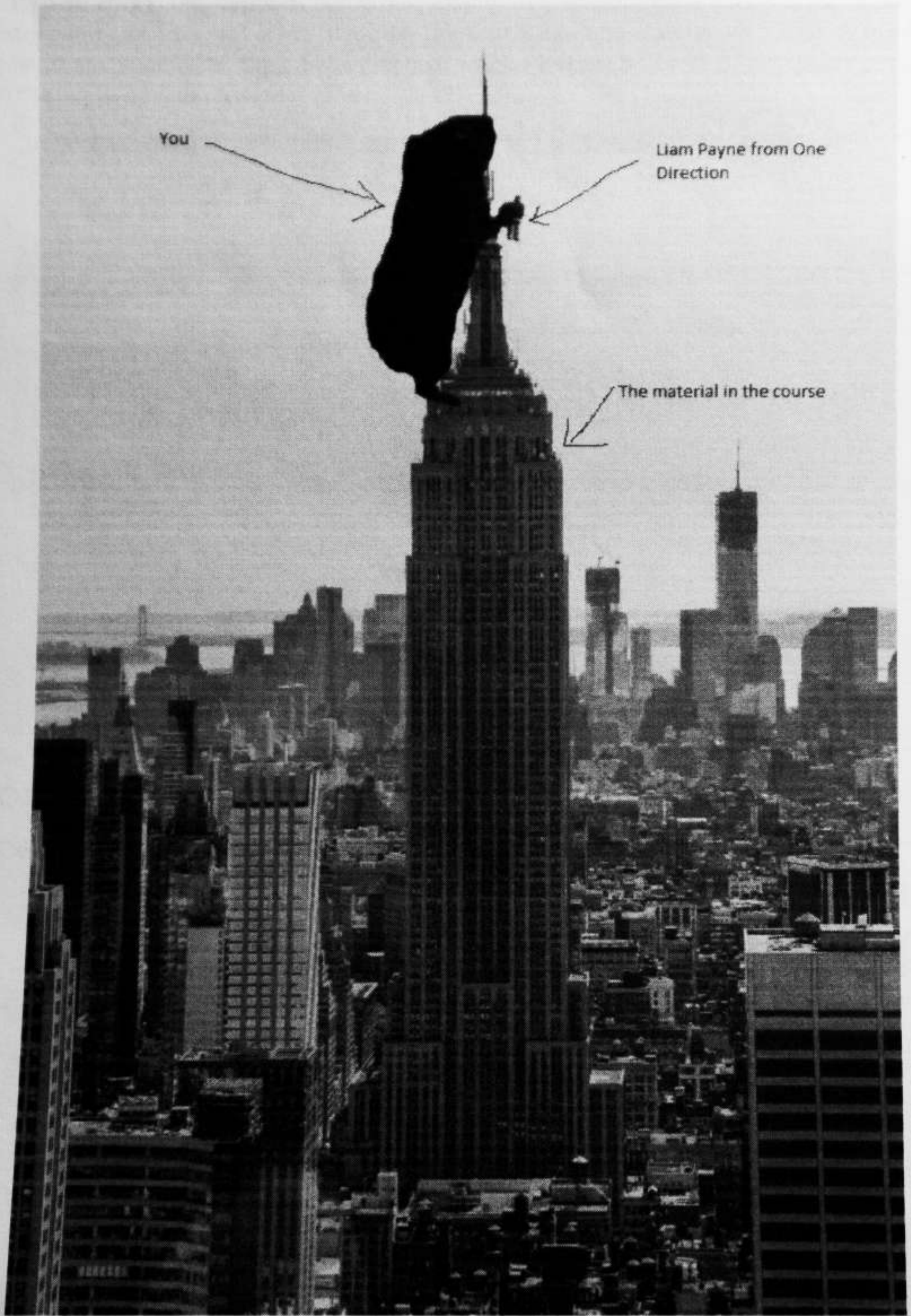
MATH1005/MATH6005 Discrete Mathematical Models
Midsemester Exam, Semester 1, 2021




Artwork by FABP.

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Throughout this exam, we adopt the following conventions: We write $\mathbb{N}$ for the set of positive integers and $\mathbb{N}^*$ for the set of non-negative integers.

**Problem 1 (5 marks)** (a) Explain, using new examples, the difference between statements and predicates, and two different ways in which predicates become statements. Your examples should be new in the sense that they do not appear in the lecture notes or in any other course materials. Write no more than five sentences.

Statement: a sentence that is true or false but not both, e.g. 10 is equal to 9.

Predicate: a sentence containing one or more variables from a domain that it can become statement when a value is given to each variable, e.g. In the domain of integers, $x < 0$.

Ways: Give each variable of the predicate a value.

(b) Let $P$ denote the set of primes. Each of the following is a famous conjecture (a statement that has not been proved or disproved, but that many suspect to be true) concerning $P$.

**(Conjecture 1)** $\forall n \in \mathbb{N}\ \ (((n \geq 4) \wedge (\exists k \in \mathbb{Z}\ (n = 2k))) \rightarrow (\exists p_1, p_2 \in P\ \ n = p_1 + p_2))$.

**(Conjecture 2)** For every positive integer $n$, there exists a prime $p$ such that $n^2 < p < (n+1)^2$.

(i) Restate Conjecture 1 as a statement in English. Make the statement as brief as you can.

For all positive integers n, if n is not smaller than 4 and there exists an integer k that n equals to 2k, then there exist primes $p_1$ and $p_2$ that n equals to $p_1$ plus $p_2$.

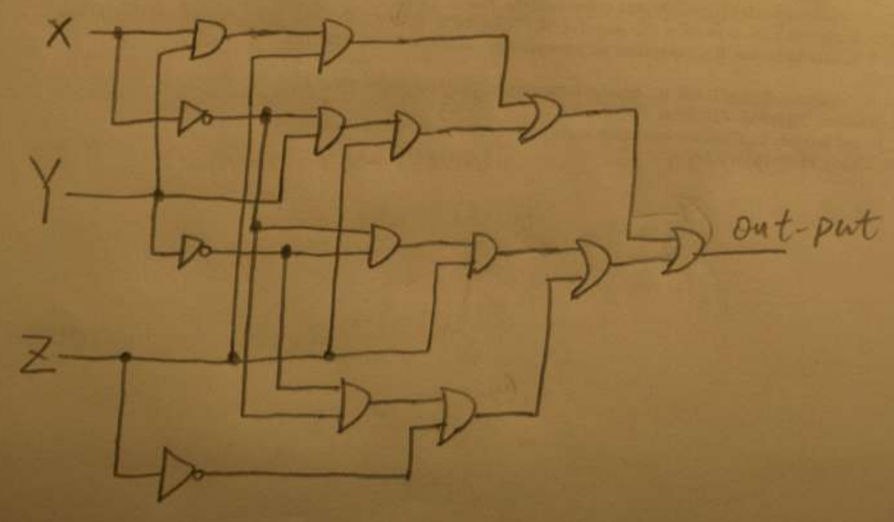
(ii) Use symbolic notation, like that used to state Conjecture 1 above, to state the **negation** of Conjecture 2 without using the symbol $\neg$.

$\forall n \in \mathbb{N}, \exists p \in P\ \ n^2 < p < (n+1)^2$

(c) Complete the following input-output table and then draw a circuit diagram for a circuit that gives the appropriate outputs.

| $X$ | $Y$ | $Z$ | $(\neg X \wedge \neg Y) \vee (Y \wedge Z)$ | | |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 | 0 |
| 0 | 0 | 0 | 1 | 1 | 0 |

$(X \wedge Y \wedge Z) \vee (\neg X \wedge Y \wedge Z) \vee (\neg X \wedge \neg Y \wedge Z) \vee (\neg X \wedge \neg Y \wedge \neg Z)$



**Problem 2 (5 marks)** (a) Give an example of a set $A$ such that $A \subseteq \mathbb{Q} \setminus \mathbb{Z}$ and $A$ has exactly two elements, and an example of a set $B$ that is a subset of the English alphabet such that $B$ has exactly three elements. Then describe each of the following sets using set-roster notation: $A \times B$, $\mathcal{P}(B)$.

$A: \{0.5, 0.6\}$ $\quad$ $B: \{a, b, c\}$

$A \times B: \{(0.5, a), (0.6, a), (0.5, b), (0.6, b), (0.5, c), (0.6, c)\}$

$P(B): \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

(b) We define a sequence by $(a_n)_{n \in \mathbb{N}}$ by

$$\begin{cases} a_1 = 1 \\ \forall n \in \mathbb{N}\ a_{n+1} = a_n + (n+1)^2 \end{cases}$$

Use mathematical induction to prove that the following is an explicit definition of the same sequence

$$\forall n \in \mathbb{N}\ a_n = \frac{n(n+1)(2n+1)}{6}.$$

Basic step: For $n=1$, formula gives $a_1 = \frac{1(1+1)(2\times 1+1)}{6} = \frac{2\times 3}{6} = 1$,

agreeing with the implicit definition.

Inductive step: Let $n \in \mathbb{N}$. Suppose the formula is correct for $a_1, a_2, \cdots a_n$. then:

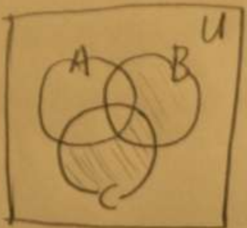
$$\begin{aligned} a_{n+1} &= a_n + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\ &= \frac{(n+1)(2n^2+n+6n+6)}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6} = \frac{(n+1)(n+1+1)(2n+2+1)}{6} \end{aligned}$$

and so the formula is also correct for $n+1$.

By the Principle of Mathematical Induction the formula is correct for all $n \in \mathbb{N}$.

(c) (i) Use Venn diagrams to help you decide whether the following statement is true or false: For any universal set $U$ and for all $A, B, C \in \mathcal{P}(U)$, we have

$$(B \setminus A) \cup (C \setminus A) = (B \cup C) \setminus A$$




True.

(ii) Use the logical equivalences below and the definitions of set operations to prove, or provide a counterexample to disprove, the statement you investigated in part (i).

$x \in (B \setminus A) \cup (C \setminus A)$

$\Leftrightarrow [x \in (B \setminus A)] \vee [x \in (C \setminus A)]$

$\Leftrightarrow [(x \in B) \wedge (x \notin A)] \vee [(x \in C) \wedge (x \notin A)]$

$\Leftrightarrow [(x \in B) \vee (x \in C)] \wedge (x \notin A)$

$\Leftrightarrow x \in (B \cup C) \wedge x \notin A$

$\Leftrightarrow x \in (B \cup C) \setminus A$

$\therefore$ the statement is true.

Given any statement variables $p$, $q$, and $r$, a tautology $t$ and a contradiction $c$, the following logical equivalences hold.

| | | |
|---|---|---|
| 1. Commutative laws: | $p \wedge q \equiv q \wedge p$ | $p \vee q \equiv q \vee p$ |
| 2. Associative laws: | $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ | $(p \vee q) \vee r \equiv p \vee (q \vee r)$ |
| 3. Distributive laws: | $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ | $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ |
| 4. Identity laws: | $p \wedge t \equiv p$ | $p \vee c \equiv p$ |
| 5. Negation laws: | $p \vee \neg p \equiv t$ | $p \wedge \neg p \equiv c$ |
| 6. Double negative law: | $\neg(\neg p) \equiv p$ | |
| 7. Idempotent laws: | $p \wedge p \equiv p$ | $p \vee p \equiv p$ |
| 8. Universal bound laws: | $p \vee t \equiv t$ | $p \wedge c \equiv c$ |
| 9. De Morgan's laws: | $\neg(p \wedge q) \equiv \neg p \vee \neg q$ | $\neg(p \vee q) \equiv \neg p \wedge \neg q$ |
| 10. Absorption laws: | $p \vee (p \wedge q) \equiv p$ | $p \wedge (p \vee q) \equiv p$ |
| 11. Negations of $t$ and $c$: | $\neg t \equiv c$ | $\neg c \equiv t$ |

**Problem 3 (5 marks)** (a) Let $A$ and $B$ be sets and let $R \subseteq A \times B$. What else must be true about $R$ for it to be a surjective function?

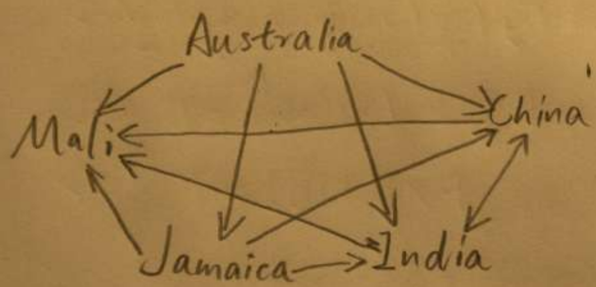
Suppose the relation $R$ is function $f$, then:

$\forall b \in B, \exists a \in A \; f(a) = b$.

(b) A relation $R$ is defined on the set $S = \{\text{Australia, China, India, Jamaica, Mali}\}$ by the rule

$$xRy \Leftrightarrow \text{The word } x \text{ has } \underline{\text{at least}} \text{ as many letters as the word } y.$$

Draw a digraph representing $R$.



(c) Does the function $f : \{1, 2, 3, 4, 5, 6\} \to \{1, 2, 3, 4, 5, 6\}$ given by $f(x) = 3^x \bmod 7$ have an inverse? Justify your answer.

$f(1) = 3^1 \bmod 7 = 3$

$f(2) = 3^2 \bmod 7 = 2$

$f(3) = 3^3 \bmod 7 = 27 \bmod 7 = 6$

$f(4) = 3^4 \bmod 7 = 148 \bmod 7 = 1$

$f(5) = 3^5 \bmod 7 = 444 \bmod 7 = 3$

$f(6) = 3^6 \bmod 7 = 1332 \bmod 7 = 2$

For $f^{-1}$, there is not a number a satisfies that $f^{-1}(5) = a$. so $f$ does not have an inverse.

(d) Let $A = \{\text{cat}, \text{dog}, \text{chicken}\}$, let $B = \{2, 5, 6, 9\}$ and let $C = \{x, y, z\}$. Prove or disprove the following: If $f : A \to B$ is injective, and $g : B \to C$ is surjective, then $g \circ f$ is surjective.

The statement is false.

Suppose that $f(\text{cat}) = 2$. $f(\text{dog}) = 5$, $f(\text{chicken}) = 6$.

$g$: $f(2) = x$. $f(5) = f(6) = y$. $f(9) = z$.

when $c = z \in C$, if $g \circ f$ is surjective, we need there is a $b \in B$ and $b = 9$. which means $\exists a \in A$. $f(a) = 9$. However, $f(a)$ can never be $9$.

Hence $g \circ f$ is not surjective.

$1110 9 8 7 6 5 4 3 2 1 0$

**Problem 4 (5 marks)** (a) Let $w$ = 110101010010. This bit string is used to represent numbers in different ways.

(i) Write in decimal the integer $x$ represented by $w$ if $w$ represents an integer in binary positional notation.

$$w = (2^1 + 2^4 + 2^6 + 2^8 + 2^{10} + 2^{11})_{10}$$
$$= (2 + 16 + 64 + 256 + 1024 + 2048)_{10}$$
$$= (2050 + 80 + 1280)_{10}$$
$$= (3410)_{10}$$

(ii) Write in decimal the integer $y$ represented by $w$ if $w$ represents an integer using the 12-bit signed integer (the two's complement) format.

$$-(001010101101+1)_2 = -1010101110$$
$$w = 2+4+8+32+128+512$$
$$= (686)_{10}$$

(iii) Write as a decimal the rational number $q$ represented by $w$ if $w$ represents a rational number $q = (-1)^s \times m \times 2^n$ (with $1 \leq m < 2$) by using: the first bit to store $s$; followed by 3 bits to store a non-negative integer representing $n + 3$; followed by 8 bits to store the bits that appear after the binary point in the mantissa $m$.

$$S=1 \quad m = 101 = 5 \quad 5-3=2.$$
$$q = -(1.01010010_2 \times 2^2)$$
$$= -101.01001$$
$$= -(5 + 2^{-2} + 2^{-5})$$
$$= -(5 + 9/32)$$
$$= -\frac{169}{32} \approx -(5.281)_{10}$$

(b) Describe the set $A = \{x \in \mathbb{Z} \mid x \bmod 5 = x \operatorname{div} 5\}$ in set-roster notation. Justify your answer.

$A = \{6, 12, 18, 24\}$

Modulo can not be negative.

Suppose $x \bmod 5 = x \operatorname{div} 5 = k$.

$k < 5 \land k \in \mathbb{N}$.

2 6 1 4 3 5

(c) The list $(X_i)_{1..6} = (G, O, B, L, I, N)$ is to be sorted using Selection sort. The sorting is to be achieved by progressively modifying an index function $\pi$, rather than by shuffling members of the list itself. So initially $(X_i)_{1..6} = (X_{\pi(i)})_{1..6}$ where $\pi(i) = i$ for $i = 1, ..., 6$, and when sorting is complete $\pi$ is sufficiently changed so that $(X_{\pi(i)})_{1..6}$ is in alphabetical order. Complete the table below, or copy what is below to your page and complete it, to show the state of the index function $\pi$ after each time the least element algorithm has been called by Selection Sort. For reference, the algorithm is shown on the next page.

| How many times LEA has been called | $\pi$ | |
|---|---|---|
| 0 | 1 2 3 4 5 6<br>1 2 3 4 5 6 | BOGLIN |
| 1 | 1 2 3 4 5 6<br>3 2 1 4 5 6 | ← write something here |
| 2 | 1 2 3 4 5 6<br>3 1 2 4 5 6 | BGOLIN<br>← write something here |
| 3 | 1 2 3 4 5 6<br>3 1 5 4 2 6 | BGILON<br>← write something here |
| 4 | 1 2 3 4 5 6<br>3 1 5 4 2 6 | BGLLON<br>← write something here |
| 5 | 1 2 3 4 5 6<br>3 1 5 4 6 2 | BGILNO<br>← write something here |
| 6 | 1 2 3 4 5 6<br>3 1 5 4 6 2 | ← write something here |

**Least Element Algorithm**

**Input**: Sequence $(x_i)_{s..f} \subseteq S$, an ordering rule "$\leq$" for $S$ and an index function $\pi$ on $\{s, \ldots, f\}$.

**Output**: Modification to $\pi$ so that $x_{\pi(s)} \leq x_{\pi(i)}$ for $i = s, \ldots, f$.

**Method:**
$i \leftarrow s + 1$.
$m \leftarrow s$.

Loop: If $i = f + 1$ stop.
    If $x_{\pi(i)} < x_{\pi(m)}$ then $m \leftarrow i$.
    $i \leftarrow i + 1$

Repeat loop
Swap the values of $\pi(s)$ and $\pi(m)$.

**Selection sort algorithm**

**Input**: Sequence $(x_i)_{1..n} \subseteq S$, an ordering rule "$\leq$" for $S$ and an index function $\pi$ on $\{1, \ldots, n\}$.

**Output**: Modification to $\pi$ so that $(x_{\pi(i)})_{1..n}$ is in non-decreasing order $x_{\pi(1)} \leq x_{\pi(2)} \leq \cdots \leq x_{\pi(n)}$.

**Method:**
$s \leftarrow 1$

Loop: If $s = n$ stop.
    Run least element algorithm on $(x_{\pi(i)})_{s..n}$
    $s \leftarrow s + 1$
Repeat loop

THIS IS THE END OF THE EXAM.