Q1.

1. A is invertable, $\therefore$ $\det(A) \neq 0$.

Suppose $A \in R^{n \times n}$,

$\det(A) = \prod_{i=1}^{n} \lambda_i \neq 0$.

$\therefore \forall i \in R, 1 \leq i \leq n, \lambda_i \neq 0$.

i.e. all the eigenvalues are non-zero.

2. $Ax = \lambda x$

$AA^{-1}x = \lambda A^{-1}x$

$x = \lambda A^{-1}x$

$\lambda^{-1}x = A^{-1}x$

$\therefore \lambda^{-1}$ is an eigenvalue of A^{-1}.

Q2. $Bx = \lambda x$. $n \geq 1$.

$\therefore B^n x = B B^{n-1} x$

$= B(B^{n-1}x)$

$= B(\lambda^{n-1}x)$

$= \lambda^{n-1}(Bx)$

$= \lambda^{n-1}\lambda x$

$= \lambda^n x$

$\therefore x$ is an eigenvector of B^n with eigenvalue λ^n.

Q3.

1. If $\{x_1, \cdots, x_n\}$ is linearly dependent.

$\exists 1 \leq p < n$, $x_{p+1} \in \text{Span}\{x_1, \cdots, x_p\}$, $\{x_1, \cdots, x_p\}$ is linearly independent.

Let $k_1 x_1 + \cdots + k_p x_p = k_{p+1} x_{p+1}$ ①

$A(k_1 x_1 + \cdots + k_p x_p) = A(k_{p+1} x_{p+1})$

$\lambda_1 k_1 x_1 + \cdots + \lambda_p k_p x_p = \lambda_{p+1} k_{p+1} x_{p+1}$ ②

② $- \lambda_{p+1} \cdot$ ① $= (\lambda_1 - \lambda_{p+1}) k_1 x_1 + \cdots + (\lambda_p - \lambda_{p+1}) k_p x_p = 0$

$\therefore \lambda_1 - \lambda_{p+1} = 0, \cdots \lambda_p - \lambda_{p+1} = 0.$

$\therefore \lambda_{p+1} = \lambda_1 = \cdots = \lambda_p$ is not distinct, which is a contradiction.

$\therefore \{x_1, \cdots, x_n\}$ should be linearly independent.

2. Suppose B has n+a distinct eigenvalues $\{\lambda_1, \cdots, \lambda_{n+1}\}$.

then eigenvectors $\{x_1, \cdots, x_{n+1}\}$ is linearly independent.

Let $x_{n+1} = k_1 x_1 + \cdots + k_n x_n$, $\{k_1, \cdots, k_n\} \neq \{0, \cdots, 0\}$.

$A x_{n+1} = A k_1 x_1 + \cdots + A k_n x_n$

$\lambda_{n+1} x_{n+1} = \lambda_1 k_1 x_1 + \cdots + \lambda_n k_n x_n$

$= \lambda_{n+1}(k_1 x_1 + \cdots + k_n x_n)$

$\therefore$ For $i \in R$, $1 \leq i \leq n$,

$\lambda_i k_i = \lambda_{n+1} k_i$

$\because \exists k_j \in \{k_1, \cdots, k_n\}$, $k_j \neq 0$.

$\therefore \exists \lambda_i$, $\lambda_i = \lambda_{n+1}$, which is not distinct.

$\therefore$ It is a contradiction.

Q4.

1. $A \in R^{n\times n}$.

For A, expansion along column j:

$$\det(A) = \sum_{k=1}^{n} (-1)^{k+j} a_{kj} \det(A_{k,j})$$

For A^T, expansion along row j:

$$\det(A^T) = \sum_{k=1}^{n} (-1)^{k+j} a_{kj} \det(A^T_{j,k})$$

$$= \sum_{k=1}^{n} (-1)^{k+j} a_{kj} \det(A_{k,j})$$

$$= \det(A)$$

2. When $n=1$, $\det(I_1) = \det([1]) = 1$.

When $n>1$, suppose for I_n, $\det(I_n)=1$.

Then for I_{n+1}, the expansion along row 1:

$$\det(I_{n+1}) = \sum_{k=1}^{n+1} (-1)^{k+1} I_{k1} \det(I_{1,k})$$

$A_{1,k} \in R^{n\times n}$ $$= (-1)^{1+1} I_{11} \det(I_{1,1})$$

$$= (-1)^2 \times 1 \times \det(I_n)$$

$$= 1$$

$\therefore$ $\det(I_n)=1$ holds for all $n \geq 1$.

Q5.

$$V_1^T A V_2 = V_1^T (A V_2) = V_1^T \lambda_2 V_2 = \lambda_2 V_1^T V_2$$

$$= V_1^T A^T V_2 = (AV_1)^T V_2 = (\lambda_1 V_1)^T V_2 = \lambda_1 V_1^T V_2.$$

$\therefore \lambda_1 V_1^T V_2 = \lambda_2 V_1^T V_2.$

$\because \lambda_1 \neq \lambda_2$

$\therefore V_1^T V_2 = 0$

$\therefore$ V_1 and V_2 are orthogonal.

Q6.

1. $$\det(A-\lambda I)=0 = \det\left(\begin{bmatrix}-1 & 2\\ 3 & 4\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix}-1-\lambda & 2\\ 3 & 4-\lambda\end{bmatrix}\right)$$

$$= (-1-\lambda)(4-\lambda) - 6$$

$$= \lambda^2 - 3\lambda - 10$$

$$= (\lambda+2)(\lambda-5)$$

$\therefore \lambda_1 = -2,\ \lambda_2 = 5.$

2. $$(A-\lambda_1 I)x = 0 = \left(\begin{bmatrix}-1 & 2\\ 3 & 4\end{bmatrix} - \begin{bmatrix}-2 & 0\\ 0 & -2\end{bmatrix}\right)x = \begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$E_2 = \text{span}\left[\begin{bmatrix}2\\ -1\end{bmatrix}\right]$$

$$(A-\lambda_2 I)x = 0 = \left(\begin{bmatrix}-1 & 2\\ 3 & 4\end{bmatrix} - \begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}\right)x = \begin{bmatrix}-6 & 2\\ 3 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$E_5 = \text{span}\left[\begin{bmatrix}1\\ 3\end{bmatrix}\right]$$

3.
$$\left[\begin{array}{cc|c} 2 & 1 & a \\ -1 & 3 & b \end{array}\right] \xrightarrow{\text{Swap } R_1 R_2} \left[\begin{array}{cc|c} -1 & 3 & b \\ 2 & 1 & a \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 + 2R_1} \left[\begin{array}{cc|c} -1 & 3 & b \\ 0 & 7 & a+2b \end{array}\right]$$

$$\xrightarrow[R_2 \leftarrow R_2/7]{R_1 \leftarrow -R_1} \left[\begin{array}{cc|c} 1 & -3 & -b \\ 0 & 1 & \frac{a+2b}{7} \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 + 3R_2} \left[\begin{array}{cc|c} 1 & 0 & \frac{3a-b}{7} \\ 0 & 1 & \frac{a+2b}{7} \end{array}\right]$$

$$\therefore \begin{cases} \lambda_1 = \frac{3a-b}{7} \\ \lambda_2 = \frac{a+2b}{7} \end{cases}$$

$\therefore$ The set of all eigenvectors of A spans R^2.

4.
$$P = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$D = P^{-1}AP = \frac{1}{7}\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -6 & 2 \\ 5 & 10 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$$

5.
$$A^n = PD^nP$$