Graduate Assignment C

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I have read the ANU Academic Skills statement regarding collusion. I have not engaged in collusion in relation to this assignment.

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27/05/2021 1pm

Question 1 [Enumerative combinatorics (counting)]

(A) Recall that a byte is a binary string of 8 bits. How many bytes contain at least six 1's?

$1 + 8 + 8 \times 7 = 9 + 56 = 65$

(B) Suppose that A is a set with 5 elements and B is a set with 7 elements.

(i) How many injections (injective functions) are there from A to B?

$P(7,5) = 7 \times 6 \times 5 \times 4 \times 3 = 2520$

(ii) How many bijections (bijective functions) are there from A to B?

0

(iii) How many bijections (bijective functions) are there from A to A?

$P(5,5) = 5 \times 4 \times 3 \times 2 \times 1 = 120$

(iv) How many bijections (bijective functions) are there from A to A with the property that no element of A is mapped to itself?

$P(5,5) - 1 - C_5^2 - C_5^2 - C_5^1 \times 9 = 120 - 1 - 10 - 10 - 45 = 54$

(C) In no more than 300 words, respond to the following claim: "It is useful for a computer scientist to be skilled in enumerative combinatorics (counting)." You may agree, or disagree or both. Your response should be informed by at least one interesting example that illustrates the point(s) you wish to make.

I think it is useful. Enumeration combinatoric problems are very common in computer science, such as password cracking, writing test cases, and so on. For example, when writing test cases for white box testing, different input parameters may correspond to different outputs, and we need to test the code completely in all cases by finding all possible combinations of input parameters. Also, when testing a system, a process may consist of multiple small processes, the order of the small processes may affect the output results. When testing, all possible permutations should be found through permutation and combination and verify whether the result can meet the requirements.

Question 2 [Probability]

(A) Five firms, F_1, F_2, F_3, F_4, F_5, are such that F_1 has greater political influence than F_2, which in turn has greater political influence than F_3, and so on. A local government has three contracts C_1, C_2, C_3 to offer. Contract C_1 is more valuable to a firm than contract C_2, which in turn in more valuable to a firm than C_3. Any one firm will be awarded at most one contract. Find the probability that, if the contracts are awarded at random, the firm awarded contract C_1 has more political influence than the firm awarded contract C_2 which, in turn, has more political influence than the firm awarded contract C_3.

$|S| = P(5,3) = 60$

If C_1 is awarded to F_1, then C_2, C_3 can be awarded to (F_2, F_3), (F_3, F_4), (F_4, F_5), (F_2, F_4), (F_2, F_5), (F_3, F_5).

If C_1 is awarded to F_2, then C_2, C_3 can be awarded to (F_3, F_4), (F_4, F_5), (F_3, F_5)

If C_1 is awarded to F_3, then C_2, C_3 can be awarded to (F_4, F_5).

$|E| = 6+3+1 = 10$

$\therefore P = \frac{|E|}{|S|} = \frac{10}{60} = \frac{1}{6}$

(B) A complex electronic system is built with a certain number of backup components in its subsystems. One subsystem has five identical components, each with a probability of .2 of failing in less than 1000 hours. The subsystem will operate if any two of the five components are operating. Assume that the components operate independently. Find the probability that the subsystem operates longer than 1000 hours.

If the subsystem operates no longer than 1000h, then there should be 5 or 4 brocken.

$P(E^c) = C_5^5 \times 0.2^5 + C_5^4 \times 0.2^4 \times 0.8 = 0.00672$

$P(E) = 1 - P(E^c) = 1 - 0.00672 = 0.99328$

(C) In no more than 300 words, respond to the following claim: "Probability is just an application of enumerative combinatorics." You may agree, or disagree or both. Your response should be informed by at least one interesting example that illustrates the point(s) you wish to make.

I partly agree with this claim, but probability is not just an application of enumerative combinatorics. Discrete probability models can usually be solved with enumerative combinatorics, for example the probability of throwing a dice a few times to get a certain point. But for the continuous probability model, because there are infinite possibilities, it is impossible to calculate the probability through enumerative combinatorics, for example, the probability of a train arriving at a station in a certain period of time.

Question 3 [Expected value in a Markov processes]

(a) Write out the transition matrix T for $\mathcal{P}$, with the i-th row and column corresponding to state ⓘ.

$$\begin{bmatrix} 1/2 & 0 & 1/4 & 0 & 0 & 0 & 0 & 1/4 \\ 0 & 0 & 1/4 & 0 & 0 & 1/4 & 0 & 1/2 \\ 1/4 & 0 & 1/2 & 0 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/4 & 0 & 1/4 \\ 1/4 & 0 & 0 & 1/4 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/4 & 0 & 1/2 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 \end{bmatrix}$$

(b) In two or three sentences, compare and contrast the terms 'sink' and 'steady state', both in general for Markov processes and specifically in relation to $\mathcal{P}$.

"Steady state" is the probabilities that the results are certain states after many repetitions, the results are not have to be 1 and 0 but the sum should be 1.

But to enter "sink" it is not necessary to take many attempts but once enter "sink" the probability of the next state being the same is 1 and being the other states is 0.

(c) Use a computer calculation to show that $\mathcal{P}$ has exactly one steady state. Explain how you do this and give your input and output from the computer.

The inputs are the transition matrix T, a list of the initial state vectors X_s, and a threshold e. The output is the steady state vectors.

For each initial state vector x_0 in X_s, calculate $x_1 = T'x_0$, then calculate the differ with differ = sum $((x_1 - x_0)^2)$. Repeat calculating $x_n = T'x_{n-1}$ and the differ of x_n and x_{n-1} until differ $< e$, then save x_n as the approximate steady state vector of this initial state vector.

For P, all the steady state vectors are $[0,0,0,0,0,0,1,0]^T$, so P has one steady state (with $e = 10^{-15}$).

(d) Use T to calculate $\mathbb{P}(X \le 10)$, the probability that $\mathcal{P}$ is in state ⑦ after ten steps.

The inputs are the initial state vector "v", the number of steps "n", and the transit matrix "T".

The output is the state vector after n steps.

First initialize "res" as v. Then for each step, calculate res = T·res. After n steps, return res.

For this question, inputs are $V = [0,0,0,0,0,0,0,1]^T$, $n = 10$, T is the one in (a).

The output is $\begin{bmatrix} 0.0652284 \\ 0.00776264 \\ 0.0652284 \\ 0.0652284 \\ 0.00776264 \\ 0.0652284 \\ 0.68232476 \\ 0.04123634 \end{bmatrix}$, so $P(X \le 10) = 0.68232476$

(e) In an attempt to estimate $\mathbb{E}(X)$, find the least n such that $\mathbb{P}(X \le n) \ge 95\%$.

The inputs are the initial state vector "v", the index of state "i", the transition matrix "T", and the probability to get to as "thr".

The output is the number of steps n.

First initialize n as 0, $p = V_i$. Calculate $V = T'V$, $p = V_i$, $n = n+1$ until $p \geqslant$ thr. Return n.

For this question, $V = [0\,0\,0\,0\,0\,0\,0\,1]^T$, $i = 7$, thr $= 0.95$, T is the one in (a).

The output $n = 26$

(f) Calculate $\mathbb{E}(X)$ exactly by using a powerful formula from the theory of Markov Processes. Let Q denote the 7×7 matrix obtained by removing row 7 (the sink row) and column 7 from the T, and let $N = (I - Q)^{-1}$. Then $\mathbb{E}(X)$ is the sum of the entries in the last row of N (the row that relates to the starting state ⑧.).

The inputs are the transition matrix T (the one in (a)), and the index i of the row and column to remove (7 for this problem).

The output is the calculated $E(X)$.

For this question the output is 8.