Australian National University

ASSIGNMENT COVER SHEET

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This coversheet must be attached to the front of your assessment

The assessment is due at 5pm unless otherwise specified in the course outline.

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Course Code	COMP6261
Course Name	Information Theory
Assignment Item	Assignment 1
Due Date	29 August 2022
Date Submitted	29 August 2022

I declare that this work:

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Signature Han Zhang

Q2.

(a) $\sum_{i=1}^{m} q_i = 1.$

$$H(X) = -\sum_{x} p(x)\log_2 p(x)$$

$$= -\sum_{i=1}^{m} q_i \log_2 q_i$$

$$H(Y) = -\left(\frac{q_1}{5}\log_2\frac{q_1}{5} + \cdots + \frac{q_m}{5}\log_2\frac{q_m}{5} + \frac{4}{5}\log_2\frac{4}{5}\right)$$

$$= -\frac{1}{5}\sum_{i=1}^{m} q_i \log_2\left(\frac{q_i}{5}\right) - \frac{4}{5}\log_2\left(\frac{4}{5}\right)$$

$$= -\frac{1}{5}\left(\sum_{i=1}^{m} q_i \log_2 q_i - \log_2(5)\right) - \frac{4}{5}\log_2\left(\frac{4}{5}\right)$$

$$= \frac{1}{5}\left(-\sum_{i=1}^{m} q_i \log_2 q_i + \log_2(5)\right) - \frac{4}{5}\log_2\left(\frac{4}{5}\right)$$

$$= \frac{H(X)}{5} + \frac{\log_2(5)}{5} - \left(\frac{4}{5}\log_2(4) - \frac{4}{5}\log_2(5)\right)$$

$$= \frac{H(X)}{5} + \frac{\log_2(5)}{5} - \frac{4}{5}\log_2(4) + \frac{4}{5}\log_2(5)$$

$$= \frac{H(X)}{5} + \log_2(5) - \frac{4}{5}\log_2(4)$$

$$= \frac{H(X)}{5} + \log_2(5) - \frac{8}{5}$$

(b) Let T be the variable of the top side,
B be the variable of the bottom side.

$$I(T;B) = H(T) - H(T|B)$$

$$H(T) = -\frac{1}{2}\log_2(\frac{1}{2}) - \frac{1}{2}\log_2(\frac{1}{2}) = -\log_2(\frac{1}{2}) = 1$$

$$H(T|B) = \sum_{b \in B} p(b) H(T|B=b)$$

$$p(T=top|B=bottom) = p(T=bottom|B=top) = 1$$
$$p(T=top|B=top) = p(T=bottom|B=bottom) = 0$$

$\therefore H(T|B) = 0.$

$\therefore I(T;B) = 1 - 0 = 1$

(C) Once rolled, for each front side, there will be 4 possible top sides.

Let T be the variable of the top side,
F be the variable of the front side.

$$I(T;F) = H(T) - H(T|F)$$
$$= -\sum_{i=1}^{6} \frac{1}{6}\log_2(\frac{1}{6}) + \sum_{j=1}^{4} \frac{1}{4}\log_2(\frac{1}{4})$$

$$= -\log_2(\frac{1}{6}) + \log_2(\frac{1}{4})$$
$$= \log_2(6) - 2$$
$$= 1 + \log_2(3) - 2$$
$$= \log_2(3) - 1$$

(d) Let A be the first distribution,
B be the second distribution.

$$H(A) = -\sum_{k=1}^{m} P_K \log_2 P_K$$

$$H(B) = -\sum_{k=1}^{m} P_K \log_2 P_K + P_i \log_2 p_i + P_j \log_2 P_j - 2\frac{P_i+P_j}{2}\log_2\frac{P_i+P_j}{2}$$

$$H(A) - H(B) = -P_i \log_2 P_i - P_j \log_2 P_j + 2\cdot\frac{P_i+P_j}{2}\log_2\frac{P_i+P_j}{2}$$

Let $f(x) = -x\log_2 x$, $P = \{P_i, P_j\}$, $P_i > 0$, $P_j > 0$.

$$f''(x) = -\frac{1}{\ln(2)x} < 0$$

According to Jensen inequality.

$$E[f(P)] \leq f(E[P])$$

$$-\frac{1}{2}(P_i \log_2 P_i + P_j \log_2 P_j) \leq -\frac{P_i+P_j}{2}\log_2\frac{P_i+P_j}{2}$$

$$-P_i \log_2 P_i - P_j \log_2 P_j \leq -2\frac{P_i+P_j}{2}\log_2\frac{P_i+P_j}{2}$$

$$\therefore H(A) - H(B) \leq 0$$

$$\therefore H(A) \leq H(B)$$

Q3.

1. (a) $P(C=\text{red}) = 0.5$

$P(f=0) = 1 - \frac{3}{12} = \frac{3}{4}$

$P(C=\text{red}, f=0) = 0.5 \times \frac{3}{4} = \frac{3}{8}$

$h(C=\text{red}, f=0) = \log_2\left(\frac{8}{3}\right) = 3 - \log_2(3)$

(b) $p(V=k) = \frac{1}{12}$

$p(f=1) = \frac{1}{4}$

$$p(V=k \mid f=1) = \frac{P(f=1 \mid V=k)\, P(V=k)}{P(f=1)} = \frac{1 \cdot \frac{1}{12}}{\frac{1}{4}} = \frac{1}{3}$$

$h(V=k \mid f=1) = \log_2(3)$

(c) $p(S) = \frac{1}{4}$.

$p(V) = \frac{1}{12}$

$p(V, S) = \frac{1}{48}$

$H(S) = -\log_2\left(\frac{1}{4}\right) = 2$

$H(V, S) = -\log_2\left(\frac{1}{48}\right) = 4 + \log_2(3)$

(d) $I(V; S) = H(V) + H(S) - H(V, S)$

$H(V) = -\log_2\left(\frac{1}{12}\right) = 2 + \log_2(3)$

$\therefore I(V; S) = 2 + \log_2(3) + 2 - 4 - \log_2(3) = 0$

(e) $H(C) = -\log_2(\frac{1}{2}) = 1$

$p(V, C) = \frac{1}{24}$

$H(V, C) = -\log_2(\frac{1}{24}) = 3 + \log_2(3)$

$I(V; C) = H(V) + H(C) - H(V, C)$

$= 2 + \log_2(3) + 1 - 3 - \log_2(3)$

$= 0$

2. (a) total $= 48 - 16 = 32$

$p(S=h) = \frac{12-4}{32} = \frac{8}{32} = \frac{1}{4}$

$p(S=d) = p(S=c) = \frac{12}{32} = \frac{3}{8}$

$H(S) = -\left(\frac{1}{4}\log_2(\frac{1}{4}) + 2 \times \frac{3}{8}\log_2(\frac{3}{8})\right)$

$= \frac{1}{2} - \frac{3}{4}\log_2(3) + \frac{9}{4}$

$= \frac{11}{4} - \frac{3}{4}\log_2(3)$

$p(V=2) = p(V=3) = p(V=4) = p(V=5) = \frac{2}{32} = \frac{1}{16}$

$p(V=6) = \cdots p(V=k) = \frac{3}{32}$

$p(V=6 | S=h) = \cdots = p(V=k | S=h) = \frac{1}{8}$

$p(V=2 | S=d) = \cdots = p(V=k | S=d) = \frac{1}{12}$

$p(V=2 | S=C) = \cdots = p(V=k | S=C) = \frac{1}{12}$

$H(V, S) = H(S) + H(V|S)$

$= \frac{11}{4} - \frac{3}{4}\log_2(3) + \frac{1}{4} \times \log_2(8) + \frac{3}{8} \times \log_2(12) \times 2$

$= \frac{11}{4} - \frac{3}{4}\log_2(3) + \frac{3}{4} + \frac{3}{2} + \frac{3}{4}\log_2(3)$

$= 5$

(b) $H(V) = -\left[4 \times \frac{1}{16}\log_2\left(\frac{1}{16}\right) + 8 \times \frac{3}{32}\log_2\left(\frac{3}{32}\right)\right]$

$$= 1 - \frac{3}{4}\cdot(\log_2(3) - 5)$$

$$= \frac{19}{4} - \frac{3}{4}\log_2(3)$$

$$I(V;S) = H(V) + H(S) - H(V,S)$$

$$= \frac{19}{4} - \frac{3}{4}\log_2(3) + \frac{11}{4} - \frac{3}{4}\log_2(3) - 5$$

$$= \frac{5}{2} - \frac{3}{2}\log_2(3)$$

The values of cards corresponding to each suit are the same befor removing, card value conveys no information about suit. But after removing the values are different, thus the card value conveys information about suit.

(c) $I(V;S|C) = H(V|C) - H(V|S,C)$

$$P(C=r) = \frac{12+8}{32} = \frac{20}{32} = \frac{5}{8}$$

$$P(C=b) = \frac{12}{32} = \frac{3}{8}$$

$$p(V=2|C=b) = \cdots = p(V=K|C=b) = \frac{1}{12}$$

$$p(V=2|C=r) = \cdots = p(V=5|C=r) = \frac{1}{20}$$

$$p(V=6|C=r) = \cdots = p(V=K|C=r) = \frac{1}{10}$$

$$H(V|C) = \frac{5}{8}\times\left[12\times\frac{1}{20}\times\log_2(20) + 8\times\frac{1}{10}\times\log_2(10)\right] + \frac{3}{8}\times\log_2(12)$$

$$= \frac{3}{8}[2+\log_2(5)] + \frac{1}{2}[1+\log_2(5)] + \frac{3}{8}[2+\log_2(3)]$$

$$= 2 + \frac{7}{8}\log_2(5) + \frac{3}{8}\log_2(3)$$

$$H(V|S,C) = H(V|S)$$

$$= \frac{1}{4} \times \log_2(8) + \frac{3}{8} \times \log_2(12) \times 2$$

$$= \frac{9}{4} + \frac{3}{4}\log_2(3)$$

$$\therefore I(V;S|C) = H(V|C) - H(V|S,C)$$

$$= 2 + \frac{7}{8}\log_2(5) + \frac{3}{8}\log_2(3) - \frac{9}{4} - \frac{3}{4}\log_2(3)$$

$$= \frac{7}{8}\log_2(5) - \frac{1}{4} - \frac{3}{8}\log_2(3)$$

$$P(X|Y) = \frac{P(X,Y)}{P(Y)} = \frac{P(Y,X)}{P(Y)}$$

Q4.

(a)
$$I(X;Y) = D_{KL}(p(X,Y) \,\|\, p(X)p(Y))$$

$$= \sum_x \sum_y p(x,y) \log_2 \frac{p(x|y)}{p(x)}$$

$$= \sum_x \sum_y p(x|y)p(y) \log_2 \frac{p(x|y)}{p(x)}$$

$$\frac{D_{KL}(p\|m) + D_{KL}(q\|m)}{2}$$

$$= \frac{\sum_x p(X|y=1) \log \frac{2 \cdot p(X|y=1)}{p(X|y=1)+p(X|y=0)} + \sum_x p(X|y=0) \log \frac{2p(X|y=0)}{p(X|y=1)+p(X|y=0)}}{2}$$

$$= \sum_x \sum_y p(X|y)p(y) \log_2 \frac{p(X|y)}{p(x)}$$

$$= I(X,Y)$$

(b) $p(Y=0) = p(Y=1) = \frac{1}{2}$.

Let $m = \frac{p+q}{2}$. $\therefore m = (\frac{2}{5}, \frac{2}{5}, \frac{1}{5})$

$$I(X;Y) = \frac{D_{KL}(p\|m) + D_{KL}(q\|m)}{2}$$

$$= \frac{\frac{3}{5}\cdot\log_2(\frac{3}{2}) + \frac{1}{5}\log_2(\frac{1}{2}) + \frac{1}{5}\log_2(1) + \frac{1}{5}\cdot\log_2(\frac{1}{2}) + \frac{3}{5}\log_2(\frac{3}{2}) + \frac{1}{5}\log_2(1)}{2}$$

$$= \frac{3}{5}\log_2(\frac{3}{2}) + \frac{1}{5}\log_2(\frac{1}{2}) + \frac{1}{5}\log_2(1)$$

$$= \frac{3}{5}\log_2(3) - \frac{4}{5}$$

(c) Let $s = (p(z=a|y=1), p(z=b|y=1), p(z=c|y=1)) = (\frac{1}{5}, \frac{3}{5}, \frac{1}{5})$

$t = (p(z=a|y=0), p(z=b|y=0), p(z=c|y=0)) = (\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$.

$n = \frac{s+t}{2} = (\frac{2}{5}, \frac{2}{5}, \frac{1}{5})$.

$$I(Z;Y) = \frac{D_{KL}(t\|n) + D_{KL}(s\|n)}{2}$$

$$= \frac{3}{5}\log_2(3) - \frac{4}{5}$$

X and Z convey the same information about Y. so their results are equal.

(d) Suppose for $Z: x \in X$,

$s = (p(Z=a|y=1), p(Z=b|y=1), p(Z=c|y=1)) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

$t = (p(Z=a|y=0), p(Z=b|y=0), p(Z=c|y=0)) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

$n = \frac{s+t}{2} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

$$I(Z;Y) = \frac{D_{KL}(s||n) + D_{KL}(t||n)}{2}$$

$$= \frac{\frac{1}{3}\log_2(1) \times 3 \times 2}{2}$$

$$= 0 < I(X;Y)$$

Givin Y. we can know less information about Z than that of X.

(e) Suppose for $Z: x \in X$,

$s = (p(Z=a|y=1), p(Z=b|y=1), p(Z=c|y=1)) = (\frac{3}{5}, 0, \frac{2}{5})$

$t = (p(Z=a|y=0), p(Z=b|y=0), p(Z=c|y=0)) = (\frac{4}{5}, \frac{1}{5}, 0)$.

$n = \frac{s+t}{2} = (\frac{7}{10}, \frac{1}{10}, \frac{1}{5})$.

$$I(Z;Y) = \frac{D_{KL}(s||n) + D_{KL}(t||n)}{2}$$

$$= \frac{\frac{3}{5}\log_2(\frac{6}{7}) + 0 + \frac{2}{5}\log_2(4) + \frac{4}{5}\log_2(\frac{8}{7}) + \frac{1}{5}\log_2(2) + 0}{2}$$

$$= 1 + \frac{3}{5} + \frac{3}{5}\log_2(3) - \frac{3}{5}\log_2(7) + \frac{12}{5} - \frac{4}{5}\log_2(7)$$

$$= 4 + \frac{3}{5}\log_2(3) - \frac{7}{5}\log_2(7) > I(X;Y)$$

Givin Y. we can know more information about Z than that of X.

Q5

(a) $g(u) = \log\left(1 - \frac{a}{a-b} + \frac{a}{a-b} \cdot e^{u}\right)$

$$e^{g(t(b-a))} = e^{\log\left(1-\frac{a}{a-b} + \frac{a}{a-b} \cdot e^{t(b-a)}\right)}$$

Let $f(x) = e^{tx}$. $f''(x) = e^{tx} t^2 > 0$.

$\therefore$ $f(x)$ is convex.