Problem set 3 Solution*

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I. Consider an exchange economy with two agents, 1 and 2, and two goods, x_1 and x_2. Suppose $u^1(x_1^1, x_2^1) = x_1^1 x_2^1$, and $u^2(x_1^2, x_2^2) = x_1^2 x_2^2$. Let the endowments of the two agents be $e^1 \equiv (e_1^1, e_2^1) = (3, 2)$ and $e^2 \equiv (e_1^2, e_2^2) = (2, 3)$. Draw an Edgeworth box representation of this economy, illustrating

(1) the endowment point $e \equiv (e^1, e^2)$,

(2) the indifference curves passing through e for both agents,

(3) the set of contract curve, and

(4) the set of barter equilibria.

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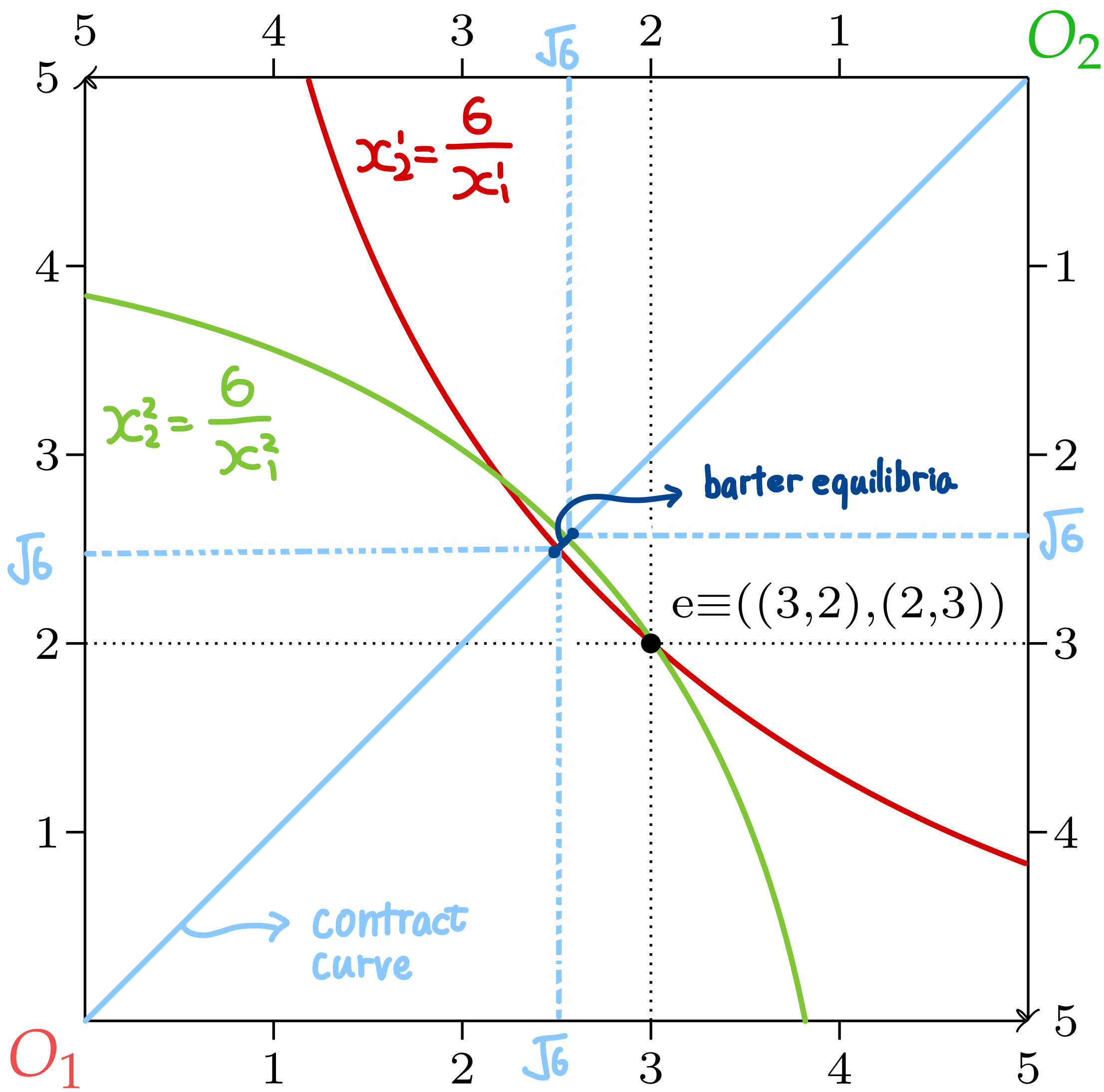
†Do not hesitate to email me when you find errors like typos, *etc*. Any updates of this problem set will be notified via Moodle announcements.

Solution:


O_1
O_2
$x_2^1 = \frac{6}{x_1^1}$
$x_2^2 = \frac{6}{x_1^2}$
barter equilibria
e≡((3,2),(2,3))
contract curve
√6
√6
√6
√6
0 1 2 3 4 5
5 4 3 2 1

II. **(JR 5.18)** In a two-good, two-consumer economy, utility functions are

$$u^1(x_1, x_2) = x_1(x_2)^2 \text{ and } u^2(x_1, x_2) = (x_1)^2 x_2.$$

Total endowments are (10, 20).

(1) A social planner wants to allocate goods to maximise consumer 1's utility while holding consumer 2's utility at $u^2 = 8000/27$. Find the assignment of goods to consumers that solves the planners problem and show that the solution is Pareto efficient.

Solution:

Note that $x_1^1 = 10 - x_2^1$ and $x_1^2 = 20 - x_2^2$. We will use x_2^1, x_2^2 as our choice variables, and cast the problem as a contstrained optimization problem:

$$\max_{x_1, x_2} \quad (10 - x_1)(20 - x_2)^2$$
$$\text{s.t} \quad (x_1)^2(x_2) = \frac{8000}{27}$$

Setting up the Lagrangian we have:

$$\mathcal{L}(x_1, x_2, \lambda) = (10 - x_1)(20 - x_2)^2 + \lambda\left(\frac{8000}{27} - (x_1)^2(x_2)\right)$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial x_1} = -(20 - x_2)^2 - 2\lambda x_1 x_2 = 0 \tag{1}$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = -2(10 - x_1)(20 - x_2) - \lambda(x_1)^2 = 0 \tag{2}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = \frac{8000}{27} - (x_1)^2(x_2) = 0 \tag{3}$$

Rearranging (1) and (2), and dividing (1) by (2), we get:

$$\frac{1}{2}\frac{20 - x_2}{10 - x_1} = 2\frac{x_2}{x_1} \tag{4}$$

Note that this is simply specifying that the MRS of $u^1(\cdot)$ is equal to the MRS of $u^2(\cdot)$, and in fact u^1 and u^2 are transformations of Cobb-Douglas utility. Therefore, we know that any allocation that satisfies the first order conditions is on the contract curve, and therefore

Pareto-optimal. Rearrange $(x_1)^2(x_2) = \frac{8000}{27}$ to get:

$$x_2 = \frac{8000}{27(x_1)^2}, \tag{5}$$

and plug into (4):

$$\begin{aligned}
4\frac{x_2}{x_1} &= \frac{20 - x_2}{10 - x_1} \\
4x_2(10 - x_1) &= x_1(20 - x_2) \\
20x_1 + 3x_1x_2 - 40x_2 &= 0 \\
20x_1 + 3x_1\frac{8000}{27(x_1)^2} - 40\frac{8000}{27(x_1)^2} &= 0 \\
540(x_1)^3 + 24000x_1 - 320000 &= 0
\end{aligned}$$

The only real solution to this cubic equation is $x_1 = \frac{20}{3}$. By (5), $x_2 = \frac{20}{3}$. Note that here the x_1 and x_2 in fact refer x_1^2 and x_2^2, respectively. We derive $x_1^1 = 10 - x_1^2 = \frac{10}{3}$, and $x_2^1 = 20 - x_2^2 = \frac{40}{3}$. We can verify that $u^2(\frac{20}{3}, \frac{20}{3}) = \frac{8000}{27}$.

(2) Suppose, instead, that the planner just divides the endowments so that $e^1 = (10, 0)$ and $e^2 = (0, 20)$ and then lets the consumers transact through perfectly competitive markets. Find the Walrasian equilibrium and show that the WEAs are the same as the solution in part (1).

Solution:

Suppose that the Walrasian equilibrium given an initial allocation of $e^1 = (10, 0)$, $e^2 = (0, 20)$ is the solution in the previous question, $x_1^1 = \frac{10}{3}$, $x_2^1 = \frac{40}{3}$. The price line must go through both points, and has a slope of $\frac{\frac{40}{3} - 20}{0 - \frac{10}{3}} = 2$. We can verify that this is equal to the MRS of both agents at the solution:

$$MRS_1 = \frac{1}{2}\frac{40/3}{10/3} = 2 \tag{6}$$

$$MRS_2 = 2\frac{20/3}{20/3} = 2 \tag{7}$$

Therefore, the price line is tangent to the indifference curves of both agents, and they are solving their utility maximization problem at the allocation. Therefore, the allocation is a Walrasian equilibrium.

III. Consider the following Robinson Crusoe economy. Robinson the consumer is endowed with zero units of coconuts, x, and 24 hours of time, h, so that $e = (0, 24)$. His preferences are defined over $\mathbb{R}^2_+$ and represented by $u(x, h) = xh$. Robinson the producer uses the consumer's labor services, l, to produce coconuts, y, according to the production function $y = \sqrt{l}$. The producer sells the coconuts to the consumer. All profits from the production and sale of coconuts are distributed to the consumer.

(1) Let p denote the price of coconuts, and normalize $p = 1$. Let w denote the price of Robinson's time. Find the Walrasian equilibrium prices and allocation of this economy.

Solution:

The firm wishes to maximize profit by choosing an amount of output y and labor l so that the firm's problem is:

$$\begin{aligned} \max_{y \geq 0, l \geq 0} \quad & py - wl \\ \text{s.t} \quad & y = \sqrt{l} \end{aligned}$$

Substituting in for output (since the constraint will hold with equality) we can make the firm's problem a function of only l so that we have:

$$\max_{l \geq 0} \quad p\sqrt{l} - wl$$

The first-order condition is:

$$\frac{d\pi}{dl} = \frac{1}{2} p l^{-\frac{1}{2}} - w \tag{8}$$

$$\frac{d\pi}{dl} = 0 \Rightarrow l = \frac{p^2}{4w^2} = \left(\frac{1}{2w}\right)^2 \quad \text{(Normalize } p = 1.) \tag{9}$$

The producer's supply is:

$$y = \sqrt{l} \tag{10}$$

$$y = \sqrt{\frac{p^2}{4w^2}} \tag{11}$$

$$y = \frac{p}{2w} = \frac{1}{2w} \quad \text{(Normalize } p = 1.) \tag{12}$$

Firm profits are (plug (9) and (12) into profit function):

$$\pi = py - wl$$
$$\pi = p\frac{p}{2w} - w\frac{p^2}{4w^2}$$
$$\pi = \frac{p^2}{4w}$$
$$\pi = \frac{1}{4w} \quad \text{(Normalize } p = 1.\text{)}$$

Notice that if $p > 0$ and $w > 0$ then $\pi > 0$ so it is okay that we imposed that the producer's first-order condition is equal to zero.
The consumer's problem is:

$$\max_{x \geq 0, 24 \geq h \geq 0} \quad xh$$
$$\text{s.t} \quad px + wh = x + wh = 24w + \frac{1}{4w} \quad \text{(Normalize } p = 1.\text{)}$$

Since the consumer's budget constraint holds with equality we can substitute in for either x or h (I will choose x), so that:

$$x = 24w + \frac{1}{4w} - wh$$

The consumer now solves:

$$\max_{24 \geq h \geq 0} \quad \left(24w + \frac{1}{4w} - wh\right) h$$
$$\max_{24 \geq h \geq 0} \quad 24wh + \frac{h}{4w} - wh^2$$

The consumer's first-order condition is:

$$\frac{du}{dh} = 24w + \frac{1}{4w} - 2wh$$
$$0 = 24w + \frac{1}{4w} - 2wh$$
$$h = 12 + \frac{1}{8w^2}$$

So that now we have that x is:

$$\begin{aligned}
x &= 24w + \frac{1}{4w} - wh \\
x &= 24w + \frac{1}{4w} - w\left(12 + \frac{1}{8w^2}\right) \\
x &= 12w + \frac{1}{8w}
\end{aligned}$$

Now, setting the equilibrium condition for the hours market we have:

$$\begin{aligned}
h + l &= 24 \\
12 + \frac{1}{8w^2} + \left(\frac{1}{2w}\right)^2 &= 24 \\
\frac{1}{8w^2} + \left(\frac{1}{2w}\right)^2 &= 12 \\
w^2 &= \frac{1}{32} \\
w &= \frac{1}{4\sqrt{2}}
\end{aligned}$$

So that the equilibrium price vector $(p^*, w^*) = \left(1, \frac{1}{4\sqrt{2}}\right)$. Now finding l, h, y, x, and π we have:

$$\begin{aligned}
l^* &= \left(\frac{1}{2w}\right)^2 = \frac{1}{4\left(\frac{1}{4\sqrt{2}}\right)^2} = 8 \\
h^* &= 12 + \frac{1}{8w^2} = 12 + \frac{1}{8\left(\frac{1}{4\sqrt{2}}\right)^2} = 16 \\
y^* &= \frac{1}{2w} = \frac{1}{2\frac{1}{4\sqrt{2}}} = 2\sqrt{2} \\
x^* &= 12w + \frac{1}{8w} = 12\frac{1}{4\sqrt{2}} + \frac{1}{8\frac{1}{4\sqrt{2}}} = 2\sqrt{2} \\
\pi^* &= \frac{1}{4w} = \frac{1}{4\frac{1}{4\sqrt{2}}} = \sqrt{2}
\end{aligned}$$

(2) Now suppose that Robinson does not think about a market, but simply chooses to enjoy h hours of leisure and spend $24 - h$ hours collecting coconuts. What is his optimal choice of h? How many coconuts does he get? Compare your answer to the answer to part (1).

Solution:

Robinson's problem is

$$\begin{aligned}
\max_{x\geq 0, 24\geq h\geq 0} \quad & xh \\
\text{s.t} \quad & x=\sqrt{24-h} \\
\max_{24\geq h\geq 0} \quad & h\sqrt{24-h}
\end{aligned}$$

Taking the derivative with respect to h we have:

$$\begin{aligned}
\frac{du}{dh} &= \sqrt{24-h}+h\frac{1}{2}(24-h)^{-\frac{1}{2}}(-1) \\
0 &= \sqrt{24-h}+h\frac{1}{2}(24-h)^{-\frac{1}{2}}(-1) \\
h\frac{1}{2}(24-h)^{-\frac{1}{2}} &= (24-h)^{\frac{1}{2}} \\
h &= 2(24-h) \\
h &= 16
\end{aligned}$$

Since $h=16$, we have $x=\sqrt{24-h}=\sqrt{24-16}=2\sqrt{2}$. So the answer is the same as in question 1.

IV. **(JR 7.10)** Calculate the set of Nash equilibria in the following games.

(1) Game 1:

	L	R
U	1,1	0,0
D	0,0	0,0

Also show that there are two Nash equilibria, but only one in which neither player plays a weakly dominated strategy.

Solution:

There are two pure NE, (U,L) and (D,R). (U,L) is one where neither player plays a weakly dominated strategy.
Let's find the set of mixed strategy NE. Let p be Player 1's probability of playing U, and q be Player 2's probability of playing L. The expected payoffs to pure strategies of each player are:

$$E_1(U) = q, E_1(D) = 0$$

$$E_2(L) = p, E_2(R) = 0$$

If $E_1(U) < E_1(D)$, then Player 1's best response is $p = 0$. If $E_1(U) = E_1(D)$, all $p \in [0, 1]$ is a best response. If $E_1(U) > E_1(D)$, Player 1's best response is $p = 1$.

$E_1(U) = E_1(D)$ when $q = 0$ (therefore all $p \in [0, 1]$ is a best response of Player 1); when $q > 0$, Player 1's best response is $p = 1$. Likewise, $E_2(L) = E_2(R)$ when $p = 0$ (therefore all $q \in [0, 1]$ is a best response of Player 2); when $p > 0$, Player 2's best response is $q = 1$. The only intersections are $p = q = 0$ and $p = q = 1$, which are the two pure NE we have already found.

(2) Game 2:

	L	R
U	1,1	0, 1
D	1,0	-1,-1

Also show that there are infinitely many Nash equilibria, only one of which has neither player playing a weakly dominated strategy.

Solution:

There are three pure NE, (U,L), (U,R), and (D,L). Only (U,L) has neither player playing a weakly dominated strategy.

Let p be Player 1's probability of playing U, and q be Player 2's probability of playing L. The expected payoffs to pure strategies of each player are:

$$E_1(U) = q, E_1(D) = 2q - 1$$

$$E_2(L) = p, E_2(R) = 2p - 1$$

$E_1(U) = E_1(D)$ when $q = 1$ (therefore all $p \in [0,1]$ is a best response of Player 1); when $q < 1$, Player 1's best response is $p = 1$. Likewise, $E_2(L) = E_2(R)$ when $p = 1$ (therefore all $q \in [0,1]$ is a best response of Player 2); when $p < 1$, Player 2's best response is $q = 1$.

The set of intersections is: $p = 1$, $q \in [0,1]$ and $p \in [0,1]$, $q = 1$. Any completely mixed strategy (i.e. not a pure strategy) is weakly dominated by the pure strategy U (for Player 1) or L (for Player 2). Therefore, the only NE where no player plays a weakly dominated strategy is (U,L), which we have already found.

(3) Game 3:

	L	l	m	M
U	1,1	1,2	0,0	0,0
C	1,1	1,1	10,10	-10,-10
D	1,1	-10,-10	10,-10	1,-10

Also show that there is a unique strategy determined by iteratively eliminating weakly dominated strategies.

Solution:

M is strictly dominated by L and can be eliminated. The pure NE are (U, l), (C,m), and (D,L).

V. Suppose that there are 2 hunters, Fred and Barney. The hunters can go after big game or small game. If a hunter goes after small game then he catches small game for a payoff of 1. If he goes after big game and he hunts alone he fails to catch anything, for a payoff of 0. However, if a hunter hunts for big game and both hunters are hunting big game, then they have a hunting party and catch the big game for a payoff of 3 each.

(1) Write down the normal form version of this 2-player game.

Solution:

		Barney: Small game	Barney: Big game
Fred	Small game	1,1	1,0
	Big game	0,1	3,3

(2) What is (are) the pure strategy Nash equilibria (PSNE) of this game?

Solution:

Using the normal form of the game from part (1) and the method of best responses we can see that there are two PSNE to this game:

		Barney: Small game	Barney: Big game
Fred	Small game	$\boxed{1,1}$	1,0
	Big game	0,1	$\boxed{3,3}$

(a) Fred hunt small game and Barney hunt small game and (b) Fred hunt big game and Barney hunt big game.

(3) Is there a mixed strategy Nash equilibrium to this game? If so find it.

Solution:

Yes, there is. To find Barney's probabilities we need to set the expected value of Fred choosing to hunt small game or big game equal:

$$\begin{aligned}
E_F[Small] &= E_F[Big] \\
1 &= 0 * \sigma_{Small} + 3 * (1 - \sigma_{Small}) \\
1 &= 3 - 3\sigma_{Small} \\
\sigma_{Small} &= \frac{2}{3}
\end{aligned}$$

Note that I didn't use the probabilities for $E_F[Small]$ because Fred always receives 1 if he hunts small regardless of what Barney does.

For Fred's probabilities we will go through the exact same process because Barney has identical payoffs to Fred ($E_B[Small] = 1$ and $E_F[Big] = 0 * \sigma_{Small} + 3 * (1 - \sigma_{Small})$), so we have the following MSNE:

Fred chooses to hunt small game with probability $\frac{2}{3}$ and big game with probability $\frac{1}{3}$, while Barney chooses to hunt small game with probability $\frac{2}{3}$ and big game with probability $\frac{1}{3}$.

VI. **(JR 7.18)** Reconsider the two countries from the previous exercise, but now suppose that country 1 can be one of two types, 'aggressive' or 'non-aggressive'. Country 1 knows its own type. Country 2 does not know country 1's type, but believes that country 1 is aggressive with probability $\varepsilon > 0$. The aggressive type places great importance on keeping its weapons. If it does so and country 2 spies on the aggressive type this leads to war, which the aggressive type wins and justifies because of the spying, but which is very costly for country 2. When country 1 is non-aggressive, the payoffs are as before (*i.e.*, as in the previous exercise). The payoff matrices associated with each of the two possible types of country 1 are given below.

Country 1 is 'aggressive'
Probablility ε

	Spy	Don't Spy
Keep	10,-9	5,-1
Destroy	0,2	0,2

Country 1 is 'non-aggressive'
Probablility $1-\varepsilon$

	Spy	Don't Spy
Keep	-1,1	1,-1
Destroy	0,2	0,2

(1) What action must the aggressive type of country 1 take in any Bayesian-Nash equilibrium?

Solution:

For the 'aggressive' type, *Keep* is strictly dominant, so it will always be played.

(2) Assuming that $\varepsilon < \frac{1}{5}$, find the unique Bayes-Nash equilibrium. (Can you prove that it is unique?)

Solution:

Let p be the probability that the 'non-aggressive' type of Player 1 plays *Keep*. Player 2's expected payoff to pure strategies are:

$$E_2(Spy) = \epsilon(-9) + (1-\epsilon)(p + 2(1-p))$$

$$E_2(Don'tSpy) = \epsilon(-1) + (1-\epsilon)(-p + 2(1-p))$$

$$E_2(Spy) = E_2(Don'tSpy) \Rightarrow p = \frac{4\epsilon}{1-\epsilon}$$

Let us call the 'non-agressive' type as Player 3. Let q be the probability that Player 2 plays Spy. Player 3's expected payoffs to pure strategies are:

$$E_3(Keep) = -q + (1-q) = 1 - 2q$$

$$E_3(Destroy) = 0$$

$E_3(Keep) = E_3(Destroy)$ when $q = \frac{1}{2}$. Therefore, a Bayesian-Nash equilibrium is when $p = \frac{4\epsilon}{1-\epsilon}$, $q = \frac{1}{2}$.