Graduate Assignment A

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I have read the ANU Academic Skills statement regarding collusion. I have not engaged in collusion in relation to this assignment.

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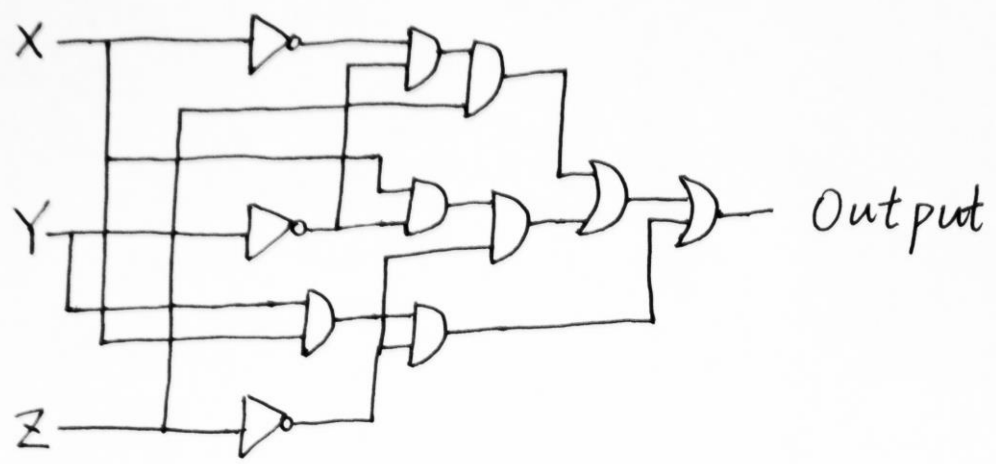
09/04/2021 1pm

Question 1

(A) Construct a circuit diagram corresponding to the input-output table below.

X	Y	Z	output
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	1
0	0	0	0

$(X \wedge Y \wedge \neg Z) \vee (X \wedge \neg Y \wedge \neg Z) \vee (\neg X \wedge \neg Y \wedge Z)$



(B) Determine whether the following statement is true or false, and explain your reasoning:

> Every compound statement is logically equivalent to one in which the only symbols used are statement variables, '(', ')', '→' and '¬'.

True.

$\because\ p \to q \equiv \neg p \vee q,$

$\neg(p \to q) \equiv \neg(\neg p \vee q) \equiv p \wedge \neg q$

$\therefore\ p \vee q \equiv \neg p \to q$

$p \wedge q \equiv \neg(p \to \neg q)$.

We can use "(", ")", "$\to$", "$\neg$" and statement variables to replace "$\vee$" and "$\wedge$", so the statement is true.

(C) Determine whether the following statement is true or false, and explain your reasoning:

> Every compound statement is logically equivalent to one in which the only symbols used are statement variables, '(', ')', '$\wedge$' and '$\vee$'.

False.

There is no way to replace "$\neg$".

Question 2

(A) Each of the variables in the following predicates is quantified over $\mathbb{Z}^+$:

$p(x)$: x is prime	$d(t,x)$: t divides x
$o(t)$: $t = 1$	$q(t,x)$: $t = x$.

Using only quantifications, parentheses, logical connectives, variables and the predicates $d(t,x)$ and $o(t)$ and $q(t,x)$, write something in place of ... in the following to make a true statement.

$$\forall x \in \mathbb{Z}^+ \left[p(x) \leftrightarrow \dots\right].$$

$\forall x \in Z^+ [p(x) \longleftrightarrow (x \notin o(x)) \wedge \forall t \in d(t,x)\ t \in (o(t) \cup q(t,x))$

(B) Using only quantifications, parentheses, logical connectives, variables and the predicates $d(t,x)$ and $o(t)$ and $q(t,x)$ defined in part (A),write something in place of $\ldots$ in the following to make a true statement.

$$\forall x \in \mathbb{Z}^+ \left[\neg p(x) \leftrightarrow \ldots\right].$$

$(x \in o(x)) \vee \exists t \in d(t,x)\ t \notin (o(t) \cup q(t,x))$

(C) Let $g\colon \mathbb{Z} \to \mathbb{Z}$ be a function. Consider the following two statements, both assuming the universe of integers.

Statement 1: $\forall x\ \exists y\big([x \le y] \wedge [g(x) \ge g(y)]\big)$

Statement 2: $\exists y\ \forall x\big([x \le y] \wedge [g(x) \ge g(y)]\big)$

Without knowing any more about the function g, are you able to determine whether or not Statement 1 is true? How about Statement 2? Explain your answers.

Statement 1: True.

For all $x \in \mathbb{Z}$, there can always find a $y = x$ that always satisfies

$[x \le y] \wedge [g(x) \ge g(y)] \equiv [x \le x] \wedge [g(x) \ge g(x)]$

$\therefore \forall x\ \exists y\,([x \le y] \wedge [g(x) \ge g(y)])$

Statement 2: False

There is no $y \in \mathbb{Z}$ that is larger than all $x \in \mathbb{Z}$

Question 3

(A) Establish or refute the validity of the following argument:
If the Raiders are playing a home match, the traffic will be bad.
If the traffic is bad, we will be late to the Tina Arena concert.
$\therefore$ If we are late to the Tina Arena concert, it will be because the Raiders are playing a home match.

Let p = the Raiders are playing a home match,

q = the traffic is bad,

r = We are late to the Tina Arena concert.

So $p \to q$. $q \to r$, that is:

$$\begin{array}{l} p \to q \\ q \to r \\ \quad\ \ p \\ \hline \quad\ \ r \end{array}$$

However, if $p \to q$ is true, $q \to p$ may not be true.
The argument is $p \to q$, $q \to r$, when r is true, q may not be true. So the argument is false.

(B) Establish or refute the validity of the following argument:
Vika is a mathematics major or Vika is a computer science major.
If Vika is a computer science major, then Vika is required to take MATH1005.
$\therefore$ Vika is a mathematics major or Vika is required to take MATH1005.

Let p = Vika is a mathmatics major, q = Vika is a computer science major,

r = Vika is required to take MATH1005.

So the argument is $((p \vee q) \wedge (q \rightarrow r)) \rightarrow (p \vee r)$

p	q	r	$((p \vee q) \wedge (q \rightarrow r)) \rightarrow (p \vee r)$			
T	T	T	T	T	T	T
T	T	F	T	F	T	T
T	F	T	T	T	T	T
T	F	F	T	F	T	T
F	T	T	T	T	T	T
F	T	F	T	F	T	F
F	F	T	F	T	T	T
F	F	F	F	T	T	F

According to the truth table the argument is true.

(C) Sharky, a leader of the underworld, was killed by one of his own band of four minions. Detective Sharp interviewed the minions and determined that all were lying except for one. The detective's notes from the interviews included the following:

- Socko said "Lefty killed Sharky."
- Fats said"Muscles didn't kill Sharky."
- Lefty said "Muscles was shooting dice with Socko when Sharky was killed."
- Muscles said " Lefty didn't kill Sharky."

Who killed Sharky? Justify your answer.

Let p = Lefty killed Sharky, q = Muscles killed Sharky,

r = Socko killed Sharky, s = Fats killed Sharky.

Now need $p \oplus q \oplus r \oplus s$ is true.

For Socko: p. For Fats: $\neg q$. For Lefty: $(\neg q) \wedge (\neg r)$. For Muscles: $\neg p$.

p	q	r	s	p	$\neg q$	$(\neg q)\wedge(\neg r)$	$\neg p$	
T	F	F	F	T	T	T	F	Only Muscles lied.
F	T	F	F	F	F	F	T	All lied except Muscles.
F	F	T	F	F	T	F	T	Socko and Lefty lied.
F	F	F	T	F	T	F	T	Socko and Lefty lied.

$\therefore$ Muscles is telling the truth, and Muscles killed Sharky.

Question 4 A relation $\mathcal{R}$ on a set S is said to be *transitive* if, and only if,

$$\forall x, y, z \in S \quad x \mathcal{R} y \wedge y \mathcal{R} z \implies x \mathcal{R} z.$$

Relations $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ are defined on the power set $\mathcal{P}(\{a,b,c,d,e,f\})$ by the rules below. In each case, prove or disprove that the relation is transitive.

(A) $A \mathcal{R}_1 B \Leftrightarrow A \backslash B = \varnothing$.

$A \backslash B = \emptyset \Leftrightarrow A = \emptyset \cup B \Leftrightarrow A = B$.

$\therefore A R_1 B \Leftrightarrow A = B$

$\forall x, y, z \in P$, $x R_1 y \Leftrightarrow x = y$, $y R_1 z \Leftrightarrow y = z$.

$x R_1 z \Leftrightarrow x = z$.

$x R_1 y \wedge y R_1 z \Leftrightarrow x = y \wedge y = z \Leftrightarrow x = z \Leftrightarrow x R_1 z$.

$\therefore A R_1 B \Leftrightarrow A \backslash B = \emptyset$ is transitive.

(B) $A \mathcal{R}_2 B \Leftrightarrow$ there exists a bijective function $f : A \to B$.

$\forall x, y, z \in P$. Since function f is bijection.

$x R_2 y \Leftrightarrow$ there exists a function $f : x \to y$ and $f^{-1} : y \to x$.

$y R_2 z \Leftrightarrow$ there exists a function $g : y \to z$ and $g^{-1} : z \to y$.

$x R_2 z \Leftrightarrow$ there exists a function $h : x \to z$ and $h^{-1} : z \to x$.

$x R_2 y \wedge y R_2 z \Leftrightarrow (g \circ f : x \to z) \wedge (g^{-1} \circ f^{-1} : z \to x)$

So there is a $h = g \circ f : x \to z$, and $h^{-1} = g^{-1} \circ f^{-1} : z \to x$.

$\therefore$ The relation is transitive.

(C) $A \mathcal{R}_3 B \Leftrightarrow$ there exists an injective function $f : A \backslash B \to B \backslash A$.

Not transitive.

Question 5 Review the definition of countable and uncountable sets given in lectures. Prove or disprove each of the following statements:

(A) Every subset of a countable set is countable.

(A) Let S be a countable set, so there is a bijection from S to a subset of the set N, which means every element in set S can be mapped to a unique natural number, vice versa. A subset of S is a collection of elements in S. so with the same mapping function the elements in the subset of S can also be mapped to a unique natural number, vice versa. So the is also a bijection from every subset of S to a subset of N, which means the statement is true.

(B) If A and B are disjoint sets and both A and B are countable, then $A \cup B$ is countable.

(B) Since A and B are both countable, each element in A and B can be mapped to a natural number by a bijective function. A and B are disjoint sets so no two elements in $A \cup B$ will be mapped to the same natural number so the mapping function can still be bijetive. So the statement is true.

(C) If $A \subseteq \mathbb{R}$ and

$$\forall a, b \in A \left((a < b) \to (\exists c \in A \; a < c < b) \right),$$

then A is uncountable.

(C) There is always a $c \in A$ between a and b where $a, b \in A$ and $a \neq b$, so A is continuous, which is uncountable. So the statement is true.