The Australian National University
School of Computing
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COMP3670/6670: Introduction to Machine Learning

Release Date. Aug 4th, 2021
Due Date. 11:59pm, Aug 22th, 2021
Maximum credit. 100

Exercise 1 **Solving Linear Systems** (4+4 credits)

Find the set $\mathcal{S}$ of all solutions $\mathbf{x}$ of the following inhomogenous linear systems $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ and $\mathbf{b}$ are defined as follows. Write the solution space $\mathcal{S}$ in parametric form.

(a)

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 5 \\ 1 & 4 & 3 \\ 2 & 7 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -4 \\ -2 \\ -2 \end{bmatrix}$$

(b)

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 0 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$$

Exercise 2 **Inverses** (4 credits)

For what values of $[a, b, c]^T \in \mathbb{R}^3$ does the inverse of the following matrix exist?

$$\begin{bmatrix} 1 & a & b \\ 1 & 1 & c \\ 1 & 1 & 1 \end{bmatrix}$$

Exercise 3 **Subspaces** (3+3+3+3 credits)

Which of the following sets are subspaces of $\mathbb{R}^3$? Prove your answer. (That is, if it is a subspace, you must demonstrate the subspace axioms are satisfied, and if it is not a subspace, you must show which axiom fails.)

(a) $A = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$

(b) $B = \{(x, y, z) : x + y + z = 0\}$.

(c) $C = \{(x, y) \in \mathbb{R}^2 : x = 0 \text{ or } y = 0\}$

(d) $D =$ The set of all solutions $\mathbf{x}$ to the matrix equation $\mathbf{Ax} = \mathbf{b}$, for some matrix $\mathbf{A}$ and some vector $\mathbf{b}$. (Hint: Your answer may depend on $\mathbf{A}$ and $\mathbf{b}$.)

Exercise 4 **Linear Independence** (4+8+8 credits)

Let V and W be vector spaces. Let $T : V \to W$ be a linear transformation.

(a) Prove that $T(\mathbf{0}) = \mathbf{0}$.

(b) For any integer $n \geq 1$, prove that given a set of vectors $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ in V and a set of coefficients $\{c_1, \ldots, c_n\}$ in $\mathbb{R}$, that

$$T(c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n) = c_1T(\mathbf{v}_1) + \ldots + c_nT(\mathbf{v}_n)$$

(c) Let $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ be a set of linearly **dependent** vectors in V.
Define $\mathbf{w}_1 := T(\mathbf{v}_1), \ldots, \mathbf{w}_n := T(\mathbf{v}_n)$.
Prove that $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is a set of linearly **dependent** vectors in W.

Exercise 5 **Inner Products** (4+8 credits)

(a) Show that if an inner product $\langle \cdot, \cdot \rangle$ is symmetric and linear in the first argument, then it is bilinear.

(b) Define $\langle \cdot, \cdot \rangle$ for all $\mathbf{x} = [x_1, x_2]^T \in \mathbb{R}^2$ and $\mathbf{y} = [y_1, y_2]^T \in \mathbb{R}^2$ as

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2 + 2(x_1y_2 + x_2y_1)$$

Which of the three inner product axioms does $\langle \cdot, \cdot \rangle$ satisfy?

Exercise 6 **Orthogonality** (8+6 credits)

Let V denote a vector space together with an inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$.
Let $\mathbf{x}, \mathbf{y}$ be **non-zero** vectors in V.

(a) Prove or disprove that if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal, then they are linearly independent.

(b) Prove or disprove that if $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, then they are orthogonal.

Exercise 7 **Properties of Norms** (4+4+10 credits)

Given a vector space V with two norms $\|\cdot\|_a : V \to \mathbb{R}_{\geq 0}$ and $\|\cdot\|_b : V \to \mathbb{R}_{\geq 0}$, we say that the two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ are *ε-equivalent* if for any $\mathbf{v} \in V$, we have that

$$\varepsilon\|\mathbf{v}\|_a \leq \|\mathbf{v}\|_b \leq \frac{1}{\varepsilon}\|\mathbf{v}\|_a.$$

where $\varepsilon \in (0, 1]$.

If $\|\cdot\|_a$ is ε-equivalent to $\|\cdot\|_b$, we denote this as $\|\cdot\|_a \overset{\varepsilon}{\sim} \|\cdot\|_b$.

(a) Is ε-equivalence reflexive for all $\varepsilon \in (0, 1]$?
(Is it true that $\|\cdot\|_a \overset{\varepsilon}{\sim} \|\cdot\|_a$?)

(b) Is ε-equivalence symmetric for all $\varepsilon \in (0, 1]$?
(Does $\|\cdot\|_a \overset{\varepsilon}{\sim} \|\cdot\|_b$ imply $\|\cdot\|_b \overset{\varepsilon}{\sim} \|\cdot\|_a$?)

(c) Assuming that $V = \mathbb{R}^2$, prove that $\|\cdot\|_1 \overset{\varepsilon}{\sim} \|\cdot\|_2$. for the largest ε possible.

Exercise 8 **Projections** (3+3+3+3 credits)

Consider the Euclidean vector space $\mathbb{R}^3$ with the dot product. A subspace $U \subset \mathbb{R}^3$ and vector $\mathbf{x} \in \mathbb{R}^3$ are given by

$$U = \text{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}\right\}, \mathbf{x} = \begin{bmatrix}12\\12\\18\end{bmatrix}$$

(a) Show that $\mathbf{x} \notin U$.

(b) Determine the orthogonal projection of $\mathbf{x}$ onto U, denoted $\pi_U(\mathbf{x})$.

(c) Show that $\pi_U(\mathbf{x})$ can be written as a linear combination of $[1, 1, 1]^T$ and $[2, 1, 0]^T$.

(d) Determine the distance $d(\mathbf{x}, U) := \min_{\mathbf{y} \in U} \|\mathbf{x} - \mathbf{y}\|_2$.