Lecture 6: Interval estimation

Statistical Methods for Data Science

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Today

1. Central limit theorem
   - Terminology
   - Standardization
   - Central limit theorem
2. Interval estimation
3. Summary

Learning outcome

- Be able to explain the following terminology:
    - Sample statistic, sampling distribution, sample mean, sample variance, standardization, z-table, t-table
    - Point estimation, interval estimation
    - Confidence interval, credible interval
- Be able to explain the central limit theorem (CLT)
- Be able to construct the following interval estimates:
    - Confidence interval for
        - sample mean of i.i.d. sample with unknown σ
        - unknown sampling distribution using bootstrap
    - Credible interval for a given posterior function

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  - **Sample mean**:

$$\bar{X} = \frac{1}{N} \sum_{i=1}^{N} X_i$$

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Note: capital letters and small letters are used to denote random variables and the values, respectively.

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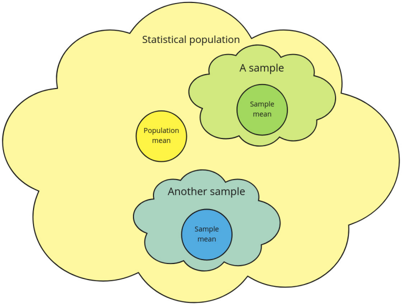

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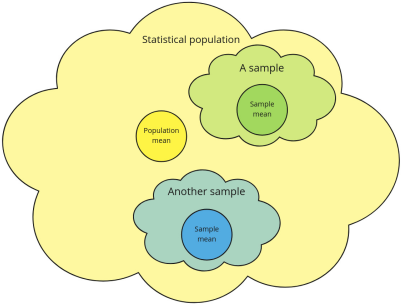

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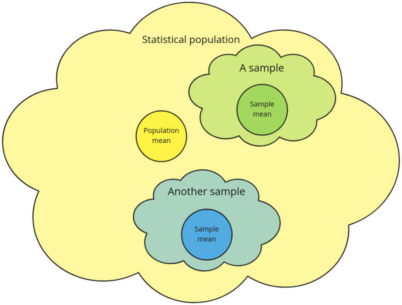

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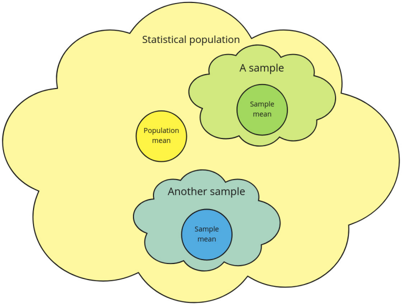

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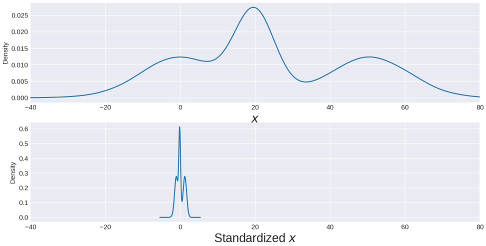

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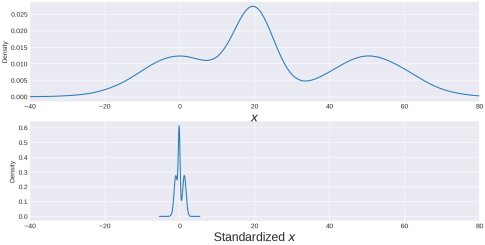

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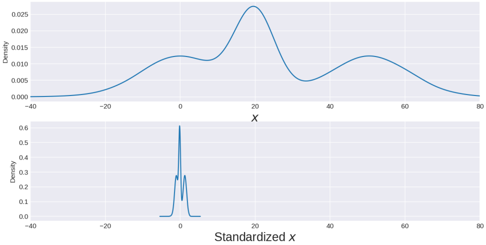
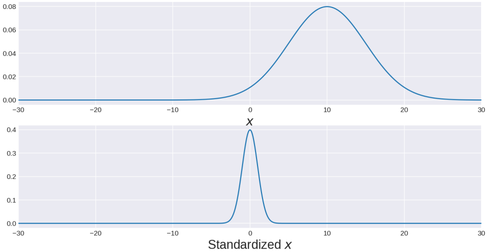


Question: what is the mean and standard deviation of Y? Random variable Y has **mean 0 and standard deviation 1**

Standardization

- Let X be a random variable following a **Gaussian distribution** with mean μ and standard deviation σ, i.e. $X \sim \mathcal{N}(\mu, \sigma^2)$; the standardization of X is

$$Z = \frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1) \tag{1}$$



The distribution $\mathcal{N}(0, 1)$ is called a **standard Gaussian (normal) distribution**

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z	+ 0.00	+ 0.01	+ 0.02	+ 0.03	+ 0.04	+ 0.05	+ 0.06	+ 0.07	+ 0.08	+ 0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56360	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91308	0.91466	0.91621	0.91774
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1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
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\begin{aligned}
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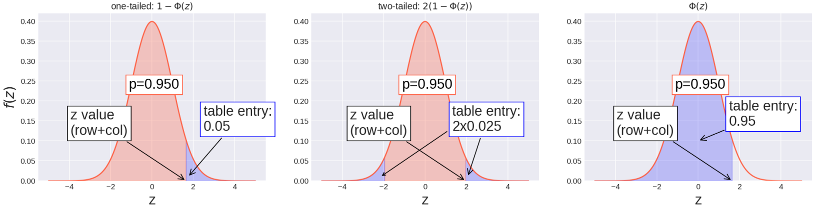
&= \textbf{stats.norm.cdf(x=row + column, loc=0, scale=1)}
\end{aligned}
$$

Standard Gaussian distribution (cont.)

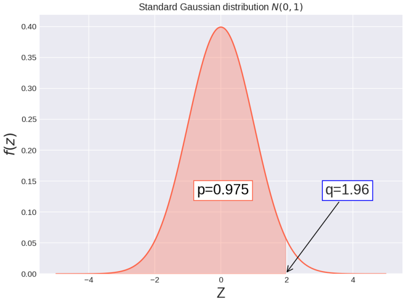
- There are different representations of the z-table; the difference is what is inside each cell, e.g. $\Phi(\text{row} + \text{column})$, $2(1 - \Phi(\text{row} + \text{column}))$, $1 - \Phi(\text{row} + \text{column})$ or $\frac{1}{2}(1 - \Phi(\text{row} + \text{column}))$; but the principle is the same; for now we use the version with $\Phi(\text{row} + \text{column})$



- Due to symmetry, there are only positive values for z in the z-table

Standard Gaussian distribution (cont.)

Exercise:



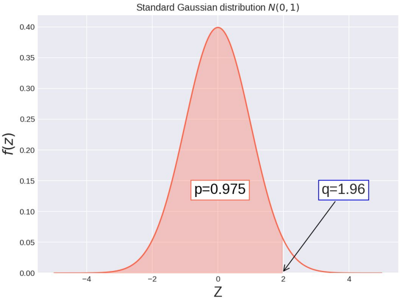
z-table

z	+ 0.00	+ 0.01	+ 0.02	+ 0.03	+ 0.04	+ 0.05	+ 0.06	+ 0.07	+ 0.08	+ 0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56360	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91308	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670

Try to find the corresponding pair $(p, q) = (0.975, 1.96)$ in the z-table (60 secs).

Standard Gaussian distribution (cont.)

Answer:



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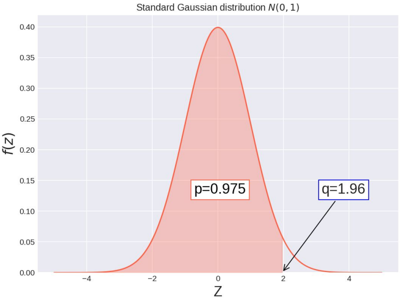

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Note: the table itself is not important (we use a computer these days); the point is to reflect on the meaning of z values (quantiles) and the related probabilities (CDFs)

Central limit theorem

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- In general, we are often interested in how things work "on average"

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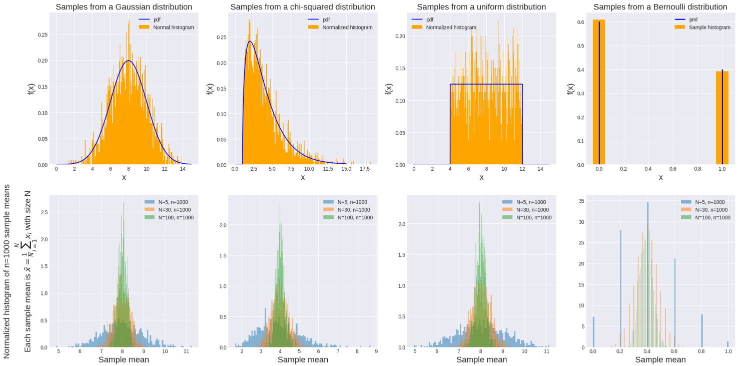
Distribution of the sample mean (cont.)

- In fact, this is true for i.i.d. samples drawn from ANY probability distribution



- The larger the sample size N (in the previous example $N = 30$), the "more Gaussian" it becomes
- A rule of thumb: $N \geq 30$
- If the data distribution is Gaussian-like (bell-shaped, symmetric), only a small sample size is needed for the sample mean to be Gaussian

Central limit theorem

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- Given an **i.i.d. sample** $X_1, X_2, \cdots, X_N$ from **ANY probability distribution** with *finite mean* μ *and variance* σ^2 (most distributions satisfy this!), when the sample size N is sufficiently large, the **sample mean** approximately follows a Gaussian distribution with mean μ and variance $\frac{\sigma^2}{N}$, i.e.,

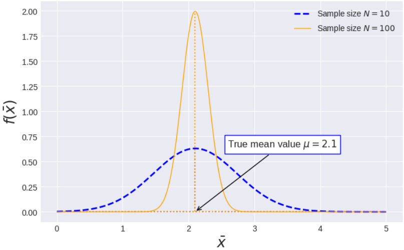
$$\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{N}) \tag{2}$$

where $\bar{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$ is the sample mean

Central limit theorem (cont.)

How to interpret this?

$$\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{N})$$



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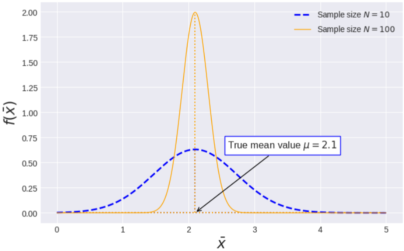

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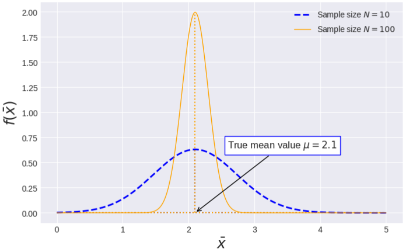

- The sample mean $\bar{X}$ is around the true mean value μ
- The "deviation" of $\bar{X}$ from μ is $\frac{\sigma^2}{N}$; the larger N, the smaller the deviation

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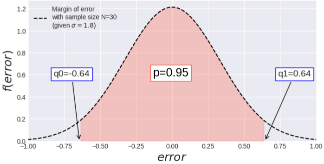

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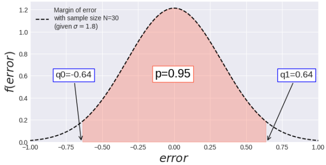

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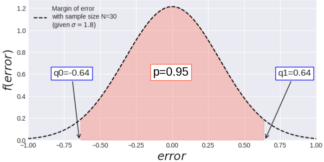

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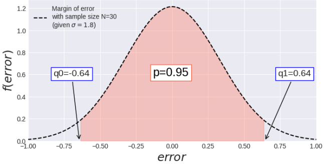

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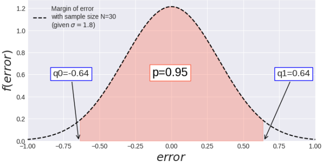

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- This **95%** is called the **confidence level**; for a given confidence level, we can find a corresponding **interval** $(q0, q1)$

Calculate the margin of error

- For a given confidence level, denoted as $1 - \alpha$, how do we find this interval for the error in Python?

Calculate the margin of error

- For a given confidence level, denoted as $1 - \alpha$, how do we find this interval for the error in Python? We can use the function **ppf** from **scipy.stats**

```
std = 1.8 # standard deviation of data
N = 30
alpha = 0.05
confidence_level = 1 - alpha # 95% confidence level
q0 = stats.norm.ppf(alpha/2,
                    0, std/math.sqrt(N))
q1 = stats.norm.ppf(confidence_level+alpha/2,
                    0, std/math.sqrt(N))
>> (-0.6441098917381766, 0.6441098917381766)
```

Find a standardized expression for the margin of error

- Standardize (cf. page 11) $\mathcal{E}$ by $\frac{\mathcal{E}}{\sigma/\sqrt{N}} = \frac{\bar{X}-\mu}{\sigma/\sqrt{N}} \sim \mathcal{N}(0,1)$

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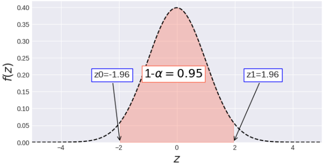

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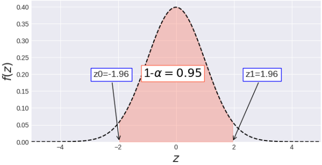

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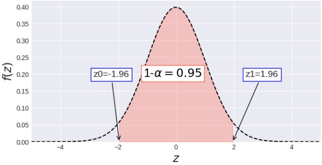


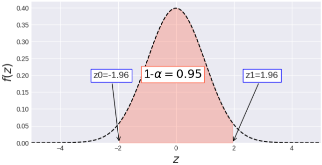
- We can use a two-tailed z-table (cf. page 13) to find the values for $z0$ and $z1$
- In order to find an interval for $\mathcal{E}$, we just need to look at

$$(z0\frac{\sigma}{\sqrt{N}}, z1\frac{\sigma}{\sqrt{N}})$$

Find a standardized expression for the margin of error (cont.)

- For example, with $1-\alpha = 95\%$ confidence level, the error is within

$$\left(-1.96\frac{\sigma}{\sqrt{N}},\ 1.96\frac{\sigma}{\sqrt{N}}\right)$$



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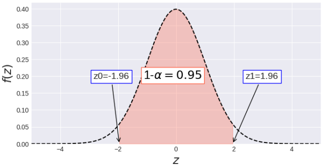

- Generally speaking, the value $z1$ (denoted by $z_{\alpha/2}$) is the quantile at $1-\alpha/2$; the value of $z_{\alpha/2}$ is called the **(right) critical value**; $\frac{\sigma}{\sqrt{N}}$ is called the **standard error**; in this example, we have $z_{\alpha/2} = z1 = -z0 = 1.96$
- Why **two-tailed** z-table: there are two tails $z \leq -z_{\alpha/2}$ and $z \geq z_{\alpha/2}$

Find a standardized expression for the margin of error (cont.)

- In Python

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alpha = 0.05
confidence_level = 1 - alpha # 95% confidence level
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Find a standardized expression for the margin of error (cont.)

- For a given sample with an estimate $\bar{x}$ (note: here the small letter $\bar{x}$ denotes the value of the estimate itself instead of a random variable), it's more convenient to have this margin of error around $\bar{x}$ instead - so that we can say: the estimated mean is $\bar{x}$ with this uncertainty:

$$\left(\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{N}},\ \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{N}}\right)$$

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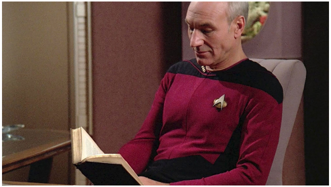
- This is called the **confidence interval**
- The confidence interval for the sample mean is *exact* when the data distribution is Gaussian, otherwise it is an approximation under the central limit theorem
- This calculation is called **interval estimation**, because it gives an interval estimate $\left(\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{N}},\ \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{N}}\right)$ instead of a single value estimate as in MAP or MLE

To Be Continued...



Today

1. Central limit theorem
2. Interval estimation
3. Summary

Today