Lecture 11: Hypothesis testing

Statistical Methods for Data Science

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Today

1. Terminology
   - Experiment and parameter of interest
   - Null hypothesis and alternative hypothesis
   - Test statistic
   - Null distribution $f(s \mid H_0)$
   - Significance level α, power and p-value
2. Example
3. p-hacking
4. Summary

Learning outcome

- Be able to explain the following terminology
  - Null hypothesis H_0 and alternative hypothesis H_A
  - Test statistic s
  - Null distribution $f(s \mid H_0)$
  - Significance level α and power
  - p-value
- Be able to design and interpret the one-sample z-test
- Be able to explain the concept of p-hacking

Today

Important example

If you control the diet of your ducks, they lose 2.1 kg after one month on average

- Company A has developed a drug D (aka. Duckyphanomin) to help duckies lose weight. They claim that **on average** the drug works better than diet control
- Company B has developed a drug E (aka. Everyduckyslim) and they claim that drug E is more effective than drug D **on average**

You NEED to help your chonker ducks lose weight. Which drug should you buy? Or should you just control their diet without drugs?

- If company A tested drug D on 30 ducks and the average weight loss after one month is 2.2 kg, would you buy drug D instead of regular diet control?
- What if company A tested drug D on 30 ducks and the average weight loss after one month is 2.3 kg? Would you buy drug D instead of regular diet control in this case?
- What if company A tested drug D on 100 ducks and the average weight loss after one month is 2.3 kg?
- Now company B tested drug E on 30 ducks and the average weight loss after one month is 2.5 kg, while drug D results in 2.3 kg weight loss with the same setup, would you buy drug E instead of drug D?

What would you do?

Statistical hypothesis

- **Hypothesis:**
  - A proposed explanation for a phenomenon (Wikipedia)
  - An idea or explanation of something that is based on a few known facts but that has not yet been proved to be true or correct (Oxford dictionary)
- **Statistical hypothesis:** A proposed distribution that explains a set of random variables
- **Hypothesis testing in statistics:** The goal is to determine whether it is likely that a random variable follows the proposed distribution
  - This is done using sample statistics derived from data
  - The process involves combining the hypothesis with data to make a decision on whether to reject or not reject the hypothesis

Hypothesis testing: a list to go through

- A default statement
- Experiment
- Data x, random variable X
- Parameter of interest θ
- Parameter estimate $\hat{\theta}$
- Null hypothesis H_0
- Alternative hypothesis H_A
- Test statistic s
- Null distribution $f(s \mid H_0)$
- Significance level α
- p-value

Experiment and parameter of interest

Experiment design

- Before formulating the statistical hypothesis, we need to propose a **default statement**: a "boring" and unsurprising claim that we would like to **test**, e.g.,
  - Drug D is **not more effective** than a regular diet on average
  - Drug E works the same as drug D on average

  In science, we are hoping for new discoveries and excitement, but we need to earn it by showing that the **trivial explanation** does not hold with evidence (**data**)
- Example 1: recall if you control the diet of your ducks, they lose 2.1 kg after one month on average
  - **A default statement**: drug D is not more effective than a regular diet on average What experiments can we run to test whether this statement is true?
  - **Experiment** (5 sec): give drug D to N chonker ducks and record the average weight loss after one month
  - **Data** and **random variable** (5 sec):
    - Data: x_i weight loss after one month for $i = 1, \cdots, N$
    - Random variable: X_i i.i.d.
  - **Parameter of interest** (5 sec): the mean of the weight loss μ_D
  - **Parameter estimate** (5 sec): the sample mean $\hat{\mu_D} = \bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$

  Then we test whether $\bar{x}$ is greater than diet control (2.1 kg)

Experiment design (cont.)

- Example 2:
  - **A default statement**: drug E and drug D work the same on average
  - **Experiment** (5 sec): give drug D to N_D chonker ducks and record the average weight loss after one month; test drug E on another N_E chonker ducks and record the average weight loss after one month
  - **Data** and **random variable** (5 sec): data - x_i weight loss using drug D after one month for $i = 1, \cdots, N_D$; random variable - X_i i.i.d.; likewise, we have data y_j and random variable Y_j for drug E
  - **Parameter of interest** (5 secs): the mean μ_D and μ_E for drug D and E, respectively
  - **Parameter estimate** (5 secs): the sample mean $\hat{\mu}_D = \bar{x} = \frac{1}{N_D} \sum_{i=1}^{N_D} x_i$ and $\hat{\mu_E} = \bar{y} = \frac{1}{N_E} \sum_{j=1}^{N_E} y_j$

  Then we test whether $\bar{x}$ and $\bar{y}$ are the same

Experiment design (cont.)

- We make our decision by observing data; if the evidence does not support the default statement, we **reject the statement**; otherwise, we **do not reject the statement**
- However, we can never definitively prove or accept the statement; we can only *reject* it by providing counterexamples.
- Intuition: "If the statement is true, then the evidence should support it", which is equivalent to saying ($\iff$) "if the evidence does not support the statement, the statement is considered false" , which is not the same as claiming ($\not\Leftrightarrow$) "if the evidence supports the statement, the statement must be true"

Null hypothesis and alternative hypothesis

Hypotheses H_0 and H_A

- **Statistical hypothesis**: a proposed **distribution** - typically a statement about the **parameter of interest**
- **Null hypothesis** H_0: the default statement translated into a mathematical expression
  - Example 1: drug D is not more effective than regular diet control on average

$$H_0 : \mu_D = 2.1$$

  - Example 2: drug E and drug D work the same on average (5 sec)

$$H_0 : \mu_D = \mu_E$$

- **Alternative hypothesis** H_A: an alternative hypothesis that is complementary (the opposite) to the null hypothesis
  - Example 2 (5 sec): drug E and drug D do not work the same on average (5 sec)

$$H_A : \mu_D \neq \mu_E$$

  - Example 1 (5 sec): drug D is more effective than regular diet control on average (5 sec)

$$H_A : \mu_D > 2.1$$

Hypotheses H_0 and H_A (cont.)

Questions:

- Question 1: Why are $H_A : \mu_D > 2.1$ and $H_0 : \mu_D = 2.1$ complementary to each other? What about $H_A : \mu_D < 2.1$?
  Answer: One implicit assumption here is that μ_D will not be smaller than 2.1
  Question 1.1: Do I need to make this assumption?
  Answer: No
  Question 1.2: Could you elaborate on that?
  Answer: Yes
  Question 1.3: When?
  Answer: In a few slides
  Okay
- Question 2: Can H_0 and H_A be ANYTHING I want? Like a magic mirror!?
  Answer: No
  Question 2.2: What are the choices for H_0 and H_A then?

Choices for H_0

- In this course, we only deal with null hypotheses **with an equal sign** in them - only one fixed choice for the distribution proposed by H_0
- **Null hypothesis H_0**: two cases
  - **One-sample test**: to test a data distribution against a theoretical probability distribution, i.e. for a given constant c

    $$H_0 : \theta = c$$

    For example, is this (binary) classifier more accurate than random? $H_0 : p = 50\%$
  - **Two-sample test**: to test a data distribution against another data distribution, i.e.

    $$H_0 : \theta_1 = \theta_2$$

    For example, is classifier A better than classifier B? $H_0 : p_A = p_B$
- We have seen one-sample test and two-sample test in the Q-Q plot lecture
- In practice, you can narrow down your choice of hypotheses by looking at Q-Q plots

Choices for H_A

Given

$$H_0 : \theta = \beta$$

where β can be either a constant (one-sample test) or a parameter from another data distribution (two-sample test)

- **Alternative hypothesis H_A**: H_A can be **one-tailed** or **two-tailed**
  - **One-tailed**:

$$H_A : \quad \theta > \beta$$

or

$$H_A : \quad \theta < \beta$$

  - **Two-tailed**:

$$H_A : \quad \theta \neq \beta \iff \theta < \beta \text{ or } \theta > \beta$$

Summary: choices for H_0 and H_A

Putting everything together,

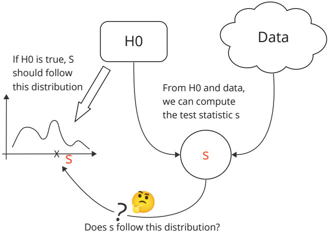
	One-sample test	Two-sample test
Two-tailed	$H_0 : \theta = c$, $H_A : \theta \neq c$	$H_0 : \theta_1 = \theta_2$, $H_A : \theta_1 \neq \theta_2$
One-tailed	$H_0 : \theta = c$, $H_A : \theta > c$	$H_0 : \theta_1 = \theta_2$, $H_A : \theta_1 > \theta_2$
	$H_0 : \theta = c$, $H_A : \theta < c$	$H_0 : \theta_1 = \theta_2$, $H_A : \theta_1 < \theta_2$

where θ, θ_1, θ_2 are the parameters of interest and c is a constant
Note: this is the answer to question 1.1 (cf. page 14): if you choose the one-tailed test, then you are making the assumption $H_A : \mu_D > 2.1$; if you choose the two-tailed test, then you are not making this assumption

Test statistic

Test statistic

- **Test statistic s** (random variable S): a (typically standardized) statistic computed from data
- **Purpose:** to determine how plausible the null hypothesis H_0 is by observing s



- What is needed for the expression of the test statistic?
  - Assumptions on random variables X_i's
  - We only need the null hypothesis H_0 (not H_A) to express the test statistic

Disclaimer: in this course, we only deal with null hypothesis where we are able to express the PDF/PMF $f(s \mid H_0)$, i.e. H_0 with an equal sign in them

Test statistic (cont.)

Example 1. one-sample test (is drug D more effective than diet control)

- **Data**: $x_1, \cdots, x_N$
- **Random variable**: $X_1, \cdots, X_N$ i.i.d. Gaussian with known σ
- **Parameter of interest**: μ_D
- **Parameter estimate**: $\bar{x}$ ($\sim \mathcal{N}(\mu_D, \frac{\sigma^2}{N})$ - CLT)
- **Null hypothesis**: $H_0 : \mu_D = 2.1$
- **Test statistic**: $s =$ **standardized** $\bar{x}$ **assuming the null hypothesis**
  - What are we trying to do here? - To decide whether we can reject the null hypothesis - if the null hypothesis is true, we should be able to see evidence that supports it - if we do not see evidence, we reject the null hypothesis
  - What is "evidence"? It is the value of the test statistic s assuming the **the distribution described by the null hypothesis** (we need H_0 to compute s)
  - What is **the distribution described by the null hypothesis**?
    - Gaussian distribution with (known) standard deviation σ and **mean** $\mu_D = 2.1$
  - **Assuming the null hypothesis**: data are assumed to be generated from the distribution described by the null hypothesis - $X_i \sim \mathcal{N}(2.1, \sigma^2)$
  - Recall: what is **standardization**?
    - Random variable X: $\frac{X-\mu_X}{\sigma_X}$
    - Data x: $\frac{x-\mu_X}{\sigma_X}$
  - Why are we standardizing the statistic $\bar{x}$? We want to use standard tools for our analysis

Standardize $\bar{x}$ (15 sec)

$$s = z = \frac{\bar{x} - 2.1}{\sigma/\sqrt{N}}$$

Test statistic (cont.)

Example 2. two-sample test

- **Data**: $x_1, \cdots, x_{N_D}$ and $y_1, \cdots, y_{N_E}$
- **Random variable**: $X_1, \cdots, X_{N_D}$ i.i.d. **Gaussian with known σ_D**; $Y_1, \cdots, Y_{N_E}$ i.i.d. **Gaussian with known σ_E**; X_i and Y_j independent
- **Parameter of interest**: μ_D, μ_E
- **Parameter estimate**: $\bar{x}$, $\bar{y}$
- **Null hypothesis**: $H_0 : \mu_D = \mu_E \iff H_0 : \mu_D - \mu_E = 0$
- **Test statistic**: standardized $\bar{x} - \bar{y}$ assuming the null hypothesis

$$z = \frac{\bar{x} - \bar{y}}{\sqrt{\sigma_D^2 / N_D + \sigma_E^2 / N_E}}$$

Note: $\bar{x}$, $\bar{y}$ Gaussian (CLT); awesome properties of Gaussian from Lecture 6

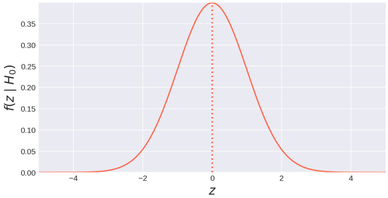
Null distribution $f(s \mid H_0)$

Null distribution

- **Null distribution** $f(s \mid H_0)$: the distribution of the **test statistic** S given the null hypothesis H_0
- Example:
  - **Data**: $x_1, \cdots, x_N$
  - **Random variable**: $X_1, \cdots, X_N$ i.i.d. Gaussian with known σ
  - **Parameter of interest**: μ
  - **Parameter estimate**: $\bar{x}$
  - **Null hypothesis**: $H_0 : \mu = \mu_0$
  - **Test statistic**:

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{N}}$$

  - **Null distribution**: standard Gaussian distribution



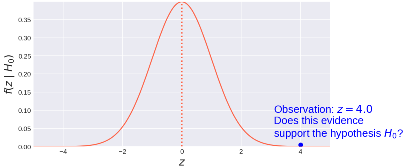
Significance level α, power and p-value

Significance level

Given a null hypothesis $H_0 : \mu = 2.1$ and the null distribution $f(s \mid H_0)$, we decide whether we reject the hypothesis or not by observing data

- Run some experiments and collect data $x_1, \cdots, x_N$
- Compute the test statistic from data, e.g.

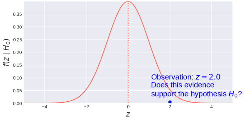
$$z = \frac{\bar{x} - 2.1}{\sigma/\sqrt{N}} = 4.0$$



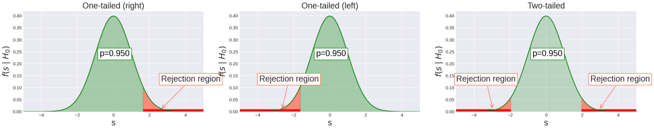
- Does this evidence support the hypothesis H_0? Probably not since it's so far away from the center?

Significance level (cont.)

- What about this observation?



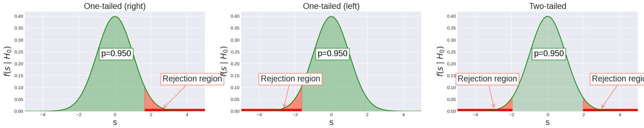
- To be able to answer the question, you need to decide where you draw the line (quite literally) - define a **rejection region** by choosing a significance level
- **Significance level** α: red area under the curve

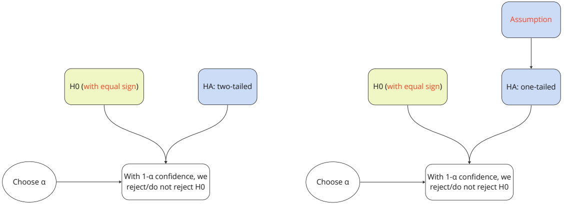


  In these three images, $\alpha = 0.05$
- What is needed for choosing a meaningful α?
  - Null distribution
  - H_A one-tailed or two-tailed

Significance level (cont.)

- **Significance level $\alpha = 0.05$**: red area under the curve

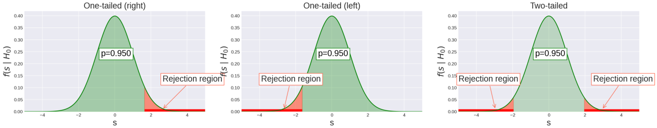




- More conservative $\Rightarrow$ less probable to reject H_0, which indicates a smaller rejection region
- Two-tailed H_A is more conservative

Interpretation of α

- $\alpha = P(\textit{reject } H_0 \mid H_0 \textit{ is true})$ - the probability of making such a mistake



  - The rejection region indicates that H_0 is **unlikely**, but the probability is not zero
  - It is possible that H_0 is true, but our observation happens to fall in the rejection region
  - If H_0 is true and our observation falls in the rejection region, we will **mistakenly** reject H_0
  - The probability of making this type of mistakes is α
- Similar to the confidence interval, $1 - \alpha$ is called the **confidence level** - "with 95% confidence, rejecting H_0 is the right thing to do"
- Define the significance level **before you run the experiments** so that you can't cheat!

Significance level and power

- Contingency table:

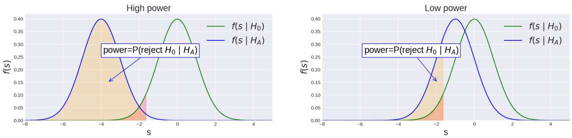
	$y = H_A$	$y = H_0$
$\hat{y}$ = reject H_0	**TP**	**FP** (Type I error)
$\hat{y}$ = do not reject H_0	**FN** (Type II error)	**TN**

- **Significance level** α: incorrectly rejecting H_0

$$\alpha = P(\text{type I error})$$

- **Power**: correctly rejecting H_0

$$\text{power} = P(\text{reject } H_0 \mid H_A) = 1 - P(\text{type II error})$$



- What is needed for computing power (20 sec)? $f(s \mid H_0)$, $f(s \mid H_A)$, α

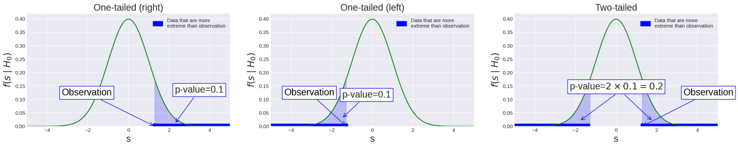
p-value

The p-value tells us how unlikely it would be to see our results (or something more extreme) by pure chance alone if the null hypothesis is true.

- ***p*-value**:
  - One-tailed:
    - Right tail: $p = P(S \geq s \mid H_0)$, e.g. 1-stats.norm.cdf(s, 0, 1)
    - Left tail: $p = P(S \leq s \mid H_0)$, e.g. stats.norm.cdf(s, 0, 1)
  - Two-tailed:
    - $p = 2\min\left(P(S \leq s \mid H_0), P(S \geq s \mid H_0)\right)$, e.g. 2*min(stats.norm.cdf(s, 0, 1), 1-stats.norm.cdf(s, 0, 1))

  Note: for example, if $f(s \mid H_0)$ is symmetric around zero and $s < 0$,

$$p = 2P(S \leq s \mid H_0)$$



- What is needed for computing the p-value? (10 sec)
  - Null distribution
  - Alternative hypothesis H_A to know one-tailed or two-tailed
  - Observation - test statistic computed from data

Summary: steps for hypothesis testing

- Step 1 Make a default statement
- Step 2 Design an **experiment**
- Step 3 Describe the **data** generated from the experiment and the corresponding random variables
- Step 4 Describe the parameter of interest and their estimates
- Step 5 Translate the default statement into a statistical hypothesis and call it the **null hypothesis** H_0
- Step 6 Find the expression for the **test statistic** s
- Step 7 Find the expression for the **null distribution**
- Step 8 Define **an alternative hypothesis** H_A: one-tailed or two-tailed
- Step 9 Choose a **significance level** α (the tail), which defines the **rejection region**
- Step 10 Run experiments and collect **data**
- Step 11 Compute the test statistic from data
- Step 12 Compute the p-value
- Step 13 If p-value$< \alpha$, i.e. the test statistic falls in the rejection region of the null distribution, then we reject the hypothesis H_0; otherwise, we fail to reject H_0.

Today

Example

Recall example: if you control the diet of your ducks, they lose 2.1 kg after one month on average. Company A has developed a drug D (Duckyphanomin!) to help ducks lose weight. They claim that on average the drug works better than diet control.

- Step 1 Make a default statement (5 secs): **drug D works the same as diet control**
- Step 2 Design an **experiment** (choose $N = 30$) (10 secs): **feed drug D to 30 chonker ducks and measure their weight loss after one month**
- Step 3 Describe the **data** and **random variables** along with assumptions about their distributions (5 secs): **weight loss** $x_1, \cdots, x_{30}$; $X_1, \cdots, X_{30}$ **i.i.d. Gaussian random variables** - let's make an additional assumption to simplify the problem - the standard deviation of X_i $\sigma = 0.6$ is known
- Step 4 Describe the parameter of interest and their estimates (10 secs): **the mean value** μ_D **and** $\hat{\mu}_D = \bar{x}$
- Step 5 Translate the default statement into a statistical hypothesis and call it the **null hypothesis** H_0 (10 secs): $H_0 : \mu_D = 2.1$
- Step 6 Find the expression for the **test statistic** s (60 secs):

$$s = z = \frac{\bar{x} - 2.1}{\sigma/\sqrt{30}}$$

- Step 7 Find the expression for the **null distribution** $f(s \mid H_0)$ (10 secs):

$$f(z \mid H_0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Example (cont.)

Step 8 Define **an alternative hypothesis** H_A (10 secs):

$$H_A : \mu_D \neq 2.1 \textbf{ or } H_A : \mu_D > 2.1$$

One-tailed or two-tailed

- **Two-tailed (5 secs):** $H_A : \mu_D \neq 2.1$
- **One-tailed (5 secs):** $H_A : \mu_D > 2.1$

Step 9 Choose a **significance level** α (the tail), which defines the rejection region (5 secs): e.g. $\alpha = 0.05$

Step 10 Collect 30 ducks in 20 secs and feed them drugs - great job! Weights measured after one month $x_1, \cdots, x_{30}$

Say $\frac{1}{30}\sum_{i=1}^{30} x_i = 2.2$

Step 11 Compute the test statistic from data (5 secs):

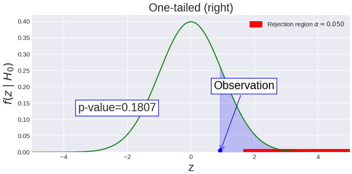
$$z_0 = \frac{2.2 - 2.1}{0.6/\sqrt{30}} = 0.91$$

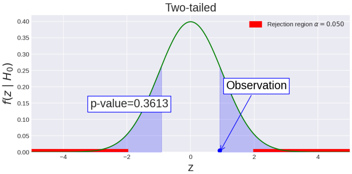
Example (cont.)

Step 12 Compute the p-value (20 secs):

- **For** $H_A : \mu_D > 2.1$ **(one-tailed):** $p = P(Z \geq z_0 \mid H_0) = 0.1807 > \alpha$
- **For** $H_A : \mu_D \neq 2.1$ **(two-tailed):** $p = 2P(Z \geq z_0 \mid H_0) = 0.3613 > \alpha$

Step 13 If p-value$< \alpha$, i.e. the test statistic falls in the rejection region of the null distribution, then we reject the hypothesis H_0





Do not reject H_0 for both one-tailed and two-tailed H_A

What does it mean? - Based on this test, you will stick to diet control instead of buying Duckyphanomin

Example (cont.)

What if $\bar{x} = 2.3$?

Step 11 Compute the test statistic from data (5 secs):

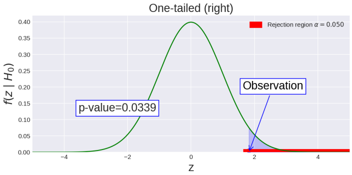
$$z_0 = \frac{2.3 - 2.1}{0.6/\sqrt{30}} = 1.826$$

Step 12 Compute the p-value (20 secs):

- **For** $H_A : \mu_D > 2.1$ **(one-tailed):** $p = P(Z \geq z_0 \mid H_0) = 0.0339 < \alpha$
- **For** $H_A : \mu_D \neq 2.1$ **(two-tailed):** $p = 2P(Z \geq z_0 \mid H_0) = 0.0679 > \alpha$

Example (cont.)

Step 13 If p-value$< \alpha$, i.e. the test statistic falls in the rejection region of the null distribution, then we reject the hypothesis H_0



Reject H_0 for one-tailed H_A; do not reject H_0 for two-tailed H_A for the same confidence level $1 - \alpha = 95\%$

Note: the two-tailed test is more conservative - if the data passes a two-tailed test, it is more conclusive than one-tailed test for the same confidence level

Example (cont.)

What if $\bar{x} = 2.3$ with $N = 100$?

Step 11 Compute the test statistic from data (5 secs):

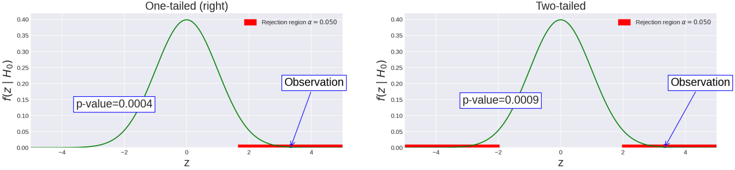
$$z_0 = \frac{2.3 - 2.1}{0.6/\sqrt{100}} = 3.33$$

Step 12 Compute the p-value (20 secs):

- **For** $H_A : \mu_D > 2.1$ **(one-tailed):** $p = P(Z \geq z_0 \mid H_0) = 0.0004 < \alpha$
- **For** $H_A : \mu_D \neq 2.1$ **(two-tailed):** $p = 2P(Z \geq z_0 \mid H_0) = 0.0009 < \alpha$

Example (cont.)

Step 13 If p-value$< \alpha$, i.e. the test statistic falls in the rejection region of the null distribution, then we reject the hypothesis H_0



Reject H_0 for both one-tailed and two-tailed H_A

Note:

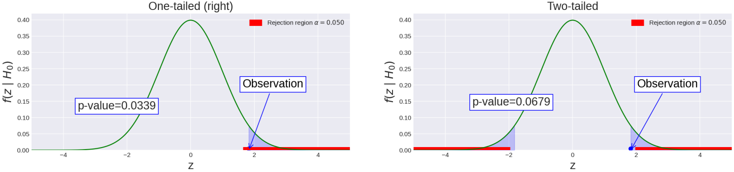
- With more data, it becomes more certain that we should reject H_0 in favor of H_A given the observation $\bar{x} = 2.3$

This test is called **one-sample z-test** (one of the established tests you choose from)

Today

Recall: one-tailed vs two-tailed tests

- *p*-value indicates how "surprising" the observation is
- In this context, "surprising" observations usually mean potential novelty
- In one of the examples, we have shown that we reject the null hypothesis for the one-tailed test but we fail to reject the null hypothesis for the two-tailed test given the same significance level



- In this example, if we use the two-tailed test, we will not claim that we have observed potential novelty with the experiment, whereas if we use the one-tailed test, we claim that we do observe potential novelty
- The conclusion we draw depends on which test we conduct

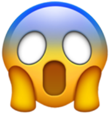
Variation of the p-value

- *p*-value is computed from data
- Data is random - ***p*-value is random**
- With the same experiment set up, if we switch to a different sample, *p*-value will be different

p-hacking

- Many factors can result in a different *p*-value
- *p*-hacking refers to situations where researchers are **trying multiple things until they get the desired result**
- This action can be a conscious decision, a subconscious decision or even an accident
- *p*-hacking can be tricky to identify
- Suggestions to avoid *p*-hacking, e.g. one should always **report effect sizes and confidence intervals**
- Reference:
  - `https://www.nature.com/news/scientific-method-statistical-errors-1.14700`
  - Why Most Published Research Findings Are False?

p-hacking (cont.)

What should I do!? 

- Be honest and explicit about your assumptions
- Be "conservative"
- Be skeptical about your result - **don't let go of any doubt**!
- Assume the first success is always **too good to be true** - **try to prove yourself wrong** - be a proper scientist

Today

Summary

So far:

- Data types and data containers
- Descriptive data analysis: descriptive statistics, visualization
- Probability distributions, events, random variables, PMF, PDF, parameters
- CDF, Q-Q plot, how to compare two distributions (data vs theoretical, data vs data)
- Modeling
- Parameter estimation: maximum likelihood estimation (MLE) and maximum a posteriori estimation (MAP)
- Classification, multinomial naive Bayes classifier, Gaussian naive Bayes classifier
- Central limit theorem, interval estimation
- Clustering, cluster tendency
- Centroid clustering, k-means, parameter estimation, SSE, Silhouette score
- Gaussian Mixture Models, AIC/BIC
- The EM algorithm
- Hypothesis test

Next:

- More examples and test statistics

Before next lecture:

- Steps for hypothesis testing

Screw diet! I'm perfect $p = 100\%$!

That's not how p-value works...