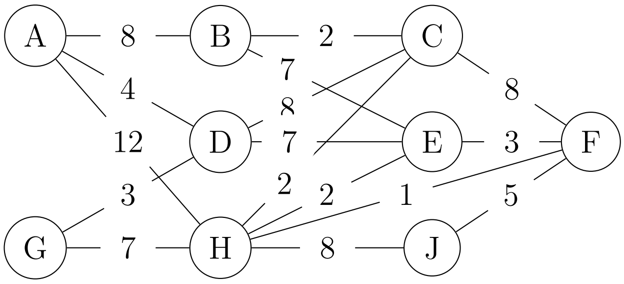
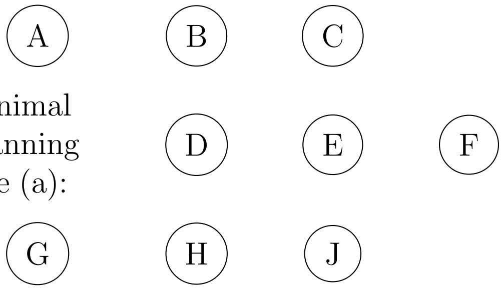
With all question sets, but *especially* with this one, you will gain much more by attempting the questions *before* looking at the solutions.

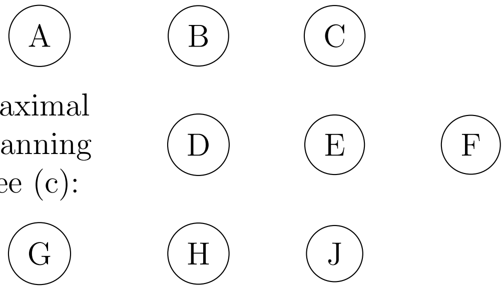
1. Let G be the weighted graph at right:
(a) Find a minimal spanning tree for G.
(b) How many minimal spanning trees does G have? Justify your answer.
(c) Find a maximal spanning tree for G.
(d) How many maximal spanning trees does G have? Justify your anser.



Minimal spanning tree (a):



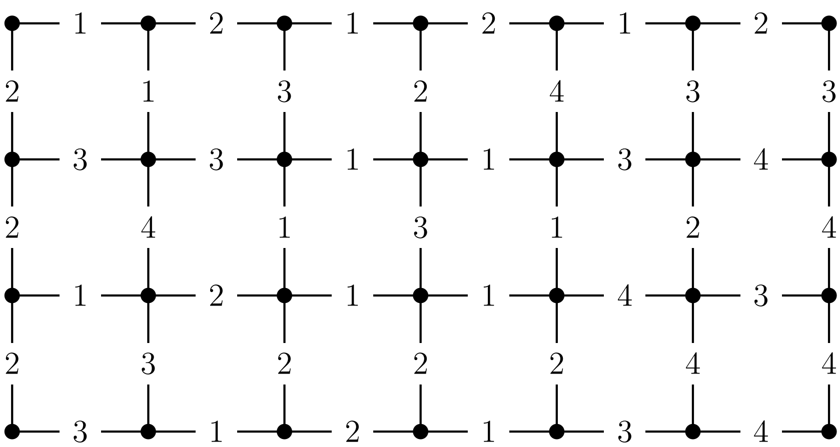
Maximal spanning tree (c):



(b)

(d)

2. Use Kruskal's algorithm to find a minimal spanning tree for the network below.
Of course there are are lots of minimal spanning trees. To get the same answer as me (so you can verify your answer) proceed as follows, in all steps working from left to right:
1. Mark all edges of weight 1 in the top line.
2. Mark all vertical edges of weight 1 between the top line and line 2 below.
3. Mark each edge of weight 1 in line 2 unless including the edge creates a circuit.
4. Continue like this, working across and down, until you reach the bottom right corner.
5. Now repeat all the above for weight 2, then again for weight 3 and weight 4.
At all times refrain from marking an edge if that edge would create a circuit.



3. A 'ring circuit' is to be installed with six power outlets $A, \ldots, F$. (As its name implies, a ring circuit with 6 outlets connects each outlet to two 'neighbours', to produce a circuit of 6 links.) The builder supplies the electrician with cost estimates for each of the 15 potential links in the circuit. These are shown in the chart at right.

A	17	24	26	12	10
	B	7	12	14	20
		C	10	20	26
			D	17	24
				E	7
					F

(a) Apply the nearest neighbour algorithm starting at A to find a low cost circuit.

(b) Does the algorithm yield a cheaper circuit if you start at a different outlet?

(c) Is any circuit produced by the algorithm the cheapest possible circuit?

4. The nearest neighbour algorithm for TSP can fail in just about the simplest of situations, including when the weights are physical distances. As a demonstration, apply the algorithm to two complete graphs K_4, where the edge weights are the distances between vertices when placed at points with coordinates as follows:

(a) $A(-3,0)$, $B(0,4)$, $C(3,0)$, $D(0,-4)$. ($|AB| = 5$ by Pythagoras.)

(b) $A(-5,0)$, $B(0,12)$, $C(5,0)$, $D(0,-12)$. ($|AB| = 13$ by Pythagoras.)

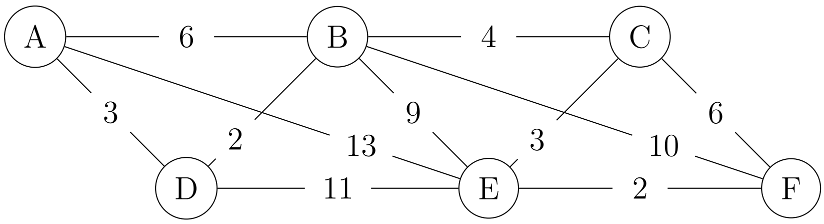
For each weighted graph answer the following:

(i) What is the tour length produced by the algorithm starting from A?

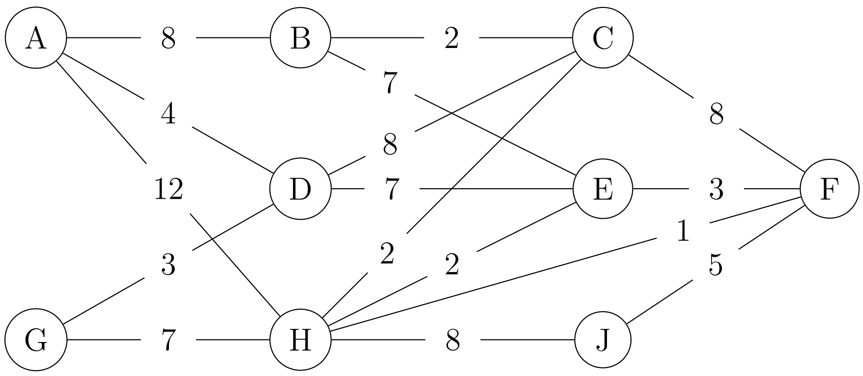
(ii) What is the tour length produced by the algorithm starting from B?

(iii) What is the shortest *possible* tour?

5. Use Dijkstra's algorithm to find the shortest path from A to F in the network below. Show all, and only, annotations generated by the algorithm. Write annotations above vertices. Put a line through any annotation that needs updating and write the new annotation above it. Using heavy lines, mark out the spanning tree generated by the algorithm. This tree is a minimal spanning tree for the graph; is that inevitable?



6. Use Dijkstra's algorithm to find the shortest path from A to J in the network of Q1, repeated below. Show all, and only, annotations generated by the algorithm. Write annotations above vertices. Put a line through any annotation that needs updating and write the new annotation above it. Using heavy lines, mark out the spanning tree generated by the algorithm. Is this tree a minimal spanning tree for the graph? (Refer to your answer to Q1.)



7. The matrix below shows the capacities for a transport network with vertices A,B,C,D,E.

$$\begin{bmatrix} 0 & 0 & 0 & 5 & 0 \\ 3 & 0 & 4 & 0 & 5 \\ 0 & 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

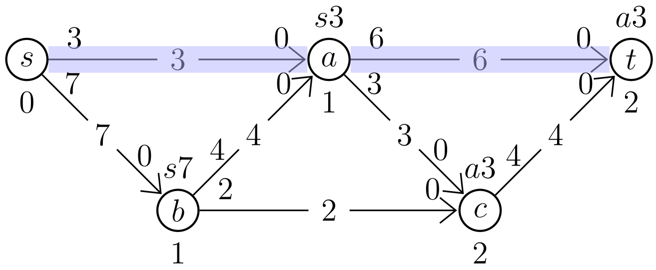
(a) Draw the network.
Identify source and sink.

(b) Specify a partition of the vertices that creates a minimum cut.

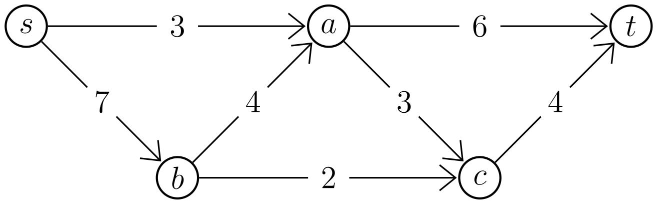
(c) Find a maximum flow that proves your cut to be minimum.
Specify the flow by a flow matrix, using the same ordering of the vertices as used in the capacities matrix. There is no need to use the labelling algorithm; find the flow 'by eye'.

8. Demonstrate the application of the vertex labelling algorithm by applying it to the transport network below. Flow F_0 has been marked up for you, and also Flow F_2 as a check on your progress. On each diagram first write level numbers below the nodes then write labels above the nodes. Finally highlight the incremental flow that will be added to produce the flow shown in the next diagram. Levels, labels and incremental flow have been indicated on the first diagram as an example.

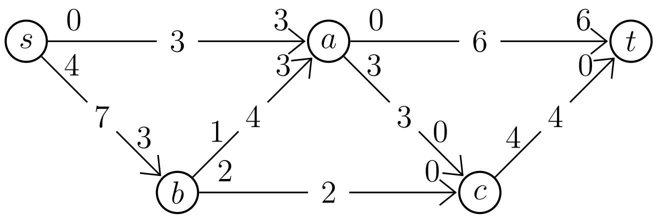
Flow F_0:



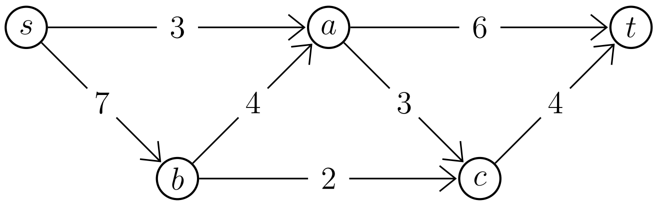
Flow F_1:



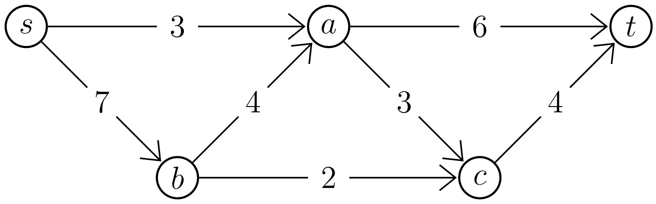
Flow F_2:



Flow F_3:



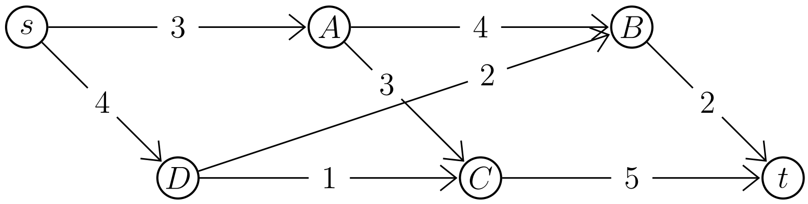
Flow F_4:



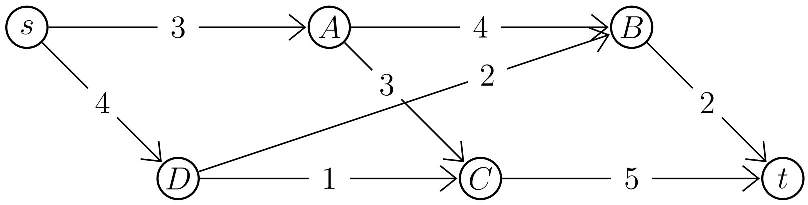
Use your max flow to help find a min cut which supports your answer.

9. Here is another transport network to give you practice with the labelling algorithm. As for Q8, mark each diagram with levels, labels and a highlighted incremental flow. Stage 4, finding the incremental flow f_4 so that $F_4 = F_3 + f_4$, requires the use of a virtual flow. Check that your flow F_3 agrees with mine below before you attempt stage 4.

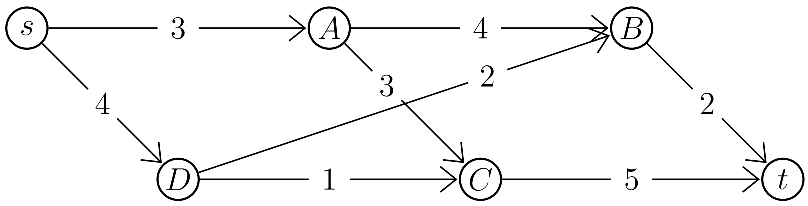
Flow F_0:



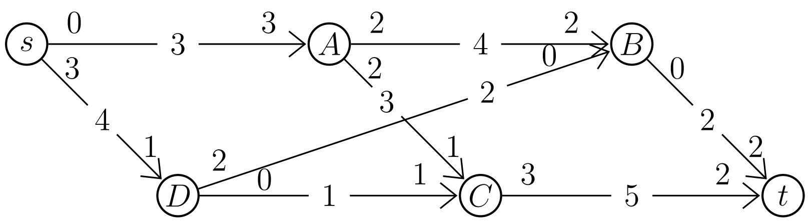
Flow F_1:



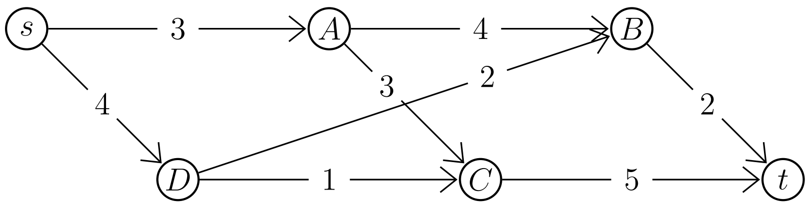
Flow F_2:



Flow F_3:



Flow F_4:

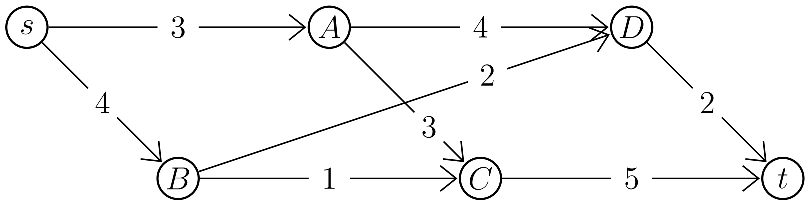


What step in the algorithm precipitates termination? Specify the node(s) involved.

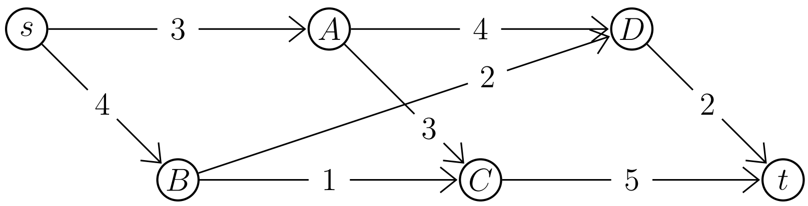
Verify your max flow by specifying an appropriate cut.

10. Because alphabetical order is used at several steps, the progress of the labelling algorithm is affected by the choice of names for the nodes. To see this, apply the algorithm to the transport network below, which is the same as for Q9 except that two nodes have had their names swapped. You should find that only three stages are required to arrive at the same maximum flow, and that virtual flows are not required to get there.

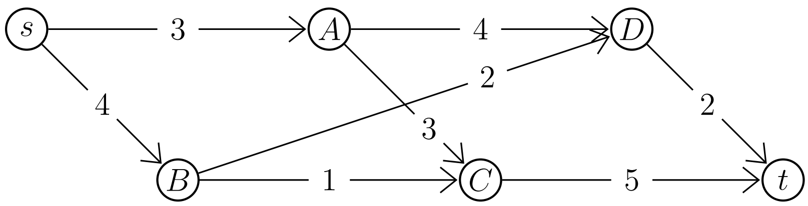
Flow F_0:



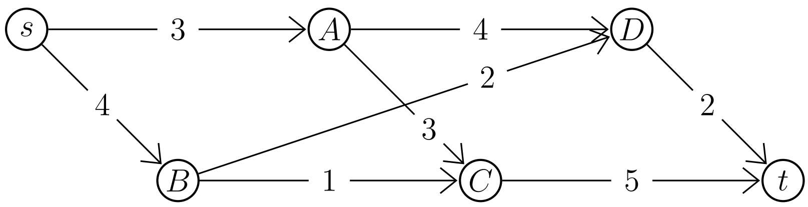
Flow F_1:



Flow F_2:

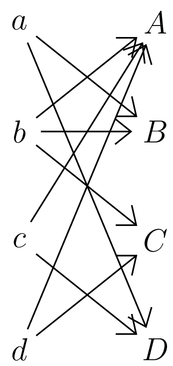


Flow F_3:



11. Briefly describe how a matching problem can be converted to a max flow problem for a transport network. In particular, explain the terms ‘supersource’ and ‘supertarget’ and say how capacities are assigned to the network.

12. For $S = \{a, b, c, d\}$ and $T = \{A, B, C, D\}$ a relation $R \subseteq S \times T$ is represented by the the arrow diagram at right.

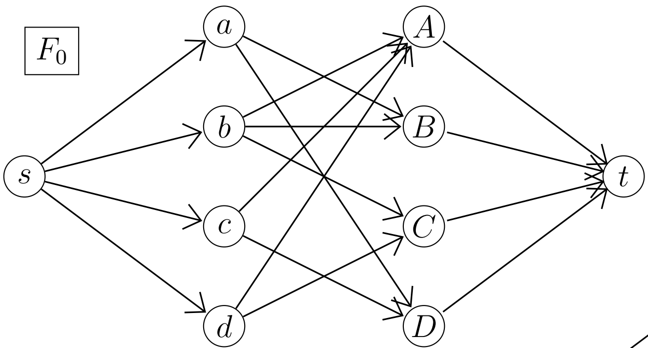


(a) List all possible full matchings $m : S \to T$. Do this without the aid of any particular algorithm; just use logical reasoning. [*e.g.* If $(a, B) \in m$ then $(c, D) \in m$ since otherwise nothing will be matched with D.]

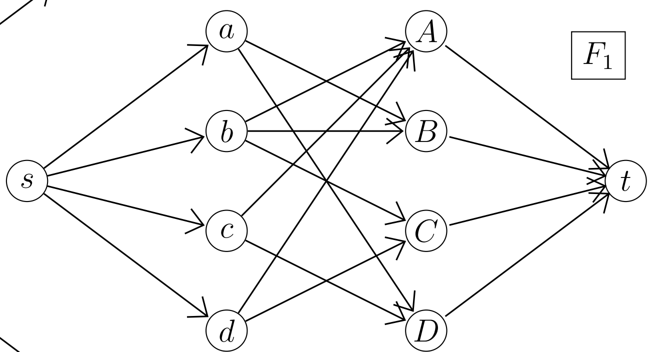
(b) Which matching is found by the vertex labelling algorithm?

Justify your answer by marking up the flow diagrams below with levels, labels and incremental flows. List the resulting matching.

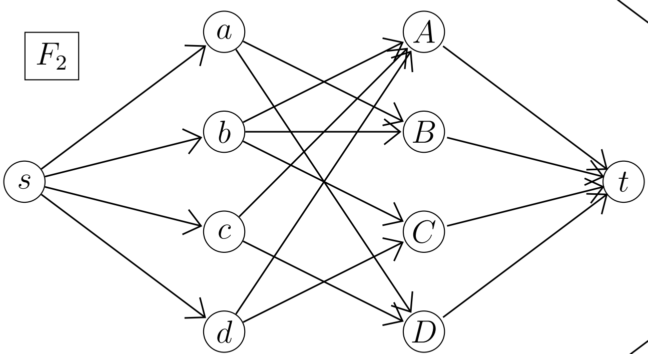
F_0



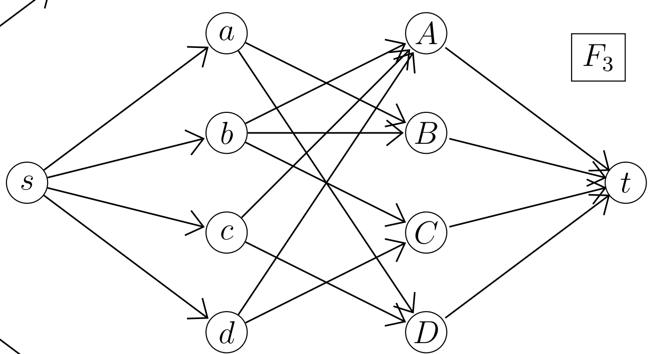
F_1



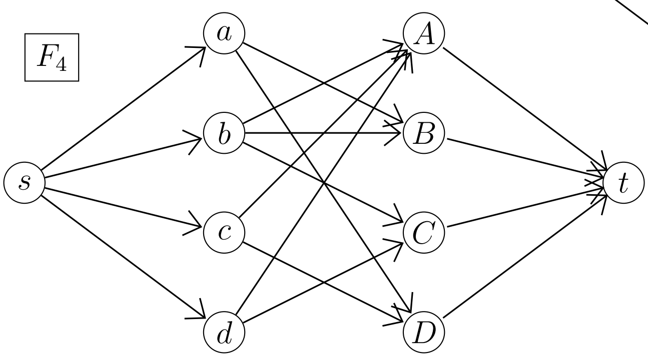
F_2



F_3

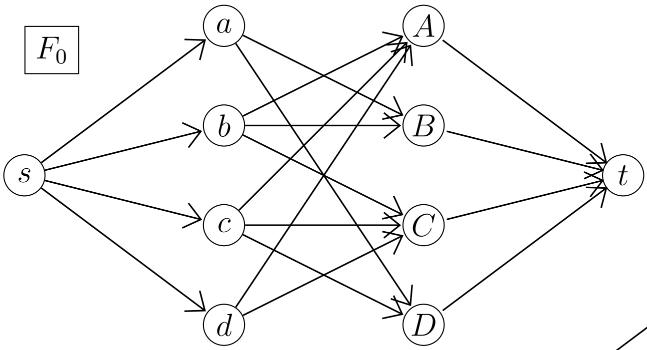


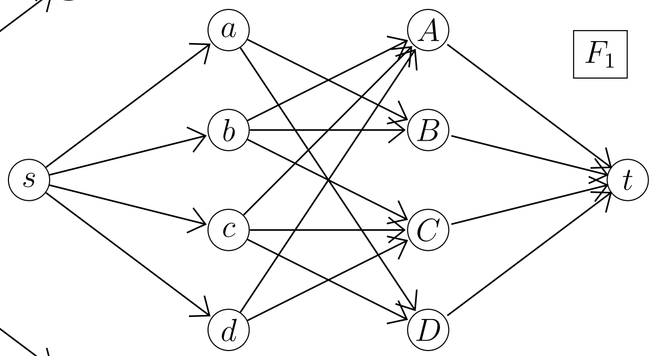
F_4

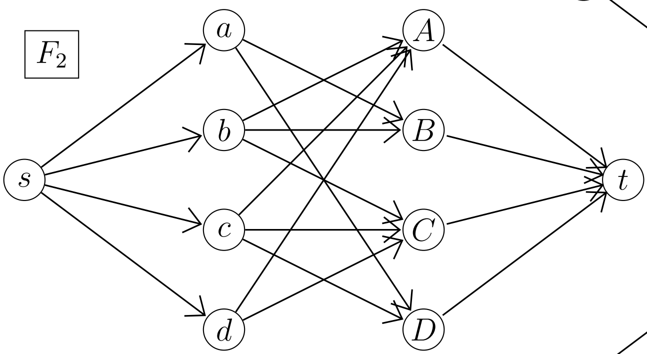


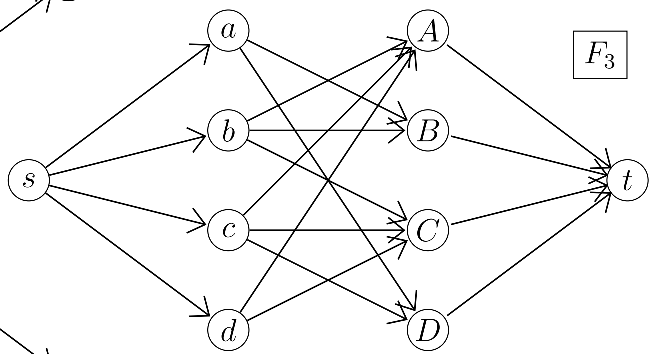
x	a	b	c	d
$m(x)$				

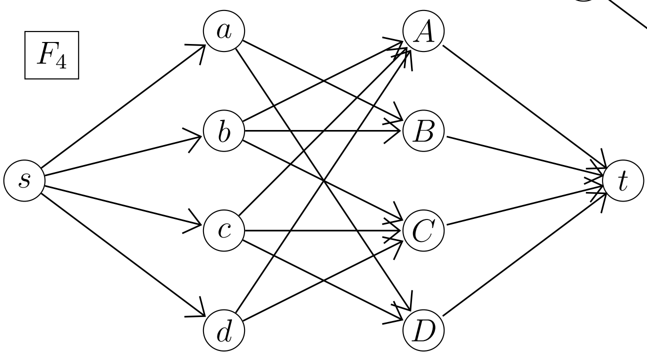
13. Repeat Q12 with the addition of one extra pair (c, C) to R. You should end up with the same matching, but the extra choice actually makes the last stage a bit harder, with a virtual flow being needed.





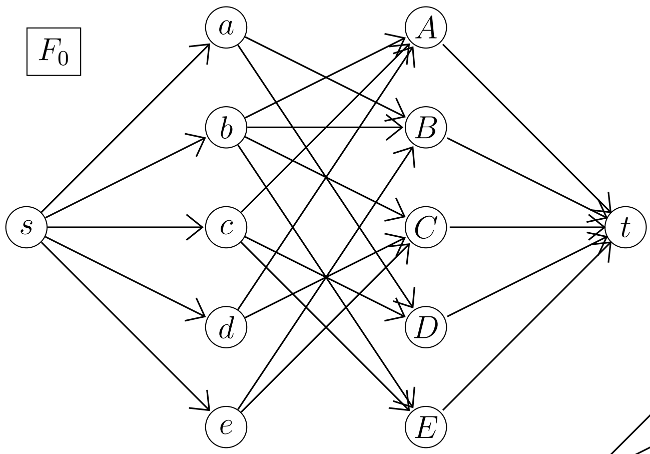




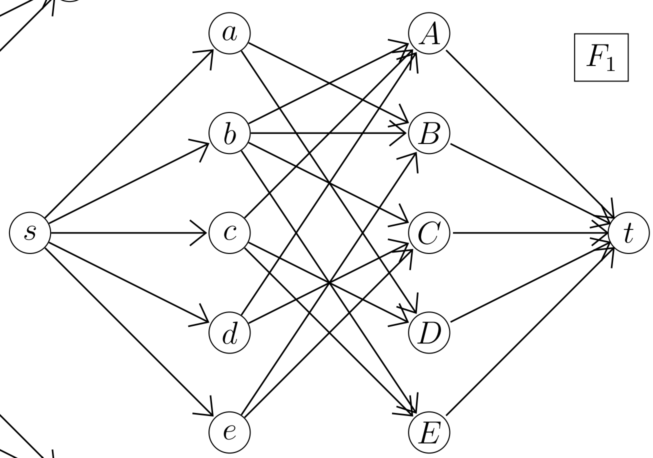


14. Now repeat Q12 again with S augmented by an extra element e, T augmented by E and R augmented by the pairs $(e, B), (e, C), (b, E), (c, E)$. Does the algorithm lead to a full matching? If so list it, if not, say how the algorithm terminates.

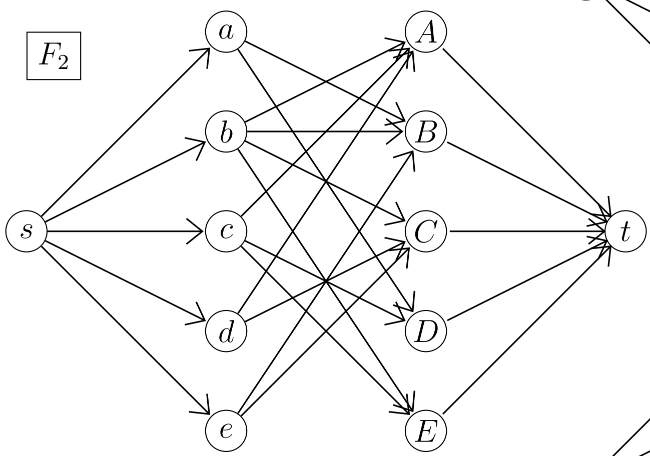
F_0



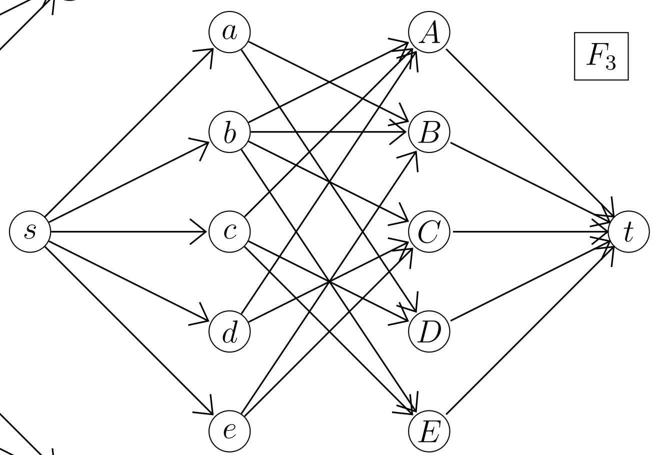
F_1



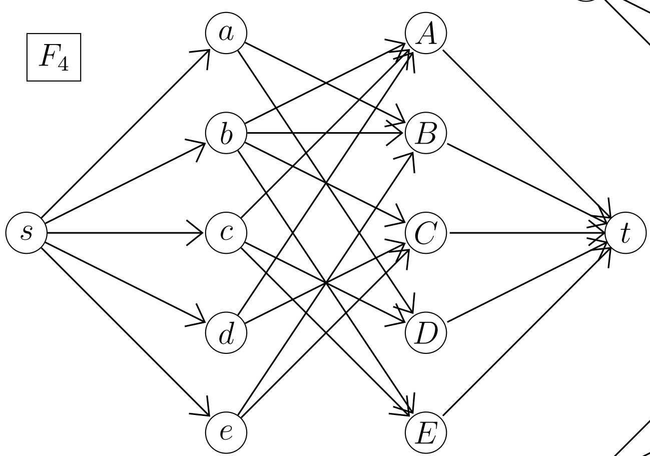
F_2



F_3



F_4



F_5