COMP2610 / COMP6261 Information Theory
Lecture 15: Shannon-Fano-Elias and Interval Coding

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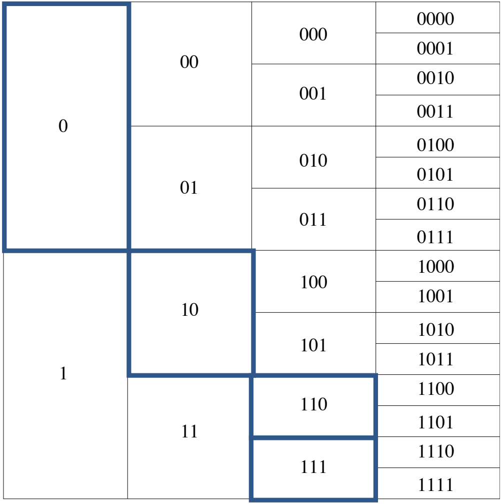
Acknowledgement: These slides were originally developed by Professor Robert C. Williamson.

Prefix Codes as Trees (Recap)

$$C_2 = \{0, 10, 110, 111\}$$



The Source Coding Theorem for Symbol Codes

Source Coding Theorem for Symbol Codes

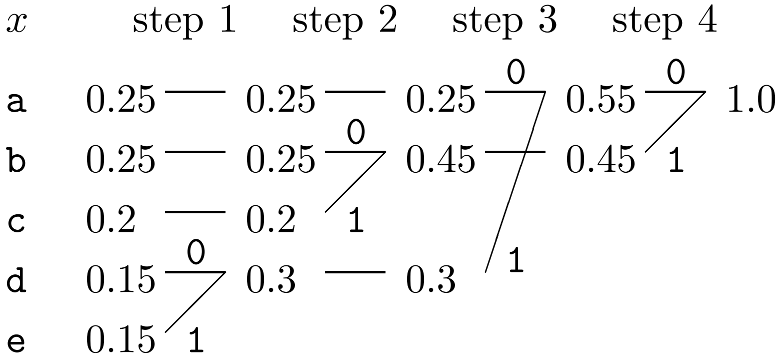
For any ensemble X there exists a *prefix code* C such that

$$H(X) \leq L(C, X) < H(X) + 1.$$

In particular, **Shannon codes** C — those with lengths $\ell_i = \left\lceil \log_2 \frac{1}{p_i} \right\rceil$ — have *expected code length within 1 bit of the entropy*.

Huffman Coding: Recap

$\mathcal{A}_X = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$ and $\mathcal{P}_X = \{0.25, 0.25, 0.2, 0.15, 0.15\}$



From Example 5.15 of MacKay

$C = \{00, 10, 11, 010, 011\}$

Huffman Coding: Advantages and Disadvantages

Advantages:

- Huffman Codes are provably optimal amongst prefix codes
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Disadvantages:

- Assumes a fixed distribution of symbols
- The extra bit in the SCT
  - If $H(X)$ is large – not a problem
  - If $H(X)$ is small (e.g., $\sim$ 1 bit for English) codes are $2\times$ optimal

Huffman codes are the best possible symbol code
but symbol coding is not always the best type of code

This time

A different way of coding (interval coding)

Shannon-Fano-Elias codes

Worse guarantee than Huffman codes, but will lead us to the powerful arithmetic coding procedure

Coding via Cumulative Probabilities

Suppose X is an ensemble with probabilities $(p_1, \ldots, p_{|X|})$

Define the cumulative distribution function by

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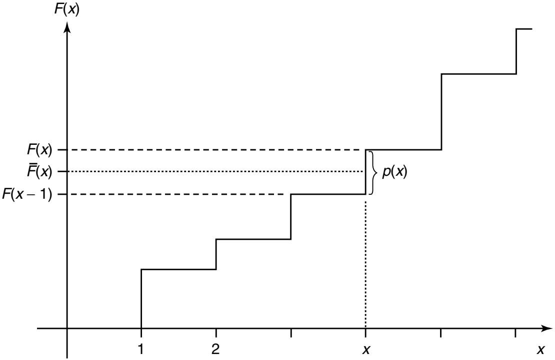
We can losslessly code outcomes based on $\overline{F}$!

Coding via Cumulative Probabilities



$\overline{F}(x)$ will uniquely determine each outcome x (lossless code)

Example

Suppose X has outcomes (a_1, a_2, a_3, a_4) and probabilities $(2/9, 1/9, 1/3, 1/3)$

Define the midpoint $\overline{F}(a_i) = F(a_i) - \frac{1}{2}p_i$

x	$p(x)$	$F(x)$	$\overline{F}(x)$
a_1	$2/9$	$2/9$	$1/9$
a_2	$1/9$	$1/3$	$5/18$
a_3	$1/3$	$2/3$	$1/2$
a_4	$1/3$	1	$5/6$

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How do we code $\overline{F}(x)$ in binary though?

Real Numbers in Binary

Real numbers are commonly expressed in decimal:

$$
\begin{aligned}
12_{10} &\rightarrow 1 \times 10^{1} + 2 \times 10^{0} \\
3.7_{10} &\rightarrow 3 \times 10^{0} + 7 \times 10^{-1} \\
0.94_{10} &\rightarrow + 9 \times 10^{-1} + 4 \times 10^{-2}
\end{aligned}
$$

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Some real numbers have infinite, repeating decimal expansions:

$$\frac{1}{3} = 0.33333\ldots_{10} = 0.\overline{3}_{10} \quad \text{and} \quad \frac{22}{7} = 3.14285714\ldots_{10} = 3.\overline{142857}_{10}$$

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Real numbers can also be similarly expressed in binary:

$$3_{10} = 11_2 \rightarrow 1 \times 2^1 + 1 \times 2^0$$

$$1.5_{10} = 1.1_2 \rightarrow 1 \times 2^0 + 1 \times 2^{-1}$$

$$0.75_{10} = 0.11_2 \rightarrow + 1 \times 2^{-1} + 1 \times 2^{-2}$$

$$\frac{1}{3} = 0.010101\ldots_2 = 0.\overline{01}_2 \quad \text{and} \quad \frac{22}{7} = 11.001001\ldots_2 = 11.\overline{001}_2$$

Converting Decimal Fractions to Binary

To convert a fraction (e.g. $3/4$) to binary:

1. Multiply the fraction by 2. Take the whole number part of the result; this is the first bit of the binary expansion.
2. Throw away the whole number part of the result, and just retain the part after the decimal point.
3. Repeat step 1. Stop when either:
   - what remains after the decimal point is zero, or
   - you detect an infinite loop

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Example: for 0.625_{10},

- $2 \cdot 0.625 = 1.25$, so first bit is 1
- $2 \cdot 0.25 = 0.5$, so second bit is 0
- $2 \cdot 0.5 = 1.0$, so third bit is 1
- decimal part is zero, so stop

Shannon-Fano-Elias Coding: To Infinity and Beyond

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Shannon-Fano-Elias Coding: To Infinity and Beyond

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- e.g. if $\overline{F}(x) = \frac{1}{3}$

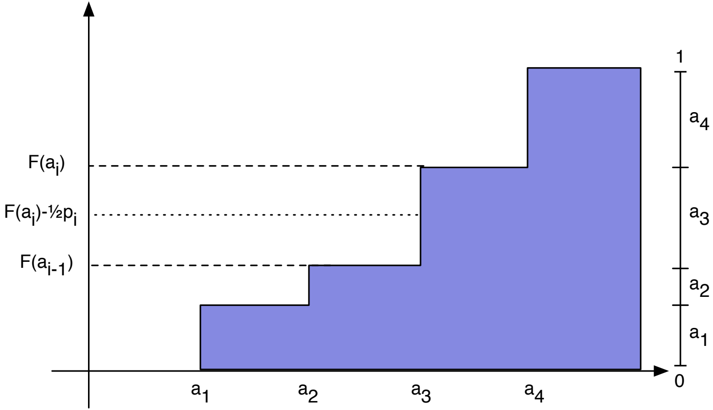
Fortunately, we can get away with only storing $\overline{F}(x)$ approximately

Shannon-Fano-Elias coding: code using the first $\ell(x) = \lceil \log_2 \frac{1}{p(x)} \rceil + 1$ bits of $\overline{F}(x)$

- (Almost) Constructive procedure for a Shannon code

Cumulative Distribution

Example



Cumulative distribution for $\mathbf{p} = (\frac{2}{9}, \frac{1}{9}, \frac{1}{3}, \frac{1}{3})$

Shannon-Fano-Elias Coding

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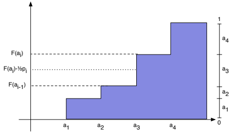

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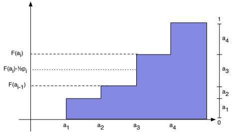

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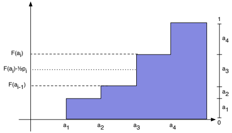
Shannon-Fano-Elias Coding: code $x \in \mathcal{A}$ using first $\ell(x)$ bits of $\overline{F}(x)$.

x	$p(x)$	$F(x)$	$\overline{F}(x)$	$\overline{F}(x)_2$	$\ell(x)$	Code
a_1	$2/9$	$2/9$	$1/9$	$0.\overline{000111}_2$	4	0001
a_2	$1/9$	$1/3$	$5/18$	$0.01\overline{000111}_2$	5	01000
a_3	$1/3$	$2/3$	$1/2$	0.1_2	3	100
a_4	$1/3$	1	$5/6$	$0.1\overline{10}_2$	3	110

Shannon-Fano-Elias Coding

Example



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Example: Sequence $\mathbf{x} = a_3 a_3 a_1$ coded as 100 100 0001.

Remaining questions

Encoding with a Shannon-Fano-Elias code is simple

But we have to check:

- is the code lossless?
- is the code prefix-free?
- how do we decode a given codeword?

Shannon-Fano-Elias Coding: Is it lossless?

Denote the Shannon-Fano-Elias code for an outcome x by

$$\lfloor \overline{F}(x) \rfloor_{\ell(x)},$$

where $\lfloor \cdot \rfloor_\ell$ means truncate to first ℓ bits

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No, because (homework exercise!)

$$F(x-1) < \lfloor \overline{F}(x) \rfloor_{\ell(x)} < F(x)$$

i.e. the codeword lies entirely in the interval between $x-1$ and x

- These intervals don't overlap for different outcomes
- The code is lossless!

Prefixes and Binary Strings

What is the set of binary strings that begin with $\mathbf{b} = \mathtt{b_1 \ldots b_n}$?

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$\mathtt{b}_1 \dots \mathtt{b}_n 0, \mathtt{b}_1 \dots \mathtt{b}_n 1, \mathtt{b}_1 \dots \mathtt{b}_n 01, \mathtt{b}_1 \dots \mathtt{b}_n 11, \dots$

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$\mathtt{b_1 \ldots b_n 0}, \mathtt{b_1 \ldots b_n 1}, \mathtt{b_1 \ldots b_n 01}, \mathtt{b_1 \ldots b_n 11}, \ldots$

Basically, anything ranging from

$$\mathtt{b_1 \ldots b_n 000 \ldots} \text{ to } \mathtt{b_1 \ldots b_n 111 \ldots}$$

These are the strings having $\mathtt{b_1 \ldots b_n}$ as a prefix

Prefixes and Binary Strings

We could equally associate $\mathtt{b_1 \ldots b_n}$ with the fraction $\mathtt{0.b_1 \ldots b_n}$

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i.e.

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What is the set of binary strings that begin with $\mathbf{b} = \mathtt{b_1 \ldots b_n}$?

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i.e.

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Note that

$$0.\mathtt{b_1 \ldots b_n}\overline{1} = 0.\mathtt{b_1 \ldots b_n} + \frac{1}{2^n} = 0.\mathtt{b_1 \ldots b_n} + 0.\mathtt{0 \ldots 1},$$

just like $0.1\overline{9}_{10} = 0.2$

Intervals: Definition

It will be useful to analyse the prefix property in terms of intervals

An interval $[a, b)$ is the set of all the numbers at least as big as a but smaller than b. That is,

$$[a, b) = \{x : a \leq x < b\}.$$

Examples: $[0, 1)$, $[0.3, 0.6)$, $[0.2, 0.4)$.

Intervals in Binary

The set of numbers in $[0, 1)$ that start with a given sequence of bits $\mathbf{b} = b_1 \ldots b_n$ form the interval

$$\left[0.\mathtt{b_1} \ldots \mathtt{b_n}, 0.\mathtt{b_1} \ldots \mathtt{b_n} + \frac{1}{2^n}\right) = [0.\mathtt{b_1} \ldots \mathtt{b_n}, 0.\mathtt{b_1} \ldots \mathtt{b_n} + 0.\mathtt{0} \ldots \mathtt{1})$$

- $\mathtt{1} \rightarrow [0.\mathtt{1}, 1.0)$ $\qquad [0.5, 1]_{10}$
- $\mathtt{01} \rightarrow [0.\mathtt{01}, 0.10)$ $\qquad [0.25, 0.5)_{10}$
- $\mathtt{1101} \rightarrow [0.\mathtt{1101}, 0.1110)$ $\qquad [0.8125, 0.875)_{10}$

Prefix Property and Intervals

Prefix property (tree form): Once you pick a node in the binary tree, you cannot pick any of its descendants

Prefix property (interval form): Once you pick a codeword $\mathtt{b_1 b_2 \ldots b_n}$, you cannot pick any codeword in

$$\left[0.\mathtt{b_1 b_2 \ldots b_n}, 0.\mathtt{b_1 b_2 \ldots b_n} + \frac{1}{2^n}\right)$$

Why? This contains all binary strings for which $\mathtt{b_1 b_2 \ldots b_n}$ is a prefix

e.g. If we pick 0110, we cannot pick anything from

$$\begin{aligned}[0.0110, 0.0111) &= [0.0110\overline{0}, 0.0110\overline{1}) \\ &= \{0.0110, 0.01101, 0.011001, 0.011011, \ldots,\}\end{aligned}$$

Prefix Property and Intervals

If $\mathbf{b}'$ is a prefix of $\mathbf{b}$, the interval for $\mathbf{b}$ is contained in the interval for $\mathbf{b}'$

e.g. $\mathbf{b}' = 01$ is prefix of $\mathbf{b} = 0101$ so $\underbrace{[0.0101, 0.0110)}_{[0.3125, 0.375)_{10}} \subset \underbrace{[0.01, 0.10)}_{[0.25, 0.5)_{10}}$

Why? Because interval for $\mathbf{b}'$ contains all strings for which $\mathbf{b}'$ is a prefix

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Implication: If intervals for $\mathbf{b}, \mathbf{b}'$ are disjoint, one cannot be a prefix of another

Shannon-Fano-Elias Coding is Prefix-Free

We already know $\lfloor \overline{F}(x) \rfloor_{\ell(x)} > F(x-1)$. We also have

$$\begin{aligned} \lfloor \overline{F}(x) \rfloor_{\ell(x)} + \frac{1}{2^\ell} &\leq \overline{F}(x) + \frac{1}{2^\ell} \\ &\leq \overline{F}(x) + \frac{p(x)}{2} \\ &= F(x), \end{aligned}$$

and so

$$\left[\lfloor \overline{F}(x) \rfloor_{\ell(x)}, \lfloor \overline{F}(x) \rfloor_{\ell(x)} + \frac{1}{2^\ell} \right) \subset [F(x-1), F(x))$$

The intervals for each codeword are thus trivially disjoint, since we know each of the $[F(x-1), F(x))$ intervals is disjoint

The SFE code is prefix-free!

Two Types of Interval

The **symbol interval** for some outcome x_i is (assuming $F(x_0) = 0$)

$$[F(x_{i-1}), F(x_i))$$

These intervals are disjoint for each outcome

The **codeword interval** for some outcome x_i is

$$\left[\lfloor \overline{F}(x_i) \rfloor_{\ell(x_i)}, \lfloor \overline{F}(x_i) \rfloor_{\ell(x_i)} + \frac{1}{2^{\ell(x_i)}}\right)$$

This is a strict subset of the symbol interval

All strings in the codeword interval start with the same prefix

- This is **not true** in general for the symbol interval

Shannon-Fano-Elias Decoding

To decode a given bitstring:

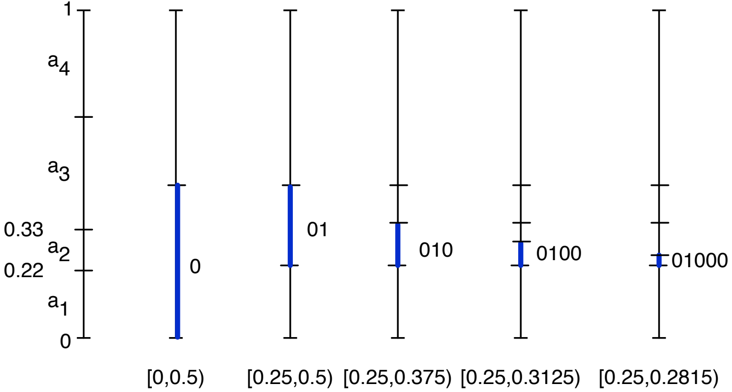
1. start with the first bit, and compute the corresponding binary interval
2. if the interval is strictly contained within that of a codeword:
   1. output the codeword
   2. skip over any redundant bits for this codeword
   3. repeat (1) for the rest of the bitstring
3. else include next bit, and compute the corresponding binary interval
4. $\vdots$

We might be able to stop early owing to redundancies in SFE

Shannon-Fano-Elias Decoding

Let $\mathbf{p} = \{\frac{2}{9}, \frac{1}{9}, \frac{1}{3}, \frac{1}{3}\}$. Suppose we want to *decode* 01000:

Find symbol interval containing codeword interval for $\texttt{01000} = [0.25, 0.28125)_{10}$



We could actually stop once we see $\texttt{0100}$, since $[0.25, 0.3125) \subset [0.22, 0.33]$

Expected Code Length of SFE Code

The extra bit for the code lengths is because we code $\frac{p_i}{2}$ and

$$\log_2 \frac{2}{p_i} = \log_2 \frac{1}{p_i} + \log_2 2 = \log_2 \frac{1}{p_i} + 1$$

What is the expected length of a SFE code C for ensemble X with probabilities $\mathbf{p}$?

$$\begin{aligned} L(C, X) = \sum_{i=1}^{K} p_i \ell(a_i) &= \sum_{i=1}^{K} p_i \left(\left\lceil \log_2 \frac{1}{p_i} \right\rceil + 1 \right) \\ &\leq \sum_{i=1}^{K} p_i \left(\log_2 \frac{1}{p_i} + 2 \right) \\ &= H(X) + 2 \end{aligned}$$

Similarly, $H(X) + 1 \leq L(C, X)$ for the SFE codes.

Why bother?

Let X be an ensemble, C_{SFE} be a Shannon-Fano-Elias code for X and C_H be a Huffman code for X

$$\underbrace{H(X) \leq L(C_H, X) \leq H(X) + 1}_{\text{Source Coding Theorem}} \leq L(C_{SFE}, X) \leq H(X) + 2$$

so why not just use Huffman codes?

SFE is a stepping stone to a more powerful type of codes

- Roughly, try to apply SFE to a block of outcomes

Summary and Reading

Main points:

- Problems with Huffman coding symbol distribution
- Binary strings to/from intervals in $[0, 1]$
- Shannon-Fano-Elias Coding:
  - Code C via cumulative distribution function for p
  - $H(X) + 1 \leq L(C, X) \leq H(X) + 2$
- Extra bit guarantees interval containment

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Next time:

Extending SFE Coding to sequences of symbols