MATH1005/MATH6005 Semester 1 2021

Assignment 7

Workshop Details:

Number	Day	Time	Demonstrator name
16 B	Friday	2pm 07/05	Cai Yang

Student Details:

ID	Surname	Given name	Preferred name
u7235649	Zhang	Han	

Instructions:

This assignment has four questions. Write your solutions in the spaces provided. Hand-writing is preferable to typesetting unless you are fast and accurate with LaTeX, and even then typesetting will take you longer. Also, be aware that typesetting will not be allowed on exams.

Except for multiple-choice questions, or where answer boxes are provided (one question of this type each week), **show all working.**

Question numbers indicate the corresponding questions from the workshop. Since the workshop has six questions, questions on this assignment may not be sequentially numbered.

Declaration:

I declare that while I may have discussed some or all of the questions in this assignment with other people, the write-up of my answers herein is entirely my own work. I have not copied or modified the written-out answers of anyone else, nor allowed mine to be so used.

Signature: Han Zhang Date: 07/05/2021

This document must be submitted by 11pm on the THURSDAY following your workshop.

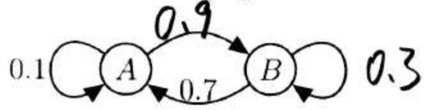
Sign, date, then scan this completed document (5 pages) and save as a pdf file with name format uXXXXXXXAssXX.pdf (*e.g.* u6543210Ass01.pdf).
Upload the file via the link from which you downloaded this document.

If copying is detected, and/or the document is not signed, no marks will be awarded.

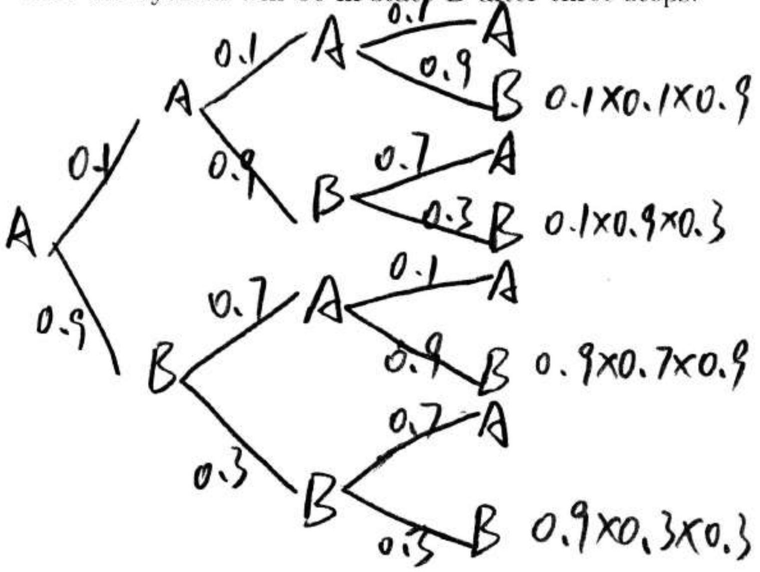
This document has five pages in total.

Question 1$^{\#}$ A Markov process has two states A and B with transition graph below.

(a) Write in the two missing probabilities.



(b) Suppose the system is in state A initially. Use a tree diagram to find the probability that the system will be in state B after three steps.



$P = 0.1\times0.1\times0.9 + 0.1\times0.9\times0.3$
$+ 0.9\times0.7\times0.9 + 0.9\times0.3\times0.3$
$= 0.009 + 0.027 + 0.567 + 0.081$
$= 0.684$

(c) The transition matrix for this process is $T = \begin{bmatrix} 0.1 & 0.9 \\ 0.7 & 0.3 \end{bmatrix}$

(d) Use T to recalculate the probability found in (b).

$T' = \begin{bmatrix} 0.1 & 0.7 \\ 0.9 & 0.3 \end{bmatrix} \quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$x_1 = T' x_0 = \begin{bmatrix} 0.1 \\ 0.9 \end{bmatrix} \quad x_2 = T' x_1 = \begin{bmatrix} 0.01 + 0.63 \\ 0.09 + 0.27 \end{bmatrix} = \begin{bmatrix} 0.64 \\ 0.36 \end{bmatrix}$

$x_3 = T' x_2 = \begin{bmatrix} 0.064 + 0.252 \\ 0.576 + 0.108 \end{bmatrix} = \begin{bmatrix} 0.316 \\ 0.684 \end{bmatrix}$

$P = 0.684$

Question 2† Write the correct values in the boxes, and for (e) circle the correct answer.

For this question, working is not required and will not be marked.

A Markov process has the transition matrix T shown at right at right.

Using a computer tool, such as the MatrixCalc online matrix calculator (`http://matrixcalc.org/en/`), calculate powers of T to **9 decimal places accuracy**, as specified in (a) - (c) below. Then answer the follow-up questions (d) - (f).

$$\begin{bmatrix} .2 & .3 & .5 \\ .1 & .3 & .6 \\ 0 & .3 & .7 \end{bmatrix}$$

(a) To 9dp the top left entry in T^4 is $\boxed{0.0388}$.

(b) To 9dp the top left entry in T^8 is $\boxed{0.037502 08}$

(c) To 9dp the top left entry in T^{16} is $\boxed{0.0375}$.

(d) Based on the values in the powers of T that you have calculated, you can conclude that the process has a steady state vector $S = \frac{1}{80}\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$ where s_3 is the integer $\boxed{53}$.

(e) To verify that this S is indeed a steady state vector for the process we just need to check that: (circle one)

$$TS = S \quad / \quad T'S = S \quad / \quad TS' = S' \quad / \quad T'S' = S'$$

(f) Suppose that the underlying system of the process starts in state 1. In the long term, the probability that the system will be in state 2 is $\boxed{0.3}$.

Question 3†

(a) By solving the relevant system of equations find the steady state vector for the Markov process with transition matrix $T = \begin{bmatrix} .6 & .4 \\ .1 & .9 \end{bmatrix}$. Do this by hand calculation, using matrix inverse.

$$T' = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix} \quad S = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$(T' - I)S = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -0.4 & 0.1 \\ 0.4 & -0.1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -0.4 & 0.1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -0.4 & 0.1 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{-0.4-0.1}\begin{bmatrix} 1 & -0.1 \\ -1 & -0.4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -2\begin{bmatrix} -0.1 \\ -0.4 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}$$

(b) Using Matrix Reshish[1] to solve the relevant equations by Gauss-Jordan Elimination, find the steady state vector for the Markov process with transition matrix

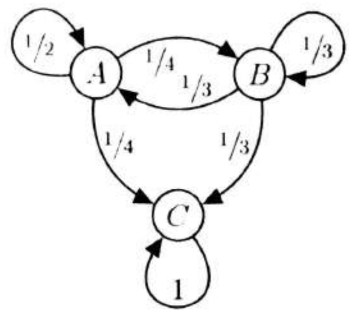
$T = \begin{bmatrix} 2/5 & 1/5 & 2/5 \\ 0 & 4/5 & 1/5 \\ 1/2 & 3/10 & 1/5 \end{bmatrix}$. Use the 'Fractional' input style.

$$T' = \begin{bmatrix} 2/5 & 0 & 1/2 \\ 1/5 & 4/5 & 3/10 \\ 2/5 & 1/5 & 1/5 \end{bmatrix} \quad [T' - I \mid 0] = \begin{bmatrix} -3/5 & 0 & 1/2 & 0 \\ 1/5 & -1/5 & 3/10 & 0 \\ 2/5 & 1/5 & -4/5 & 0 \end{bmatrix}$$

$$\text{steady state vector} = \begin{bmatrix} 1/5 \\ 14/25 \\ 6/25 \end{bmatrix}$$

[1] https://matrix.reshish.com/

Question 5* Consider the Markov process $\mathcal{P}$ with transition diagram at right. Missing arrows represent zero probabilities.



(a) Write out the transition matrix T for this Markov process.

$$T = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Notice that state C is a 'sink'; once in this state there is no escape from it. Deduce the steady state vector for $\mathcal{P}$ and verify your deduction.

$$S = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad T'S = \begin{bmatrix} 1/2 & 1/3 & 0 \\ 1/4 & 1/3 & 0 \\ 1/4 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(c) Starting from state A, suppose $\mathcal{P}$ enters state C after exactly X steps. Calculate:

(i) $\mathbb{P}(\{X=1\}) = 1/4$

(ii) $\mathbb{P}(\{X=2\}) = 1/2 \times 1/4 + 1/4 \times 1/3 = 5/24$

(iii) $\mathbb{P}(\{X=3\}) = 1/2 \times 1/2 \times 1/4 + 1/2 \times 1/4 \times 1/3 + 1/4 \times 1/3 \times 1/3 + 1/4 \times 1/3 \times 1/4$

$$= \frac{1}{16} + \frac{1}{24} + \frac{1}{36} + \frac{1}{48} = \frac{22}{144} = \frac{11}{72}$$

(d) Since the possible values of X are unlimited, the average number of steps from A to C will be $\mathbb{E}(X) = \sum_{n=1}^{\infty} \mathbb{P}(X=n)n$. To evaluate this apparently requires first calculating an infinite number of probabilities, a daunting task. Fortunately a powerful formula from the theory of Markov Processes does most of the work for us.
We first need the 'fundamental matrix' $N = (I-Q)^{-1}$ for $\mathcal{P}$, where, Q is the 2×2 matrix obtained by removing the last row and last column from T. Calculate N.

$$Q = \begin{bmatrix} 1/2 & 1/4 \\ 1/3 & 1/3 \end{bmatrix} \quad N = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/2 & 1/4 \\ 1/3 & 1/3 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1/2 & -1/4 \\ -1/3 & 2/3 \end{bmatrix}^{-1}$$

$$= \frac{1}{1/3 - 1/12} \begin{bmatrix} 2/3 & 1/4 \\ 1/3 & 1/2 \end{bmatrix} = 4 \cdot \begin{bmatrix} 2/3 & 1/4 \\ 1/3 & 1/2 \end{bmatrix} = \begin{bmatrix} 8/3 & 1 \\ 4/3 & 2 \end{bmatrix}$$

(e) The sum of the entries in the first row of N is the average number of steps from (A) to (C). Show that this answer is consistent with your answers to (c).

$$1 + 8/3 = 11/3$$