1. 

A fair coin is tossed four times.

(a) Write out the sample space S.

$$\boldsymbol{S = \{\text{HHHH,HHHT,HHTH,HHTT,HTHH,HTHT,HTTH,HTTT,}}$$
$$\boldsymbol{\text{THHH,THHT,THTH,THTT,TTHH,TTHT,TTTH,TTTT}\}}.$$

(b) Write out the event E: 'exactly half the tosses show heads'.

$$\boldsymbol{E = \{\text{HHTT,HTHT,HTTH,THHT,THTH,TTHH}\}}.$$

(c) Evaluate $|E|$ and $|S|$ and hence find the probability $\mathbb{P}(E)$ of E.

$$\boldsymbol{|S| = 16,\ |E| = 6,\ \mathbb{P}(E) = \frac{6}{16} = \frac{3}{8}}.$$

(d) If the fair coin is tossed six times instead of four, would you expect $\mathbb{P}(E)$ to increase, decrease or stay the same?
To test you intuition, calculate the probability.

Decrease.

$$\boldsymbol{|S| = 2^6 = 64,}$$
$$\boldsymbol{|E| = \binom{6}{3} = \frac{6\times5\times4}{3\times2\times1} = 20.}$$
$$\boldsymbol{\mathbb{P}(E) = \frac{20}{64} = \frac{5}{16} < \frac{6}{16}}.$$

2. Two fair D4 dice are thrown, showing values $a, b \in \{1,2,3,4\}$.

(a) Write out the sample space S.

$$\boldsymbol{S = \{(1,1),(1,2),(1,3),(1,4),}$$
$$\boldsymbol{(2,1),(2,2),(2,3),(2,4),}$$
$$\boldsymbol{(3,1),(3,2),(3,3),(3,4),}$$
$$\boldsymbol{(4,1),(4,2),(4,3),(4,4)\}}.$$

(b) Write out the event E: $\{|a-b| = 1\}$

$$\boldsymbol{E = \{(1,2),(2,1),(2,3),(3,2),(3,4),(4,3)\}}.$$

(c) Evaluate $|E|$ and $|S|$ and hence find the probability $\mathbb{P}(E)$ of E.

$$\boldsymbol{|S| = 4^2 = 16,\quad |E| = 6,\quad \mathbb{P}(E) = \frac{6}{16} = \frac{3}{8}}.$$

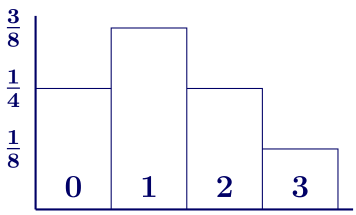
(d) Draw a histogram for the probability density of $X = |a-b|$.
Is it symmetrical?
(Technically, X is a random variable $X : \{1,2,3,4\}^2 \to \mathbb{Q}_+$; $X(a,b) = |a-b|$.)

$\boldsymbol{\{X=0\}=\{(1,1),(2,2),(3,3),(4,4)\}}$ $\boldsymbol{\mathbb{P}(\{X=0\})=\frac{4}{16}}$

$\boldsymbol{\{X=1\}=E}$ **above** $\boldsymbol{\mathbb{P}(\{X=1\})=\frac{6}{16}}$

$\boldsymbol{\{X=2\}=\{(1,3),(3,1),(2,4),(4,2)\}}$ $\boldsymbol{\mathbb{P}(\{X=2\})=\frac{4}{16}}$

$\boldsymbol{\{X=3\}=\{(1,4),(4,1)\}}$ $\boldsymbol{\mathbb{P}(\{X=3\})=\frac{2}{16}}$



Not symmetrical.

3. A pack of playing cards contains fours suits (♣, ♢, ♡, ♠) each of thirteen cards of which four are honours (A,K,Q,J). A card is drawn at random from the pack. What is the probability it is:

(a) a heart?; $\boldsymbol{\frac{13}{52} = \frac{1}{4}}$.

(b) an honour?; $\boldsymbol{\frac{4\times4}{52} = \frac{4}{13}}$.

(c) a heart *and* an honour?; $\boldsymbol{\frac{4}{52} = \frac{1}{13}}$.

(d) a heart *or* an honour? $\boldsymbol{\frac{13}{52} + \frac{16}{52} - \frac{4}{52} = \frac{25}{52}}$.

4. [Challenge] At a Queanberra school two thirds of the girls, and seven eighths of the boys, are Australian-born.
A student is selected at random from the school.
The probability that this student is an overseas-born girl is 20%.

What is the probability that this student is an an overseas-born boy?

Let b be the number of boys at the school.
Let g be the number of girls at the school.
Since only a third of the girls are overseas-born:

$$\mathbb{P}(\text{random student is an overseas-born girl}) = \frac{g/3}{b+g} = \frac{1}{5} \quad \text{(given)}.$$

Thus $\frac{g}{b+g} = \frac{3}{5}$ **and so** $\frac{b}{b+g} = \frac{2}{5}$. **As an eighth of the boys are overseas-born:**

$$\mathbb{P}(\text{random student is an overseas-born boy}) = \frac{b/8}{b+g} = \left(\frac{1}{8}\right)\left(\frac{2}{5}\right) = \frac{1}{20} = 5\%.$$

5. If you pick a random student at the Havana Acadamy there is a 32% chance that she/he is good at maths, a 27% chance that her/his favourite drink is a mojito, and a 6% chance that she/he is good at maths and favours mojitos.

(a) Are the events "the student is good at maths", and "the student's favourite drink is a mojito" independent?

$\mathbb{P}$(Good at Maths)= $\mathbb{P}$(GM)= 0.32.
$\mathbb{P}$(Favours Mojitos)= $\mathbb{P}$(FM)= 0.27.
$\mathbb{P}(GM)\times\mathbb{P}(FM) = 0.32\times 0.27 = 0.0864$.
But $\mathbb{P}(GM\&FM) = 0.06 < 0.0864$ and so GM and FM are *not* independent.

(b) Should you conclude that drinking mojitos makes you good at maths, or that maths makes you drink mojitos?

Neither. Dependence does not imply causality (even though causality does imply dependence).

6. A 'poker hand' is a set of five cards drawn from a pack of playing cards. (See Q3.)

(a) What is the probability, to two significant figures, that a poker hand:

(i) is a 'flush'? *i.e* all five cards are in the same suit;

$$\frac{4\binom{13}{5}}{\binom{52}{5}} = 0.0020 \text{ approx. (by calculator).}$$

(ii) is a 'straight'? *i.e* five consecutive values from list A,K,Q,J,10,9,8,7,6,5,4,3,2,A (suits irrelevant);

There are ten kinds of straights, from AKQJ10 down to 5432A. Each kind can be made in 4^5 ways, so the probability is

$$\frac{10\times 4^5}{\binom{52}{5}} = 0.0039 \text{ approx. (by calculator).}$$

(iii) is a straight flush?

Each kind of straight can now be made in only 4 ways, giving probability

$$\frac{10\times 4}{\binom{52}{5}} = 0.000\,015 \text{ approx. (by calculator).}$$

(b) Are the events 'flush', and 'straight', independent?

$0.0020 \times 0.0039 = 0.000\,007\,8 < 0.000\,015$, so the events are *not* independent.

7. A fair D8 die is tossed eight times.
Use the binomial density function to compute the probability that:

(a) half the tosses give an even result (2, 4, 6 or 8);

$$\boldsymbol{n=8,\ p=\tfrac{1}{2},\ k=4.}\ \textbf{So}$$
$$\mathbb{P}(\boldsymbol{k}=\mathbf{4}) = \binom{8}{4}\left(\tfrac{1}{2}\right)^4\left(1-\tfrac{1}{2}\right)^4 = \binom{8}{4}2^{-8} = \frac{8\times7\times6\times5}{4\times3\times2\times1\times128} = \frac{70}{256} = \mathbf{0.27\ approx.}$$

(b) exactly one of the tosses give result 8;

$$\boldsymbol{n=8,\ p=\tfrac{1}{8},\ k=1.}\ \textbf{So}$$
$$\mathbb{P}(\boldsymbol{k}=\mathbf{1}) = \binom{8}{1}\left(\tfrac{1}{8}\right)^1\left(1-\tfrac{1}{8}\right)^7 = \binom{8}{1}\frac{7^7}{8^8} = \left(\tfrac{7}{8}\right)^7 = \mathbf{0.39\ approx.}$$

(c) exactly three of the tosses give a result of 3 or less.

$$\boldsymbol{n=8,\ p=\tfrac{3}{8},\ k=3.}\ \textbf{So}$$
$$\mathbb{P}(\boldsymbol{k}=\mathbf{3}) = \binom{8}{3}\left(\tfrac{3}{8}\right)^3\left(1-\tfrac{3}{8}\right)^5 = \binom{8}{3}\frac{3^3 5^5}{8^8} = \frac{8\times7\times6}{3\times2\times1}\,\frac{3^3 5^5}{8^8} = \mathbf{0.28\ approx.}$$

8. The owner of an ice cream shop was asked about the proportions of customers buying 1-, 2- and 3-scoop ice creams with wafer or waffle cones. The table records his estimates. Based on these, is the choice of cone type independent of the number of scoops?

Cone	scoops 1	2	3
wafer	7%	8%	20%
waffle	13%	15%	37%

ℙ(1 scoop)=(7+13)%=20%
ℙ(2 scoop)=(8+15)%=23%
ℙ(3 scoop)=(20+37)%=57%
ℙ(wafer)=(7+8+20)%=35%
ℙ(waffle)=(13+15+37)%=65%

ℙ(1 scoop)×ℙ(wafer)= 0.2 × 0.35 = 0.07 = ℙ(1 scoop&wafer), so the events "1 scoop" and "wafer" are independent. Similar calculations for other pairs of events show them all to be independent.

9. Stephan has capital invested in a portfolio of highly volatile shares on the stock market. In his current monthly performance model there is a 10% chance that the capital will grow by 20%, a 20% chance that it will grow by 10%, a 20% chance that it will grow by 5%, a 30% chance that it will lose 5%, and a 20% chance that it will lose 20%.

(a) Let $S = \{1,2,3,4,5\}$. Define a probability density function $\mathbb{P} : S \to \mathbb{Q}_+$ and a random variable $X : S \to \mathbb{Q}$ modelling this situation, such that $\mathbb{E}(X)$ is the expected growth (or loss) after a month. Evaluate $\mathbb{E}(X)$.

$\boldsymbol{s}$	**1**	**2**	**3**	**4**	**5**
$\mathbb{P}(\boldsymbol{s})$	**0.1**	**0.2**	**0.2**	**0.3**	**0.2**
$\boldsymbol{X(s)}$	**20%**	**10%**	**5%**	**−5%**	**−20%**

$$\mathbb{E}(\boldsymbol{X}) = \mathbf{0.1\times20\%+0.2\times10\%+0.2\times5\% -0.3\times5\%-0.2\times20\% = -0.5\% = 0.005}$$

(b) The random variable $Y = 1 + X$ represents the monthly 'multiplier' for Stephan's capital. (*e.g.* if he has \$10 000 invested at the beginning of the month and $Y = 1.05$ then he will have $1.05\times\$10\,000=\$10\,500$ at the begining of next month.)
Prove that $\mathbb{E}(Y) = 1 + \mathbb{E}(X)$. Hence evaluate $\mathbb{E}(Y)$.

$$\mathbb{E}(\boldsymbol{Y})=\mathbb{E}(\mathbf{1}+\boldsymbol{X})=\sum_{s\in S}\mathbb{P}(\boldsymbol{s})(\mathbf{1}+\boldsymbol{X(s)})=\sum\mathbb{P}(\boldsymbol{s})+\sum\mathbb{P}(\boldsymbol{s})\boldsymbol{X(s)}=\mathbf{1}+\mathbb{E}(\boldsymbol{X})=\mathbf{1-0.005=0.995}$$

(c) The random variable $Z : S^2 \to \mathbb{Q}$ defined by $Z(s,t) = Y(s)Y(t)$ represents the two-monthly multiplier.
Prove that $\mathbb{E}(Z) = (\mathbb{E}(Y))^2$, stating any required assumptions. Hence evaluate $\mathbb{E}(Z)$.

$$\mathbb{E}(\boldsymbol{Z})=\sum_{(s,t)\in S^2}\mathbb{P}((\boldsymbol{s},\boldsymbol{t}))\boldsymbol{Z}((\boldsymbol{s},\boldsymbol{t}))=\sum_{(s,t)\in S^2}\mathbb{P}(\boldsymbol{s})\mathbb{P}(\boldsymbol{t})\boldsymbol{Y(s)Y(t)}[\ \textbf{ass. month-to-month independence}\ ]$$
$$=\sum_{s\in S}\mathbb{P}(\boldsymbol{s})\boldsymbol{Y(s)}\sum_{t\in S}\mathbb{P}(\boldsymbol{t})\boldsymbol{Y(t)}=\mathbb{E}(\boldsymbol{Y})\mathbb{E}(\boldsymbol{Y})=(\mathbf{0.995})^2=\mathbf{0.990025}$$

(d) Is it true that $(\mathbb{E}(Y))^2 = \mathbb{E}(Y^2)$? Prove or disprove.

$$\mathbb{E}(\boldsymbol{Y}^2)=\sum_{s\in S}\mathbb{P}(\boldsymbol{s})(\boldsymbol{Y(s)})^2=\sum_{s\in S}\mathbb{P}(\boldsymbol{s})(\mathbf{1}+\boldsymbol{X(s)})^2=\sum_{s\in S}\mathbb{P}(\boldsymbol{s})(\mathbf{1}+\mathbf{2}\boldsymbol{X(s)})+(\boldsymbol{X(s)})^2)$$
$$=\sum\mathbb{P}(\boldsymbol{s})+\mathbf{2}\sum\mathbb{P}(\boldsymbol{s})\boldsymbol{X(s)}+\sum\mathbb{P}(\boldsymbol{s})(\boldsymbol{X(s)})^2=\mathbf{1}+\mathbf{2}\mathbb{E}(\boldsymbol{X})+\mathbb{E}(\boldsymbol{X}^2)$$
$$=\mathbf{1-0.01+0.01525}[\ \textbf{calculated as in (a)}\ ]=\mathbf{1.00525}\neq(\mathbf{0.995})^2$$