Problem set 1 Solution*

October 21,2022
Last updated: October 25, 2022†

Solution to Question I

Recall:

Axiom of completeness of $\succsim$: either $x \succsim y$ or $y \succsim x$ (or both) (for all x, y).
Axiom of transitivity of $\succsim$: if $x \succsim y$ and $y \succsim z$, then $x \succsim z$ (for all x, y, z).

(1)	a.	$x \succsim x$	by axiom of completeness of $\succsim$
	b.	$x \succsim x$ and $x \succsim x$	from a, by logic
	$\therefore$	$x \sim x$	from b, by definition of $\sim$
(2)	a.	$x \sim y$	by assumption
	b.	$x \succsim y$ and $y \succsim x$	from a, by definition of $\sim$
	c.	$y \succsim x$ and $x \succsim y$	from b, by logic
	d.	$y \sim x$	from c, by definition of $\sim$
	$\therefore$	$x \sim y \rightarrow y \sim x$	from a to d, by definition of $\sim$
(3)	a.	$x \sim y$ and $y \sim z$	by assumption
	b.	$x \succsim y$ and $y \succsim x$	from a, by definition of $\sim$
	c.	$y \succsim z$ and $z \succsim y$	from a, by definition of $\sim$
	d.	$x \succsim z$	from b and c, by axiom of transitivity of $\succsim$
	e.	$z \succsim x$	from b and c, by axiom of transitivity of $\succsim$
	f.	$z \sim x$	from d and e, by definition of $\sim$
	$\therefore$	$x \sim y$ and $y \sim z \rightarrow x \sim z$	from a to e, by logic

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†Do not hesitate to email me when you find errors like typos, *etc*. Any updates of this solution set will be notified via Moodle announcements.

(4)	a.	$x \succ y$ and $y \succ z$	by assumption
	b.	$x \succsim y$	from a, by definition of $\succ$
	c.	$y \succsim z$	from a, by definition of $\succ$
	d.	$x \succsim z$	from b and c, by axiom of transitivity of $\succsim$
	e.	suppose $z \succsim x$ is true	prove by contradiction (we aim to show $z \succsim x$ is false)
	f.	$z \succsim y$	from b and e, by axiom of transitivity of $\succsim$
	g.	$z \not\succsim y$	from a, by definition of $\succ$
	h.	$z \not\succsim x$	g contradicts with f, thus, $z \succsim x$ in e is false
	i.	$x \succ z$	from d and h, by definition of $\succ$
	$\therefore$	$x \succ y$ and $y \succ z \rightarrow x \succ z$	from a to i, by logic

(5)	a.	$x \succ y$	by assumpion
	b.	$x \succsim y$	from a, by definition of $\succ$
	c.	suppose $y \succ x$ is true	prove by contradiction (we aim to show $y \succ x$ is false)
	d.	$x \not\succsim y$	from c, by definition of $\succ$
	e.	$y \nsucc x$	d contradicts with b, thus, $y \succ x$ in c is false
	$\therefore$	$x \succ y \rightarrow y \nsucc x$	from a to e, by logic

(6)	a.	suppose $x \succ x$ is true	prove by contradiction (we aim to show $x \succ x$ is false)
	b.	$x \succsim x$	from a, by definition of $\succ$
	c.	$x \not\succsim x$	from a, by definition of $\succ$
	$\therefore$	not $x \succ x$	c contradicts with b, thus, $x \succ x$ is false

(7)	a.	$x \succ y$ and $y \succsim z$	by assumption
	b.	$x \succsim y$	from a, by definition of $\succ$
	c.	$y \not\succsim x$	from a, by definition of $\succ$
	d.	$x \succsim z$	from a and b, by axiom of transitivity of $\succsim$
	e.	suppose $z \succsim x$ is true	prove by contradiction (we aim to show $z \succsim x$ is false)
	f.	$y \succsim x$	from a and e, by by definition of $\succsim$
	g.	$z \not\succsim x$	g contradicts with f, thus, $z \succsim x$ in c is false
	h.	$x \succ z$	from d and g, by definition of $\succ$
	$\therefore$	$x \succ y \succsim z \rightarrow x \succ z$	from a to h, by logic

Solution to Question II

Recall:

A utility function representing a preference relation $\succsim$ is not unique. For any strictly increasing function $f : \mathbb{R} \rightarrow \mathbb{R}$, $v(x) = f(u(x))$ is a new utility function representing the same preference as $u(x)$.

(1) $f(x) = u(x) + [u(x)]^3$

$$\frac{\partial f(x)}{\partial u(x)} = 1 + 3[u(x)]^2 > 0$$

This is a strictly increasing function, and it represents the same $\succsim$.

(2) $f(x) = u(x) - [u(x)]^2$

$$\frac{\partial f(x)}{\partial u(x)} = 1 - 2[u(x)]$$

Situation 1: When $u(x) \geq \frac{1}{2}$, $\frac{\partial f(x)}{\partial u(x)} \leq 0$, thus, $f(x)$ does not represent the same $\succsim$.
Situation 2: When $u(x) < \frac{1}{2}$, $\frac{\partial f(x)}{\partial u(x)} > 0$, thus, $f(x)$ represents the same $\succsim$.

(3) ~~$f(x) = u(x) + \sum_{i=1}^{n} x_i$~~
~~Not necessarily. $f(x)$ represents the same $\succsim$ as $u(x)$ if and only if $u(x)$ is a monotonic transformation of $\sum_{i=1}^{n} x_i$. Let us take an counter example. Suppose $u(x) = x_1$.~~
~~Then, u(1,0, . . . ,0) > u(0,3, . . . ,0). However, f(1,0, . . . ,0)<f(0,3, . . . ,0).~~

(2022.10.25) Correction on Question II-(3):

The solution to Question II-(3) presented above (as well as presented in the seminar on last Friday by me) is not correct. The reason is that, this question considers monotonic preferences — however, the example consumption bundle $x' = \{1, 0, \ldots, 0\}$ and $x' = \{0, 3, \ldots, 0\}$ I raised in Friday's seminar does not satisfy this condition.

Thanks @ Yudong QIU (20518076) <hmxyq1@nottingham.edu.cn> for pointing out this issue. He also contributes the correct solution as below:

$f(x)$ represents the same $\succsim$ as $u(x)$ in Question II-(3) because:

(i) Suppose $x = \{x_1, x_2, ..., x_n\}$, $y = \{y_1, y_2, ..., y_n\}$ are two consumption bundles. We further assume $y \geq x$ (*i.e.*, bundle y contains at least as much of every commodity as bundle x). Then by Strict Monotonicity (Axiom 4 in JR, pp.10), $y \succ x$.
(ii) Given $y \succ x$, by DEFINITION 1.5 (JR, pp.13), $u(y) \geq u(x)$.
(iii) Also note that $\sum_{i=1}^{n} x_i$ (or, $\sum_{i=1}^{n} y_i$) is the total amount of the commodity in bundle x (or, bundle y). $y \geq x$ in (i) also implies $\sum_{i=1}^{n} y_i \geq \sum_{i=1}^{n} x_i$.
(iv) By point (ii) and (iii), we can see $u(y) + \sum_{i=1}^{n} y_i \geq u(x) + \sum_{i=1}^{n} x_i$. It implies $f(y) > f(x)$, which is consistent with $u(y) \geq u(x)$ in point (ii). This means that $f(x)$ represents the same $\succsim$ as $u(x)$.

Solution to Question III

(1) There are four possible bundles: $(2,2)$, $(2,1)$, $(1,2)$, $(1,1)$.
Lexicographic preferences require $(2,2) \succ (2,1) \succ (1,2) \succ (1,1)$.
Therefore, any utility representation that satisfies $u(2,2) > u(2,1) > u(1,2) > u(1,1)$ would be a valid representation of Daniel's preferences.

(2) To show $u(a,b) = a + 1 - \frac{1}{b}$ is a vaild numerical representation of Daniel's preferences:

(i) Suppose $a > a'$.
Show $u(a,b) > u(a',b')$ for $a > a'$ and any b, b'.
This is true if $a + 1 - \frac{1}{b} > a' + 1 - \frac{1}{b'}$.
So we have to verify $a - a' > \frac{1}{b} - \frac{1}{b'}$
Because we only consider all possible natural numbers: $1, 2, 3, \ldots$.
So, if $a > a' \Rightarrow a - a' \geq 1$
and $b \geq 1 \Rightarrow 0 < \frac{1}{b} \leq 1$, same for $\frac{1}{b'}$, $\Rightarrow |\frac{1}{b} - \frac{1}{b'}| < 1$.
This proves that if $a > a' \Leftrightarrow u(a,b) > u(a',b')$ for any b, b'.

(ii) Suppose $a = a', b > b'$.
Show $u(a,b) > u(a',b')$ for $a = a'$ and $b > b'$.
This is true if $a + 1 - \frac{1}{b} > a' + 1 - \frac{1}{b'}$.
Since $a = a' \Rightarrow$ need to verify $\frac{1}{b'} > \frac{1}{b}$.
Since $b > b'$, b, b' are positive natural numbers, $\Rightarrow \frac{1}{b'} > \frac{1}{b}$
This proves that if $a = a', b > b' \Leftrightarrow u(a,b) > u(a',b')$.

(3) No, it cannot. Because $a - a' \geq 1$ in (2)-(i) would not be true if we consider all positive real numbers.

Solution to Question IV

(1) Marshallian demand functions.
Utility maximazation problem is defined as below:

$$\begin{aligned} \max \quad & u(x_1, x_2) = x_1^a x_2^{1-a} \\ \text{s.t} \quad & p_1 x_1 + p_2 x_2 \le w \end{aligned}$$

Setting up the Lagrangian:

$$\mathcal{L} = x_1^a x_2^{1-a} + \lambda(w - p_1 x_1 - p_2 x_2)$$

The first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial x_1} = a x_1^{a-1} x_2^{1-a} - \lambda p_1 = 0 \Rightarrow \lambda = \frac{a x_1^{a-1} x_2^{1-a}}{p_1} \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = (1-a) x_1^a x_2^{-a} - \lambda p_2 = 0 \Rightarrow \lambda = \frac{(1-a) x_1^a x_2^{-a}}{p_2} \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = w - p_1 x_1 - p_2 x_2 = 0 \Rightarrow x_2 = \frac{w - p_1 x_1}{p_2} \tag{3}$$

Solving x_1^* and x_2^*:

$$\text{use (1) and (2)}: \frac{a x_1^{a-1} x_2^{1-a}}{p_1} = \frac{(1-a) x_1^a x_2^{-a}}{p_2} \Rightarrow x_2 = \frac{1-a}{a} \frac{p_1}{p_2} x_1 \tag{4}$$

$$\text{use (3) and (4)}: \frac{w - p_1 x_1}{p_2} = \frac{1-a}{a} \frac{p_1}{p_2} x_1 \Rightarrow x_1^* = \frac{aw}{p_1} \tag{5}$$

$$\text{use (4) and (5)}: x_2^* = \frac{(1-a)w}{p_2} \tag{6}$$

(2) Indirect utility function.
Recall:
The indirect utility function is the maximum-value function corresponding to the consumer's

utility maximization problem.

$$\begin{aligned} v(p,w) &= u(x_1^*, x_2^*) \\ &= \left(\frac{aw}{p_1}\right)^a \left(\frac{(1-a)w}{p_2}\right)^{1-a} \\ &= w\left(\frac{a}{p_1}\right)^a \left(\frac{1-a}{p_2}\right)^{1-a} \end{aligned}$$

(3) Recall Roy's identity:

$$\frac{\partial v/\partial p_i}{\partial v/\partial w} = -x_i(p,w)$$

Consider good 1 as an example:

$$\frac{\frac{\partial v(p,w)}{\partial p_i}}{\frac{\partial v(p,w)}{\partial w}} = \frac{-\frac{a^2}{p_1^2} w \left(\frac{a}{p_1}\right)^{a-1} \left(\frac{1-a}{p_2}\right)^{1-a}}{\left(\frac{a}{p_1}\right)^a \left(\frac{1-a}{p_2}\right)^{1-a}} = -\frac{a}{p_1} w = -x_1(p,w)$$

(4) Dual consumer problem is also named as "cost minimization problem".

$$\begin{aligned} \min \quad & p_1x_1 + p_2x_2 \\ \text{s.t} \quad & x_1^a x_2^{1-a} \geq \overline{u} \end{aligned}$$

(5) Hicksian demand functions.
Setting up the Lagrangian:

$$\mathcal{L} = p_1x_1 + p_2x_2 + \lambda(\overline{u} - x_1^a x_2^{1-a})$$

The first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial x_1} = -p_1 - \lambda a x_1^{a-1} x_2^{1-a} = 0 \Rightarrow \lambda = -\frac{a x_1^{a-1} x_2^{1-a}}{p_1} \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = -p_2 - \lambda(1-a) x_1^a x_2^{-a} = 0 \Rightarrow \lambda = -\frac{(1-a) x_1^a x_2^{-a}}{p_2} \tag{8}$$

$$\frac{\partial \mathcal{L}}{\lambda} = \overline{u} - x_1^a x_2^{1-a} = 0 \Rightarrow \overline{u} = \left(\frac{x_1}{x_2}\right)^{a-1} x_1 \tag{9}$$

Solving h_1 and h_2:

$$\text{use (7) and (8)}: \frac{a x_1^{a-1} x_2^{1-a}}{p_1} = \frac{(1-a) x_1^a x_2^{-a}}{p_2} \Rightarrow \frac{x_1}{x_2} = \frac{p_2}{p_1}\frac{a}{1-a} \tag{10}$$

$$\text{use (9) and (10)}: \overline{u} = \left(\frac{p_2}{p_1}\frac{a}{1-a}\right)^{a-1} x_1 \Rightarrow h_1 = \overline{u}\left(\frac{p_2}{p_1}\frac{a}{1-a}\right)^{1-a} \tag{11}$$

$$\text{use (10) and (11)}: h_2 = \overline{u}\left(\frac{p_1}{p_2}\frac{1-a}{a}\right)^{a} \tag{12}$$

(6) Expenditure function.

$$\begin{aligned}
e(p,\overline{u}) &= p_1 h_1(p,\overline{u}) + p_2 h_2(p,\overline{u}) \\
&= p_1\overline{u}\left(\frac{p_2}{p_1}\frac{a}{1-a}\right)^{1-a} + p_2\overline{u}\left(\frac{p_1}{p_2}\frac{1-a}{a}\right)^{a} \\
&= \overline{u} p_1^a p_2^{1-a}\left[\left(\frac{a}{1-a}\right)^{1-a} + \left(\frac{1-a}{a}\right)^{a}\right] \\
&= \overline{u} p_1^a p_2^{1-a}\left[(1-a)^a a^{1-a}\frac{1}{(1-a)a}\right] \\
&= \overline{u} p_1^a p_2^{1-a}\left[\frac{1}{a^a(1-a)^{1-a}}\right]
\end{aligned}$$

(7) Indirect utility function.
We know that,

$$e(p, v(p,w)) = w.$$

Therefore,

$$v(p,w) p_1^a p_2^{1-a}\left[\frac{1}{a^a(1-a)^{1-a}}\right] = w \Rightarrow v(p,w) = w\left(\frac{a}{p_1}\right)^a\left(\frac{1-a}{p_2}\right)^{1-a}$$

(8) Price elasticity of good 1.

$$\begin{aligned}
\epsilon_{12} &= \frac{\partial x_1(p,w)}{\partial p_2}\frac{p_2}{x_1(p,w)} \\
&= \frac{\partial(aw/p_1)}{\partial p_2}\frac{p_2}{aw/p_1} \\
&= 0
\end{aligned}$$

(2022.10.25) A comment with respect to Question IV:

Question IV askes us to set up the consumer problem (or, utility maximization problem) and the dual consumer problem (or, cost minimization problem), then askes us to explore the relationships among Marshallian demand function(s), Hicksian demand function(s), Indirect utility function, and Expenditure function. The graph below summarizes those relationships — although not all of them are reflected in Question IV, and you might refer to text book for detailed explanations.