Problem 1

1a) A **statement** is a sentence that is true or false but not both.

eg. Bruce Springsteen is an American musician.

see slide # 7 for more examples.

A **predicate** is a sentence containing one or more variables, with the property that, when a value from a specified domain is given to each variable, the sentence becomes a statement.

eg. $P(x): \quad x^2 > 7$

is a predicate with x taking values from the domain $\mathbb{Z}$

see slide # 69

An argument is a sequence of statements, one of which is flagged as the conclusion. The other statements, called premises, are alleged to justify the conclusion.

eg. If it is sunny, I wear a hat.
It is sunny, so I will wear a hat.

This is an argument of the form

s: It is sunny
h: I wear a hat

$$\frac{s \to h \qquad s}{h}$$

or $[s \to h, s \therefore h]$

Not needed but may be helpful

eg. See slide #51

Key: Make sure that the conclusion is "flagged" by your language (in the example above, it is the word "so" doing this)

1b) Working:

$\neg\big(\forall a \in \mathbb{Z}\ \forall b \in \mathbb{Z}\ (a<b \rightarrow$ there exists a unique soln to the equation $\mathcal{E}(a,b))\big)$

$\equiv \exists a \in \mathbb{Z}\ \exists b \in \mathbb{Z}\ \neg(a<b \rightarrow$ there exists a unique soln to the equation $\mathcal{E}(a,b))$

$\equiv \exists a \in \mathbb{Z}\ \exists b \in \mathbb{Z}\ (a<b \wedge \neg($there exists a unique soln to the eqn $\mathcal{E}(a,b)))$

$\equiv \exists a \in \mathbb{Z}\ \exists b \in \mathbb{Z}\ (a<b \wedge [$there is no soln to the equation $\mathcal{E}(a,b)$ $\vee$ there is more than one solution to the eqn $\mathcal{E}(a,b)])$

A: There exist integers a and b such that $a<b$ and either there is no solution to the equation $\mathcal{E}(a,b)$ or there is more than one solution to the equation $\mathcal{E}(a,b)$.

(see slides #46, 47, 48, #64, 65

1(c) Let

w: There is water on Planet X

l: There is life on Planet X

i: Planet X will be invited to join the Confederation of Lively planets.

The argument has the form

$[\neg w,\ l \to w,\ i \to l \ \therefore\ \neg i]$

The argument is valid when

$[\neg w \wedge (l \to w) \wedge (i \to l)] \to \neg i$

is a tautology. We use a truth table to check.

w	l	i	$[\neg w$	$\wedge (l \to w)$	$\wedge (i \to l)]$	$\to$	$\neg i$
T	T	T	F			T	
T	T	F	F			T	
T	F	T	F			T	
T	F	F	F			T	
F	T	T		F		T	
F	T	F		F		T	
F	F	T			F	T	
F	F	F	T	T	T	T	T

Since the compound statement is true in every row, it is a tautology.
Hence the argument is valid

(See slides #54-57.
Note, as soon as one premise is false, the implication is true)

1d) Yes. In class we established that $\{\uparrow\}$, the set containing only NAND, is functionally complete.

p	q	$p \uparrow q$	$\neg p \vee \neg q$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

The truth table shows that every compound statement that $\uparrow$s can be replaced by a combination of $\neg$ and $\vee$.

(see slides # 90, 99)

Problem 2

2a) $A = \{1, 2, 3, 4, 5, 6\}$

$B = \{z \in \mathbb{Z} \mid 1 \leq z \leq 27\}$

The following is a partition of $\mathbb{Z}$ into exactly four sets

$\{\{0\}, \{1\}, \{2, 3\}, \{z \in \mathbb{Z} \mid z < 0 \vee z \geq 4\}\}$

(see slides # 138, 139)

2b) $\{(d, 17, b, 7/3), (x, -3, x, 1/2), (t, 21, u, 3/2)\}$

2c) Let

$p(n): \quad a_n = (n-1)2^{n+1} + 2$.

We shall use mathematical induction to prove $\forall n \in \mathbb{N} \; p(n)$.

Basis case:

The Left-hand side (LHS) of $p(1)$ is $a_1 = 2$ (by definition)

The right-hand side (RHS) of $p(1)$ is $(1-1)2^{1+1} + 2 = 0 \times 2^2 + 2 = 2$

Hence $p(1)$ is true.

Let $n \in \mathbb{N}$, Suppose that $p(1), p(2), \ldots, p(n)$ are all true.

LHS of $p(n+1)$

$= a_{n+1}$

$= a_n + (n+1)2^{n+1}$ (by definition)

$= (n-1)2^{n+1} + 2 + (n+1)2^{n+1}$ (because $p(n)$ is true)

$= [(n-1)+(n+1)]2^{n+1} + 2$

$= (2n)2^{n+1} + 2$

$= n\,2^{n+2} + 2$

$= [(n+1)-1]2^{(n+1)+1} + 2$

$= $ RHS of $p(n+1)$.

See slides # 247, 248, 254, 255, 256, 257. and the companion to lecture 14

Hence $p(n+1)$ is true.

By the principle of mathematical induction, $\forall n \in \mathbb{N}\ p(n)$.

2d)

$x \in (A \cup B) \setminus C$

$\Leftrightarrow \quad x \in A \cup B \wedge x \notin C$ (defn of $\setminus$)

$\Leftrightarrow \quad x \in A \cup B \wedge \neg(x \in C)$ (defn of $\notin$)

$\Leftrightarrow \quad (x \in A \vee x \in B) \wedge \neg(x \in C)$ (defn $\cup$)

$\Leftrightarrow \quad \neg(x \in C) \wedge (x \in A \vee x \in B)$ (using L of 1)

$\Leftrightarrow \quad (\neg(x \in C) \wedge x \in A) \vee (\neg(x \in C) \wedge x \in B)$ (using L of 3)

$\Leftrightarrow \quad (x \notin C \wedge x \in A) \vee (x \notin C \wedge x \in B)$ (defn $\notin$)

$\Leftrightarrow \quad (x \in A \wedge x \notin C) \vee (x \in B \wedge x \notin C)$ (using L of 1 twice)

$\Leftrightarrow \quad x \in A \setminus C \vee x \in B \setminus C$ (defn $\setminus$)

$\Leftrightarrow \quad x \in (A \setminus C) \cup (B \setminus C)$ (defn $\cup$)

(see slides #112)

Problem 3

3a) Any subset R of $A \times B$ is a relation from A to B

A relation from A to A is called a relation on A.

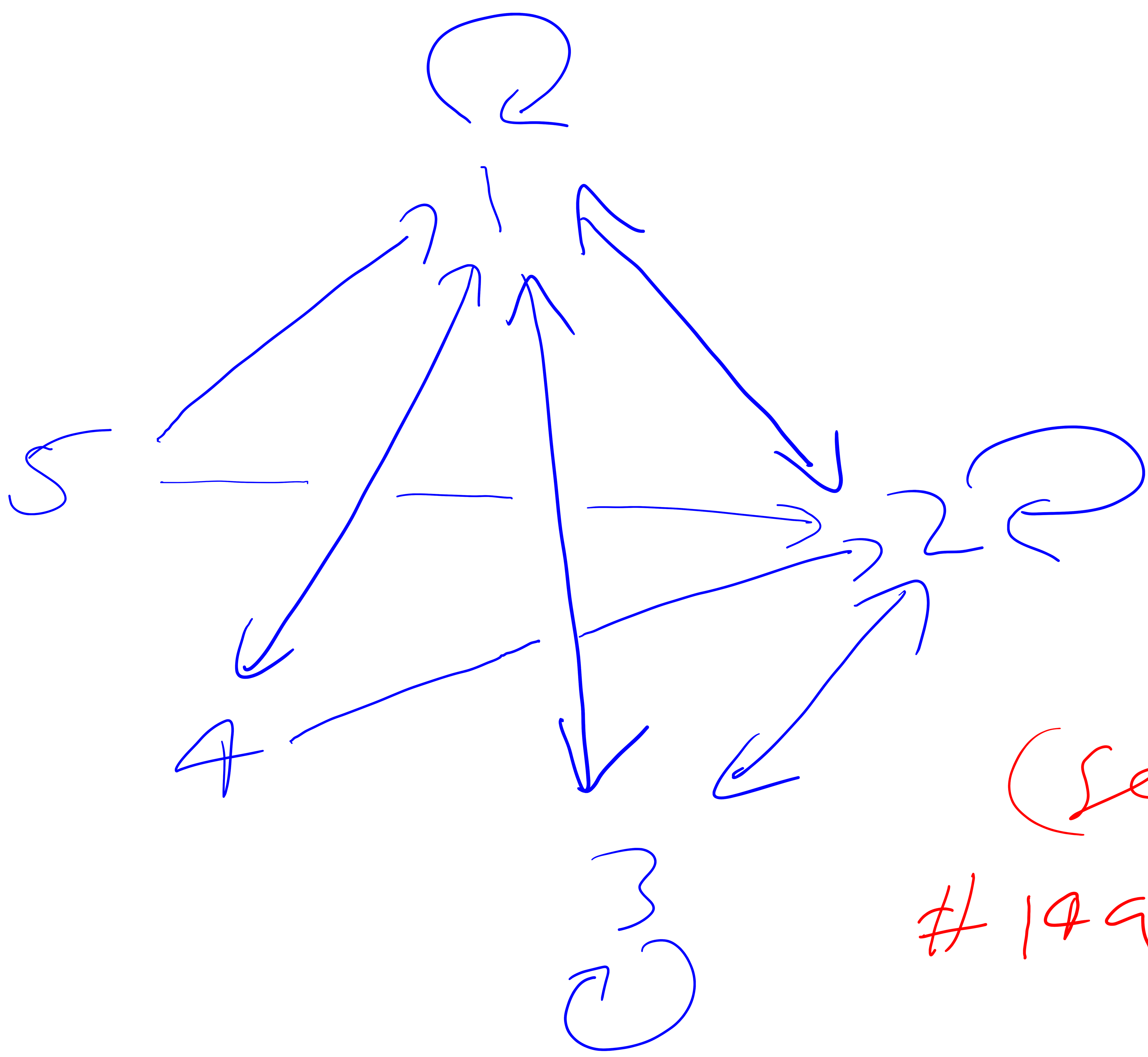
If R is a relation from A to B, then

$$R^{-1} = \{ (b,a) \in B \times A \mid (a,b) \in R \}$$

is the inverse relation of R.

(See slide #142)

2b)



(See slides #149, 150)

3c). To be a function, we need that each element of A is related to a unique element of B, and that any elements of A that are different are related to different elements of B.

(See slides #155-165)

d). No, because $3 \notin$ range f.

$$f: \begin{array}{cccccccc} 1 & 2. & 3 & 4 & 5 & 6 & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 5 & 2 & 1 & 2 & 5 & 10 & \end{array}$$

(See slide #166)

P.S.: I will not use the notation $\mathbb{Z}^+$ in the exam.

e) The statement is true

<u>Proof</u>: Sps that f and g are injective. We must show that $g \circ f$ is injective

Let $a_1, a_2 \in A$. Sps that $a_1 \neq a_2$.

Since f is injective, $f(a_1) \neq f(a_2)$.

Since g is injective and $f(a_1) \neq f(a_2)$,

$(g \circ f)(a_1) = g(f(a_1)) \neq g(f(a_2)) = (g \circ f)(a_2)$

(~~See~~ slides # 165, 122-124) $\square$.

Problem 4

a) $(1101001101)_2$

$= (\underbrace{0011}_{(3)_{10}}\ \underbrace{0100}_{(4)_{10}}\ \underbrace{1101}_{(13)_{10}})_2$

$= (34D)_{16}$

$= (3 \times 16^2 + 4 \times 16^1 + 13 \times 16^0)_{10}$

$= (845)_{10}$ (See slides # 130-138)

b)

$z = 11 \times q + r$ (See slides # 223 and 225)

$q = z \text{ div } 11 = -4$

and $r = z \bmod 11 = 3$

so $z = 11 \times (-4) + 3 = -41$

c)

(see workshop 3 question 4+5)

$(B3A)_{16}$

represents the binary string

$$\boxed{\underbrace{1011}_{(11)_{10}}\ \underbrace{0011}_{(3)_{10}}\ \underbrace{1010}_{(10)_{10}}}$$

So

$$\underbrace{1}_{s}\ \underbrace{0111}_{n+3}\ \underbrace{00111010}_{\text{bits after 1. in the mantissa.}}$$

represents the number

$$(-1)^1 \times 2^0 \times (1.00111010)_2$$

$$= -(1.00111010)_2$$

$$= -\left(1+\frac{0}{2}+\frac{0}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{0}{64}+\frac{1}{128}+\frac{0}{256}\right)_{10}$$

$$= -\left(\frac{128+0+0+16+8+4+0+1+0}{128}\right)$$

$$= -\frac{157}{128}$$

$$\approx -1.2265625$$

(see slides #179-181 and workshop 3 question 4)

d)

(see workshop 4 Q5 and ECM360 video on sorting)

Start

π $\qquad\qquad$ $(X_{\pi(i)})_{i=1 \dots 8}$

	π	$(X_{\pi(i)})$
Start	1 2 3 4 5 6 7 8 1 2 3 4 5 6 7 8	D I S C R E T E
After CEA called once	1 2 3 4 5 6 7 8 4 2 3 1 5 6 7 8	C I S D R E T E
After CEA called twice	1 2 3 4 5 6 7 8 4 1 3 2 5 6 7 8	C D S I R E T E
After CEA called three times	1 2 3 4 5 6 7 8 4 1 6 2 5 3 7 8	C D E I R S T E

THOUGHTS:

This practice exam was a little too long.