# COMP2610 / COMP6261 Information Theory
# Lecture 8: Some Fundamental Inequalities

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# Announcements

Assignment 1

- Available via Wattle
- Worth 10% of Course total
- Due Monday 29 August 2022, 5:00 pm
- Answers could be typed or handwritten

You can use latex LaTeX primer:
http://tug.ctan.org/info/lshort/english/lshort.pdf

# Last time

- Decomposability of entropy
- Relative entropy (KL divergence)
- Mutual information

# Review

Relative entropy (KL divergence):

$$D_{\mathrm{KL}}(p\|q) = \sum_{x\in\mathcal{X}} p(x) \log \frac{p(x)}{q(x)}$$

Mutual information:

$$\begin{aligned} I(X;Y) &= D_{\mathrm{KL}}\left(p(X,Y)\|p(X)p(Y)\right) \\ &= H(X) + H(Y) - H(X,Y) \\ &= H(X) - H(X|Y). \end{aligned}$$

- Average reduction in uncertainty in $X$ when $Y$ is known
- $I(X;Y) = 0$ when $X, Y$ statistically independent

Conditional mutual information of $X, Y$ given $Z$:

$$I(X;Y|Z) = H(X|Z) - H(X|Y,Z)$$

# This time

Mutual information chain rule

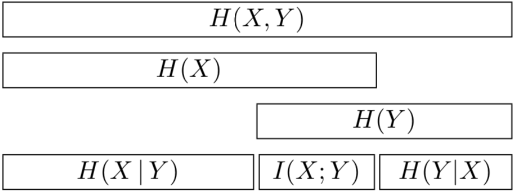
Jensen's inequality

"Information cannot hurt"

Data processing inequality

# Outline

# Breakdown of Joint Entropy




(From Mackay, p140; see his exercise 8.8)

# Recall: Joint Mutual Information

Recall the mutual information between $X$ and $Y$:

$$I(X;Y) = H(X) + H(Y) - H(X,Y) = I(Y;X).$$

We can also compute the mutual information between $X_1,\ldots,X_N$ and $Y_1,\ldots,Y_M$:

$$\begin{aligned} I(X_1,\ldots,X_N;Y_1,\ldots,Y_M) &= H(X_1,\ldots,X_N) + H(Y_1,\ldots,Y_M) \\ &\quad - H(X_1,\ldots,X_N,Y_1,\ldots,Y_M) \\ &= I(Y_1,\ldots,Y_M;X_1,\ldots,X_N). \end{aligned}$$

Note that $I(X,Y;Z) \neq I(X;Y,Z)$ in general

- Reduction in uncertainty of $X$ and $Y$ given $Z$ versus reduction in uncertainty of $X$ given $Y$ and $Z$

# Chain Rule for Mutual Information

Let $X, Y, Z$ be r.v. and recall that:

$$p(Z, Y) = p(Z|Y)p(Y)$$
$$H(Z, Y) = H(Z|Y) + H(Y)$$

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$$I(X; Y, Z) = I(Y, Z; X) \quad \text{symmetry}$$

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I(X; Y, Z) &= I(Y, Z; X) \quad \text{symmetry} \\
&= H(Z, Y) - H(Z, Y|X) \quad \text{definition of mutual info.}
\end{aligned}
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&= H(Z, Y) - H(Z, Y|X) && \text{definition of mutual info.} \\
&= H(Z|Y) + H(Y) - H(Z|X, Y) - H(Y|X) && \text{entropy's chain rule}
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&= H(Z|Y) + H(Y) - H(Z|X, Y) - H(Y|X) \quad \text{entropy's chain rule} \\
&= \underbrace{H(Y) - H(Y|X)}_{I(Y;X)} + \underbrace{H(Z|Y) - H(Z|X, Y)}_{I(Z;X|Y)}
\end{aligned}
$$

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I(X; Y, Z) &= I(X; Y) + I(X; Z|Y) && \text{definition of mutual info and conditional mutual info}
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I(X; Y, Z) &= I(X; Y) + I(X; Z|Y) && \text{definition of mutual info and conditional mutual info}
\end{aligned}
$$

Similarly, by symmetry:

$$I(X; Y, Z) = I(X; Z) + I(X; Y|Z)$$

# Chain Rule for Mutual Information

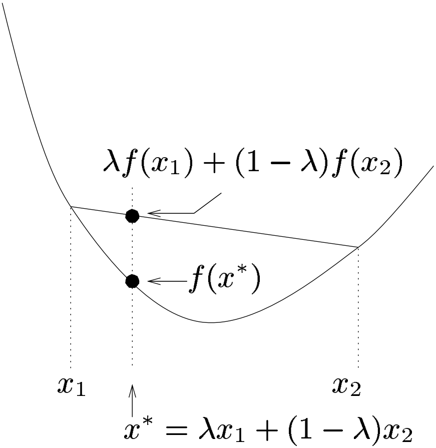
## General form

For any collection of random variables $X_1, \ldots, X_N$ and $Y$:

$$\begin{aligned}
I(X_1, \ldots, X_N; Y) &= I(X_1; Y) + I(X_2, \ldots, X_N; Y|X_1) \\
&= I(X_1; Y) + I(X_2; Y|X_1) + I(X_3, \ldots, X_N; Y|X_1, X_2) \\
&= \ldots \\
&= \sum_{i=1}^{N} I(X_i; Y|X_1, \ldots, X_{i-1}) \\
&= \sum_{i=1}^{N} I(Y; X_i|X_1, \ldots, X_{i-1}).
\end{aligned}$$

# Convex Functions:

## Introduction




$0 \leq \lambda \leq 1$ (Figure from Mackay, 2003)

A function is convex $\smile$ if every chord of the function lies above the function

# Convex and Concave Functions

## Definitions

### Definition

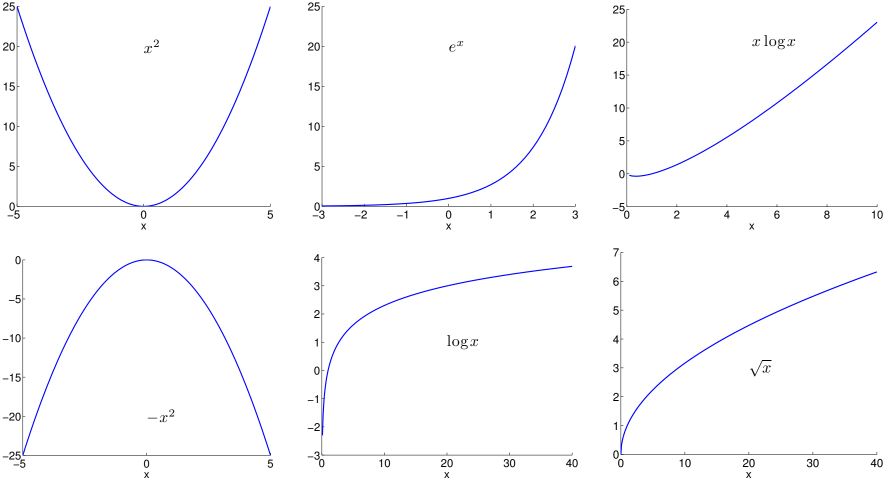
A function $f(x)$ is convex $\smile$ over $(a, b)$ if for all $x_1, x_2 \in (a, b)$ and $0 \leq \lambda \leq 1$:

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$$

We say $f$ is strictly convex $\smile$ if for all $x_1, x_2 \in (a, b)$ equality holds only for $\lambda = 0$ and $\lambda = 1$.

Similarly, a function $f$ is concave $\frown$ if $-f$ is convex $\smile$, i.e. if every chord of the function lies below the function.

# Examples of Convex and Concave Functions



# Verifying Convexity

**Theorem (Cover & Thomas, Th 2.6.1)**

If a function $f$ has a second derivative that is non-negative (positive) over an interval, the function is convex $\smile$(strictly convex $\smile$) over that interval.

*This allows us to verify convexity or concavity.*

Examples:

- $x^2$: $\frac{d}{dx}\left(\frac{d}{dx}(x^2)\right) = \frac{d}{dx}(2x) = 2$
- $e^x$: $\frac{d}{dx}\left(\frac{d}{dx}(e^x)\right) = \frac{d}{dx}(e^x) = e^x$
- $\sqrt{x}, x > 0$: $\frac{d}{dx}\left(\frac{d}{dx}(\sqrt{x})\right) = \frac{1}{2}\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = -\frac{1}{4}\frac{1}{\sqrt{x^3}}$

# Convexity, Concavity and Optimization

If $f(x)$ is concave $\frown$ and there exists a point at which

$$\frac{df}{dx} = 0,$$

then $f(x)$ has a maximum at that point.

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Note: the converse does not hold: if a concave $\frown$ $f(x)$ is maximized at some $x$, it is not necessarily true that the derivative is zero there.

- $f(x) = -|x|$: is maximized at $x = 0$ where its derivative is undefined
- $f(p) = \log p$ with $0 \leq p \leq 1$, is maximized at $p = 1$ where $\frac{df}{dp} = 1$

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- $f(p) = \log p$ with $0 \leq p \leq 1$, is maximized at $p = 1$ where $\frac{df}{dp} = 1$
- Similarly for minimisation of convex functions

# Jensen's Inequality for Convex Functions

**Theorem: Jensen's Inequality**

If $f$ is a convex $\smile$ function and $X$ is a random variable then:

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)].$$

Moreover, if $f$ is strictly convex $\smile$, equality implies that $X = \mathbb{E}[X]$ with probability 1, i.e $X$ is a constant.

In other words, for a probability vector **p**,

$$f\left(\sum_{i=1}^{N} p_i x_i\right) \leq \sum_{i=1}^{N} p_i f(x_i).$$

Similarly for a concave $\frown$ function: $\mathbb{E}[f(X)] \leq f(\mathbb{E}[X])$ .

# Jensen's Inequality for Convex Functions

## Proof by Induction

(1) $K = 2$:

- Two-state random variable $X \in \{x_1, x_2\}$
- With $\mathbf{p} = (p_1, p_2) = (p_1, 1 - p_1)$
- $0 \leq p_1 \leq 1$

we simply follow the definition of convexity:

$$\underbrace{p_1 f(x_1) + p_2 f(x_2)}_{\mathbb{E}[f(X)]} \geq f(\underbrace{p_1 x_1 + p_2 x_2}_{\mathbb{E}[X]})$$

# Jensen's Inequality for Convex Functions

## Proof by Induction — Cont'd

(2) $(K-1) \to K$: Assuming the theorem is true for distributions with $K-1$ states, and writing: $p_i' = p_i/(1-p_K)$ for $i = 1, \ldots, K-1$:

$$\sum_{i=1}^{K} p_i f(x_i) = p_K f(x_K) + (1-p_K) \sum_{i=1}^{K-1} p_i' f(x_i)$$

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$$\sum_{i=1}^{K} p_i f(x_i) = p_K f(x_K) + (1-p_K) \sum_{i=1}^{K-1} p'_i f(x_i)$$

$$\geq p_K f(x_K) + (1-p_K) f\left(\sum_{i=1}^{K-1} p'_i x_i\right)$$ Induction hypothesis

# Jensen's Inequality for Convex Functions

## Proof by Induction — Cont'd

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$$
\begin{aligned}
\sum_{i=1}^{K} p_i f(x_i) &= p_K f(x_K) + (1-p_K) \sum_{i=1}^{K-1} p'_i f(x_i) \\
&\geq p_K f(x_K) + (1-p_K) f\left(\sum_{i=1}^{K-1} p'_i x_i\right) \qquad \text{Induction hypothesis} \\
&\geq f \underbrace{\left(p_K x_K + (1-p_K) \sum_{i=1}^{K-1} p'_i x_i\right)}_{\sum_{i=1}^{K} p_i x_i} \qquad \text{definition of convexity}
\end{aligned}
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# Jensen's Inequality for Convex Functions

## Proof by Induction — Cont'd

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$$\geq f\underbrace{\left(p_K x_K + (1-p_K) \sum_{i=1}^{K-1} p_i' x_i\right)}_{\sum_{i=1}^{K} p_i x_i} \quad \text{definition of convexity}$$

$$\sum_{i=1}^{K} p_i f(x_i) \geq f\left(\sum_{i=1}^{K} p_i x_i\right) \Rightarrow \mathbb{E}[f(X)] \geq f(\mathbb{E}[x]) \quad \text{equality case?}$$

# Jensen's Inequality Example: The AM-GM Inequality

Recall that for a concave $\frown$ function: $\mathbb{E}[f(X)] \leq f(\mathbb{E}[X])$.

Consider $X \in \{x_1, \ldots, x_N\}$, $X \geq 0$ with uniform probability distribution $\mathbf{p} = (\frac{1}{N}, \ldots, \frac{1}{N})$ and the strictly concave $\frown$ function $f(x) = \log x$:

$$\frac{1}{N} \sum_{i=1}^{N} \log x_i \leq \log \left( \frac{1}{N} \sum_{i=1}^{N} x_i \right)$$

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$$\frac{1}{N}\sum_{i=1}^{N} \log x_i \leq \log\left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)$$

$$\log\left(\prod_{i=1}^{N} x_i\right)^{\frac{1}{N}} \leq \log\left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)$$

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$$\left( \prod_{i=1}^{N} x_i \right)^{\frac{1}{N}} \leq \frac{1}{N} \sum_{i=1}^{N} x_i$$

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$$\log\left(\prod_{i=1}^{N} x_i\right)^{\frac{1}{N}} \leq \log\left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)$$

$$\left(\prod_{i=1}^{N} x_i\right)^{\frac{1}{N}} \leq \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$\sqrt[N]{x_1 x_2 \ldots x_N} \leq \frac{x_1 + x_2 \ldots + x_N}{N}$$

# Gibbs' Inequality

**Theorem**

The relative entropy (or KL divergence) between two distributions $p(X)$ and $q(X)$ with $X \in \mathcal{X}$ is non-negative:

$$D_{\text{KL}}(p \| q) \geq 0$$

with equality if and only if $p(x) = q(x)$ for all x.

# Gibbs' Inequality

## Proof (1 of 2)

Recall that: $D_{\mathrm{KL}}(p \| q) = \sum_{x \in \mathcal{X}} p(x) \log \frac{p(x)}{q(x)} = \mathbb{E}_{p(X)} \left[ \log \frac{p(X)}{q(X)} \right]$

Let $\mathcal{A} = \{x : p(x) > 0\}$. Then:

# Gibbs' Inequality

## Proof (1 of 2)

Recall that: $D_{\mathrm{KL}}(p\|q) = \sum_{x\in\mathcal{X}} p(x)\log\frac{p(x)}{q(x)} = \mathbb{E}_{p(X)}\left[\log\frac{p(X)}{q(X)}\right]$

Let $\mathcal{A} = \{x : p(x) > 0\}$. Then:

$$-D_{\mathrm{KL}}(p\|q) = \sum_{x\in\mathcal{A}} p(x)\log\frac{q(x)}{p(x)}$$

# Gibbs' Inequality

## Proof (1 of 2)

Recall that: $D_{\text{KL}}(p \| q) = \sum_{x \in \mathcal{X}} p(x) \log \frac{p(x)}{q(x)} = \mathbb{E}_{p(X)} \left[ \log \frac{p(X)}{q(X)} \right]$

Let $\mathcal{A} = \{x : p(x) > 0\}$. Then:

$$\begin{aligned} -D_{\text{KL}}(p \| q) &= \sum_{x \in \mathcal{A}} p(x) \log \frac{q(x)}{p(x)} \\ &\leq \log \sum_{x \in \mathcal{A}} p(x) \frac{q(x)}{p(x)} \qquad \text{Jensen's inequality} \end{aligned}$$

# Gibbs' Inequality

## Proof (1 of 2)

Recall that: $D_{\mathrm{KL}}(p\|q) = \sum_{x \in \mathcal{X}} p(x) \log \frac{p(x)}{q(x)} = \mathbb{E}_{p(X)}\left[\log \frac{p(X)}{q(X)}\right]$

Let $\mathcal{A} = \{x : p(x) > 0\}$. Then:

$$
\begin{aligned}
-D_{\mathrm{KL}}(p\|q) &= \sum_{x \in \mathcal{A}} p(x) \log \frac{q(x)}{p(x)} \\
&\leq \log \sum_{x \in \mathcal{A}} p(x) \frac{q(x)}{p(x)} && \text{Jensen's inequality} \\
&= \log \sum_{x \in \mathcal{A}} q(x)
\end{aligned}
$$

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## Proof (1 of 2)

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&\le \log\sum_{x\in\mathcal{A}} p(x)\frac{q(x)}{p(x)} \qquad \text{Jensen's inequality} \\
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&\le \log\sum_{x\in\mathcal{X}} q(x)
\end{aligned}$$

# Gibbs' Inequality

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-D_{\mathrm{KL}}(p\|q) &= \sum_{x\in\mathcal{A}} p(x)\log\frac{q(x)}{p(x)} \\
&\leq \log\sum_{x\in\mathcal{A}} p(x)\frac{q(x)}{p(x)} && \text{Jensen's inequality} \\
&= \log\sum_{x\in\mathcal{A}} q(x) \\
&\leq \log\sum_{x\in\mathcal{X}} q(x) \\
&= \log 1
\end{aligned}$$

# Gibbs' Inequality

## Proof (1 of 2)

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Let $\mathcal{A} = \{x : p(x) > 0\}$. Then:

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\begin{aligned}
-D_{\mathrm{KL}}(p\|q) &= \sum_{x\in\mathcal{A}} p(x)\log\frac{q(x)}{p(x)} \\
&\leq \log\sum_{x\in\mathcal{A}} p(x)\frac{q(x)}{p(x)} && \text{Jensen's inequality} \\
&= \log\sum_{x\in\mathcal{A}} q(x) \\
&\leq \log\sum_{x\in\mathcal{X}} q(x) \\
&= \log 1 \\
&= 0
\end{aligned}
$$

# Gibbs' Inequality

## Proof (2 of 2)

Since $\log u$ is strictly convex we have equality if $\dfrac{q(x)}{p(x)} = c$ for all $x$. Then:

$$\sum_{x\in\mathcal{A}} q(x) = c \sum_{x\in\mathcal{A}} p(x) = c$$

Also, the last inequality in the previous slide becomes equality only if:

$$\sum_{x\in\mathcal{A}} q(x) = \sum_{x\in\mathcal{X}} q(x).$$

Therefore $c = 1$ and $D_{\mathrm{KL}}(p\|q) = 0 \Leftrightarrow p(x) = q(x)$ for all $x$.

Alternative proof: Use the fact that $\log x \leq x - 1$.

# Non-Negativity of Mutual Information

**Corollary**

For any two random variables $X$, $Y$:

$$I(X;Y) \geq 0,$$

with equality if and only if $X$ and $Y$ are statistically independent.

Proof: We simply use the definition of mutual information and Gibbs' inequality:

$$I(X;Y) = D_{\mathrm{KL}}(p(X,Y) \| p(X)p(Y)) \geq 0,$$

with equality if and only if $p(X,Y) = p(X)p(Y)$, i.e. $X$ and $Y$ are independent.

# Conditioning Reduces Entropy

## Information Cannot Hurt — Proof

### Theorem

For any two random variables $X, Y$,

$$H(X|Y) \leq H(X),$$

with equality if and only if $X$ and $Y$ are independent.

# Conditioning Reduces Entropy

## Information Cannot Hurt — Proof

### Theorem

For any two random variables $X$, $Y$,

$$H(X|Y) \leq H(X),$$

with equality if and only if $X$ and $Y$ are independent.

Proof: We simply use the non-negativity of mutual information:

$$\begin{aligned} I(X;Y) &\geq 0 \\ H(X) - H(X|Y) &\geq 0 \\ H(X|Y) &\leq H(X) \end{aligned}$$

with equality if and only if $p(X,Y) = p(X)p(Y)$, i.e $X$ and $Y$ are independent.

Data are helpful, they don't increase uncertainty on average.

# Conditioning Reduces Entropy

## Information Cannot Hurt — Example (from Cover & Thomas, 2006)

Let $X$, $Y$ have the following joint distribution:

| $p(X, Y)$ | | $X = 1$ | $X = 2$ |
|---|---|---|---|
| $Y$ | 1 | 0 | 3/4 |
| | 2 | 1/8 | 1/8 |

# Conditioning Reduces Entropy

## Information Cannot Hurt — Example (from Cover & Thomas, 2006)

Let $X$, $Y$ have the following joint distribution:

| $p(X, Y)$ | | $X$ = 1 | $X$ = 2 |
|---|---|---|---|
| $Y$ | 1 | 0 | 3/4 |
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$$\mathbf{p}(X) = (1/8, 7/8)$$
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## Information Cannot Hurt — Example (from Cover & Thomas, 2006)

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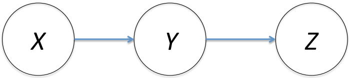
Information cannot hurt on average

# Markov Chain

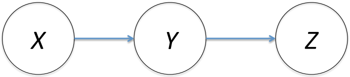



## Definition

Random variables $X, Y, Z$ are said to form a Markov chain in that order (denoted by $X \to Y \to Z$) if their joint probability distribution can be written as:

$$p(X, Y, Z) = p(X)p(Y|X)p(Z|Y) = p(Z|Y)p(Y|X)p(X)$$

# Markov Chain

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Consequences (prove these facts!):

- $X \to Y \to Z$ if and only if $X$ and $Z$ are conditionally independent given $Y$.
- $X \to Y \to Z$ implies that $Z \to Y \to X$.
- If $Z = f(Y)$, then $X \to Y \to Z$

□ $X \to Y \to Z$ ($X$, $Y$ and $Z$ form a Markov chain.

$\therefore P(X, Y, Z) = P(Z|Y)\,P(Y|X)\,P(X)$ — (1)

---

□ $X$ and $Z$ are conditionally independent given $Y$

$\Rightarrow P(X, Z|Y) = P(X|Y)\,P(Z|Y)$ — (2)

---

Start with (2)

$$P(X, Z|Y) = \frac{P(X, Z, Y)}{P(Y)} \quad \text{— (3)}$$

(2), (3) $\Rightarrow$

$$P(X, Z, Y) = P(Z|Y)\,\underbrace{P(Y)\;P(X|Y)}_{P(Y|X)\,P(X)}$$

$$= P(Z|Y)\,P(Y|X)\,P(X)$$

Same as (1).

**Theorem**

if $X \rightarrow Y \rightarrow Z$ then: $I(X; Y) \geq I(X; Z)$

- $X$ is the state of the world, $Y$ is the data gathered and $Z$ is the processed data
- No “clever” manipulation of the data can improve the (best-possible) inferences that can be made from the data
- No processing of $Y$, deterministic or random, can increase the information that $Y$ contains about $X$

# Data-Processing Inequality

## Proof

Recall that the chain rule for mutual information states that:

$$\begin{aligned} I(X;Y,Z) &= I(X;Y) + I(X;Z|Y) \\ &= I(X;Z) + I(X;Y|Z) \end{aligned}$$

Therefore:

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Therefore:

$$I(X;Y) + \underbrace{I(X;Z|Y)}_{0} = I(X;Z) + I(X;Y|Z)$$ Markov chain assumption

$$I(X; Z|Y) = H(Z|Y) - H(Z$$

$$= E\left\{ \log_2 \frac{P(X, Z|Y)}{P(X|Y)\, P(Z|Y)} \right\}$$

$$= E\left[ \log_2 \frac{P(X, Z, Y)}{\underbrace{P(Y)\, P(X|Y)}_{P(Y|X)\,P(X)}\, P(Z|Y)} \right]$$

$$= E\left[ \log_2 \frac{P(X, \cancel{Z}Y, Z)}{\underbrace{P(Z|Y)\, P(Y|X)\, P(X)}_{P(X, Y, Z)}} \right\}$$

$$= E\{ \underbrace{\log_2 1}_{0} \}$$

$$= 0$$

$\therefore$ For $X \to Y \to Z$

$$I(X; Z|Y) = 0$$

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$$I(X;Y) = I(X;Z) + I(X;Y|Z) \quad \text{but } I(X;Y|Z) \geq 0$$

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I(X;Y) + \underbrace{I(X;Z|Y)}_{0} &= I(X;Z) + I(X;Y|Z) && \text{Markov chain assumption} \\
I(X;Y) &= I(X;Z) + I(X;Y|Z) && \text{but } I(X;Y|Z) \geq 0 \\
I(X;Y) &\geq I(X;Z)
\end{aligned}
$$

# Data-Processing Inequality

## Functions of the Data

**Corollary**

*In particular*, if $Z = g(Y)$ we have that:

$$I(X;Y) \geq I(X;g(Y))$$

Proof: $X \rightarrow Y \rightarrow g(Y)$ forms a Markov chain.

Functions of the data $Y$ cannot increase the information about $X$

# Data-Processing Inequality

## Observation of a "Downstream" Variable

**Corollary**

If $X \to Y \to Z$ then $I(X;Y|Z) \leq I(X;Y)$

Proof: We again use the chain rule for mutual information:

$$\begin{aligned} I(X;Y,Z) &= I(X;Y) + I(X;Z|Y) \\ &= I(X;Z) + I(X;Y|Z) \end{aligned}$$

Therefore:

$$I(X;Y) + \underbrace{I(X;Z|Y)}_{0} = I(X;Z) + I(X;Y|Z) \quad \text{Markov chain assumption}$$

$$I(X;Y|Z) = I(X;Y) - I(X;Z) \quad \text{but } I(X;Z) \geq 0$$

$$I(X;Y|Z) \leq I(X;Y)$$

The dependence between $X$ and $Y$ cannot be increased by the observation of a "downstream" variable.

# Summary & Conclusions

- Chain rule for mutual information
- Convex Functions
- Jensen's inequality, Gibbs' inequality
- Important inequalities regarding information, inference and data processing
- Reading: Mackay §2.6 to §2.10, Cover & Thomas §2.5 to §2.8

# Next time

- Law of large numbers

- Markov's inequality

- Chebychev's inequality

# Acknowledgement

These slides were originally developed by Professor Robert C. Williamson.