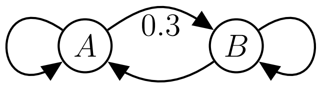
Instructions: *See Worksheets 1 and 2*

Some ideas for these questions came from
http://www.dartmouth.edu/~chance/teaching_aids/books_articles/probability_book/Chapter11.pdf

Question 1 A Markov process has two states A and B with transition graph below.



(a) Write in the two missing probabilities.

(b) Suppose the system is in state A initially. Use a tree diagram to find the probability that the system will be in state B after three steps.

(c) The transition matrix for this process is $T =$?

(d) Use T to recalculate the probability found in (b).

Question 2 Let $T = \begin{bmatrix} .9 & .1 \\ .7 & .3 \end{bmatrix}$.

(a) Use the 'Matrix Calculator' computer application `https://matrixcalc.org/en/` to calculate T^2, T^4, T^8 and T^{16} to 3dp accuracy. (You can progressively insert the results back into an input matrix, so there is no need to physically enter anything more than the four entries of T.)

(b) Based on (a) guess a steady state vector for the Markov process with transition matrix T.

(c) Use the transpose matrix T' to verify that your guess from (b) is correct. Do the calculation by hand.

Question 3

(a) By solving the relevant system of equations find the steady state vector for the Markov process with transition matrix $T = \begin{bmatrix} .7 & .3 \\ .2 & .8 \end{bmatrix}$. Do this by hand calculation, using matrix inverse.

(b) Using Matrix Reshish `https://matrix.reshish.com/` to solve the relevant equations by Gauss-Jordan Elimination, find the steady state vector for the Markov process with transition matrix

$T = \begin{bmatrix} 2/5 & 0 & 3/5 \\ 3/5 & 2/5 & 0 \\ 1/5 & 1/2 & 3/10 \end{bmatrix}$. Use the 'Fractional' input style.

Question 4

(a) Carefully prove that the steady state vector for the Markov process with transition matrix $T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$ is $S = \dfrac{1}{t_{12}+t_{21}} \begin{bmatrix} t_{21} \\ t_{12} \end{bmatrix}$ (providing $t_{12}+t_{21} \neq 0$).

(b) What happens if $t_{12}+t_{21}=0$?

(c) Check that the formula proved for (a) gives the correct result for Q2c and/or Q3a.

Question 5 The *Ehrenfest urns model* is used as part of an explanation of how gas diffusion works. Two urns A and B contain between them a fixed number of balls. At each time step one ball is selected at random (equal probabilities), removed from its current urn and placed into the other one. For the case of four balls the model has five states corresponding to A containing 0, 1, 2, 3 or 4 balls.

(a) Explain why the transition matrix for this Markov process is $T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$.

(b) Explain why, for any n, T^n will always contain some zeroes.
(Hint: Can the number of balls in A change by 1 in an even number of steps?)

(c) The property of T stated in (b) makes this Markov process *non-regular*. Non-regular Markov processes do not necessarily converge towards a steady state. Find the steady state vector for this process (use the computer as in 3(b)) and explain why this will never be reached starting from any *known* (*i.e.* probability 1) starting state.

Question 6 In a simple but modified Ehrenfest model there are three urns, A, B and C, and just two balls. At every time step one ball is selected at random (equal probabilities), removed from its current urn and randomly placed into one of the other urns (again equal probability). Focussing on only urn A, there are three sates: it contains 0, 1 or 2 balls.

(a) Find the transition matrix for this Markov process.

(b) Suppose A is initially empty. What is the probability that it will also be empty after two time steps?

(c) There is no very simple formula for the steady state vector of a 3-state Markov process with $T = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix}$. The best I can come up with is

$$S = \frac{1}{\sigma_1+\sigma_2+\sigma_3} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} \quad \text{where, for any } \{i,j,k\} = \{1,2,3\},\ \sigma_i = t_{kj}t_{ji} + t_{ji}t_{ki} + t_{ki}t_{jk}.$$

Use this formula to find the steady state vector for this example.