Chapter 5
Continuous Random Variables

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Recap

X: the number of trials performed until we get r success, where p is the probability of success on each trial.

$$p(k) = \binom{k-1}{r-1}(1-p)^{k-r}p^r, \quad k = r, r+1, \ldots$$

- Negative Binomial distribution $X \sim \text{NB}(r, p)$
- Mean $\mu = \frac{r}{p}$, variance $\sigma^2 = \frac{r(1-p)}{p^2}$.
- If $r = 1$, Geometric distribution $X \sim \text{NB}(1, p) = \text{Geometric}(p)$
- Geometric distribution is memoryless.

Review: discrete distributions

Name	Range	pmf $p(x)$	mean	variance
Ber(p)	$\{0,1\}$	$p^x(1-p)^{1-x}$	p	$p(1-p)$
Bin(n,p)	$\{0,1,\dots,n\}$	$\binom{n}{x}p^x(1-p)^{n-x}$	np	$np(1-p)$
Pois(λ)	$\{0,1,2,\dots\}$	$e^{-\lambda}\ \frac{\lambda^x}{x!}$	λ	λ
Geometric(p)	$\{1,2,\dots\}$	$(1-p)^{x-1}p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
NegBin(r,p)	$\{r,r+1,\dots\}$	$\binom{x-1}{r-1}(1-p)^{x-r}p^r$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$

Continuous random variables

☞ A *continuous random variable* X can take any real value in $(-\infty, \infty)$.

Definition

X is a continuous random variable if there exists a **nonnegative** *function f defined for any $x \in (-\infty, \infty)$, such that for any set B of real numbers,*

$$P(X \in B) = \int_B f(x)dx$$

This function f is called the probability density function (pdf) of the random variable X.

Examples of continuous rv

- Rainfall amount for a year.
- Lifetime of your first car.
- Amount of beer consumed on a game day.

Pdf and cdf

- For pdf to be valid, in additional to being non-negative,

$$\int_{-\infty}^{\infty} f(x)dx = P(-\infty < X < \infty) = 1$$

- Cdf function of continuous rv

$$F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)dx$$

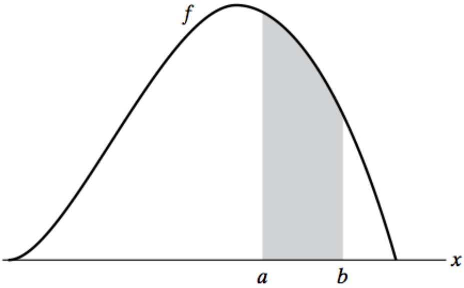
- For continuous rv, probability to be a single point is zero.

$$P(X = a) = \int_{a}^{a} f(x)dx = 0$$

$$P(X < a) = P(X \leq a) - P(X = a) = F(a)$$

- Probability on an interval

$$P(a \leq X \leq b) = F(b) - F(a) = P(a < X < b)$$



$P(a \le X \le b)$ = area of shaded region

Connection between pdf and cdf of continuous rv. If we know the pdf,

$$F(a) = P(X \le a) = \int_{-\infty}^{a} f(x)dx$$

How to find the pdf if we know the cdf?

$$f(x) = \frac{d}{dx}F(x)$$

Example: suppose that X is a continuous random variable whose pdf is

$$f(x) = \begin{cases} c(8x - 4x^3) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

- What's the value of c?

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f(x)dx \\ &= \int_0^1 c(8x - 4x^3)dx \\ &= c\left(4x^2 - x^4\Big|_0^1\right) \end{aligned}$$

$$3c = 1 \Longrightarrow c = \frac{1}{3}$$

Question

Suppose that X is a continuous random variable whose pdf is

$$f(x) = \begin{cases} c(8x - 4x^3) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

- Find $P(X > 0.5)$.

$$\begin{aligned} P(X > 0.5) &= \int_{0.5}^{\infty} f(x)dx \\ &= \int_{0.5}^{1} \frac{8x - 4x^3}{3} dx \\ &= \frac{1}{3}\left(4x^2 - x^4\Big|_{0.5}^{1}\right) \\ &= \frac{1}{3}\left[3 - 4 \times \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4\right] = \frac{1}{3} \times \frac{33}{16} = \frac{11}{16} \end{aligned}$$

Suppose that X is a continuous random variable whose pdf is

$$f(x) = \begin{cases} c(8x - 4x^3) & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- Find the cdf function $F(x)$.

$$\int_{-\infty}^{\infty} f(x)dx = 1 \Longrightarrow c = 1/3$$

$$F(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} 0 & x < 0 \\ \frac{4x^2 - x^4}{3} & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

Note that $F(0) = 0$ and $F(1) = 1$.

Interpretation of the pdf

For some small value $h > 0$,

$$
\begin{aligned}
P\left(x - \frac{h}{2} \leq X \leq x + \frac{h}{2}\right) &= \int_{x-\frac{h}{2}}^{x+\frac{h}{2}} f(t)dt \\
&\approx \left[(x + \frac{h}{2}) - (x - \frac{h}{2})\right] f(x) \\
&= h \cdot f(x)
\end{aligned}
$$

The larger $f(x)$ is, the more likely X is to be "near" x.

Recap

A continuous rv X can take more than countable number of values in $\mathbb{R}$.

- We defined continuous rv using pdf

$$P(X \in B) = \int_B f(x)dx$$

- Cdf function of continuous rv

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$$

$$f(x) = \frac{d}{dx}F(x)$$

Expectation and variance of continuous rv

- Recall that expected value of the discrete rv X with pmf $f(x)$,

$$E[X] = \sum_{\text{all } x} x f(x)$$

- We define expected value of a continuous rv X with pdf $f(x)$ as

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

- Definitions of variance and standard deviation are the same.

$$Var(X) = EX^2 - (EX)^2, \quad SD(X) = \sqrt{VarX}$$

Properties of $E(X)$ for continuous rv

- Recall that expected value of the discrete rv X with pmf $f(x)$,

$$E[g(X)] = \sum_{\text{all } x} g(x)f(x)$$

- Similarly, expected value of a continuous rv X with pdf $f(x)$ as

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

Example: show that for non-negative continuous rv X,

$$E(X) = \int_0^{\infty} P(X > x)dx$$

$$\begin{aligned}\int_0^{\infty} P(X > x)dx &= \int_0^{\infty} \left[\int_x^{\infty} f(y)dy\right] dx \\ &= \int_0^{\infty} \int_0^{y} f(y)dxdy = \int_0^{\infty} yf(y)dy = E(X)\end{aligned}$$

Properties of $E(X)$ for continuous rv

Similarly as discrete rv, for continuous rv X and Y

- Sum of two rv's

$$E[X+Y] = E[X] + E[Y]$$

- If a and b are constants, then

$$E[aX+b] = aE[X] + b$$

- If a and b are constants, then

$$Var(aX+b) = a^2 Var(X)$$

Question

Find the mean and variance of the continuous rv X, whose pdf is

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_0^1 x \cdot 2x dx = \frac{2}{3} x^3 \Big|_0^1 = \frac{2}{3} \end{aligned}$$

$$Var(X) = E[X^2] - (EX)^2 = \int_{-\infty}^{\infty} x^2 f(x) dx - (2/3)^2$$

$$= \int_0^1 x^2 \cdot 2x dx - 4/9 = \frac{2}{4} x^4 \Big|_0^1 - 4/9 = \frac{1}{18}$$

Question

X is a continuous rv. Its pdf $f(x)$ is an even function, i.e.,

$$f(-x) = f(x), \quad \text{for any } x > 0$$

What is $E[X]$?

(a) Cannot decide. Need more information.
(b) 0
(c) 1
(d) e

$$\begin{aligned}
EX &= \int_{-\infty}^{0} xf(x)dx + \int_{0}^{\infty} xf(x)dx \quad (\text{let } y = -x) \\
&= \int_{-\infty}^{0} (-y)f(-y)d(-y) + \int_{0}^{\infty} xf(x)dx \\
&= \int_{\infty}^{0} yf(-y)dy + \int_{0}^{\infty} xf(x)dx = -\int_{0}^{\infty} yf(-y)dy + \int_{0}^{\infty} xf(x)dx
\end{aligned}$$

Uniform Distribution

Let's define a continuous probability distribution that has some constant value c between α and β where $\alpha \leq \beta$. What is the pdf?

$$f(x) = \begin{cases} c & \text{if } \alpha \leq x \leq \beta \\ 0 & \text{otherwise} \end{cases}$$

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_{\alpha}^{\beta} c\,dx = c(\beta - \alpha) \Longrightarrow c = \frac{1}{\beta - \alpha}$$

Definition

A continuous rv X has a Uniform distribution on the interval (α, β) if its pdf is

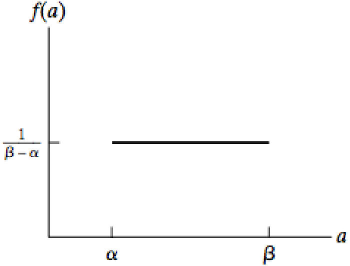
$$X \sim \mathit{Unif}(\alpha, \beta) \Longleftrightarrow f(x) = \frac{1}{\beta - \alpha} \cdot \mathbf{1}_{(\alpha,\beta)}(x) = \begin{cases} \frac{1}{\beta-\alpha} & \textit{if } \alpha < x < \beta \\ 0 & \textit{otherwise} \end{cases}$$

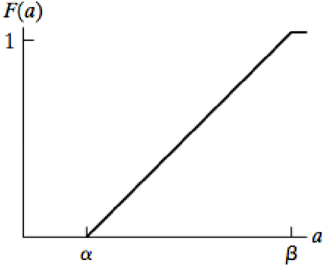
Cdf of Uniform distribution

For any $x \in (\alpha, \beta)$,

$$F(x) = \int_{-\infty}^{x} f(x)\ dx = \int_{\alpha}^{x} \frac{1}{\beta - \alpha}\ dt = \frac{x - \alpha}{\beta - \alpha}$$

- $X \sim$ Unif(α, β), then its cdf is $F(x) = \begin{cases} 0 & \text{if } x \leq \alpha \\ \frac{x-\alpha}{\beta-\alpha} & \text{if } \alpha < x < \beta \\ 1 & \text{if } x \geq \beta \end{cases}$





Mean of a Uniform distribution

- Expected value of $X \sim \text{Unif}(\alpha, \beta)$ is

$$E(X) = \frac{\alpha + \beta}{2}$$

Variance of a Uniform distribution

- Expected value of $X \sim \text{Unif}(\alpha, \beta)$ is

$$Var(X) = \frac{(\beta - \alpha)^2}{12}$$

Uniform distribution: probability calculation

If $X \sim \text{Unif}(\alpha, \beta)$, then $P(X \in B) = \dfrac{\text{length}(B)}{\beta - \alpha}$

Example: I want to take a orange route bus to the Fike gym this afternoon. The bus arrives at my nearest bus stop at 10-minute intervals starting at 4pm. Suppose I arrive the bus stop at a time uniformly distributed between 4pm to 5pm. What's the probability I will wait more than 7 minutes?

Let X denote the time I arrive the bus stop.

$$X \sim \text{Unif}(0, 60)$$

The bus arrives at time $t = 0, 10, 20, 30, 40, 50, 60.$
So the intervals of my arriving time such that I will wait more than 7 min:

$$B = (0,3) \cup (10,13) \cup (20,23) \cup (30,33) \cup (40,43) \cup (50,53)$$

$$P(X \in B) = \frac{3 \times 6}{60} = 0.3$$

Recap

Expectation for continuous rv X and a function of it $g(X)$

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx$$

Uniform distribution $X \sim \text{Unif}(\alpha, \beta)$

- Pdf
$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \text{if } \alpha < x < \beta \\ 0 & \text{otherwise} \end{cases}$$

- Cdf
$$F(x) = \begin{cases} 0 & \text{if } x \leq \alpha \\ \frac{x-\alpha}{\beta-\alpha} & \text{if } \alpha < x < \beta \\ 1 & \text{if } x \geq \beta \end{cases}$$

- Mean and variance
$$E[X] = \frac{\alpha+\beta}{2}, \quad Var(X) = \frac{(\beta-\alpha)^2}{12}$$

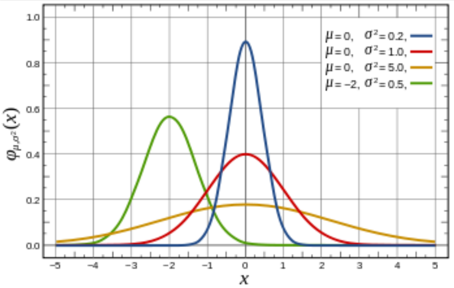
Normal Distribution

Definition

A continuous rv X has a Normal distribution distribution with mean μ and variance σ^2 if its pdf is

$$X \sim N(\mu, \sigma^2) \iff f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \text{ where } x \in \mathbb{R}$$

Pdf: unimodal and symmetric, bell shaped curve



Get yourself a 10 Deutsche Mark bill

The normal pdf is well-defined

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}\ e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$$

Example: use the fact that normal pdf is well define, calculate the integral

$$\int_{-\infty}^{\infty} e^{-\frac{(x-1)^2}{10}} dx = ?$$

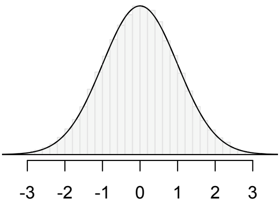
Suppose we have a rv $X \sim \mathsf{N}(\mu = 1, \sigma^2 = 5)$, then

$$1 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}\ e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\cdot 5}}\ e^{-\frac{(x-1)^2}{10}} dx$$

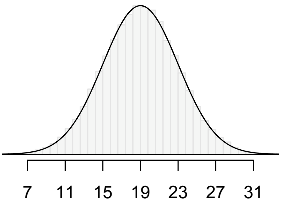
$$\Longrightarrow \int_{-\infty}^{\infty} e^{-\frac{(x-1)^2}{10}} dx = \sqrt{10\pi}$$

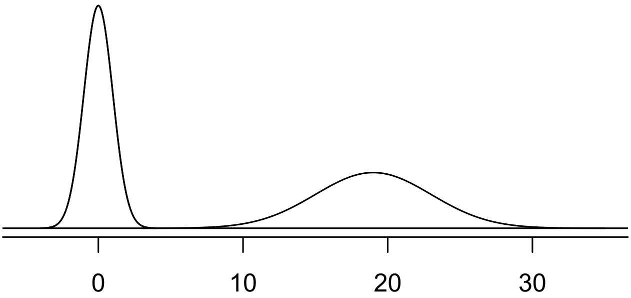
Normal distributions with different parameters

$N(\mu = 0, \sigma^2 = 1)$



$N(\mu = 19, \sigma^2 = 16)$





Mean and variance of Normal rv $X \sim \mathrm{N}(\mu, \sigma^2)$

If $X \sim \mathrm{N}(\mu, \sigma^2)$, and $Z = \frac{X-\mu}{\sigma}$, then Z has a standard normal distribution.

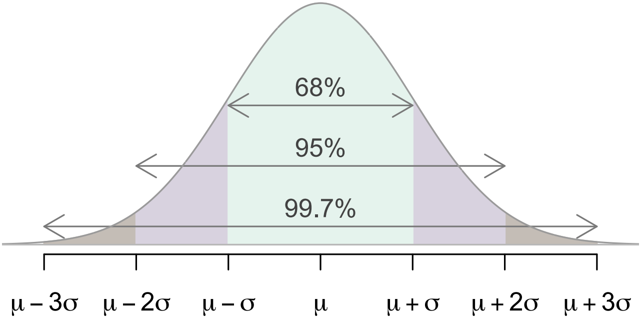
$$Z \sim \mathrm{N}(0, 1)$$

We are going to show this later, when we learn Chapter 5.7.

- $E[X] = \mu$
- $Var(X) = \sigma^2$

68-95-99.7 Rule

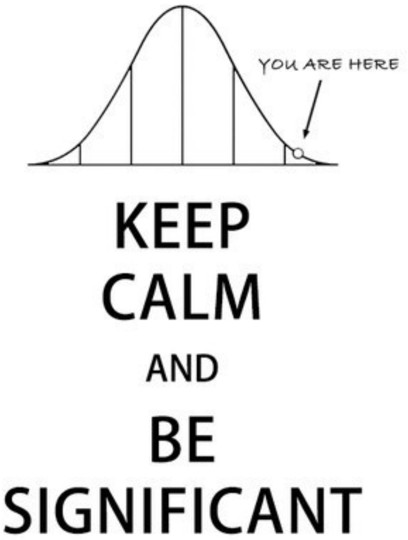
- A rv X has a normal distribution,
  - about 68% probability X falls within 1 SD of the mean,
  - about 95% probability X falls within 2 SD of the mean,
  - about 99.7% probability X falls within 3 SD of the mean.
- The probability of X falls 4, 5, or more standard deviations away from the mean is very low.



Normal probability calculation

- We denote $\phi(x)$ and $\Phi(x)$ as pdf and cdf of the standard normal distribution respectively.
- Probability calculations for X in terms of Z:

$$\begin{aligned} P(a < X \leq b) &= P\left(\frac{a-\mu}{\sigma} < Z \leq \frac{b-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right) \end{aligned}$$



Question

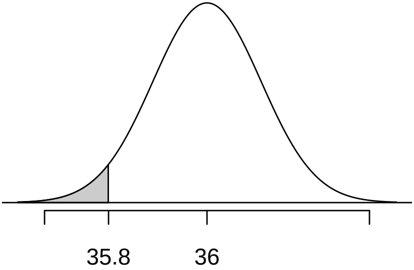
Using the 68-95-99.7 rule, find the standard normal cdf $\Phi(-2)$ and $\Phi(2)$

$$\begin{aligned}\Phi(-2) =& P(Z \leq -2) \\ =& \frac{1 - P(-2 < Z \leq 2)}{2} \\ =& \frac{1-0.95}{2} = 0.025\end{aligned}$$

$$\Phi(2) = 1 - \Phi(-2) = 0.975$$

At Heinz ketchup factory the amounts which go into bottles of ketchup are supposed to be normally distributed with mean 36 oz. and standard deviation 0.11 oz. Once every 30 minutes a bottle is selected from the production line, and its contents are noted precisely. If the amount of ketchup in the bottle is below 35.8 oz. or above 36.2 oz., then the bottle will fails the quality control inspection. What's the probability that the amount of ketchup in a randomly selected bottle is less than 35.8 ounces?

Let X = amount of ketchup in a bottle: $X \sim N(36, 0.11^2)$



$$z = \frac{x-\mu}{\sigma} = \frac{35.8-36}{0.11} = -1.82$$

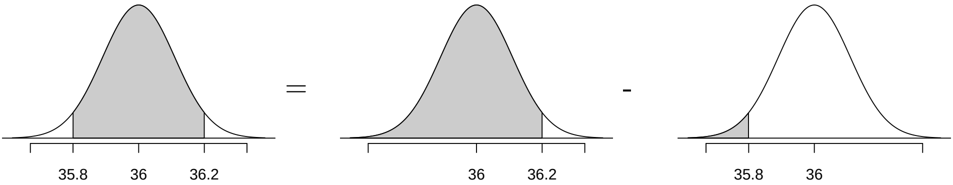
$$P(X < 35.8) = P(Z < -1.82) = 0.0344$$

0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.00	z
0.0143	0.0146	0.0150	0.0154	0.0158	0.0162	0.0166	0.0170	0.0174	0.0179	−2.1
0.0183	0.0188	0.0192	0.0197	0.0202	0.0207	0.0212	0.0217	0.0222	0.0228	−2.0
0.0233	0.0239	0.0244	0.0250	0.0256	0.0262	0.0268	0.0274	0.0281	0.0287	−1.9
0.0294	0.0301	0.0307	0.0314	0.0322	0.0329	0.0336	0.0344	0.0351	0.0359	−1.8
0.0367	0.0375	0.0384	0.0392	0.0401	0.0409	0.0418	0.0427	0.0436	0.0446	−1.7
0.0455	0.0465	0.0475	0.0485	0.0495	0.0505	0.0516	0.0526	0.0537	0.0548	−1.6
0.0559	0.0571	0.0582	0.0594	0.0606	0.0618	0.0630	0.0643	0.0655	0.0668	−1.5

Question

In the previous example, what percent of bottles pass the quality control inspection?

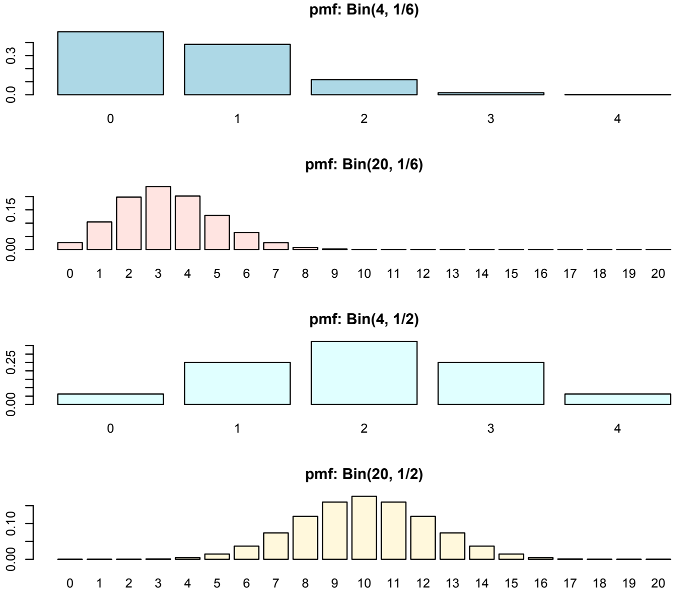
$$X \sim \mathsf{N}(36, 0.11^2), \quad P(35.8 < X < 36.2) = ?$$



$$z_1 = \frac{35.8 - 36}{0.11} = -1.82, \quad z_2 = \frac{36.2 - 36}{0.11} = 1.82$$

$$\begin{aligned} P(35.8 < X < 36.2) &= P(-1.82 < Z < 1.82) \\ &= P(Z < 1.82) - P(Z < -1.82) \\ &= 0.9656 - 0.0344 = 0.9312 \end{aligned}$$

Binomial densities



Normal approximation to the Binomial distribution

Let $X \sim \mathsf{Bin}(n,p)$. When n is large enough, or more specifically, both $np(1-p) \geq 10$, the binomial distribution can be approximated by the normal distribution

$$P(X=i) \approx P(i-0.5 < Y < i+0.5), \quad Y \sim \mathsf{N}(\mu, \sigma^2)$$

with parameters $\mu = np$ and $\sigma^2 = np(1-p)$.

- Probability calculation (actually, approximation)

$$\begin{aligned} P(a \leq X \leq b) &\approx P(a-0.5 < Y < b+0.5) \\ &= P\left(\frac{a-0.5-np}{\sqrt{np(1-p)}} < \frac{Y-\mu}{\sigma} < \frac{b+0.5-np}{\sqrt{np(1-p)}}\right) \\ &= \Phi\left(\frac{b+0.5-np}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{a-0.5-np}{\sqrt{np(1-p)}}\right) \end{aligned}$$

A recent study found that "Facebook users get more than they give". For example:

- Users in our sample pressed the like button next to friends' content an average of 14 times, but had their content "liked" an average of 20 times
- 12% of users tagged a friend in a photo, but 35% were themselves tagged in a photo

This is because there are "power users" who contribute much more content than the typical user. The same study found that approximately 25% of Facebook users are considered power users. It also found that the average Facebook user has 245 friends. What is the probability that the average Facebook user with 245 friends has 70 or more friends who would be considered power users?

- We are given that $X \sim \text{Bin}(n = 245, p = 0.25)$, and we are asked for the probability

$$P(X \geq 70) = p(70)+p(71)+\cdots+p(245) = 1-p(0)-p(1)-\cdots-p(69)$$

- To use normal approximation, first check conditions

$$np(1-p) = 245 \times 0.25 \times 0.75 = 45.94 \geq 10$$

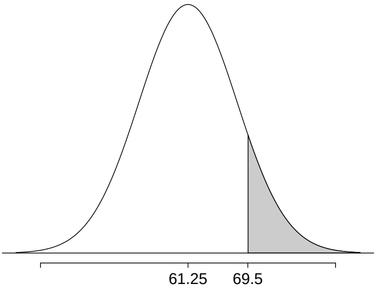
http://www.pewinternet.org/Reports/2012/Facebook-users/Summary.aspx

What is the probability that the average Facebook user with 245 friends has 70 or more friends who would be considered power users?

Use Normal approximation.

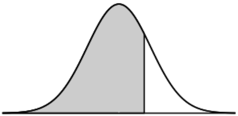
$$\mu = 245 \times 0.25 = 61.25, \quad \sigma = \sqrt{245 \times 0.25 \times 0.75} = 6.78$$

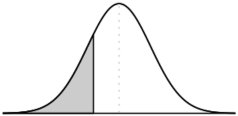
$$Y \sim \mathsf{N}(\mu, \sigma^2), \quad P(X \geq 70) \approx P(Y > 70 - 0.5)$$



$$\begin{aligned} P(X \geq 70) &\approx P(Y > 70 - 0.5) \\ &= P\left(Z > \frac{69.5 - 61.25}{6.78}\right) \\ &= P(Z > 1.22) \\ &= 1 - 0.8888 = 0.1112 \end{aligned}$$

Recap: Normal distribution $X \sim \mathsf{N}(\mu, \sigma^2)$





- Mean μ, variance σ^2
- Symmetry

$$f(\mu - x) = f(\mu + x), F(\mu - x) = 1 - F(\mu + x)$$

- Standard normal distribution $\mu = 0, \sigma^2 = 1.$

$$X \sim \mathsf{N}(\mu, \sigma^2) \Longleftrightarrow Z = \frac{X - \mu}{\sigma} \sim \mathsf{N}(0, 1)$$

- Find probability using $\Phi(\cdot)$ table

$$P(a < X < b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

- Normal approximation to Binomial $X \sim \mathsf{Bin}(n, p)$

$$Y \sim \mathsf{N}(\mu = np, \sigma^2 = np(1-p))$$

$$P(X = i) \approx P(i - 0.5 < Y < i + 0.5)$$

Exponential distribution

Definition

A continuous rv X has a Exponetial distribution with rate λ if its pdf

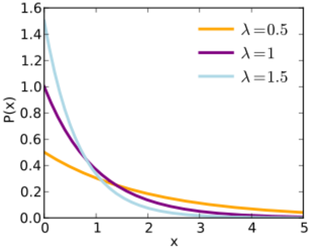
$$X \sim \mathit{Exp}(\lambda) \iff f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

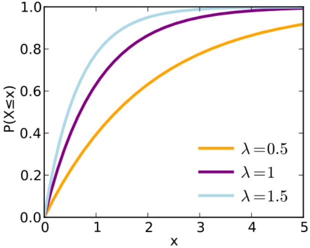
- Parameter $\lambda > 0$
- Range (possible values X can take): $[0, \infty)$
- Exponential rv: amount of time until some specific event occurs.

Examples

- Amount of time until a lightbulb dies.
- Amount of time until it rains.
- Amount of time until the next earthquake occurs in California.

Example: random variable $X \sim \text{Exp}(\lambda)$. Find its cdf.





$$F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Example: random variable $X \sim \text{Exp}(\lambda)$. Find its mean and variance.

$$E(X) = \frac{1}{\lambda}, \quad Var(X) = \frac{1}{\lambda^2}$$

- Parameter λ: rate.

Question

There are on average 3 earthquakes in California with Richter magnitude 6.0 or more occur per century. Let X denote the time (in century) until the next such earthquake occurs. Suppose $X \sim$Exp(λ).
(1) What's the value of λ?
(2) What's the probably that it occurs within 10 years?

$$E(X) = 1/\lambda = 1/3 \Longrightarrow \lambda = 3$$

$$P(X \leq 0.1) = F(0.1) = 1 - e^{-\lambda \times 0.1} = 1 - e^{-0.3} = 0.2592$$

Memoryless

$$P(X > t + s \mid X > t) = P(X > s)$$

Question

Quick review: which discrete distribution is memoryless?

(a) Binomial

(b) Poisson

(c) *Geometric*

(d) Negative Binomial

- Exponential distribution is memoryless

Question

Suppose no such earthquake occurs in 10 years, then what's the probability no such earthquake occurs in 20 years?

Since exponential distribution is memoryless,

$$P(X > 0.2 \mid X > 0.1) = P(X > 0.1) = 1 - P(X \leq 0.1) = 0.7408$$

Example: two light bulbs each have a lifetime that is Exp(λ). Both light bulbs are plugged in at time 0. How long do we expect light to last?

Let X_1, X_2 denote the lengths of the light bulbs, and L the length of light.

$$L = \max(X_1, X_2)$$

$$\begin{aligned} P(L \leq x) &= P[\max(X_1, X_2) \leq x] \\ &= P(X_1 \leq x \cap X_2 \leq x) \\ &= P(X_1 \leq x)P(X_2 \leq x) \end{aligned}$$

For any $x > 0$, $P(X_1 \leq x) = P(X_2 \leq x) = 1 - e^{-\lambda x}$, so

$$F_L(x) = \left(1 - e^{-\lambda x}\right)^2 \Longrightarrow f_L(x) = \frac{d}{dx} F_L(x) = 2\lambda e^{-\lambda x}\left(1 - e^{-\lambda x}\right)$$

$$E[L] = \int_0^\infty x f_L(x) dx = \frac{2}{\lambda} - \frac{1}{2\lambda} = \frac{3}{2\lambda}$$

Question

Let continuous random variables

- X_1 denote the amount of time until the next female student step in the Cooper library,
- X_2 denote the amount of time until the next male student step in the library, and
- Y denote the amount of time until the next student step into the library.

Which of the following is TRUE?

(a) *X_1 and X_2 have Exponential distributions; $Y = \min(X_1, X_2)$*

(b) X_1 and X_2 have Normal distributions; $Y = |X_1 - X_2|$

(c) X_1 and X_2 have Exponential distributions; $Y = X_1 + X_2$

(d) X_1 and X_2 have Exponential distributions; $Y = \max(X_1, X_2)$

Minimal of Independent Exponential Rv's

In the previous example, suppose that

$$X_1 \sim \text{Exp}(\lambda_1), \quad X_2 \sim \text{Exp}(\lambda_2),$$

independently, find the distribution of $Y = \min(X_1, X_2)$.

We first find the cdf $F_Y(\cdot)$. For any $x \geq 0$,

$$\begin{aligned} P(Y > x) &= P(X_1 > x \cap X_2 > x) \\ &= P(X_1 > x)P(X_2 > x) \\ &= e^{-\lambda_1 x} \cdot e^{-\lambda_2 x} = e^{-(\lambda_1+\lambda_2)x} \end{aligned}$$

So cdf of Y is $F_Y(x) = 1 - e^{-(\lambda_1+\lambda_2)x}$, therefore

$$Y \sim \text{Exp}(\lambda_1 + \lambda_2)$$

Gamma function

Definition

The *Gamma function* $\Gamma(\alpha)$ *is defined as*

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt, \text{ for } \alpha > 0$$

- $\Gamma(\alpha) < \infty$ for all $\alpha > 0$.
- Connection between $\Gamma(\alpha+1)$ and $\Gamma(\alpha)$:

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

- $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$.
- For any $n \in \mathbb{N}^+$,

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = \cdots = (n-1)!$$

Gamma distribution

Definition

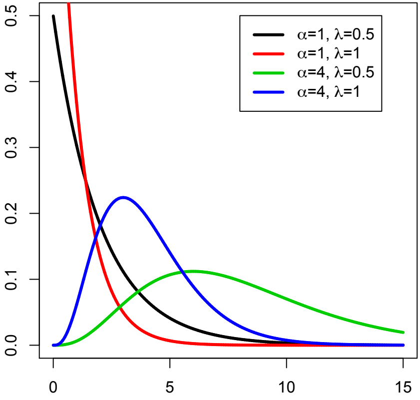
A continuous rv X has a Gamma distribution if its pdf

$$X \sim G(\alpha, \lambda) \iff f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

- Shape parameter $\alpha > 0$, and rate parameter $\lambda > 0$.
- Range: $[0, \infty)$
- Special cases
    - Exponential distribution

$$\text{Exp}(\lambda) = \text{G}(1, \lambda)$$

Random varible $X \sim \mathrm{G}(\alpha, \lambda)$. Find its mean and variance.



$$E(X) = \frac{\alpha}{\lambda}, \quad Var(X) = \frac{\alpha}{\lambda^2}$$

Question

The chi-square distribution with degrees of freedom n is a special case of Gamma distribution with parameters $\alpha = n/2$ and $\lambda = 1/2$. What's the mean and variance of $X \sim \chi^2(n)$?

$$X \sim \mathsf{G}\left(\alpha = \frac{n}{2}, \lambda = \frac{1}{2}\right)$$

$$E(X) = \frac{\alpha}{\lambda} = n, \quad Var(X) = \frac{\alpha}{\lambda^2} = 2n$$

Review: continuous distributions

Name	Range	pdf $f(x)$	mean	variance
G(α, λ)	$[0, \infty)$	$\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$
Unif(α, β)	$[\alpha, \beta]$	$\frac{1}{\beta-\alpha}$	$\frac{\alpha+\beta}{2}$	$\frac{(\beta-\alpha)^2}{12}$
N(μ, σ^2)	$(-\infty, \infty)$	$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$	μ	σ^2
Exp(λ)	$[0, \infty)$	$\lambda e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$

A (important!) theorem on finding pdf of $g(X)$

Suppose X is a continuous rv with pdf $f_X(x)$. If a function $g(x)$ is

1. monotonic (increasing or decreasing), and
2. differentiable (and thus continuous),

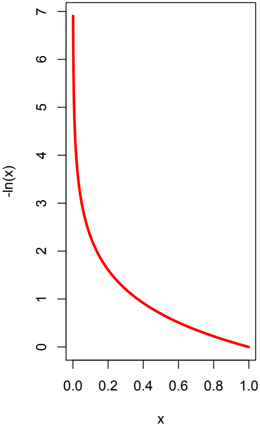
then the rv defined by $Y = g(X)$ has pdf

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

Or more rigorously,

$$f_Y(y) = \begin{cases} f_X\left[g^{-1}(y)\right] \cdot \left|\frac{d}{dy} g^{-1}(y)\right| & \text{if } y = g(x) \text{ for some } x \\ 0 & \text{if } y \neq g(x) \text{ for all } x \end{cases}$$

Example: let $X \sim$ Unif$(0,1)$, what distribution does $Y = -\ln(X)$ have?



In order to use the previous theorem, need to check the function $g(x) = -\ln(x)$

1. monotonic ✓
2. differentiable ✓

Inverse function $g^{-1}(y)$:

$$y = -\ln(x) \Longleftrightarrow x = e^{-y}$$

$$\frac{dx}{dy} = -e^{-y}$$

Range of Y: $y \in [0, \infty)$.

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = 1 \cdot \left|-e^{-y}\right| = e^{-y}$$

Therefor, Y has an exponential distribution,

$$Y \sim \text{Exp}(1)$$

Question

Example: let $X \sim \mathrm{N}(\mu, \sigma^2)$, what distribution does $Y = (X - \mu)/\sigma$ have?

In order to use the previous theorem, follow these steps:

1. Check if $g(x) = (x - \mu)/\sigma$ is monotonic and differentiable ✓
2. Compute the inverse function $y = g(x) \iff x = g^{-1}(y)$

$$x = \sigma y + \mu$$

3. Compute the derivative $\frac{dx}{dy} = \sigma$
4. Identify the range of the new random variable $Y = g(X)$. $y \in \mathbb{R}$
5. Apply the formula

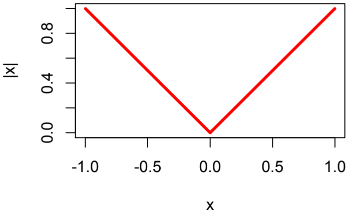
$$\begin{aligned} f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot |\sigma| \\ &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\sigma y+\mu-\mu)^2}{2\sigma^2}} \cdot \sigma = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \end{aligned}$$

Question

Let X have pdf $f_X(x)$, where $-\infty < x < \infty$. We want to find the pdf for $Y = |X|$. Can we use the formula $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$?

(a) Yes
(b) *No*

$g(x) = |x|$ is not monotonic; also not differentiable at 0.



First find cdf $F_Y(y)$, then find $f_Y(y)$.
For any $y \geq 0$,

$$
\begin{aligned}
F_Y(y) &= P(-y < X < y) \\
&= F_X(y) - F_X(-y)
\end{aligned}
$$

$$
\begin{aligned}
f_Y(y) &= \frac{d}{dy}[F_X(y) - F_X(-y)] \\
&= f_X(y) + f_X(-y)
\end{aligned}
$$

Another example when we can not apply the theorem

Let X have a standard normal distribution. What is the pdf of $Y = X^2$?

Two steps: (1) write the cdf $F_Y(y)$ in the form of $F_X(\cdot)$. For any $y \geq 0$,

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(X^2 \leq y) \\ &= P(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned}$$

2) Take derivative to get pdf $f_Y(y) = \frac{d}{dy} F_Y(y)$.

$$\begin{aligned} f_Y(y) &= \frac{d}{dy}\left[F_X(\sqrt{y}) - F_X(-\sqrt{y})\right] \\ &= f_X(\sqrt{y})\frac{1}{2\sqrt{y}} - f_X(-\sqrt{y})\left(-\frac{1}{2\sqrt{y}}\right) \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{y})^2}{2}} \frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{(-\sqrt{y})^2}{2}} \frac{1}{2\sqrt{y}} \\ &= \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-\frac{y}{2}} = \frac{\left(\frac{1}{2}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} y^{\frac{1}{2}-1} e^{-\frac{y}{2}} \Longrightarrow Y \sim \mathsf{G}\left(\frac{1}{2}, \frac{1}{2}\right) = \chi^2(1) \end{aligned}$$