COMP2610 / COMP6261 — Information Theory: Self-assessment Quiz

Australian National University

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Instructions

This quiz is designed to test your background knowledge in elementary probability theory for the course. It does not count in the assessment of the course in any way. And it is not obligatory. It will not be marked but you will be provided with solutions in early week 2 which you can use to evaluate how well you did. I encourage you to team up with a friend and "mark" each others answers. You should write out answers as if this was an assessable piece of work, meaning that you should show all your working. If you are "marking" a friend's attempt, you should deduct marks for incomplete or unclear working, regardless of whether the final answer is correct because that is what will be done on the actual assignments.

1. [8 points] Two fair, six-sided die are rolled. Compute the probability that the sum of the outcomes of the two rolls is:

   (a) (3 pt) Equal to 1

   (b) (3 pt) Equal to 4

   (c) (2 pt) Less than 13

2. [15 points] Suppose that X, Y are discrete random variables with joint probability

	$Y=-1$	$Y=0$	$Y=1$
$X=0$	0	$\frac{1}{3}$	0
$X=1$	$\frac{1}{3}$	0	$\frac{1}{3}$

   (a) (3 pt) Compute the conditional distributions $p(X|Y=y)$ for all possible values of y.

   (b) (2 pt) Compute the conditional distributions $p(Y|X=x)$ for all possible values of x.

   (c) (3 pt) Compute $E[X|Y=y]$ for all values of y.

   (d) (2 pt) Plot $E[X|Y=y]$ against y.

   (e) (5 pt) Are X and Y dependent? If yes, explain why. If no, how do you think X and Y are related?

3. [20 points] You have a large jar containing 999 fair coins and one two-headed coin (i.e. a coin that is guaranteed to come up heads). Suppose you pick one coin at random out of the jar and flip it 10 times.

   (a) (2 pt) What is the probability you selected a fair coin?

   (b) (2 pt) What is the probability of the first flip being heads, assuming you selected a fair coin?

   (c) (2 pt) What is the probability of the first flip being heads, assuming you selected the two-headed coin?

   (d) (3 pt) What is the overall probability of the first flip being heads?

   (e) (3 pt) What is the probability of the first *and* second flip being heads, assuming you selected a fair coin?

   (f) (8 pt) What is the probability you selected a fair coin if *all* ten flips turn up heads?

4. [32 points] A kidnapping recently occurred in the city of Twin Peaks. Dale, an FBI agent, has been assigned to catch the person who did it. He has determined that the guilty person must be one of the 1,000 members of a club called the White Lodge.

   (a) (2 pt) Leland is a member of the White Lodge. Based on only this information, what is the probability that Leland is guilty?

   (b) (6 pt) Dale has an expensive DNA test that he can use to help his investigation. Suppose the probability of a DNA match given that a person is innocent is 0.1%, and the probability of a DNA match given that a person is guilty is 99%. What is the probability of a DNA match for a randomly chosen member of the White Lodge?

(c) (9 pt) Dale has found DNA evidence at the scene of the kidnapping. An analysis of the DNA matches a sample that Leland provides. What is the probability that Leland is guilty given the positive outcome of the DNA test? Explain intuitively why this number is higher or lower than that of part (a).

(d) (12 pt) Bob is trying to sell Dale a lie detector machine that he claims is very accurate. He says: *"If my machine starts beeping, you can be 99% sure the person really is guilty"*. What must be the relative fraction of times the machine starts beeping for a guilty person, over the times it starts beeping for an innocent person?

(e) (3 pt) Given just the above information, is it possible to determine the fraction of times that Bob's machine starts beeping for a guilty person? If yes, compute the fraction. If no, explain why not.

5. [25 points] Suppose that X is a random variable with values $\{0,1\}$, and $p(X=1)=\theta$. In the following, express your answer in terms of θ.

(a) (3 pt) Define and calculate the expectation of X.

(b) (4 pt) Define and calculate the variance of X.

(c) (6 pt) Calculate the quantity $\phi(t) := E[e^{tX}]$ for a fixed $t > 0$.

(d) (6 pt) Obtain an expression for the derivative $\phi'(t)$. Explain the relation of $\phi'(0)$ to the result of (a).

(e) (6 pt) Obtain an expression for the second derivative $\phi''(t)$. Explain the relation of $\phi''(0)$ to the result of (b).