E1.

(a)

$$\begin{bmatrix} 0 & 1 & 5 & -4 \\ 1 & 4 & 3 & -2 \\ 2 & 7 & 1 & -2 \end{bmatrix} \xrightarrow{R_3 = R_3 - 2R_2} \begin{bmatrix} 0 & 1 & 5 & -4 \\ 1 & 4 & 3 & -2 \\ 0 & -1 & -5 & 2 \end{bmatrix} \xrightarrow{\text{Swap } R_1 R_2} \begin{bmatrix} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & -1 & -5 & 2 \end{bmatrix} \xrightarrow{R_3 = R_2 + R_3}$$

$$\begin{bmatrix} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & -2 \end{bmatrix} \xrightarrow{R_1 = R_1 - 4R_2} \begin{bmatrix} 1 & 0 & -17 & 14 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = -2$, $0 = -2$, invalid. $\therefore$ No solution.

(b)

$$\begin{bmatrix} 2 & 3 & 1 & 6 \\ 4 & 0 & 3 & 12 \end{bmatrix} \xrightarrow{R_2 = R_2 - 2R_1} \begin{bmatrix} 2 & 3 & 1 & 6 \\ 0 & -6 & 1 & 0 \end{bmatrix} \xrightarrow[R_2 = -\frac{1}{6}R_2]{R_1 = \frac{1}{2}R_1} \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} & 3 \\ 0 & 1 & -\frac{1}{6} & 0 \end{bmatrix} \xrightarrow{R_1 = R_1 - \frac{3}{2}R_2}$$

$$\begin{bmatrix} 1 & 0 & \frac{3}{4} & 3 \\ 0 & 1 & -\frac{1}{6} & 0 \end{bmatrix}$$

x_1, x_2: basic. x_3: free.

Let $x_3 = 0$:

$x_2 = 0$. $x_1 = 3$. $\therefore$ A particular solution: $\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

Let $x_3 = 1$:

$\begin{bmatrix} 1 & 0 & \frac{3}{4} & 0 \\ 0 & 1 & -\frac{1}{6} & 0 \end{bmatrix}$, $x_2 - \frac{1}{6} = 0$, $x_2 = \frac{1}{6}$. $x_1 + \frac{3}{4} = 0$, $x_1 = -\frac{3}{4}$.

$\therefore$ All solutions to $Ax = b$: $\left\{ x \in R^3 : x = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{6} \\ 1 \end{bmatrix}, \lambda \in R \right\}$

E2.

$$\left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 1 & 1 & c & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2-R_1 \to R_2]{R_3-R_2 \to R_3} \left[\begin{array}{cccccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1-a & c-b & -1 & 1 & 0 \\ 0 & 0 & 1-c & 0 & -1 & 1 \end{array}\right].$$

$\therefore 1-a \neq 0$, $a \neq 1$. $1-c \neq 0$, $c \neq 1$.

$\therefore a \neq 1$, $c \neq 1$, b can be any real number.

E3.

(a) No. Closure does not satisfy. $-1 \times (1,1) = (-1,-1) \notin A$.

(b) Yes. B is a subset of R^3, $B \neq \emptyset$ and $0 \in B$.

For any $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in B$, $k \in R$.

$$ku = \begin{bmatrix} ku_1 \\ ku_2 \\ ku_3 \end{bmatrix} \quad ku_1 + ku_2 + ku_3 = k(u_1+u_2+u_3) = k \times 0 = 0. \in B$$

$$u + v = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}. \quad (u_1+v_1)+(u_2+v_2)+(u_3+v_3) = (u_1+u_2+u_3)+(v_1+v_2+v_3) = 0 \in B$$

$\therefore B$ satisfies closure. $\therefore B$ is a subspace of R^3.

(c) No. Closure not satisfy.

For $u = (1,0)$, $v = (0,1)$, $u, v \in C$.

$u + v = (1,1) \notin C$.

an ements in in X

(d) It is when $X \in R^3$, $X \neq \emptyset$, and elements in X are either only 0 or any real number so X can satisfy closure.

Otherwise D is not.

E4.

(a) Suppose the transformation matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & - & - & a_{nn} \end{bmatrix} \in R^n$

$$T(0) = A \cdot 0 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & - & - & a_{nn} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}\times 0 + a_{12}\times 0 + \cdots + a_{1n}\times 0 \\ a_{21}\times 0 + a_{22}\times 0 + \cdots + a_{2n}\times 0 \\ \vdots \\ a_{n1}\times 0 + a_{n2}\times 0 + \cdots + a_{nn}\times 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\therefore T(0) = 0$

(b) Suppose the transformation matrix is A.

$$T(C_1V_1 + \cdots + C_nV_n) = A(C_1V_1 + \cdots + C_nV_n)$$
$$= AC_1V_1 + \cdots + AC_nV_n$$
$$= C_1AV_1 + \cdots + C_nAV_n$$
$$= C_1(AV_1) + \cdots + C_n(AV_n)$$
$$= C_1T(V_1) + \cdots C_nT(V_n).$$

$\therefore T(C_1V_1 + \cdots + C_nV_n) = C_1T(V_1) + \cdots C_nT(V_n).$

(c) Suppose the transformation matrix is A.

For $\{V_1, \cdots, V_n\}x_1 = 0$, $\{V_1, \cdots, V_n\}$ is linearly dependent, so x_1 has non-zero solution.

According to (b), $\{W_1, \cdots, W_n\} = T(\{V_1, \cdots, V_n\}) = A\{V_1, \cdots, V_n\}$.

So $\{W_1, \cdots, W_n\} \cdot x_2 = 0 \equiv A\{V_1, \cdots, V_n\}x_2 = 0$

$A^{-1}A\{V_1, \cdots, V_n\} \cdot x_2 = 0 \cdot A^{-1}$. $I\{V_1, \cdots, V_n\}x_2 = 0$. $\{V_1, \cdots, V_n\}x_2 = 0$.

$\therefore x_2 = x_1$. $\therefore x_2$ has non-zero solution, $\{W_1, \cdots, W_n\}$ is linearly dependent.

E5.

(a) For a vector space V. for all $x, y, z \in V$, $\lambda, \varphi \in \mathbb{R}$.

$\because \langle \cdot, \cdot \rangle$ is linear in the first argument.

$\therefore \langle \lambda x + \varphi y, z \rangle = \lambda \langle x, z \rangle + \varphi \langle y, z \rangle$

$\because \langle \cdot, \cdot \rangle$ is symmetric,

$$\begin{aligned} \therefore \langle z, \lambda x + \varphi y \rangle &= \langle \lambda x + \varphi y, z \rangle \\ &= \lambda \langle x, z \rangle + \varphi \langle y, z \rangle \\ &= \lambda \langle z, x \rangle + \varphi \langle z, y \rangle \end{aligned}$$

$\therefore \langle \cdot, \cdot \rangle$ is also linear in the second argument.

$\therefore \langle \cdot, \cdot \rangle$ is bilinear.

(b) $\langle x, y \rangle = x_1 y_1 + x_2 y_2 + 2(x_1 y_2 + x_2 y_1)$

$\langle y, x \rangle = y_1 x_1 + y_2 x_2 + 2(y_1 x_2 + y_2 x_1) = \langle x, y \rangle$

$\therefore \langle \cdot, \cdot \rangle$ is symmetric.

$$\begin{aligned} \langle \lambda x + \varphi y, z \rangle &= (\lambda x_1 + \varphi y_1) z_1 + (\lambda x_2 + \varphi y_2) z_2 + 2[(\lambda x_1 + \varphi y_1) z_2 + (\lambda x_2 + \varphi y_2) z_1] \\ &= (\lambda x_1 + \varphi y_1)(z_1 + 2 z_2) + (\lambda x_2 + \varphi y_2)(z_2 + 2 z_1) \end{aligned}$$

$$\begin{aligned} \lambda \langle x, z \rangle + \varphi \langle y, z \rangle &= \lambda [x_1 z_1 + x_2 z_2 + 2(x_1 z_2 + x_2 z_1)] + \varphi [y_1 z_1 + y_2 z_2 + 2(y_1 z_2 + y_2 z_1)] \\ &= \lambda x_1 z_1 + \lambda x_2 z_2 + 2\lambda x_1 z_2 + 2\lambda x_2 z_1 + \varphi y_1 z_1 + \varphi y_2 z_2 + 2\varphi y_1 z_2 + 2\varphi y_2 z_1 \\ &= (\lambda x_1 + \varphi y_1)(z_1 + 2 z_2) + (\lambda x_2 + \varphi y_2)(z_2 + 2 z_1) \\ &= \langle \lambda x + \varphi y, z \rangle. \end{aligned}$$

$\therefore \langle \cdot, \cdot \rangle$ is linear in the first argument. According to (a), $\langle \cdot, \cdot \rangle$ is bilinear.

Let $x = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, $y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\begin{aligned} \langle x, y \rangle &= (-1) \times 1 + (-1) \times 1 + 2[(-1) \times 1 + (-1) \times 1] \\ &= -1 - 1 - 4 = -6 < 0. \end{aligned}$$

$\therefore \langle \cdot, \cdot \rangle$ is not positive definit.

$\therefore \langle \cdot, \cdot \rangle$ is symmetric and bilinear. but not positive definit.

E 6.

(a) Suppose x, y are linearly dependent. Then $ax+by=0$. $a, b \in R$ and $a \neq 0$ or $b \neq 0$.

$\langle x, ax+by \rangle = \langle x, 0 \rangle = 0$

$= \langle x, ax \rangle + \langle x, by \rangle$

$= a\langle x, x \rangle + b\langle x, y \rangle$

$\because x, y$ are orthogonal, $\therefore \langle x, y \rangle = 0$.

$\therefore \langle x, ax+by \rangle = a\langle x, x \rangle + b \times 0$

$= a\langle x, x \rangle = 0$

$\because x \neq 0$, $\therefore \langle x, x \rangle \neq 0$ $\therefore a=0$

$\langle y, ax+by \rangle = \langle y, 0 \rangle = 0$

$= \langle y, ax \rangle + \langle y, by \rangle$

$= a\langle y, x \rangle + b\langle y, y \rangle$

$= b\langle y, y \rangle$

$\because y \neq 0$, $\therefore \langle y, y \rangle \neq 0$ $\therefore b=0$.

$\therefore a=0$ and $b=0$, which is contradicts the assumption.

$\therefore$ If x and y are orthogonal, then they are linearly independent.

(b) If $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\{x, y\} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \Rightarrow \begin{cases} a=0 \\ b=0 \end{cases}$, so x and y are linearly independent.

If x and y are orthogonal, using dot product as inner product, $\langle x, y \rangle = 0$.

However $x \cdot y = [1 \ 0] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1+0 = 1 \neq 0$

$\therefore$ If x and y are linearly independent, they are not always orthogonal.

E7

(a) Assume it is true that $\|\cdot\|_a \overset{\varepsilon}{\sim} \|\cdot\|_a$. For any $v \in V$.

$\|v\|_a \overset{\varepsilon}{\sim} \|v\|_a$

$\equiv \varepsilon\|v\|_a \leq \|v\|_a \leq \frac{1}{\varepsilon}\|v\|_a$.

$\|v\|_a$ should be positive definite, so $\|v\|_a \geq 0$.

When $\|v\|_a = 0$,

$\varepsilon\|v\|_a \leq \|v\|_a \leq \frac{1}{\varepsilon}\|v\|_a \equiv \varepsilon \times 0 \leq 0 \leq \frac{1}{\varepsilon}\times 0$. which is true..

When $\|v\|_a > 0$. $\because \varepsilon \in (0,1]$,

$\therefore \varepsilon\|v\|_a \leq \|v\|_a \leq \frac{1}{\varepsilon}\|v\|_a \equiv \varepsilon \leq 1 \leq \frac{1}{\varepsilon}$ is true.

$\therefore$ ε-equivalence is reflexive for all $\varepsilon \in (0,1]$.

(b) For $v \in V$. suppose $\|v\|_a \overset{\varepsilon}{\sim} \|v\|_b \equiv \varepsilon\|v\|_a \leq \|v\|_b \leq \frac{1}{\varepsilon}\|v\|_a$ holds.

If it is symmetric, then need to prove that

$\|v\|_b \overset{\varepsilon}{\sim} \|v\|_a \equiv \varepsilon\|v\|_b \leq \|v\|_a \leq \frac{1}{\varepsilon}\|v\|_b$ also holds.

$\varepsilon\|v\|_a \leq \|v\|_b \leq \frac{1}{\varepsilon}\|v\|_a \equiv \frac{1}{\varepsilon}\times\varepsilon\|v\|_a \leq \frac{1}{\varepsilon}\|v\|_b \leq \frac{1}{\varepsilon}\times\frac{1}{\varepsilon}\|v\|_a$

$\equiv \|v\|_a \leq \frac{1}{\varepsilon}\|v\|_b \leq \frac{1}{\varepsilon^2}\|v\|_a$.

$\therefore \|v\|_a \leq \frac{1}{\varepsilon}\|v\|_b$ holds.

$\varepsilon\|v\|_a \leq \|v\|_b \leq \frac{1}{\varepsilon}\|v\|_a \equiv \varepsilon\times\varepsilon\|v\|_a \leq \varepsilon\|v\|_b \leq \varepsilon\times\frac{1}{\varepsilon}\|v\|_a$

$\equiv \varepsilon^2\|v\|_a \leq \varepsilon\|v\|_b \leq \|v\|_a$.

$\therefore \varepsilon\|v\|_b \leq \|v\|_a$ holds.

$\therefore \varepsilon\|v\|_b \leq \|v\|_a \leq \frac{1}{\varepsilon}\|v\|_b \equiv \|v\|_b \overset{\varepsilon}{\sim} \|v\|_a$ also holds.

$\therefore$ ε-equivalence is symmetric for all $\varepsilon \in (0,1]$.

(c) For any $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in V$, $\|v\|_1 = |v_1| + |v_2|$, $\|v_2\| = \sqrt{v_1^2 + v_2^2}$.

$\therefore \|v\|_1 \overset{\varepsilon}{\sim} \|v\|_2 \equiv \varepsilon\|v\|_1 \leq \|v\|_2 \leq \frac{1}{\varepsilon}\|v\|_1$

$\equiv \varepsilon(|v_1| + |v_2|) \leq \sqrt{v_1^2 + v_2^2} \leq \frac{1}{\varepsilon}(|v_1| + |v_2|)$

Let $l = \sqrt{v_1^2 + v_2^2}$, $|v_1| = l \cdot \cos\theta$, $|v_2| = l \cdot \sin\theta$. $0 \leq \theta \leq \frac{\pi}{2}$.

$\therefore \|v\|_1 \overset{\varepsilon}{\sim} \|v\|_2 \equiv \varepsilon \cdot l \cdot (\cos\theta + \sin\theta) \leq l \leq \frac{1}{\varepsilon} \cdot l \cdot (\cos\theta + \sin\theta)$

When $l = 0$, $\varepsilon \in (0, 1]$

When $l > 0$,

$\|v\|_1 \overset{\varepsilon}{\sim} \|v\|_2 \equiv \varepsilon \leq \frac{1}{\cos\theta + \sin\theta} \leq \frac{1}{\varepsilon}$.

$0 \leq \theta \leq \frac{\pi}{2}$, $1 \leq \cos\theta + \sin\theta \leq \sqrt{2}$. $\frac{\sqrt{2}}{2} \leq \frac{1}{\cos\theta + \sin\theta} \leq 1$.

$\therefore \varepsilon \leq \frac{\sqrt{2}}{2}$ and $\frac{1}{\varepsilon} \geq 1 \equiv \varepsilon \leq 1$.

$\therefore$ The largest value of ε is $\frac{\sqrt{2}}{2}$.

E8

(a) If $x \in U$, there exists a vector $w \in R^2$ that

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} w = \begin{bmatrix} 12 \\ 12 \\ 18 \end{bmatrix}.$$

To solve w:

$$\begin{bmatrix} 1 & 2 & 12 \\ 1 & 1 & 12 \\ 1 & 0 & 18 \end{bmatrix} \xrightarrow[R_2 := R_2 - R_3]{R_1 := R_1 - 2R_2 + R_3} \begin{bmatrix} 0 & 0 & 6 \\ 0 & 1 & -6 \\ 1 & 0 & 18 \end{bmatrix} \xrightarrow{\text{Swap } R_1, R_3} \begin{bmatrix} 1 & 0 & 18 \\ 0 & 1 & -6 \\ 0 & 0 & 6 \end{bmatrix}$$

From R_3: $0 + 0 = 6$. which is always false.

$\therefore w$ not exists.

$\therefore x \notin U$.

(b) The basis matrix $B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$

$$B^T B \lambda = B^T x$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \lambda = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 12 \\ 12 \\ 18 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \lambda = \begin{bmatrix} 42 \\ 36 \end{bmatrix}$$

$$\lambda = \begin{bmatrix} 17 \\ -3 \end{bmatrix}$$

$$\pi_u(x) = B\lambda = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 17 \\ -3 \end{bmatrix} = \begin{bmatrix} 11 \\ 14 \\ 17 \end{bmatrix}$$

(C) $\pi_u(x) = \begin{bmatrix} 11 \\ 14 \\ 17 \end{bmatrix} = 17 \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-3) \times \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

(d) $d(x, U) = \min_{y \in U} \|x - y\|_2$

$$= \|x - \pi_u(x)\|_2$$

$$= \left\| \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\|_2$$

$$= \sqrt{1^2 + (-2)^2 + 1^2}$$

$$= \sqrt{6}$$