Throughout these questions the notation $\mathbb{N}^*$ *will be used as a shorthand for* $\mathbb{N} \cup \{0\}$.

1. Let $a_n = 2n + 3 \quad \forall n \in \mathbb{N}$. Compute the following.

(a) $\sum_{n=1}^{4} a_n$.

(b) $\prod_{n=1}^{4} a_n$.

2. Let $(x_n)_{n\in\mathbb{N}^*} \subseteq \mathbb{Q}$ be such that $\quad x_0 = 1 \quad$ and $\quad x_{n+1} = \dfrac{x_n}{2} \quad \forall n \in \mathbb{N}^*$.

Prove by mathematical induction that $x_n = 2^{-n} \quad \forall n \in \mathbb{N}^*$.

3. Let $x \in \mathbb{Q}$. Prove that $\quad \forall N \in \mathbb{N}^* \quad (1 + x) \sum_{n=0}^{N} (-x)^n = 1 - (-x)^{N+1}$.

Use mathematical induction.

4. (Population dynamics: Hassel's model). Let $(p_n)_{n\in\mathbb{N}^*} \subseteq \mathbb{Q}$ be the size (in hundreds of individuals) of a population of ponies at time check n. Assume that $p_0 = x$ for some x with $0 < x < 1$ and that $\forall n \in \mathbb{N}^* \ \ p_{n+1} = \dfrac{2p_n}{1 + p_n}$. Prove that $p_n \leq 1 \quad \forall n \in \mathbb{N}$.

5. Let $v_n \in \mathbb{Q}$ represent the quantity, at time $n \in \mathbb{N}^\star$, of a certain virus in the bloodstream. Let a_n represent the number of antibodies produced to fight the virus at the same time n. Assume that a_0 and v_0 are positive and that, $\forall n \in \mathbb{N}^\star$, a_n and v_n are related by the equations

$$\text{A:}\quad a_{n+1} = a_n + 2v_n, \qquad\qquad \text{V:}\quad v_{n+1} = 4v_n - a_n.$$

(a) Explain what these equations model.

(b) Prove that $v_{n+2} = 5v_{n+1} - 6v_n \quad \forall n \in \mathbb{N}^*$.

(c) Use mathematical induction to prove that $v_n = (2v_0 - a_0)3^n + (a_0 - v_0)2^n \quad \forall n \in \mathbb{N}^*$.

(d) Under what initial conditions does the model predict that the virus will eventually be eliminated?

6. Compute the following sums:

(a) $\sum\limits_{n=1}^{30} n$.

(b) $\lfloor \sum\limits_{n=1}^{30} (\left(\frac{5}{4}\right) n + 3) \rfloor$

(c) $\lfloor \sum\limits_{n=1}^{30} \left(\frac{5}{4}\right)^n \rfloor$

(d) $\lfloor \sum\limits_{n=1}^{30} (\left(\frac{5}{4}\right)^n + 3) \rfloor$

7. Selection sort is used to sort the letters of the word SELECTION into alphabetical order. This requires eight applications using the Least element algorithm. Write out the state of the word after each of these eight applications.

8. Merge sort is used to sort the letters of the word TRANSFORMATIONAL into alphabetical order. This requires fifteen applications of the Merge algorithm, spread over four iterations of the Merge sort algorithm. Write out the state of the word after each of these four iterations.

9. The list $(1, 3, 5, 7, 8, 6, 4, 2)$ is to be sorted into ascending order.

(a) How many comparisons will be required using the Selection sort algorithm?

(b) Is the number of comparisons required by Selection sort affected by the order of the original list? If so, what is the most that would ever be required (for list length 8)?

(c) How many comparisons will be required using the Merge sort algorithm?

(d) Is the number of comparisons required by Merge sort affected by the order of the original list? If so, what is the most that would ever be required (for list length 8)?

10. Merge sort is to be used to sort a list of 8 distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ into ascending order. Give examples of list orders that give the algorithm the

(a) least trouble; (best case complexity)

(b) most trouble. (worst case complexity)

"Trouble" is to be measured by the number of comparisons required. Calculate this number in each case.

(c) What should we mean by an average amount of trouble (average case complexity), and how could it be calculated? [Do not attempt to actually do the calculation, unless you have a lot of spare time to kill.]