

A Positivity-preserving Strategy for Entropy Stable Discretizations of the Compressible Euler and Navier-Stokes equations

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Compressible Euler and Navier-Stokes equations

- Compressible Euler and Navier-Stokes equations

$$\frac{\partial U}{\partial t} + \underbrace{\sum_{i=1}^3 \frac{\partial f_i(U)}{\partial x_i}}_{\text{inviscid flux}} = \underbrace{\sum_{i=1}^3 \frac{\partial g_i(U)}{\partial x_i}}_{\text{viscous flux}}$$

- Entropy variables symmetrizes the viscous fluxes:

$$\sum_{i=1}^d \frac{\partial g_i}{\partial x_i} = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(K_{ij} \frac{\partial v}{\partial x_j} \right),$$
$$K = \begin{bmatrix} K_{11} & \dots & K_{1d} \\ \vdots & \ddots & \vdots \\ K_{d1} & \dots & K_{dd} \end{bmatrix} = K^T, \quad K \succeq 0$$

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$$\int_{\Omega} \frac{\partial \eta(u)}{\partial t} + \sum_{i=1}^d \int_{\partial \Omega} n_i \left(F_i(u) - \frac{1}{c_v T} \kappa \frac{\partial T}{\partial x_i} \right) = - \int_{\Omega} \sum_{i,j=1}^d \left(\frac{\partial \mathbf{v}}{\partial x_i} \right)^T \left(\kappa_{ij} \frac{\partial \mathbf{v}}{\partial x_j} \right)$$

Integration by parts and chain rule

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- Loss of chain rule at discrete level (discrete effects, inexact quadrature)
 \implies Loss of entropy stability

High order entropy stable DG discretization

- Discretize the variational form of invscid term

$$\int_{\hat{D}} \frac{\partial f}{\partial x} \vec{l} \quad \xrightarrow{\text{Flux Differencing}} \quad 2(\mathbf{Q} \circ \mathbf{F}_S) \mathbf{1}, \quad (\mathbf{F}_S)_{ij} = f_S(u_i, u_j)$$

Summation-by-parts $\mathbf{Q} = \mathbf{M}\mathbf{D}, \quad \mathbf{Q} + \mathbf{Q}^T = \mathbf{B}, \quad \mathbf{Q}\mathbf{1} = 0$

Entropy conservation $(\mathbf{v}_L - \mathbf{v}_R)^T \mathbf{f}_S(\mathbf{u}_L, \mathbf{u}_R) = \psi(\mathbf{u}_L) - \psi(\mathbf{u}_R)$

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- Discretize the symmetrized viscous term

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x} \left(\mathbf{K} \frac{\partial \mathbf{v}}{\partial x} \right) \quad \xrightarrow{\text{Discretize}} \quad \begin{cases} (\boldsymbol{\Theta}, \varphi)_{\Omega} = \left(\frac{\partial \mathbf{v}}{\partial x}, \varphi \right)_{\Omega} + \langle \llbracket \mathbf{v} \rrbracket n_i, \varphi \rangle_{\partial \Omega} \\ (\boldsymbol{\sigma}, \eta)_{\Omega} = (\mathbf{K} \boldsymbol{\Theta}, \eta)_{\Omega} \\ (\mathbf{G}_{\text{visc}}, \psi)_{\Omega} = - \left(\boldsymbol{\sigma}, \frac{\partial \psi}{\partial x} \right)_{\Omega} + \langle \{\{\boldsymbol{\sigma}\}\} n_i, \psi \rangle_{\partial \Omega} \end{cases}$$

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- The scheme is entropy stable

$$\sum_k \left(\frac{\partial \eta(\mathbf{u})}{\partial t}, \mathbf{v} \right)_{D^k} = \sum_k \sum_{i,j=1}^d -(\mathbf{K}_{ij} \boldsymbol{\Theta}_j, \boldsymbol{\Theta}_i)_{D^k} \leq 0$$

Current work: Positivity Limiting for nodal ESDG

- The entropy is well-defined only if densities and pressures are positive.

$$\mathbf{v}_1 = (\gamma + 1 - s) - \frac{(\gamma - 1)E}{p}, \quad s = \log \left(\frac{p}{\rho^\gamma} \right)$$

Current work: Positivity Limiting for nodal ESDG

- Strong shock forms - Negative densities

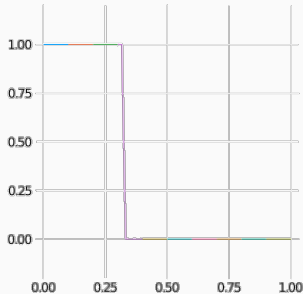


Figure 1: Exact solution

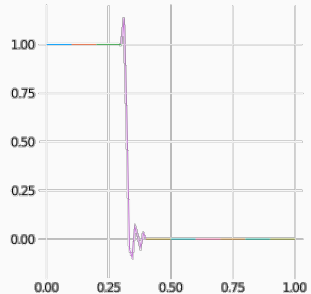


Figure 2: Solution in polynomial basis

- Oscillation by Gibbs phenomenon leads to negative density

Limiting strategy

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- Choose suitable parameter $l_i \in [0, 1]$ to satisfy positivity

$$m_i u_i^{n+1} = m_i u_i^{L,n+1} + l_i \tau (r_{L,i} - r_{H,i})$$

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 $l_i = 0 \implies$ recovers low order positivity-preserving scheme.

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- $l_i = 1 \implies$ recovers ESDG.
- $l_i = 0 \implies$ recovers low order positivity-preserving scheme.
- Find largest possible l_i that satisfy positivity (or relaxed bounds).
Elementwise limiting parameter $l = \min_{i \in D^k} l_i$

Positivity preserving discretization

- Low order positivity preserving method could be written as

$$\underbrace{m_i \frac{\partial u}{\partial t} + \sum Q_{ij} (f(u_j) - \sigma_j)}_{\text{low order nodal DG on LGL nodes}} - \underbrace{\sum d_{ij} (u_j - u_i)}_{\text{graph viscosity}} = 0$$

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- Weighted differentiation matrix \mathbf{Q} is a sparse low order (SBP) operator:

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- Define the **graph viscosity coefficients**:

$$d_{ij} = \max \{ \beta(u_i, u_j, n_{ij}) \|Q_{ij}\|, \beta(u_j, u_i, n_{ji}) \|Q_{ji}\| \}, n_{ij} = Q_{ij} / \|Q_{ij}\|$$

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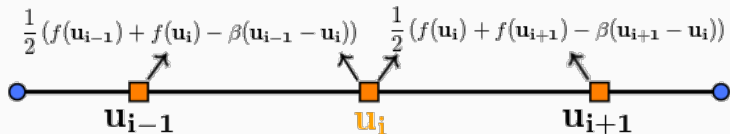
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- Compressible Euler - Maximum wavespeed (Lax-Friedrichs flux)
Compressible Navier-Stokes - Zhang's positivity preserving flux

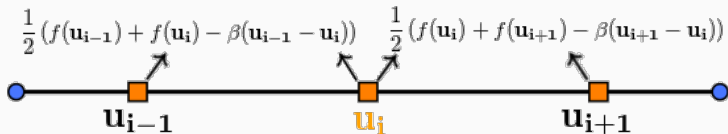
Positivity preserving discretization

- Interpretation: subcell Lax-Friedrichs type dissipation

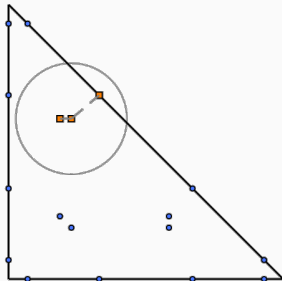
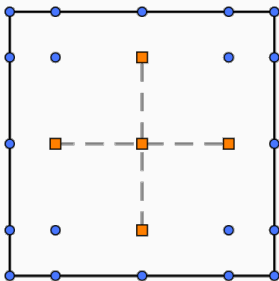


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- Extension to 2D (tensor product and simplex elements)



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- Shock capturing: replace l with a shock indicator.

Double Mach Reflection - Compressible Euler

- $N = 3$, 1000×250 elements, $T = 0.2$, limiting with the nodal bound $\rho > \zeta \rho^L, \rho e > \zeta (\rho e)^L$

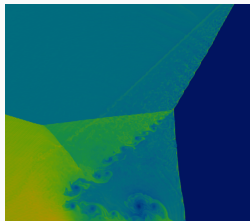


Figure 3: $\zeta = 0.1$ without shock capturing

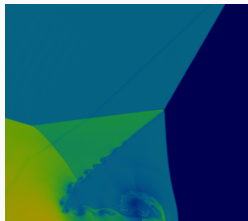


Figure 4: $\zeta = 0.1$ with shock capturing

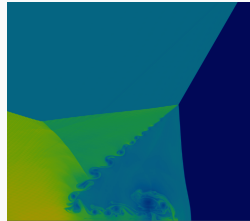


Figure 5: $\zeta = 0.5$ without shock capturing

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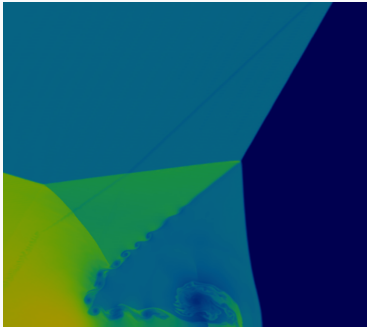


Figure 3: (ESDG) $\zeta = 0.1$ with shock capturing

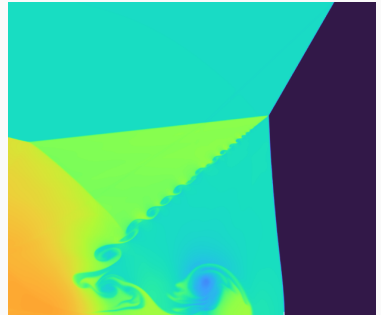


Figure 4: (Pazner, Sprase IDP, enforced minimum entropy principle) $N = 3$, 2400×600 elements, $T \approx 0.275$

Daru-Tenaud shocktube - Compressible Navier-Stokes

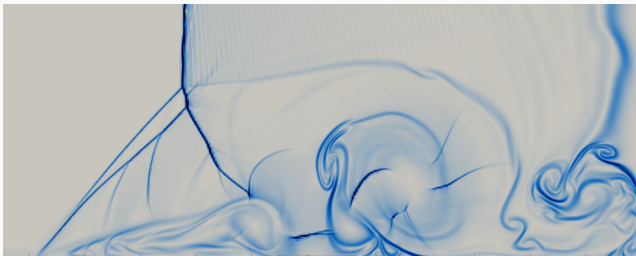


Figure 5: $Re = 1000, T = 1.0$ Result: $N = 3, 300 \times 150$ uniform quad elements without shock capturing

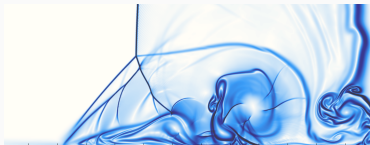


Figure 6: Invariant domain discretization, 2048×1024 elements

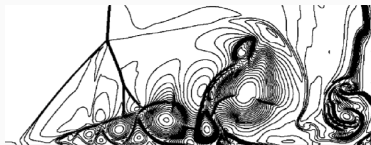


Figure 7: WENO5-RK4, 1000×500 grid points

Daru-Tenaud shocktube - Compressible Navier-Stokes

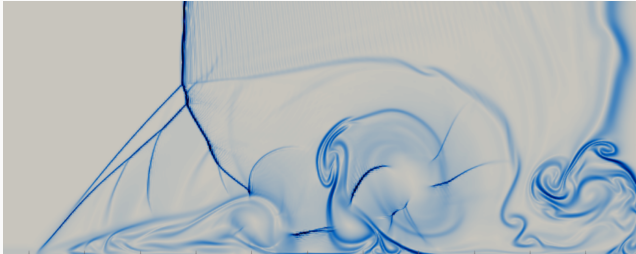


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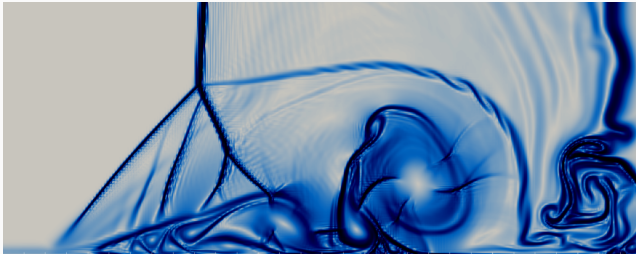


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Summary and future works

- We present a positivity limiting strategy for nodal ESDG based on graph viscosity.
- Future work: Positivity limiting for modal ESDG. Implicit timestepping.

Thank you!