B2: Sequences

Text Reference (Epp)

3ed: Sections	4.1-4, 8.1-3 (Sequences and induction), 9.3,5 (Sorting)
4ed: Sections	5.1-4,6-8, (Sequences and induction), 11.3,5 (Sorting)
5ed: Sections	5.1-4,6-7, (Sequences and induction), 11.3,5 (Sorting)

Sequences

Let S be a set and $I \subseteq \mathbb{N}^\star$. A function $a : I \to S$ is called a **sequence in** S. Special **sequence notation** is often used:

Function notation	Sequence notation
$\begin{aligned} a : I &\to S \\ n &\mapsto a(n). \end{aligned}$	$(a_n)_{n\in I} \subseteq S$

The notation $(a_n)_{n\in I}$ indicates that the function can be represented as an *ordered -tuple* or, more simply, as a *list*.

(Unlike a *set*, a list has an order, and can have repeated entries.)

Examples

- $I = \{1,2,3\}: \ (a_n)_{n\in I} = (a_1, a_2, a_3)$.
- $I = \mathbb{N}^\star: \ (a_n)_{n\in I} = (a_0, a_1, a_2, \ldots\ldots)$.

In practice we usually leave out the parentheses and speak of "the sequence a_1, a_2, a_3" or "the sequence $a_0, a_1, a_2, \ldots\ldots$."

An agreed upon abuse of notation

The "$\subseteq S$" part of the sequence notation $(a_n)_{n\in I} \subseteq S$ indicates that the sequence members belong to S; *i.e.* that the range of the sequence function $a : I \to S$ is a subset of its codomain S.

The sequence *itself* is **not** a subset of S, since it is not a *set*.

Examples

1. Suppose n represents time (in months since January 1, 2000) and a_n is the standard savings account interest rate offered by bank X at time n. Then $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{Q}$ is a sequence of interests rates since 2000 and into the future!

   For example, a_{17} is the standard savings account interest rate offered by bank X on 1 June, 2001.

Examples

2. Suppose n represents time (in months since January 1, 2000) and a_n, f_n, z_n represent the populations of amphibians, fish and zooplankton in a particular lake ecosystem at time n. Let $p_n = (a_n, f_n, z_n)$. Then $(p_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}^\star \times \mathbb{N}^\star \times \mathbb{N}^\star$ is a sequence of states of the ecosystem since 2000 and into the future!

Examples

3. For each $n \in \mathbb{N}$, let a_n denote the amplitude of the harmonic of frequency $n \times f$ (where f is the fundamental frequency). Then $(a_n)_{n \in \mathbb{N}} \subseteq \mathbb{N}^\star$ is a sequence of amplitudes.

4. Let U be a set of users, then $(u_n)_{n \in \{1,2,3,4,5\}} \subseteq U$ is a list of 5 users.

In examples 1, 2, 3 the indexing variable n had some intuitive meaning; in example 4 the indexing variable did not necessary have an intuitive meaning other than we have ordered the 5 interesting users into the first, second, third, fourth and fifth user.

Describing sequences: explicit definitions

An **explicit definition** of a sequence is a formula for a_n.

Examples:

1. For all $n \in \mathbb{N}$, let $a_n = 2^n$. Then

$$(a_n)_{n\in\mathbb{N}} = 2, 4, 8, 16, \ldots$$

2. Let a_1 = Pierre, a_2 = Julie, a_3 = Paul. Then $(a_n)_{n\in\{1,2,3\}}$ $=$ Pierre, Julie, Paul.

Describing sequences: Implicit definitions

An **implicit definition** of a sequence comprises starting value(s) and a relationship between the a_n's.

Examples: Let $(a_n)_{n\in\mathbb{N}}$ be the sequence such that:

$$\begin{cases} a_1 = 2, \text{ and} \\ \forall n \in \mathbb{N}\ a_{n+1} = 2a_n. \end{cases}$$

This defines the sequence

$$(a_n)_{n\in\mathbb{N}} \ = \ 2, 4, 8, 16, \ldots,$$

Another example

Let $(a_n)_{n\in\mathbb{N}}$ be the sequence such that:

$$\begin{cases} a_1 = 0, \\ a_2 = 1, \text{ and} \\ \forall n \in \{2, 3, 4, \dots\}\ a_{n+1} = -a_n + a_{n-1}. \end{cases}$$

Defines the sequence

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 1 \\ a_3 &= -1+0=-1 \\ a_4 &= -(-1)+1=2 \\ a_5 &= -2+(-1)=-3 \\ \vdots &\quad \vdots \end{aligned}$$

Proofs about sequences

Mathematical induction

Let $P(n)$ be a predicate with variable $n \in \mathbb{N}$.
How to prove that $\forall n \in \mathbb{N} \;\; P(n)$?

METHOD 1:

Introduce a fixed but arbitrary variable: Let $n \in \mathbb{N}$.
you are now working with a fixed but arbitrary value of n.

Deduce $P(n)$ from what you know: *Insert mathemagic here.*

Victory lap: Since $P(n)$ hold for a fixed but arbitrary choice $n \in \mathbb{N}$, $P(n)$ holds for all $n \in \mathbb{N}$. *No one write this, but this is why the method works.*

Method 2:

The basis step Prove $P(1)$.

The inductive step Prove

$$\forall n \in \mathbb{N} \quad \Big(P(1) \wedge P(2) \wedge P(3) \wedge ... \wedge P(n)) \Rightarrow P(n+1)\Big)$$

Let $n \in \mathbb{N}$. Suppose that all of the statements $P(1)$, $P(2)$, ..., $P(n)$ are true. *Now deduce* $P(n+1)$ *making use somewhere of one or more of the facts* $P(1), \ldots, P(n)$.

The victory lap By the Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$.

(This is also known as *strong* mathematical induction.)

Why does induction work?

Suppose that you have completed the base step and the inductive step.

From the basis step, we know that $P(1)$ is true.

From the inductive step, we know that $P(1) \implies P(2)$. Since $P(1)$ is true and $P(1) \implies P(2)$, we deduce that $P(2)$ is true.

From the inductive step, we know that $P(2) \implies P(3)$. Since $P(2)$ is true and $P(2) \implies P(3)$, we deduce that $P(3)$ is true.

Continuing to argue in this manner gives $P(n)$ for all $n \in \mathbb{N}$.

From implicit to explicit definitions; Example 1

A sequence is defined implicitly by

$$\begin{cases} \boldsymbol{a_{n+1} = 3a_n} & \forall n \in \mathbb{N}, \\ \boldsymbol{a_1 = 3.} \end{cases}$$

Can we get an explicit definition?
First generate some values:

$a_1 = 3$, $a_2 = 9$, $a_3 = 27$, $a_4 = 81$,...

Now we make a claim/hypothesis/informed guess:
$\forall n \in \mathbb{N}\ a_n = 3^n$.

Proof that the claim is correct

We shall prove the claim using mathematical induction.
Basis step: For $n=1$, formula gives $a_1=3^1=3$, agreeing with the implicit definition.

Inductive step: Let $n \in \mathbb{N}$. Suppose that the formula is correct for $a_1, a_2, \ldots, a_n$. Then

$$\begin{aligned} a_{n+1} &= 3a_n \quad \text{(from the implicit definition)} \\ &= 3(3^n) \quad \text{(by the inductive assumption)} \\ &= 3^{n+1} \end{aligned}$$

and so the formula is also correct for $n+1$.

By the Principle of Mathematical Induction, the formula is correct for all $n \in \mathbb{N}$.

Sum and products of terms

Terms of a sequence can be summed: $a_1 + a_2 + a_3 + ...$ or multiplied: $a_1 \times a_2 \times a_3 \times$ .... We use the special notation

$$\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + ... + a_k,$$

$$\prod_{n=1}^{k} a_n = a_1 \times a_2 \times a_3 \times ... \times a_k.$$

Examples

1. $$\sum_{n=1}^{10} n = 1 + 2 + 3 + 4 + ... + 9 + 10 = 55.$$

2. $$\sum_{n=0}^{7} 2^n = 1 + 2 + 4 + 8 + ... + 128 = 255.$$

3. $$\prod_{n=1}^{5} n = 1 \times 2 \times 3 \times 4 \times 5 = 5! = 120.$$

4. $$\prod_{n=1}^{8} n^2 = 4 \times 9 \times 16 \times ... \times 64 = 1\,625\,702\,400.$$

Geometric sequences

Given a set of integers $K = \{n \in \mathbb{Z} \mid ngeqk\}$, a sequence $(a_n)_{n \in K} \subseteq \mathbb{R}$ is a **geometric sequence** when there exist $a, r \in \mathbb{R}$ such that

$$\begin{cases} a_k = a, \text{ and} \\ \forall k \in K a_{k+1} = r a_k \end{cases}$$

We call a the **first term** and r the **common ratio** of the geometric sequence.

A geometric sequence can also be defined explicitly:

$$\forall n \in K \quad a_n = ar^{n-k}$$