

## COMP3670: Introduction to Machine Learning

### Question 1 Systems of Linear Equations

Let

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

for some constants  $b_1, \dots, b_5 \in \mathbb{R}$ .

1. Show that  $\mathbf{A}$  is non-invertible.
2. Find the set of solutions  $\{\mathbf{x} : \mathbf{Ax} = \mathbf{b}\}$ .
3. Hence, or otherwise, find a non-zero value for  $\mathbf{x}$  such that  $\mathbf{Ax} = \mathbf{0}$ .

### Question 2 Matrix Inverses

Let

$$\mathbf{A} = \begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

for some constants  $a, b, c \in \mathbb{R}$ .

1. For what values of  $a, b, c$  is the inverse of  $\mathbf{A}$  defined?
2. Find  $\mathbf{A}^{-1}$  assuming the properties on  $a, b, c$  to ensure the inverse exists.

### Question 3 Which matrices commute?

Let

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Find all matrices  $\mathbf{B} \in \mathbb{R}^{2 \times 2}$  such that  $\mathbf{AB} = \mathbf{BA}$ .

### Question 4 Proving Properties of Matrix Operations

For each of the following statements, if it is true, prove it. If it is false, give a counter-example.

1. Let  $\mathbf{A} \in \mathbb{R}^{n \times n}, \mathbf{B} \in \mathbb{R}^{n \times n}$ . Assume that both  $\mathbf{A}$  and  $\mathbf{B}$  are invertible.  
Does  $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$  hold?
2. Let  $\mathbf{A} \in \mathbb{R}^{n \times n}, \mathbf{B} \in \mathbb{R}^{n \times n}$ . Assume that both  $\mathbf{A}$  and  $\mathbf{B}$  are invertible.  
Does  $(\mathbf{A} + \mathbf{B})^{-1} = \mathbf{A}^{-1} + \mathbf{B}^{-1}$  hold?
3. Let  $\mathbf{A} \in \mathbb{R}^{n \times m}$ . Both  $\mathbf{AA}^T$  and  $\mathbf{A}^T\mathbf{A}$  are well-defined<sup>1</sup> and symmetric<sup>2</sup> matrices.

---

<sup>1</sup>as in, the matrix product is defined

<sup>2</sup>A *symmetric* matrix is one equal to its own transpose.

4. Let  $\mathbf{A} \in \mathbb{R}^{n \times m}$ . If  $\mathbf{A}$  is non-invertible, then there must exist two different vectors  $\mathbf{u} \in \mathbb{R}^n$  and  $\mathbf{v} \in \mathbb{R}^n$  such that  $\mathbf{A}\mathbf{u} = \mathbf{A}\mathbf{v}$ .
5. Let  $\mathbf{A} \in \mathbb{R}^{n \times m}$ . If there exists two different vectors  $\mathbf{u} \in \mathbb{R}^n$  and  $\mathbf{v} \in \mathbb{R}^n$  such that  $\mathbf{A}\mathbf{u} = \mathbf{A}\mathbf{v}$ , then  $\mathbf{A}$  is non-invertible.
6. If there exists two different vectors  $\mathbf{u} \in \mathbb{R}^n$  and  $\mathbf{v} \in \mathbb{R}^n$  such that  $\mathbf{A}\mathbf{u} = \mathbf{A}\mathbf{v}$ , prove that there exists a non-zero vector  $\mathbf{x}$  such that  $\mathbf{A}\mathbf{x} = \mathbf{0}$ .