Lecture 2: Probability Distribution

Statistical Methods for Data Science

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November 3, 2022

Today

1. Probability distribution
   - Why probability distributions?
   - Terminology
   - Some probability distributions that you should know by heart
2. Summary

Learning outcome

- Be able to explain the following concepts: experiment, event, random variable, probability distribution
- Given the PDF/PMF, be able to describe the probability distribution of a continuous/discrete random variable
- Be able to compute conditional probability
- Understand Gaussian distribution and Bernoulli distribution: 1) PDF/PMF; 2) what are the parameters? 3) what happens to the shape of the distribution if we change the parameters?
- Be able to apply the learning routine to study a new probability distribution yourself

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Why probability distributions?

Histogram vs probability distribution

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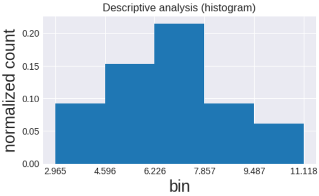

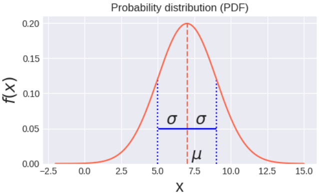

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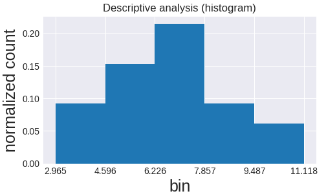

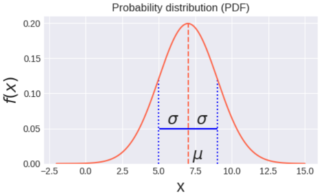

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  To address this question, let's describe the data distribution using a **histogram** and a **Gaussian distribution** to see the difference.

Histogram vs probability distribution

Here are the weights of the 20 ducks in kg

duck id	1	2	3	4	$\cdots$	19	20
weight	6.98	5.43	2.97	7.07	$\cdots$	4.63	7.27

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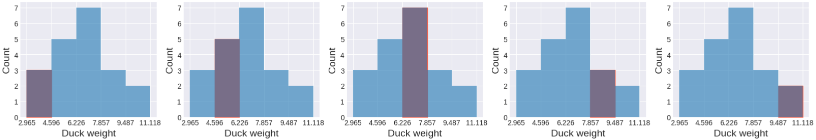

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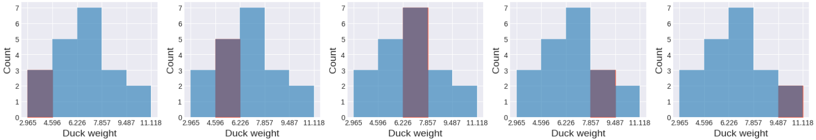

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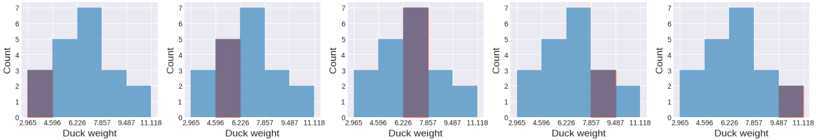

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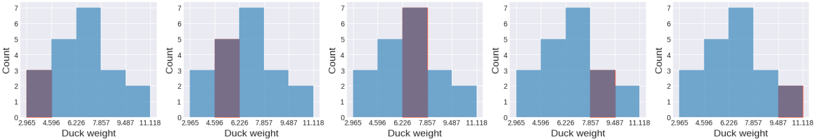

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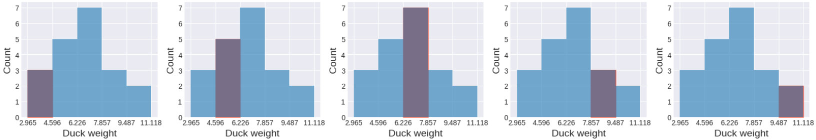

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  How about between 3.1 kg and 3.4 kg?

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- Resolution: the number of bins per kilogram

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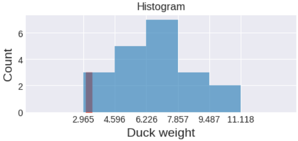

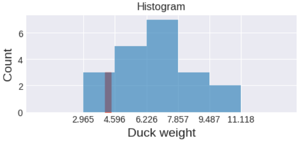

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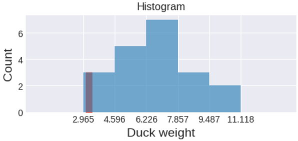

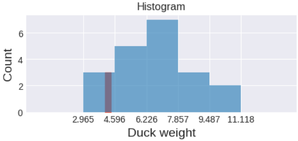

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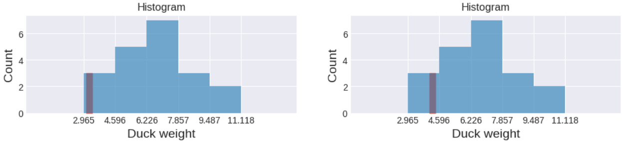

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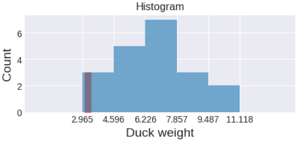

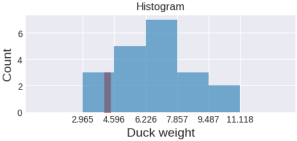

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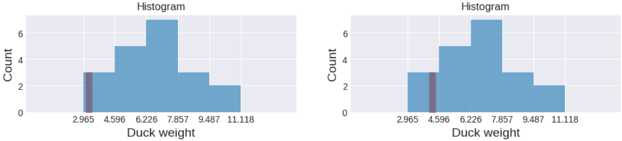

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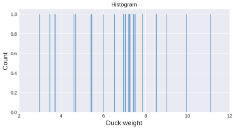

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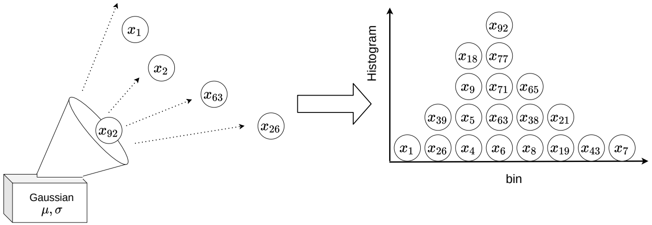
- First, we **assume** that data is **generated** from a Gaussian distribution



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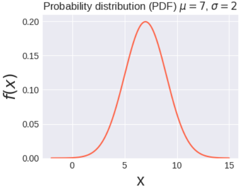

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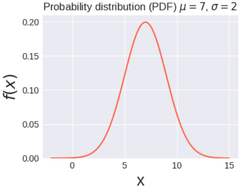

- We will try to use this **function** instead of the histogram to describe the data.

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- Describe the distribution:
  - Histogram (using 0.61 bins to describe 1 kg):
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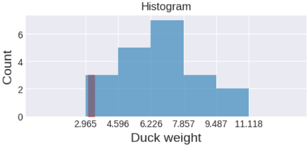

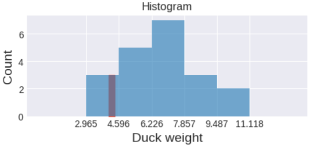

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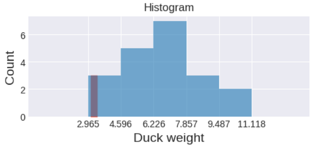
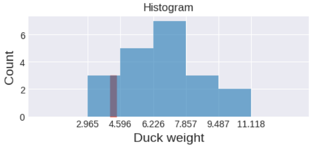
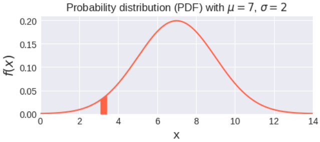
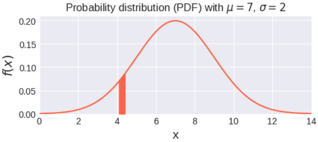




  - Gaussian distribution (using **infinite** bins to describe 1 kg):
    - The chance of *weight* $\in [3.1, 3.4]$: $\int_{3.1}^{3.4} f(t)dt = 0.010$
    - The chance of *weight* $\in [4.1, 4.4]$: $\int_{4.1}^{4.4} f(t)dt = 0.023$





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Representation	M values		mathematical function f	
Number of parameters	M	-	2 (μ and σ)	+
Resolution	$\frac{M}{\max(x)-\min(x)}$	-	infinity	+
Analytical properties	No	-	Yes	+
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- Statistical modelling using probability distributions (e.g. a Gaussian distribution) can help you with that!

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- Statistical modelling using probability distributions (e.g. a Gaussian distribution) can help you with that!
- This concludes the question why we use probability distributions instead of histograms to describe your ducks.

Histogram vs probability distribution

- Descriptive analysis: a *histogram* with M bins (e.g. $M = 5$)
- Gaussian distribution: a mathematical function $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$
- Comparison

	Histogram		Gaussian distribution	
Representation	M values		mathematical function f	
Number of parameters	M	-	2 (μ and σ)	+
Resolution	$\frac{M}{\max(x)-\min(x)}$	-	infinity	+
Analytical properties	No	-	Yes	+
Assumptions	No	+	Yes	-
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Disclaimer: we used a continuous distribution to illustrate the comparison between a histogram and a probability distribution. A discrete probability distribution differs from a continuous distribution.

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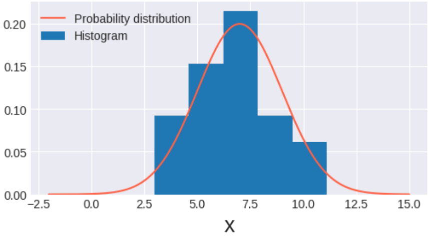

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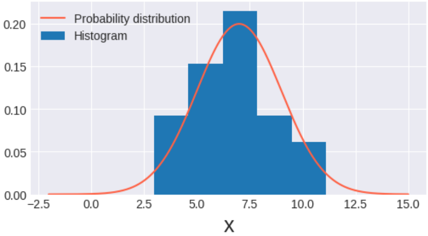
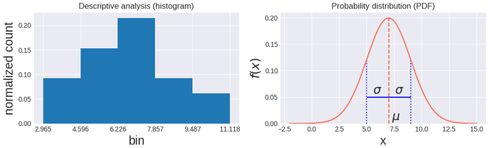


    - Long answer will be given in lecture 3.

Parameter estimation and evaluation

- Question 3: Okay fine, let's say we describe the data with a Gaussian distribution. How do I know what the parameters μ and σ are?



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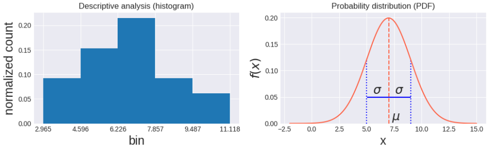

  - This is done by **parameter estimation**. In lecture 3 & 4, we will talk about the **maximum likelihood estimation (MLE)** and the **maximum a posteriori estimation (MAP)**.

Terminology

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- **Random variable** X:
  - Heuristically, X assigns a numerical value to each outcome of the experiment:

$$X : \text{weight} \rightarrow \mathbb{R}$$

  - X follows some underlying probability distribution.
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- **Data** x: a value drawn from the **underlying distribution of X**.
  - We use a **capital letter** (e.g. X) to denote a random variable and the corresponding lower case letter (e.g. x) to denote the data generated from the underlying distribution of X.
  - Discrete random variable: categorical data or discrete numerical data
  - Continuous random variable: continuous numerical data

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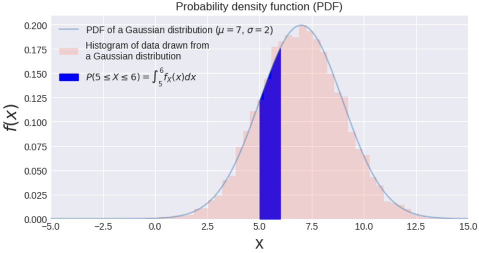
$\boxed{P(\text{event})}$ is the probability of the **event** occurring.

Example: continuous random variables and PDF

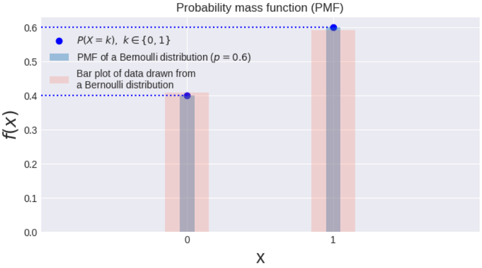
- **Experiment**: you weigh a duck and look at its weight
- **Sample space**: $0 < weight < \infty$
- **Random variable** X : $weight \to \mathbb{R}$
  - $X = x$ if the duck weighs x kg for $0 < x < \infty$
  - X follows a **Gaussian distribution** with parameters μ and σ; denoted as $X \sim \mathcal{N}(\mu, \sigma^2)$
- **PDF**: $f_X(x)$

$$P(a \leq X \leq b) = \underbrace{\int_a^b f_X(x)dx}_{\textbf{Integral = area under the PDF curve}} \qquad \forall a, b \in \mathbb{R}, a \leq b$$



Example: discrete random variables and PMF

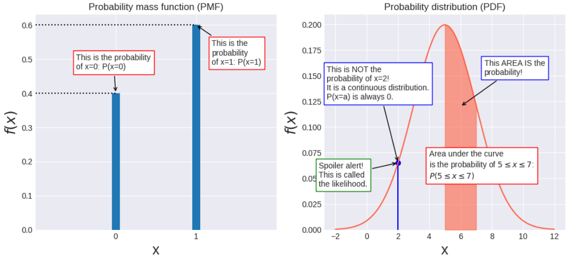
- **Experiment**: you measure the color of the duck.
- **Sample space**: the color can be either red or blue
- **Random variable** X : $color \rightarrow \mathbb{Z}$
  - $X = \begin{cases} 0, & \text{if a duck is red} \\ 1, & \text{if a duck is blue} \end{cases}$
  - X follows a **Bernoulli distribution** with parameter p; denoted as $X \sim Bernoulli(p)$
- **PMF**: $f_X(x_i) = P(X = x_i)$



Probability distribution

Differences between PMF and PDF

- Discrete distribution $f_X(x_i) = P(X = x_i)$:
  - y-axis represents the probability itself
- Continuous distribution:
  - $P(a \leq X \leq b) = \int_a^b f_X(x)dx$: **y-axis $f(x)$** DOES NOT represent the probability itself.
  - For continuous distributions, **the probability at any given value is always 0**, i.e. $P(X = a) = P(a \leq X \leq a) = \int_a^a f_X(x)dx \equiv 0$. Example: what is the probability of a duck weighing exactly 4.32028374... kg?



Conditional probability

Given events A and B,

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{P(A \cap B)}{P(B)}$$

The probability of event A given event B.

Conditional probability example

Example:

- **Experiment**: You ask your ducks to stand in a row again and look at their colors and sizes.
- **Sample space**: The color can be either red or blue; the size can be either slim or chonker.
- **Data**:

duck id	1	2	3	4	5	6
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Conditional probability example (cont.)

An alternative way to estimate $P(A \mid B)$:

- Count the chonkers: 3
- Count blue ducks within the chonker gang: 2
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Side note: from this calculation, can you make a bold statement of the probability distribution of all the ducks in the world? No. It is only an estimation based on the data you got.

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$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)} = P(X = 1 \mid Y = 1) = \frac{P(X = 1 \text{ and } Y = 1)}{P(Y = 1)}$$

Independent events

Two events A and B are independent if and only if

$$P(A \text{ and } B) := P(A \cap B) = P(A)P(B)$$

$$\iff \quad P(A \mid B) = P(A),\ P(B \mid A) = P(B) \text{ (conditional probability)}$$

$$\iff \quad log\,(P(A \text{ and } B)) = \log\,(P(A \cap B)) = \log(P(A)) + \log(P(B))$$

Bayes' rule

Given events A and B,

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

Just a heads-up!

Summary: Terminology

- Experiment
- Sample space
- Event
- Random variable:
  - Discrete random variable
  - Continuous random variable
- Data
- Probability distribution:
  - Discrete distribution: P(event) is described by the probability mass function (PMF)
  - Continuous distribution: P(event) is described by the probability density function (PDF)

  What are their differences?
- Conditional probability of events
- Independent events
- Bayes' rule

Some probability distributions that you should know by heart

Probability distributions

Probability distribution	Continuous/discrete	Apply to data type
Bernoulli distribution	Discrete	Categorical (nominal)
Categorical distribution	Discrete	Categorical (nominal)
Discrete uniform	Discrete	Numerical (discrete)
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For each distribution, you need to know:

- its PMF or PDF: the equation and the shape
- its parameters
- its applications

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- how to estimate the parameters (next lecture)

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  Merge these two equations:

$$P(X = k) = f_X(k) \equiv f_X(k \mid p) = pk + (1-p)(1-k), \; k \in \{0, 1\}, p \in [0, 1]$$

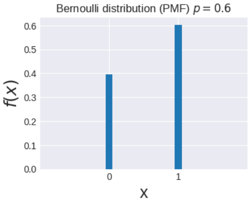
Note: here we use a | to indicate that the parameter p is given.

Bernoulli distribution

- Discrete distribution
- Applies to nominal data with 2 categories
- PMF:
  - Equation

$$f_X(k \mid p) = pk + (1-p)(1-k), k \in \{0,1\}, p \in [0,1]$$

  - Shape



- Parameters: p

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Categorical distribution

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- What is the PMF?

$$P(X = k) = f_X(k) \equiv f_X(k \mid p_1, p_2, p_3, p_4) = p_k, \ \sum_{i=1}^{4} p_i = 1, p_i \geq 0, \ k \in \{1, \cdots, 4\}$$

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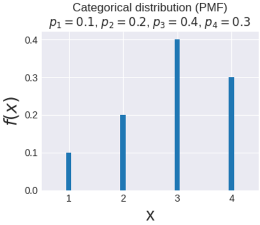
Note: categorical distribution is also called multinoulli distribution. It is a generalization of the Bernoulli distribution.

Categorical distribution

- Discrete distribution
- Applies to nominal data with $n > 0$ categories
- PMF:
  - Equation

$$f_X(k \mid p_1, p_2, \cdots, p_n) = p_k, \ \sum_{i=1}^{n} p_i = 1, p_i \geq 0, \ k \in \{1, \cdots, n\}$$

  - Shape



- Parameters: $p_k, \ k \in \{1, \cdots, n\}$ for given n; $n-1$ parameters $(\sum_{i=1}^{n} p_i = 1)$.

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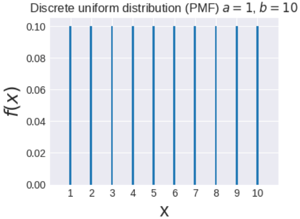
Discrete uniform distribution

- Discrete distribution
- Applies to discrete numerical data
- PMF:
  - Equation

$$f_X(k \mid a, b) = \frac{1}{b - a + 1}, \quad a \leq b, \quad a, b \text{ integers}$$

  - Shape



- Parameters: integers a, b

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- What is the PDF of a Gaussian distribution with $\mu = 7$ and $\sigma = 2$?

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  $P(a \leq X \leq b) = \int_a^b f_X(x)dx$ for all $a < b$
- What is the PDF of a Gaussian distribution with $\mu = 7$ and $\sigma = 2$?

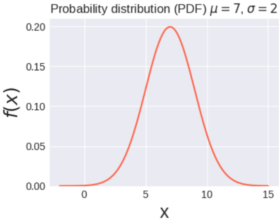
$$f_X(x \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{1}{\sqrt{8\pi}} e^{-\frac{1}{2}\left(\frac{x-7}{2}\right)^2}$$

Gaussian (normal) distribution

- Continuous distribution
- Applies to continuous numerical data
- PDF:
  - Equation

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \ \mu \in \mathbb{R}, \ \sigma \in \mathbb{R}^+$$

  - Shape



- Parameters: $\mu, \ \sigma$

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Hooray!

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Check out what data types they apply to!

We are going to talk about more applications in the future

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Check out what data types they apply to!

We are going to talk about more applications in the future (even though they won't be as important as ducks)

Today

So far:

- Data types, data containers, descriptive statistics (e.g. sample mean, sample variance, data quantile), visualization (e.g. histogram)
- Probability distributions, sample space, events, random variables, PMF, PDF, parameters

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Before next lecture:

- Quantile
- Probability distributions from today and their parameters
- PMF and PDF

Stay safe!