Problem set 2 Solution*

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I. An individual has wealth w. Her von Neumann-Morgenstern utility function over non-negative levels of wealth is

$$u(w) = w^{\rho},$$

where $0 < \rho < 1$. The individual is offered the following bet. If she pays x, with probability $\frac{1}{2}$ she receives nothing, and with probability $\frac{1}{2}$ she receives $x(1+s)$, where $s > 1$. How much will she bet (as a function of s)?

Solution:

The individual maximizes expected utility. If she wins, she receives $(w - x) + x(1+s) = w + xs$. If she loses, she receives $w - x$ because she pays x. Her expected utility is then:

$$u(g) = \frac{1}{2}(w + xs)^{\rho} + \frac{1}{2}(w - x)^{\rho}$$

She chooses x, the amount to pay, to maximize utility so we have:

$$\max_{x} \quad \frac{1}{2}(w + xs)^{\rho} + \frac{1}{2}(w - x)^{\rho}$$

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†Do not hesitate to email me when you find errors like typos, *etc*. Any updates of this problem set will be notified via Moodle announcements.

Assume that $0 \leq x \leq w$ so that:

$$\begin{aligned}
\frac{du}{dx} &= \frac{1}{2}\rho(w+xs)^{\rho-1}s + \frac{1}{2}\rho(w-x)^{\rho-1}(-1) \\
0 &= \frac{1}{2}\rho(w+xs)^{\rho-1}s + \frac{1}{2}\rho(w-x)^{\rho-1}(-1) \\
0 &= (w+xs)^{\rho-1}s - (w-x)^{\rho-1} \\
(w-x)^{\rho-1} &= (w+xs)^{\rho-1}s \\
s &= \frac{(w-x)^{\rho-1}}{(w+xs)^{\rho-1}} \\
s^{\frac{1}{\rho-1}} &= \frac{w-x}{w+xs} \\
ws^{\frac{1}{\rho-1}} + xss^{\frac{1}{\rho-1}} &= w-x \\
x + xss^{\frac{1}{\rho-1}} &= w - ws^{\frac{1}{\rho-1}} \\
x\left(1+s^{\frac{\rho}{\rho-1}}\right) &= w\left(1-s^{\frac{1}{\rho-1}}\right) \\
x &= w\frac{\left(1-s^{\frac{1}{\rho-1}}\right)}{\left(1+s^{\frac{\rho}{\rho-1}}\right)}
\end{aligned}$$

A brief discussion with respect to the result $x = w\frac{\left(1-s^{\frac{1}{\rho-1}}\right)}{\left(1+s^{\frac{\rho}{\rho-1}}\right)}$:

i. $x > 0$ since $w > 0$, $1+s^{\frac{\rho}{\rho-1}} > 0$, and $1-s^{\frac{1}{\rho-1}} > 0$. $w > 0$ and $1+s^{\frac{\rho}{\rho-1}} > 0$ are easy to show (or have been given in the question). Here is an explanation of $1-s^{\frac{1}{\rho-1}} > 0$. Given $0 < \rho < 1$:
   - As $\rho \to 0$, we have $\frac{1}{\rho-1} \to -1$, and so $s^{\frac{1}{\rho-1}} \to \frac{1}{s} < 1$ since $s > 1$.
   - As $\rho \to 1$, we have $\frac{1}{\rho-1} \to -\infty$, and so $s^{\frac{\rho}{\rho-1}} \to 0 < 1$ since $s > 1$.

ii. $x < w$ since $w > 0$ and $\frac{\left(1-s^{\frac{1}{\rho-1}}\right)}{\left(1+s^{\frac{\rho}{\rho-1}}\right)} < 1$.

II. **(JR 2.25)** Consider the quadratic Bernoulli utility function $u(w) = a + bw + cw^2$.

(1) What restrictions, if any, must be placed on parameters a, b, and c for this function to display risk aversion?

Solution:

First, a simply shifts utility up or down so we do not need to place any restriction on it other than it is some finite real number. Next, we know that we need:

$$u'(w) > 0$$
$$u''(w) < 0$$

for risk aversion. We have:

$$u'(w) = b + 2cw$$
$$u''(w) = 2c$$

Since we need $2c < 0$, we need $c < 0$. Since we need $b + 2cw > 0$, and $c < 0$, we need $b > -2cw$. Note that b is a positive number, this is because $c < 0$ makes $-2cw$ to be positive. We can altenatively write the restriction on b as $b > 2|c|w$.

(2) Over what domain of wealth is the quadratic Bernoulli utility function defined?

Solution:

Using our restriction on b and c above, the quadratic utility function can be defined for $0 \leq w \leq \frac{b}{2|c|}$.
Note that once $w > \frac{b}{2|c|}$, our quadratic utility function is no longer increasing.

(3) Given the gamble

$$g = \left(\frac{1}{2} \circ (w + h) \, , \frac{1}{2} \circ (w - h) \right),$$

show that the certainty equivalent, CE, is strictly smaller than the expected value of the gamble, $E(g)$.

Solution:

We know that:

$$u(g) = \frac{1}{2}u(w+h) + \frac{1}{2}u(w-h)$$

Using $u(w) = a + bw + cw^2$, we have:

$$\begin{aligned}
u(g) &= \frac{1}{2}\left[a + b(w+h) + c(w+h)^2\right] + \frac{1}{2}\left[a + b(w-h) + c(w-h)^2\right] \\
u(g) &= \frac{1}{2}\left[a + bw + bh + cw^2 + 2cwh + ch^2\right] + \frac{1}{2}\left[a + bw - bh + cw^2 - 2cwh + ch^2\right] \\
u(g) &= a + bw + cw^2 + ch^2
\end{aligned}$$

By definition, the certainty equivalent is the certain amount of money which gives the same utility as the gamble, so we have:

$$\begin{aligned}
u(CE) &= u(g) \\
u(CE) &= a + bw + cw^2 + ch^2
\end{aligned}$$

Now, we need to find $E(g)$:

$$\begin{aligned}
E(g) &= \frac{1}{2}(w+h) + \frac{1}{2}(w-h) \\
E(g) &= w
\end{aligned}$$

Thus, the utility of the expected value of the gamble is:

$$\begin{aligned}
u(E(g)) &= u(w) \\
u(E(g)) &= a + bw + cw^2
\end{aligned}$$

Now we can compare $u(E(g))$ with $u(CE)$. We want:

$$u(E(g)) > u(CE)$$

If this is the case, then we can say that $E(g) > CE$ since $u(\cdot)$ is increasing. Comparing we have:

$$\begin{aligned}
a + bw + cw^2 &> a + bw + cw^2 + ch^2 \\
0 &> ch^2
\end{aligned}$$

Recall that $c < 0$ and $h > 0$, so $ch^2 < 0$ is true.
Since $P = E(g) - CE$, then $P > 0$.

(4) Show that this function, satisfying the restrictions in part (1), cannot represent preferences that display *decreasing* absolute risk aversion.

Solution:

For the quadratic function satisfying the restrictions in part (1) we cannot have decreasing absolute risk aversion because we would need to have that the Arrow-Pratt risk aversion parameter decreases when wealth increases. We can find that the Arrow-Pratt risk aversion coefficient for this quadratic is:

$$R_a(w) = -\frac{u''(w)}{u'(w)}$$
$$R_a(w) = -\frac{2c}{b + 2cw}$$

Now, differentiate with respect to w:

$$R_a(w) = -2c(b + 2cw)^{-1}$$
$$\frac{dR_a(w)}{dw} = -2c(-1)(b + 2cw)^{-2}$$
$$\frac{dR_a(w)}{dw} = \frac{(2c)^2}{(b + 2cw)^2}$$

So $R_a(w)$ increases in w.

III. Consider a firm that uses only one factor of production, x, with a production technology

$$y = 70\sqrt{x}.$$

Let w denote the price of input x. Compute the marginal cost and the average cost of producing y. Verify that the average cost is less than the marginal cost for all values of y. Explain why this is so.

Solution:

With only one input x, the cost of producing y units is wx, where $x = \frac{y^2}{4900}$, or $c(x, y) = \frac{wy^2}{4900}$. The marginal cost is:

$$\frac{dc(x,y)}{dy} = \frac{2wy}{4900}$$

and the average cost is:

$$\frac{c(w,y)}{y} = \frac{wy}{4900}$$

Average cost is less than marginal cost if:

$$\begin{aligned} \frac{c(w,y)}{y} &\leq \frac{dc(x,y)}{dy} \\ \frac{wy}{4900} &\leq \frac{2wy}{4900} \\ wy &\leq 2wy \end{aligned}$$

This inequality is strict if $y > 0$ and the two functions are equal when $y = 0$.

We can see that average cost is an increasing function of y for $y \in [0, \infty)$. If average cost is increasing over the entire range, then marginal cost must be greater than or equal to average cost by definition (if an average is increasing then the marginal must be above the average to pull it up).

IV. **(JR 3.54)** Consider a firm with the cost function

$$c(y, w_1, w_2) = y^2(w_1 + w_2),$$

where w_i denotes the price of input i for $i = 1, 2$. Let p denote the output price. Derive the output supply function $y(p, w_1, w_2)$, and the input demand functions $x_i(p, w_1, w_2)$ for $i = 1, 2$.

Solution: The firm's profit function is:

$$\pi(y, w_1, w_2) = py - y^2(w_1 + w_2)$$

Finding the first order condition we have:

$$\begin{aligned} \frac{d\pi}{dy} &= p - 2y^*(w_1 + w_2) = 0 \\ p &= 2y^*(w_1 + w_2) \\ \frac{p}{2(w_1 + w_2)} &= y^* \end{aligned}$$

To find the conditional input demand functions we can differentiate $c(y, w_1, w_2)$ with respect to w_1 and w_2:

$$\begin{aligned} \frac{\partial c(y, w_1, w_2)}{\partial w_1} &= x_1(y, w_1, w_2) \\ \frac{\partial c(y, w_1, w_2)}{\partial w_1} &= y^2 \end{aligned}$$

We also have $x_2(y, w_1, w_2) = y^2$. To find input demand functions, which are $x_i(y, w_1, w_2)$, we subsitute in y^* so that we have:

$$x_1(y, w_1, w_2) = x_2(y, w_1, w_2) = \left(\frac{p}{2(w_1 + w_2)}\right)^2$$

V. Consider a firm with production function

$$y = (x_1^\rho + x_2^\rho)^\alpha,$$

where $0 < \rho < 1$ and $\alpha > 0$.

(1) For what value of ρ and α are there

(i) increasing returns to scale;

(ii) constant return to scale;

(iii) decreasing returns to scale?

Solution:

For each of the following returns to scale we need:

$$\begin{aligned} f(tx_1, tx_2) &< tf(x_1, x_2) \quad \text{decreasing} \\ f(tx_1, tx_2) &= tf(x_1, x_2) \quad \text{constant} \\ f(tx_1, tx_2) &> tf(x_1, x_2) \quad \text{increasing} \end{aligned}$$

So we have:

$$\begin{aligned} f(tx_1, tx_2) &= ((tx_1)^\rho + (tx_2)^\rho)^\alpha \\ f(tx_1, tx_2) &= (t^\rho x_1^\rho + t^\rho x_2^\rho)^\alpha \\ f(tx_1, tx_2) &= t^{\rho\alpha}(x_1^\rho + x_2^\rho)^\alpha \\ f(tx_1, tx_2) &= t^{\rho\alpha} f(x_1, x_2) \end{aligned}$$

Therefore, if:

$$\begin{aligned} \rho\alpha &< 1 \quad \text{decreasing} \\ \rho\alpha &= 1 \quad \text{constant} \\ \rho\alpha &> 1 \quad \text{increasing} \end{aligned}$$

(2) Suppose that there are decreasing return to scale.
  (i) Find the long-run cost function.
  (ii) Derive the output supply function and the input demand functions for this long-run cost function.

Solution to (i):

We need to find x_1^* and x_2^* that solves the cost-minimisation problem. Assuming w_1 and w_2 are the input prices of x_1 and x_2, respectively. Setting up the cost-minimization problem:

$$\begin{aligned}\min_{x_1,x_2} \quad & w_1x_1 + w_2x_2 \\ \text{s.t} \quad & y \leq (x_1^\rho + x_2^\rho)^\alpha\end{aligned}$$

Setting up the Lagrangian:

$$\min_{x_1,x_2,\lambda} \quad \mathcal{L}(x_1,x_2,\lambda) = w_1x_1 + w_2x_2 + \lambda(y - (x_1^\rho + x_2^\rho)^\alpha)$$

The first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial x_1} = w_1 - \lambda\alpha(x_1^\rho + x_2^\rho)^{\alpha-1}\rho x_1^{\rho-1} \tag{1}$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = w_2 - \lambda\alpha(x_1^\rho + x_2^\rho)^{\alpha-1}\rho x_2^{\rho-1} \tag{2}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = y - (x_1^\rho + x_2^\rho)^\alpha \tag{3}$$

Assuming an interior solution we would have:

$$\text{by (1): } \frac{\partial \mathcal{L}}{\partial x_1} = 0 \quad \Rightarrow \quad w_1 = \lambda\alpha(x_1^\rho + x_2^\rho)^{\alpha-1}\rho x_1^{\rho-1} \tag{4}$$
$$\text{by (2): } \frac{\partial \mathcal{L}}{\partial x_2} = 0 \quad \Rightarrow \quad w_2 = \lambda\alpha(x_1^\rho + x_2^\rho)^{\alpha-1}\rho x_2^{\rho-1} \tag{5}$$

Therefore, by (4) and (5):

$$\frac{w_1}{w_2} = \frac{\lambda\alpha(x_1^\rho + x_2^\rho)^{\alpha-1}\rho x_1^{\rho-1}}{\lambda\alpha(x_1^\rho + x_2^\rho)^{\alpha-1}\rho x_2^{\rho-1}} \tag{6}$$
$$\frac{w_1}{w_2} = \frac{x_1^{\rho-1}}{x_2^{\rho-1}} \tag{7}$$
$$x_2\left(\frac{w_1}{w_2}\right)^{\frac{1}{\rho-1}} = x_1 \tag{8}$$

Plugging (8) into the production function:

$$y = (x_1^\rho + x_2^\rho)^\alpha \tag{9}$$

$$y = \left(\left(x_2 \left(\frac{w_1}{w_2} \right)^{\frac{1}{\rho-1}} \right)^\rho + x_2^\rho \right)^\alpha \tag{10}$$

$$y = \left(x_2^\rho \left(\frac{w_1}{w_2} \right)^{\frac{\rho}{\rho-1}} + x_2^\rho \right)^\alpha \tag{11}$$

$$y = \left(x_2^\rho \left(\left(\frac{w_1}{w_2} \right)^{\frac{\rho}{\rho-1}} + 1 \right) \right)^\alpha \tag{12}$$

$$y = x_2^{\rho\alpha} \left(\left(\frac{w_1}{w_2} \right)^{\frac{\rho}{\rho-1}} + 1 \right)^\alpha \tag{13}$$

$$y = x_2^{\rho\alpha} \left(\left(\frac{w_1}{w_2} \right)^{\frac{\rho}{\rho-1}} + \left(\frac{w_2}{w_2} \right)^{\frac{\rho}{\rho-1}} \right)^\alpha \tag{14}$$

$$y = x_2^{\rho\alpha} \left(\frac{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}}{w_2^{\frac{\rho}{\rho-1}}} \right)^\alpha \tag{15}$$

$$y^{\frac{1}{\alpha}} = x_2^\rho \left(\frac{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}}{w_2^{\frac{\rho}{\rho-1}}} \right) \tag{16}$$

$$y^{\frac{1}{\alpha}} \frac{w_2^{\frac{\rho}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}} = x_2^\rho \tag{17}$$

$$y^{\frac{1}{\rho\alpha}} \frac{w_2^{\frac{1}{\rho-1}}}{\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \right)^{\frac{1}{\rho}}} = x_2^* \tag{18}$$

By a similar method we can find $x_1(y, w)$ to be:

$$x_1^* = y^{\frac{1}{\rho\alpha}} \frac{w_1^{\frac{1}{\rho-1}}}{\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \right)^{\frac{1}{\rho}}} \tag{19}$$

Now the cost function, $c(w, y)$ is (make use of (18) and (19)):

$$c(w,y) = w_1 x_1^* + w_2 x_2^* \tag{20}$$

$$c(w,y) = w_1 \left(y^{\frac{1}{\rho\alpha}} \frac{w_1^{\frac{1}{\rho-1}}}{\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}} \right) + w_2 \left(y^{\frac{1}{\rho\alpha}} \frac{w_2^{\frac{1}{\rho-1}}}{\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}} \right) \tag{21}$$

$$c(w,y) = y^{\frac{1}{\rho\alpha}} \left(\frac{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}}{\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}} \right) \tag{22}$$

$$c(w,y) = y^{\frac{1}{\rho\alpha}} \left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}} \tag{23}$$

Solution to (ii):

Assuming the market price of each unit of output is p. Now we set up the profit function so that the firm maximizes profit (make use of (23)) :

$$\max_{y} \quad py - y^{\frac{1}{\rho\alpha}} \left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \right)^{\frac{\rho-1}{\rho}}$$

Taking the first derivative with respect to y:

$$p - \frac{1}{\rho\alpha} y^{\frac{1-\rho\alpha}{\rho\alpha}} \left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \right)^{\frac{\rho-1}{\rho}} = 0 \tag{24}$$

$$\frac{1}{\rho\alpha} y^{\frac{1-\rho\alpha}{\rho\alpha}} \left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \right)^{\frac{\rho-1}{\rho}} = p \tag{25}$$

$$y^{\frac{1-\rho\alpha}{\rho\alpha}} \left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \right)^{\frac{\rho-1}{\rho}} = \rho\alpha p \tag{26}$$

$$y^{\frac{1-\rho\alpha}{\rho\alpha}} = \frac{\rho\alpha p}{\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \right)^{\frac{\rho-1}{\rho}}} \tag{27}$$

$$y = \left(\frac{\rho\alpha p}{\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \right)^{\frac{\rho-1}{\rho}}} \right)^{\frac{\rho\alpha}{1-\rho\alpha}} \tag{28}$$

$$y = \left(\rho\alpha p \left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \right)^{\frac{1-\rho}{\rho}} \right)^{\frac{\rho\alpha}{1-\rho\alpha}} \tag{29}$$

Now to find the input demands (not conditional) we can use $x_i(y, w)$ and substitute in for y (make use of (29)). Recall (19):

$$x_1(y,w) = y^{\frac{1}{\rho\alpha}} \frac{w_1^{\frac{1}{\rho-1}}}{\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}} \tag{30}$$

$$x_1(y,w) = x_1(y(p,w),w) = x_1(p,w) \tag{31}$$

$$x_1(p,w) = \left(\left(\rho\alpha p\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}}\right)^{\frac{\rho\alpha}{1-\rho\alpha}}\right)^{\frac{1}{\rho\alpha}} \frac{w_1^{\frac{1}{\rho-1}}}{\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}} \tag{32}$$

$$x_1(p,w) = \left(\rho\alpha p\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}}\right)^{\frac{1}{1-\rho\alpha}} \frac{w_1^{\frac{1}{\rho-1}}}{\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}} \tag{33}$$

$$x_1(p,w) = (\rho\alpha p)^{\frac{1}{1-\rho\alpha}} \frac{\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho-\rho^2\alpha}}}{\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}} w_1^{\frac{1}{\rho-1}} \tag{34}$$

$$x_1(p,w) = (\rho\alpha p)^{\frac{1}{1-\rho\alpha}} \left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\alpha-1}{1-\rho\alpha}} w_1^{\frac{1}{\rho-1}} \tag{35}$$

For $x_2(p, w)$ we have something similar (just flipping the subscripts 1 and 2):

$$x_2(p,w) = (\rho\alpha p)^{\frac{1}{1-\rho\alpha}} \left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\alpha-1}{1-\rho\alpha}} w_2^{\frac{1}{\rho-1}} \tag{36}$$