

# Lecture 12: Hypothesis testing part II

## Statistical Methods for Data Science

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# Today

- ① Test statistics and hypothesis tests
  - z-test
  - One-sample t-test
  - Two-sample t-test (Welch's t-test - unequal variances)
  - Paired t-test
  - Binomial test
  - McNemar's test
  - Summary
- ② Compare two classifiers
- ③ More exercises
- ④ Summary

## Learning outcome

- Be able to explain the following hypothesis tests
  - One-sample and two-sample z-test
  - One-sample and two-sample t-test
  - Paired t-test
  - Binomial test (exact, approximate)
  - McNemar's test (exact, approximate)

For each of these tests, be able to describe the typical set up for the experiment, the general purpose of the test, data produced by the experiment, random variables, parameter of interest, null hypothesis, alternative hypothesis, test statistic, null distribution, the computation of  $p$ -value

- Be able to generalize the learning routine to new hypothesis tests
- Be able to compare two classifiers using the paired t-test and McNemar's test for different scenarios

# Today

## 1 Test statistics and hypothesis tests

- z-test
- One-sample t-test
- Two-sample t-test (Welch's t-test - unequal variances)
- Paired t-test
- Binomial test
- McNemar's test
- Summary

## 2 Compare two classifiers

## 3 More exercises

## 4 Summary



## Remark (cont.)

- Recall that in this course, we only consider  $H_0$  **with an equal sign in them**, i.e. the **null distribution is fully specified**; the description of  $H_0$  is based on this assumption
- For **symmetric null distributions**, e.g. **standard Gaussian distribution**, **student's t distribution**, **binomial distribution with  $p = 0.5$** , etc, without loss of generality, we only illustrate examples with the two-tailed alternative hypothesis  $H_A$  in this lecture; the one-tailed version can be easily derived
- For the **exact binomial test with  $p \neq 0.5$** , the null distribution is not symmetric; in this case, the computation of the two-tailed  $p$ -value is not uniquely defined; in this lecture, we will not go into details for these cases; we will only look at the one-tailed tests for asymmetric binomial null distributions
- For each hypothesis test, the purpose of the Python code snippet is to provide a better understanding of the calculation; in practice, there are alternative libraries and built-in functions for these tests that might result in a more compact implementation

## Remark (cont.)

For each of the hypothesis tests we introduce, we present the following components:

- **Typical set up for the experiment**
  - **Test subjects**, e.g. number of samples, number of groups, etc
  - Description of the **experiment** and the **result**
  - Description of the **data type** produced in the result
- **Purpose**: the general purpose of the test
- **Data**: description of data produced by the experiment
- **Random variables** and **assumptions**
- **Parameter of interest** and the **estimates**
- **Hypotheses**  $H_0$  and  $H_A$
- **Test statistic**
- **Null distribution**
  - PDF/PMF: description of the PDF/PMF
  - Python: code snippet of the PDF/PMF
- **p-value**
  - Definition: an expression of  $p$ -value in terms of a probability
  - Python: code snippet to illustrate the computation of the  $p$ -value (see page 5)

## Test statistics and hypothesis tests

Compare two classifiers

More exercises

Summary

### z-test

One-sample t-test

Two-sample t-test (Welch's t-test - unequal variances)

Paired t-test

Binomial test

McNemar's test

Summary

# z-test



# One-sample z-test

- **Typical set up for the experiment:**
  - **One sample** of independent test subjects, e.g. a sample of patients, a sample of customers, etc
  - Run the same experiment on each subject and collect the outcomes, e.g. give a new drug to a sample of patients and measure the effect on each individual patient; test a new web design on a sample of customers and record the time they spend on the web page, etc
  - The outcomes contain **one i.i.d. sample** with **continuous numerical values**
- **Purpose:** to test if the mean of the outcomes differs from a predefined constant
- **Data:**  $x_1, \dots, x_N$ , e.g. blood pressure after taking a new drug
- **Random variable** and **assumption:**  $X_1, \dots, X_N$ 
  - $X_i$  i.i.d.
  - $X_i$  Gaussian or large  $N$  (CLT)
  - $X_i$  standard deviation  $\sigma$  known
- **Parameter of interest:**  $\mu$
- **Parameter estimate:**  $\bar{x}$ ,  $\bar{X} \sim \mathcal{N}(\mu, \sigma^2/N)$
- **Hypotheses**  $H_0$  and  $H_A$ : given  $c$  a constant

$$H_0 : \mu = c$$

$$H_A : \mu \neq c$$



Note: only two-tailed  $H_A$  is illustrated here.

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## One-sample z-test (cont.)

- **Test statistic:**

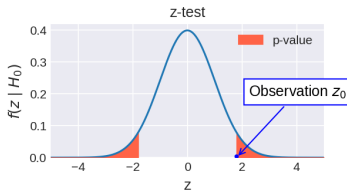
$$z_0 = \frac{\bar{x} - c}{\sigma / \sqrt{N}}$$

- **Null distribution:** standard normal distribution

- PDF:  $f(z | H_0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$
- Python: `stats.norm.pdf(z, 0, 1)`

- **p-value**

- Definition:  $p = 2 \min(P(Z \leq z_0 | H_0), P(Z \geq z_0 | H_0))$
- Python: `2 * min(stats.norm.cdf(z_0, 0, 1), 1 - stats.norm.cdf(z_0, 0, 1))`



# Two-sample z-test

- **Typical set up for the experiment:**
  - **Two samples** of independent test subjects, where the two samples  $\mathcal{X}$  and  $\mathcal{Y}$  *letters with a calligraphic font are typically used to denote sets* are independent from one another, e.g. two samples of independent patients, two samples of independent customers, etc
  - Run two sets of experiments A and B on the test subjects from the two samples  $\mathcal{X}$  and  $\mathcal{Y}$ , respectively, and collect the outcomes, e.g. give drug D to patient sample  $\mathcal{X}$  and drug E to patient sample  $\mathcal{Y}$  and measure the effect on each individual patient; test two web designs on two samples of customers and record the time they spend on the web page, etc
  - The outcomes contain two i.i.d. samples with **continuous numerical values**
- **Purpose:** to test if two alternative options have different effects by testing if the means differ by a constant
- **Data:**  $x_1, \dots, x_{N_X}$  and  $y_1, \dots, y_{N_Y}$ , e.g. blood pressure measured after taking two different drugs
- **Random variable** and **assumption:**  $X_1, \dots, X_{N_X}$ ,  $Y_1, \dots, Y_{N_Y}$ 
  - $X_i$  and  $Y_j$  independent
  - $X_i$  i.i.d.;  $Y_j$  i.i.d.
  - $X_i$  Gaussian or large  $N_X$ ;  $Y_j$  Gaussian or large  $N_Y$
  - $X_i$  and  $Y_j$  have known standard deviation  $\sigma_X$  and  $\sigma_Y$ , respectively
- **Parameter of interest:**  $\mu_X, \mu_Y$
- **Parameter estimate:**  $\bar{x}, \bar{y}$
- **Hypotheses**  $H_0$  and  $H_A$ : given  $c$  a constant

$$H_0: \mu_X - \mu_Y = c$$

$$H_A: \mu_X - \mu_Y \neq c$$

## Two-sample z-test (cont.)

- **Test statistic:**

$$z_0 = \frac{\bar{x} - \bar{y} - c}{\sqrt{\frac{\sigma_X^2}{N_X} + \frac{\sigma_Y^2}{N_Y}}}$$

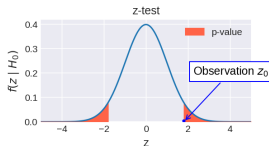
Hint:  $\bar{X} - \bar{Y} \sim \mathcal{N}(\mu_X - \mu_Y, \sigma_X^2/N_X + \sigma_Y^2/N_Y)$

- **Null distribution:** standard normal distribution

- PDF:  $f(z | H_0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$
- Python: `stats.norm.pdf(z, 0, 1)`

- **p-value**

- Definition:  $p = 2 \min(P(Z \leq z_0 | H_0), P(Z \geq z_0 | H_0))$
- Python: `2 * min(stats.norm.cdf(z_0, 0, 1), 1-stats.norm.cdf(z_0, 0, 1))`



Test statistics and hypothesis tests

Compare two classifiers

More exercises

Summary

z-test

**One-sample t-test**

Two-sample t-test (Welch's t-test - unequal variances)

Paired t-test

Binomial test

McNemar's test

Summary

## One-sample t-test



# One-sample t-test

- **Typical set up for the experiment** (same as one-sample z-test):
  - One sample of independent test subjects, e.g. a sample of patients, a sample of customers, etc
  - Run the same experiment on each subject and collect the outcomes, e.g. give a new drug to a sample of patients and measure the effect on each individual patient; test a new web design on a sample of customers and record the time they spend on the web page, etc
  - The outcomes contain one i.i.d. sample with **continuous numerical values**
- **Purpose**: to test if the mean of the outcomes differs from a predefined constant
- **Data**:  $x_1, \dots, x_N$ , e.g. blood pressure after taking a new drug
- **Random variable** and **assumption**:  $X_1, \dots, X_N$ 
  - $X_i$  i.i.d.
  - $X_i$  Gaussian or large  $N$
  - $X_i$  standard deviation  $\sigma$  **unknown**
- **Parameter of interest**:  $\mu$
- **Parameter estimate**:  $\bar{x}$
- **Hypotheses**  $H_0$  and  $H_A$ : given  $c$  a constant

$$H_0 : \mu = c$$

$$H_A : \mu \neq c$$

## One-sample t-test (cont.)

- **Test statistic:**

$$t_0 = \frac{\bar{x} - c}{s/\sqrt{N}}$$

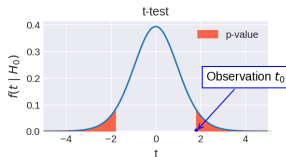
where  $s = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (x_i - \bar{x})^2}$  is the sample standard deviation

- **Null distribution:**

- Student's-t distribution with degrees of freedom  $df = N - 1$
- Python: `stats.t.pdf(t, df = N - 1)`

- **p-value:**

- Definition:  $p = 2 \min(P(T \leq t_0 | H_0), P(T \geq t_0 | H_0))$
- Python: `2 * min(stats.t.cdf(t_0, df = N - 1), 1 - stats.t.cdf(t_0, df = N - 1))`



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Summary

## Two-sample t-test (Welch's t-test - unequal variances)



## Two-sample t-test

- **Typical set up for the experiment** (same as the two-sample z-test):
  - Two samples of independent test subjects, where the two samples  $\mathcal{X}$  and  $\mathcal{Y}$  are independent from one another, e.g. two samples of independent patients, two samples of independent customers, etc
  - Run two sets of experiments A and B on the test subjects from the two samples  $\mathcal{X}$  and  $\mathcal{Y}$ , respectively, and collect the outcomes, e.g. give drug D to patient sample  $\mathcal{X}$  and drug E to patient sample  $\mathcal{Y}$  and measure the effect on each individual patient; test two web designs on two samples of customers and record the time they spend on the web page, etc
  - The outcomes contain two i.i.d. samples with continuous numerical values
- **Purpose:** to test if two alternative options have different effects by testing if the means differ by a constant
- **Data:**  $x_1, \dots, x_{N_X}$  and  $y_1, \dots, y_{N_Y}$ , e.g. blood pressure measured after taking two different drugs
- **Random variable** and **assumption:**  $X_1, \dots, X_{N_X}, Y_1, \dots, Y_{N_Y}$ 
  - $X_i$  and  $Y_j$  independent
  - $X_i$  i.i.d.;  $Y_j$  i.i.d.
  - $X_i$  Gaussian or large  $N_X$ ;  $Y_j$  Gaussian or large  $N_Y$
  - $X_i$  and  $Y_j$  have **unknown** standard deviation  $\sigma_X$  and  $\sigma_Y$ , respectively
- **Parameter of interest:**  $\mu_X, \mu_Y$
- **Parameter estimate:**  $\bar{x}, \bar{y}$
- **Hypotheses**  $H_0$  and  $H_A$ : given  $c$  a constant

$$H_0 : \mu_X - \mu_Y = c$$

$$H_A : \mu_X - \mu_Y \neq c$$



## Two-sample t-test (cont.)

- **Test statistic:**

$$t_0 = \frac{\bar{x} - \bar{y} - c}{\sqrt{\frac{s_X^2}{N_X} + \frac{s_Y^2}{N_Y}}}$$

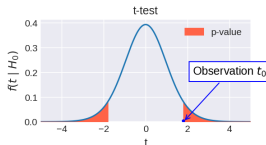
with degrees of freedom  $df = \frac{(s_X^2/N_X + s_Y^2/N_Y)^2}{(\frac{s_X^2}{N_X})^2/(N_X-1) + (\frac{s_Y^2}{N_Y})^2/(N_Y-1)}$

- **Null distribution:**

- Student's-t distribution with degrees of freedom  $df$
- Python: `stats.t.pdf(t, df = df)`

- **p-value:**

- Definition:  $p = 2 \min(P(T \leq t_0 | H_0), P(T \geq t_0 | H_0))$
- Python: `2 * min(stats.t.cdf(t_0, df=df), 1-stats.t.cdf(t_0, df=df))`



## Test statistics and hypothesis tests

Compare two classifiers

More exercises

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z-test

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Two-sample t-test (Welch's t-test - unequal variances)

**Paired t-test**

Binomial test

McNemar's test

Summary

# Paired t-test



## Paired t-test

- **Typical set up for the experiment:**
  - Typically one sample of independent test subjects, e.g. one sample of independent patients; or two paired samples
  - Run two sets of experiments A and B on all subjects from the sample and collect the outcomes, e.g. measure the blood pressure of a sample of patients **before** giving them a new drug (experiment A); measure the blood pressure of these patients **after** giving them the new drug (experiment B)
  - The outcomes contain two samples with **continuous numerical values**
- **Purpose:** to test if two alternative options have different effects by testing if the mean of their differences differs from a predefined constant
- **Data:**  $x_1, \dots, x_N, y_1, \dots, y_N$
- **Random variable** and **assumption:**  $X_1, \dots, X_N, Y_1, \dots, Y_N$ 
  - $X_i - Y_i$  i.i.d.
  - $X_i - Y_i \sim \mathcal{N}(\mu_{X-Y}, \sigma_{X-Y}^2)$  or large  $N$  (CLT)
  - standard deviation unknown
- **Parameter of interest:**  $\mu_{X-Y}$
- **Parameter estimate:**  $m_{X-Y} = \frac{1}{N} \sum_{i=1}^N (x_i - y_i)$
- **Hypotheses**  $H_0$  and  $H_A$ : given  $c$  a constant

$$H_0 : \mu_{X-Y} = c$$

$$H_A : \mu_{X-Y} \neq c$$

## Paired t-test

- **Test statistic:**

$$t_0 = \frac{m_{X-Y} - c}{s_{X-Y} / \sqrt{N}}$$

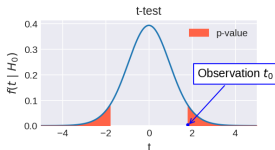
where  $s_{X-Y} = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (x_i - y_i - m_{X-Y})^2}$

- **Null distribution:**

- Student's t distribution with degrees of freedom  $N - 1$
- Python: `stats.t.pdf(t, df = N - 1)`

- **p-value:**

- Definition:  $p = 2 \min(P(T \leq t_0 | H_0), P(T \geq t_0 | H_0))$
- Python: `2 * min(stats.t.cdf(t_0, df = N - 1), 1 - stats.t.cdf(t_0, df = N - 1))`



## Exercise 1

- A company claims that a new drug E they have developed can increase the average sleeping hours of people with insomnia. Design three different hypothesis tests to test this statement.

Let's design experiments for running the one-sample t-test, two-sample t-test and paired t-test

## Test 1: one-sample t-test

- **Statement:** drug E does not increase the average sleeping hours of people with insomnia; for the one-sample t-test, the average sleeping hours of people with insomnia is a known constant - say, it is 4.5 hours
- **Experiment:** let  $N = 40$  people with insomnia take drug E and observe the amount of their sleep
- **Data:**  $x_1, \dots, x_N$  the sleeping hours of people who have taken drug E; **random variable**  $X_1, \dots, X_N$  i.i.d.
- **Parameter of interest:** the mean value  $\mu$ ; **estimate:** sample mean

$$\hat{\mu} = \bar{x} = \frac{1}{N} \sum_{i=1}^N x_i$$

- **Null hypothesis  $H_0$ :**  $H_0 : \mu = 4.5$
- **Significance level  $\alpha$ :** set to 0.05

## Test 1: one-sample t-test (cont.)

- **Test statistic:**

$$t_0 = \frac{\bar{x} - 4.5}{s/\sqrt{N}}$$

where  $s = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (x_i - \bar{x})^2}$

- **Null distribution:**

- Student's-t distribution with degrees of freedom  $df = N - 1$
- Python: `stats.t.pdf(t, df = N - 1)`

- **Alternative hypothesis  $H_A$ :**  $H_A : \mu \neq 4.5$  - two tailed test

## Test 1: one-sample t-test (cont.)

- Run the experiment and collect data

# Data in this example is generated using the following command

$N = 40$

```
x = stats.norm.rvs(loc=5.2, scale=1.2, size=N, random_state=1)
```

```
>> x = [7.14921444 4.4658923 4.5661939 3.91243765  
        6.23848916 2.43815356 7.29377412 4.28655172  
        5.58284692 4.90075555 6.95452952 2.72783115  
        4.81309936 4.73913477 6.56052333 3.88013048  
        4.99308615 4.1465699 5.2506565 5.89937826  
        3.87925699 6.57366845 6.28190886 5.80299321  
        6.28102714 4.37952657 5.05253173 4.07707668  
        4.8785343 5.83642656 4.3700071 4.72389577  
        4.37539276 4.18575323 4.39450464 5.18480248  
        3.85922758 5.48129884 7.19176261 6.09045299]
```

$\Rightarrow \bar{x} = 5.092$



## Test 1: one-sample t-test (cont.)

- Compute the test statistic  $t_0$  from data:
  - First, estimate the **nuisance parameter** - the parameter that is **not the parameter of interest**: standard deviation

$$s = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (x_i - \bar{x})^2} = 1.172$$

- Then compute the test statistic

$$t_0 = \frac{\bar{x} - 4.5}{s/\sqrt{N}} = \frac{5.09 - 4.5}{1.172/\sqrt{40}} = 3.197$$

## Test 1: one-sample t-test (cont.)

- Compute the  $p$ -value:

$$p = 2 \min(P(T \leq t_0 \mid H_0), P(T \geq t_0 \mid H_0)) = 0.003$$

- $p < \alpha$ : reject  $H_0$

Example implementation in Python: `stats.ttest_1samp(x, 4.5)`

- $x$  is specified on page 24

## Test 2: two-sample t-test

- **Statement:** drug E does not increase the average sleeping hours of people with insomnia
- **Experiment:** let  $N_X = 40$  people with insomnia take drug E and observe their amount of sleep; observe the sleeping hours of  $N_Y = 50$  people with insomnia without taking drug E
- **Data:**
  - $x_1, \dots, x_{N_X}$  sleeping hours of people with insomnia who have taken drug E; **random variable**  $X_1, \dots, X_{N_X}$  i.i.d.
  - $y_1, \dots, y_{N_Y}$  sleeping hours of people with insomnia who have not taken drug E; **random variable**  $Y_1, \dots, Y_{N_Y}$  i.i.d.
  - $X_i$  and  $Y_j$  independent, for  $i = 1, \dots, N_X, j = 1, \dots, N_Y$

## Test 2: two-sample t-test (cont.)

- **Parameter of interest:**

- The mean value of the sleeping hours of people with insomnia after taking drug E  $\mu_E$ ; **estimate**: sample mean  $\hat{\mu}_E = \bar{x} = \frac{1}{N_X} \sum_{i=1}^{N_X} x_i$
- The mean value of the sleeping hours of people with insomnia without taking drug E  $\mu_0$ ; **estimate**: sample mean  $\hat{\mu}_0 = \bar{y} = \frac{1}{N_Y} \sum_{i=1}^{N_Y} y_i$

- **Null hypothesis**  $H_0$ :  $\mu_E - \mu_0 = 0$

- **Test statistic:**

$$t_0 = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{s_X^2}{N_X} + \frac{s_Y^2}{N_Y}}}$$

with degrees of freedom  $df = \frac{(s_X^2/N_X + s_Y^2/N_Y)^2}{(\frac{s_X^2}{N_X})^2/(N_X-1) + (\frac{s_Y^2}{N_Y})^2/(N_Y-1)}$ , where

$$s_X = \sqrt{\frac{1}{N_X-1} \sum_{i=1}^{N_X} (x_i - \bar{x})^2} \text{ and } s_Y = \sqrt{\frac{1}{N_Y-1} \sum_{i=1}^{N_Y} (y_i - \bar{y})^2}$$

## Test 2: two-sample t-test (cont.)

- **Null distribution:**
  - Student's-t distribution with degrees of freedom **df** (cf. page 28)
  - Python: `stats.t.pdf(t, df = df)`
- **Alternative hypothesis  $H_A$ :**  $H_A : \mu_E - \mu_0 \neq 0$  - two tailed test
- **Significance level  $\alpha$ :** set to 0.05

## Test 2: two-sample t-test (cont.)

- Run the experiment and collect data:  $x$  is the same data as page 24

# Data  $y$  in this example is generated using the following command

```
y = stats.norm.rvs(loc=4.5, scale=0.9, size=50, random_state=2)
```

```
>> y = [4.12491794 4.44935986 2.57742351 5.97624373 2.88590797  
        3.74242737 4.95259328 3.37924072 3.547843    3.68189315  
        4.99630864 6.56298721 4.53738545 3.4938671    4.98515249  
        3.96345627 4.48278255 5.5575011    3.82691615 4.50812273  
        3.7097029    4.35920925 4.73091341 3.61009886 4.19506023  
        4.28743437 3.92611049 3.43114894 3.2209045    4.36185432  
        4.25784874 6.50823011 2.30870918 4.60145385 4.83340008  
        5.72367048 4.95167149 3.74020767 4.50000879 4.98811731  
        4.21784262 5.19391056 2.81871841 6.0580662    5.82091021  
        4.1978904    5.0502067    4.54317353 3.75377824 4.5789392 ]
```

Parameter estimate:

- Parameter of interest:**  $\bar{x} = 5.092$ ,  $\bar{y} = 4.374$
- Nuisance parameter:**

$$s_x = \sqrt{\frac{1}{N_x - 1} \sum_{i=1}^{N_x} (x_i - \bar{x})^2} = 1.172, \quad s_y = \sqrt{\frac{1}{N_y - 1} \sum_{i=1}^{N_y} (y_i - \bar{y})^2} = 0.946$$

## Test 2: two-sample t-test (cont.)

- Compute the test statistic  $t_0$  from data:
  - Then compute the test statistic

$$t_0 = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{s_x^2}{N_x} + \frac{s_y^2}{N_y}}} = 3.142$$

- Compute the  $p$ -value:

$$p = 2 \min(P(T \leq t_0 \mid H_0), P(T \geq t_0 \mid H_0)) = 0.002$$

- $p < \alpha$ : reject  $H_0$

## Test 2: two-sample t-test (cont.)

- In this two-sample t-test, we do not assume equal variance for  $X_i$  and  $Y_j$ ; this type of two-sample t-test is also called **Welch's t-test**
- Example implementation in Python:

`stats.ttest_ind(x, y, equal_var=False)`

where `equal_var=False` means we do not assume equal variance for  $x$  and  $y$



## Test 3: paired t-test

- **Statement:** drug E does not increase the average sleeping hours of people with insomnia
- **Experiment:** let  $N = 40$  people with insomnia take drug E and observe their amount of sleep before and after taking drug E
- **Data:** let  $z_1, \dots, z_N$  and  $x_1, \dots, x_N$  be the sleeping hours of people before and after taking drug E, respectively; **random variable**  $X_1 - Z_1, \dots, X_N - Z_N$  i.i.d.
- **Parameter of interest:** the mean value of the difference  $\mu_{X-Z}$ ;  
**estimate:** sample mean  $\hat{\mu}_{X-Z} = \frac{1}{N} \sum_{i=1}^N x_i - z_i$
- **Null hypothesis  $H_0$ :**  $H_0 : \mu_{X-Z} = 0$

## Test 3: paired t-test (cont.)

- **Test statistic:**

$$t_0 = \frac{\hat{\mu}_{X-Z}}{s_{X-Z}/\sqrt{N}}$$

where  $s_{X-Z} = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (x_i - z_i - \hat{\mu}_{X-Z})^2}$

- **Null distribution:**

- Student's-t distribution with degrees of freedom  $df = N - 1$
- Python: `stats.t.pdf(t, df = N - 1)`

- **Alternative hypothesis  $H_A$ :**  $H_A : \mu_{X-Z} \neq 0$  - two tailed test

- **Significance level  $\alpha$ :** set to 0.05

## Test 3: paired t-test (cont.)

- Run the experiment and collect data:  $x$  is the same data as page 24  
# Data  $z$  in this example is generated using the following command  
 $N = 40$   
 $z = \text{stats.norm.rvs}(loc=4.5, scale=0.9, size=N, random\_state=0)$   
>>  $z =$  [6.08764711 4.86014149 5.38086419 6.51680388 6.18080219  
3.62044991 5.35507958 4.36377851 4.40710303 4.86953865  
4.62963921 5.80884616 5.18493395 4.60950751 4.89947691  
4.80030689 5.84467117 4.31535756 4.78176093 3.73131383  
2.20230917 5.08825674 5.27799258 3.83205148 6.54277916  
3.19107089 4.54118267 4.33153453 5.87950129 5.82242289  
4.63945268 4.84034627 3.70099283 2.71728318 4.18687907  
4.64071407 5.60726161 5.58214186 4.15140586 4.22792752]

Parameter estimate:

- Parameter of interest:**  $\Rightarrow \hat{\mu}_{X-Z} = 0.311$
- Nuisance parameter:**  $s_{X-Z} = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (x_i - z_i - \hat{\mu}_{X-Z})^2} = 1.313$

## Test 3: paired t-test (cont.)

- Compute the test statistic  $t_0$  from data:

$$t_0 = \frac{\hat{\mu}_{X-Z}}{s_{X-Z}/\sqrt{N}} = 1.499$$

- Compute the  $p$ -value:

$$p = 2 \min(P(T \leq t_0 \mid H_0), P(T \geq t_0 \mid H_0)) = 0.142$$

- $p > \alpha$ : fail to reject  $H_0$

Example implementation in Python: `stats.ttest_rel(x, z)`

## Exercise 2

- One of the tests you have designed is a two-sample test. After the experiments, you realized the test subjects being selected in the second group are twins of the first group (and they both have insomnia. Duh!). Would that be a problem? Can you still use the result somehow?
- Solution:
  - The two-sample test is the two-sample t-test (cf. page 27); cannot use the result as is since the two samples are not independent
  - As a potential solution, we can match related subjects in the first group and the second group to create a paired data set  $(x_1, y_1), \dots, (x_N, y_N)$ , i.e.  $x_i$  and  $y_i$  in each pair are related to each other
  - Apply the paired t-test on the new data set  $(x_1, y_1), \dots, (x_N, y_N)$

## Test statistics and hypothesis tests

Compare two classifiers

More exercises

Summary

z-test

One-sample t-test

Two-sample t-test (Welch's t-test - unequal variances)

Paired t-test

**Binomial test**

McNemar's test

Summary

# Binomial test



# Binomial distribution

- Discrete distribution
- Applies to discrete numerical data - the number of successes from  $n$  independent Bernoulli trials with probability of success  $p$
- Example: You try to catch 10 ducks one by one (they need their cuddles!); the success rate of catching a duck is  $p = 20\%$ ; what is the probability of catching  $k$  ducks successfully, where  $k = 0, 1, \dots, 10$ ?
- PMF:

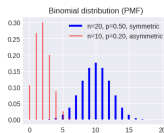
- Equation

$$f_X(k | n, p) = P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0, \dots, n, \quad p \in [0, 1]$$

where  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$  is the binomial coefficient (choose  $k$  from  $n$ )

- Shape

- When  $p = 0.5$ , the PMF is symmetric
- When  $p \neq 0.5$ , the PMF is asymmetric



- Parameters:  $p$  and  $n$ ;  $n$  is typically known

# Binomial test

- **Typical set up for the experiment:**
  - One sample of independent test subjects, e.g. one sample of independent patients
  - Run the same experiment on all subjects from the sample and collect the outcomes, e.g. give a new drug to a sample of patients and measure how many patients are cured
  - The outcomes contain one sample with **nominal categorical values with two categories**, which are then summarized into one **discrete numerical value** - the number of "success" cases
- **Purpose:** to test if the proportion of "success" differs from a predefined constant
- **Data:**  $N$  independent Bernoulli trials  $x_i$  with  $k_0$  "success" outcomes, e.g. the number of cured patients within the sample of size  $N$
- **Random variable** and **assumption:**  $X_i \sim \text{Bernoulli}(p)$ ,  $K \sim \text{Binomial}(N, p)$  with known  $N$  and unknown success rate  $p$
- **Parameter of interest:**  $p$
- **Parameter estimate:**  $\hat{p} = \frac{k_0}{N}$
- **Null hypothesis:** given  $\pi$  a constant,

$$H_0 : p = \pi$$



## (exact) Binomial test (cont.)

- **Test statistic:**  $k_0$
- **Null distribution:**

$$P(X = k) = \binom{N}{k} \pi^k (1 - \pi)^{N-k}$$

- Binomial distribution with parameters  $N$  and  $\pi$
- Python: `stats.binom.pmf(k, N,  $\pi$ )`
- As discussed in the remarks (cf. page 5), we only introduce the following scenarios:
  - One-tailed (left) binomial test with any  $\pi \in (0, 1)$
  - One-tailed (right) binomial test with any  $\pi \in (0, 1)$
  - Two-tailed binomial test with  $\pi = 0.5$ , where the null distribution is symmetric

## (exact) One-tailed (left) binomial test

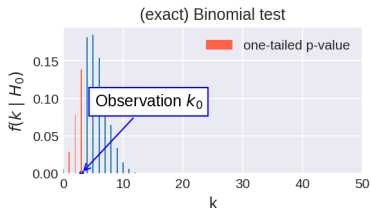
- Hypotheses  $H_0$  and  $H_A$ :

$$H_0 : p = \pi$$

$$H_A : p < \pi$$

- $p$ -value:

- Definition:  $P(K \leq k_0 \mid H_0)$
- Python: `stats.binom.cdf(k0, n=N, p=π)`



## (exact) One-tailed (right) binomial test

- Hypotheses  $H_0$  and  $H_A$ :

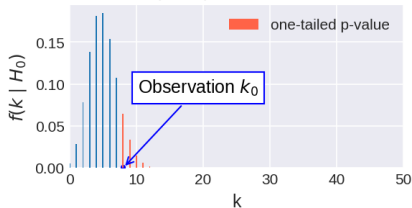
$$H_0 : p = \pi$$

$$H_A : p > \pi$$

- $p$ -value:

- Definition:  $P(K \geq k_0 \mid H_0)$
- Python: `1 - stats.binom.cdf(k0, n = N, p =  $\pi$ ) + stats.binom.pmf(k0, n = N, p =  $\pi$ )`

(exact) Binomial test



## (exact) Two-tailed binomial test

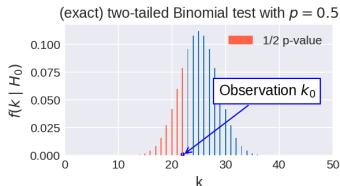
- Hypotheses  $H_0$  and  $H_A$ :

$$H_0 : p = 0.5$$

$$H_A : p \neq 0.5$$

- $p$ -value:

- Definition:  $2 \min (P(K \leq k_0 | H_0), P(K \geq k_0 | H_0))$
- Python:
  - `c = stats.binom.cdf(k0, n = N, p = 0.5)`
  - `2 * min (c, 1 - c + stats.binom.pmf(k0, n = N, p = 0.5))`



## (large $N$ ) Binomial test

Same set up as page 40, but for large  $N$ , the Binomial distribution can be approximated using a Gaussian distribution  $K \sim \mathcal{N}(N\pi, N\pi(1 - \pi))$

- **Test statistic:**

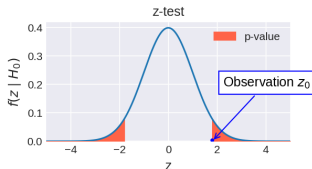
$$z_0 = \frac{k_0 - N\pi}{\sqrt{N\pi(1 - \pi)}}$$

- **Null distribution:** standard Gaussian distribution

- PDF:  $f(z | H_0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$
- Python: `stats.norm.pdf(z, 0, 1)`

- **p-value:**

- Definition:  $p = 2 \min(P(Z \leq z_0 | H_0), P(Z \geq z_0 | H_0))$
- Python: `2 * min(stats.norm.cdf(z_0, 0, 1), 1 - stats.norm.cdf(z_0, 0, 1))`



## Test statistics and hypothesis tests

Compare two classifiers

More exercises

Summary

z-test

One-sample t-test

Two-sample t-test (Welch's t-test - unequal variances)

Paired t-test

Binomial test

**McNemar's test**

Summary

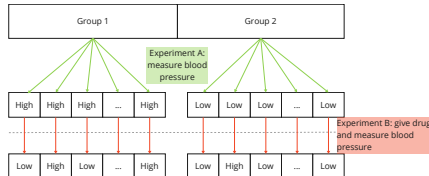
# McNemar's test



# McNemar's test

- **Typical set up for the experiment:**

- **One sample** of independent test subjects, e.g. one sample of independent patients
- Within the sample, there are **two groups**; each subject belongs to one and only one group, e.g. within the sample of patients, we have one group with high blood pressure and another group with normal blood pressure
- Run two sets of experiments A and B on all test subjects from the sample and collect the outcomes, e.g. measure the blood pressure (high or normal) of the patients **before** giving them a new drug (experiment A); measure the blood pressure (high or normal) of the patients **after** giving them the new drug (experiment B)
- The result contains two paired samples with **nominal categorical values with two categories** measured from each test subject, e.g. high blood pressure and normal blood pressure, which are summarized in a contingency table



## McNemar's test (cont.)

- **Purpose:** to test if an action have different effects on two different **groups**
- **Data:**  $N$  independent Bernoulli trials with outcomes  $x_1, \dots, x_N$  and  $y_1, \dots, y_N$  for the two experiments, respectively;  $x_i, y_i \in \{0, 1\}$

	$x_i = 0$	$x_i = 1$	
$y_j = 0$	$n_{00}$	$n_{10}$	$n_{00} + n_{10}$
$y_j = 1$	$n_{01}$	$n_{11}$	$n_{01} + n_{11}$
	$n_{00} + n_{01}$	$n_{10} + n_{11}$	$N$

where  $n_{mn}$  is the count of  $x_i = m$  and  $y_j = n$

- **Random variable** and **assumption:** i.i.d.  $X_i \sim \text{Bernoulli}(p_X)$  and i.i.d.  $Y_i \sim \text{Bernoulli}(p_Y)$



## McNemar's test (cont.)

### Example

- A company is trying to determine the effectiveness of a drug on lowering blood pressure
- The company tested the drug on a sample of 229 **independent** patients
- There are **two groups** within this sample: a high blood pressure group (112 patients) and a normal blood pressure group (117 patients); each patient belongs to one and only one of these two groups
- The blood pressure of each patient is measured **before** (to determine the group) and **after** (to determine the effect) taking the drug
- The data is summarized as follows:

	Before (high blood pressure)	Before (normal blood pressure)	
After (high blood pressure)	90	15	105
After (normal blood pressure)	22	102	124
	112	117	229

## (small discordance $n_{01} + n_{10}$ ) McNemar's test (cont.)

	$x_i = 0$	$x_i = 1$	
$y_j = 0$	$n_{00}$	$n_{10}$	$n_{00} + n_{10}$
$y_j = 1$	$n_{01}$	$n_{11}$	$n_{01} + n_{11}$
	$n_{00} + n_{01}$	$n_{10} + n_{11}$	$N$

- Parameter of interest: **discordance**

$$p = \min(P(X_i = 0, Y_i = 1 \mid X_i \neq Y_i), P(X_i = 1, Y_i = 0 \mid X_i \neq Y_i))$$

- Parameter estimate:  $\hat{p} = \frac{\min(n_{01}, n_{10})}{n_{01} + n_{10}}$
- Hypotheses  $H_0$  and  $H_A$ :

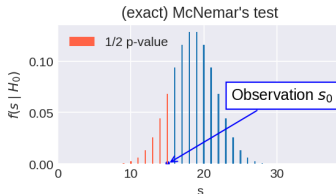
$$H_0 : p = 0.5$$

$$H_A : p \neq 0.5$$

- $H_0$ : the drug does not have any effect on blood pressure control
- $H_1$ : the drug has effect on blood pressure control

## (small discordance $n_{01} + n_{10}$ ) McNemar's test (cont.)

- **Test statistic:**  $s_0 = \min(n_{01}, n_{10})$
- **Null distribution:**
  - Binomial distribution with parameters  $(n_{01} + n_{10}, 0.5)$
  - Python: `stats.binom.pmf(s,  $n_{01} + n_{10}$ , 0.5)`
- **p-value:**
  - Definition:  $p = 2P(S \leq s_0 \mid H_0)$
  - Python: `2 * stats.binom.cdf(s_0,  $n_{01} + n_{10}$ , 0.5)`



## (large discordance $n_{01} + n_{10}$ ) McNemar's test

	$x_i = 0$	$x_i = 1$	
$y_j = 0$	$n_{00}$	$n_{10}$	$n_{00} + n_{10}$
$y_j = 1$	$n_{01}$	$n_{11}$	$n_{01} + n_{11}$
	$n_{00} + n_{01}$	$n_{10} + n_{11}$	$N$

Same set up as page 48, but with large  $n_{01} + n_{10}$ , e.g.  $n_{01} + n_{10} > 25$

- **Parameter of interest:** **discordance** (note: it is different from the previous definition (cf. page 50))  $p_{01} = P(X = 0, Y = 1)$  and  $p_{10} = P(X = 1, Y = 0)$
- **Parameter estimate:**  $\hat{p}_{01} = \frac{n_{01}}{N}$  and  $\hat{p}_{10} = \frac{n_{10}}{N}$
- **Hypotheses**  $H_0$  and  $H_A$ :

$$H_0 : p_{01} = p_{10}$$

$$H_A : p_{01} \neq p_{10}$$

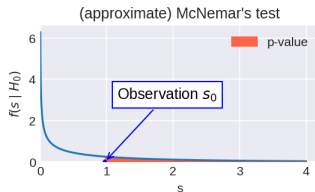
- **Test statistic:**

$$s_0 = \frac{(|n_{01} - n_{10}| - 1)^2}{n_{01} + n_{10}}$$

**Note:** "-1" is called the **continuity correction** ([https://en.wikipedia.org/wiki/Continuity\\_correction](https://en.wikipedia.org/wiki/Continuity_correction))

## (large discordance $n_{01} + n_{10}$ ) McNemar's test (cont.)

- **Null distribution:**
  - Chi-squared distribution with  $df = 1$
  - Python: `stats.chi2.pdf(s, df = 1)`
- **p-value:**
  - Definition:  $P(S \geq s_0 \mid H_0)$
  - Python: `1-stats.chi2.cdf(s_0, df = 1)`



## Exercise 3

- Run both the exact McNemar's test and the approximate McNemar's test on the data set provided on page 49

## Test 1: exact McNemar's test

	Before (high blood pressure)	Before (normal blood pressure)	
After (high blood pressure)	$n_{00}$ (90)	$n_{10}$ (15)	105
After (normal blood pressure)	$n_{01}$ (22)	$n_{11}$ (102)	124
	112	117	229

- Data:** contingency table

- $n_{01}$ : high blood pressure  $\xrightarrow{\text{take drug}}$  normal blood pressure
- $n_{10}$ : normal blood pressure  $\xrightarrow{\text{take drug}}$  high blood pressure
- We want to test if there is a significant difference between  $\frac{n_{01}}{n_{01}+n_{10}}$  and  $\frac{n_{10}}{n_{01}+n_{10}}$

- Parameter estimate:**  $\hat{p} = \frac{\min(n_{01}, n_{10})}{n_{01}+n_{10}}$

- Hypotheses**  $H_0$  and  $H_A$ :

$$H_0 : p = 0.5$$

$$H_A : p \neq 0.5$$

- Significance level**  $\alpha$ : 0.05

- Collected data:**  $n_{01} = 22$ ,  $n_{10} = 15$

- Test statistic:**  $s_0 = \min(n_{01}, n_{10}) = \min(22, 15) = 15$

## Test 1: exact McNemar's test (cont.)

- **Null distribution:**
  - Binomial distribution with parameters  $(n_{01} + n_{10}, 0.5)$
  - Python: `stats.binom.pmf(s,  $n_{01} + n_{10}$ , 0.5)`
- **p-value:**  $2P(S \leq 15 \mid H_0) = 0.3239$
- **p-value**  $> \alpha$ : fail to reject  $H_0$

Example implementation in Python:

```
from statsmodels.stats import contingency_tables
contingency_tables.mcnemar(table=[[n00, n10], [n01, n11]],
                           exact=True)
```

Args:

- **table:** the contingency table
- **exact=True:** exact test



## Test 2: approximate McNemar's test

- **Data:** contingency table
  - $n_{01}$ : high blood pressure  $\xrightarrow{\text{take drug}}$  normal blood pressure
  - $n_{10}$ : normal blood pressure  $\xrightarrow{\text{take drug}}$  high blood pressure
  - We want to test if there is a significant difference between  $\frac{n_{01}}{N}$  and  $\frac{n_{10}}{N}$
- **Parameter estimate:**  $\hat{p}_{01} = \frac{n_{01}}{N}$  and  $\hat{p}_{10} = \frac{n_{10}}{N}$
- **Hypotheses**  $H_0$  and  $H_A$ :

$$H_0 : p_{01} = p_{10}$$

$$H_A : p_{01} \neq p_{10}$$

- **Significance level:**  $\alpha = 0.05$
- **Collected data:**  $n_{01} = 22$ ,  $n_{10} = 15$ ,  $n_{00} = 90$ ,  $n_{11} = 102$
- **Test statistic:**

$$s_0 = \frac{(|n_{01} - n_{10}| - 1)^2}{n_{01} + n_{10}} = \frac{(|22 - 15| - 1)^2}{22 + 15} = 0.973$$

## Test 2: approximate McNemar's test (cont.)

- **Null distribution:**
  - Chi-squared distribution with  $df = 1$
  - Python: `stats.chi2.pdf(s, df = 1)`
- **p-value:**  $P(S \geq s_0 \mid H_0) = P(S \geq 0.973 \mid H_0) = 0.324$
- $p\text{-value} > \alpha$ : fail to reject  $H_0$

Example implementation in Python:

```
from statsmodels.stats import contingency_tables
contingency_tables.mcnemar(table=[[n00, n10], [n01, n11]],
                           exact=False,
                           correction=True)
```

Args:

- `table`: the contingency table
- `exact=False`: approximate test
- `correction=True`: continuity correction (cf. page 52)

## Test statistics and hypothesis tests

Compare two classifiers

More exercises

Summary

z-test

One-sample t-test

Two-sample t-test (Welch's t-test - unequal variances)

Paired t-test

Binomial test

McNemar's test

Summary

# Summary



# Statistical tests

	Data discrete/continuous	No. of samples collected for the test	Remark	Test statistic	Null distribution
One-sample z-test	Continuous	1	$\sigma$ known	$\frac{\bar{x}-c}{\sigma/\sqrt{N}}$	Standard Gaussian
One-sample t-test	Continuous	1	$\sigma$ unknown	$\frac{\bar{x}-c}{s/\sqrt{N}}$	Student's-t distribution
Two-sample z-test	Continuous	2	$\sigma_X, \sigma_Y$ known	$\frac{\bar{x}-\bar{y}-c}{\sqrt{\frac{\sigma_X^2}{N_X} + \frac{\sigma_Y^2}{N_Y}}}$	Standard Gaussian
Two-sample t-test	Continuous	2	$\sigma_X, \sigma_Y$ unknown	$\frac{\bar{x}-\bar{y}-c}{\sqrt{\frac{s_X^2}{N_X} + \frac{s_Y^2}{N_Y}}}$	Student's-t distribution
Paired t-test	Continuous	1 or 2 (paired)	$\sigma_X, \sigma_Y$ unknown	$\frac{\bar{m}_{X-Y}-c}{s_{X-Y}/\sqrt{N}}$	Student's-t distribution
Binomial test (exact)	Discrete	1	Small $N$	$k_0$	Binomial distribution
Binomial test (approximate)	Discrete	1	Large $N$	$\frac{k_0 - N\pi}{\sqrt{N\pi(1-\pi)}}$	Standard Gaussian
McNemar's test (exact)	Discrete	1 (2 groups)	Small $n_{01} + n_{10}$	$\min(n_{01}, n_{10})$	Binomial distribution
McNemar's test (approximate)	Discrete	1 (2 groups)	Large $n_{01} + n_{10}$	$\frac{( n_{01} - n_{10}  - 1)^2}{n_{01} + n_{10}}$	Chi-squared distribution

# Today

- 1 Test statistics and hypothesis tests
- 2 Compare two classifiers
- 3 More exercises
- 4 Summary

## K-fold cross validation

- **Classifiers:** A and B
- **Data:** evaluation metric; **continuous numerical data**, e.g. accuracies  $p_1^A, \dots, p_K^A$  and  $p_1^B, \dots, p_K^B$  on the  $K$  validation sets

	fold 1	fold 2	...	fold $K$
classifier A	$p_1^A$	$p_2^A$	...	$p_K^A$
classifier B	$p_1^B$	$p_2^B$	...	$p_K^B$
$p_i^A - p_i^B$	$p_1^A - p_1^B$	$p_2^A - p_2^B$	...	$p_K^A - p_K^B$

- **Random variable** and **assumption:**  $p_1^A, \dots, p_K^A, p_1^B, \dots, p_K^B$ 
  - $p_i^A - p_i^B$  i.i.d.
  - $p_i^A - p_i^B \sim \mathcal{N}(\mu_{A-B}, \sigma_{A-B}^2)$  with unknown standard deviation
- **Parameter of interest:**  $\mu_{A-B}$
- **Parameter estimate:**  $m_{A-B} = \frac{1}{K} \sum_{i=1}^K (p_i^A - p_i^B)$
- **Hypotheses**  $H_0$  and  $H_A$ :

$$H_0 : \mu_{A-B} = 0$$

$$H_A : \mu_{A-B} \neq 0$$

- **Test statistic:**  $t = \frac{m_{A-B}}{s_{A-B}/\sqrt{K}}$ , where  $s_{A-B} = \sqrt{\frac{1}{K-1} \sum_{i=1}^K (p_i^A - p_i^B - m_{A-B})^2}$
- **Hypothesis test:** paired t-test

## Training-validation split and leave-one-out cross validation

- **Classifiers:** A and B
- **Data:** classifiers A and B tested on the validation data; the outcome  $x_i^A$  and  $x_i^B$  can be either correct (0) or incorrect (1); **nominal categorical data** with two categories correct or incorrect

	classifier A correct	classifier A incorrect
classifier B correct	$n_{00} = \text{count}(\text{A correct, B correct})$	$n_{10} = \text{count}(\text{A incorrect, B correct})$
classifier B incorrect	$n_{01} = \text{count}(\text{A correct, B incorrect})$	$n_{11} = \text{count}(\text{A incorrect, B incorrect})$

- **Random variable** and **assumption:**  $X_i^A \sim \text{Bernoulli}(p_A)$ ,  $X_i^B \sim \text{Bernoulli}(p_B)$
- **Test statistic:**
  - Small discordance (e.g.  $n_{01} + n_{10} < 25$ ):  $s_0 = \min(n_{01}, n_{10})$
  - Large discordance:  $s_0 = \frac{(|n_{01} - n_{10}| - 1)^2}{n_{01} + n_{10}}$
- **Hypothesis test:** McNemar's test

## Exercise 4

- You have a labeled data set  $\mathcal{D} = \{(x_1, y_1), \dots, (x_N, y_N)\}$
- You developed two classifiers using the 10-fold validation
- Construct a table of the resulting F1 scores for these two classifiers
- Design a hypothesis test to compare these two classifiers



## Exercise 5

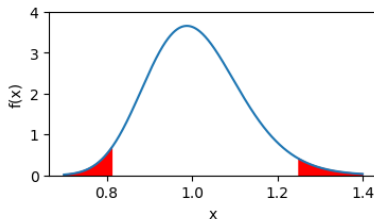
- You have a labeled data set  $\mathcal{D} = \{(x_1, y_1), \dots, (x_N, y_N)\}$
- You developed two classifiers using the training-validation split
- Construct a table of the results for these two classifiers
- Design a hypothesis test to compare these two classifiers

# Today

- 1 Test statistics and hypothesis tests
- 2 Compare two classifiers
- 3 More exercises**
- 4 Summary

## Exercise 6

- **Setup:** To test equality of variances given two samples from *Gaussian distributions*, we can use the F-test, where the test statistic is  $F = \frac{S_X^2}{S_Y^2}$ . Let us assume  $s_X \geq s_Y$ , i.e., we place the larger variance on the numerator. The null distribution is an F-distribution (`scipy.stats.f`) with parameters  $d_1 = N_X - 1$  and  $d_2 = N_Y - 1$ . Let us perform a two-tailed F-test for  $\alpha = 0.05$ . The rejection region for each tail is defined by the area corresponding to  $\alpha/2$ .
- **Questions:** What is the null and alternative hypothesis? Plot the null distribution and indicate the rejection region. Draw a conclusion for a given sample.
- **Answers:**
  - Null hypothesis:  $H_0 : \sigma_X^2 = \sigma_Y^2$
  - Alternative hypothesis:  $H_A : \sigma_X^2 \neq \sigma_Y^2$
  - Test statistics:  $F = \frac{S_X^2}{S_Y^2}$ , where  $S_X$  and  $S_Y$  are sample standard deviations of  $X$  and  $Y$ , respectively
  - Null distribution: F-distribution (`scipy.stats.f`) with parameters  $d_1 = N_X - 1$  and  $d_2 = N_Y - 1$
  - Rejection region: x-axis of the red area (you can show only the right tail)

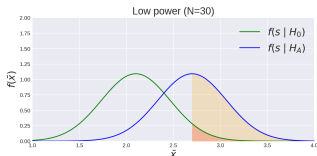
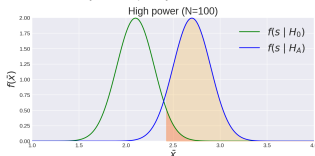


## Exercise 7

- Question: company A is trying to sell Duckyphanomin (drug D); what is the sample size needed to achieve 90% power at the alternative  $\mu_D = 2.7$  compared to regular diet  $\mu_D = 2.1$  (one-sample z-test)
- Hint:
  - Power: correctly rejecting  $H_0$

$$\text{power} = P(\text{reject } H_0 \mid H_A) = 1 - P(\text{type II error})$$

- CLT:  $\bar{X} \sim \mathcal{N}(\mu, \sigma^2/N)$



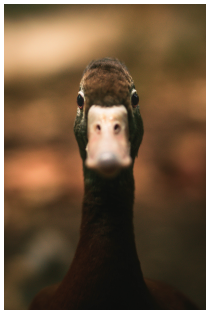
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# Summary

So far:

- Data types and data containers
- Descriptive data analysis: descriptive statistics, visualization
- Probability distributions, events, random variables, PMF, PDF, parameters
- CDF, Q-Q plot, how to compare two distributions (data vs theoretical, data vs data)
- Modeling
- Parameter estimation: maximum likelihood estimation (MLE) and maximum a posteriori estimation (MAP)
- Classification, multinomial naive Bayes classifier, Gaussian naive Bayes classifier
- Central limit theorem, interval estimation
- Clustering, cluster tendency
- Centroid clustering, k-means, parameter estimation, SSE, Silhouette score
- Gaussian Mixture Models, AIC/BIC
- The EM algorithm
- Hypothesis tests, comparison of two classifiers



I will miss you all...

Happy holidays!