1. Let $B = \{0, 1\}$ and $n \in \mathbb{N}$.

(a) Prove that $|B| = 2$. Of course it is completely obvious that $|B| = 2$, *i.e.* that B has cardinality (size) 2. But cardinality is defined in terms of bijections (one-to-one correspondences); in particular B has cardinality 2 if and only if there is a bijection between B and $\{1, 2\}$. Specify such a bijection, giving its signature and rule.

$f : \{1,2\} \to \{0,1\}$; $f(n) = n-1$. Since $f(1) = 0$ and $f(2) = 1$, f is a bijection.

(b) Compute $|B^n|$. **2^n**

(c) Compute the number of n digit binary numbers that start with a 1. **2^{n-1}**

2. You have to deliver flyers to one side of part of a long street. Your houses have consecutive odd numbers starting at 37 and finishing at 251. How many flyers do you need? Prove your answer is correct by specifying a suitable function as in Q1(a) above, and justify that this function is a bijection.

Let S be the set of your house numbers; $S = \{37, 39, \ldots, 249, 251\}$. Subtracting 35 from each member of S gives a new set $T = \{2, 4, \ldots, 214, 216\}$. Halving each member of T gives the set $\{1, 2, \ldots, 107, 108\}$. The function $f : \{1, \ldots, 108\} \to S, f(n) = 2n + 35$ is a bijection, so you need 108 flyers.

3. Let S be any subset of $\{1, 2, \ldots, 12\}$ with $|S| = 7$. Prove that $\exists a, b \in S : \ a - b = 3$.

Think of the numbers in S as pigeons and create six pigeon holes labelled $(1, 4)$, $(2, 5)$, $(3, 6)$, $(7, 10)$, $(8, 11)$, $(9, 12)$. Note that each number from 1 to 12 is part of exactly one label. Now let the the 'pigeons' fly off to settle in the holes whose label contains their number. Since there are more pigeons than holes, at least one hole must contain two pigeons, and the corresponding numbers must differ by three.

4. The digits 1, 2, . . . , 9 are divided into three groups. Prove that the product of the numbers in one of the groups must exceed 71.

Suppose no group product exceeds 71. Then the product of three group products does not exceed $71^3 = 357911$. But the product of the three group products is the product of all the digits 1,...,9. This product is $9! = 362880 > 357911$. So our supposition is false, and at least one of the group products *must* exceed 71.

5. Trying to break an 8-character password by intercepting internet packets, you have found all eight characters used in the password, but not their order. Your program can try one possible password every 2 seconds. How long do you need (at most) to break the password?

There are $8! = 40320$ different permutations of the 8 characters, so we need at most $2(40320) = 80640$ seconds. Dividing by 60×60 gives 22.4 hours.

6. Jane has to choose 2 out of 6 maths subjects and 3 out of 10 computer science subjects. How many different combinations of maths and computer science subjects are there for Jane to choose from?

$\binom{6}{2} = \frac{6\times 5}{2\times 1} = 15$ choices for maths, $\binom{10}{3} = \frac{10\times 9\times 8}{3\times 2\times 1} = 120$ for science; $15 \times 120 = 1800$ all up.

7. Using only the formula $\displaystyle\binom{a}{b} = \frac{a!}{b!(a-b)!}$ prove

(a) $\forall n \in \mathbb{N} \quad \forall r \in \{0, \ldots, n\} \quad \displaystyle\binom{n}{r} = \binom{n}{n-r}$. $\displaystyle\frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!r!} = \frac{n!}{(n-r)!(n-(n-r))!}$

(b) $\forall n \in \mathbb{N} \quad \forall r \in \{1, \ldots, n\} \quad \displaystyle\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$.

$$\frac{(n-1)!}{r!(n-1-r)!} + \frac{(n-1)!}{(r-1)!(n-1-(r-1))!} = \frac{(n-1)!}{(r-1)!(n-r-1)!}\left(\frac{1}{r} + \frac{1}{n-r}\right) = \frac{(n-1)!}{(r-1)!(n-r-1)!}\left(\frac{n}{r(n-r)}\right) = \frac{n!}{r!(n-r)!}$$

8. A PIN is a number with 4 decimal digits (*e.g.* 2357, 0922 etc).

(a) How many (different) PINs are there?

$$\mathbf{10^4 = 10\,000.}$$

(b) How many (different) PINs contain the digit 0 at least once?

$$\mathbf{10^4 - 9^4 = 3439}$$

(Remove all PINs that contain no 0.)

(c) How many (different) PINs contain at least one even digit?

$$\mathbf{10^4 - 5^4 = 9375}$$

(Remove all PINs containing only odd digits.)

(d) How many (different) PINs contain at least one even digit and at least one odd digit?

$$\mathbf{10^4 - 2(5^4) = 8750}$$

(Remove all PINs containing only odd, or only even, digits.)

(e) How many (different) PINs start or end with 0 (or both)?

$$\mathbf{10^3 + 10^3 - 10^2 = 1900}$$

(Use inclusion-exclusion principle.)

(f) How many (different) PINs contain no digit more than once?

$$\boldsymbol{P(10,4) = 10 \times 9 \times 8 \times 7 = 5040.}$$

(g) How many (different) PINs have their digits in increasing order (*e.g.* 0458)?

$$\boldsymbol{C(10,4) = \binom{10}{4} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210.}$$

(Choose four distinct digits. They have only one ascending order arrangement.)

(h) How many (different) PINs have their digits in nowhere-decreasing order (*e.g.* 0448)?

$$\boldsymbol{C(9+4,4) = \binom{13}{4} = \frac{13 \times 12 \times 11 \times 10}{4 \times 3 \times 2 \times 1} = 715.}$$

(Nine bars and four stars. *E.g.* $*||||**||||*|$ represents 0448.)

(i) How many (different) PINs have digit sum 9?

$$\boldsymbol{C(3+9,9) = \binom{12}{9} = \binom{12}{3} = \frac{12 \times 11 \times 10}{3 \times 2 \times 1} = 220.}$$

(Three bars, and nine stars worth 1 each. *E.g.* $***||*|*****$ represents 3015.)

(j) How many (different) PINs have digit sum 10?

$$\boldsymbol{C(3+10,10) - 4 = \binom{13}{3} - 4 = \frac{13 \times 12 \times 11}{3 \times 2 \times 1} - 4 = 286 - 4 = 282.}$$

(As in (i) above with an extra star. But PINs A000,0A00,00A0,000A not allowed.)

(k) How many (different) PINs involve only two different digits?

$$\boldsymbol{C(10,2) \times (2^4 - 2) = \binom{10}{2}(16-2) = \frac{10 \times 9}{2 \times 1} \times 14 = 630.}$$

(Each choice of two different digits gives 2^4 PINs, two of which use only one digit.)

(l) How many (different) PINs involve only two different digits, each used twice.

$$\boldsymbol{C(10,2) \times C(4,2) = \binom{10}{2}\binom{4}{2} = \frac{10 \times 9}{2 \times 1} \times \frac{4 \times 3}{2 \times 1} = 45 \times 6 = 270.}$$

(For each choice of two different digits, choose two positions for the greater digit.)