1. Let $a_{i,j} = i + j \quad \forall i, j \in \{1, .., 5\}$.
Write $(a_{i,j})_{1 \le i,j \le 5}$ as an array of numbers, *i.e.* as a 5×5 matrix.

2. Define a function $a : \{1, 2, 3, 4\}^2 \to \{-1, 1\}$ representing the matrix

$$A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

That is, give a formula for $a_{i,j}$. Hint: One way is to use power of (-1).

3. Let R be the relation defined by

$$R = \{(1, 2), (1, 3), (3, 4), (2, 1)\} \subseteq \{1, 2, 3, 4\}^2.$$

Define $(a_{i,j})_{1 \le i,j \le 4} \in M_n(\{0, 1\})$ by $a_{i,j} = 1 \iff iRj$.
Write $(a_{i,j})_{1 \le i,j \le 4}$ as a matrix.

4. Let $(q_1, q_2, q_3) \in \mathbb{Q}^3$ represent the quantities (in ml) of three ingredients required to produce one glass of a cocktail.

(a) Which vector $(r_1, r_2, r_3) \in \mathbb{Q}^3$ represents the quantities required to produce five glasses of the cocktail?

(b) If q_1, q_2 correspond to alcohols, and q_3 to juice, are you making the cocktail stronger or weaker by replacing (q_1, q_2, q_3) by $(q_1, q_2, q_3) + (-10, -20, 30)$?

5. Compute the following.

(a) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} + \begin{pmatrix} 6 & 2 & 1 \\ 4 & 5 & 3 \end{pmatrix}$.

(b) $3 \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 2 \end{pmatrix}$.

(c) $\alpha \begin{pmatrix} x & y \\ z & w \end{pmatrix} + \beta \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

6. Are the following functions linear?

(a) $f : \mathbb{Q} \to \mathbb{Q}$ defined by $f(x) = 2x + 1$.

(b) $g : \mathbb{Q} \to \mathbb{Q}$ defined by $f(x) = x^2 + 1$.

(c) $k : \mathbb{Q}^3 \to \mathbb{Q}^3$ defined by $f(x_1, x_2, x_3) = M \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, for $M \in M_3(\mathbb{Q})$.

7. Compute the following.

(a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

(c) $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

8. Compute the following.

(a) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 4 & 5 \\ 3 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 1 \\ 3 & 1 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x & y & z \\ a & b & c \\ 2 & 1 & 0 \end{pmatrix}$

9. Let $(x_n, y_n) \in \mathbb{Q}^2$ represent, at time $n \in \mathbb{N}^* = \mathbb{N} \cup \{0\}$, the quantity x_n of a certain plant in an ecosystem, and y_n the quantity of a pollutant. Assume that they are related in the following way: $x_0 = a \in \mathbb{Q},\ y_0 = b \in \mathbb{Q},\ \ \forall n \in \mathbb{N}^*\ x_{n+1} = 2x_n - 3y_n,\ y_{n+1} = y_n/2.$

(a) Explain the meaning of these equations.

(b) Prove that $\forall n \in \mathbb{N}\quad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2^n & 0 \\ 0 & 2^{-n} \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}.$

(c) Prove that, if $a > 2b$, then the plant will survive.

10. A portfolio is to contain three types of shares; A, B and C. To hedge certain risks, the investor wants twice as many C shares as the combined number of A and B shares and only a third as many B shares as the combined number of A and C shares. The numbers of A, B and C shares are to be a, b and c with a total of 1200.

(a) Show that $\begin{pmatrix} 2 & 2 & -1 \\ 1 & -3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1200 \end{pmatrix}$.

(b) Verify that $\begin{pmatrix} 2 & 2 & -1 \\ 1 & -3 & 1 \\ 1 & 1 & 1 \end{pmatrix}^{-1} = \dfrac{1}{12} \begin{pmatrix} 4 & 3 & 1 \\ 0 & -3 & 3 \\ -4 & 0 & 8 \end{pmatrix}$.

(c) Find a, b and c.

11. Compute $\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

12. Prove that $\forall A, B \in M_2(Q) \quad \det(AB) = \det(A)\det(B)$.

13. Let $P \in M_2(\mathbb{Q})$ be such that $P^2 = I$. Prove that $\det(P) \in \{-1, 1\}$.