

COMP3670/6670: Introduction to Machine Learning

Exercises with a ! denote harder ones, !! denotes very difficult, and !!! denotes optional challenge exercises.

This tutorial will be primarily about proofs in analytic geometry. There are far too many exercises to do in the 2 hours, so you should choose some particular ones to work on. Your tutor will present some in class, and feel free to post partial solutions on Piazza if you get stuck.

Question 1 Properties of the zero vector

Show that for any vector space V with any inner product $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$, we have that $\mathbf{0}$ (the zero vector) is orthogonal to every vector $\mathbf{v} \in V$.

Also, show that for any vector $\mathbf{v} \in V$, that $\{\mathbf{v}, \mathbf{0}\}$ forms a linearly dependent set.

Question 2 Inner products

Prove that the standard Euclidean inner product on \mathbb{R}^2 given by

$$\mathbf{x} \cdot \mathbf{y} := x_1 y_1 + x_2 y_2$$

is an inner product.

Question 3 Pythagorus

We have that any inner product induces a norm,

$$\|\mathbf{x}\| := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$$

Show that for two orthogonal vectors \mathbf{x} and \mathbf{y} (that is $\langle \mathbf{x}, \mathbf{y} \rangle = 0$) that the following holds

$$\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 = \|\mathbf{x} + \mathbf{y}\|^2$$

(This is an extension of Pythagorus' Theorem, that for a right angled triangle with hypotenuse of length c , and two other sides of length a and b , that $a^2 + b^2 = c^2$.)

Question 4 Parseval's Identity

Let V be a vector space, together with an inner product $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$. Given a set of orthogonal vectors $\{x_1, \dots, x_n\}$, show that

$$\left\| \sum_{i=1}^n x_i \right\|^2 = \sum_{i=1}^n \|x_i\|^2$$

Question 5 Norms

1. Prove that the Manhattan norm (l_1 norm) on \mathbb{R}^2 defined by

$$\|\mathbf{x}\|_1 := |x_1| + |x_2|$$

is a norm. (You will need the triangle inequality on \mathbb{R} , $|a + b| \leq |a| + |b|$, to help you.)

2. ! Prove that the supremum norm (l_∞ norm) on \mathbb{R}^2 defined by

$$\|\mathbf{x}\|_\infty := \max(|x_1|, |x_2|)$$

is a norm. (Hint: You will need triangle inequality on \mathbb{R} , and the property that if $A \subseteq B$, then $\max_{x \in A} f(x) \leq \max_{x \in B} f(x)$.)

Question 6**! Basis of a vector space**

Let V be a finite dimensional vector space, and let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for V . Suppose that for any two basis vectors \mathbf{b}_i and \mathbf{b}_j , we can compute the inner product $\langle \mathbf{b}_i, \mathbf{b}_j \rangle$. Then, show that for any two vectors \mathbf{u}, \mathbf{v} in V , we can express the inner product $\langle \mathbf{u}, \mathbf{v} \rangle$ in terms of the inner product of basis vectors $\langle \mathbf{b}_i, \mathbf{b}_j \rangle$

(Hint: Use the fact that B spans the space V .)

Question 7**Orthogonal matrices preserve angles and norms**

Suppose we are in the vector space \mathbb{R}^n , together with the standard Euclidean dot product, that is

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} := \mathbf{x}^T \mathbf{y}$$

Let

$$\|\mathbf{x}\|_2 = \sqrt{\mathbf{x} \cdot \mathbf{x}}.$$

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be an orthogonal matrix (that is, $\mathbf{A}^{-1} = \mathbf{A}^T$.)

Show that for any vector $\mathbf{x} \in \mathbb{R}^n$ that

$$\|\mathbf{A}\mathbf{x}\|_2 = \|\mathbf{x}\|_2$$

Using the above result (or otherwise), show that if the angle between two vectors \mathbf{x} and \mathbf{y} is θ then the angle between $\mathbf{A}\mathbf{x}$ and $\mathbf{A}\mathbf{y}$ is either θ , or $-\theta$ (modulo 2π).

Given an example of an orthogonal matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ and vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, such that the angle between \mathbf{x} and \mathbf{y} is not the same as the angle between $\mathbf{A}\mathbf{x}$ and $\mathbf{A}\mathbf{y}$.

Question 8**Rotation matrices preserve norms**

Given a vector $\mathbf{x} \in \mathbb{R}^2$ and the rotation matrix

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Show that for any angle of rotation θ , we have

$$\|\mathbf{x}\|_2 = \|\mathbf{R}(\theta)\mathbf{x}\|_2$$

Question 9**Gram-Schmidt**

Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, the standard basis vectors for \mathbb{R}^2 . Let \mathbf{v} be any vector in \mathbb{R}^2 .

Define the projection operator

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) := \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}$$

(if $\mathbf{u} = \mathbf{0}$, then we define $\text{proj}_{\mathbf{0}}(\mathbf{v}) = \mathbf{0}$.)

The Gram-Schmidt algorithm takes a set of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ and proceeds as follows:

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{v}_1 \\ \mathbf{u}_2 &= \mathbf{v}_2 - \text{proj}_{\mathbf{u}_1}(\mathbf{v}_2) \\ \mathbf{u}_3 &= \mathbf{v}_3 - \text{proj}_{\mathbf{u}_1}(\mathbf{v}_3) - \text{proj}_{\mathbf{u}_2}(\mathbf{v}_3) \\ &\dots = \dots \\ \mathbf{u}_n &= \mathbf{v}_n - \sum_{j=1}^{n-1} \text{proj}_{\mathbf{u}_j}(\mathbf{v}_n) \end{aligned}$$

The output $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is a set of orthonormal vectors that spans the same set as $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ (If the dimension of the space spanned by the \mathbf{v}_i 's is less than n , then some of the \mathbf{u}_i 's will be zero.)

Suppose we are considering vectors in the vector space of \mathbb{R}^2 .

Show that if we input $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{v}\}$ to the Gram-Schmidt algorithm, the output is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{0}\}$

Question 10

!!! Cauchy-Schwartz

Prove the Cauchy-Schwartz inequality for a general inner product and corresponding induced norm:

$$\langle \mathbf{u}, \mathbf{v} \rangle \leq \|\mathbf{u}\| \|\mathbf{v}\|$$

(Hint: Let $\mathbf{z} = \mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v}$, and start with the fact that $\langle \mathbf{z}, \mathbf{z} \rangle \geq 0$.)