ECE 227 Final Report

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Abstract

The modern financial system has become more and more complicated and interdependent, which helps risk sharing and also contributes to systemic fragility. This is because the network connections not only diversify firms' risk exposures but also create channels through which shocks can spread by contagion. We first review the financial network model proposed by Elliott et al. (2014), and then present some stylized facts about the modern international intermediate bank holding network. Next, we relax the model assumptions posted in that model, to match a more realistic setting, and explore the model implications. We also provide some suggested directions for future research.

1 Introduction

The assessment of interdependence in financial institutions has evolved significantly, especially after the 2007-2008 global financial crises which started in the U.S. market. This crisis highlighted that both the solvency and network of financial institutions are important. Therefore, incorporating network analysis, has become more and more popular and crucial for understanding the dynamics of financial crises and how a shock is propagated within a complex system.

Financial crises can originate both externally and within the financial sector itself. For instance, the hyperinflation in Weimar Germany during 1921-1923, caused by excessive printing of money to pay war reparations, exemplifies an externally triggered crisis in which the banking sector played a minor role. However, crises like the one in 2007-2008 begin during economic booms characterized by rising asset prices and speculative investments, only to end disastrously when inflated values collapse. This leads to a downward spiral where financial institutions, in an attempt to stabilize their market value, must liquidate assets at a loss, and withhold short-term loans, which exacerbates the decline of asset prices and amplifies the progress of the crisis.

Therefore, the concept of contagion is central to understanding these crises, illustrating how financial distress can spread across the network of institutions. If one bank faces stress, it can quickly transmit this to others through its financial contracts, creating a cascading effect of instability. It is good to leverage the network models to analyze this self-reinforcing mechanism. The report is organized as follows. In section 2, we review a classic financial network model and point out the important assumptions they made. In section 3, we present what a financial network looks like in the real world and give some stylized facts. In section 4 we add the heterogeneity. e.g. distinguish between the small organizations and large organizations in simulation, and show how the results are changed. Section 5 concludes with the future directions.

2 A Classic Financial Network Model

In this section we review the financial network model proposed by Elliott et al. (2014). Suppose there are n organizations and m primitive assets of production. Let p_k denote the present value or market price of asset $k \in M = \{1, ..., m\}$. Let $D_{ik} \in [0, 1]$ denote the share of the value of asset k held by organization $i \in N = \{1, ..., n\}$, and $\mathbf{D} = [D_{ik}]$ the matrix representing the fractional holding of assets among these organizations.

The value of an organization consists not only of its holding on assets, but also on its shares in other organizations. For any $i, j \in N$, let $C_{ij} \in [0, 1]$ denote the share of company j owned by company i, with C_{ii} defined to be 0 for all i. Matrix $\mathbf{C} = [C_{ij}]$, then, denotes the cross-holdings among all the organizations and indicates the interdependencies between them.

The equity value V_i of organization i the total values that this organization has and is given by:

$$V_i = \sum_k D_{ij} p_k + \sum_j C_{ij} V_j$$

As indicated by the equation, its value consists of its share of each asset (according to their present market prices) and of each other company. Rewriting this formula in matrix form gives us:

$$\mathbf{V} = \mathbf{Dp} + \mathbf{CV}$$

Assuming $\mathbf{I} - \mathbf{C}$ to be nonsingular, and we get the closed-form solution of V:

$$\mathbf{V} = (\mathbf{I} - \mathbf{C})^{-\mathbf{1}} \mathbf{Dp}$$

. A particular thing to notice is that the sum of all equity values exceeds the total value of all assets. As a result, while the equity value well represents how much each organization is worth, it is inflated and does not well capture the true importance or contribution that the organization has towards the economy. To avoid this caveat, economists introduce another concept called "market" value. Let $\hat{C_{ii}}$ denote the share of company i that is held by outside investors. The market value, denoted by v_i, is given by

$$v_i = \hat{C_{ii}} V_i$$

In matrix form, we have

$$\mathbf{v} = \hat{\mathbf{C}} \mathbf{V} = \hat{\mathbf{C}} (\mathbf{I} - \mathbf{C})^{-\mathbf{1}} \mathbf{Dp}$$

Let $\mathbf{A} := \hat{\mathbf{C}}(\mathbf{I}-\mathbf{C})^{-\mathbf{1}}$. A is called *the dependency matrix* and captures the relationship between the value of organizations and the value of assets.

Finally, introduce failure cost into the equation. Let $\underline{v}_i$ be the threshold market value of organization i. When the market value of company i falls below $\underline{v}_i$, a penalty β_i is applied to represent the failure cost. Thus, the formula for equity value of organization i becomes:

$$V_i = \sum_j C_{ij}V_j + \sum_k D_{ik}p_k - \beta_i \mathbf{1}_{v_i<\underline{v}_i} \tag{1}$$

where $\mathbf{1}_{v_i<\underline{v}_i}$ is the indicator function of whether $v_i < \underline{v}_i$. In matrix form, we have:

$$\mathbf{V} = (\mathbf{I}-\mathbf{C})^{-\mathbf{1}}(\mathbf{Dp}-\mathbf{b}(\mathbf{v})) \tag{2}$$

with $b_i(\mathbf{v}) = \beta_i \mathbf{1}_{v_i<\underline{v}_i}$.
The matrix form for market value then becomes:

$$\mathbf{v} = \hat{\mathbf{C}}(\mathbf{I}-\mathbf{C})^{-\mathbf{1}}(\mathbf{Dp}-\mathbf{b}(\mathbf{v})) = \mathbf{A}(\mathbf{Dp}-\mathbf{b}(\mathbf{v})) \tag{3}$$

An important thing to note here is that the dependency matrix $\mathbf{A}$ here captures the relationship of losses that each organization shares with each other, based on their cross-holdings. For example, if organization j incurs a failure cost of β_j, the decrease of organization i's value will be given by $A_{ij}\beta_j$.

2.1 The importance of diversification

The model above can be easily converted into a network graph: Let each node represent a distinct organization and a directed edge is applied if the source node holds some share of the end node. When running simulations, they apply the following assumptions:
Assumption 1: Each organization owns one and only one propriety asset, i.e., $m = n$ and $\mathbf{D} = \mathbf{I}$;
Assumption 2: All organizations have the same normalized asset value, i.e., $\forall i \in [N]$, $p_i = 1$;
Assumption 3: All organizations share an identical failure threshold factor, i.e., $\underline{v}_i = \theta v_i$ for some $\theta \in (0, 1)$;
Assumption 4: All co-owners of an organization have the same share, i.e., $C_{ji} = C_{ki}\ \forall j \neq k$ s.t. $C_{ji} \neq 0, C_{ki} \neq 0$.

Define diversification to be a number of other companies holding shares of each other. When we treat the model as a random graph, diversification is identical to each node's expected node degree d. Economically, diversification is an important indicator of "how spread-out cross-holdings are" and it helps to answer the following question: are the shares owned by few other organizations, or are they owned by many organizations, each with a lesser share?

The original paper concludes that, if we increase d, we first observe a major increase in failure numbers. However, as d is continually increased, this spike disappears, as shown in Figure 6. Economically, this indicates that when the system is little connected, the failure of one organization can only reach a few others, rendering it ineffective. As it increases, the organizations become more interdependent, and one failure can thus lead to cascades of failures. However, with increasingly larger d, though the system becomes even more interdependent, each organization only has a small share of another ownership, and thus one failure is collectively handled by many others, each dealing with only a tiny fraction of the loss. As a result, cascades of failures are not likely, again.

3 Stylized Facts

While the model by Elliott et al. (2014) gives us insightful results, we also want to explore the data to understand what the real world looks like and give a descriptive analysis of cross-debt holding in the international banking sector.

3.1 Data

We leverage the Consolidated Banking Statistics collected by the Bank of International Settlements (BIS)[1]. The data are the sum of cross-border and local claims (loans or other liability contracts) of institutions in one country on the debt obligation of another country. We follow the data cleaning procedure in Sargent and Stachurski (2022). The data source looks at the immediate borrower but is aggregated at the country level. What's more, this data is updated quarterly and therefore we can get a time series of the total debt obligation of one country's financial institutions claimed by other countries' financial institutions.

(a) 2000 (b) 2010 (c) 2020

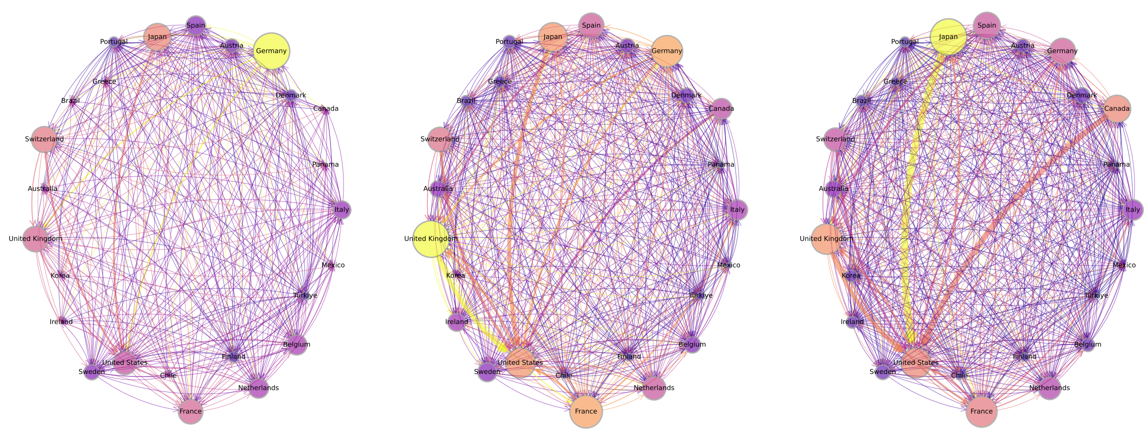


Figure 1: Debt Claims between Financial Institutions in 2000, 2010 and 2020

[1]The web URL is https://data.bis.org/topics/CBS.

Figure 7 shows the loans between financial institutions (banks) grouped by country of origin. An arrow from Canada to the US means the total claims held by Canadian banks on all US-registered banks. The size of each node is increasing in the total foreign claims of all other nodes on this node. The widths of the arrows are proportional to the foreign claims they represent. The foreign claim of a node to itself is set to zero. Figure 1 shows a similar network relationship in 2000, 2010, and 2020. We can have a first impression that the network itself is becoming denser and denser, indicating that the size and frequency of cross-border holding increases over time. Also, The largest creditor of the U.S. financial institutions changed over decades: from Germany to the United Kingdom, and to Japan. The proportion of the United States' financial institutions borrowing from other countries' financial institutions is becoming larger.

3.2 Centrality

We continue to analyze the centrality of the network above. There are two types of centrality in a network: hub centrality and authority centrality, as shown in 2.

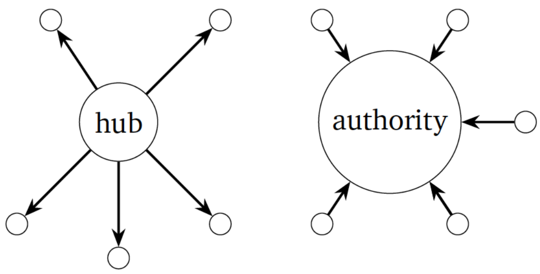


Figure 2: Different Centrality

Formally, let $\mathcal{G}$ be the set of weighted digraphs. A centrality measure associates a vector $m(\mathcal{G})$, where the ith element of $m(\mathcal{G})$ is interpreted as the centrality (or rank) of the vertex i. In the financial networks we study above, high hub centrality is related to an important role in lending: the institutions in such countries tend to be the creditors/lenders to those in other countries. Conversely, a high authority ranking will coincide with borrowing.

3.2.1 Degree Centrality

Two of the most elementary measures of the centrality in a digraph $\mathcal{G}$ are its in-degree and out-degree. To compute them, we can first convert the adjacency matrix A to an unweighted version U. For example:

$$A = \begin{pmatrix} 0 & a_{12} & 0 & 0 & 0 \\ a_{21} & 0 & a_{23} & 0 & 0 \\ 0 & 0 & 0 & a_{34} & 0 \\ 0 & a_{42} & 0 & 0 & a_{45} \\ a_{51} & 0 & a_{53} & a_{54} & 0 \end{pmatrix} \rightarrow U = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix} \tag{4}$$

Then the out-degree centrality measure $o(\mathcal{G})$ can be simply calculated by summing the rows of U and the in-degree centrality $i(\mathcal{G})$ measure can be given by summing the columns of U:

$$o(\mathcal{G}) = U\mathbb{I} \quad \text{and} \quad i(\mathcal{G}) = U^\top \mathbb{I}, \tag{5}$$

The out-degree centrality measure is a hub-based ranking, while the vector of in-degrees is an authority-based ranking. A high out-degree for a given institution means that it lends to many other institutions. A high in-degree indicates that many institutions lend to it.

3.2.2 Eigenvector Centrality

We can also define a more informative centrality measure based on the eigenvector. Given a weighted digraph $\mathcal{G}$ with adjacency matrix A. The hub-based eigenvector centrality of $\mathcal{G}$ is defined as the vector $e \in \mathbb{R}^n_+$ that solves

$$e = \frac{1}{r(A)} Ae \tag{6}$$

where the $r(A)$ is the spectral radius of A. Element-by-element, it is

$$e_i = \frac{1}{r(A)} \sum_j a_{ij} e_j \quad \text{for all } i$$

This definition has a recursive nature: the centrality obtained by vertex i is proportional to a weighted sum of the centrality of all vertices. A vertex i is highly ranked if (a) there are many edges leaving i, (b) these edges have large weights, and (c) the edges point to other highly ranked vertices.

Similarly, we can define the authority-based eigenvector centrality of $\mathcal{G}$ as the $e \in \mathbb{R}^n_+$ solving

$$e = \frac{1}{r(A)} A^\top e \tag{7}$$

Element-by-element, this is

$$e_j = \frac{1}{r(A)} \sum_i a_{ij} e_i \quad \text{for all } j.$$

We see e_j will be high if many nodes with high authority rankings link to j.

3.2.3 Results

The top panels of Figure 8 are the degree centrality measures. Because they are simply the summation of the in and out edges of each node, the maximum value is 25, which is the total number of countries in this system. In Figure 9, we also show the degree centrality measures in 2000, 2010, and 2020. It is clear that the international credit connections became more comprehensive in 2020 compared to 2000.

The bottom left panel of Figure 8 shows the hub-based eigenvector centrality. Countries that are rated highly tend to be important players in terms of the supply of credit, which means that Japanese banks loan most funds to or deposit most funds in other countries. Other countries with large financial sectors such as the United Kingdom, Canada, the United States, and France are not far behind.

The bottom right panel of Figure 8 shows the authority-based eigenvector centrality ranking. Highly ranked countries are those that attract large inflows of funds, or funds inflows from other major players. The US clearly dominates other countries as a target of intermediate bank networks.

In Figure 10, we also show both hub-based and authority-based eigenvector centrality measures in 2000, 2010, and 2020. It is interesting that the largest supplier of the funds changes over time, from Germany, to the United Kingdom, and to Japan, but the top 5 players are relatively consistent. In 2000 and 2010, the United Kingdom was also a relatively important target of funds flows, but in 2020 the concentration increased to the United States.

4 A Revised Approach: Adding Heterogeneity

As shown above, there is a lot of heterogeneity among different countries: the financial institutions in the United States, the United Kingdom, Japan, and other advanced economies own larger assets and larger debt. It is clear that the total amount of shares owned by an organization is positively correlated with the value of its primitive assets, or with its market value. This result makes intuitive sense, because an organization holding an important primitive asset is more likely to have a greater and more stable cash flow, and is more able to afford to invest more in other companies' businesses. Based on this understanding, we propose revising the assumptions as below:

New assumptions: We keep assumption 1 and 3 unchanged. However, we no longer assume that all organizations have the same normalized asset value. Instead, we assume that the asset values of these organizations are randomly distributed. Moreover, let $\mathbf{N_i} = \{n_1, ..., n_k\}$ represent the companies that co-own company i, then we assume that their shares are proportional to the values of their primitive assets, i.e.,

$$\forall j, k \in \mathbf{N_i}, \frac{C_{ji}}{C_{ki}} = \frac{p_j}{p_k}$$

. This assumption wants to make sure that the more "powerful" organizations own more shares of other organizations than the smaller organizations.

According to this new assumption, we can generate network graphs in the following manner: Assume there are n organizations in total, and each completely holds one and only one primitive asset. This implies $m = n$ and $\mathbf{D} = \mathbf{I}$. For each organization i, we generate the value of its primitive asset p_i based on some distribution. Then, generate a directed random graph $\mathbf{G}$ with expected node degree d, with G being the adjacency matrix. For each node,

$$C_{ij} = c\frac{p_i}{\sum_{jG_{ji}=1} p_j}$$

, where $c = 1 - \hat{C}_{ii}$. Matrix C is the cross-holding matrix. With $\mathbf{C}, \hat{\mathbf{C}}, \mathbf{p}$ determined, we can, using equation (2), calculate $\mathbf{v}$, the market value matrix. Finally, we set the market value of some organization i to 0. Then again by equation 2, we can iteratively calculate the market values of each company after cascades of failures.

4.1 Comparisons between Various Values of Variance

We first set the values of each primitive asset to be of normal distribution. By setting $\mathbb{E}[p] = 1$ and $\mathbf{std}(p) = 0$, we replicate the scenario simulated in the original paper.
Fix θ and c at certain values and change only d and $\mathbf{std}(p)$, we observe the average number of failures with respect to only d and σ.
In Figure 3, we show two sets of results after simulation. We notice that increasing σ does not affect the general relationship between d and the number of failures, as the overlapping of these functions shows. For a relatively small c ($c = 0.7$), the number of failures remains stable and low; as we increase c to 0.9, we observe a rapid increase in the number of failures as we increase d, followed by a sharp decline and a long tail converging to 1.

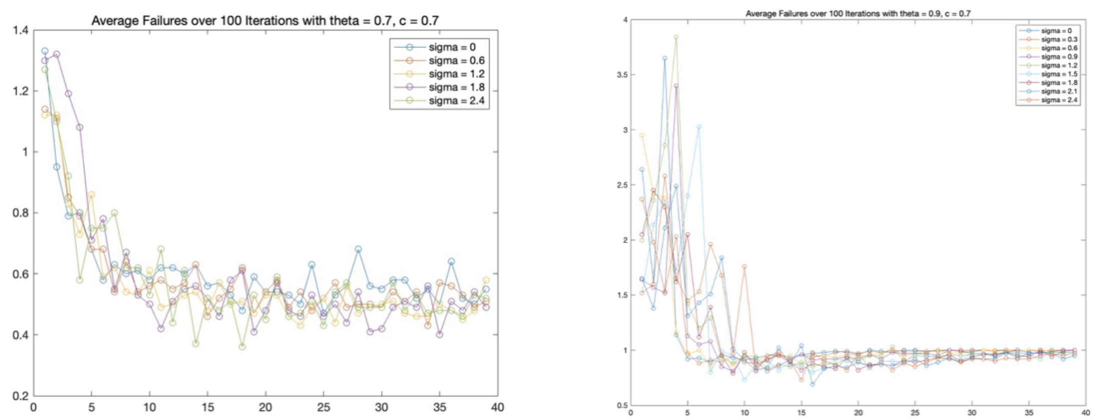


Figure 3: Results for Various σ

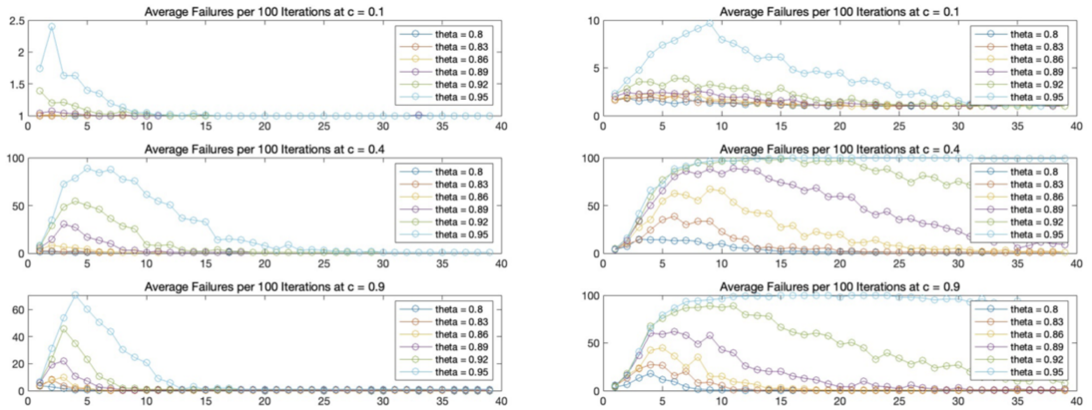


Figure 4: Average Failures when the Largest Organization Fails Initially

4.2 When the biggest or smallest company fails

A major difference between our simulation and that proposed by Elliott et al. (2014) is that we have, essentially, given organizations shares of other organizations based on their economic status. This determines that wealthier organizations should have a greater impact on the system. A natural question that arises is: Is there any difference, in terms of consequences, between the failure of a large organization and a small one?

Suppose that each primitive asset p_i is randomly generated based on normal distribution, with $\mathbb{E}[p_i] = 5$. For purposes of comparison, we consider two vastly different values of σ: $\sigma = 1$, and $\sigma = 10$.

Now, in each case, we let the largest company fail and propagate the failure.

We observe that, for small θ and large θ, the general pattern is the same as proposed in Elliott et al. (2014). However, for large θ, the spike as we increase d is much more flat, which means that for even large values of d, the average number of failures can still be very large in comparison to the cases when σ are small. Moreover, for relatively large θ, for example, $\theta = 0.95$, the average number of failures may reach 100 and stay there as d changes, which is only observed in the original paper at $\theta = 0.99$. These results effectively show us that, when we let the largest company fail first, while the general patterns with respect to diversification still hold, the number of average failures can become much larger than when a random organization fails. In short, the failure of a large company leads to greater crisis than a random company.

Another phenomenon we get to observe is that, for either $\sigma = 1$ or $\sigma = 10$, at $c = 0.4$, the maximal average failure is greater than that at $c = 0.9$, and the spike is observably wider, in the sense that it decreases much more slowly as d increases.

For the purpose of comparison, we now let the smallest organization fail at first. We similarly observed that the general pattern remains observably the same for small σ, except that the average number of failures at every position is smaller than when the largest company fails. This result makes sense: while a smaller company's failure exerts an influence in a similar fashion as that of a large company, its impact should be lesser in extent. Moreover, again, cascades of failures are more severe and persistent at $c = 0.4$ than at $c = 0.9$.

On the other hand, when σ is large, the average number of failures is fixed about $[0, 1]$. This suggests that, when a company is so insignificant in terms of value, its failure cannot generate a significant effect on any other organizations.

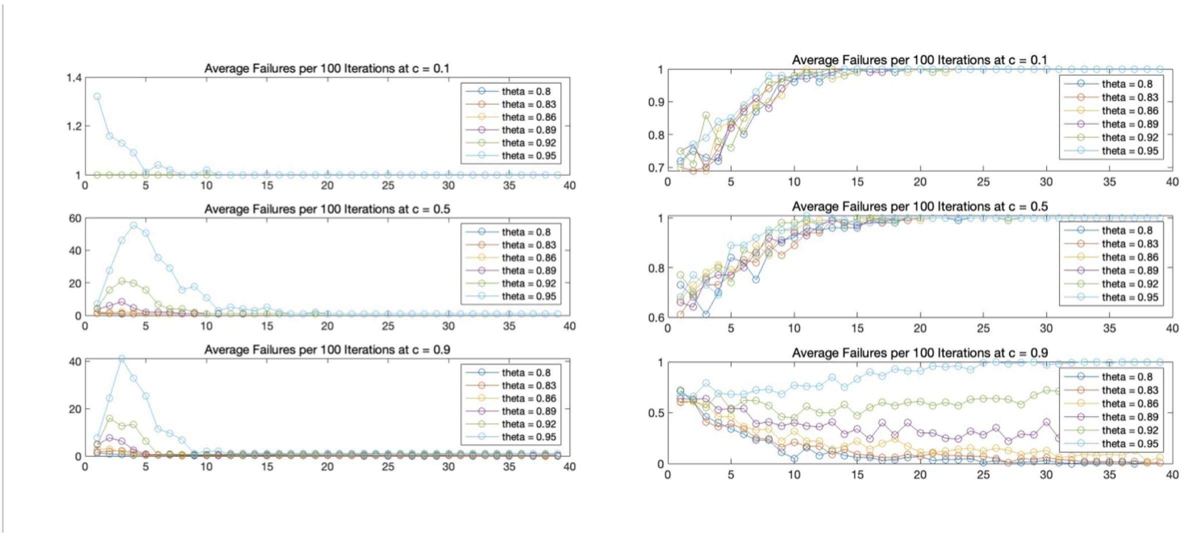


Figure 5: Average Failures when the Smallest Company Fails Initially

5 Conclusion

In this project, we have drawn various qualitative conclusions about the economic network, based on our new assumptions. While these results can give us insights into the possible patterns behind economic activities, more quantitative conclusions might be drawn. for example, we observe that when the largest companies fail at first, for sufficiently large σ, c, and θ, all organizations might fail and the entire system becomes paralyzed. Does there exist a particular hyperspace that acts as a threshold for this "shutdown"? Or when σ is sufficiently small, failing the smallest company first will cause absolutely no effect on any other organizations. Is there also a threshold value for θ? These questions might be of practical value and thus worthy of future explorations.

References

Elliott, Matthew, Benjamin Golub, and Matthew O. Jackson, "Financial Networks and Contagion," *American Economic Review*, October 2014, *104* (10), 3115–3153.

Sargent, Thomas J. and John Stachurski, “Economic Networks: Theory and Computation,” July 2022. arXiv:2203.11972 [econ, q-fin].

Appendix A Figures

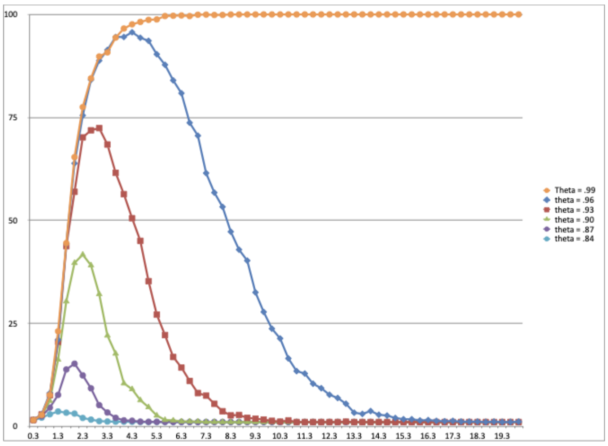


Figure 6: The Results of Diversification in Original Paper

Figure 7: Debt Claims between Financial Institutions in 2023 Quarter 4

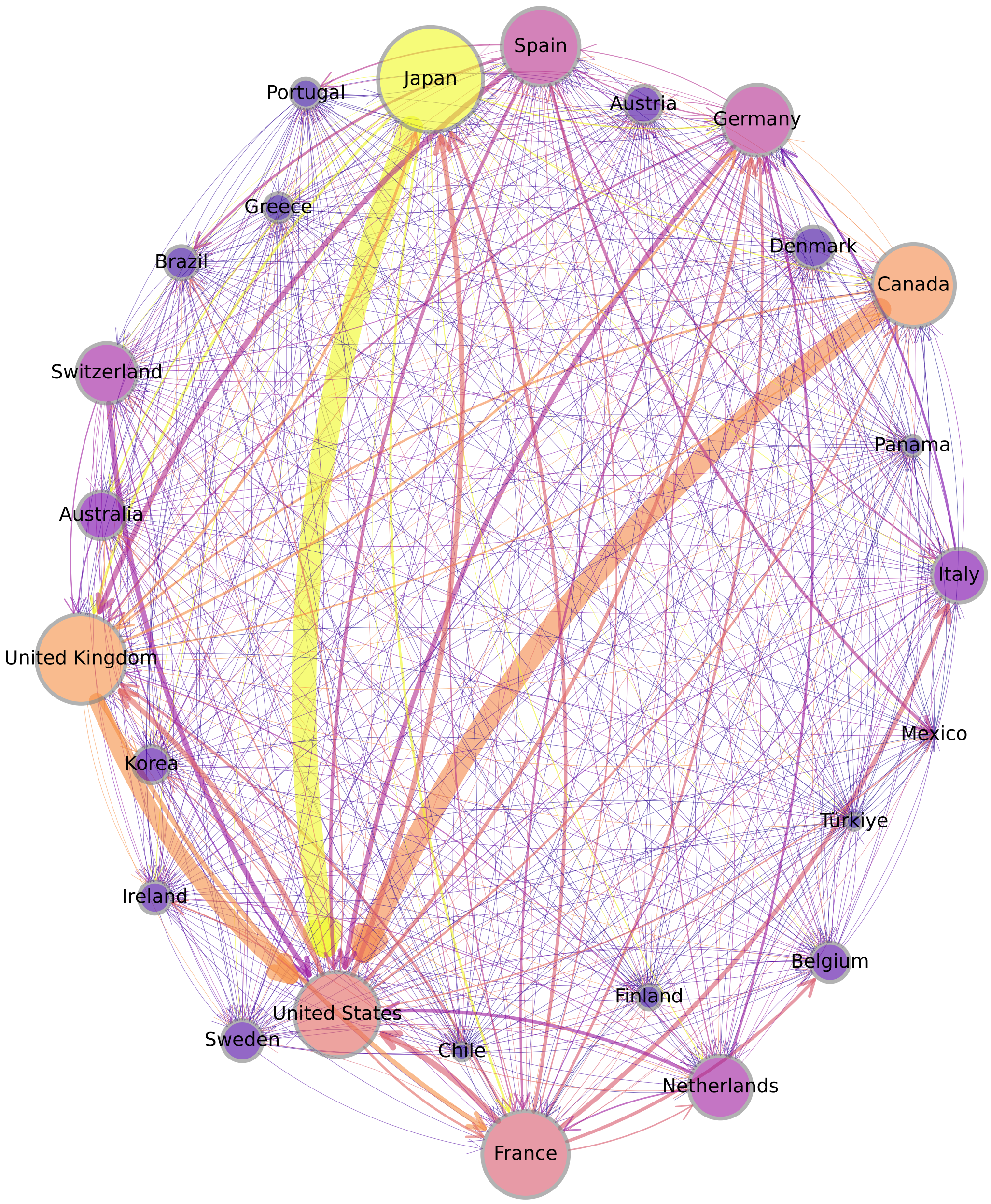


Figure 8: Centrality Measure in 2023 Quarter 4

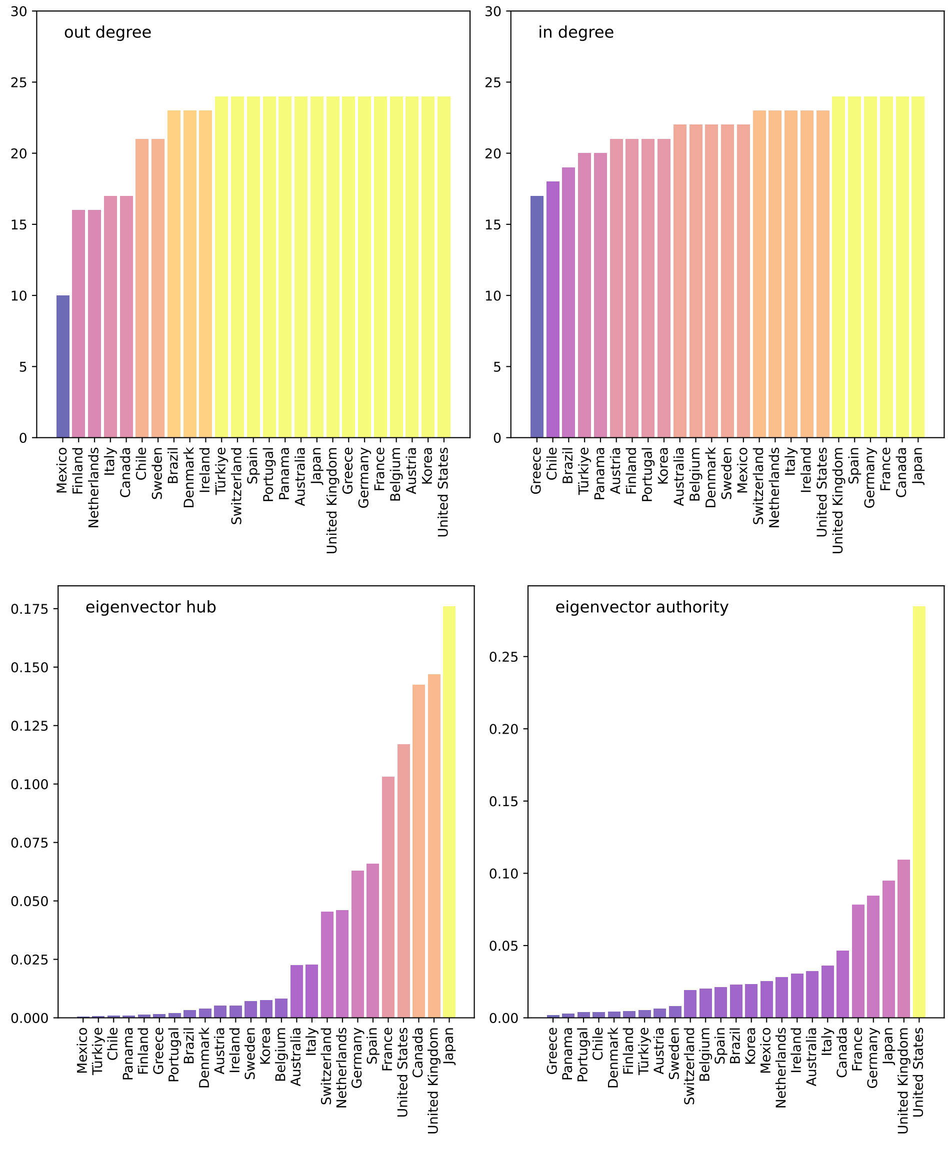
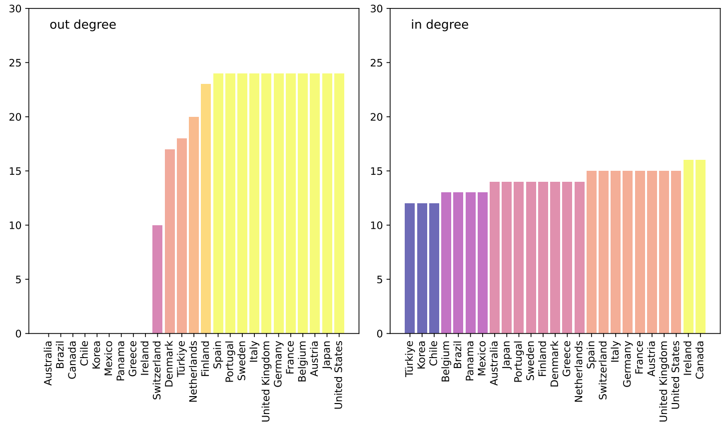
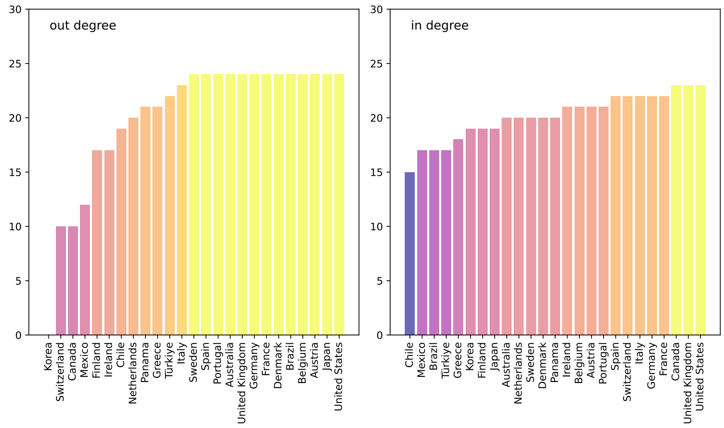


Figure 9: Degree Centrality in 2000, 2010 and 2020 years

(a) 2000



(b) 2010



(c) 2020

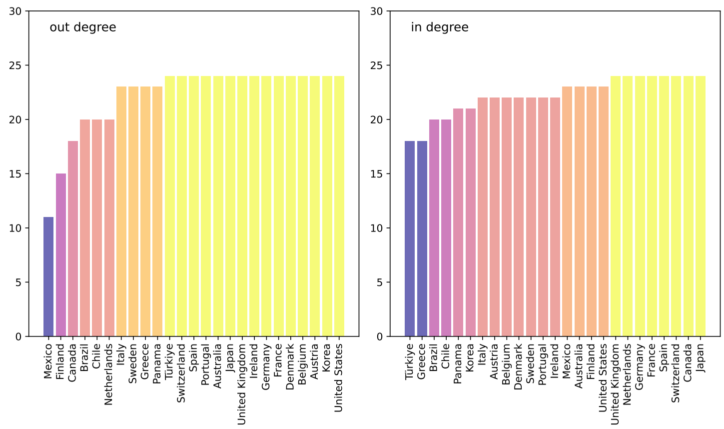
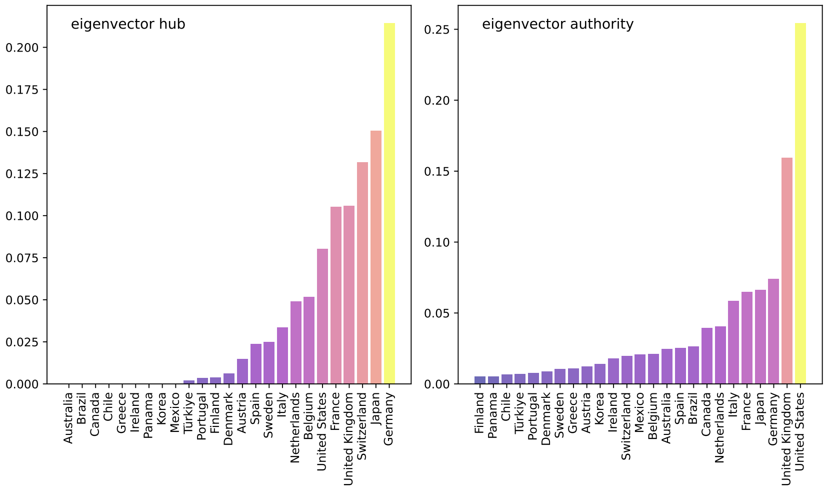
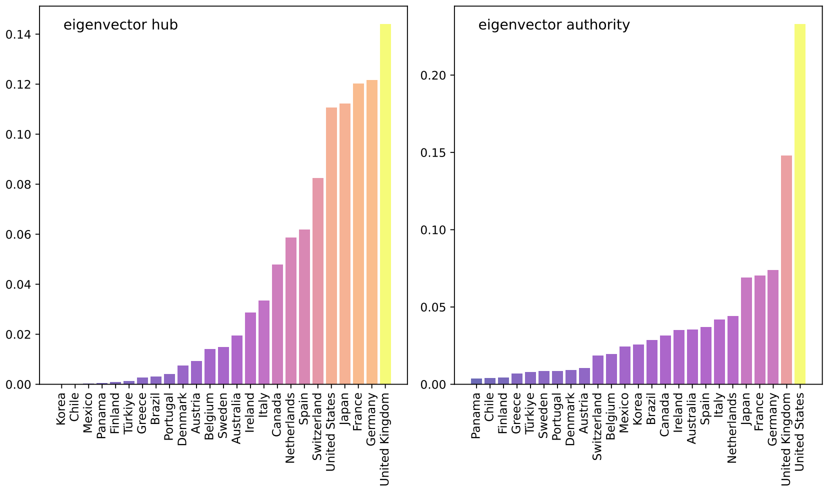


Figure 10: Eigenvector Centrality in 2000, 2010 and 2020 years

(a) 2000



(b) 2010



(c) 2020