Limiting techniques for high order, entropy stable, and positivity-preserving discontinuous Galerkin discretizations

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Compressible Euler and Navier-Stokes equations

- Compressible Euler and Navier-Stokes equations:

$$\frac{\partial \boldsymbol{U}}{\partial t} + \underbrace{\sum_{i=1}^{3} \frac{\partial \boldsymbol{f}_i(\boldsymbol{U})}{\partial \boldsymbol{x}_i}}_{\text{inviscid flux}} = \underbrace{\sum_{i=1}^{3} \frac{\partial \boldsymbol{g}_i(\boldsymbol{U})}{\partial \boldsymbol{x}_i}}_{\text{viscous flux}}$$

- Entropy variables symmetrizes the viscous fluxes:

$$\sum_{i=1}^{d} \frac{\partial \boldsymbol{g}_i}{\partial x_i} = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(K_{ij} \frac{\partial \boldsymbol{v}}{\partial x_j} \right),$$

$$K = \begin{bmatrix} K_{11} & \dots & K_{1d} \\ \vdots & \ddots & \vdots \\ K_{d1} & \dots & K_{dd} \end{bmatrix} = K^T, \qquad K \succeq 0$$

Continuous Entropy Balance

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Test by $\boldsymbol{v}$

$$\int_\Omega \frac{\partial \eta(\boldsymbol{u})}{\partial t} + \sum_{i=1}^{d} \int_{\partial\Omega} n_i \left(F_i(\boldsymbol{u}) - \frac{1}{c_v T} \kappa \frac{\partial T}{\partial x_i} \right) = - \int_\Omega \sum_{i,j=1}^{d} \left(\frac{\partial \boldsymbol{v}}{\partial x_i} \right)^T \left(K_{ij} \frac{\partial \boldsymbol{v}}{\partial x_j} \right)$$

Integration by parts and chain rule

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Periodic

- Loss of chain rule at discrete level (discrete effects, inexact quadrature) $\implies$ Loss of entropy stability.

Goal

Develop a $\overbrace{\text{high-order}}^{\substack{\text{Efficiency (less DOFs)}\\ \text{low dissipative/dispersion error}}}$, $\overbrace{\text{entropy-stable}}^{\substack{\text{provable nonlinear stability}\\ \text{Convergence to entropic solution}}}$, and $\overbrace{\text{positivity-preserving}}^{\text{guaranteed robustness}}$ numerical solver for compressible flows.

Outline

Positivity limiting for nodal entropy stable DG method for the compressible Navier-Stokes equation

High order entropy stable discontinuous Galerkin spectral element methods through subcell limiting

1. Positivity limiting for nodal entropy stable DG method for the compressible Navier-Stokes equation

2. High order entropy stable discontinuous Galerkin spectral element methods through subcell limiting

Goal

Develop a $\overbrace{\text{high-order, entropy-stable}}^{\text{Entropy stable discretization}}$, and $\overbrace{\text{positivity-preserving}}^{\text{provable limiting}}$ numerical solver for compressible flows.

High order entropy stable DG discretization

- Discretize the variational form of invscid term (LGL quadrature):

$$\int_{\widehat{D}} \frac{\partial \boldsymbol{f}}{\partial x} \vec{l} \quad \xrightarrow{\text{Flux Differencing}} \quad 2\left(\mathsf{Q} \circ \mathsf{F}_S\right) \mathbf{1}, \quad \left(\mathsf{F}_S\right)_{ij} = \boldsymbol{f}_S\left(\boldsymbol{u}_i, \boldsymbol{u}_j\right)$$

Summation-by-parts $\quad \mathsf{Q} = \mathsf{M}\mathsf{D}, \quad \mathsf{Q} + \mathsf{Q}^T = \mathsf{B}, \quad \mathsf{Q}\mathbf{1} = 0$

Entropy conservation $\quad \left(\boldsymbol{v}_L - \boldsymbol{v}_R\right)^T \boldsymbol{f}_S\left(\boldsymbol{u}_L, \boldsymbol{u}_R\right) = \psi\left(\boldsymbol{u}_L\right) - \psi\left(\boldsymbol{u}_R\right)$

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- Discretize the symmetrized viscous term:

$$\frac{\partial \boldsymbol{g}}{\partial x} = \frac{\partial}{\partial x}\left(K \frac{\partial \boldsymbol{v}}{\partial x}\right) \xrightarrow{\text{Discretize}} \begin{cases} \left(\boldsymbol{\Theta}, \varphi\right)_\Omega = \left(\frac{\partial \boldsymbol{v}}{\partial x}, \varphi\right)_\Omega + \langle [\![\boldsymbol{v}]\!] n_i, \varphi \rangle_{\partial\Omega} \\ \left(\boldsymbol{\sigma}, \eta\right)_\Omega = \left(K\boldsymbol{\Theta}, \eta\right)_\Omega \\ \left(\boldsymbol{G}_{\text{visc}}, \psi\right)_\Omega = -\left(\boldsymbol{\sigma}, \frac{\partial \psi}{\partial x}\right)_\Omega + \langle \{\{\boldsymbol{\sigma}\}\} n_i, \psi \rangle_{\partial\Omega} \end{cases}$$

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- The scheme is entropy stable [1]:

$$\sum_k \left(\frac{\partial \eta\left(\mathbf{u}\right)}{\partial t}, \mathbf{1}\right)_{D^k} = \sum_k \sum_{i,j=1}^{d} -\left(K_{ij}\boldsymbol{\Theta}_j, \boldsymbol{\Theta}_i\right)_{D^k} \leq 0$$

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  - Low order positivity-preserving and ESDG:

$$\frac{\boldsymbol{m}_i}{\tau}(\boldsymbol{u}_i^{L,n+1} - \boldsymbol{u}_i^n) + \boldsymbol{r}^{\mathrm{L},i} = 0$$
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  - Find largest possible l_i that satisify positivity (or relaxed bounds). Elementwise limiting parameter $l = \min_{i \in D^k} l_i$.

Positivity preserving discretization

- Low order positivity preserving method could be written as:

$$\underbrace{\boldsymbol{m}_i \frac{\partial \boldsymbol{u}}{\partial t} + \sum \mathbf{Q}_{ij} \left(\boldsymbol{f}(\boldsymbol{u}_j) - \boldsymbol{\sigma}_j\right)}_{\text{low order nodal DG on LGL nodes}} - \underbrace{\sum d_{ij}(\boldsymbol{u}_j - \boldsymbol{u}_i)}_{\text{graph viscosity}} = 0$$

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- Weighted differentiation matrix $\boldsymbol{Q}$ is a sparse low order (SBP) operator:

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & \dots & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

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- Define the graph viscosity coefficients:

$$d_{ij} = \max\left\{\beta(\boldsymbol{u}_i, \boldsymbol{u}_j, \boldsymbol{n}_{ij})\|\boldsymbol{Q}_{ij}\|, \beta(\boldsymbol{u}_j, \boldsymbol{u}_i, \boldsymbol{n}_{ji})\|\boldsymbol{Q}_{ji}\|\right\}, \boldsymbol{n}_{ij} = \boldsymbol{Q}_{ij}/\|\boldsymbol{Q}_{ij}\|$$

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- Zhang's positivity preserving flux [2]:

$$\beta\left(u_i, u_j, \sigma_i, \sigma_j, n_{ij}\right) = |n \cdot u| + \frac{1}{2\rho^2 e}\left(\sqrt{\rho^2\left(q \cdot n\right)^2 + 2\rho^2 e\,\|n \cdot \tau - pn\|} + \rho\,|q \cdot n|\right)$$

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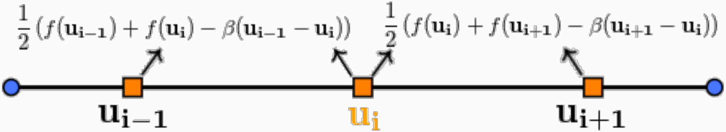
Positivity preserving discretization - Multidimension

- Interpretation: subcell Lax-Friedriches type dissipation.



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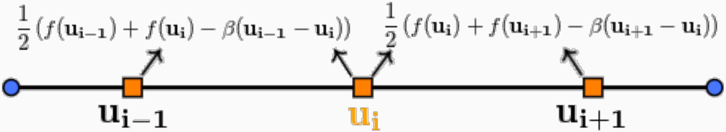
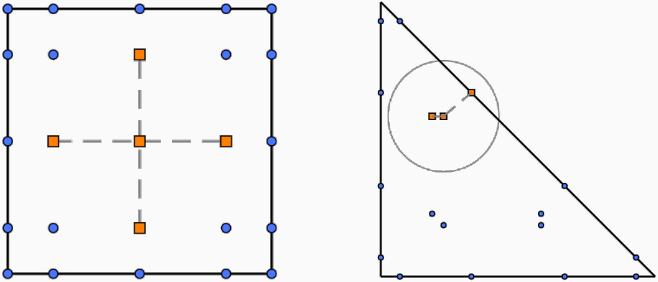


- Extension to tensor product and simplicial elements.



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- Shock capturing: replace l with a shock indicator.

- 2D viscous shocktube convergence test, elementwise limiting with the nodal bound $\rho > 0.5\rho^{\mathrm{L}}, \rho e > 0.5\left(\rho e\right)^{\mathrm{L}}$.

	$N=1$		$N=2$		$N=3$		$N=4$	
K	L^2 error	Rate	L^2 error	Rate	L^2 error	Rate	L^2 error	Rate
10	7.368×10^{-2}		4.751×10^{-2}		1.796×10^{-2}		5.448×10^{-3}	
20	3.204×10^{-2}	1.20	1.007×10^{-2}	2.24	2.168×10^{-3}	3.05	1.203×10^{-3}	2.18
40	1.145×10^{-2}	1.48	1.349×10^{-3}	2.90	3.533×10^{-4}	2.61	1.011×10^{-4}	3.57
80	2.921×10^{-3}	1.97	1.976×10^{-4}	2.77	3.882×10^{-5}	3.19	3.231×10^{-6}	4.97

Figure 1: Quadrilateral mesh

	$N=1$		$N=2$		$N=3$		$N=4$	
K	L^2 error	Rate	L^2 error	Rate	L^2 error	Rate	L^2 error	Rate
10	6.372×10^{-2}		3.053×10^{-2}		1.834×10^{-2}		4.957×10^{-3}	
20	2.384×10^{-2}	1.42	5.663×10^{-3}	2.43	3.805×10^{-3}	2.27	9.617×10^{-4}	2.37
40	7.324×10^{-3}	1.70	1.308×10^{-3}	2.11	4.876×10^{-4}	2.96	7.997×10^{-5}	3.59
80	2.020×10^{-3}	1.86	2.163×10^{-4}	2.60	5.127×10^{-5}	3.25	2.807×10^{-6}	4.74

Figure 2: Simplicial mesh

Double Mach Reflection - Compressible Euler

- $N = 3,\ 1000 \times 250$ elements, $T = 0.2$, limiting with the nodal bound $\rho > \zeta \rho^{\mathrm{L}}, \rho e > \zeta \left(\rho e\right)^{\mathrm{L}}$.

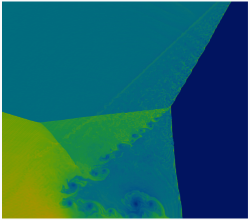

Figure 3: $\zeta = 0.1$ without shock capturing

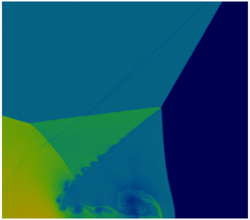

Figure 4: $\zeta = 0.1$ with shock capturing

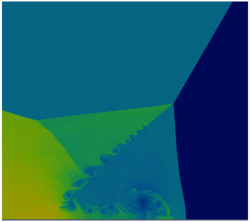

Figure 5: $\zeta = 0.5$ without shock capturing

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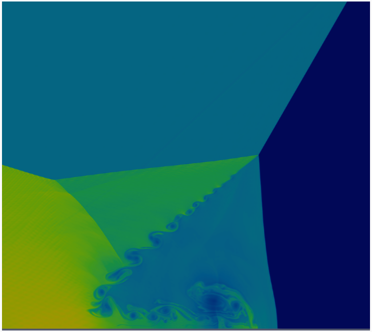


Figure 3: (ESDG) $\zeta = 0.5$ without shock capturing

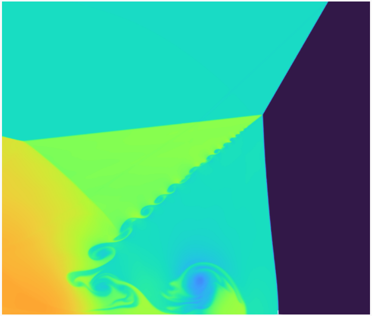


Figure 4: (Pazner, Sprase IDP, enforced minimum entropy principle) $N = 3, 2400 \times 600$ elements, $T \approx 0.275$[3]

[3] Pazner, Will. "Sparse invariant domain preserving discontinuous Galerkin methods with subcell convex limiting."

Daru-Tenaud shocktube - Compressible Navier-Stokes

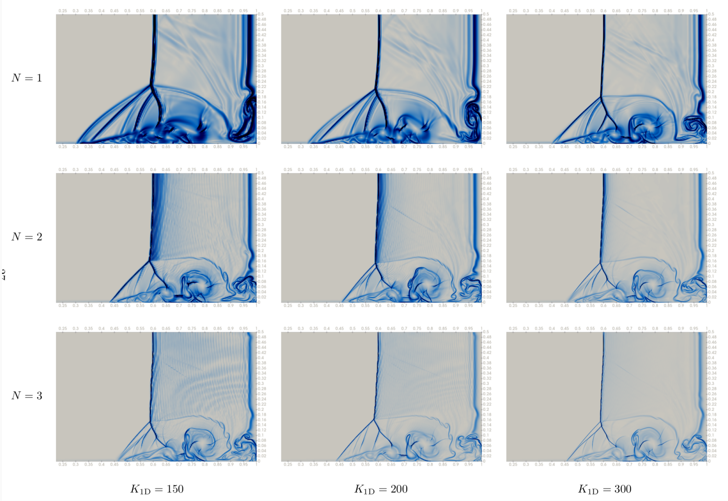


Figure 5: $Re = 1000$, uniform quad mesh, $\eta = 0.1$ without shock capturing

Daru-Tenaud shocktube - Compressible Navier-Stokes

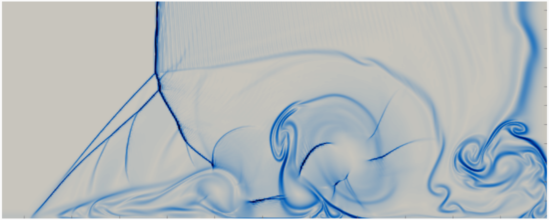


Figure 5: $Re = 1000$, $T = 1.0$ Result: $N = 3,300 \times 150$ uniform quad elements without shock capturing

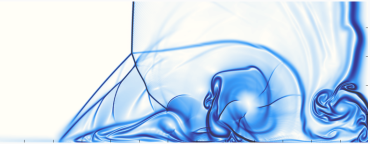


Figure 6: Invariant domain discretization, 2048×1024 elements[4]

Figure 7: WENO5-RK4, 1000×500 grid points[5]

[4]Guermond et al., "Second-Order Invariant Domain Preserving Approximation of the Compressible Navier–Stokes Equations."

[5]Sjögreen et al., "Grid Convergence of High Order Methods for Multiscale Complex Unsteady Viscous Compressible Flows."

Daru-Tenaud shocktube - Compressible Navier-Stokes

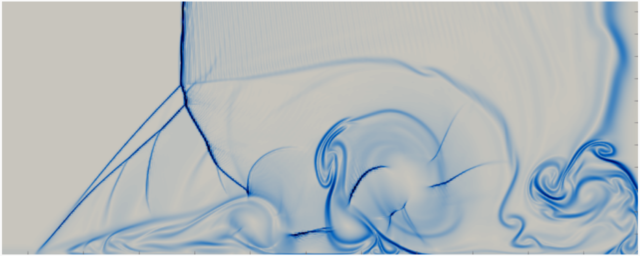


Figure 5: $Re = 1000, T = 1.0$ Result: $N = 3,300 \times 150$ uniform quad elements without shock capturing

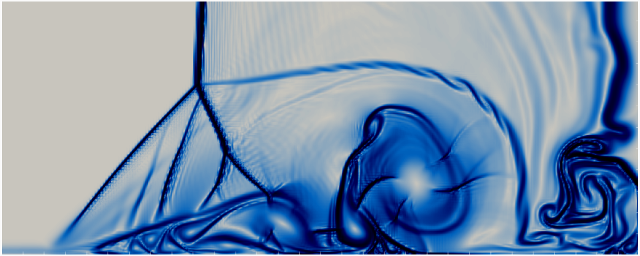


Figure 6: $Re = 1000, T = 1.0$ Result: $N = 3,300 \times 150$ uniform quad elements with shock capturing

1. Positivity limiting for nodal entropy stable DG method for the compressible Navier-Stokes equation

2. High order entropy stable discontinuous Galerkin spectral element methods through subcell limiting

Goal

Develop a $\overbrace{\text{high-order}}^{\text{Nodal DG (DGSEM)}}$, $\overbrace{\text{entropy-stable, and positivity-preserving}}^{\text{Provable limiting}}$ numerical solver for compressible flows.

Discontinuous Galerkin spectral element methods

- DG collocation on LGL nodes:

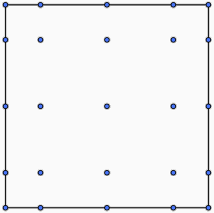


Figure 7: Degree $N = 4$ LGL nodes

Discontinuous Galerkin spectral element methods

- DG collocation on LGL nodes:

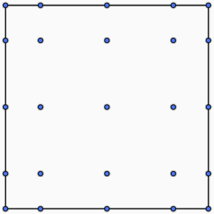


Figure 7: Degree $N = 4$ LGL nodes

- DGSEM for nonlinear conservation laws.

$$\mathsf{M}\frac{\mathrm{d}\mathsf{u}}{\mathrm{d}t} + \mathsf{Q}\mathsf{f} + \mathsf{B}\left(\mathsf{f}^{*}\left(\mathsf{u}_{f}, \mathsf{u}_{f}^{+}\right) - \boldsymbol{f}\left(\mathsf{u}_{f}\right)\right) = 0$$

Discontinuous Galerkin spectral element methods

- DG collocation on LGL nodes:

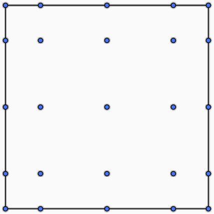


Figure 7: Degree $N = 4$ LGL nodes

- DGSEM for nonlinear conservation laws.

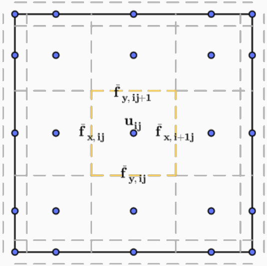
$$\mathsf{M}\frac{\mathrm{d}\mathsf{u}}{\mathrm{d}t} + \mathsf{Qf} + \mathsf{B}\left(\mathsf{f}^*\left(\mathsf{u}_f, \mathsf{u}_f^+\right) - \boldsymbol{f}\left(\mathsf{u}_f\right)\right) = 0$$

- Do not provably satisfy entropy stability and positivity constraints.

Subcell limiting strategy for DGSEM

- Blend high and low order schemes with algebraic subcell fluxes [4].

$$\mathsf{m}_i \frac{\mathrm{d}\mathbf{u}_i}{\mathrm{d}t} = \underbrace{\left[l_i \bar{\mathsf{f}}_i^{\mathrm{H}} + (1 - l_i)\, \bar{\mathsf{f}}_i^{\mathrm{L}} \right]}_{\bar{\mathsf{f}}_i} - \underbrace{\left[l_{i-1} \bar{\mathsf{f}}_{i-1}^{\mathrm{H}} + (1 - l_{i-1})\, \bar{\mathsf{f}}_{i-1}^{\mathrm{L}} \right]}_{\bar{\mathsf{f}}_{i-1}},$$



[4] Rueda-Ramírez, Andrés et al., “Subcell limiting strategies for discontinuous Galerkin spectral element methods.”

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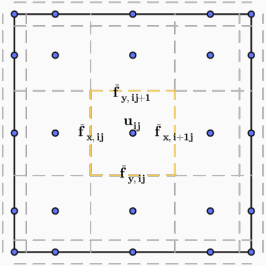

- $l_i = 1 \implies$ recovers nodal DG.
  $l_i = 0 \implies$ recovers low order structure-preserving scheme.

[4] Rueda-Ramírez, Andrés et al., "Subcell limiting strategies for discontinuous Galerkin spectral element methods."

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$$
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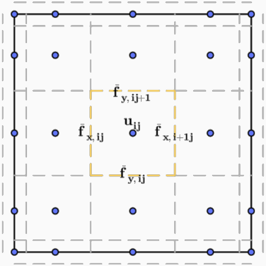
$$



- $l_i = 1 \implies$ recovers nodal DG.
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- Pick suitable l_i that satisfies positivity constraints.

[4] Rueda-Ramírez, Andrés et al., "Subcell limiting strategies for discontinuous Galerkin spectral element methods."

Subcell limiting strategy for DGSEM

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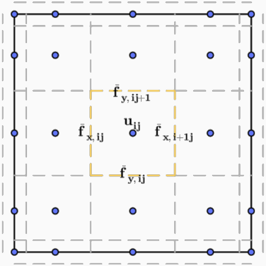
$$\mathsf{m}_i \frac{\mathrm{d}\mathbf{u}_i}{\mathrm{d}t} = \underbrace{\left[l_i \bar{\mathsf{f}}_i^{\mathrm{H}} + (1 - l_i)\, \bar{\mathsf{f}}_i^{\mathrm{L}} \right]}_{\bar{\mathsf{f}}_i} - \underbrace{\left[l_{i-1} \bar{\mathsf{f}}_{i-1}^{\mathrm{H}} + (1 - l_{i-1})\, \bar{\mathsf{f}}_{i-1}^{\mathrm{L}} \right]}_{\bar{\mathsf{f}}_{i-1}},$$



- $l_i = 1 \implies$ recovers nodal DG.
  $l_i = 0 \implies$ recovers low order structure-preserving scheme.
- Pick suitable l_i that satisfies positivity constraints.
- Subcell resolution. Retain high order as much as possible.
  High order accurate when enforcing positivity.

[4] Rueda-Ramírez, Andrés et al., "Subcell limiting strategies for discontinuous Galerkin spectral element methods."

- Minimum entropy principle [5].

$$s_i \geq \min_{j \in \mathcal{N}(i)} s_j^n.$$

[5] Tadmor, Eitan. ”A minimum entropy principle in the gas dynamics equations.”

[6] Kuzmin, Dmitri et al., ”Limiter-based entropy stabilization of semi-discrete and fully discrete schemes for nonlinear hyperbolic problems.”

Subcell limiting strategy for DGSEM - Entropy stabilization

- Minimum entropy principle [5].

$$s_i \geq \min_{j \in \mathcal{N}(i)} s_j^n.$$

- Tadmor's condition on subcell fluxes [6].

$$(\mathsf{v}_i - \mathsf{v}_{i-1})^T \bar{\mathsf{f}}_i \leq \psi(\mathsf{u}_i) - \psi(\mathsf{u}_{i-1})$$

[5] Tadmor, Eitan. "A minimum entropy principle in the gas dynamics equations."

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Subcell limiting strategy for DGSEM - Entropy stabilization

- Minimum entropy principle [5].

$$s_i \geq \min_{j \in \mathcal{N}(i)} s_j^n.$$

- Tadmor's condition on subcell fluxes [6].

$$\left(\mathbf{v}_i - \mathbf{v}_{i-1}\right)^T \overline{\mathbf{f}}_i \leq \psi\left(\mathbf{u}_i\right) - \psi\left(\mathbf{u}_{i-1}\right)$$

- Unable to maintain high order accuracy.

	$N=2$		$N=3$		$N=4$	
K	L^2 error	Rate	L^2 error	Rate	L^2 error	Rate
5	7.498×10^{-1}		4.499×10^{-1}		3.135×10^{-1}	
10	3.343×10^{-1}	1.17	2.109×10^{-1}	1.09	1.486×10^{-1}	1.08
20	1.894×10^{-1}	0.82	1.092×10^{-1}	0.95	7.509×10^{-2}	0.98
40	9.718×10^{-2}	0.96	5.956×10^{-2}	0.87	4.160×10^{-2}	0.85
80	5.116×10^{-2}	0.93	3.186×10^{-2}	0.90	2.157×10^{-2}	0.95

Table 1: 2D isentropic vortex, subcell limiting, minimum entropy principle

[5] Tadmor, Eitan. "A minimum entropy principle in the gas dynamics equations."

[6] Kuzmin, Dmitri et al., "Limiter-based entropy stabilization of semi-discrete and fully discrete schemes for nonlinear hyperbolic problems."

Subcell limiting strategy for DGSEM - Cell entropy inequality

- Enforce cell entropy inequality.

$$\mathsf{v}^T \mathsf{M} \frac{\mathrm{d}\mathsf{u}^{\text{limited}}}{\mathrm{d}t} \leq \underbrace{\left[\boldsymbol{\psi}\left(\mathsf{u}_{N+1}\right) - \boldsymbol{\psi}\left(\mathsf{u}_1\right)\right]}_{\mathsf{P}^{\text{vol}}} - \underbrace{\left[\mathsf{v}_{N+1}^T \mathsf{f}^*\left(\mathsf{u}_{N+1}, \mathsf{u}_{N+1}^+\right) - \mathsf{v}_1^T \mathsf{f}^*\left(\mathsf{u}_1, \mathsf{u}_1^+\right)\right]}_{\mathsf{P}^{\text{surf}}}.$$

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- Separate volume and surface contributions of the limited solution.

$$\mathsf{M} \frac{\mathrm{d}\mathsf{u}^{\text{limited}}}{\mathrm{d}t} = \mathsf{M} \frac{\mathrm{d}\mathsf{u}^{\text{limited,vol}}}{\mathrm{d}t} + \mathsf{M} \frac{\mathrm{d}\mathsf{u}^{\text{limited,surf}}}{\mathrm{d}t}.$$

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- Enforce cell entropy inequality.

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- High and low order surface fluxes conincide.

$$\mathsf{v}^T \mathsf{M} \frac{\mathrm{d}\mathsf{u}^{\text{limited,surf}}}{\mathrm{d}t} \leq -\mathsf{P}^{\text{surf}}.$$

Subcell limiting strategy for DGSEM - Cell entropy inequality

- Enforce cell entropy inequality.

$$\mathsf{v}^T \mathsf{M} \frac{\mathrm{d}\mathsf{u}^{\text{limited}}}{\mathrm{d}t} \leq \underbrace{\left[\psi\left(\mathsf{u}_{N+1}\right) - \psi\left(\mathsf{u}_1\right)\right]}_{\mathsf{P}^{\text{vol}}} - \underbrace{\left[\mathsf{v}_{N+1}^T \mathsf{f}^*\left(\mathsf{u}_{N+1}, \mathsf{u}_{N+1}^+\right) - \mathsf{v}_1^T \mathsf{f}^*\left(\mathsf{u}_1, \mathsf{u}_1^+\right)\right]}_{\mathsf{P}^{\text{surf}}}.$$

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- High and low order surface fluxes conincide.

$$\mathsf{v}^T \mathsf{M} \frac{\mathrm{d}\mathsf{u}^{\text{limited,surf}}}{\mathrm{d}t} \leq -\mathsf{P}^{\text{surf}}.$$

- Goal: Enforce volume entropy estimate:

$$\mathsf{v}^T \mathsf{M} \frac{\mathrm{d}\mathsf{u}^{\text{limited,vol}}}{\mathrm{d}t} \leq \mathsf{P}^{\text{vol}}.$$

Enforcing cell entropy inequality via linear program

- Enforce volume entropy estimate by solving subcell limiting factors l_i:

$$\mathsf{v}^T \mathsf{M} \frac{\mathrm{d}\mathsf{u}^{\text{limited,vol}}}{\mathrm{d}t} \leq \mathsf{P}^{\text{vol}}, \qquad \mathsf{u}^{\text{limited,vol}} \text{ depends on } l_i$$

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- Over each element, find subcell limited solution close to DGSEM, satisfies cell entropy stability and positivity.

Enforcing cell entropy inequality via linear program

- Enforce volume entropy estimate by solving subcell limiting factors l_i:

$$\mathsf{v}^T \mathsf{M} \frac{\mathrm{d}\mathsf{u}^{\mathrm{limited,vol}}}{\mathrm{d}t} \leq \mathsf{P}^{\mathrm{vol}}, \qquad \mathsf{u}^{\mathrm{limited,vol}} \text{ depends on } l_i$$

- Over each element, find subcell limited solution close to DGSEM, satisfies cell entropy stability and positivity.

$$\begin{aligned} \max_{l_i} \quad & \sum_{i=1}^{N} l_i \\ \text{s.t.} \quad & \sum_{i=1}^{N} \left(\mathsf{v}_i - \mathsf{v}_{i+1}\right)^T \bar{\mathsf{f}}_i \left(l_i\right) \leq \mathsf{P}^{\mathrm{vol}} \\ & 0 \leq l_i \leq l_i^{\mathrm{pos}} \end{aligned}$$

Deterministic greedy algorithm

- The LP is a continuous Knapsack problem.

$$\begin{aligned}\max_{l_i} \quad & \sum_{i=1}^{N} l_i \\ \text{s.t.} \quad & \sum_{i=1}^{N} \left(\mathsf{v}_i - \mathsf{v}_{i+1}\right)^T \bar{\mathsf{f}}_i\left(l_i\right) \leq \mathsf{P}^{\mathrm{vol}} \\ & 0 \leq l_i \leq l_i^{\mathrm{C}}\end{aligned} \quad \Longrightarrow \quad \begin{aligned}\max_{\mathsf{x}} \quad & \sum_{i=1}^{M} \mathsf{x}_i \\ \text{s.t.} \quad & \mathsf{a}^T\mathsf{x} \leq b \\ & \mathsf{0} \leq \mathsf{x} \leq \mathsf{U}\end{aligned}$$

Deterministic greedy algorithm

- The LP is a continuous Knapsack problem.

$$
\begin{aligned}
\max_{l_i} \quad & \sum_{i=1}^{N} l_i \\
\text{s.t.} \quad & \sum_{i=1}^{N} \left(\mathsf{v}_i - \mathsf{v}_{i+1}\right)^T \bar{\mathsf{f}}_i \left(l_i\right) \leq \mathsf{P}^{\mathrm{vol}} \\
& 0 \leq l_i \leq l_i^{\mathrm{C}}
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
\max_{\mathsf{x}} \quad & \sum_{i=1}^{M} \mathsf{x}_i \\
\text{s.t.} \quad & \mathsf{a}^T \mathsf{x} \leq b \\
& \mathsf{0} \leq \mathsf{x} \leq \mathsf{U}
\end{aligned}
$$

- Deterministic greedy algorithm: $O\left(N_{elem} \log N_{elem}\right)$.
  - Step 1: Sort a.
  - Step 2: Start at upper bound $\mathsf{x} = \mathsf{U}$ (preserve positivity).
  - Step 3: Traverse a in decreasing order, set the corresponding limiting factor to be the maximal value satisfying constraint, or 0.

$$
\begin{array}{rcccccccc}
 & \mathsf{a}_1 \geq & \mathsf{a}_2 \geq & \cdots \geq & \mathsf{a}_i \geq & \cdots \geq & \mathsf{a}_N \geq & \mathsf{a}_{N+1} \\
x_i = & 0, & 0, & \ldots, & l_i, & \ldots, & \mathsf{U}_N, & \mathsf{U}_{N+1}
\end{array}
$$

Multidimension, shock capturing, and timestepping

- Extension to d dimension in a dimension-by-dimension fashion (d LPs per element).

Multidimension, shock capturing, and timestepping

- Extension to d dimension in a dimension-by-dimension fashion (d LPs per element).
- Shock capturing for entropy dissipation.

$$
\begin{aligned}
\max_{l_i} \quad & \sum_{i=1}^{N} l_i \\
\text{s.t.} \quad & \sum_{i=1}^{N} \left(\mathsf{v}_i - \mathsf{v}_{i+1}\right)^T \bar{\mathsf{f}}_i\left(l_i\right) \le \left(1-\epsilon\right) \mathsf{P}^{\text{vol}} + \epsilon \mathsf{P}^{\text{vol,low}} \\
& 0 \le l_i \le l_i^{\mathrm{C}}
\end{aligned}
$$

Multidimension, shock capturing, and timestepping

- Extension to d dimension in a dimension-by-dimension fashion (d LPs per element).

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$$
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\text{s.t.} \quad & \sum_{i=1}^{N} \left(\mathsf{v}_i - \mathsf{v}_{i+1}\right)^T \bar{\mathsf{f}}_i\left(l_i\right) \leq \left(1-\epsilon\right) \mathsf{P}^{\mathrm{vol}} + \epsilon \mathsf{P}^{\mathrm{vol,low}} \\
& 0 \leq l_i \leq l_i^{\mathrm{C}}
\end{aligned}
$$

- SSPRK is not necessary for enforcing cell entropy stability.

Convergence test - Isentropic vortex

- 2D isentropic vortex, uniform quadrilateral mesh, subcell limiting by enforcing cell entropy inequality.

	$N = 2$		$N = 3$		$N = 4$	
K	L^2 error	Rate	L^2 error	Rate	L^2 error	Rate
5	6.935×10^{-1}		2.498×10^{-1}		1.587×10^{-1}	
10	1.785×10^{-1}	1.96	7.083×10^{-2}	1.82	2.000×10^{-2}	2.99
20	4.126×10^{-2}	1.11	8.898×10^{-3}	2.99	9.557×10^{-4}	4.39
40	6.714×10^{-3}	2.62	8.163×10^{-4}	3.45	3.142×10^{-5}	4.93
80	1.210×10^{-3}	2.74	4.208×10^{-5}	4.28	1.530×10^{-6}	4.36

2D KPP problem

- $N = 3,\ 128 \times 128$ elements, modal shock capturing.

Figure 8: DGSEM with shock capturing

Figure 9: DGSEM with shock capturing and cell entropy inequality

Mach 2000 Astrophysical jet - Compressible Euler

- $N = 3,\ 150 \times 150$ elements, relaxed positivity bound $\rho > 0.5\rho^{\mathrm{L}}, \rho e > 0.5\,(\rho e)^{\mathrm{L}}$.

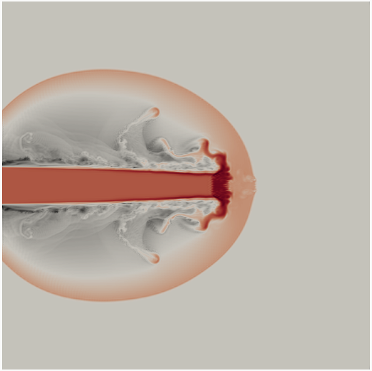

Figure 10: Subcell limited DGSEM: relaxed positivity bound and cell entropy inequality

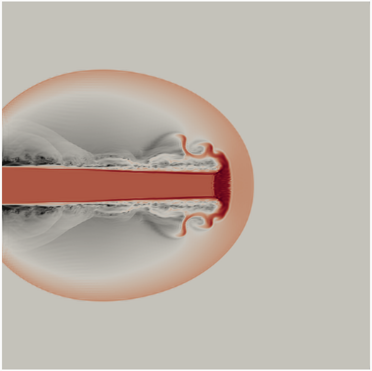

Figure 11: Elementwise limited ESDG: relaxed positivity bound

Mach 2000 Astrophysical jet - Compressible Euler

- $N = 3,\ 150 \times 150$ elements, relaxed positivity bound $\rho > 0.5\rho^{\mathrm{L}}, \rho e > 0.5\,(\rho e)^{\mathrm{L}}$.
  ϵ : max percentage of low order entropy dissipation to be blended in.

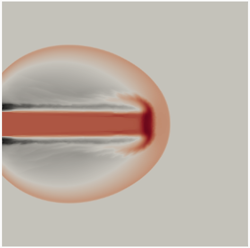

Figure 10: $\beta = 1.0$

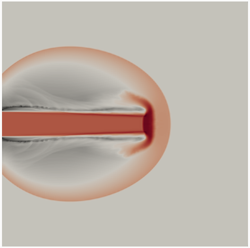

Figure 11: $\beta = 0.1$

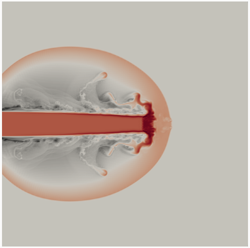

Figure 12: $\beta = 0.0$

Summary and future works

- We present a positive limiting strategy for nodal ESDG[7].
- We present a subcell limiting strategy for enforcing cell entropy stability[8].
- Code: `P2DE.jl` [9], `Trixi.jl` [10].
- Future work: Implicit timestepping. Extending the subcell limiting framework to the compressible Navier-Stokes equations, Gauss collocation, and unstructured meshes.

Thank you!

---

[7] Lin, Yimin, Jesse Chan, and Ignacio Tomas. "A positivity preserving strategy for entropy stable discontinuous Galerkin discretizations of the compressible Euler and Navier-Stokes equations.", 2023

[8] Lin, Yimin, Jesse Chan. "Subcell limiting strategies for discontinuous Galerkin spectral element methods.", arXiv

[9] `https://github.com/yiminllin/P2DE.jl`

[10] `https://github.com/trixi-framework/Trixi.jl`