COMP2610 / COMP6261 Information Theory
Lecture 6: Entropy

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Announcements

Assignment 1

- Available via Wattle
- Worth 10% of Course total
- Due Friday 26th August 2022, 5:00 pm

Last time

- The Bernoulli and Binomial distributions
- Maximum likelihood estimation
- Bayesian parameter etimation

This time

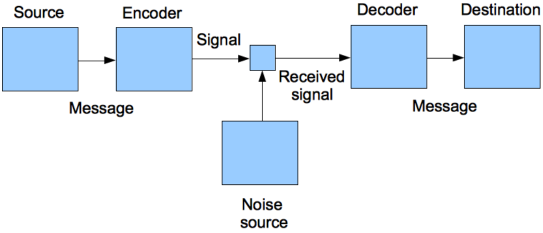
- Information content and entropy

- Examples and intuition

- Some basic properties of entropy

Outline

Recap: A General Communication System



How informative is a message?

Information Content: Informally

Say that a message comprises a single bit (one binary random variable)

- Whether or not a coin comes up heads
- Whether or not my favourite horse wins a race

Informally, the amount of information in such a message is:

- How unexpected or "surprising" an outcome of random variable is
  - If a coin comes up Heads 99.99% of the time, the message "Tails" is much more informative than "Heads"
  - If I believe my favourite horse will win with 99.99% probability, it is surprising to find out it did not

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  - If I believe my favourite horse will win with 99.99% probability, it is surprising to find out it did not
- How predictable a random variable is
  - If a coin comes up Heads 99.99% of the time, we can predict the next message as "Heads" and be right most of the time
  - If I believe my favourite horse will win with 99.99% probability, then I believe predicting so to be right most of the time

Information Content: Formally

Intuitively, we measure information of a message in relation to the other messages we could have seen

- For binary messages, we either see `0` or `1`
- The message `1` is informative when there is a good chance I might have seen `0`

How can we *formalise* and thus *measure* information content?

- Information content of an outcome must depend on its probability
- Information content of a random variable must depend on its probability distribution

Information Content of an Outcome: Definition

Let X be a discrete r.v. with possible outcomes $\mathcal{X}$

- e.g. $\mathcal{X} = \{\texttt{0}, \texttt{1}\}$

- e.g. $\mathcal{X} = \{\texttt{Yes}, \texttt{No}, \texttt{Maybe}\}$

Let $p(x)$ denote the probability of outcome $x \in \mathcal{X}$

The information content of an outcome $x \in \mathcal{X}$ is:

$$h(x) = \log_2 \frac{1}{p(x)}$$

Information Content of an Outcome: Properties

The information content of x grows as $p(x)$ shrinks

- Outcomes that are rare are deemed to contain more information

Choice of logarithm basis is arbitrary

- If we use $\log_2$ we measure information in *bits*

What about other functions of $p(x)$, e.g. $\frac{1}{p(x)^2} - 1$?

Entropy of a Random Variable: Definition

Let X be a discrete r.v. with possible outcomes $\mathcal{X}$.

The entropy of the random variable X is the average information content of the outcomes:

$$
\begin{aligned}
H(X) &= \mathbb{E}_x\left[h(x)\right] \\
&= \sum_x p(x) \cdot \log_2 \frac{1}{p(x)} \\
&= -\sum_x p(x) \log_2 p(x)
\end{aligned}
$$

where we define $0 \log 0 \equiv 0$, as $\lim_{p \to 0} p \log p = 0$.

Entropy of a Random Variable

Some Basic Properties

- Non-negativity:

$$
\begin{aligned}
0 \leq p(x) \leq 1 \Rightarrow \log \frac{1}{p(x)} &\geq 0 \\
\Rightarrow \sum_x p(x) \log \frac{1}{p(x)} &\geq 0 \\
\Rightarrow H(X) &\geq 0
\end{aligned}
$$

Entropy of a Random Variable

Some Basic Properties

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$$\Rightarrow \sum_x p(x) \log \frac{1}{p(x)} \geq 0$$
$$\Rightarrow H(X) \geq 0$$

- Change of base:

$$H_b(X) = -\sum_x p(x) \log_b p(x)$$
$$= \sum_x p(x) \log_a p(x) \log_b a$$
$$H_b(X) = \log_b a \, H_a(X)$$

  - If we use $\log_2$ the units are called *bits*
  - If we use natural logarithm the units are called *nats*

Unrolling the Definition

The entropy of X is

$$H(X) = -\sum_{x} p(x) \log_2 p(x).$$

Pick a random outcome x, and see how large its probability is

- Average information content of each outcome

Does not depend on the values of the outcomes

- Only on their probabilities

- Contrast with expectation $\mathbb{E}[X] = \sum_x x \cdot p(X = x)$.

What Does Entropy “Mean”?

Not a well posed question.
Entropy does match some intuitive properties of our informal notion of “information content”

- Rare outcomes provide more information

But other functions also seem plausible, e.g.

$$G(X) = \sum_x p(x) \frac{1}{p(x)^2} = \sum_x \frac{1}{p(x)}.$$

We will see some examples where our definition of entropy arises naturally
The main justification is the results we can obtain with it.

Entropy of a Random Variable

Example 1 — Bernoulli Distribution

Let $X \in \{0, 1\}$ with $X \sim \text{Bern}(X|\theta)$

Then,

$$
\begin{aligned}
p(X = 0) &= 1 - \theta \\
p(X = 1) &= \theta
\end{aligned}
$$

So, the entropy of a Bernoulli random variable is

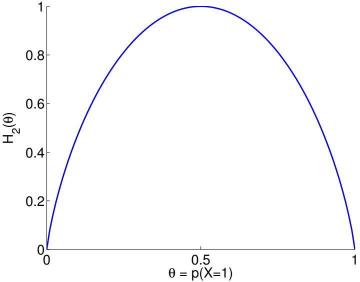
$$
\begin{aligned}
H(X) &= -\sum_{x \in \{0,1\}} p(x) \cdot \log_2 p(x) \\
&= -\theta \log_2 \theta - (1 - \theta) \log_2 (1 - \theta)
\end{aligned}
$$

Entropy of a Random Variable

Example 1 — Bernoulli Distribution

Let $X \in \{0, 1\}$ with $X \sim \text{Bern}(X|\theta)$ and $\theta = p(X = 1)$

$$H(X) = -\theta \log_2 \theta - (1 - \theta) \log_2 (1 - \theta)$$



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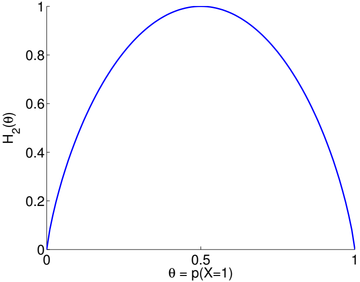

- *Concave* function of the distribution

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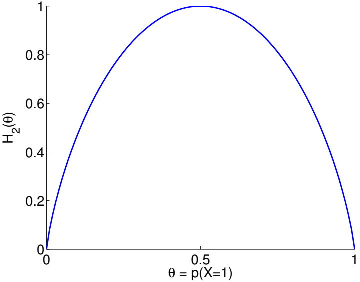

- *Concave* function of the distribution
- Minimum entropy $\rightarrow$ no uncertainty about X, i.e. $\theta = 1$ or $\theta = 0$

Entropy of a Random Variable

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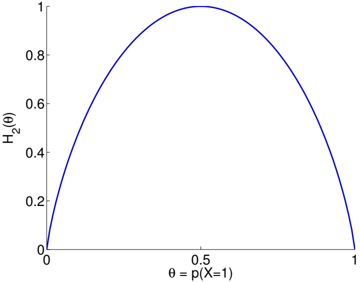

- *Concave* function of the distribution
- Minimum entropy → no uncertainty about X, i.e. $\theta = 1$ or $\theta = 0$
- Maximum when → complete uncertainty about X, i.e. $\theta = 0.5$

Entropy of a Random Variable

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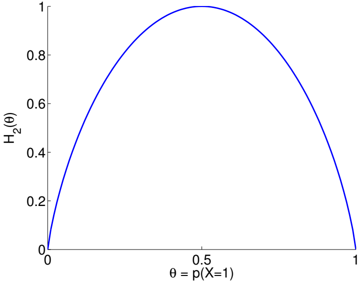

- *Concave* function of the distribution
- Minimum entropy → no uncertainty about X, i.e. $\theta = 1$ or $\theta = 0$
- Maximum when → complete uncertainty about X, i.e. $\theta = 0.5$
- For $\theta = 0.5$ (e.g. a fair coin) $H_2(X) = 1$ bit.

Entropy of a Random Variable

Example 2

Consider a random variable X with uniform distribution over 32 outcomes:

The entropy of this rv is given by:

$$H(X) = -\sum_{i=1}^{32} p(i) \log_2 p(i) = -\sum_{i=1}^{32} \frac{1}{32} \log_2 \frac{1}{32} = \log_2 32 = 5 \text{ bits.}$$

Entropy of a Random Variable

Example 3 — Categorical Distribution

Categorical distributions with 30 different states:

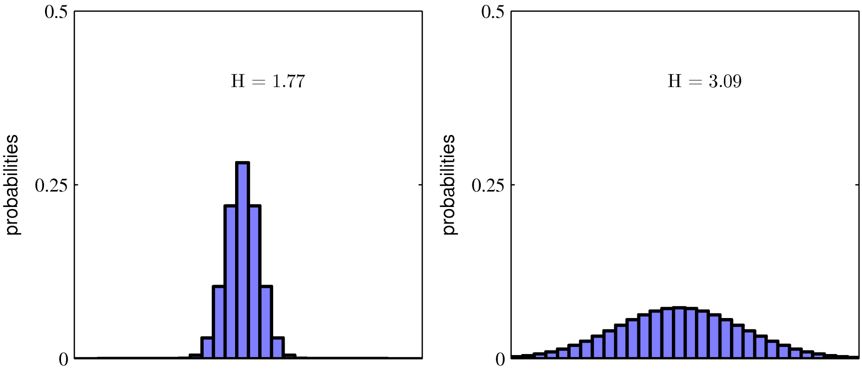

Figure from Bishop, PRML, 2006)

Entropy of a Random Variable

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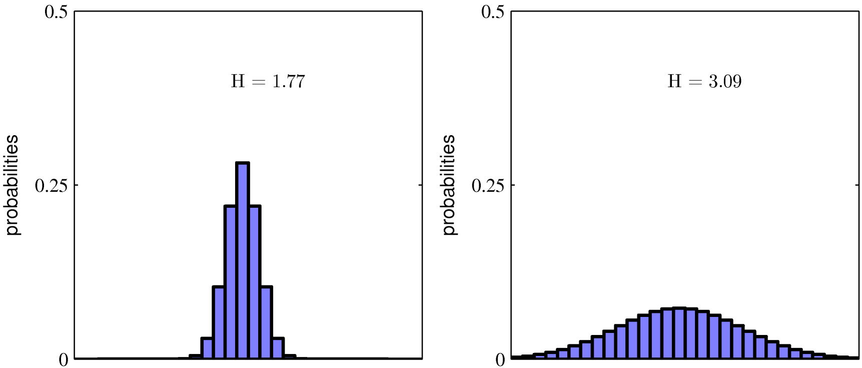

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- The more sharply peaked the lower the entropy

Entropy of a Random Variable

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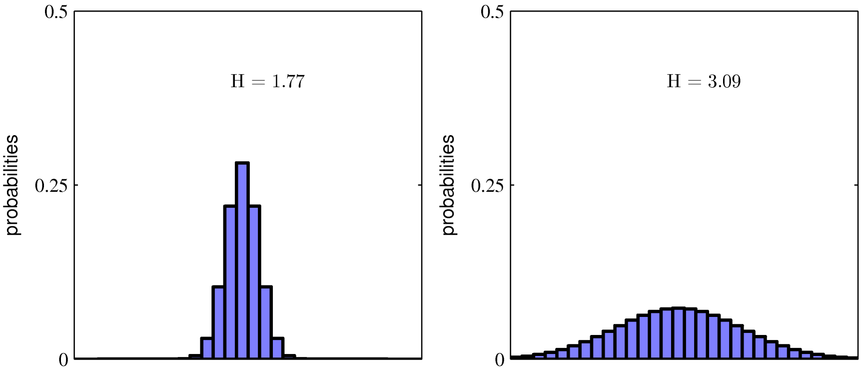

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- The more sharply peaked the lower the entropy
- The more evenly spread the higher the entropy

Entropy of a Random Variable

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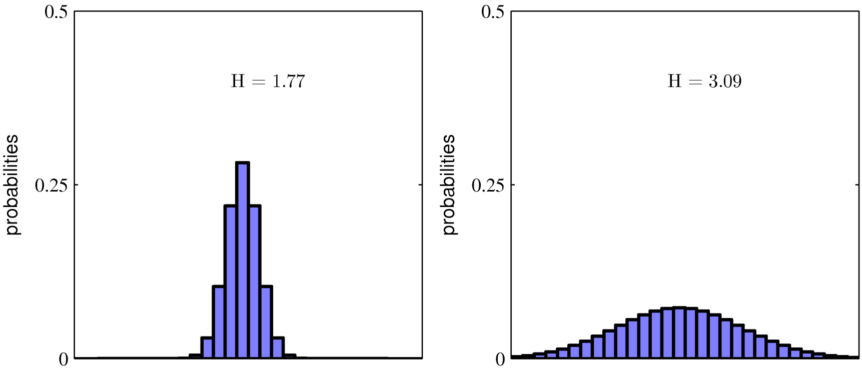

Figure from Bishop, PRML, 2006)

- The more sharply peaked the lower the entropy
- The more evenly spread the higher the entropy
- Maximum for *uniform* distribution: $H(X) = -\log \frac{1}{30} \approx 3.4$ nats (5 bits)
    - When will the entropy be minimum?

Entropy of a Random Variable

Maximum Entropy

Consider a discrete variable X taking on values from the set $\mathcal{X}$

- Let p_i be the probability of each state, with $i = 1, \ldots, |\mathcal{X}|$

Entropy of a Random Variable

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- Denote the vector of probabilities with $\mathbf{p}$

Entropy of a Random Variable

Maximum Entropy

Consider a discrete variable X taking on values from the set $\mathcal{X}$

- Let p_i be the probability of each state, with $i = 1, \ldots, |\mathcal{X}|$
- Denote the vector of probabilities with $\mathbf{p}$

The entropy is maximized if $\mathbf{p}$ is uniform:

$$H(X) \leq \log_2 |\mathcal{X}|$$

with equality iff $p_i = \frac{1}{|\mathcal{X}|}$ for all i

Note $\log_2 |\mathcal{X}|$ is the number of bits needed to describe an outcome of X

Proof (1)

We can prove the above statement by maximizing the entropy wrt each p_i. Our objective function to maximize is:

$$H(X) = -\sum_{i=1}^{|\mathcal{X}|} p_i \log p_i, \tag{1}$$

subject to the constraint $\sum_{i=1}^{|\mathcal{X}|} p_i = 1$. This is a constrained optimization problem and therefore we can use Lagrange multipliers. Thus, we have the Lagrangian:

$$\mathcal{L} = -\sum_i p_i \log p_i + \lambda \left(\sum_i p_i - 1 \right). \tag{2}$$

Computing the derivatives of $\mathcal{L}$ wrt λ, p_j and setting them to zero we have that:

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \sum_i p_i = 0 \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial p_j} = -(\log p_j + 1) + \lambda = 0 \tag{4}$$

$$\log p_j = \lambda - 1. \tag{5}$$

Proof (1)

Summing over all p_j:

$$\sum_{j=1}^{|\mathcal{X}|} p_j = \sum_{j=1}^{|\mathcal{X}|} 2^{\lambda-1} \tag{6}$$

$$1 = 2^{\lambda-1}|\mathcal{X}| \tag{7}$$

$$\lambda - 1 = \log \frac{1}{|\mathcal{X}|} \tag{8}$$

$$\lambda = 1 + \log \frac{1}{|\mathcal{X}|}. \tag{9}$$

Replacing (9) in (5):

$$\log p_j = 1 + \log \frac{1}{|\mathcal{X}|} - 1 \tag{10}$$

$$p_j = \frac{1}{|\mathcal{X}|}. \tag{11}$$

With this we have that the entropy is given by:

$$H(X) = -\sum_i p_i \log p_i \tag{12}$$

$$= -\sum_{i=1}^{|\mathcal{X}|} \frac{1}{|\mathcal{X}|} \log \frac{1}{|\mathcal{X}|} \tag{13}$$

$$= -\log \frac{1}{|\mathcal{X}|} = \log|\mathcal{X}|. \tag{14}$$

Entropy of a Random Variable

Example 4 (from Cover & Thomas, 2006) — 1 of 3

Consider a horse race with 8 horses participating:

$$\{acer, babe, cactus, daisy, epic, fancy, gem, hairy\}$$

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  Note that the entropy of the corresponding random variable, say X, is:

$$H(X) = 8 \times \frac{1}{8} \log_2 8 = 3 \text{ bits.}$$

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- Now say that the probabilities of each horse winning are:

  $$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{64}, \frac{1}{64}, \frac{1}{64}, \frac{1}{64}\right)$$

Entropy of a Random Variable

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  What is the average code-length to transmit the identity of the winning horse?

Entropy of a Random Variable

Example 4 (from Cover & Thomas, 2006) — 2 of 3

We see that some horses have higher probability of winning:

- We can still use a 3-bit representation
  - However, this would be wasteful as some horses are more likely to win

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- Let us try representing the horses (states) using the following codes

$$\{0, 1, 10, 11, 100, 101, 110, 111, 1000\}?$$

Decode 010 into 'aba' or 'ac'? Ambiguous.

Entropy of a Random Variable

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  Decode 010 into 'aba' or 'ac'? Ambiguous.
- We should be able to disambiguate a concatenation of these strings into the corresponding components.
- Represent the horses (states) using the following codes:

$$\{0, 10, 110, 1110, 111100, 111101, 111110, 111111\}$$

  - E.g. 11001110 $\rightarrow$??

Entropy of a Random Variable

Example 4 (from Cover & Thomas, 2006) — 3 of 3

What is the average code length that has to be transmitted?

Entropy of a Random Variable

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$$\text{Average code-length} = \frac{1}{2}\times 1 + \frac{1}{4}\times 2 + \frac{1}{8}\times 3 + \frac{1}{16}\times 4 + 4\times\frac{1}{64}\times 6 = 2 \text{ bits}$$

Entropy of a Random Variable

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Entropy of a Random Variable

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What is the entropy of the corresponding random variable?

$$\begin{aligned} H(X) &= -\left(\frac{1}{2}\log_2\frac{1}{2}+\frac{1}{4}\log_2\frac{1}{4}+\frac{1}{8}\log_2\frac{1}{8}+\frac{1}{16}\log_2\frac{1}{16}+\frac{4}{64}\log_2\frac{1}{64}\right) \\ &= 2 \text{ bits} \end{aligned}$$

Entropy of a Random Variable:

Example 5 (from Cover & Thomas, 2006)

Let $X \in \{1, 2, 3\}$ and $p(X = 1) = p(X = 2) = p(X = 3) = \frac{1}{3}$

Given the corresponding codeword:

$$\{\overbrace{\ 0\ }^{1}, \overbrace{\ 10\ }^{2}, \overbrace{\ 11\ }^{3}\}$$

Then $H(X) = 1.58$, and average code length = 1.66

Entropy of a Random Variable:
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Given the corresponding codeword:

$$\{\overbrace{0}^{1}, \overbrace{10}^{2}, \overbrace{11}^{3}\}$$

Then $H(X) = 1.58$, and average code length = 1.66

In general, Entropy is a lower bound on the average number of bits to transmit the state of a random variable.

As we shall see later, we can construct descriptors with average length within 1 bit of the entropy.

Entropy of a Random Variable:

What Questions Should We Ask? (From Cover & Thomas, 2006)

Assume that only the following horses participated in the last race: $\{$*acer, babe, cactus, daisy*$\}$.

The corresponding probabilities of winning are give by:

$$p(X = a) = \frac{1}{2} \quad p(X = b) = \frac{1}{4} \quad p(X = c) = \frac{1}{8} \quad p(X = d) = \frac{1}{8}.$$

You want to determine which horse won the race with the minimum number of yes/no questions:

(a) What binary questions should you ask?

(b) What is the minimum expected number of binary questions for this?

Entropy of a Random Variable:

What Questions Should We Ask? (From Cover & Thomas, 2006) — Cont'd

As acer is more likely to have won the race I first ask about him: has $X = a$ won the race?

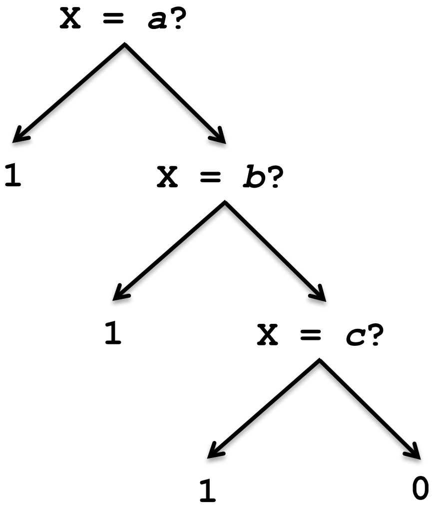
If the answer is no, I then ask about the second most probable winner: has $X = b$ won the race?

Then $X = c$?, and $X = d$?

Note that the series of questions corresponding to an outcome can be seen as a code!

Entropy of a Random Variable:

What Questions Should We Ask? (From Cover & Thomas, 2006) — Cont'd



a	1
b	01
c	001
d	000

Entropy of a Random Variable:

What Questions Should We Ask? (From Cover & Thomas, 2006) — Cont'd

The entropy of this random variable determines a lower bound for the minimum number of binary questions:

$$H_2(X) = -\left(\frac{1}{2}\log_2\frac{1}{2} + \frac{1}{4}\log_2\frac{1}{4} + \frac{1}{8}\log_2\frac{1}{8} + \frac{1}{8}\log_2\frac{1}{8}\right) = 1.75 \text{ bits.}$$

This is in fact the minimum expected number of binary questions. In general, this number lies between $H(X)$ and $H(X) + 1$

Intuitively, each question reduces our amount of uncertainty in the outcome by attempting to eliminate (or validate) the hard to predict outcomes

Joint Entropy

The joint entropy $H(X, Y)$ of a pair of discrete random variables with joint distribution $p(X, Y)$ is given by:

$$
\begin{aligned}
H(X, Y) &= \mathbb{E}_{X,Y}\left[\log \frac{1}{p(X, Y)}\right] \\
&= \sum_{x\in\mathcal{X}} \sum_{y\in\mathcal{Y}} p(x, y) \log \frac{1}{p(x, y)}
\end{aligned}
$$

Joint Entropy:

Independent Random Variables

If X and Y are statistically independent we have that:

$$H(X, Y) = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log \frac{1}{p(x, y)}$$

Joint Entropy:

Independent Random Variables

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\begin{aligned}
H(X,Y) &= \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log\frac{1}{p(x,y)} \\
&= -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x)p(y)\big[\log p(x) + \log p(y)\big] \text{ as } p(x,y) = p(x)p(y)
\end{aligned}
$$

Joint Entropy:

Independent Random Variables

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\end{aligned}
$$

Joint Entropy:

Independent Random Variables

If X and Y are statistically independent we have that:

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&= -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x)p(y)\big[\log p(x)+\log p(y)\big] \text{ as } p(x,y)=p(x)p(y) \\
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&= \sum_{x\in\mathcal{X}} p(x)\log\frac{1}{p(x)} + \sum_{y\in\mathcal{Y}} p(y)\log\frac{1}{p(y)}
\end{aligned}$$

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&= H(X) + H(Y)
\end{aligned}$$

Entropy is additive for independent random variables

Conditional Entropy

The conditional entropy of Y given $X = x$ is the entropy of the probability distribution $p(Y|X = x)$:

$$H(Y|X = x) = \sum_{y \in \mathcal{Y}} p(y|X = x) \log \frac{1}{p(y|X = x)}$$

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The conditional entropy of Y given X, is the average over X of the conditional entropy of Y given $X = x$:

$$\begin{aligned} H(Y|X) &= \sum_{x \in \mathcal{X}} p(x) H(Y|X = x) \\ &= \sum_{x \in \mathcal{X}} p(x) \sum_{y \in \mathcal{Y}} p(y|x) \log \frac{1}{p(y|x)} \end{aligned}$$

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Average uncertainty that remains about Y when X is known.

Conditional Entropy — Cont'd

We can re-write the conditional entropy as follows:

$$H(Y|X) = \sum_{x \in \mathcal{X}} p(x) \sum_{y \in \mathcal{Y}} p(y|x) \log \frac{1}{p(y|x)}$$

Conditional Entropy — Cont'd

We can re-write the conditional entropy as follows:

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Conditional Entropy — Cont'd

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&= \sum_{x\in\mathcal{X}} \sum_{y\in\mathcal{Y}} p(x, y) \log \frac{1}{p(y|x)}
\end{aligned}
$$

Conditional Entropy — Cont'd

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&= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x) p(y|x) \log \frac{1}{p(y|x)} \\
&= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log \frac{1}{p(y|x)} \\
&= \mathbb{E}_{X,Y} \left[\log \frac{1}{p(Y|X)} \right]
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&= \mathbb{E}_{X,Y}\left[\log \frac{1}{p(Y|X)}\right]
\end{aligned}
$$

Note the expectation is not wrt the conditional distribution but wrt the joint distribution $p(X, Y)$

Chain Rule

The joint entropy can be written as:

$$H(X, Y) = -\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log p(x, y)$$

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H(X,Y) &= -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log p(x,y) \\
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&= -\sum_{x\in\mathcal{X}} \log p(x) \underbrace{\sum_{y\in\mathcal{Y}} p(x,y)}_{p(x)} - \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log p(y|x)
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\end{aligned}$$

$$H(X, Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$$

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\end{aligned}
$$

$$
H(X, Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)
$$

The joint uncertainty of X and Y is the uncertainty of X plus the uncertainty of Y given X

An Axiomatic Characterisation

Suppose we want a measure H of “information” in a random variable X such that

1. H depends on the distribution of X, and not the outcomes themselves
2. The H for the combination of two variables X, Y is at most the sum of the corresponding H values
3. The H for the combination of two independent variables X, Y is the sum of the corresponding H values
4. Adding outcomes with probability zero does not affect H
5. The H for an unbiased Bernoulli is 1
6. The H for a Bernoulli with parameter p tends to 0 as $p \to 0$

Then, the only possible choice for H is

$$H(X) = -\sum_{x} p(x) \log_2 p(x)$$

Outline

Summary

- Entropy as a measure of information content
- Computation of entropy of discrete random variables
- Entropy and average code length
- Entropy and minimum expected number of binary questions
- Joint and conditional entropies, chain rule
- Reading: Mackay § 1.2 – § 1.5, § 8.1; Cover & Thomas § 2.1; Bishop § 1.6

Next time

- More properties of entropy
- Relative entropy
- Mutual information

Acknowledgement

These slides were originally developed by Professor Robert C. Williamson.