COMP2610 / COMP6261 - Information Theory

Tutorial 3: Entropy and Information

Akram Shafie

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1. Let X be a random variable with possible outcomes $\{1,2,3\}$. Let the probabilities of the outcomes be

$$p(X=1)=\frac{\theta}{2}$$
$$p(X=2)=\frac{\theta}{2}$$
$$p(X=3)=1-\theta$$

for some parameter $\theta \in [0,1]$.

Suppose we see N observations of the random variable, $\{x_1,\ldots,x_N\}$. Let n_i denote the number of times that we observe the outcome $X = i$, i.e.

$$n_i = \sum_{k=1}^{N} \begin{cases} 1 & \text{if } x_k = i \\ 0 & \text{else.} \end{cases}$$

(a) Write down the likelihood function of θ given the observations $\{x_1,\ldots,x_N\}$ in terms of n_1,n_2,n_3.

(b) Suppose the observations are

$$\{3,3,1,2,3,2,2,1,3,1\}.$$

Compute the maximum likelihood estimate of θ. (*Hint*: Compute the log-likelihood function, and check when the derivative is zero.)

2. Consider the following joint distribution over X,Y:

$p(X,Y)$		X			
		1	2	3	4
Y	1	0	0	1/8	1/8
	2	1/8	1/16	1/16	0
	3	1/8	1/8	0	0
	4	0	1/16	1/16	1/8

(a) Show that X and Y are not statistically independent. (*Hint*: You need only show that for at least one specific x,y pair, $p(X = x,Y = y)$ not equal to $p(X = x)p(Y = y)$.)

(b) Compute the following quantities:

(i) $H(X)$

(ii) $H(Y)$

(iii) $H(X|Y)$

(iv) $H(Y|X)$

(v) $H(X,Y)$

3. A standard deck of cards contains 4 *suits* — ♥,♦,♣,♠("hearts", "diamonds", "clubs", "spades") — each with 13 *values* — A,2,3,4,5,6,7,8,9,10,J,Q,K (The A,J,Q,K are called "Ace", "Jack", "Queen", "King"). Each card has a *colour*: hearts and diamonds are coloured red; clubs and spades are black. Cards with values J, Q, K are called *face cards*.

   Each of the 52 cards in a deck is identified by its value v and suit s and denoted vs. For example, 2♥, J♣, and 7♠ are the "two of hearts", "Jack of clubs", and "7 of spades", respectively. The variable c will be used to denote a card's colour. Let $f = 1$ if a card is a face card and $f = 0$ otherwise.

   A card is drawn at random from a thoroughly shuffled deck. Calculate:

   (a) The information in observing a red King, i.e., $h(c = \text{red}, v = \text{K})$

   (b) The conditional information in observing a King given a face card was drawn, i.e., $h(v = \text{K} | f = 1)$

   (c) The entropies $H(S)$ and $H(V,S)$.

4. Let X be a random variable taking on a finite number of values. What is the (general) inequality relationship of H(X) and H(Y) if

   a. $Y = 2^X$?
   b. $Y = \cos X$?