Two-View Geometry

Outline

- Review
- Epipolar Geometry
- Fundamental Matrix
- Triangulation

Review: Normalized DLT for Camera Calibration

Normalized DLT algorithm

<u>Objective</u>

Given $n \geq 6$ 2D-to-3D point correspondences $\{\mathbf{x}_i \leftrightarrow \mathbf{X}_i\}$, determine the 3x4 projection matrix $\mathbf{P}$ such that $\mathbf{x}_i = \mathbf{P}\mathbf{X}_i$

<u>Algorithm</u>

(i) Normalize points: $\tilde{\mathbf{x}}_i = \mathbf{T}_{\text{norm}}\mathbf{x}_i, \tilde{\mathbf{X}}_i = \mathbf{S}_{\text{norm}}\mathbf{X}_i$

(ii) Apply the DLT algorithm to $\{\tilde{\mathbf{x}}_i \leftrightarrow \tilde{\mathbf{X}}_i\}$

(iii) Denormalize the recovered solution $\tilde{\mathbf{P}}$ using $\mathbf{P} = \mathbf{T}^{-1}_{\text{norm}}\tilde{\mathbf{P}}\mathbf{S}_{\text{norm}}$

- Example normalizations:

$$\mathbf{T}_{\text{norm}} = \begin{bmatrix} w+h & 0 & w/2 \\ 0 & w+h & h/2 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

$$\mathbf{S}_{\text{norm}} = \begin{bmatrix} \mathbf{V}\,\text{diag}(\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1})\mathbf{V}^{-1} & -\mathbf{V}\,\text{diag}(\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1})\mathbf{V}^{-1}\boldsymbol{\mu}_{\mathbf{X}_i} \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{V}\,\text{diag}(\lambda_1, \lambda_2, \lambda_3)\mathbf{V}^{-1} = \text{eig}\left(\sum_i \left(\mathbf{X}_{i,\text{nonhom}} - \boldsymbol{\mu}_{\mathbf{X}_i}\right)\left(\mathbf{X}_{i,\text{nonhom}} - \boldsymbol{\mu}_{\mathbf{X}_i}\right)^T\right)$$

Review – vanishing points, vanishing line

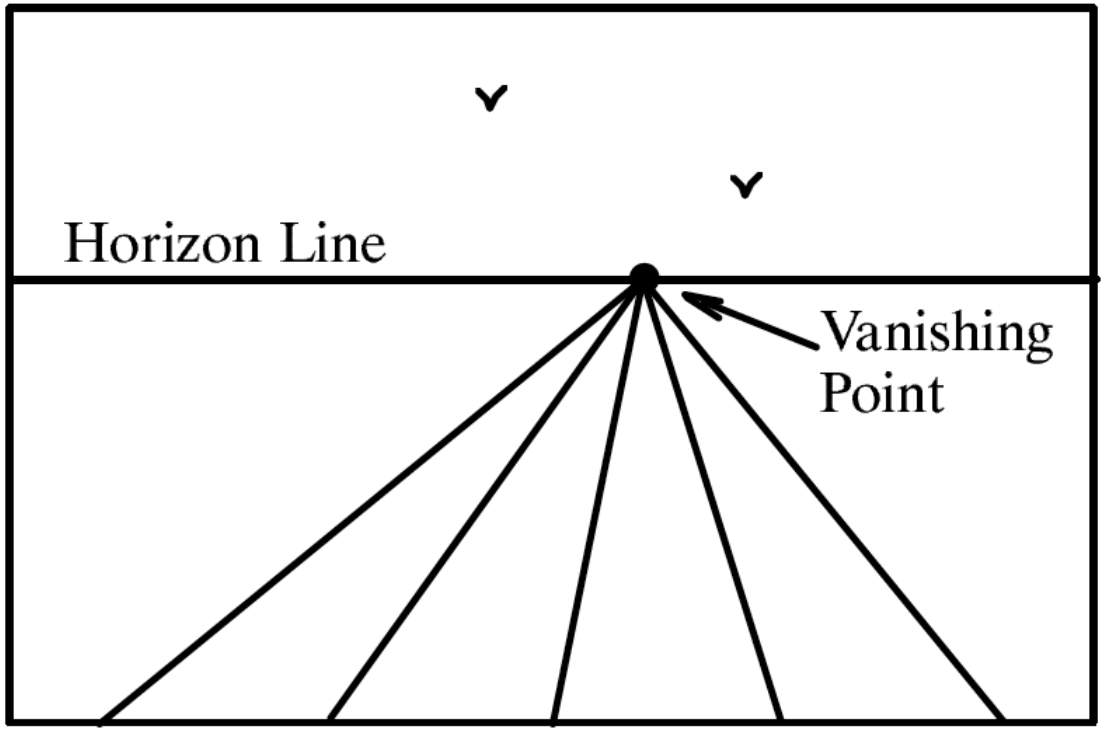
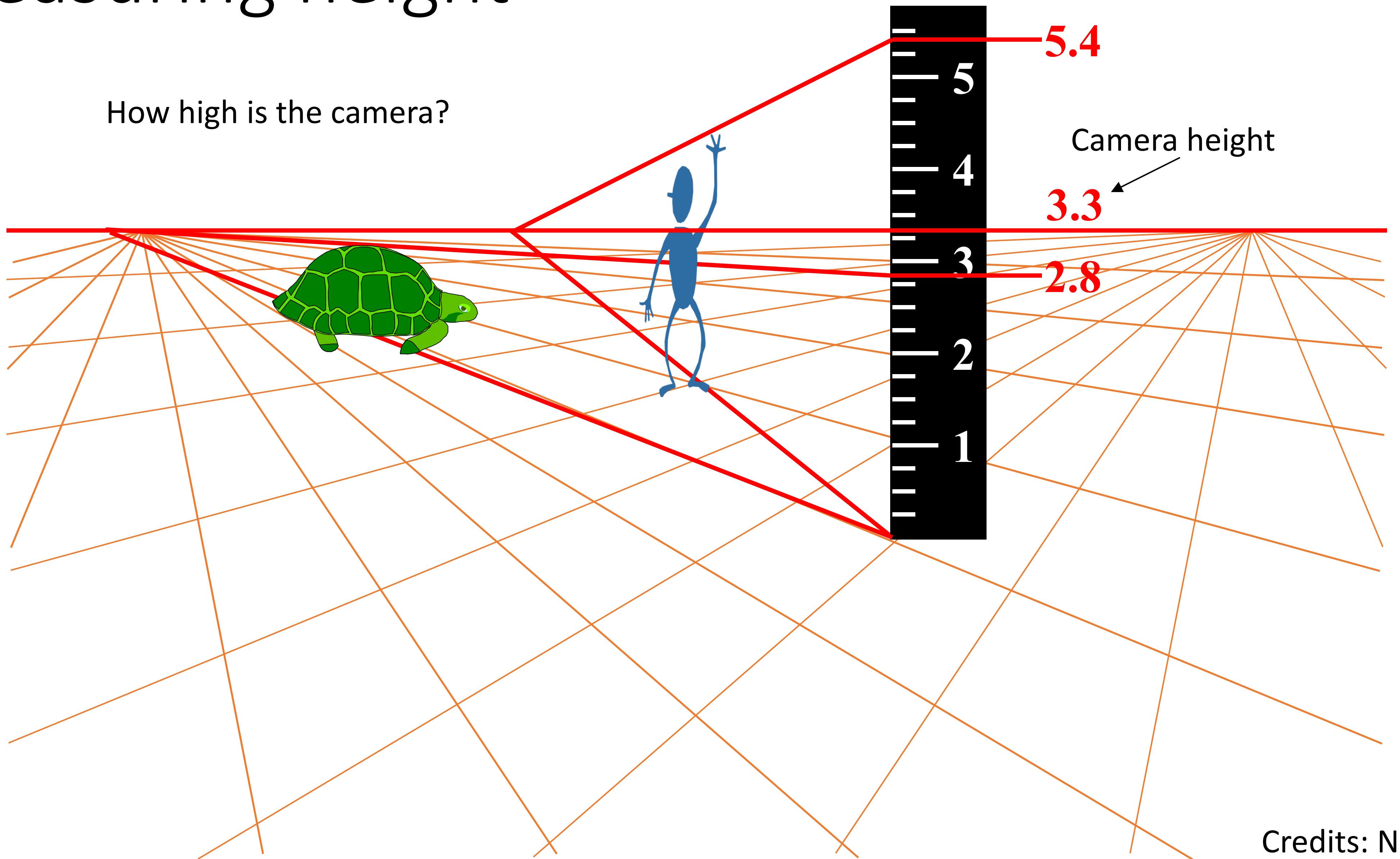


Figure 23.4
A perspective view of a set of parallel lines in the plane. All of the lines converge to a single vanishing point.

Measuring height

How high is the camera?



Credits: Noah Snavely

The DLT algorithm – computing a homography

Problem Statement:

Given n correspondences $\mathbf{x}'_i \leftrightarrow \mathbf{x}_i$, points in two images.

Compute homography $\mathbf{H}$ such that $\mathbf{x}'_i \approx \mathbf{H}\mathbf{x}_i$.

Each correspondence generates two equations

$$x'_i = \frac{h_{11}x_i + h_{12}y_i + h_{13}}{h_{31}x_i + h_{32}y_i + h_{33}} \qquad y'_i = \frac{h_{21}x_i + h_{22}y_i + h_{23}}{h_{31}x_i + h_{32}y_i + h_{33}}$$

Multiplying out gives equations linear in the matrix elements of $\mathbf{H}$

$$\begin{aligned} x'_i(h_{31}x_i + h_{32}y_i + h_{33}) &= h_{11}x_i + h_{12}y_i + h_{13} \\ y'_i(h_{31}x_i + h_{32}y_i + h_{33}) &= h_{21}x_i + h_{22}y_i + h_{23} \end{aligned}$$

These equations can be rearranged as

$$\begin{pmatrix} x & y & 1 & 0 & 0 & 0 & -x'x & -x'y & -x' \\ 0 & 0 & 0 & x & y & 1 & -y'x & -y'y & -y' \end{pmatrix} \mathbf{h} = \mathbf{0}$$

with $\mathbf{h} = (h_{11}, h_{12}, h_{13}, h_{21}, h_{22}, h_{23}, h_{31}, h_{32}, h_{33})^\top$ a 9-vector.

Computing homography continued ...

Solving for $\mathbf{H}$

(i) Concatenate the equations from ($n \geq 4$) correspondences to generate $2n$ simultaneous equations, which can be written: $\mathbf{A}\mathbf{h} = \mathbf{0}$, where $\mathbf{A}$ is a $2n \times 12$ matrix.

(ii) In general this will not have an exact solution, but a (linear) solution which minimizes $|\mathbf{A}\mathbf{h}|$, subject to $|\mathbf{h}| = 1$ is obtained from the eigenvector with least eigenvalue of $\mathbf{A}^\top \mathbf{A}$. Or equivalently from the vector corresponding to the smallest singular value of the SVD of $\mathbf{A}$.

(iii) This linear solution is then used as the starting point for a non-linear minimization of the difference between the measured and projected point:

$$\min_{\mathbf{H}} \sum_i \left((x_i', y_i') - \mathrm{dehom}(\mathbf{H}(x_i, y_i, 1)) \right)^2$$

where

(i) hom is the homogenizing operation $(x, y) \mapsto (x, y, 1)$.

(ii) dehom is the dehomogenizing operation $(x, y, w) \mapsto (x/w, y/w)$.

DLT algorithm

<u>Objective</u>

Given $n \geq 4$ 2D to 2D point correspondences $\{x_i \leftrightarrow x_i'\}$, determine the 2D homography matrix H such that $x_i' = Hx_i$

<u>Algorithm</u>

(i) For each correspondence $x_i \leftrightarrow x_i'$ compute A_i. Usually only two first rows needed.

(ii) Assemble n 2x9 matrices A_i into a single $2n$x9 matrix A

(iii) Obtain SVD of A. Solution for h is last column of V

(iv) Determine H from h

Importance of normalization

$$\begin{bmatrix} 0 & 0 & 0 & -x_i & -y_i & -1 & y_i'x_i & y_i'y_i & y_i' \\ x_i & y_i & 1 & 0 & 0 & 0 & -x_i'x_i & -x_i'y_i & -x_i' \end{bmatrix} \begin{pmatrix} h^1 \\ h^2 \\ h^3 \end{pmatrix} = 0$$

Orders of magnitude!

Check Page 110, Multiple view geometry in computer Vision.

Normalized DLT algorithm

Objective

Given $n \geq 4$ 2D to 2D point correspondences $\{x_i \leftrightarrow x_i'\}$, determine the 2D homography matrix H such that $x_i' = Hx_i$

Algorithm

(i) Normalize points $\tilde{x}_i = T_{norm} x_i, \tilde{x}_i' = T'_{norm} x_i'$

(ii) Apply DLT algorithm to $\tilde{x}_i \leftrightarrow \tilde{x}_i'$,

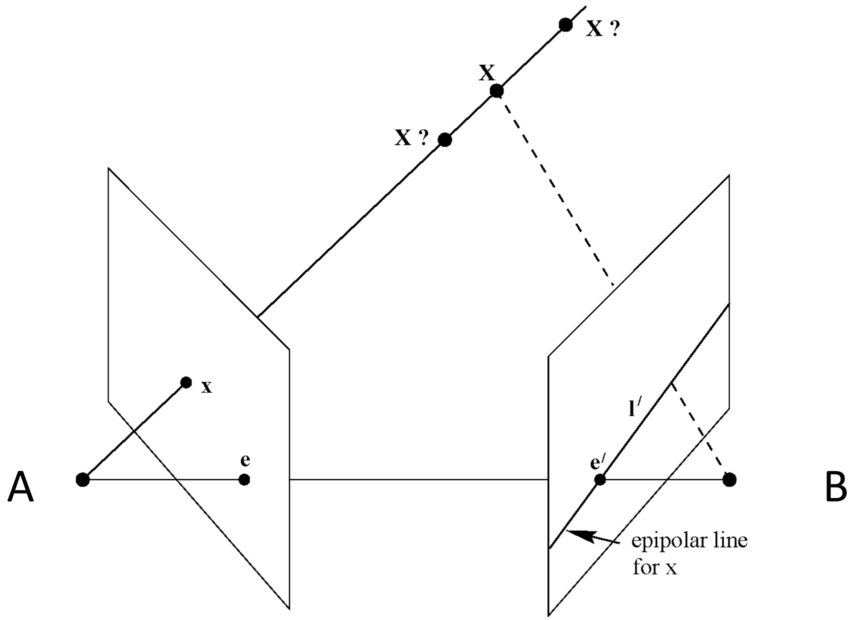
(iii) Denormalize solution $H = T'^{-1}_{norm} \tilde{H} T_{norm}$

$$T_{norm} = \begin{bmatrix} w+h & 0 & w/2 \\ 0 & w+h & h/2 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

Epipolar Geometry

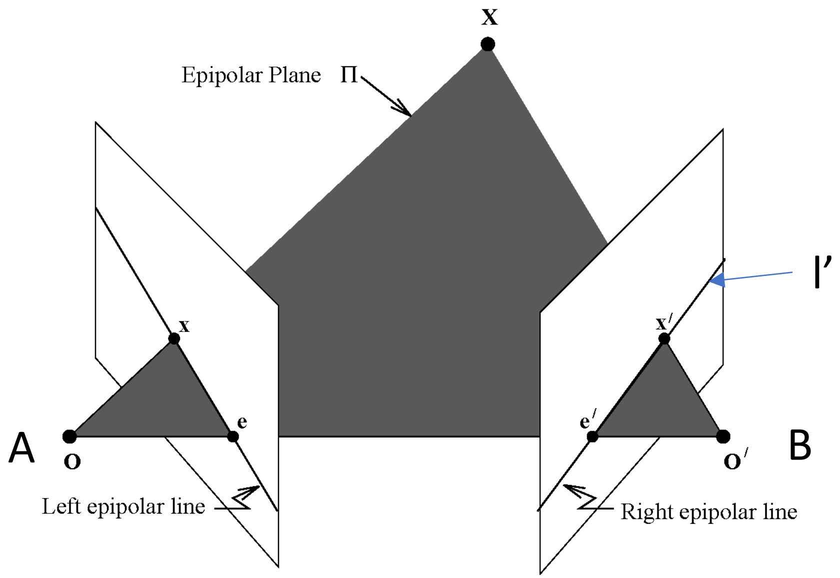
Correspondence Geometry

Given the image of a point in one view, what can we say about its position in another?



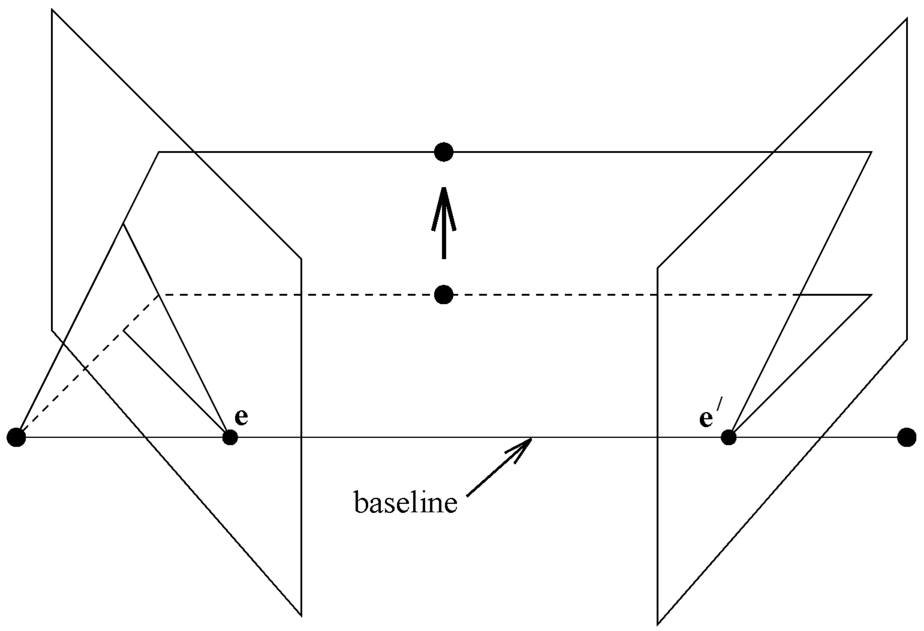
- A point in one image "generates" a line in the other image.
- This line is known as an epipolar line, and the geometry which gives rise to it is known as epipolar geometry.

Epipolar Geometry



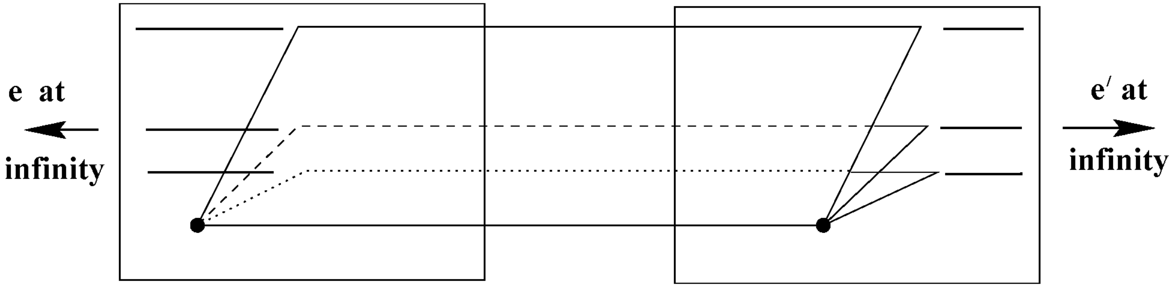
- The epipolar line $\mathbf{l}'$ is the image of the ray through $\mathbf{x}$.
- The epipole $\mathbf{e}'$ is the point of intersection of the line joining the camera centres—the baseline—with the image plane.
- The epipole is also the image in one camera of the centre of the other camera.
- All epipolar lines intersect in the epipole.

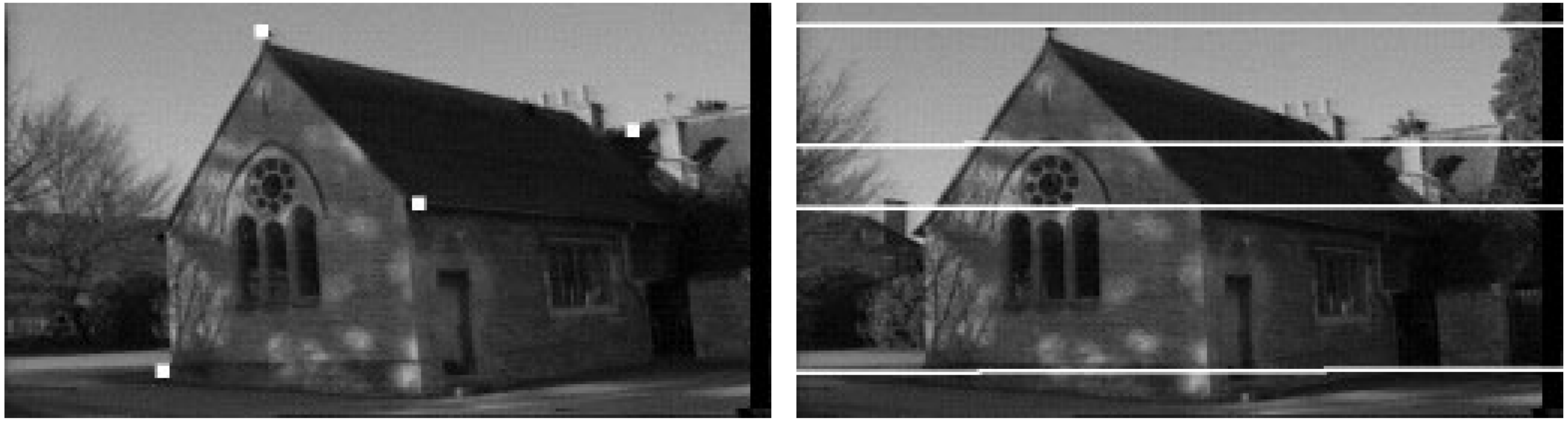
Epipolar pencil



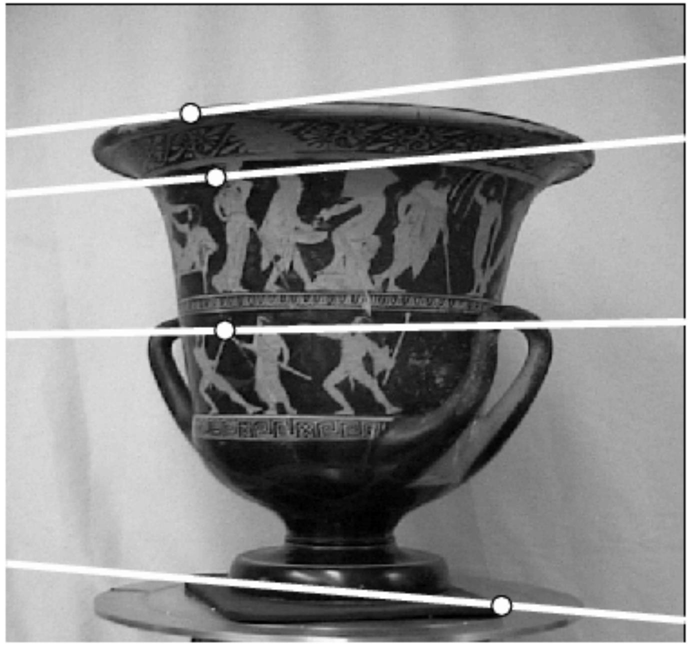
As the position of the 3D point $\mathbf{X}$ varies, the epipolar planes "rotate" about the baseline. This family of planes is known as an epipolar pencil. All epipolar lines intersect at the epipole.

Epipolar geometry example





Epipolar geometry depends only on the relative pose (position and orientation) and internal parameters of the two cameras, i.e. the position of the camera centres and image planes. It does not depend on structure (3D points external to the camera).

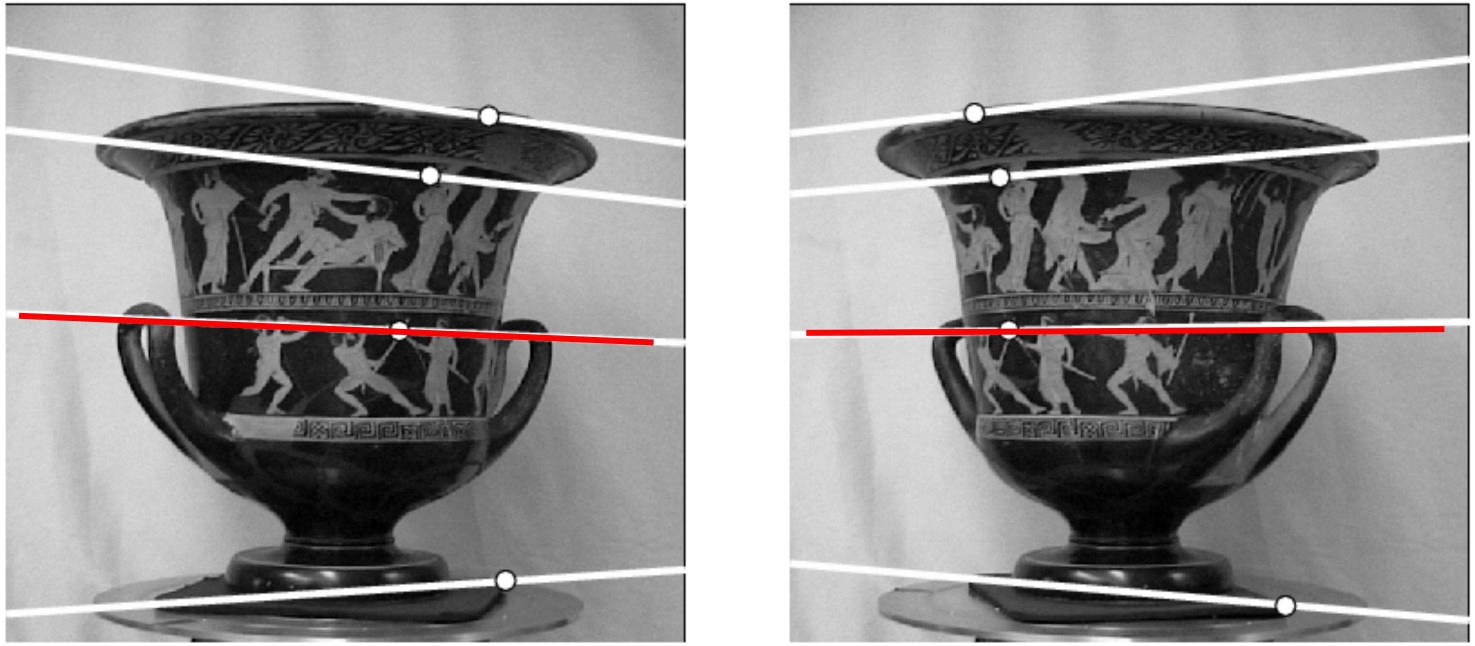


Examples of epipolar lines:

These are the intersections of the epipolar planes with the images.

Corresponding points lie on corresponding lines.

Point matching becomes a 1-dimensional search.



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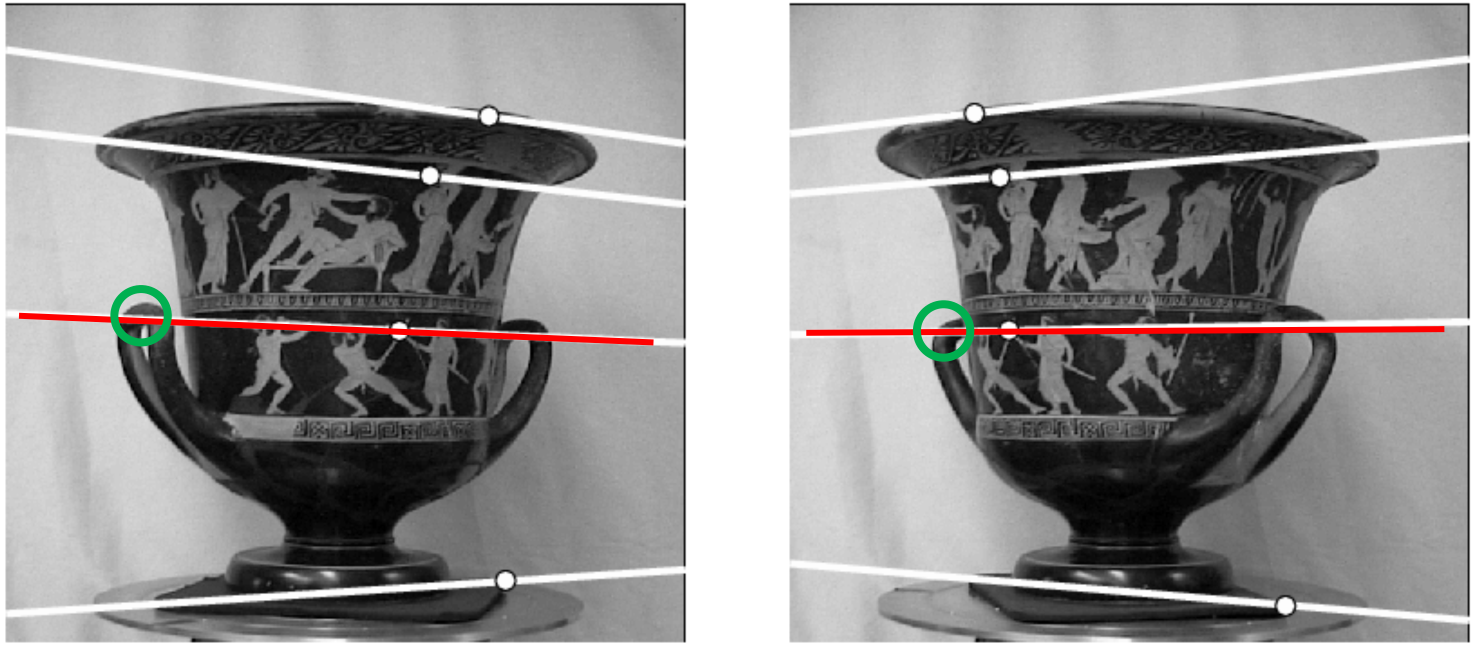

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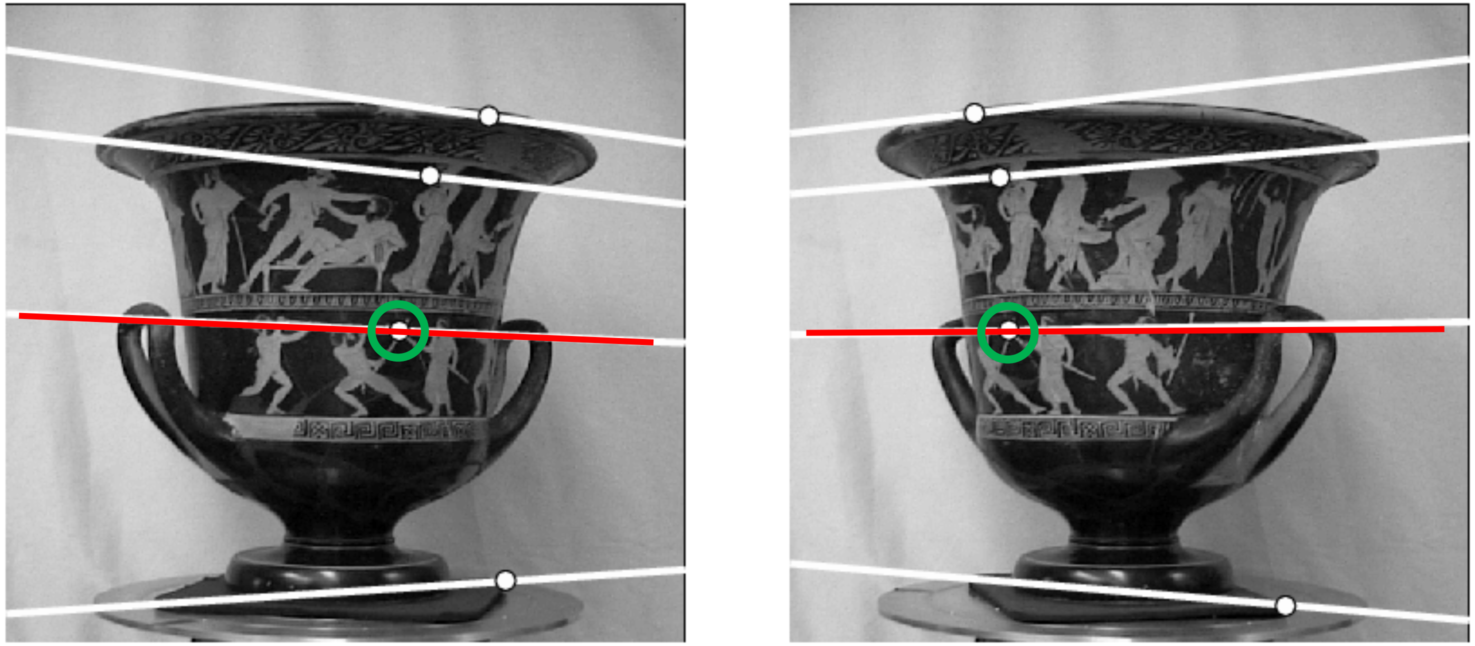

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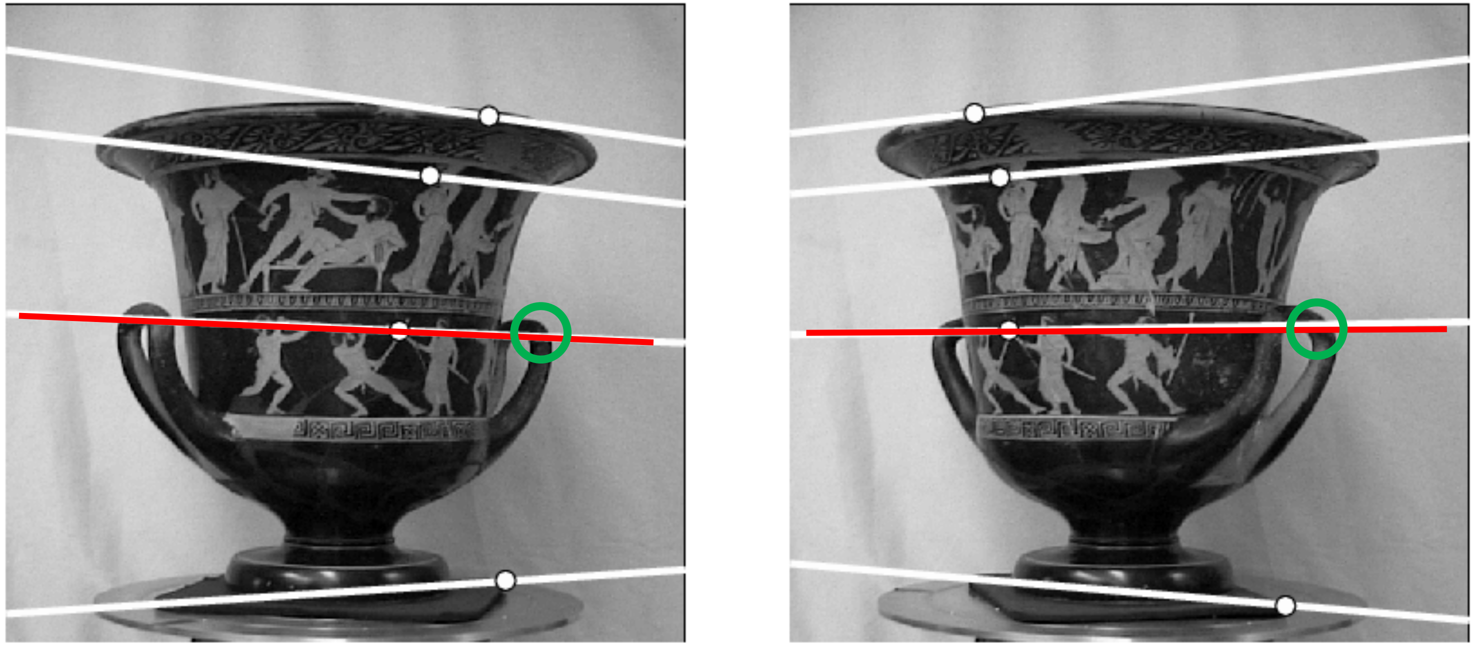

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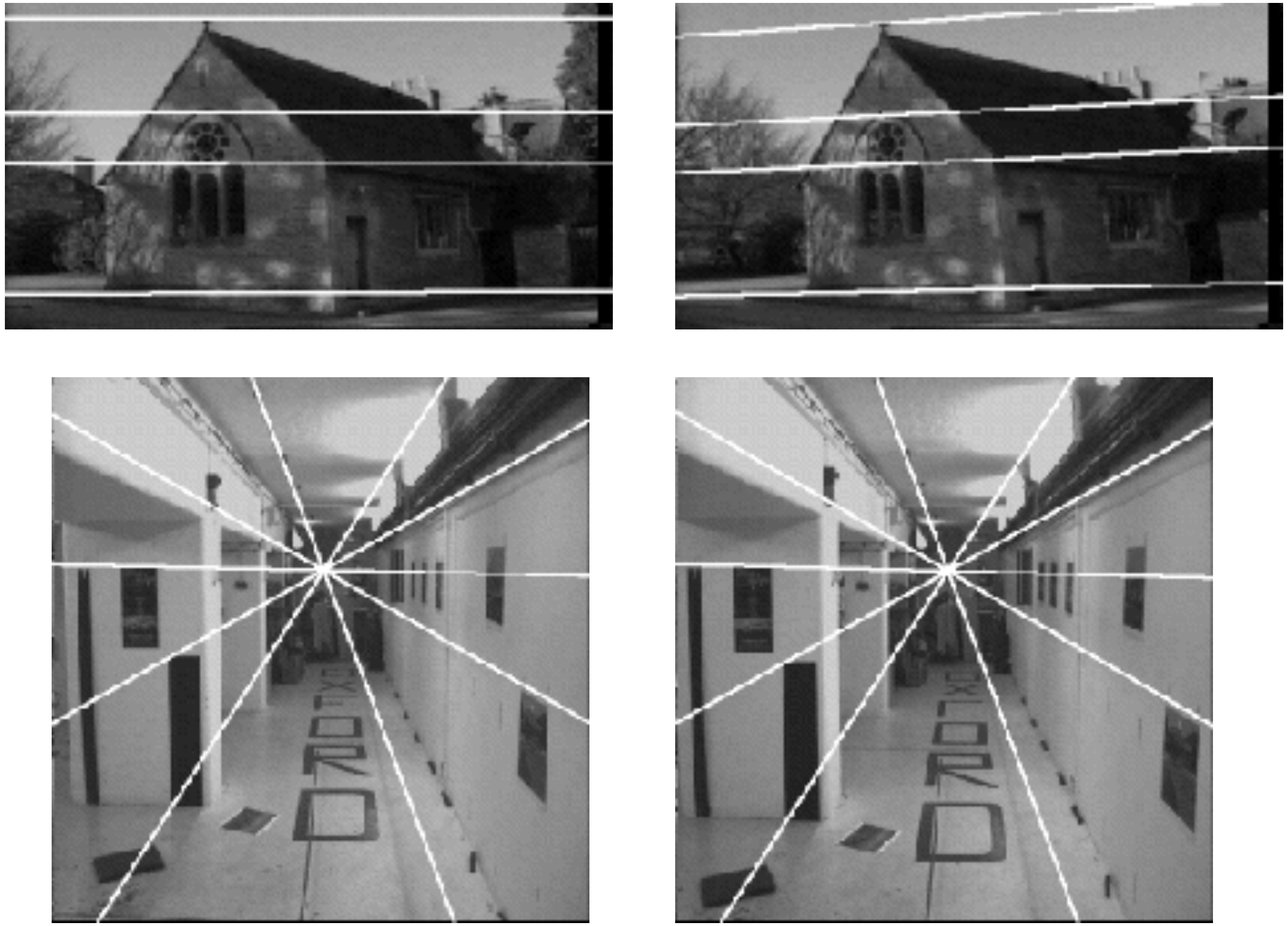
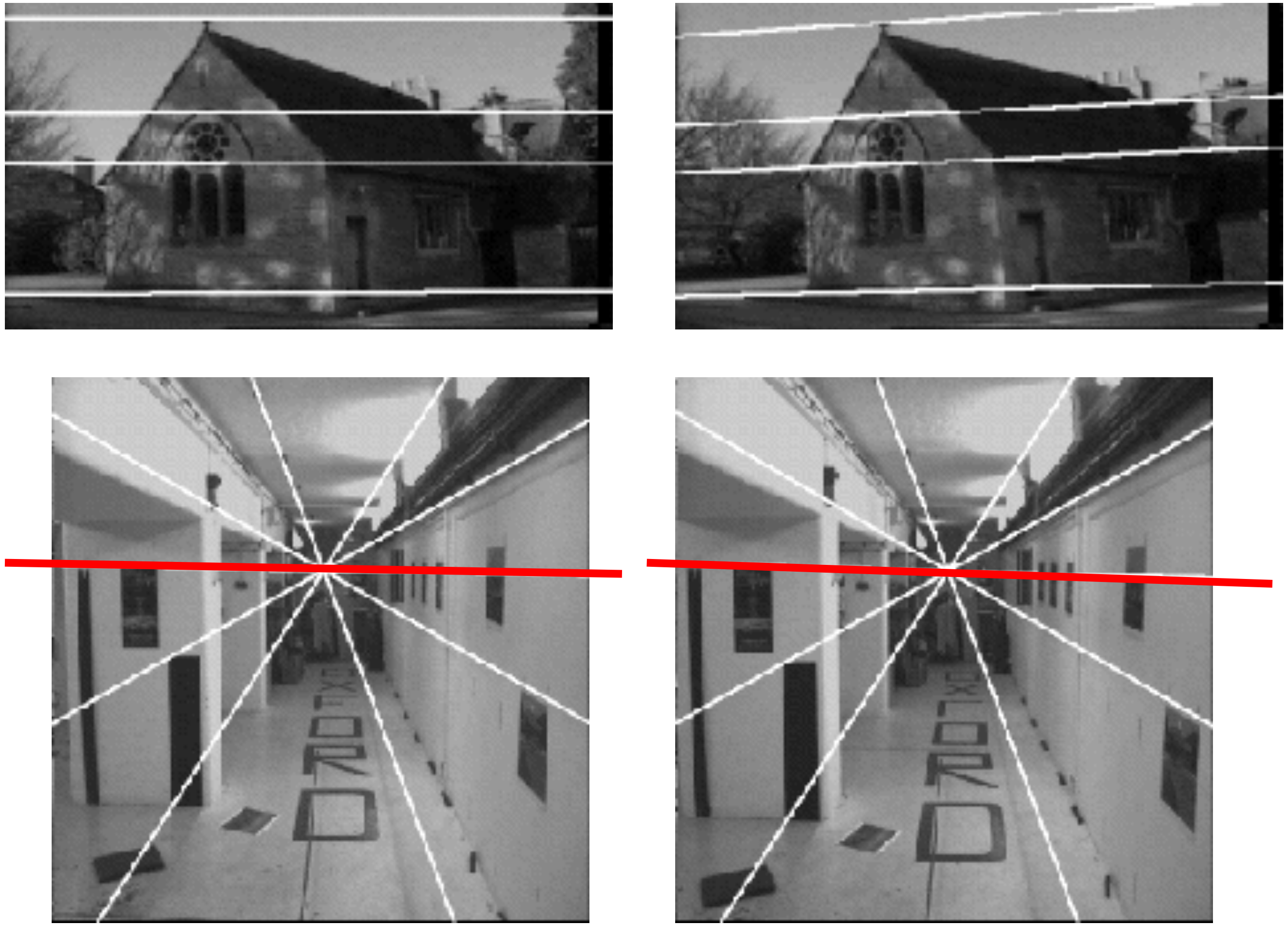


More examples of epipolar lines.



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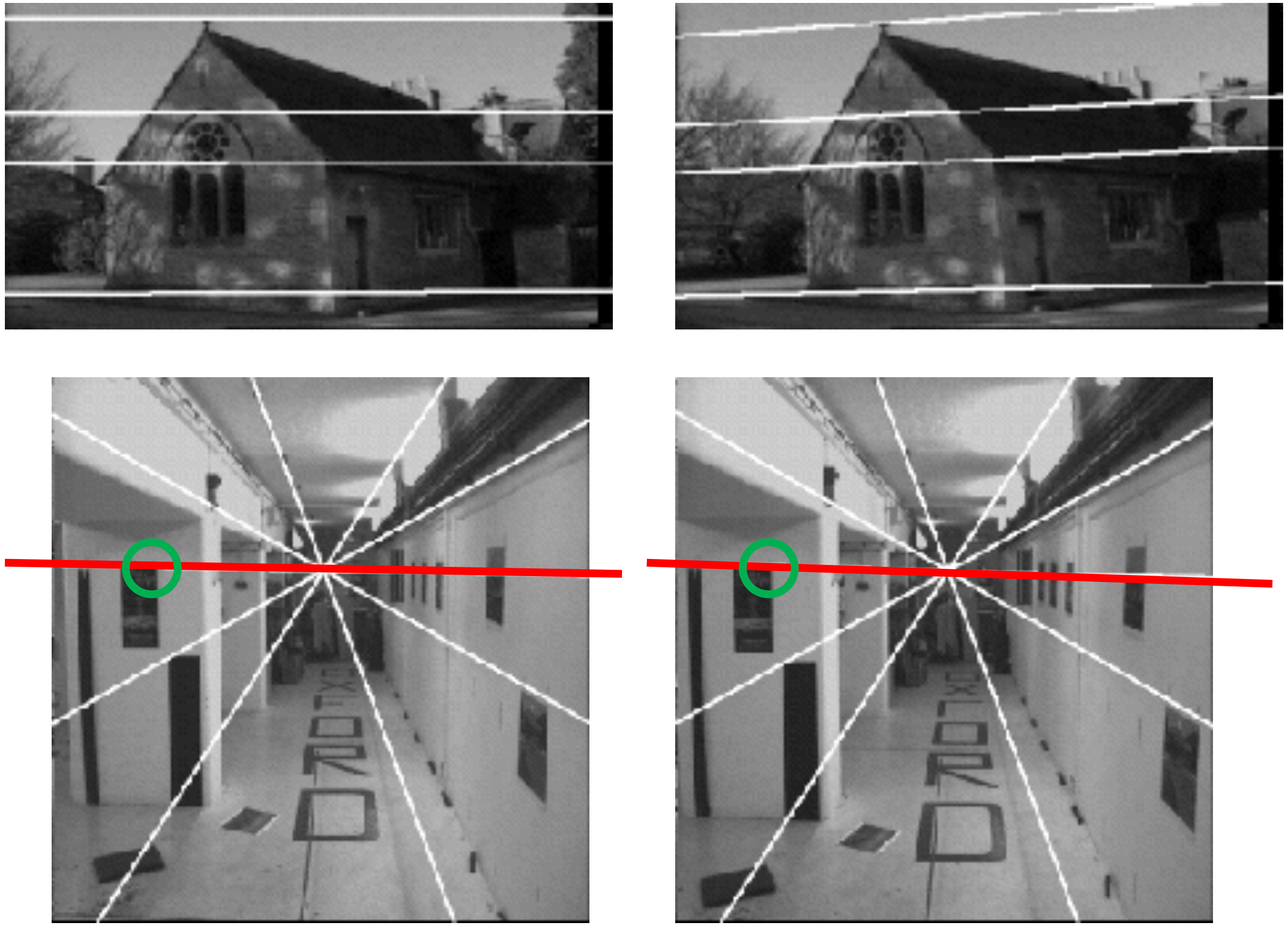

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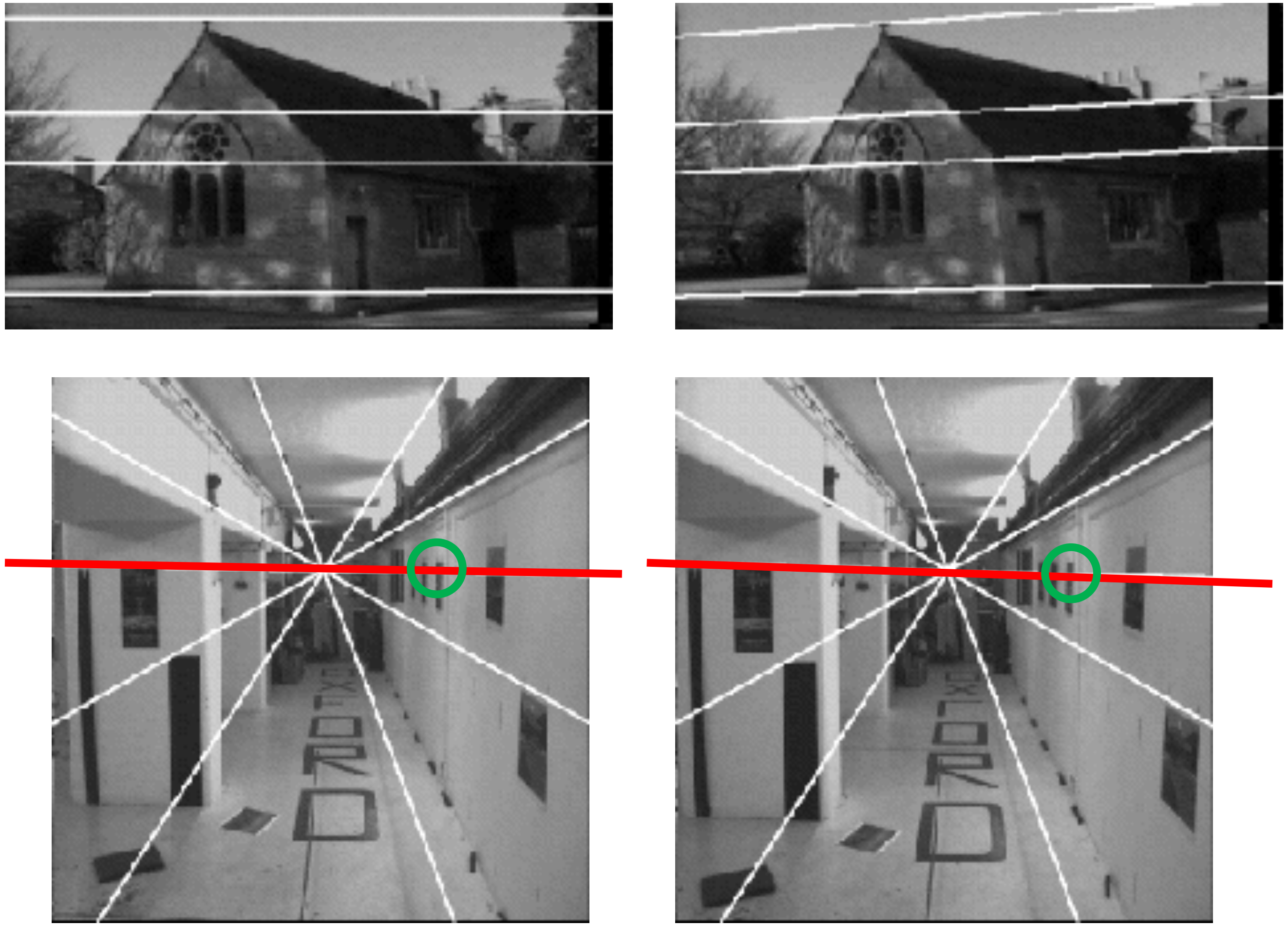

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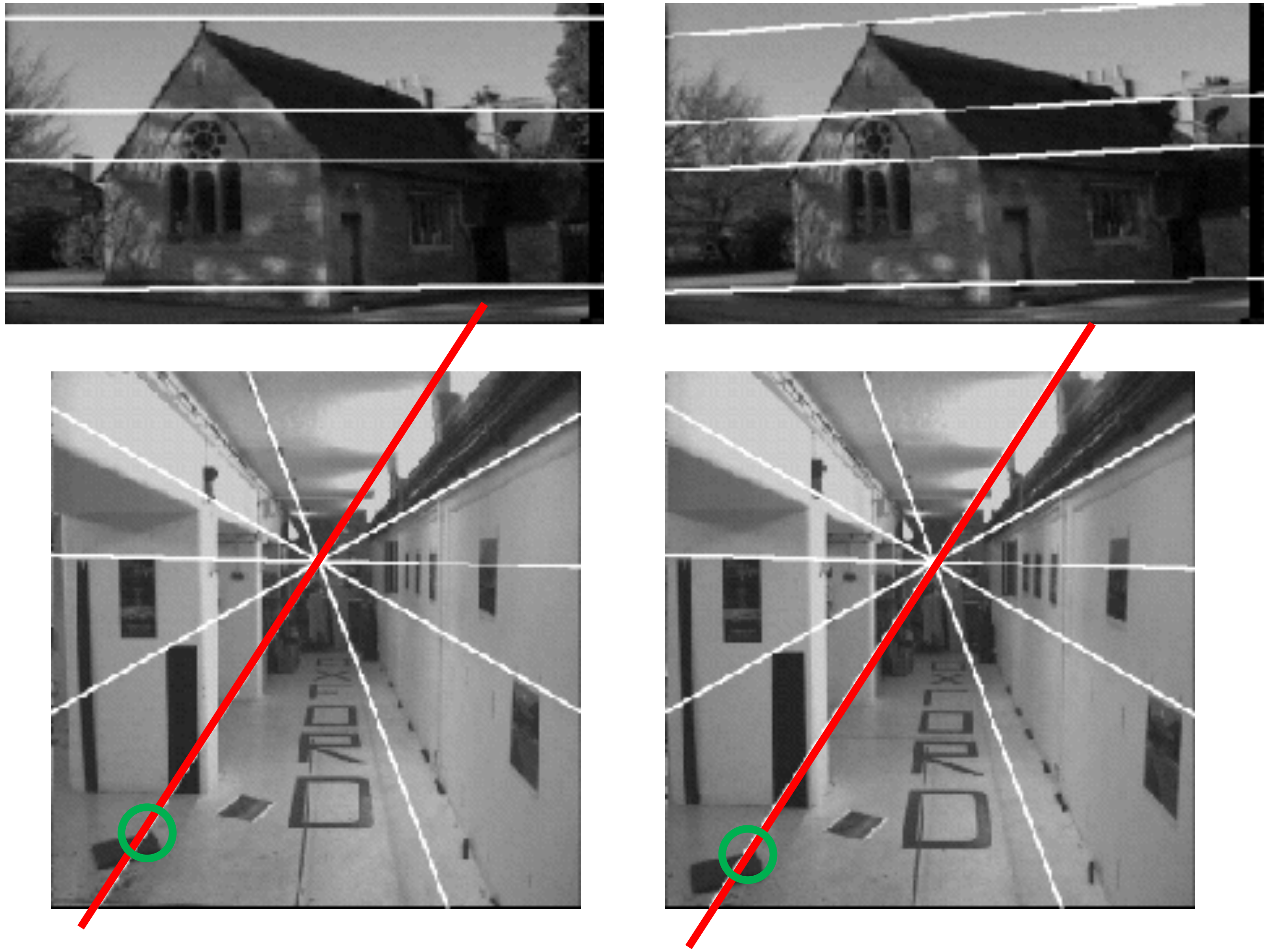

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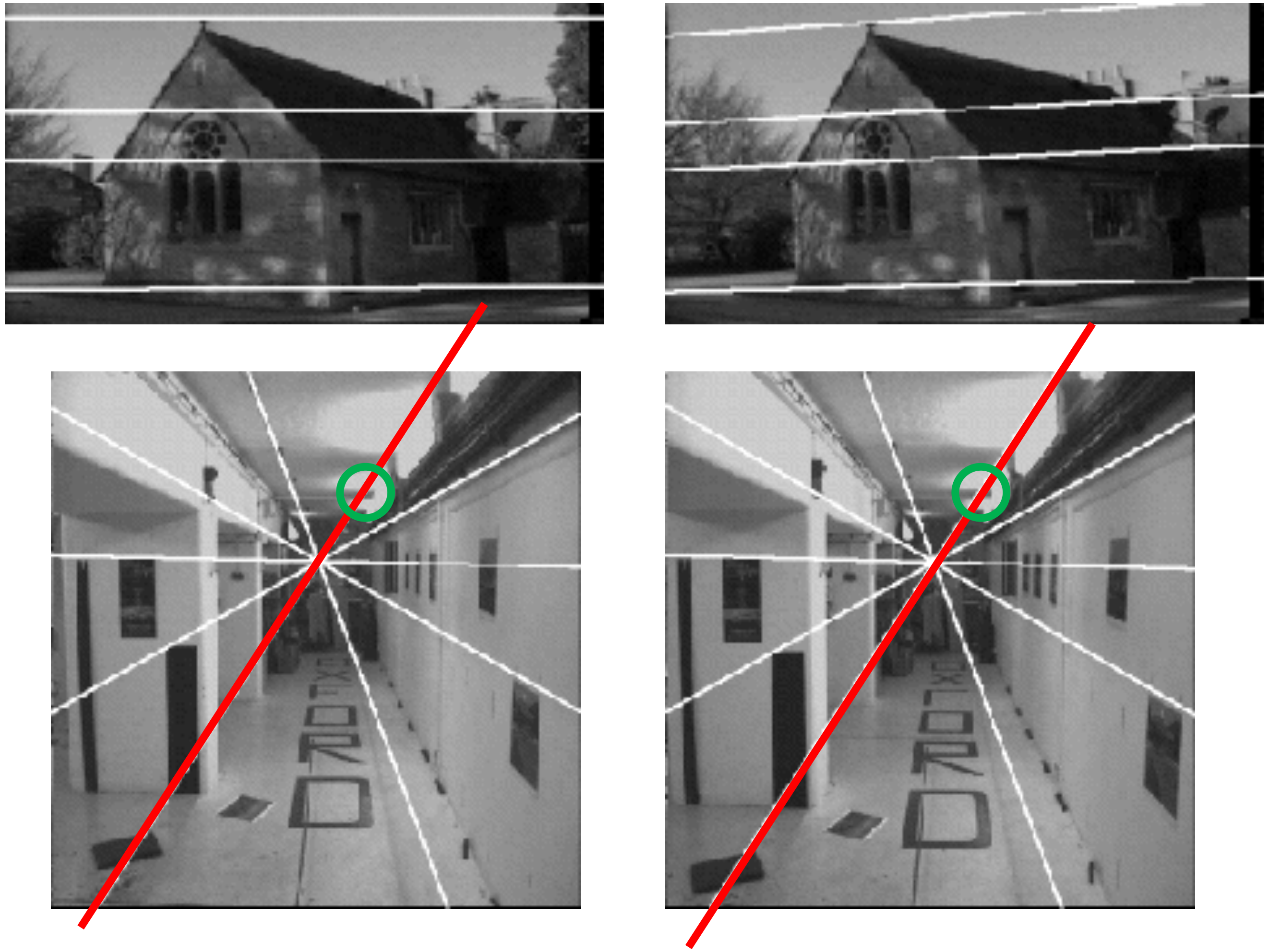
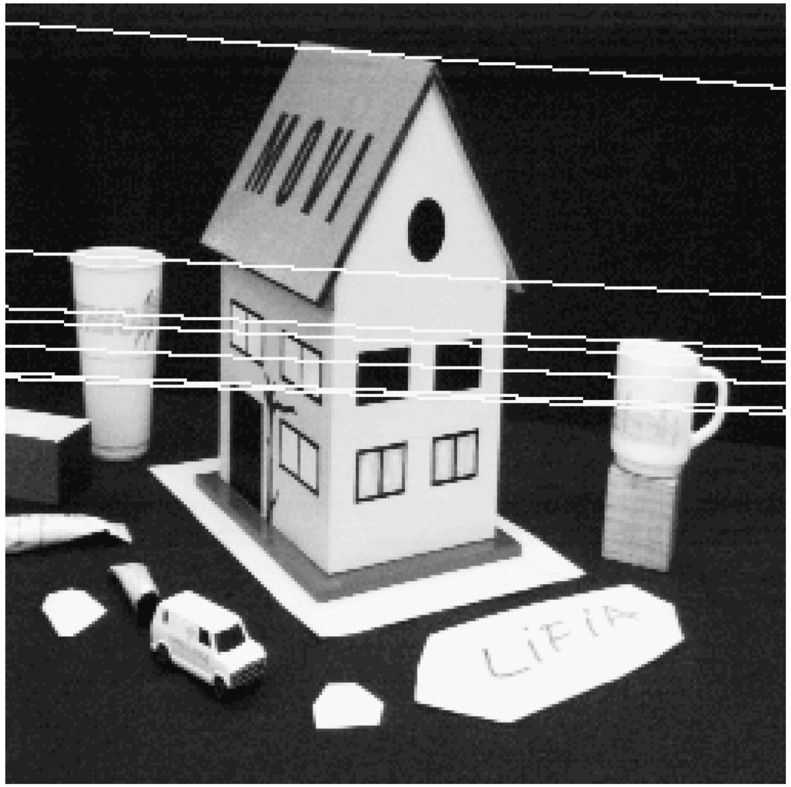


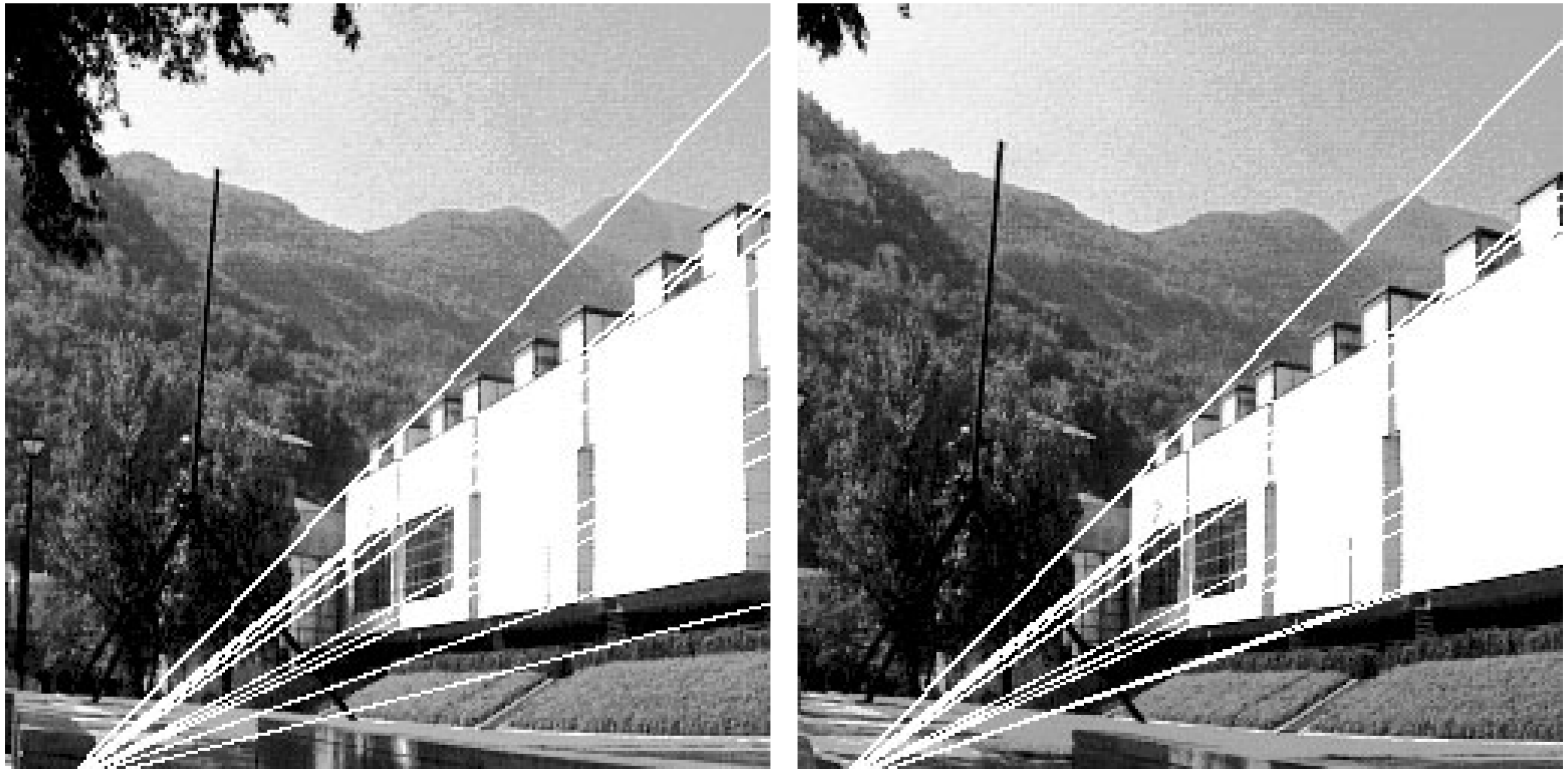
More examples of epipolar lines.

Lifia House images

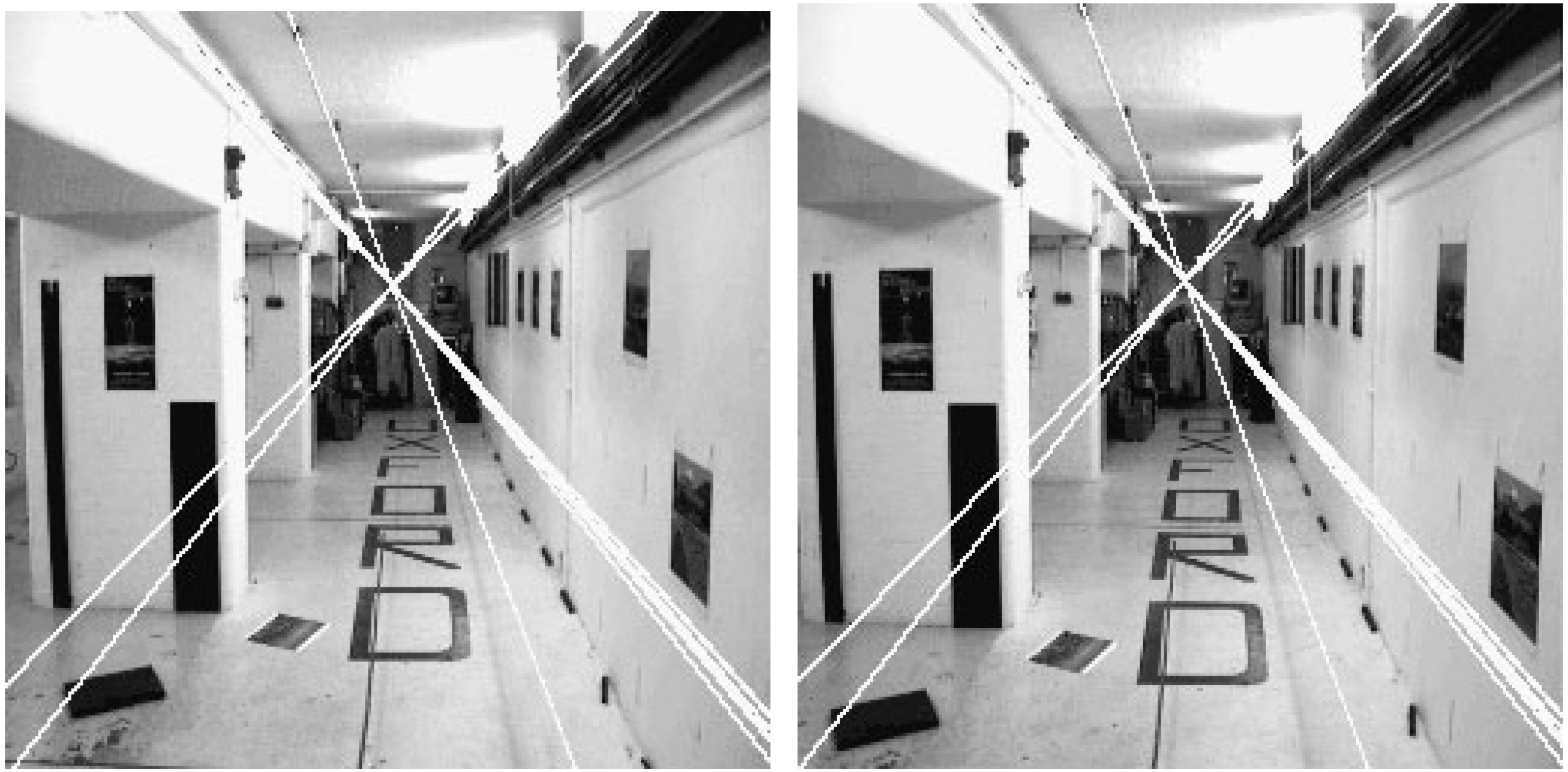


Statue images

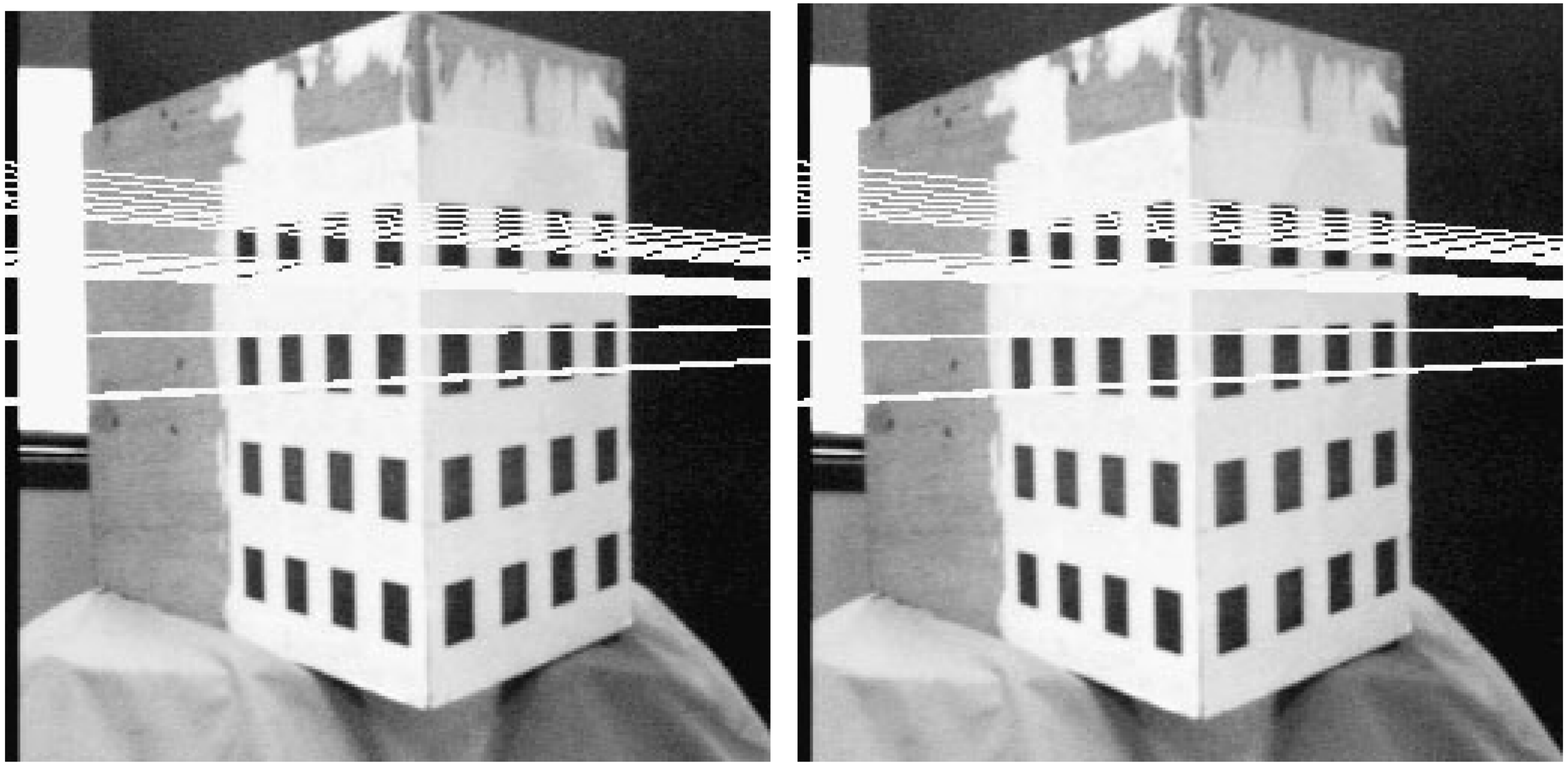
Grenoble Museum



Oxford Basement



Calibration object



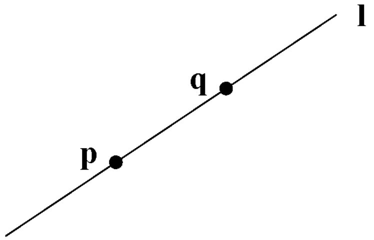
Homogeneous Notation Interlude

- A line $\mathbf{l}$ is represented by the homogeneous 3-vector

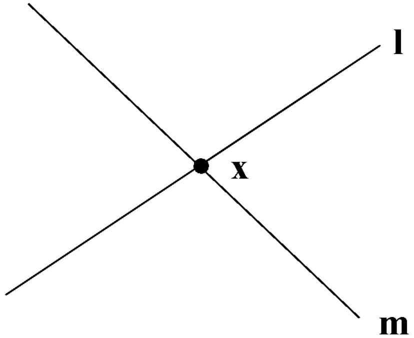
$$\mathbf{l} = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}$$

for the line $l_1 x + l_2 y + l_3 = 0$. Only the ratio of the homogeneous line coordinates is significant.

- point on line: $\mathbf{l}.\mathbf{x} = 0$ or $\mathbf{l}^\top \mathbf{x} = 0$ or $\mathbf{x}^\top \mathbf{l} = 0$



- two points define a line: $\mathbf{l} = \mathbf{p} \times \mathbf{q}$



- two lines define a point: $\mathbf{x} = \mathbf{l} \times \mathbf{m}$

Matrix notation for vector product

The vector product $\mathbf{v} \times \mathbf{x}$ can be represented as a matrix multiplication

$$\mathbf{v} \times \mathbf{x} = [\mathbf{v}]_\times \mathbf{x}$$

where

$$[\mathbf{v}]_\times = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix}$$

- $[\mathbf{v}]_\times$ is a 3×3 skew-symmetric matrix of rank 2.
- $\mathbf{v}$ is the null-vector of $[\mathbf{v}]_\times$, since $\mathbf{v} \times \mathbf{v} = [\mathbf{v}]_\times \mathbf{v} = \mathbf{0}$.

Algebraic representation - the Fundamental Matrix

$$\mathbf{x}'^{\top} \mathtt{F}\, \mathbf{x} = 0 \qquad \mathbf{l}' = \mathtt{F}\, \mathbf{x}$$

- $\mathtt{F}$ is a 3×3 rank 2 homogeneous matrix
- $\mathtt{F}^{\top}\mathbf{e}' = \mathbf{0}$
- It has 7 degrees of freedom

- Compute from 7 image point correspondences

Derivations are from following slides.

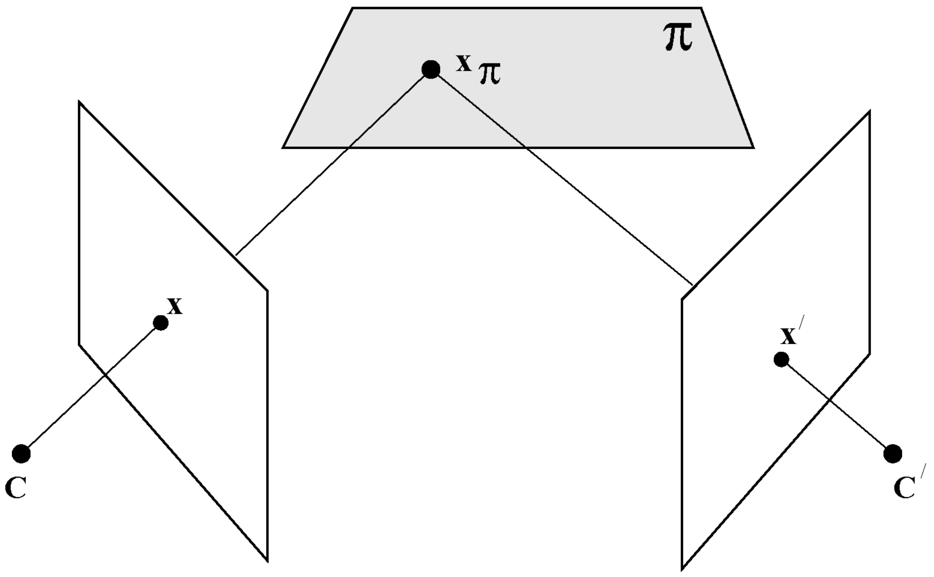
Images of Planes

Projective transformation between images induced by a plane

$$\mathbf{x} = \mathtt{H}_{1\pi}\mathbf{x}_\pi \qquad \mathbf{x}' = \mathtt{H}_{2\pi}\mathbf{x}_\pi$$

$$\mathbf{x}' = \mathtt{H}_{2\pi}\mathbf{x}_\pi$$

$$= \mathtt{H}_{2\pi}\mathtt{H}_{1\pi}^{-1}\mathbf{x} = \mathbf{H}_\pi\mathbf{x}$$



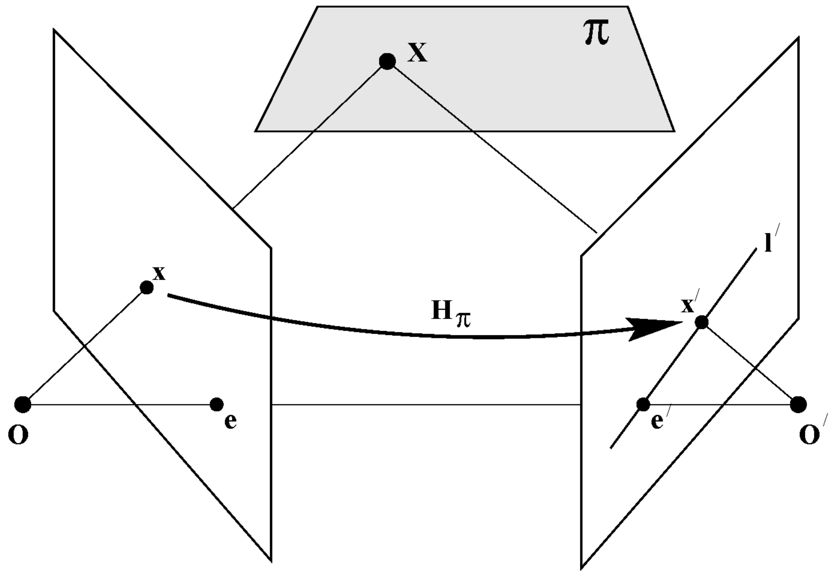
- $\mathbf{H}_\pi$ can be computed from the correspondence of four points on the plane.

$$X_2 = R_2 X_1 + t_2.$$

$$w_1 \begin{pmatrix} u_1 \\ v_1 \\ 1 \end{pmatrix} = K_1(R_1 X_1 + t_1) = K_1(-R_1 R_2^{-1} t_2 + t_1)$$

$$X_1 = R_2^{-1}(X_2 - t_2) = -R_2^{-1} t_2$$

Fundamental matrix - sketch derivation



Step 1: Point transfer via a plane $\mathbf{x}' = \mathtt{H}_\pi \mathbf{x}$

Step 2 : Construct the epipolar line $\mathbf{l}' = \mathbf{e}' \times \mathbf{x}' = [\mathbf{e}']_\times \mathbf{x}'$

$\mathbf{l}' = [\mathbf{e}']_\times \mathtt{H}_\pi \mathbf{x} = \mathtt{F}\mathbf{x}$

$\mathtt{F} = [\mathbf{e}']_\times \mathtt{H}_\pi$

This shows that $\mathtt{F}$ is a 3×3 rank 2 matrix.

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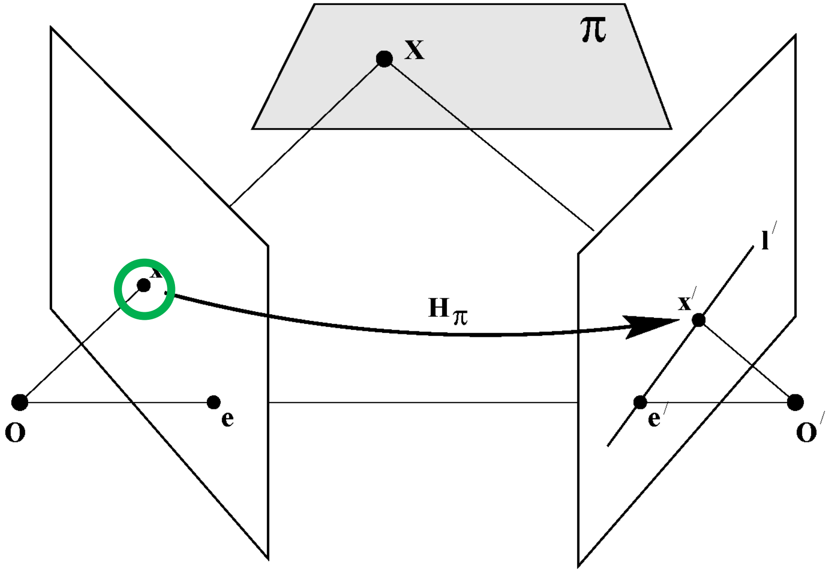

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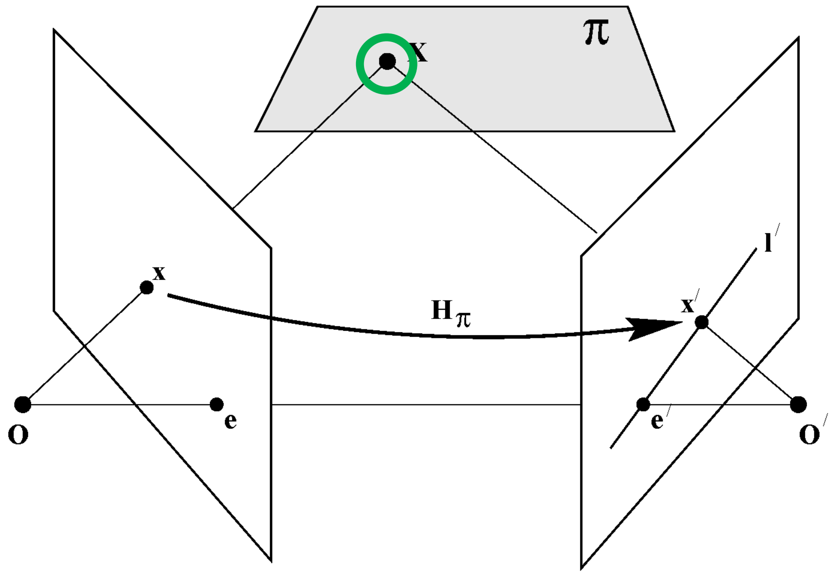

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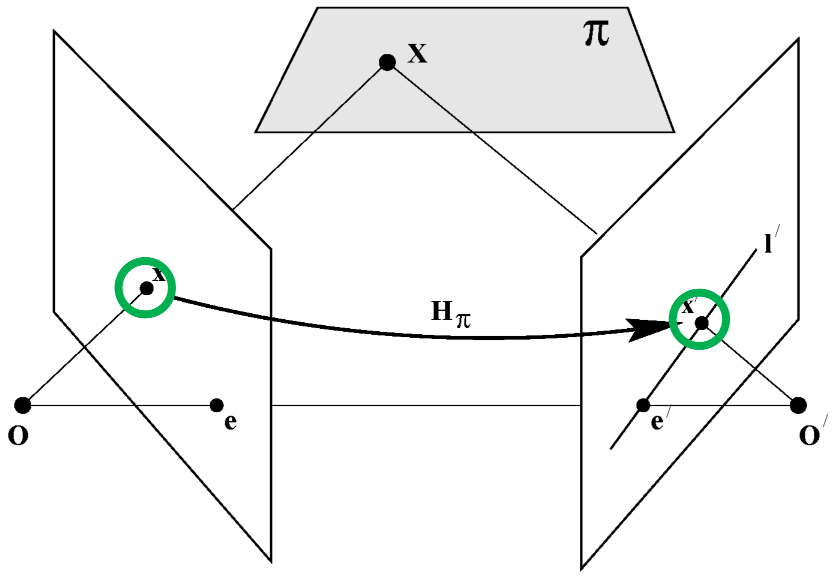

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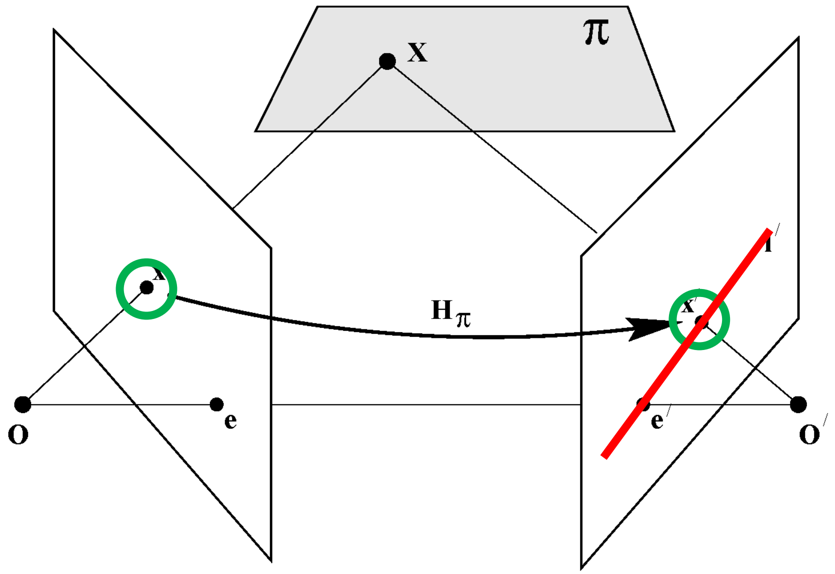

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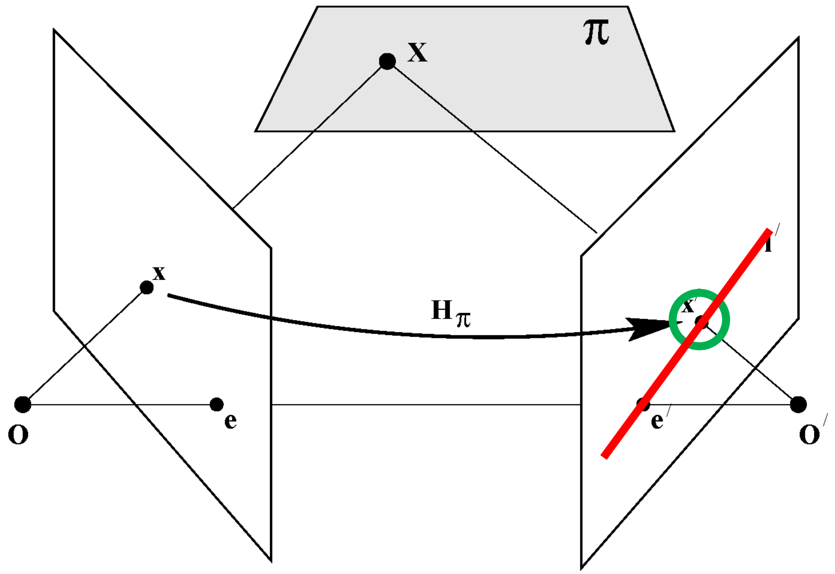

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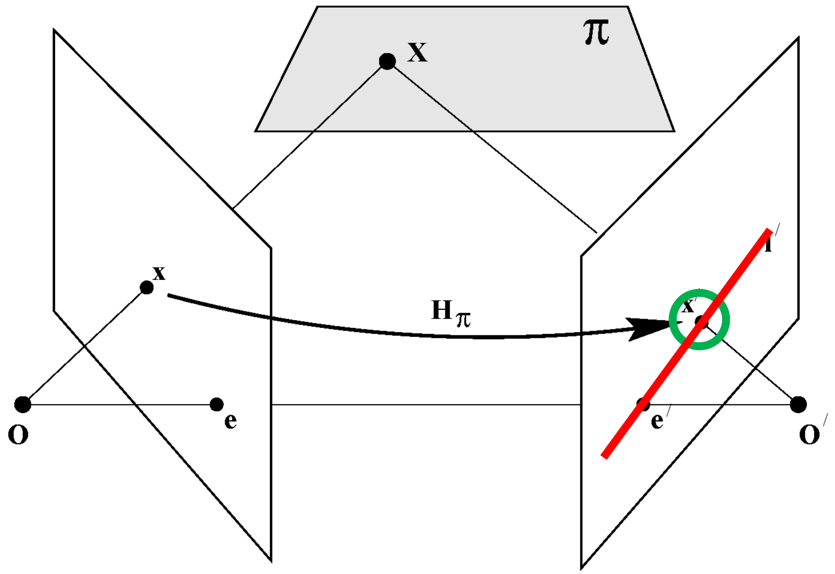

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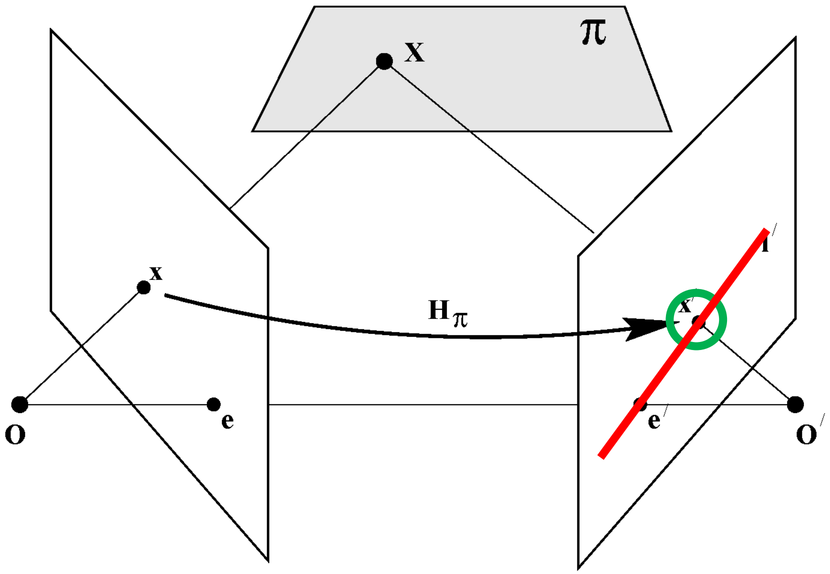

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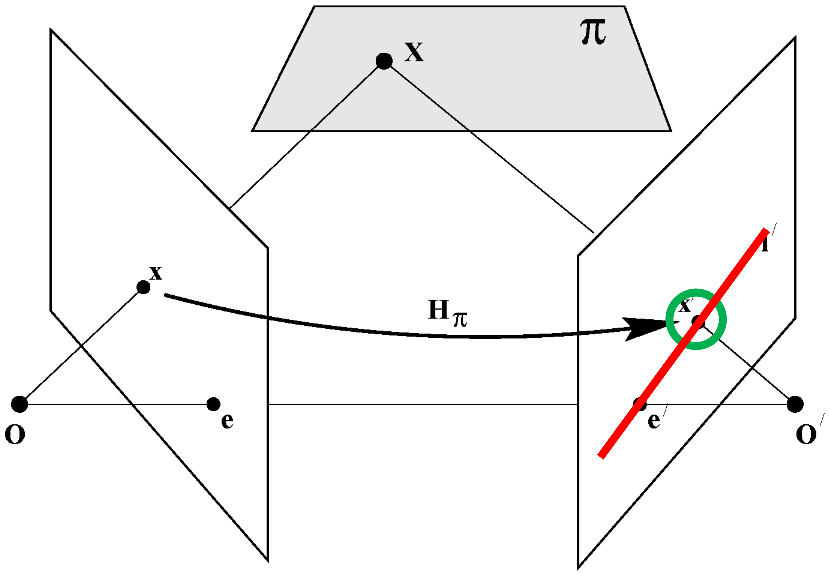

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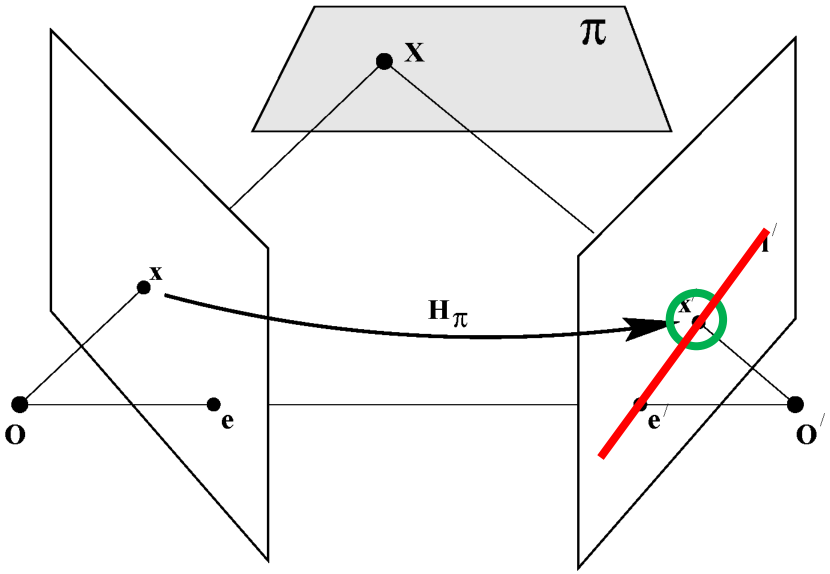
This shows that $\mathtt{F}$ is a 3×3 rank 2 matrix.

Fundamental matrix - sketch derivation



Step 1: Point transfer via a plane $\mathbf{x}' = \mathtt{H}_\pi \mathbf{x}$

Step 2 : Construct the epipolar line $\mathbf{l}' = \mathbf{e}' \times \mathbf{x}' = [\mathbf{e}']_\times \mathbf{x}'$

Point $\mathbf{x}'$ lies on line $\mathbf{l}'$. Hence:

$$\mathbf{x}'^\top \mathbf{F} \mathbf{x} = 0 \ .$$

Properties of $\mathtt{F}$

- $\mathtt{F}$ is a rank 2 homogeneous matrix with 7 degrees of freedom.
- **Point correspondence**: If $\mathbf{x}$ and $\mathbf{x}'$ are corresponding image points, then $\mathbf{x}'^{\top}\mathtt{F}\mathbf{x} = 0$.
- **Epipolar lines**:
  - ◇ $\mathbf{l}' = \mathtt{F}\mathbf{x}$ is the epipolar line corresponding to $\mathbf{x}$.
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- **Epipoles**:
  - ◇ $\mathtt{F}\mathbf{e} = \mathbf{0}$ $\qquad \mathtt{F}^{\top}\mathbf{e}' = \mathbf{0}$
- **Computation from camera matrices** $\mathtt{P}, \mathtt{P}'$:
  - ◇ $\mathtt{F} = [\mathtt{P}'\mathbf{C}]_{\times}\mathtt{P}'\mathtt{P}^{+}$, where $\mathtt{P}^{+}$ is the pseudo-inverse of $\mathtt{P}$, and $\mathbf{C}$ is the centre of the first camera. Note, $\mathbf{e}' = \mathtt{P}'\mathbf{C}$.
  - ◇ Canonical cameras, $\mathtt{P} = [\mathtt{I} \mid \mathbf{0}], \mathtt{P}' = [\mathtt{M} \mid \mathbf{m}]$, $\mathtt{F} = [\mathbf{e}']_{\times}\mathtt{M} = \mathtt{M}^{-\top}[\mathbf{e}]_{\times}$, where $\mathbf{e}' = \mathbf{m}$ and $\mathbf{e} = \mathtt{M}^{-1}\mathbf{m}$.

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Derivation

If $\mathtt{F}$ factorizes as

$$\mathtt{F} = [\mathbf{e}']_\times \mathtt{H}$$

Then a pair of cameras that correspond to this fundamental matrix are

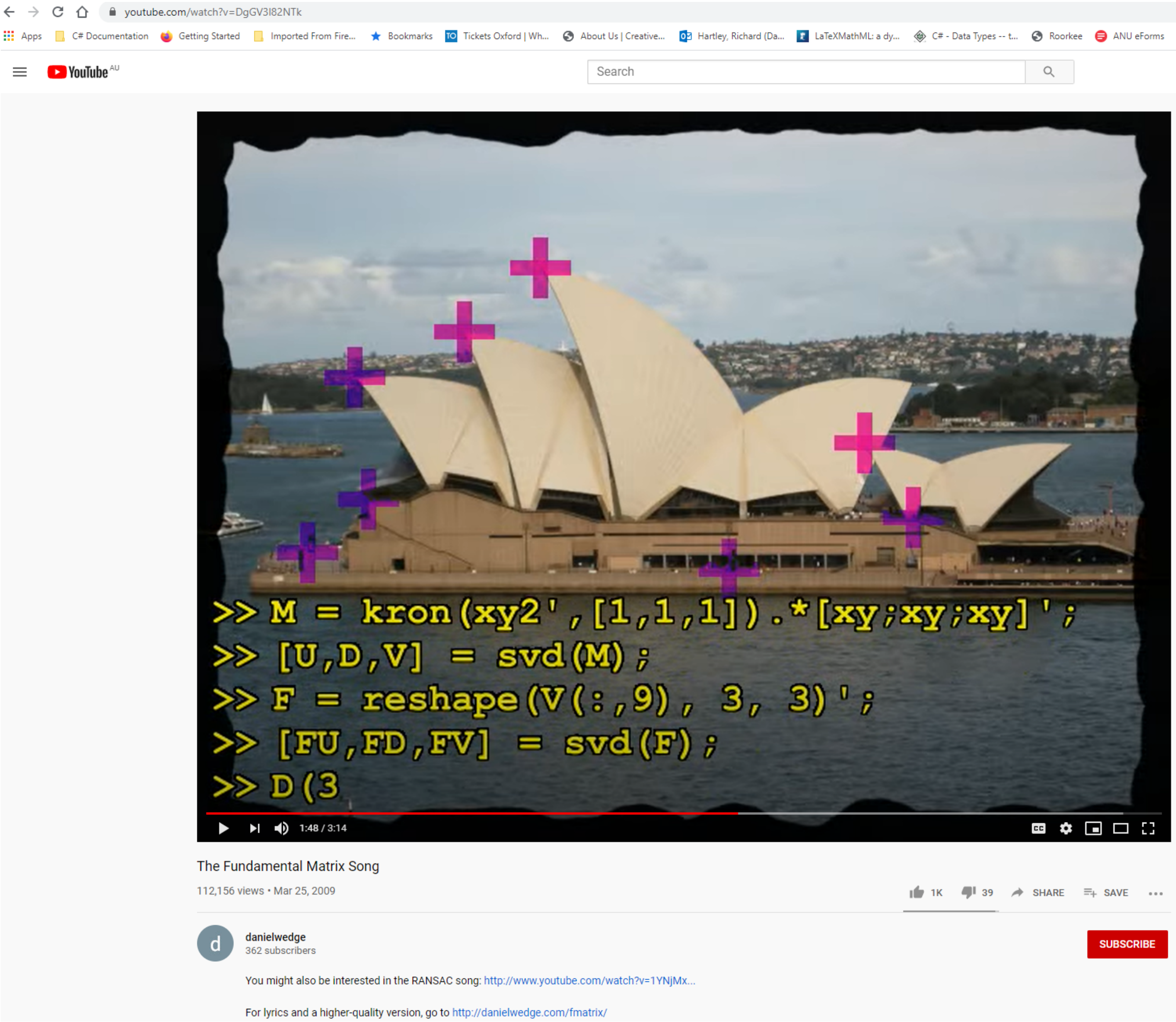
$$\boxed{\begin{aligned} \mathtt{P} &= [\mathtt{I} \mid \mathbf{0}] \\ \mathtt{P}' &= [\mathtt{H} \mid \mathbf{e}'] \end{aligned}}$$

Check:

1. Centre of camera $\mathtt{P}$ is the origin, with coordinates $(0, 0, 0, 1)^\top$.
2. Point projects via camera $\mathtt{P}' = [\mathtt{H} \mid \mathbf{e}']$ to point $\mathbf{e}'$.
3. This is what it should be, since epipole $\mathbf{e}'$ is defined as the image of centre of $\mathtt{P}$ as viewed by camera $\mathbf{P}'$.
4. Verify that $\mathtt{H}$ is the image-to-image homography for the plane at infinity.
   (a) Let point $\mathbf{x} = (x, y, 1)$ be in image of $\mathtt{P}'$.
   (b) Point $\mathbf{X} = (x, y, 1, 0)$ on plane at infinity satisfies $\mathtt{P}\mathbf{X} = \mathbf{x}$.
   (c) $\mathbf{X}$ maps to point in camera $\mathtt{P}'$ given by

$$\mathbf{x}' = [\mathtt{H} \mid \mathbf{e}']\mathbf{X} = \mathtt{H}\mathbf{x} \ .$$

The fundamental matrix song
Daniel Wedge, youtube or
http://danielwedge.com/fmatrix



Computation of the Fundamental Matrix

Basic equations

Given a correspondence

$$\mathbf{x} \leftrightarrow \mathbf{x}'$$

The basic incidence relation is

$$\mathbf{x}'^{\top} \mathrm{F} \mathbf{x} = 0$$

May be written

$$x'xf_{11} + x'yf_{12} + x'f_{13} + y'xf_{21} + y'yf_{22} + y'f_{23} + xf_{31} + yf_{32} + f_{33} = 0 \quad .$$

Single point equation - Fundamental matrix

Gives an equation :

$$(x'x, x'y, x', y'x, y'y, y', x, y, 1) \begin{pmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{pmatrix} = 0$$

where

$$\mathbf{f} = (f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, f_{33})^\top$$

holds the entries of the Fundamental matrix

Total set of equations

$$\mathtt{Af} = \begin{bmatrix} x'_1x_1 & x'_1y_1 & x'_1 & y'_1x_1 & y'_1y_1 & y'_1 & x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x'_nx_n & x'_ny_n & x'_n & y'_nx_n & y'_ny_n & y'_n & x_n & y_n & 1 \end{bmatrix} \begin{pmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{pmatrix} = \mathbf{0}$$

Solving the Equations

- Solution is determined up to scale only.
- Need 8 equations $\Rightarrow$ 8 points
- 8 points $\Rightarrow$ unique solution
- > 8 points $\Rightarrow$ least-squares solution.

Least-squares solution

(i) Form equations $\mathtt{A}\mathbf{f} = \mathbf{0}$.

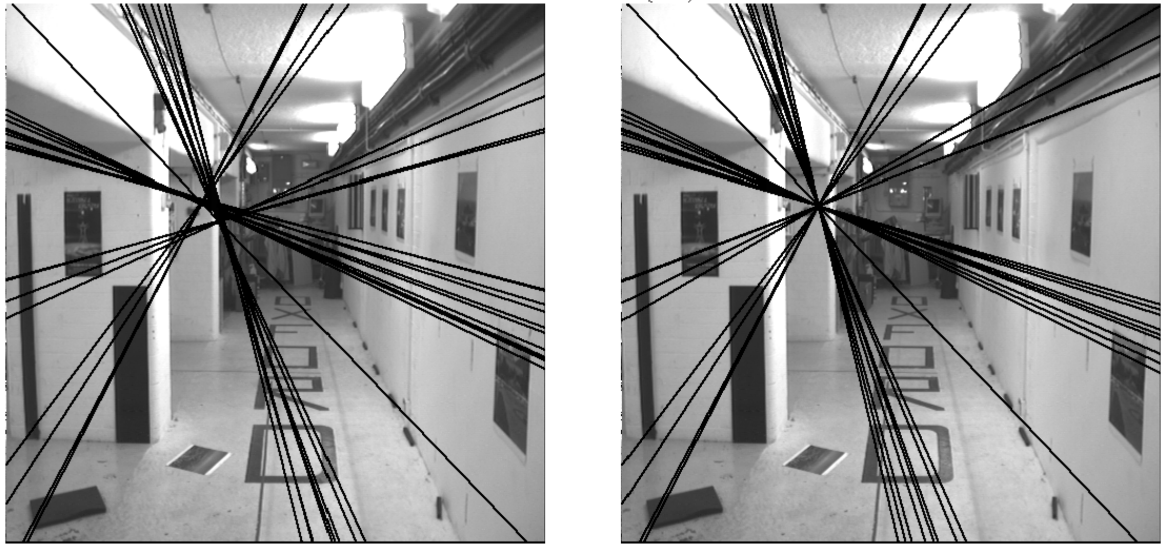
(ii) Take SVD : $\mathtt{A} = \mathtt{UDV}^\top$.

(iii) Solution is last column of $\mathtt{V}$ (corresp : smallest singular value)

(iv) Minimizes $||\mathtt{A}\mathbf{f}||$ subject to $||\mathbf{f}|| = 1$.

The singularity constraint

Fundamental matrix has rank 2 : $\det(\mathtt{F}) = 0$.



Left : Uncorrected $\mathtt{F}$ – epipolar lines are not coincident.

Right : Epipolar lines from corrected $\mathtt{F}$.

Computing $\mathtt{F}$ from 7 points

- $\mathtt{F}$ has 9 entries but is defined only up to scale.
- Singularity condition $\det \mathtt{F} = 0$ gives a further constraint.
- $\mathtt{F}$ has 3 rows $\implies \det \mathtt{F} = 0$ is a cubic constraint.
- $\mathtt{F}$ has only 7 degrees of freedom.
- It is possible to solve for $\mathtt{F}$ from just 7 point correspondences.

Computation of $\mathtt{F}$ from 7 point correspondences

(i) Form the 7×9 set of equations $\mathtt{A}\mathbf{f} = 0$.

(ii) System has a 2-dimensional solution set.

(iii) General solution (use SVD) has form

$$\mathbf{f} = \lambda \mathbf{f}_0 + \mu \mathbf{f}_1$$

(iv) In matrix terms

$$\mathtt{F} = \lambda \mathtt{F}_0 + \mu \mathtt{F}_1$$

(v) Condition $\det \mathtt{F} = 0$ gives cubic equation in λ and μ.

(vi) Either one or three real solutions for ratio $\lambda : \mu$.

Correcting $\mathtt{F}$ using the Singular Value Decomposition

If $\mathtt{F}$ is computed linearly from 8 or more correspondences, singularity condition does not hold.

SVD Method

(i) SVD : $\mathtt{F} = \mathtt{UDV}^\top$

(ii) $\mathtt{U}$ and $\mathtt{V}$ are orthogonal, $\mathtt{D} = \mathrm{diag}(r, s, t)$.

(iii) $r \geq s \geq t$.

(iv) Set $\mathtt{F}' = \mathtt{U}\,\mathrm{diag}(r, s, 0)\,\mathtt{V}^\top$.

(v) Resulting $\mathtt{F}'$ is singular.

(vi) Minimizes the Frobenius norm of $\mathtt{F} - \mathtt{F}'$

(vii) $\mathtt{F}'$ is the "closest" singular matrix to $\mathtt{F}$.

SVD method

(i) Define

$$\mathtt{Z} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(ii) Compute the SVD

$$\mathtt{F} = \mathtt{UDV}^\top \text{ where } \mathtt{D} = \mathrm{diag} r, s, 0$$

(iii) Factorization is

$$\mathtt{F} = (\mathtt{UZU}^\top)(\mathtt{UZDV}^\top)$$

- Simultaneously corrects $\mathtt{F}$ to a singular matrix.

Non-uniqueness of factorization

- Factorization of the fundamental matrix is not unique.
- General formula : for varying $\mathbf{v}$ and λ

$$\mathtt{P} = [\mathtt{I} \mid 0] \;\; ; \;\; \mathtt{P}' = [\mathtt{M} + \mathbf{e}'\mathbf{v}^\top \mid \lambda\mathbf{e}']$$

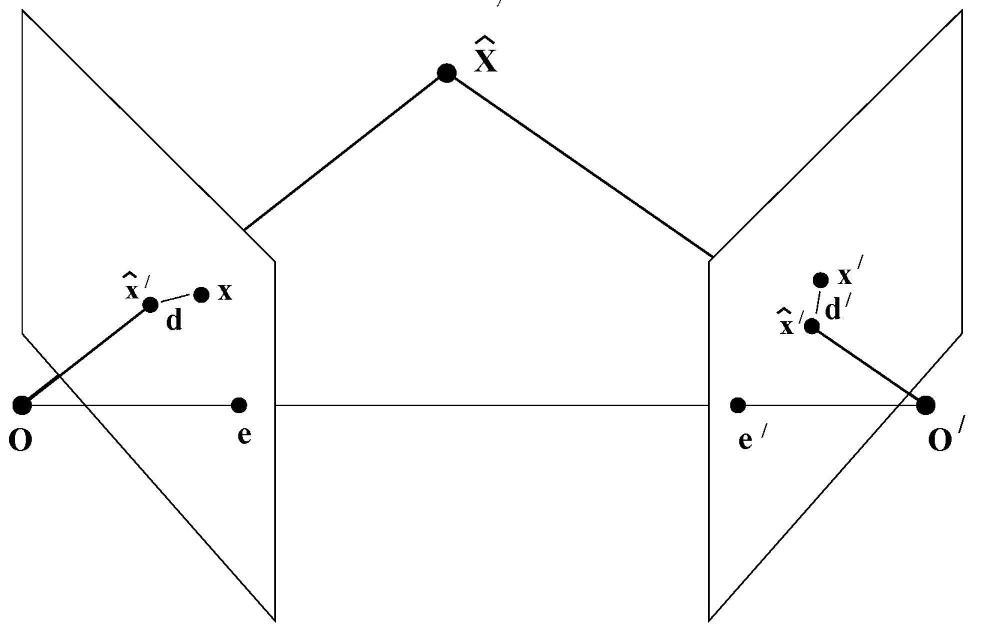
- Difference factorizations give configurations varying by a projective transformation.
- 4-parameter family of solutions with $\mathtt{P} = [\mathtt{I} \mid 0]$.

Triangulation

Triangulation :

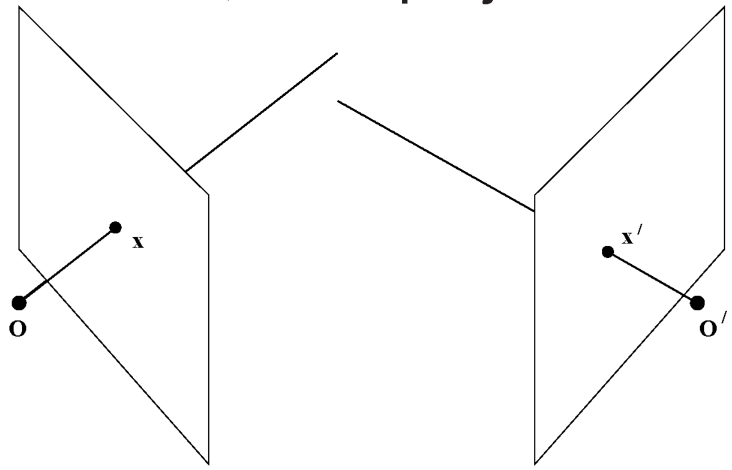
- Knowing $\mathtt{P}$ and $\mathtt{P}'$
- Knowing $\mathtt{x}$ and $\mathtt{x}'$
- Compute $\mathtt{x}'$ such that

$$\mathbf{x} = \mathtt{P}\mathtt{x} \quad ; \quad \mathbf{x}' = \mathtt{P}'\mathtt{x}$$



Triangulation in presence of noise

- In the presence of noise, back-projected lines do not intersect.



Rays do not intersect in space

Measured points do not lie on corresponding epipolar lines

Problem statement

- Assume camera matrices are given without error, up to projective distortion.
- Hence $\mathtt{F}$ is known.
- A pair of matched points in an image are given.
- Possible errors in the position of matched points.
- Find 3D point that minimizes suitable error metric.
- Method must be invariant under 3D projective transformation.

Linear triangulation methods

- Direct analogue of the linear method of camera resectioning.
- Given equations

$$\begin{aligned} \mathbf{x} &= \mathtt{P}\mathrm{X} \\ \mathbf{x}' &= \mathtt{P}'\mathrm{X} \end{aligned}$$

- $\mathbf{p}^{i\top}$ are the rows of $\mathtt{P}$.
- Write as linear equations in X

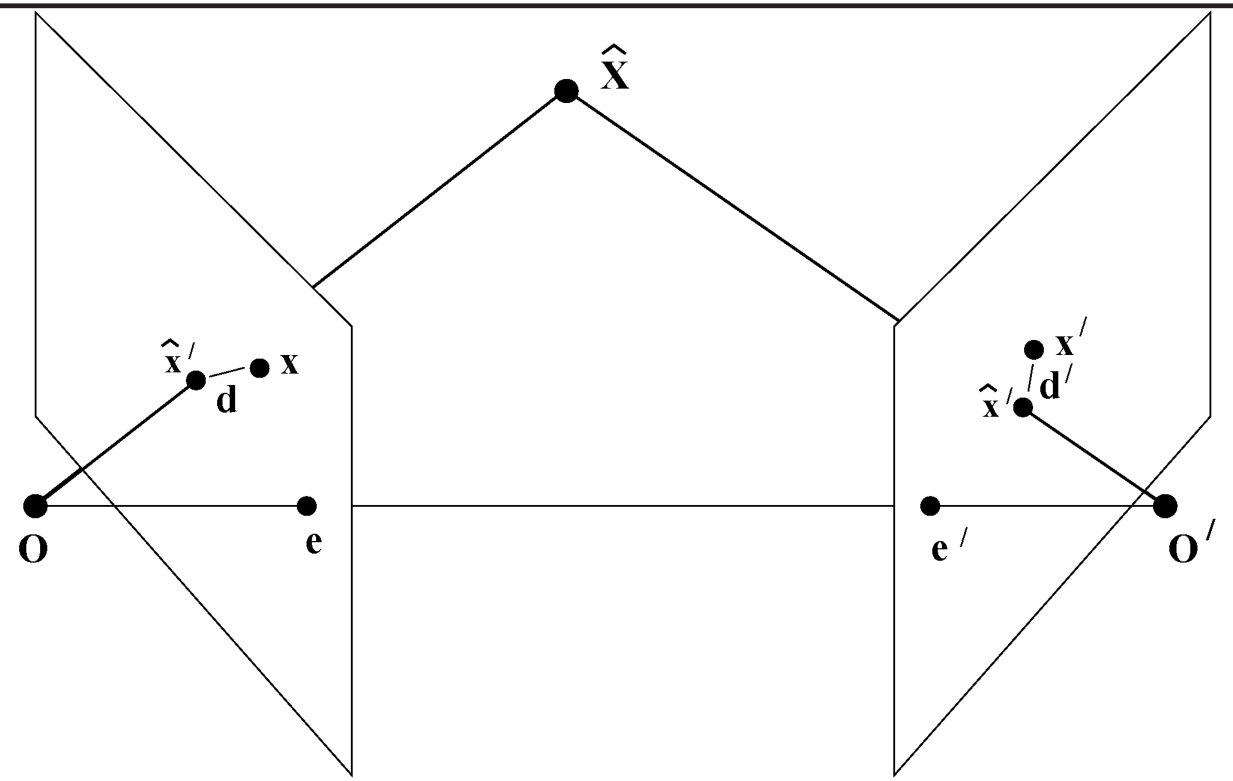
$$\begin{bmatrix} x\mathbf{p}^{3\top} - \mathbf{p}^{1\top} \\ y\mathbf{p}^{3\top} - \mathbf{p}^{2\top} \\ x'\mathbf{p}'^{3\top} - \mathbf{p}'^{1\top} \\ y\mathbf{p}'^{3\top} - \mathbf{p}'^{2\top} \end{bmatrix} \mathrm{X} = 0$$

- Solve for X.
- Generalizes to point match in several images.
- Minimizes no meaningful quantity – not optimal.

Minimizing geometric error

- Point x in space maps to projected points $\hat{\mathrm{x}}$ and $\hat{\mathrm{x}}'$ in the two images.
- Measured points are x and x'.
- Find x that minimizes difference between projected and measured points.

Geometric error . . .



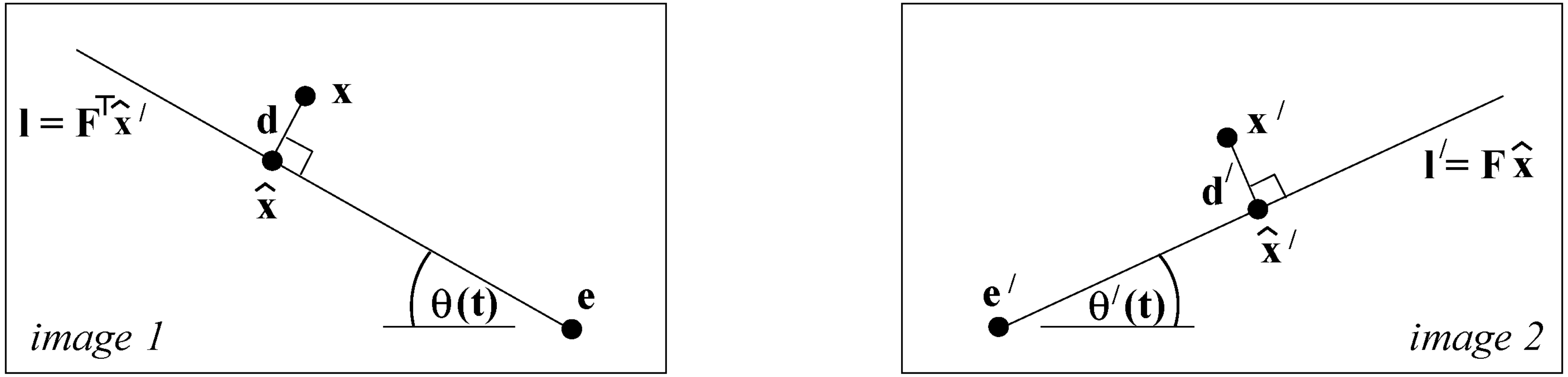
Cost function

$$\mathcal{C}(\mathrm{x}) = d(\mathbf{x}, \hat{\mathbf{x}})^2 + d(\mathbf{x}', \hat{\mathbf{x}}')^2$$

Different formulation of the problem

Minimization problem may be formulated differently:

- Minimize
$$d(\mathbf{x}, \mathbf{l})^2 + d(\mathbf{x}', \mathbf{l}')^2$$
- $\mathbf{l}$ and $\mathbf{l}'$ range over all choices of corresponding epipolar lines.
- $\hat{\mathbf{x}}$ is the closest point on the line $\mathbf{l}$ to $\mathbf{x}$.
- Same for $\hat{\mathbf{x}}'$.



Complete 8-point algorithm

8 point algorithm has two steps :

(i) **Linear solution.** Solve $\mathtt{A}\mathbf{f} = 0$ to find $\mathtt{F}$.

(ii) **Constraint enforcement.** Replace $\mathtt{F}$ by $\mathtt{F}'$.

Warning This algorithm is unstable and should never be used with unnormalized data (see next slide).

The normalized 8-point algorithm

Raw 8-point algorithm performs badly in presence of noise.

Normalization of data

- 8-point algorithm is sensitive to origin of coordinates and scale.
- Data must be translated and scaled to “canonical” coordinate frame.
- Normalizing transformation is applied to both images.
- Translate so centroid is at origin
- Scale so that RMS distance of points from origin is $\sqrt{2}$.
- “Average point” is $(1, 1, 1)^\top$.

Normalized 8-point algorithm

(i) **Normalization:** Transform the image coordinates :

$$\hat{\mathbf{x}}_i = \mathrm{T}\mathbf{x}_i$$
$$\hat{\mathbf{x}}'_i = \mathrm{T}'\mathbf{x}'_i$$

(ii) **Solution:** Compute $\mathtt{F}$ from the matches $\hat{\mathbf{x}}_i \leftrightarrow \hat{\mathbf{x}}'_i$

$$\hat{\mathbf{x}}'^{\top}_i \widehat{\mathrm{F}} \hat{\mathbf{x}}_i = 0$$

(iii) **Singularity constraint :** Find closest singular $\widehat{\mathtt{F}}'$ to $\widehat{\mathtt{F}}$.

(iv) **Denormalization:** $\mathtt{F} = \mathtt{T}'^{\top}\widehat{\mathtt{F}}'\mathtt{T}$.

Readings

- Chapter 9.1, 9.2, 11.1, 11.2, 11.3, 12.1, 12.2
- Epipolar geometry and Fundamental matrix, triangulation