1. *This question is is mostly just about definitions; almost no calculations are required.* A graph G has $V(G) = \{a, b, c, d\}$ and $E(G) = \{\{a, b\}, \{a, d\}, \{b, b\}, \{b, d\}, \{d, b\}, \{d, d\}\}$.

(a) Draw the graph

(b) What is the order of the graph?

(c) Write out a table of edges for G.

(d) Write out a vertex adjacency listing for G.

(e) What extra information is included in the vertex adjacency listing that is not in the table of edges?

(f) Write down the adjacency matrix for G.

(g) Does G contain loops? How can you tell from the adjacency matrix?

(h) Does G contain parallel edges? How can you tell from the adjacency matrix?

(i) Is G a simple graph? Why?

(j) Name all the isolated vertices in G. How can you tell from the adjacency matrix?

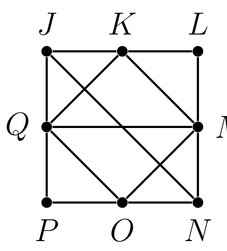
(k) Is the adjacency matrix symmetric? Why?

(l) How many paths of length 2 are there from a to b? Is it possible to get this information from the adjacency matrix? If so, How?

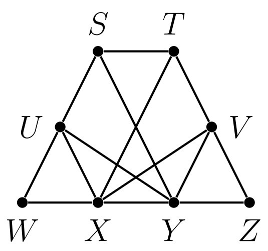
(m) Is G connected? Is it possible to tell this from the adjacency matrix? If so, How?

(n) What are the degrees of a, b, c and d?

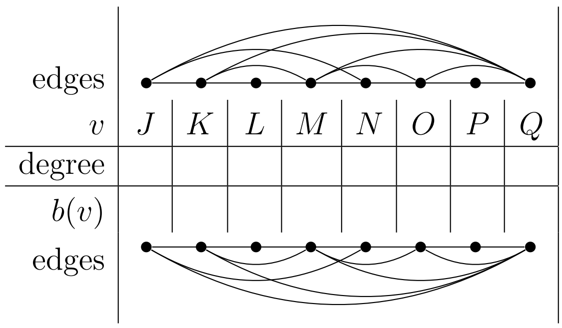
2. Although it may seem implausible, the graphs G and H below are isomorphic. Find an isomorphism $b : V(G) \to V(H)$ that verifies this.
Hint: For every vertex v of G, the degree of $b(v)$ must be the same as the degree of v.



Graph G

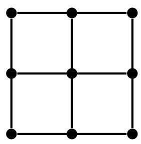


Graph H

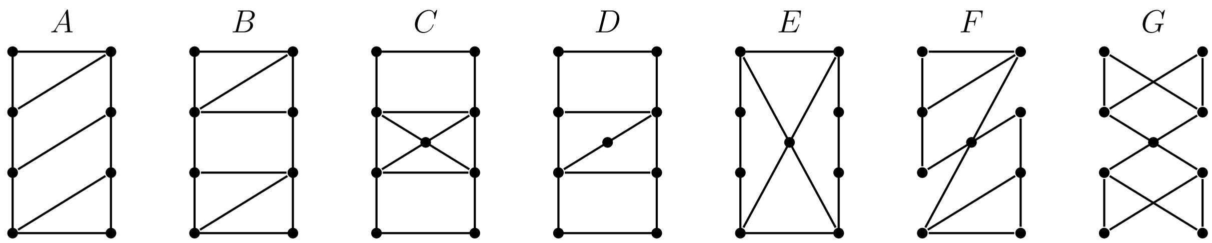


3. How many non-isomorphic simple graphs have exactly four vertices? Draw one of each.

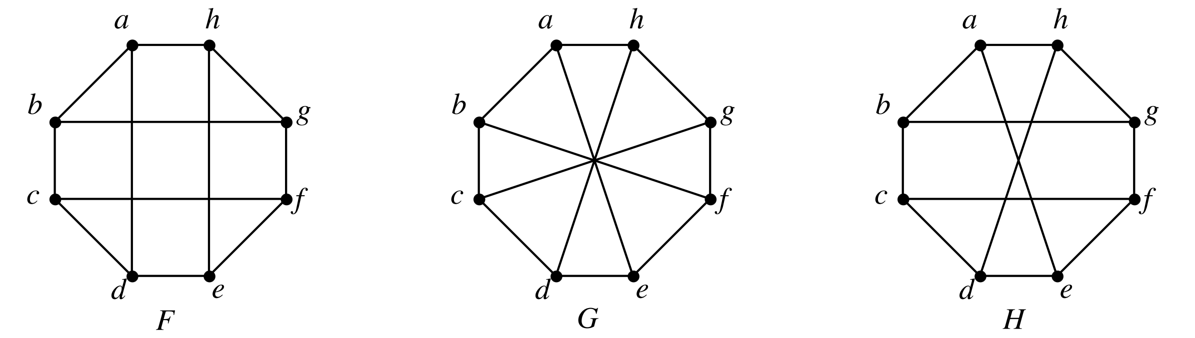
4. Γ :



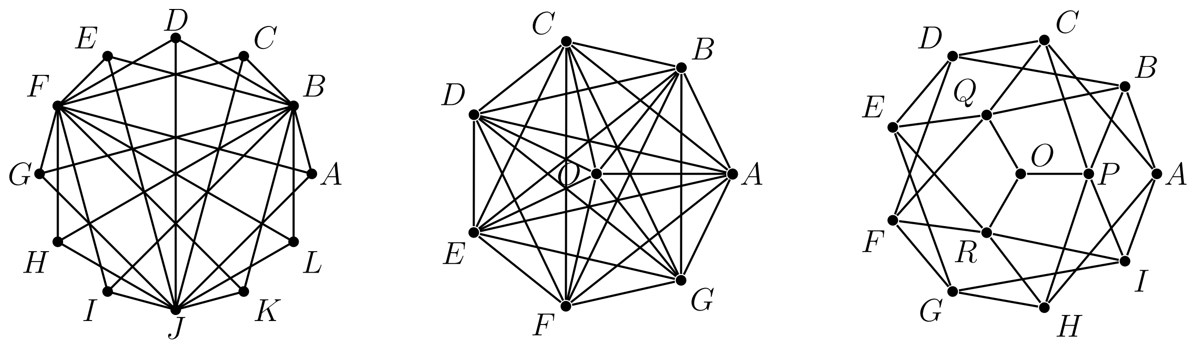
The graph Γ at left is not isomorphic to any of the graphs A - G below. In each case state an invariant property of Γ that is not possessed by the other graph. (An 'invariant' property is one that that does not depend on any particular labelling of the vertices.)



5. [*Challenge*] For each pair of the graphs F, G, H below, decide whether or not the two graphs are isomorphic. For two graphs that are isomorphic, give an isomorphism between them, as well as a convincing argument showing why your function is an isomorphism. For two graphs that are not isomorphic, state an invariant property possessed by one but not by the other.



6. For each of the graphs G_1, G_2, G_3 below decide whether it is bipartite and/or complete and/or complete bipartite, or none of these. In the case of bipartite graphs, specify the two partite sets of vertices. If appropriate, give the standard K-names for the graphs.



Graph G_1 Graph G_2 Graph G_3

7. Recall that a graph H is a subgraph of a graph G if, and only if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Don't forget that every set has itself and the empty set as subsets.

(a) The complete graph K_2 with $V(K_2) = \{a, b\}$ and $E(K_2) = \{\{a, b\}\}$ has 5 subgraphs. Draw them.

(b) The complete graph K_3 has 18 subgraphs. Provide a counting argument to justify this. Ensure that your counting method does not miss any subgraphs nor count any more than once.

(c) How many subgraphs has the complete graph K_4? (No argument required.)

8. Prove or disprove that there exists a simple graph with:

(a) One vertex of degree 1, two vertices of degree 2, three vertices of degree 3 and four vertices of degree 4.

(b) One vertex of degree 1, two vertices of degree 2, three vertices of degree 3, four vertices of degree 4 and five vertices of degree 5.

9. For the 'boxer' graph below give an example of each of the following:

(a) A simple path.

(b) A non-simple path.

(c) A walk that is not a path.

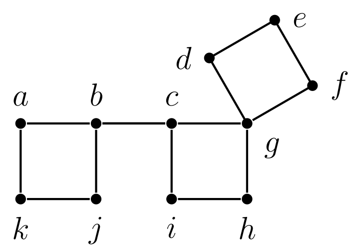
(d) A simple circuit.

(e) A non-simple circuit.

(f) A closed walk that is not a circuit.

(g) A bridge.

(h) A cut vertex that is not part of a bridge.



10. For the 'boxer' graph of Q9, find those of the following walks that exist. Justify any claim that a particular walk does not exist.

(a) An Euler path (c) An Euler circuit

(b) A Hamilton path (d) A Hamilton circuit

11. Which, if any, of the graphs A - G of Q4 have an Euler circuit? Justify your answer, but there is no need to actually specify any circuits.

12. [*Challenge*] For each of the graphs G_1 - G_3 of Q6 prove or disprove that the graph has a Hamilton circuit.

13. How many different walks of length 6 on the complete graph K_3:

(a) return to their starting point; (b) do not return to their starting point.

Hint: Use powers of the adjacency matrix. Take into account all the different start and end vertices of the walk.

14. Find the connected components of the graph whose adjacency matrix A is shown at right.

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

15. A graph G has 100 vertices labelled $0, 1, \ldots, 98, 99$ and a total of 179 edges comprising two sets

$$\begin{aligned} E_1 &= \{\{i, i{+}10\} : i = 0, \ldots, 89\} \\ E_2 &= \{\{i, i{+}11\} : i = 0, \ldots, 88\} \end{aligned}$$

Prove that G is connected.
[Hint: As an example, a path from 35 to 0 is 35 - 45 - 55 - 44 - 33 - 22 - 11 - 0.]

16. A simple graph G has ten vertices and ten edges. Prove or disprove:

(a) G must be connected. (b) G must contain a circuit.

17. For the 'boxer' graph of Q9:

(a) Draw a spanning tree. (b) How many spanning trees does the graph have?

18. [*Challenge*] A tree T has 30 vertices, of which 10 have degree 3. What is the greatest number of vertices of degree 2 that T could have?