Total Marks: 15 Value: 5% of final grade

Due: 2 pm Friday 28 May 2021.

Please upload your solutions in PDF format, using the link provided. If you write the solutions by hand, you will need to scan your work and save it as a pdf file.

Page 1 of your solutions document should be a 'cover page' containing **only**:

1. Title: "Graduate Assignment C"
2. Your full name, with surname in upper case.
3. Your ANU ID
4. The declaration: "I have read the ANU Academic Skills statement regarding collusion." (`https://www.anu.edu.au/students/academic-skills/academic-integrity/plagiarism/collusion`) "I have not engaged in collusion in relation to this assignment".
5. Your signature. (If you are typesetting rather than scanning a hand-written document, you can type your name and it will be deemed a signature.)
6. The date and approximate time of your submission.

Regarding item 4, I emphasise the last paragraph of the Academic Skills statement:
The best way people can help each other to understand the material is to discuss the ideas, questions, and potential solutions in general terms. However, ***students should not draw up a detailed plan of their answers together. When it comes to writing up the assignment, it should be done separately. If collusion is detected, all students involved will receive no marks.***

There are three questions. You may find some questions more difficult/time-consuming than others, but nevertheless each question is worth the same (5 marks) and assessed against the same marking criteria. The marking criteria is detailed on the next page.

The following marking criteria will be applied to each question in this assignment.

Score	Description
5	Solutions are correct and complete; solutions are written in complete sentences; solutions are succinct and clearly communicated; notation is used accurately; statements to be justified are justified so well that the explanation or counterexample given constitutes a proof; any hypotheses/assumptions made are explicitly identified; any examples/counterexamples constructed are described effectively and how they serve the purpose at hand is made clear; any new variables used are introduced explicitly.
4	Solutions are correct and complete, except perhaps a minor error; solutions are written in complete sentences almost always; solutions are clearly communicated; notation is used accurately, except perhaps a minor misuse; statements to be justified are justified effectively; any hypotheses/assumptions made are explicitly identified; any examples/counterexamples constructed are described effectively and how they serve the purpose at hand is made clear; any new variables used are introduced explicitly.
3	Solutions are correct and complete, except for several minor errors or omissions; explanation is given for solutions; notation is used accurately most of time; statements to be justified are justified effectively; any hypotheses/assumptions made are identifiable; any examples/counterexamples constructed are described effectively; new variables may be used without introduction, but the role they play is discernible from the context.
2	Solutions do not meet the criteria for 3 points, but they provide evidence of partial understanding of the material and evidence of a substantial effort to answer the question.
1	Solutions do not meet the criteria for 2 points, but they provide evidence of a substantial effort to answer the question.
0	Solutions do not meet the criteria for 1 point.

Question 1 [Enumerative combinatorics (counting)]

(A) Recall that a byte is a binary string of 8 bits. How many bytes contain at least six 1's?

(B) Suppose that A is a set with 5 elements and B is a set with 7 elements.

(i) How many injections (injective functions) are there from A to B?

(ii) How many bijections (bijective functions) are there from A to B?

(iii) How many bijections (bijective functions) are there from A to A?

(iv) How many bijections (bijective functions) are there from A to A with the property that no element of A is mapped to itself?

(C) In no more than 300 words, respond to the following claim: "It is useful for a computer scientist to be skilled in enumerative combinatorics (counting)." You may agree, or disagree or both. Your response should be informed by at least one interesting example that illustrates the point(s) you wish to make.

Question 2 [Probability]

(A) Five firms, F_1, F_2, F_3, F_4, F_5, are such that F_1 has greater political influence than F_2, which in turn has greater political influence than F_3, and so on. A local government has three contracts C_1, C_2, C_3 to offer. Contract C_1 is more valuable to a firm than contract C_2, which in turn in more valuable to a firm than C_3. Any one firm will be awarded at most one contract. Find the probability that, if the contracts are awarded at random, the firm awarded contract C_1 has more political influence than the firm awarded contract C_2 which, in turn, has more political influence than the firm awarded contract C_3.

(B) A complex electronic system is built with a certain number of backup components in its subsystems. One subsystem has five identical components, each with a probability of .2 of failing in less than 1000 hours. The subsystem will operate if any two of the five components are operating. Assume that the components operate independently. Find the probability that the subsystem operates longer than 1000 hours.

(C) In no more than 300 words, respond to the following claim: "Probability is just an application of enumerative combinatorics." You may agree, or disagree or both. Your response should be informed by at least one interesting example that illustrates the point(s) you wish to make.

Question 3 [Expected value in a Markov processes]

A "Stepping Stone" model is a special kind of Markov process which has applications in several areas, including genetics. The concept is an $m\times n$ grid of squares (the stepping stones) each of which can be one of k colours, so there are k^{mn} states. At each time step a square is chosen at random (equal probabilities) and takes on the colour of one of its (up to eight) neighbours, also chosen at random. If you wait long enough, all squares will be the same colour and remain so for ever. (A state in a Markov process with this property is called a *sink*.) A 2-colour simulator can be found at `https://math.dartmouth.edu/archive/m20x11/public_html/test.html`

As our tiny example, consider the 1×3 grid shown at right. Stones Left and R ight each have only one neighbour M iddle whereas M has two neighbours L and R. We will use only two 'colours', black(B) and white(W).

L	M	R

There are $2^3 = 8$ states ① to ⑧. For convenience let ⓝ designate the state determined as follows: Write n as a 3-bit binary number lmr (represent ⑧ by `000`), with l, m, r representing the colours of L,M,R respectively using code 1 = black, 0 = white. For example ⑤ $\leftrightsquigarrow$ `101` $\leftrightsquigarrow$ BWB, *i.e.* L and R black, and M white.

As with all stepping stone grids, once all stones are the same colour there can be no further change. But for this question I make one extra rule, so that only the all-black state ⑦ is a sink. The extra rule is that from the all-white state ⑧ the next state is equally likely to be any of the states ①,...,⑥. So, for example, a possible sequence of transitions is as follows, with annotations above the arrows indicating stone and neighbour chosen:

$$\underset{\text{BBW}}{⑥} \xrightarrow{\text{M,R}} \underset{\text{BWW}}{④} \xrightarrow{\text{R,M}} \underset{\text{BWW}}{④} \xrightarrow{\text{L,M}} \underset{\text{WWW}}{⑧} \xrightarrow{\text{extra}} \underset{\text{WBW}}{②} \xrightarrow{\text{R,M}} \underset{\text{WBB}}{③}$$

Let $\mathcal{P}$ denote our 2-colur 3-stepping-stone process with the extra rule.

(a) Write out the transition matrix T for $\mathcal{P}$, with the i-th row and column corresponding to state ⓘ.

(b) In two or three sentences, compare and contrast the terms 'sink' and 'steady state', both in general for Markov processes and specifically in relation to $\mathcal{P}$.

(c) Use a computer calculation to show that $\mathcal{P}$ has exactly one steady state.
Explain how you do this and give your input and output from the computer.

Since ⑦ (all-black) is a sink, from which there is no escape, it is natural to ask how soon $\mathcal{P}$ gets stuck there. To make this question precise we define the random variable X to be the number of steps before $\mathcal{P}$ first enters state ⑦, having started in state ⑧ (all-white). The question now is, 'What is the expected value of X?' (This is denoted by $\mathbb{E}(X)$.)

For each question below, say what you ask the computer to do, and show input and output.

(d) Use T to calculate $\mathbb{P}(X \le 10)$, the probability that $\mathcal{P}$ is in state ⑦ after ten steps.

(e) In an attempt to estimate $\mathbb{E}(X)$, find the least n such that $\mathbb{P}(X \le n) \ge 95\%$.

(f) Calculate $\mathbb{E}(X)$ exactly by using a powerful formula from the theory of Markov Processes. Let Q denote the 7×7 matrix obtained by removing row 7 (the sink row) and column 7 from the T, and let $N = (I - Q)^{-1}$. Then $\mathbb{E}(X)$ is the sum of the entries in the last row of N (the row that relates to the starting state ⑧.).

End of Questions for Assignment C