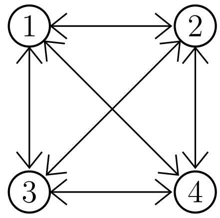
1. Jeremy is taking random walks on the digraph shown at right. At any vertex Jeremy has three choices as to where to go next:



- go to the vertex diagonally across the square.
- go to the adjacent vertex around the square in either a clockwise or an anticlockwise direction;

He takes the first option with probability one half but his probability for the other options is equal to one tenth of the destination vertex number. For example, if he is at vertex 1, his probabilities of next visiting vertices 2, 3 and 4 are 0.2, 0.3 and 0.5 respectively.

(a) List all the walks of length 2 from vertex 1 to vertex 2 and the probabilities associated with each of them. Hence find the probability that a walk of length 2 that starts at vertex 1 finishes at vertex 2.

(b) List all the walks of length 3 from vertex 1 to vertex 2 (there are a lot!) and the probabilities associated with each of them. Hence find the probability that a walk of length 3 that starts at vertex 1 finishes at vertex 2.

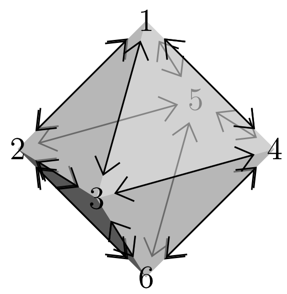
(c) Compile Jeremy's transition matrix T. Check that it is stochastic.

(d) Calculate T^2 (by hand or computer) and use it to check your answer to (a).

(e) Calculate T^3 (by hand or computer) and use it to check your answer to (b). If the answers don't agree, you probably missed some of the walks.

(f) Use T to verify that, if Jeremy keeps walking, then in the long run his visits to any vertex v will constitute a fraction $(v + 5)/30$ of his visits to all vertices. (Thus, for example, 20% of his visits will be to vertex 1 and 30% to vertex 4.)

2. Anton the ant is wandering around the structure shown at right, made from twelve length 1 struts joined at six vertices 1,...,6, making a regular octahedron with the struts as edges. He walks at a steady pace along the edges and at each vertex makes a random (equal-probability) choice as to which edge to walk next (this includes the possibility of retracing the previous edge).



(a) Anton starts at vertex 1. Explain why, after a walk of length two, he is twice as likely to be at vertex 6 than at vertex 5.

(b) Again starting from vertex 1, what is the probability that after a walk of length 3 (not necessarily using distinct edges), Anton is back at vertex 1?

(c) Write out Anton's transition matrix T. Write it in the form $T = fM$ where f is a fraction and M is a symmetric matrix of zeros and ones.

(d) Making good use of the simple nature of M to save work, calculate T^2, writing it in similar simple form. Use T^2 to check your answer to (a).

(e) Calculate $T^4 = (T^2)^2$. Use a computer if you like, but it it is not really necessary.

(f) Suppose Anton has been wandering for a long time. By comparing T, T^2 and T^4, and by considering the symmetry inherent in the situation, estimate the proportion of Anton's vertex visits that have been to vertex 1.

(g) Support your answer to (f) by demonstrating a steady state vector for T.

3. A tiny web graph has four pages A, B, C, D. Every page has a link to every other page except for pages C and D, neither of which has link to the other.

(a) Draw the web graph.

(b) Write out the transition matrix, with probabilities assigned according to the Google PageRank algorithm.

(c) Calculate the page ranks of each page.
Hint: By symmetry, pages A and B will have the same rank, as will pages C and D. So there will only be two unknowns.

4. Repeat Question 3 with the links from A and B to C removed.
It will no longer be true that C and D have the same rank, so there will now be three equations in three unknowns. However one equation gives the value of one of the unknowns immediately, leading to a 2×2 system.

5. Repeat Question 3 but this time with the links **to** A and B **from** C removed.
Again C and D will have different ranks, leading to a 3×3 system, but one equation involves only two unknowns, allowing one unknown to be eliminated from the other two equations, thus generating a 2×2 system.

6. Repeat Question 3 with 90% damping. This means solving the equation

$$(I - (1-\alpha)T')P\!R = (\alpha/n)\mathbf{1}$$

with $\alpha = 1/10$. Symmetry still applies, so $P\!R$ still involves only two unknowns, so this 4×4 system boils down to only 2×2, and hence can be solved without the need for Gaussian elimination. However the fractions get a little nastier so you may prefer to use a computer. You should find that the page rank fractions have denominator 64 (when expressed in lowest terms).

7. Repeat Question 4 with 80% damping. Considerations similar to those for the previous question regarding solution method will apply. You should find D has rank $43/140$.

8. Again repeat Question 4 with 80% damping, but this time use two steps of the iterative method

$$P_0 = (1/n)\mathbf{1}; \quad P_k = \alpha P_0 + (1-\alpha)T'P_{k-1},\ k \geq 1.$$

That is, calculate P_0, then P_1 and finally P_2, and use P_2 as an approximation for $P\!R$. Compare the result to your answer to question 7.

9. The adjacency matrix A for a twelve page web graph is given at right.

(a) Write out the corresponding transition matrix T.

(b) Using a computer, find the page ranks of each page using the Google PageRank algorithm without damping.

(c) Repeat part (b) but using 60% damping.

$$\begin{bmatrix}
0&1&0&1&1&0&0&1&0&0&0&0\\
0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&1&0&1&1&0&0&0&0\\
0&0&0&0&0&1&1&1&1&0&0&1\\
0&0&0&0&0&0&0&1&1&0&0&0\\
0&0&0&0&0&0&0&1&0&1&0&1\\
0&0&0&0&0&0&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}$$