A Positivity-preserving Strategy for Entropy Stable Discretizations of the Compressible Euler and Navier-Stokes equations

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Compressible Euler and Navier-Stokes equations

- Compressible Euler and Navier-Stokes equations

$$\frac{\partial \boldsymbol{U}}{\partial t} + \underbrace{\sum_{i=1}^{3} \frac{\partial \boldsymbol{f}_i(\boldsymbol{U})}{\partial \boldsymbol{x}_i}}_{\text{inviscid flux}} = \underbrace{\sum_{i=1}^{3} \frac{\partial \boldsymbol{g}_i(\boldsymbol{U})}{\partial \boldsymbol{x}_i}}_{\text{viscous flux}}$$

- Entropy variables symmetrizes the viscous fluxes:

$$\sum_{i=1}^{d} \frac{\partial \boldsymbol{g}_i}{\partial x_i} = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(K_{ij} \frac{\partial \boldsymbol{v}}{\partial x_j} \right),$$

$$K = \begin{bmatrix} K_{11} & \dots & K_{1d} \\ \vdots & \ddots & \vdots \\ K_{d1} & \dots & K_{dd} \end{bmatrix} = K^T, \qquad K \succeq 0$$

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$$\int_\Omega \frac{\partial \eta(\boldsymbol{u})}{\partial t} + \sum_{i=1}^d \int_{\partial\Omega} n_i \left(F_i(\boldsymbol{u}) - \frac{1}{c_v T} \kappa \frac{\partial T}{\partial x_i} \right) = -\int_\Omega \sum_{i,j=1}^d \left(\frac{\partial \boldsymbol{v}}{\partial x_i} \right)^T \left(\boldsymbol{K}_{ij} \frac{\partial \boldsymbol{v}}{\partial x_j} \right)$$

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- Loss of chain rule at discrete level (discrete effects, inexact quadrature) $\Longrightarrow$ Loss of entropy stability

High order entropy stable DG discretization

- Discretize the variational form of invscid term

$$\int_{\widehat{D}} \frac{\partial \boldsymbol{f}}{\partial x} \vec{l} \quad \xRightarrow{\text{Flux Differencing}} \quad 2\left(\mathsf{Q} \circ \mathsf{F}_S\right)\mathbf{1}, \quad \left(\mathsf{F}_S\right)_{ij} = \boldsymbol{f}_S\left(\boldsymbol{u}_i, \boldsymbol{u}_j\right)$$

Summation-by-parts $\quad \mathsf{Q} = \mathsf{M}\mathsf{D}, \quad \mathsf{Q} + \mathsf{Q}^T = \mathsf{B}, \quad \mathsf{Q}\mathbf{1} = 0$

Entropy conservation $\quad \left(\boldsymbol{v}_L - \boldsymbol{v}_R\right)^T \boldsymbol{f}_S\left(\boldsymbol{u}_L, \boldsymbol{u}_R\right) = \psi\left(\boldsymbol{u}_L\right) - \psi\left(\boldsymbol{u}_R\right)$

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- Discretize the symmetrized viscous term

$$\frac{\partial \boldsymbol{g}}{\partial x} = \frac{\partial}{\partial x}\left(K \frac{\partial \boldsymbol{v}}{\partial x}\right) \quad \xrightarrow[]{\text{Discretize}} \begin{cases} \left(\boldsymbol{\Theta}, \varphi\right)_\Omega = \left(\frac{\partial \boldsymbol{v}}{\partial x}, \varphi\right)_\Omega + \langle [\![\boldsymbol{v}]\!] n_i, \varphi \rangle_{\partial\Omega} \\ \left(\boldsymbol{\sigma}, \eta\right)_\Omega = \left(K\boldsymbol{\Theta}, \eta\right)_\Omega \\ \left(\boldsymbol{G}_{\text{visc}}, \psi\right)_\Omega = -\left(\boldsymbol{\sigma}, \frac{\partial \psi}{\partial x}\right)_\Omega + \langle \{\{\boldsymbol{\sigma}\}\} n_i, \psi \rangle_{\partial\Omega} \end{cases}$$

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- The scheme is entropy stable

$$\sum_k \left(\frac{\partial \eta\left(\mathbf{u}\right)}{\partial t}, \boldsymbol{v}\right)_{D^k} = \sum_k \sum_{i,j=1}^{d} -\left(K_{ij}\boldsymbol{\Theta}_j, \boldsymbol{\Theta}_i\right)_{D^k} \leq 0$$

Current work: Positivity Limiting for nodal ESDG

- The entropy is well-defined only if densities and pressures are positive.

$$v_1 = (\gamma + 1 - s) - \frac{(\gamma - 1)\,E}{p}, \qquad s = \log\left(\frac{p}{\rho^{\gamma}}\right)$$

Current work: Positivity Limiting for nodal ESDG

- Strong shock forms - Negative densities

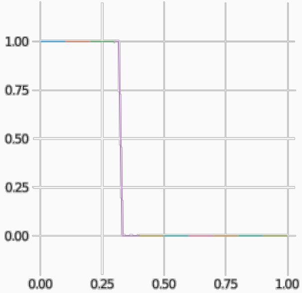

Figure 1: Exact solution

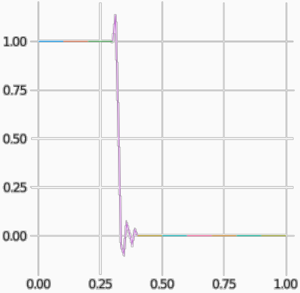

Figure 2: Solution in polynomial basis

- Oscillation by Gibbs phenomenon leads to negative density

Limiting strategy

- Step 1. Compute high order target scheme (nodal ESDG)

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    - Low order positivity-preserving and ESDG in algebraic flux form:

$$\frac{\boldsymbol{m}_i}{\tau}(\boldsymbol{u}_i^{L,n+1} - \boldsymbol{u}_i^n) + \sum \mathsf{F}_{ij}^{L,n} = 0$$
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  - Choose suitable parameter $l_{ij} \in [0,1]$ to satisfy positivity

$$\boldsymbol{m}_i\boldsymbol{u}_i^{n+1} = \boldsymbol{m}_i\boldsymbol{u}_i^{L,n+1} + \sum \tau l_{ij}(\mathbf{F}_{ij}^{L,n} - \mathbf{F}_{ij}^{H,n})$$

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  - Find largest possible l_{ij} that satisfy positivity.

Positivity preserving discretization

- Low order positivity preserving method could be written as

$$\underbrace{\boldsymbol{m}_i \frac{\partial \boldsymbol{u}}{\partial t} + \sum \mathsf{Q}_{ij} \left(\boldsymbol{f}(\boldsymbol{u}_j) - \boldsymbol{\sigma}_j \right)}_{\text{low order nodal DG on LGL nodes}} - \underbrace{\sum d_{ij}(\boldsymbol{u}_j - \boldsymbol{u}_i)}_{\text{graph viscosity}} = 0$$

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- Define the graph viscosity coefficients:

$$d_{ij} = \max\left\{ \beta(\boldsymbol{u}_i, \boldsymbol{u}_j, \boldsymbol{n}_{ij}) \|\boldsymbol{Q}_{ij}\|, \beta(\boldsymbol{u}_j, \boldsymbol{u}_i, \boldsymbol{n}_{ji}) \|\boldsymbol{Q}_{ji}\| \right\}, \boldsymbol{n}_{ij} = \boldsymbol{Q}_{ij} / \|\boldsymbol{Q}_{ij}\|$$

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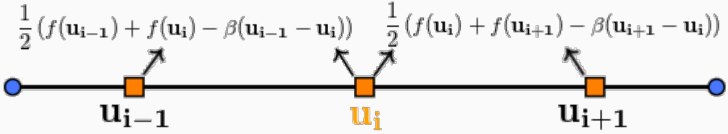
- Compressible Euler - Maximum wavespeed (Lax-Friedrichs flux)
  Compressible Navier-Stokes - Zhang's positivity preserving flux

Positivity preserving discretization

- Interpretation: subcell Lax-Friedriches type dissipation



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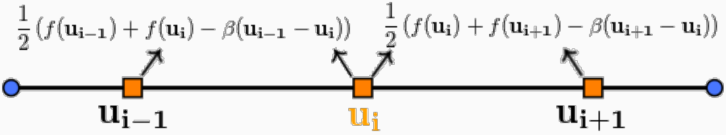
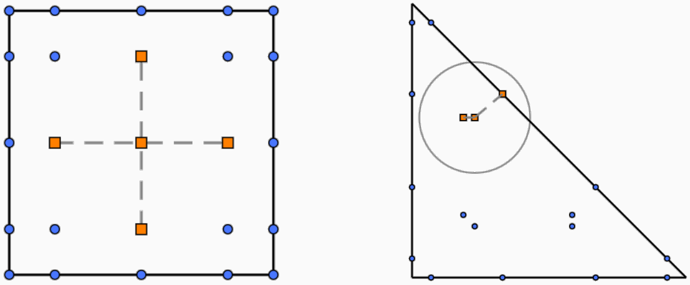


- Extension to 2D (tensor product and simplex elements)



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- Low order positive $+$ High order entropy stable $\not\Rightarrow$ positivity-preserving and entropy stable limited solution

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- The limited solution is both positivity-preserving and entropy stable.

$$\mathbf{u}_i^{n+1} = (1 - l)\, \mathbf{u}_i^{\mathrm{L},n} + l \mathbf{u}_i^{\mathrm{H},n}$$

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- Local Lax-Friedrichs flux dissipates entropy

$$\psi - \mathsf{v}^T \boldsymbol{f}_{\mathrm{LF}} \le 0$$

Double Mach Reflection - Compressible Euler

- $N = 3,\ 1000 \times 250$ elements, $T = 0.2$, element-wise and node-wise limiting

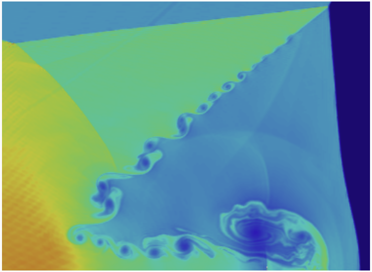

Figure 3: Element-wise

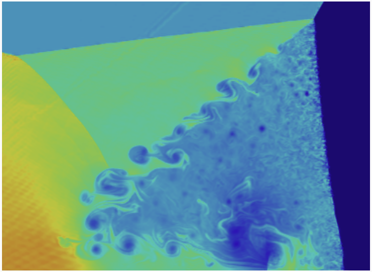

Figure 4: Node-wise

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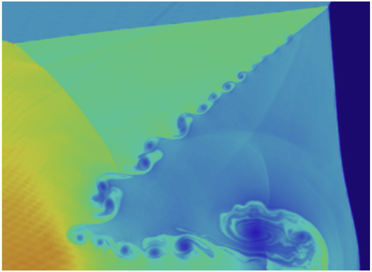

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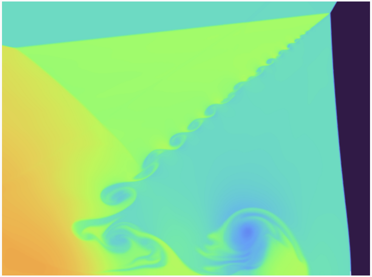
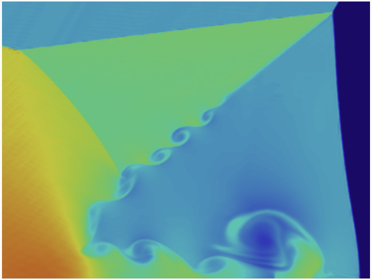


Figure 4: (Pazner) $N = 3, 2400 \times 600$ elements, $T \approx 0.275$

Double Mach Reflection - Compressible Navier-Stokes

- $N = 3,\ 250 \times 750$ elements, $Re = 500$, element-wise and node-wise limiting



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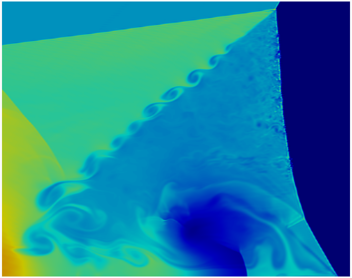

Figure 6: Node-wise

Summary and future works

- We present a positivity limiting strategy for nodal ESDG based on graph viscosity.
- Future work: Positivity limiting for modal ESDG. Implicit timestepping.

Thank you!