Revisiting sets

The first five slides were prepared by Pierre Portal and Malcolm Brooks; the next three are based very closely on the presentation in our optional text; blame Adam Piggott for the slides after that.

Text Reference (Epp) 5ed: Section 6.4

The Barber Puzzle

In a certain town, there is a barber who shaves all those townsfolk, and only those townsfolk, who do not shave themselves. Does the barber shave themselves?

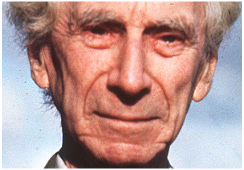
Let S denote the set of townsfolk who shave themselves. Either the barber is a member of S (they shave themselves), or the barber is not a member of S (they do not shave themselves). We consider cases.

Consider first the case that the barber is a member of S. Then they are a member of the set of townsfolk who shave themselves. But no member of this set is shaved by the barber. We have a contradiction. So this case is impossible.

Next consider the case that the barber is not a member of S. Then they are a not a member of the set of townsfolk who shave themselves. The barber shaves every member of this set. We have a contradiction. So this case is impossible too...This is a problem!

(Bertrand) Russell's paradox

(within naive set theory)



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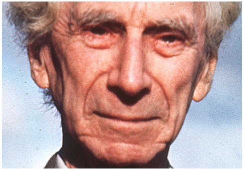

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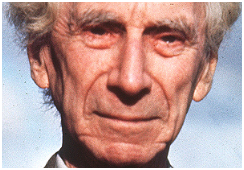

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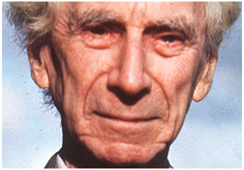

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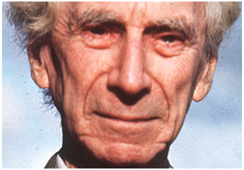

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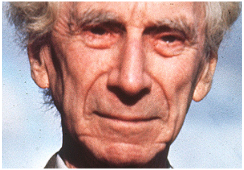

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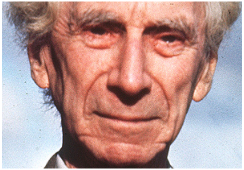

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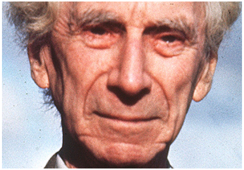

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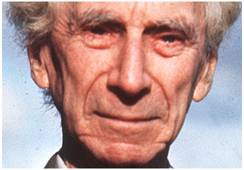

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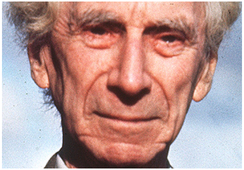

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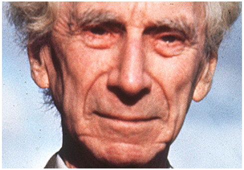

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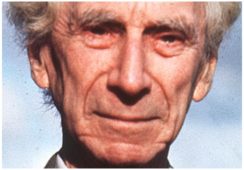

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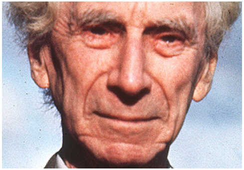

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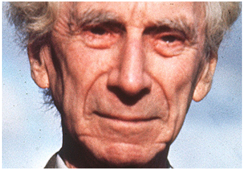

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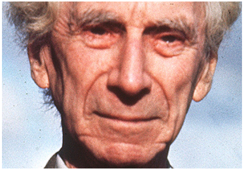

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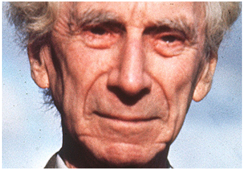

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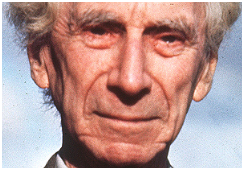

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A contradiction either way! (paradox) Naive set theory fails!

Axiomatic Set Theory

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But, if we have managed to define A as the set of all animals, then "the sets of all animals that are birds" does define a set in ZFC:

A: set of all animals. $\qquad B = \{b \in A \;;\; b \text{ is a bird}\} \subseteq A.$

Russell's paradox removed (for interest only)

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$\mathcal{R}_{\mathcal{U}}$ regular $\Longrightarrow \mathcal{R}_{\mathcal{U}}$ is not a member of itself

Russell's paradox removed (for interest only)

In ZFC we can still say that a set S is regular if and and only if it satisfies the condition (predicate) $S \notin S$.
However we cannot define $\mathcal{R}$ as the set of *all* regular sets.
Instead, for any *known set of sets* $\mathcal{U}$ we can define $\mathcal{R}_{\mathcal{U}}$ by

$$\boxed{\mathcal{R}_{\mathcal{U}} = \{S \in \mathcal{U} \ ; \ S \text{ is regular}\} = \{S \in \mathcal{U} \ ; \ S \notin S\}}$$

So we again ask: **Is $\mathcal{R}_{\mathcal{U}}$ itself regular?**

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\mathcal{R}_{\mathcal{U}} \text{ not regular} \quad &\Longrightarrow \quad \mathcal{R}_{\mathcal{U}} \text{ is a member of itself} \Longrightarrow \mathcal{R}_{\mathcal{U}} \in \mathcal{R}_{\mathcal{U}} \\
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Since $\Big[(\mathcal{R}_{\mathcal{U}} \text{ regular}) \Longrightarrow (\mathcal{R}_{\mathcal{U}} \text{ is not regular})\Big]$ is contradictory we conclude that $\mathcal{R}_{\mathcal{U}}$ is regular but not in $\mathcal{U}$.

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Since $\Big[(\mathcal{R}_\mathcal{U} \text{ regular}) \Longrightarrow (\mathcal{R}_\mathcal{U} \text{ is not regular})\Big]$ is contradictory we conclude that $\mathcal{R}_\mathcal{U}$ is regular but not in $\mathcal{U}$. No overall contradiction.

The Halting Problem

Theorem (Alan Turing, 1936): There is no algorithm that will accept any algorithm X and data set D as input and then will output "halts" or "loops forever" to indicate whether or not X terminates in a finite number of steps when X is run with data set D

Proof: We shall use a proof by contradiction. Suppose that there is an algorithm, CheckHalt, such that if algorithm X and a data set D are input, then:

- CheckHalt(X, D) prints "halts" if X terminates in a finite number of steps when run with data set D;
- CheckHalt(X, D) prints "loops forever" if X does not terminate in a finite number of steps when run with data set D.

The Halting Problem (continued)

A sequence of characters making up an algorithm X can be regarded as a data set itself. So it is possible to call CheckHalt(X, X).

Let Test be a new algorithm that takes as input an algorithm X and so that:

- Test(X) loops forever if CheckHalt(X, X) prints "halts";
- Test(X) stops if CheckHalt(X, X) prints "loop forever".

(The existence of such an algorithm follows immediately the existence of CheckHalt.)

Now consider what happens when we run the algorithm Test with the input Test. Either it terminates after a finite number of steps or it loops forever.

The Halting Problem (continued)

Consider first the case that Test(Test) terminates after a finite number of steps. Then CheckHalt(Test, Test) prints “halts” and so Test(Test) loops forever. This is a contradiction.

Consider next the case that Test(Test) loops forever. Then CheckHalt(Test, Test) prints “loops forever” and so Test(Test) terminates in a finite number of steps. This is also a contradiction.

In each case, we reached a contradiction. Our supposition that an algorithm such as CheckHalt exists allowed us to deduce a false statement. It is therefore false itself. □

An example to pull some ideas together

In parallel computing, multiple CPUs are connected into a network. Each CPU gets busy working on different parts of a problem. Occasionally, the CPUs need to communicate.

PROBLEM: How can we connect many CPUs into network so that:

1. We don't use too many connections, because connections cost money and take up physical space
2. When one CPU needs to communicate with another, the message does not need to be passed through too many intermediary CPUs
3. CPUs are labelled in a logical way that allows us to write easily write algorithms for getting messages around the network.

A possible solution

Connect your CPUs so as to make a hypercube.

For each $n \in \{1, 2, 3, \dots\}$, the **hypercube** H_n is the graph H_n such that

- The vertex set of H_n is $\{0, 1\}^n$; that is, the vertex set of H_n is the set of bit strings of length n
- Two vertices of H_n are adjacent if and only if they differ in exactly one bit.

Getting comfortable with H_n

When trying to become comfortable with a new familky of objects, it helps to spend some time building intuition...just thinking about them

Let's try drawing some hypercubes

The vertex set of H_1 is $\{0, 1\}$. We can identify the vertices of H_1 as points in $\mathbb{R}$ (the number line), and then draw the edge between them to get ... an interval.

The vertex set of H_2 is $\{00, 01, 10, 11\}$. We can identify the vertices of H_2 as the points $\{(0, 0), (0, 1), (1, 0), (1, 1\}$ in $\mathbb{R}^2$ (the Euclidean plane), and then draw the edges between them to get ... the frame of a square.

Getting comfortable with H_n (cont.)

The vertex set of H_3 is $\{000, 001, 010, 011, 100, 101, 110, 111\}$. We can identify the vertices of H_3 as the points

$$\{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$$

in $\mathbb{R}^3$, and then draw the edges between them to get ... the frame of a cube.

What do we get when represent H_4 in $\mathbb{R}^4$?

Some questions

Let $n \in \mathbb{N}$.

1. How many vertices in H_n?
2. What is the degree of each vertex in H_n?
3. How many edges in H_n?
4. What is the greatest distance between any pair of vertices in H_n?
5. Find a Hamilton circuit in H_n, if one exists.

Now suppose that a parallel computing network has the structure of H_n. In this context, what is the importance of each question and answer above?

A Hamilton Circuit in H_n

We use the n-bit reflected binary code to build a Hamilton circuit in H_n.

0, 1, 0 is a Hamilton circuit on H_1.

start with rbc for $n-1$	reflect	add 0's above and 1's below
0	0	00
1	1	01
	1	11
	0	10

Note that 00, 01, 11, 10, 00 is a Hamilton circuit in H_2.

A Hamilton Circuit in H_n (cont.)

start with rbc for $n-1$	reflect	add 0's above and 1's below
00	00	000
01	01	001
11	11	011
10	10	010
	10	110
	11	111
	01	101
	00	100

Note that 000, 001, 011, 010, 110, 111, 101, 100, 000 is a Hamilton circuit in H_3.

Pros and Cons

What do you think are the pros and cons of arranging your parallel computing network so that it has the structure of a hypercube?