A Positivity-preserving Strategy for Entropy Stable Discretizations of the Compressible Euler and Navier-Stokes equations

Yimin Lin, Ignacio Tomas, Jesse Chan

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Department of Computational and Applied Mathematics, Rice University
Sandia National Laboratories

Compressible Euler and Navier-Stokes equations

- Compressible Euler and Navier-Stokes equations

$$\frac{\partial \boldsymbol{U}}{\partial t} + \underbrace{\sum_{i=1}^{3} \frac{\partial \boldsymbol{f}_i(\boldsymbol{U})}{\partial \boldsymbol{x}_i}}_{\text{inviscid flux}} = \underbrace{\sum_{i=1}^{3} \frac{\partial \boldsymbol{g}_i(\boldsymbol{U})}{\partial \boldsymbol{x}_i}}_{\text{viscous flux}}$$

- Entropy variables symmetrizes the viscous fluxes:

$$\sum_{i=1}^{d} \frac{\partial \boldsymbol{g}_i}{\partial x_i} = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(K_{ij} \frac{\partial \boldsymbol{v}}{\partial x_j}\right),$$

$$K = \begin{bmatrix} K_{11} & \dots & K_{1d} \\ \vdots & \ddots & \vdots \\ K_{d1} & \dots & K_{dd} \end{bmatrix} = K^T, \qquad K \succeq 0$$

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- Loss of chain rule at discrete level (discrete effects, inexact quadrature) $\implies$ Loss of entropy stability

High order entropy stable DG discretization

- Discretize the variational form of invscid term

$$\int_{\widehat{D}} \frac{\partial \boldsymbol{f}}{\partial x} \vec{l} \quad \xRightarrow{\text{Flux Differencing}} \quad 2\left(\mathbf{Q} \circ \mathbf{F}_S\right) \mathbf{1}, \quad \left(\mathbf{F}_S\right)_{ij} = \boldsymbol{f}_S\left(\boldsymbol{u}_i, \boldsymbol{u}_j\right)$$

Summation-by-parts $\quad \mathbf{Q} = \mathbf{M}\mathbf{D}, \quad \mathbf{Q} + \mathbf{Q}^T = \mathbf{B}, \quad \mathbf{Q}\mathbf{1} = 0$

Entropy conservation $\quad \left(\boldsymbol{v}_L - \boldsymbol{v}_R\right)^T \boldsymbol{f}_S\left(\boldsymbol{u}_L, \boldsymbol{u}_R\right) = \psi\left(\boldsymbol{u}_L\right) - \psi\left(\boldsymbol{u}_R\right)$

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- Discretize the symmetrized viscous term

$$\frac{\partial \boldsymbol{g}}{\partial x} = \frac{\partial}{\partial x}\left(K \frac{\partial \boldsymbol{v}}{\partial x}\right) \quad \xRightarrow{\text{Discretize}} \begin{cases} \left(\boldsymbol{\Theta}, \varphi\right)_\Omega = \left(\frac{\partial \boldsymbol{v}}{\partial x}, \varphi\right)_\Omega + \langle [\![\boldsymbol{v}]\!] n_i, \varphi \rangle_{\partial\Omega} \\ \left(\boldsymbol{\sigma}, \eta\right)_\Omega = \left(K\boldsymbol{\Theta}, \eta\right)_\Omega \\ \left(\boldsymbol{G}_{\text{visc}}, \psi\right)_\Omega = -\left(\boldsymbol{\sigma}, \frac{\partial \psi}{\partial x}\right)_\Omega + \langle \{\{\boldsymbol{\sigma}\}\} n_i, \psi \rangle_{\partial\Omega} \end{cases}$$

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- The scheme is entropy stable

$$\sum_k \left(\frac{\partial \eta\left(\mathbf{u}\right)}{\partial t}, \boldsymbol{v}\right)_{D^k} = \sum_k \sum_{i,j=1}^{d} -\left(K_{ij}\boldsymbol{\Theta}_j, \boldsymbol{\Theta}_i\right)_{D^k} \leq 0$$

Current work: Positivity Limiting for nodal ESDG

- The entropy is well-defined only if densities and pressures are positive.

$$\boldsymbol{v}_1 = (\gamma + 1 - s) - \frac{(\gamma - 1)\, E}{p}, \qquad s = \log\left(\frac{p}{\rho^\gamma}\right)$$

Current work: Positivity Limiting for nodal ESDG

- Strong shock forms - Negative densities

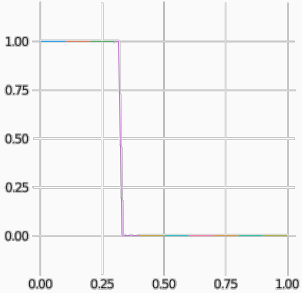

Figure 1: Exact solution

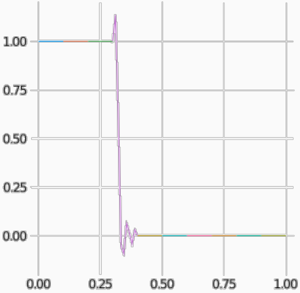

Figure 2: Solution in polynomial basis

- Oscillation by Gibbs phenomenon leads to negative density

Limiting strategy

- Step 1. Compute high order target scheme (nodal ESDG)

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  - Find largest possible l_i that satisify positivity (or relaxed bounds).
    Elementwise limiting parameter $l = \min_{i \in D^k} l_i$

Positivity preserving discretization

- Low order positivity preserving method could be written as

$$\underbrace{m_i \frac{\partial \boldsymbol{u}}{\partial t} + \sum \mathbf{Q}_{ij} \left(\boldsymbol{f}(\boldsymbol{u}_j) - \boldsymbol{\sigma}_j\right)}_{\text{low order nodal DG on LGL nodes}} - \underbrace{\sum d_{ij}(\boldsymbol{u}_j - \boldsymbol{u}_i)}_{\text{graph viscosity}} = 0$$

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- Weighted differentiation matrix $\boldsymbol{Q}$ is a sparse low order (SBP) operator:

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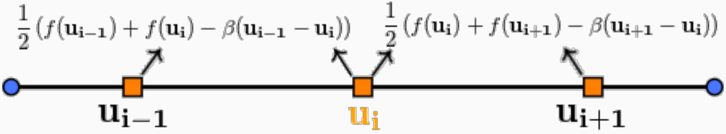
- Compressible Euler - Maximum wavespeed (Lax-Friedrichs flux)
  Compressible Navier-Stokes - Zhang's positivity preserving flux

Positivity preserving discretization

- Interpretation: subcell Lax-Friedriches type dissipation



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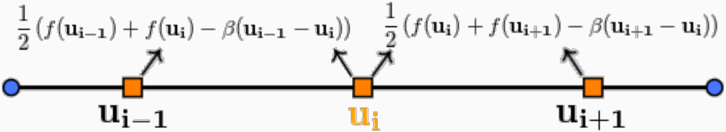
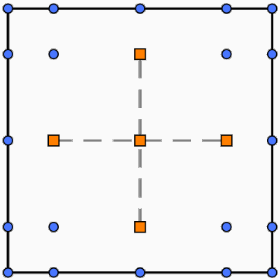
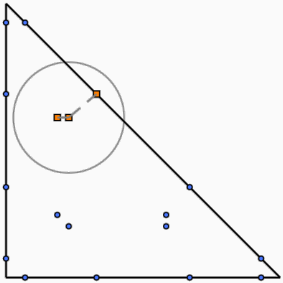


- Extension to 2D (tensor product and simplex elements)





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- Shock capturing: replace l with a shock indicator.

Double Mach Reflection - Compressible Euler

- $N = 3,\ 1000 \times 250$ elements, $T = 0.2$, limiting with the bound satisfying $\rho > 0.1\rho^{\mathrm{L}}, \rho e > 0.1\left(\rho e\right)^{\mathrm{L}}$

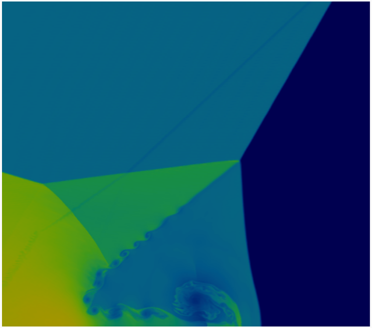

Figure 3: With shock capturing

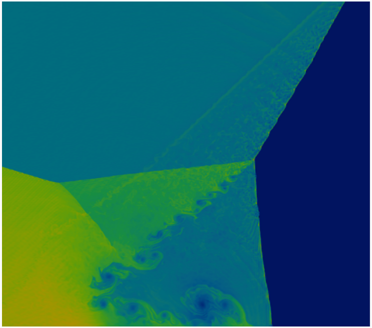

Figure 4: Without shock capturing

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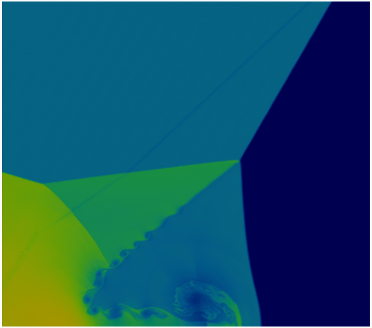

Figure 3: (ESDG) With shock capturing

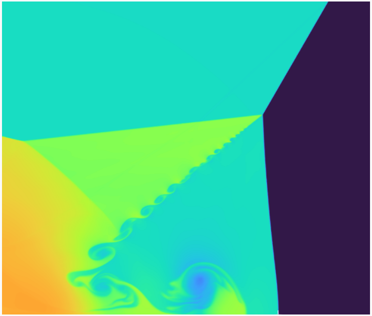

Figure 4: (Pazner, Sprase IDP, enforced minimum entropy principle) $N = 3, 2400 \times 600$ elements, $T \approx 0.275$

Daru-Tenaud shocktube - Compressible Navier-Stokes

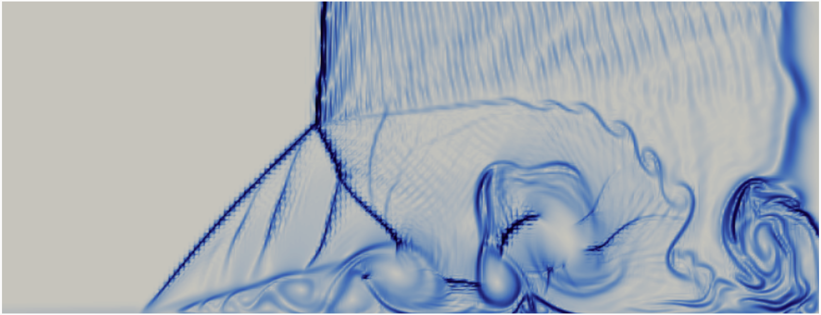

Figure 5: $Re = 1000$, $T = 1.0$ Result: $N = 3,240 \times 120$ uniform quad elements

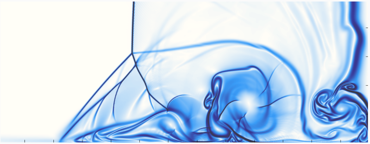

Figure 6: Invariant domain discretization, 2048×1024 elements

Figure 7: WENO5-RK4, 1000×500 grid points

Summary and future works

- We present a positivity limiting strategy for nodal ESDG based on graph viscosity.
- Future work: Positivity limiting for modal ESDG. Implicit timestepping.

Thank you!