MATH1005/MATH6005 Semester 1 2021

Assignment 4

Workshop Details:

Number	Day	Time	Demonstrator name
16 B	**Friday**	5pm 01/04	Cai Yang

Student Details:

ID	Surname	Given name	Preferred name
u7235649	Zhang	Han	

Instructions:

This assignment has four questions. Write your solutions in the spaces provided. Hand-writing is preferable to typesetting unless you are fast and accurate with LaTeX, and even then typesetting will take you longer. Also, be aware that typesetting will not be allowed on exams.

Except for multiple-choice questions, or where answer boxes are provided (one question of this type each week), **show all working**.

Question numbers indicate the corresponding questions from the workshop. Since the workshop has six questions, questions on this assignment may not be sequentially numbered.

Declaration:

I declare that while I may have discussed some or all of the questions in this assignment with other people, the write-up of my answers herein is entirely my own work. I have not copied or modified the written-out answers of anyone else, nor allowed mine to be so used.

Signature:.... Han Zhang.......................... Date: 04/04/2021

This document must be submitted by 11pm on the THURSDAY following your workshop.

Sign, date, then scan this completed document (5 pages) and save as a pdf file with name format uXXXXXXXAssXX.pdf (*e.g.* u6543210Ass01.pdf).
Upload the file via the link from which you downloaded this document.

If copying is detected, and/or the document is not signed, no marks will be awarded.

This document has five pages in total.

Question 2$^+$ A sequence $(a_n)_{n\in\mathbb{N}^*}$ defined implicitly by

$$a_0 = 1,\ a_1 = 2, \qquad \forall n \in \mathbb{N} \quad a_{n+1} = a_n + 2a_{n-1}$$

(The equation $a_{n+1} = a_n + 2a_{n-1}$ is called a *second order recurrence relation*, reflecting the fact that the value of each term depends on the values of the previous **two** terms.) By supplying the missing bits (a)–(h), complete the proof below that, for this sequence, every a_n is given by the formula $a_n = 2^n$.

Basis Step: For $n = 0$ the formula gives $a_0 = 2^0 = 1$ which agrees with the definition of (a_n).
For $n = 1$ the formula gives ... (a) ... which also agrees with the definition of (a_n).

Inductive Step: Assume the formula holds up to and including some fixed value of ... (b) ... $\in \mathbb{N}$. Then

$$\begin{aligned} a_{n+1} &= \ldots(\text{c})\ldots \text{ by definition of } a_n \\ &= \ldots(\text{d})\ldots \text{ by } \ldots(\text{e})\ldots \\ &= 2^n + 2^n = \ldots(\text{f})\ldots. \end{aligned}$$

So the formula holds for... (g) .... By mathematical this proves that the formula holds ... (h) ....

(a) $a_1 = 2^1 = 2$

(b) n

(c) $a_n + 2a_{n-1}$

(d) $2^n + 2 \times 2^{n-1}$

(e) the inductive assumption.

(f) 2^{n+1}

(g) $n+1$

(h) for all $n \in N$.

Question 3† Write the correct values in the boxes.

For this question, working is not required and will not be marked.

(a) The sum of the arithmetic series $11 + 14 + 17 + \ldots + 71$ is $\boxed{861}$.

(b) Rounded to the nearest integer,

the value of the 10th term of the geometric sequence $4, 6, 9, 13.5, \ldots$ is $\boxed{154}$.

(c) Rounded to the nearest integer,

the sum of the first 10 terms of the geometric sequence $4, 6, 9, 13.5, \ldots$ is $\boxed{\cancel{455}}$. 453

(d) A mixed geometric-arithmetic sequence has $a_0 = 6$, multiplier $1/3$, and offset 3.

The exact value, expressed as a fraction, of a_5 is $\boxed{\frac{122}{27}}$.

(e) A mixed geometric-arithmetic sequence has $a_0 = 6$, multiplier $1/3$, and offset 3.

The exact steady state value, expressed as a fraction, of this sequence is $\boxed{}$.

(f) Given that $$\sum_{k=1}^{n}(2k-1)^2 = \frac{n(4n^2-1)}{3},$$

the sum of all odd squares from 1^2 to 99^2 is $\boxed{166650}$.

Question 4*

(a) For $r \neq 1$ and any d a sequence $(a_n)_{n\in\{0,\ldots,N\}}$ is defined implicitly by

$$a_0 = a, \qquad \forall n \in \mathbb{N}^* \quad n < N \Rightarrow a_{n+1} = ra_n + d.$$

Below is a proof by mathematical induction of the explicit formula

$$\forall n \in \{0,\ldots,N\} \quad a_n = r^n a + d\left(\frac{r^n - 1}{r-1}\right).$$

Using the grid at the end, supply the missing bits in this proof.
[The blanks are deliberately all of the same length; but the answers are not.]

Basis step: For $n = \ldots(\text{a})\ldots$ the formula gives

$$\ldots(\text{b})\ldots = \ldots(\text{c})\ldots = a,$$

agreeing with …(d)….

Inductive step: Assume …(e)… up to and including some fixed value of …(f)…. Then for the next value, …(g)…, we have

$$\begin{aligned}
\ldots(\text{h})\ldots &= \ldots(\text{i})\ldots && \text{(from the } \ldots(\text{j})\ldots) \\
&= \ldots(\text{k})\ldots && \text{(by the } \ldots(\text{l})\ldots) \\
&= r^{n+1}a + d\left(\frac{r^{n+1} - r}{r-1} + 1\right) \\
&= r^{n+1}a + d\left(\frac{\ldots(\text{m})\ldots}{r-1}\right) \\
&= \ldots(\text{n})\ldots
\end{aligned}$$

And so the formula also holds for …(o)….

(a) 0	(b) a_0	(c) $r^0 a + d(\frac{r^0-1}{r-1})$
(d) the definition of (a_n)	(e) the formula holds	(f) $n \in N^*$
(g) $n+1$	(h) a_{n+1}	(i) $ra_n + d$
(j) definition of a_n	(k) $r\cdot(r^n a + d(\frac{r^n-1}{r-1})) + d$	(l) inductive assumption
(m) $r^{n+1} - r + r - 1$	(n) $r^{n+1}a + d(\frac{r^{n+1}-1}{r-1})$	(o) all $n \in N^*$

(b) I have inherited \$10 000. I have put it in the bank and I will add \$100 per month, starting at the end of the first month. The bank pays 4.5%p.a. interest, fixed for 5 years and paid into the account monthly. What will my investment be worth at the end of the 5 years? [Use the formula of part (a).]

$a_0 = 10000$, $r = 1+0.045/12 = 1.00375$, $d = 100$, $n = 12\times 5 = 60$

$$a_n = 1.00375^n \times 10000 + 100 \times \left(\frac{1.00375^n - 1}{1.00375 - 1}\right)$$

$$a_{60} = 1.00375^{60} \times 10000 + 100 \times \frac{1.00375^{60} - 1}{0.00375} \approx 13077.50$$

Question 6$^{\#}$ In lectures we saw how use the Merge sort algorithm to sort a sequence of length $n = 2^r$ into ascending order. In fact the algorithm can be applied to sequences of any length $n \in \mathbb{N}$. At each iteration the current sorted sub-sequences are merged in pairs as for the 2^r case but if there are an odd number of sub-sequences then the 'left over' one just joins, unchanged, the newly formed sub-sequences at the next iteration. This will mean that the merge algorithm will sometimes need to merge sequences of unequal lengths, but this causes no problems.

For example, if Merge sort is used to sort the letters of the word `PROVISIONAL` into alphabetical order then the subsequences at each stage will be:

after 0th iteration	`(P),(R),(O),(V),(I),(S),(I),(O),(N),(A),(L);`
after 1st iteration	`(P,R),(O,V),(I,S),(I,O),(A,N),(L);`
after 2nd iteration	`(O,P,R,V),(I,I,O,S),(A,L,N);`
after 3rd iteration	`(I,I,O,O,P,R,S,V),(A,L,N);`
after 4th iteration	`(A,I,I,L,N,O,O,P,R,S,V).`

(a) Apply the Merge sort algorithm to sort the letters of the word **STEREOPHOTOGRAPHIC** into alphabetical order, showing the results of each iteration as in the example above.

(b) How many comparison operations are used to merge sort **STEREOPHOTOGRAPHIC**? As in Worksheet Q5, remember that when the merge algorithm reaches the stage where one of its input lists is empty, it does not need any more comparisons to complete its task. For example, for `PROVISIONAL` there are only 5 comparisons during the first iteration, 8 in the 2nd, 7 in the 3rd and 5 in the last.

(c) How many comparison operations would be used if **STEREOPHOTOGRAPHIC** were sorted using the Selection sort algorithm?

(a) after 0st: S.T.E.R.E.O.P.H.O.T.O.G.R.A.P.H.I.C ;

after 1st: (S,T), (E,R), (E,O), (H,P), (O,T), (G,O), (A,R), (H,P), (C,I);

after 2nd: (E,R,S,T), (E,H,O,P), (G,O,O,T), (A,H,P,R), (C,I);

after 3rd: (E,E,H,O,P,R,S,T), (A,G,H,O,O,P,R,T), (C,I);

after 4th: (A,E,E,G,H,H,O,O,O,P,P,R,R,S,T,T), (C,I);

after 5th: (A,C,E,E,G,H,H,I,O,O,O,P,P,R,R,S,T,T);

(b) $9+(2+3+3+3)+(4+7)+15+8=54$

(c) 75