

$$\begin{bmatrix} 0 & .5 & .5 & 0 \\ 0 & 0 & .5 & .5 \\ .5 & 0 & 0 & .5 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- (b) Ari starts with six students at each of stations 1, 3 and 4, and five at station 2. The class lasts an hour. How many students will there be at each station when the class ends? [There are several possible answers here, since odd-sized groups cannot be equally subdivided. Flip a coin to decide to which station each larger subgroup goes.]

$$T' \begin{bmatrix} 6 \\ 5 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & .5 & 1 \\ .5 & 0 & 0 & 0 \\ .5 & .5 & 0 & 0 \\ 0 & .5 & .5 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 5.5 \\ 5.5 \end{bmatrix} \rightarrow \begin{bmatrix} 9 \\ 3 \\ 6 \\ 5 \end{bmatrix} \star$$

$$T' \begin{bmatrix} 9 \\ 3 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & .5 & 1 \\ .5 & 0 & 0 & 0 \\ .5 & .5 & 0 & 0 \\ 0 & .5 & .5 & 0 \end{bmatrix} \begin{bmatrix} 9 \\ 3 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 8 \\ 4.5 \\ 6 \\ 4.5 \end{bmatrix} \rightarrow \begin{bmatrix} 8 \\ 4 \\ 6 \\ 5 \end{bmatrix} \star$$

★: Total number of students stays at 23.

- (c) Verify that the steady state is eight students at station 1, four at station 2, six at station 3 and five at station 4.

$$T' \begin{bmatrix} 8 \\ 4 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & .5 & 1 \\ .5 & 0 & 0 & 0 \\ .5 & .5 & 0 & 0 \\ 0 & .5 & .5 & 0 \end{bmatrix} \begin{bmatrix} 8 \\ 4 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 6 \\ 5 \end{bmatrix}$$

4. A certain Markov Process has transition matrix  $T$  at right.

Using a computer some powers of  $T$  were calculated and are shown below to three decimal places.

$$T = \begin{bmatrix} .6 & .2 & 0 & .2 \\ .3 & .3 & .2 & .2 \\ .2 & 0 & .2 & .6 \\ .1 & .2 & .1 & .6 \end{bmatrix}$$

$$T^2 = \begin{bmatrix} .44 & .22 & .06 & .28 \\ .33 & .19 & .12 & .36 \\ .22 & .16 & .10 & .52 \\ .20 & .20 & .12 & .48 \end{bmatrix} \quad T^4 = \begin{bmatrix} .335 & .204 & .092 & .368 \\ .306 & .200 & .099 & .396 \\ .276 & .199 & .105 & .421 \\ .276 & .197 & .106 & .421 \end{bmatrix} \quad T^8 = \begin{bmatrix} .302 & .200 & .100 & .398 \\ .300 & .200 & .100 & .400 \\ .299 & .200 & .100 & .401 \\ .299 & .200 & .100 & .401 \end{bmatrix}$$

Use the powers of  $T$  to guess a steady state vector for the process, and then prove your guess is correct.

**Rows of  $T^8$  are all approximately  $[\mathbf{.3 \ .2 \ .1 \ .4}]$ .**

$$\text{So guess } S = \begin{bmatrix} .3 \\ .2 \\ .1 \\ .4 \end{bmatrix}. \quad \text{Then } T'S = \begin{bmatrix} .6 & .3 & .2 & .1 \\ .2 & .3 & 0 & .2 \\ 0 & .2 & .2 & .1 \\ .2 & .2 & .6 & .6 \end{bmatrix} \begin{bmatrix} .3 \\ .2 \\ .1 \\ .4 \end{bmatrix} = \begin{bmatrix} .3 \\ .2 \\ .1 \\ .4 \end{bmatrix} = S.$$

5. A certain Markov Process has transition matrix  $T$  at right. Use

a matrix calculation tool such as <https://matrixcalc.org/en/> to calculate  $T^{16}$  to three decimal places.

$$T = \begin{bmatrix} .4 & .1 & 0 & .5 \\ .4 & .1 & .2 & .3 \\ 0 & .3 & .4 & .3 \\ .7 & .2 & .1 & 0 \end{bmatrix}$$

Use  $T^{16}$  to guess a steady state vector for the process, and then prove your guess is correct.

(If your matrix tool doesn't have a powering function but does have a multiplication function you could first calculate  $T \times T = T^2$  then  $T^2 \times T^2 = T^4$  and so on. Depending on the tool, with lots of cut-and-paste the only matrix you may need to enter is  $T$ .)

$$T^{16} \approx \begin{bmatrix} .450 & .150 & .100 & .300 \\ .450 & .150 & .100 & .300 \\ .450 & .150 & .100 & .300 \\ .450 & .150 & .100 & .300 \end{bmatrix}; \quad S = \begin{bmatrix} .45 \\ .15 \\ .1 \\ .3 \end{bmatrix}. \quad T'S = \begin{bmatrix} .4 & .4 & 0 & .7 \\ .1 & .1 & .3 & .2 \\ 0 & .2 & .4 & .1 \\ .5 & .3 & .3 & 0 \end{bmatrix} \begin{bmatrix} .45 \\ .15 \\ .1 \\ .3 \end{bmatrix} = \begin{bmatrix} .45 \\ .15 \\ .1 \\ .3 \end{bmatrix} = S.$$

6. Calculate the steady state vector  $S$  for the Markov process of Question 1. Do this by hand, using the matrix inverse method with short cut to solve  $T'S = S$ .

$$T' - I = \begin{bmatrix} -.2 & .3 \\ .2 & -.3 \end{bmatrix}. \text{ Solve } \begin{bmatrix} -.2 & .3 \\ 1 & 1 \end{bmatrix} S = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$S = \begin{bmatrix} -.2 & .3 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{-.2-.3} \begin{bmatrix} 1 & -.3 \\ -1 & -.2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix}.$$

7. Recalculate the steady state vector for the Markov process of Question 5 by using the 'Gauss-Jordan Elimination' function in the 'Matrix Reshish'<sup>1</sup> computer application <https://matrix.resish.com>. Specify your input to, and output from, Reshish.

As for Q6, solve  $(T' - I)S = 0$  with last row replaced by all 1's.

The input augmented matrix :  $\begin{bmatrix} -0.6 & 0.4 & 0 & 0.7 & 0 \\ 0.1 & -0.9 & 0.3 & 0.2 & 0 \\ 0 & 0.2 & -0.6 & 0.1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ , Output:  $\begin{matrix} x_1=0.45 \\ x_2=0.15 \\ x_3=0.1 \\ x_4=0.3 \end{matrix}$ , so  $S = \begin{bmatrix} 0.45 \\ 0.15 \\ 0.1 \\ 0.3 \end{bmatrix}$ .

8. Let  $T$  be an  $n \times n$  stochastic matrix (rows are probability vectors) and  $\mathbf{v}$  a column probability  $n$ -vector. Prove that  $T'\mathbf{v}$  is always also a probability vector. Try this first for  $n = 2$  and then for  $n = 3$ . Do it for general  $n$  if your algebra is up to it.

For any  $n$  the entries of  $T'$  and  $\mathbf{v}$  are non-negative, so same is true for  $T'\mathbf{v}$ .

For  $n = 2$  let  $T = \begin{bmatrix} p_1 & q_1 \\ p_2 & q_2 \end{bmatrix}$ ,  $\mathbf{v} = \begin{bmatrix} p_0 \\ q_0 \end{bmatrix}$ . Then  $T'\mathbf{v} = \begin{bmatrix} p_1 p_0 + p_2 q_0 \\ q_1 p_0 + q_2 q_0 \end{bmatrix}$  and entries sum to  $(p_1 p_0 + p_2 q_0) + (q_1 p_0 + q_2 q_0) = (p_1 + q_1)p_0 + (p_2 + q_2)q_0 = p_0 + q_0 = 1$ .

For  $n = 3$ ,  $T = \begin{bmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{bmatrix}$ ,  $\mathbf{v} = \begin{bmatrix} p_0 \\ q_0 \\ r_0 \end{bmatrix}$ ,  $T'\mathbf{v} = \begin{bmatrix} p_1 p_0 + p_2 q_0 + p_3 r_0 \\ q_1 p_0 + q_2 q_0 + q_3 r_0 \\ r_1 p_0 + r_2 q_0 + r_3 r_0 \end{bmatrix}$  and entries sum to  $(p_1 p_0 + p_2 q_0 + p_3 r_0) + (q_1 p_0 + q_2 q_0 + q_3 r_0) + (r_1 p_0 + r_2 q_0 + r_3 r_0) = (p_1 + q_1 + r_1)p_0 + (p_2 + q_2 + r_2)q_0 + (p_3 + q_3 + r_3)r_0 = p_0 + q_0 + r_0 = 1$ .

9. *Hardy-Weinburg Equilibrium*: Consider a gene that has two forms, or *alleles*,  $A$  and  $a$ . Each individual has two of these genes and so has *genotype*  $AA$ ,  $Aa$  or  $aa$ .

Assume that an individual's genotype consists of a random selection of one each of its parents' alleles. So, for example, the offspring of parents who are both  $Aa$  has a 50% chance of also being  $Aa$  and a 25% chance each of being  $AA$  and  $aa$ .

Assume further that mating partners are chosen at random.

Let  $\pi_{AA}$ ,  $\pi_{Aa}$  and  $\pi_{aa}$  be the proportions of each genotype in a breeding colony.

(a) Explain why  $p = \pi_{AA} + \pi_{Aa}/2$  and  $q = \pi_{aa} + \pi_{Aa}/2$  are the probabilities that a random allele chosen from a random individual is  $A$  or  $a$  respectively.

$$\begin{aligned} \mathbb{P}(\text{allele is } A) &= \mathbb{P}(\text{genotype is } AA)\mathbb{P}(A \text{ chosen from } AA) \\ &\quad + \mathbb{P}(\text{genotype is } Aa)\mathbb{P}(A \text{ chosen from } Aa) \\ &\quad + \mathbb{P}(\text{genotype is } aa)\mathbb{P}(A \text{ chosen from } aa) \\ &= \pi_{AA} \times 1 + \pi_{Aa} \times 1/2 + \pi_{aa} \times 0 = \pi_{AA} + \pi_{Aa}/2 \end{aligned}$$

Calculation of  $\mathbb{P}(\text{allele is } a)$  is similar.

(b) Explain why the parent-to-offspring transition matrix is given by  $T = \begin{matrix} & \begin{matrix} AA & Aa & aa \end{matrix} \\ \begin{matrix} AA \\ Aa \\ aa \end{matrix} & \begin{bmatrix} p & q & 0 \\ p/2 & 1/2 & q/2 \\ 0 & p & q \end{bmatrix} \end{matrix}$ .

Here are explanations for two representative sample entries:

$AA$  parent gives  $A$  with prob. 1, mate gives  $A$  with prob.  $p \therefore t_{11} = 1p = p$ .

$Aa$  parent gives  $A$  with prob.  $1/2$ , mate gives  $a$  with prob.  $q$  and

$Aa$  parent gives  $a$  with prob.  $1/2$ , mate gives  $A$  with prob.  $p \therefore t_{22} = q/2 + p/2 = 1/2$ .

(c) Show that the steady state vector is  $S = \begin{bmatrix} p^2 \\ 2pq \\ q^2 \end{bmatrix}$ .  $T'S = \begin{bmatrix} p^3 + p^2 q \\ p^2 q + pq + pq^2 \\ pq^2 + q^3 \end{bmatrix} = \begin{bmatrix} p^2(p+q) \\ pq(p+1+q) \\ q^2(p+q) \end{bmatrix} = S$  as  $p+q=1$ .

(d) Show that  $S$  is always achieved in just one transition step.

<sup>1</sup>When entering decimal values, Reshish requires a digit before the decimal point. e.g. enter '0.4', not '.4'.

$$T' \begin{bmatrix} \pi_{AA} \\ \pi_{Aa} \\ \pi_{aa} \end{bmatrix} = \begin{bmatrix} p & \frac{p}{2} & 0 \\ q & \frac{p}{2} + \frac{q}{2} & p \\ 0 & \frac{q}{2} & q \end{bmatrix} \begin{bmatrix} \pi_{AA} \\ \pi_{Aa} \\ \pi_{aa} \end{bmatrix} = \begin{bmatrix} p(\pi_{AA} + \frac{1}{2}\pi_{Aa}) \\ q(\pi_{AA} + \frac{1}{2}\pi_{Aa}) + p(\pi_{aa} + \frac{1}{2}\pi_{Aa}) \\ q(\pi_{aa} + \frac{1}{2}\pi_{Aa}) \end{bmatrix} = \begin{bmatrix} pp \\ qp + pq \\ qq \end{bmatrix} = S.$$