1. Let $a_{i,j} = i + j \quad \forall i, j \in \{1, .., 5\}$.
Write $(a_{i,j})_{1 \le i,j \le 5}$ as an array of numbers, *i.e.* as a 5×5 matrix.

$$\begin{pmatrix} \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} \\ \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{8} \\ \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{8} & \mathbf{9} \\ \mathbf{6} & \mathbf{7} & \mathbf{8} & \mathbf{9} & \mathbf{10} \end{pmatrix}$$

2. Define a function $a : \{1, 2, 3, 4\}^2 \to \{-1, 1\}$ representing the matrix

$$A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

$$\boldsymbol{a_{i,j} = (-1)^{i+j}}$$

That is, give a formula for $a_{i,j}$. Hint: One way is to use power of (-1).

3. Let R be the relation defined by

$$R = \{(1, 2), (1, 3), (3, 4), (2, 1)\} \subseteq \{1, 2, 3, 4\}^2.$$

Define $(a_{i,j})_{1 \le i,j \le 4} \in M_n(\{0, 1\})$ by $a_{i,j} = 1 \iff iRj$.
Write $(a_{i,j})_{1 \le i,j \le 4}$ as a matrix.

$$\begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

4. Let $(q_1, q_2, q_3) \in \mathbb{Q}^3$ represent the quantities (in ml) of three ingredients required to produce one glass of a cocktail.

(a) Which vector $(r_1, r_2, r_3) \in \mathbb{Q}^3$ represents the quantities required to produce five glasses of the cocktail?
$\boldsymbol{(r_1, r_2, r_3) = 5(q_1, q_2, q_3) = (5q_1, 5q_2, 5q_3)}$

(b) If q_1, q_2 correspond to alcohols, and q_3 to juice, are you making the cocktail stronger or weaker by replacing (q_1, q_2, q_3) by $(q_1, q_2, q_3) + (-10, -20, 30)$?
weaker

5. Compute the following.

(a) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} + \begin{pmatrix} 6 & 2 & 1 \\ 4 & 5 & 3 \end{pmatrix}$.
$\begin{pmatrix} \mathbf{7} & \mathbf{4} & \mathbf{4} \\ \mathbf{7} & \mathbf{7} & \mathbf{4} \end{pmatrix}$

(b) $3 \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 2 \end{pmatrix}$.
$\begin{pmatrix} \mathbf{3} & \mathbf{6} \\ \mathbf{6} & \mathbf{3} \\ \mathbf{9} & \mathbf{6} \end{pmatrix}$

(c) $\alpha \begin{pmatrix} x & y \\ z & w \end{pmatrix} + \beta \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
$\begin{pmatrix} \boldsymbol{\alpha x + \beta a} & \boldsymbol{\alpha y + \beta b} \\ \boldsymbol{\alpha z + \beta c} & \boldsymbol{\alpha w + \beta d} \end{pmatrix}$

6. Are the following functions linear?

(a) $f : \mathbb{Q} \to \mathbb{Q}$ defined by $f(x) = 2x + 1$. **No. Counterexample to $\boldsymbol{f(\lambda x) = \lambda f(x)}$: $\boldsymbol{f(3\times 2) = f(6) = 13; \quad 3\times f(2) = 3 \times 5 = 15}$.**

(b) $g : \mathbb{Q} \to \mathbb{Q}$ defined by $f(x) = x^2 + 1$. **No. Counterexample to $\boldsymbol{f(\lambda x) = \lambda f(x)}$: $\boldsymbol{f(3\times 2) = f(6) = 37; \quad 3\times f(2) = 3 \times 5 = 15}$.**

(c) $k : \mathbb{Q}^3 \to \mathbb{Q}^3$ defined by $f(x_1, x_2, x_3) = M \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, for $M \in M_3(\mathbb{Q})$.

Yes. We have a theorem that says that every function defined by a matrix in this way is linear.

7. Compute the following.

(a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ $\qquad$ $\boldsymbol{\begin{pmatrix} x+2y \\ 3x+4y \end{pmatrix}}$

(b) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\qquad$ $\boldsymbol{\begin{pmatrix} x+z \\ y+z \\ x \end{pmatrix}}$

(c) $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\qquad$ $\boldsymbol{\begin{pmatrix} a+2b+3c \\ d+2e+3f \\ g+2h+3j \end{pmatrix}}$

8. Compute the following.

(a) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 4 & 5 \\ 3 & 1 \end{pmatrix}$ $\qquad$ $\boldsymbol{\begin{pmatrix} 10 & 7 \\ 11 & 11 \end{pmatrix}}$

(b) $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ $\qquad$ $\boldsymbol{\begin{pmatrix} 3 & 3 & 2 \\ 7 & 6 & 5 \\ 10 & 5 & 5 \end{pmatrix}}$

(c) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x & y & z \\ a & b & c \\ 2 & 1 & 0 \end{pmatrix}$ $\qquad$ $\boldsymbol{\begin{pmatrix} x+2 & y+1 & z \\ a+2 & b+1 & c \\ x+a & y+b & z+c \end{pmatrix}}$

9. Let $(x_n, y_n) \in \mathbb{Q}^2$ represent, at time $n \in \mathbb{N}^* = \mathbb{N} \cup \{0\}$, the quantity x_n of a certain plant in an ecosystem, and y_n the quantity of a pollutant. Assume that they are related in the following way: $x_0 = a \in \mathbb{Q}$, $y_0 = b \in \mathbb{Q}$, $\forall n \in \mathbb{N}^*\ x_{n+1} = 2x_n - 3y_n$, $y_{n+1} = y_n/2$.

(a) Explain the meaning of these equations.

Each timestep the plant quantity doubles by growth but is reduced by three times the quantity of pollutant. The pollutant quantity halves each time step.

(b) Prove that $\forall n \in \mathbb{N}$ $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2^n & 0 \\ 0 & 2^{-n} \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$.

The proof uses mathematical induction as in the ecology example in lectures: First note that:

(i) $\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 0 & 2^{-1} \end{bmatrix}\begin{bmatrix} x_n \\ y_n \end{bmatrix}$ **(ii)** $\begin{bmatrix} 2 & -3 \\ 0 & 2^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2^{-1} \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$.

Basis Step: **For $n=1$, using (i) & (ii) we get:**

$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 0 & 2^{-1} \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2^1 & 0 \\ 0 & 2^{-1} \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$, **which agrees with the formula.**

Inductive Step: **Assume the formula holds up to and including some fixed $n \in \mathbb{N}$ and consider the case $n+1$.**

Using (i), (ii) & (iii) and the inductive assumption we get:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 0 & 2^{-1} \end{bmatrix}\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2^{-1} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2^n & 0 \\ 0 & 2^{-n} \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2^{n+1} & 0 \\ 0 & 2^{-(n+1)} \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

and so the formula also holds for $n+1$.

(c) Prove that, if $a > 2b$, then the plant will survive.

Multiplying out the formula proved for part (b) gives $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 2^n & -2^{n+1} + 2^{-n+1} \\ 0 & 2^{-n} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$.

Thus $x_n = 2^n(a-2b) + 2^{1-n}b$, so if $a-2b > 0$, x_n increases with n, but if not then x_n will go negative for some n, meaning the plant has died out.

10. A portfolio is to contain three types of shares; A, B and C. To hedge certain risks, the investor wants twice as many C shares as the combined number of A and B shares and only a third as many B shares as the combined number of A and C shares. The numbers of A, B and C shares are to be a, b and c with a total of 1200.

(a) Show that $\begin{pmatrix} 2 & 2 & -1 \\ 1 & -3 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1200 \end{pmatrix}$.

From the given information we get that $c = 2(a+b)$, $b = \frac{1}{3}(a+c)$ and $a+b+c = 1200$. These equations can be rewritten as the system

$$\begin{aligned} 2a + \; 2b + -c &= 0 \\ a + -3b + \; c &= 0 \\ a + \; b + \; c &= 1200. \end{aligned}$$

Multiplying out the matrix equation gives precisely this system of equations.

(b) Verify that $\begin{pmatrix} 2 & 2 & -1 \\ 1 & -3 & 1 \\ 1 & 1 & 1 \end{pmatrix}^{-1} = \frac{1}{12}\begin{pmatrix} 4 & 3 & 1 \\ 0 & -3 & 3 \\ -4 & 0 & 8 \end{pmatrix}$.

It is sufficient to check that $A^{-1}A = I$:

$$\frac{1}{12}\begin{pmatrix} 4 & 3 & 1 \\ 0 & -3 & 3 \\ -4 & 0 & 8 \end{pmatrix}\begin{pmatrix} 2 & 2 & -1 \\ 1 & -3 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \frac{1}{12}\begin{pmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(c) Find a, b and c.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 0 \\ 1200 \end{pmatrix} = \frac{1}{12}\begin{pmatrix} 4 & 3 & -1 \\ 0 & -3 & 3 \\ -4 & 0 & 8 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1200 \end{pmatrix} = \frac{1}{12}\begin{pmatrix} 1200 \\ 3600 \\ 9600 \end{pmatrix} = \begin{pmatrix} 100 \\ 300 \\ 800 \end{pmatrix}.$$

So $a = 100$, $b = 300$ and $c = 800$.

11. Compute $\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. **$1\times 4 - 2\times 3 = -2$.**

12. Prove that $\forall A, B \in M_2(Q) \quad \det(AB) = \det(A)\det(B)$.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. Then:

$$\begin{aligned} \det(AB) &= \det\left(\begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix}\right) \\ &= acpq + adps + bcqr + bdrs \\ &\quad -acpq - adqr - bcps - bdrs \\ &= ad(ps-qr) + bc(qr-ps) \\ &= (ad-bc)(ps-qr) = \det(A)\det(B). \end{aligned}$$

13. Let $P \in M_2(\mathbb{Q})$ be such that $P^2 = I$. Prove that $\det(P) \in \{-1, 1\}$.

Taking the determinant of both sides of the equation, and using the result of Q12 gives:

$$(\det(P))^2 = \det(P^2) = \det(I) = 1\times 1 - 0\times 0 = 1.$$

So $\det(P) = \pm 1$.