C1. Counting.

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Text Reference (Epp)	3ed: Sections	6.1-7, 7.3
	4ed: Sections	9.1-7
	5ed: Sections	9.1-7

Principles of counting

Bijections preserve cardinality If A and B are finite sets and there exists a bijection $f : A \to B$, then $|A| = |B|$.
TO USE THIS PRINCIPLE: Count something easier, and exhibit a bijection between the set you which to count and the set you have counted.

The Pigeonhole Principle If $k + 1$ or more pigeons occupy k pigeonholes, then at least one pigeonhole must contain two or more pigeons.

The Generalised Pigeonhole Principle If N objects are classified in k disjoint categories, then at least one category must contain $\lceil \frac{N}{k} \rceil$ objects. ($\lceil \frac{N}{k} \rceil$ means the least integer that is greater than or equal to $\frac{N}{k}$)

Permutations There are $n!$ ways to arrange n distinct objects in a list.

Principles of counting

r-Permutations There are

$$P(n, r) = \frac{n!}{(n-r)!}$$

ways to select and order r out of n distinct objects.

Combinations There are

$$C(n, r) = \binom{n}{r} = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}$$

ways to choose a set of r objects from a set of n objects (that is, to select r out of n distinct objects when the order in which objects are selected in not important). The notation $\binom{n}{r}$ is read "n choose r."

Principles of counting

Multisets (Stars and Bars) There are $\binom{r+n-1}{r}$ size-r multisets with members from a set of size n. That is, there are $\binom{r+n-1}{r}$ ways to arrange a list of r stars and $n-1$ bars.

Inclusion-Exclusion If A and B are finite sets, then $|A \cup B| = |A| + |B| - |A \cap B|$.

The Sum Rule If A is a finite set and $\{A_1, A_2, \ldots, A_m\}$ is a partition of A, then $|A| = |A_1| + |A_2| + \cdots + |A_m|$.

The Product Rule If $A_1, A_2, \ldots, A_m$ are finite sets, then $|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \times |A_2| \times \cdots \times |A_m|$.

A mixed example

How many distinguishable ways can the letters of the word

MILLIMICRON

be arranged?

If we were to distinguish between like letters using labels, as in

$$\mathrm{M_1 I_1 L_1 L_2 I_2 M_2 I_3 C\,R\,O\,N}$$

there would be $11! = 39\,916\,800$ different arrangements.

Now we must compensate for the over-counting induced by this distinguishing between indistinguishable arrangements.

Since MILLIMICRON has 2 M's, 3 I's and 2 L's the true answer is :

$$\frac{11!}{2!\,3!\,2!} = 1\,663\,200.$$

This example generalises both permutations and combinations.
Can you see how?

'Stars and Bars'

Let s and b be positive integers. How many different strings of length $s+b$ can we form out of s stars ($\star$) and b bars ($|$)? For example, here are three different strings of length 13 formed out of 10 stars and 3 bars:

$$\mid \ \star \ \star \ \star \ \star \ \mid \ \star \ \star \ \mid \ \star \ \star \ \star \ \star$$
$$\star \ \mid \ \star \ \star \ \star \ \star \ \star \ \star \ \star \ \mid \ \mid \ \star \ \star$$
$$\star \ \star \ \star \ \mid \ \star \ \star \ \mid \ \star \ \star \ \mid \ \star \ \star \ \star$$

ANSWER: Consider a horizontal arrangement of cells numbered 1 through $s+b$.

1	2	3	...	$s+b$
			...	

Making a string of s stars and b bars is like choosing a set of b numbers from the set $\{1, 2, \ldots, s+b\}$, where the set of chosen numbers is the set of positions containing bars. Hence we can form $\binom{s+b}{b}$ different strings of length $s+b$ from s stars and b bars.

'Stars and Bars' example

(Epp(4ed) Q9.6.15)

For how many integers from 1 through 99 999 is the sum of their digits equal to 10?

Proof: By inserting leading zeros if necessary, all the integers to be counted can be considered to be 5-digit strings $abcde$ with $a+b+c+d+e=10$.

Each of these 5-digit strings can be represented as a length-14 pattern of 10 **stars** and 4 **bars**. For example:

$$\begin{array}{lll} \star\star|\star\star\star\star|\star|\star\star|\star & \text{represents} & 24\,121. \\ \star\star\star||\star\star\star\star|\star\star\star| & \text{represents} & 30\,430. \\ |||\star\star\star\star\star\star\star\star\star|\star & \text{represents} & 00\,091. \end{array}$$

There are $\binom{14}{4}$ ways to arrange a list of 10-stars and 4-bars. But five of the patterns have ten stars in a row and so don't count (ten is not a digit). So the number of integers is

$$\binom{14}{4} - 5 = \frac{14\times13\times12\times11}{4\times3\times2\times1} - 5 = 996. \square$$

Counting 'Multisets'

What is a 'multiset'?

It's a 'set' with multiple copies of elements allowed and acknowledged.
An example is $\{c, b, a, c, a\}$, which has 2 a's, 1 b and 2 c's.
As for ordinary sets, order is irrelevant: $\{c, b, a, c, a\} = \{a, a, b, c, c\}$.
But the multiplicities **do** matter.

Formally, a **size-r multiset** is a set S together with a 'multiplicity function' $m : S \to \mathbb{N}$, where,

$\forall s \in S \quad m(s) = \text{number of copies of } s$ and $r = \sum_{s \in S} m(s)$.

So, for example, $\{c, b, a, c, a\}$ has size $r = 2 + 1 + 2 = 5$.

Counting multisets

How many different size-r multisets can be formed from members of a set S of cardinality n?
IDEA: A size r-multiset formed from members of a set S of cardinality n can be represent by a pattern of r stars and $n-1$ bars.

For example if $S = \{a, b, c, d\}$ and $r = 5$ then $\{c, b, a, c, a\}$ is represented by $\star\star|\star|\star\star|$ $(m(a)=2, m(b)=1, m(c)=2, m(d)=0)$.

Which multisets, selectected from S, are represented by the following arrangements?

$\|\|\star\star\star\|\star\star$	$\|\star\star\star\|\star\star\|$	$\star\star\|\star\|\star\|\star$
$\{c, c, c, d, d\}$	$\{b, b, b, c, c\}$	$\{a, a, b, c, d\}$

There are $\binom{r+n-1}{r}$ size-r multisets with members from a set of size n.

Multisets example

(Epp(4ed) Q9.6.6)

If n is a positive integer, how many 5-tuples of integers from 1 through n can be formed in which the elements of the 5-tuple are written in non-increasing order?

For $n = 9$ some 5-tuples are $(8, 6, 4, 2, 1)$, $(9, 3, 3, 2, 2)$, $(6, 6, 6, 6, 6)$.

There is a bijection (one-to-one correspondence) between the set of all these 5-tuples and the set of all size-5 multisets chosen from $\{1, \ldots, n\}$, because the r 'members' of the multiset can only be arranged in one way in non-increasing order.

So by stars-and-bars, there are $\binom{5+n-1}{5} = \binom{n+4}{5}$ of these 5-tuples.

For example for $n = 3$ there are $\binom{7}{5} = \binom{7}{2} = 21$ such 5-tuples:

33333 33332 33331 33322 33321 33311 33222 33221 33211 33111
32222 32221 32211 32111 31111 22222 22221 22211 22111 21111 11111

New counts from old

- The Sum Rule
- The Product Rule
- Inclusion-Exclusion

The Sum Rule

If sets A and B are finite and *disjoint* then the cardinality of their union $A \cup B$ is the sum of the their individual cardinalities, i.e.

$$\boxed{A \cap B = \emptyset \implies |A \cup B| = |A| + |B|.}$$

More generally, if $\{A_1, A_2, ..., A_m\}$, $m \in \mathbb{N}$, is a *partition* of the finite set A then

$$\boxed{|A| = |A_1| + |A_2| + ... + |A_m|.}$$

Example:

For $U = \{-10, \ldots, 10\} \subseteq \mathbb{Z}$ and $S = \{n \in U \ : \ |20 - n^2| > 10\}$, find $|S|$.

Observe that $|20 - n^2| > 10 \iff n^2 < 10 \ \vee \ n^2 > 30$.

So $S = \{-3, -2, \ldots, 2, 3\} \cup \{6, 7, \ldots, 10\} \cup \{-10, -9, \ldots, -6\}$.

Hence $|S| = 7 + 5 + 5 = 17$.

The Product Rule

For finite sets A and B the cardinality of their cartesian product $A \times B$ is the product of the their individual cardinalities, i.e.

$$\boxed{|A \times B| = |A| \times |B|.}$$

More generally, for finite sets $A_1, A_2, ..., A_m$, $m \in \mathbb{N}$,

$$\boxed{|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \times |A_2| \times \cdots \times |A_m|.}$$

Example:

A regular **cubic die, D6,** has face set $C = \{1, 2, 3, 4, 5, 6\}$.

An **octahedral die, D8,** has face set $O = \{1, 2, \ldots, 8\}$.
A **dodecahedral die, D12,** has face set $D = \{1, 2, \ldots, 12\}$.

The number of possible outcomes from throwing the three dice together is $|C \times O \times D| = |C| \times |O| \times |D| = 6 \times 8 \times 12 = 576.$

The product rule via an outcome construction procedure

Often the product rule is implemented by designing a procedure by which we construct one of the objects to be counted, and then counting the number of different possible outcomes from each step of the procedure. When you do this you must be careful that each outcome can be constructed in only one way.

Suppose that a web-banking password is always 8 characters long and it always comprises two upper case letters, one digit, and 5 lower case characters. How many different passwords can be created that follow these rules?

Counting passwords

A: We construct a password in 11 steps.
In step 1, we choose the two positions in which the upper case letters will be placed. There are $\binom{8}{2}$ ways to make this choice.

In step 2, we choose one integers from the remaining 6 positions. This are $\binom{6}{1}$ ways to make this choice.

In step 3, we choose an upper case letter to go in the first place for an upper case letter. There are 26 ways to make this choice.

In step 4, we choose an upper case letter to go in the second place for an upper case letter. There are 26 ways to make this choice.

In step 5, we choose a digit to go in the digit place. There are 10 ways to make this choice.

In step 6, we choose a lower case letter to go in the first place for a lower case letter. There are 26 ways to make this choice.

Couning passwords (cont.)

In step 7, we choose a lower case letter to go in the first place for a lower case letter. There are 26 ways to make this choice.

In step 8, we choose a lower case letter to go in the first place for a lower case letter. There are 26 ways to make this choice.

In step 9, we choose a lower case letter to go in the first place for a lower case letter. There are 26 ways to make this choice.

In step 10, we choose a lower case letter to go in the first place for a lower case letter. There are 26 ways to make this choice.

In step 11, we choose a lower case letter to go in the first place for a lower case letter. There are 26 ways to make this choice.

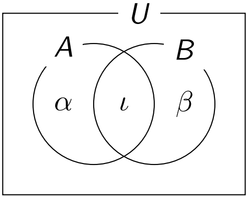
By the product rule, we can create

$$\binom{8}{2} \times \binom{6}{1} \times 26 \times 26 \times 10 \times 26^6$$

different passwords.$\square$

Inclusion-Exclusion

If A and B are finite sets which *may not be disjoint* the sum rule has to be modified to the **inclusion-exclusion rule:**



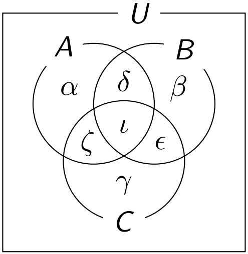
$$\boxed{|A \cup B| = |A| + |B| - |A \cap B|}$$

This is because the plain sum rule $|A \cup B| = |A| + |B|$ includes the intersection $A \cap B$ twice, so it has to be excluded once:

$$|A \cup B| = \alpha + \iota + \beta = (\alpha + \iota) + (\iota + \beta) - \iota = |A| + |B| - |A \cap B|.$$

The inclusion-exclusion rule can be generalised to deal with more than two sets, but it quickly gets very messy.

Can you figure out how to extend the rule to deal with just three sets A, B and C?



Bit-String Example of Inclusion-Exclusion

How many bytes start with '1' or end with '00'?

IDEA: Count the bytes that start with 1, then count the bytes that end in 00, then count the bytes that start with 1 and end with 0, then apply Inclusion-Exclusion.

Task 1: Construct a byte that starts with '1'.

- There is one way to choose the first bit (1)
- There are two ways to choose the second bit (0 or 1)
- There are two ways to choose the third bit (0 or 1)

$\vdots$

- There are two ways to choose the eighth bit (0 or 1)

Product Rule: Task 1 can be done in $1 \times 2^7 = 128$ ways.

Bit-String Example of Inclusion-Exclusion

Task 2

Task 2: Construct a byte that ends with ‘00’.

- There are two ways to choose the first bit (0 or 1)
- There are two ways to choose the second bit (0 or 1)

  ⋮

- There are two ways to choose the sixth bit (0 or 1)
- There is one way to choose the seventh bit (0)
- There is one way to choose the eighth bit (0)

Product Rule: Task 2 can be done in $2^6 \times 1^2 = 64$ ways.

Bit-String Example of Inclusion-Exclusion

Task 3

Is the answer $128+64 = 196$? **NO!**

That would be overcounting.
We have to subtract off the cases we counted twice.

Task 3: Construct a string of length 8 that both starts with '1' and ends with '00'.

- There is one way to choose the first bit (1)
- There are two ways to choose the second bit (0 or 1)

$\vdots$

- There are two ways to choose the sixth bit (0 or 1)
- There is one way to choose the seventh bit (0)
- There is one way to choose the eighth bit (0)

Product Rule: Task 3 can be done in $1 \times 2^5 \times 1^2 = 32$ ways.

Bit-String Example of Inclusion-Exclusion

Conclusion

Finally, the number of ways to construct a bit string of length 8 that starts with '1' or ends with '00' is equal to:

- the number of ways to do task 1, **plus**
- the number of ways to do task 2, **minus**
- the number of ways to do both at the same time (task 3), *i.e.*

$$128 + 64 - 32 = 160.$$

Note: An alternative, and quite different, way to solve this problem is to use **complementary counting**; *i.e.* calculate $|S^c|$ where S is the set of strings we are interested in and U is the set of all (8-bit) strings. Then $|S| = |U| - |S^c| = 2^8 - 1 \times 2^5 \times 3 = 256 - 96 = 160$.

Can you see how to get the $1 \times 2^5 \times 3$?

Combinations and binomial coefficients

There are $\boldsymbol{C(n,r)} = \dfrac{P(n,r)}{r!} = \dfrac{n!}{r!(n-r)!}$ ways to choose a set of r objects from a set of n candidates.
I.e. A set of cardinality n has $\frac{n!}{r!(n-r)!}$ subsets of cardinality r.

The subsets are called r-**combinations**.

We say 'n choose r' for $C(n,r)$ and often write it $\dbinom{n}{r}$.

These numbers $\binom{n}{r}$ arise as coefficients in the algebraic expansion of the n-th power of the 'binomial' $(x+y)$ and are consequently also known as **binomial coefficients**. The expansion is

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r.$$

Two important properties of $\binom{n}{r}$

1. $\forall r, n \in \mathbb{N}^\star \quad 0 \leq r \leq n \implies$ $\boxed{\binom{n}{r} = \binom{n}{n-r}.}$ *e.g.* $\binom{5}{3} = \binom{5}{2}$.

   Proof: Choosing the r elements of a subset S of U, with $|U| = n$, is exactly equivalent to choosing the $n-r$ elements of U to be left out.

2. $\forall r, n \in \mathbb{N}^\star \quad 0 < r \leq n \implies$ $\boxed{\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.}$
   e.g. $\binom{5}{3} = \binom{4}{3} + \binom{4}{2}$.
   (Pascal's Triangle Identity)

   Proof: Let u be a fixed member of U, with $|U| = n$.
   Subsets S of U with $|S| = r$ are of two types; those that don't contain u and those that do.
   There are $\binom{n-1}{r}$ of the first kind, since the r members of S are chosen from the $n-1$ members of $U \backslash u$.
   There are $\binom{n-1}{r-1}$ of the second kind, since the $r-1$ members of $S \backslash u$ are also chosen from the $n-1$ members of $U \backslash u$.

END OF SECTION C1