Chapter 2

Axioms of Probability

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Recap

- From n distinct items, number of ways to draw r of them

	without replacement	with replacement
order matters	$n!/(n-r)!$	n^r
order doesn't matter	$\binom{n}{r} = n!/(n-r)!r!$	see Ch1.6*

- Binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

- Multinomial coefficient

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}$$

Question

In a well shuffled deck of 52 cards, how many ways can the 4 Aces to be together?

Treat the 4 Aces together as one object, then the total number of objects are 49.
Also consider the number of permutations among the four aces: $4!$

$$N = 49! \times 4!$$

Sample space

Definition

A sample space S is the set of all possible outcomes of an experiment.

Examples:	3 coin tosses	S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}
	One die roll	S = {1,2,3,4,5,6}
	Sum of two rolls	S = {2,3,...,11,12}
	Seconds waiting for bus	S = $[0, \infty)$

Examples of sample spaces

- Experiment is playing five rounds of Russian roulette
  Sample space is $\{D, LD, LLD, LLLD, LLLLD, LLLLL\}$.

  Why is this different than coin flipping?

- Experiment is sequencing three nucleotides
  Sample space is $\{AAA, CCC, GGG, TTT, AAC, AAT, AAG, ...\}$.

Question

How big is this sample space? (Hint: there are four types of nucleotides.)

$$4 \times 4 \times 4 = 64$$

An event

Definition

An event E is any subset of the sample space S.

$$E \subset S$$

Examples:	2 heads	E $= \{$HHT, HTH, THH$\}$
	Even number	E $= \{$2,4,6$\}$
	$<$ 2 minutes	E $= [0, 120)$

- Impossible event: empty set $\emptyset \subset S$
- $S \subset S$

Set theory

Let A, B be two events.

Definition

1. **Intersection** $A \cap B$: *implies the event that both* A *and* B *occur*
2. **Union** $A \cup B$: *implies the event that at least one of* A *or* B *occur*
3. *The* **complement** *of an event* A *denoted* A^c *(also notated* A' *or* $\bar{A}$*):* $A^c = S \backslash A$ *- the event that* A *does not occur*
4. $A \subset B$ *implies that the occurrence of* A *implies the occurrence of* B

Venn diagram

More set theory

Definition

Two events A and B are **mutually exclusive** *or* **disjoint** *if they have no outcomes in common, i.e. $A \cap B = \emptyset$.*

Being a Clemson fan and a South Carolina fan is mutually exclusive.

Some rules

1. Commutative laws

$$A \cup B = B \cup A, \quad A \cap B = B \cap A$$

2. Associative laws

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

3. Distributive laws

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

Note: Think of union as addition and intersection as multiplication: $(A + B)C = AC + BC$

DeMorgan's Laws

$$(A \cup B)^c = A^c \cap B^c, \quad (A \cap B)^c = A^c \cup B^c$$

- Outline of proof (to the first equation): two steps
  Left $\subset$ Right $\Longleftrightarrow$ For any $x \in (A \cup B)^c$, then $x \in A^c \cap B^c$.
  Right $\subset$ Left $\Longleftrightarrow$ For any $x \in A^c \cap B^c$, then $x \in (A \cup B)^c$.

- DeMorgan's laws can be generalized to n events $A_1, \ldots, A_n$:
  $$\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c, \quad \left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c$$

Recap

- Sample space S, event E
- Set operations: intersection, union, complement, subset, disjoint
- Rules: commutative, associative, distributive, DeMorgan's laws
- Hint: Venn Diagram is helpful

What is probability?

Given an experiment and a sample space S, the objective of probability is to assign to each event A a number $P(A)$, called the probability of the event A, which will give a precise measure of the chance that A will occur.

What is $P(A)$ of the following events (A's)?

1. $A =$ Clemson wins next year's NCAA football title.
2. $A =$ Someone in this class room wins the MegaMillion.
3. $A =$ You spin a quarter and it comes up heads.
4. $A =$ You spin a quarter and it stands up.
5. $A =$ At least two people in this class room have the same birthday.

Probability: Frequentist interpretation

- There are several possible interpretation of probability.
- There is not agreement, at all, in how probabilities should be interpreted
- There is (nearly) complete agreement on the mathematical rules probability must follow (axioms)
- *Frequentist* interpretation: The probability of event A is the proportion of times (frequency) that A occurs in an infinite sequence (or very long run) of separate tries of the experiment.

$$P(A) = \lim_{n \to \infty} \frac{\text{\# times A happens}}{n}.$$

- Often associated with Jerzy Neyman and Egon Pearson who described the logic of statistical hypothesis testing.
- John Maynard Keynes (1883-1946) commented on this: *In the long run, we are all dead*.

Probability: Bayesian interpretation

- A *Bayesian* can pick whatever number they prefer for $P(A)$, based on their own personal experience and intuition, provided that number is consistent with all of the other probabilities they choose in life.
- A Bayesian interprets probability as a subjective degree of belief: *For the same event, two separate people could have differing probabilities.*
- The Bayesian's view must: (1) conform to all other personal opinions; (2) change as new data arise according to *Bayes' Rule*.
- Bayesian interpretations of probabilities avoid some of the philosophical difficulties of frequency interpretations.
- Named after the 18^{th} century Presbyterian minister and mathematician Thomas Bayes.
- Largely popularized by revolutionary advance in computational technology and methods during the last twenty years.

Axiomatic Foundations of Probability

- The foundations of modern probability theory, the axiomatic basis, are laid by Andrey Kolmogorov in 1933.

- The axiomatic approach is not concerned with the interpretation of probabilities.
- Concerned only that probabilities are defined by a function satisfying the axioms.
- Kolmogorov (1903-1987) was one of the greatest mathematicians of the 20th century. This axiomatization was one of his "trivial" accomplishment.

Axioms of probability

Definition

Let P be a function that assigns a nonnegative real number to each event E of a sample space S. We call P a probability if

1. *Axiom 1: non-negative*
$$0 \leq P(E) \leq 1$$
2. *Axiom 2: total one*
$$P(S) = 1$$
3. *Axiom 3: countable addition*
$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i), \text{ if } E_i \cap E_j = \emptyset \text{ for } i \neq j$$

In particular, for k **disjoint** events $E_1, \ldots, E_k$,

$$P\left(\bigcup_{i=1}^{k} E_i\right) = \sum_{i=1}^{k} P(E_i)$$

Propositions

☞ Complement Rule:

$$P(A^c) = 1 - P(A)$$

Proof (hint: use Axiom 2 and Axiom 3)

- $P(\emptyset) = 0$

☞ Difference Rule:

$$P(B \cap A^c) = P(B) - P(A), \text{ if } A \subseteq B$$

Proof (hint: use Axiom 3)

- $P(B) \geq P(A)$, if $A \subseteq B$

Proof (hint: use Axiom 1)

☞ Inclusion-Exclusion: two events A, B (not necessarily disjoint)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof (hint: use Axiom 3 and Venn-diagram)

Inclusion-Exclusion example

Suppose that for a randomly selected student in a probability class,

- $P(\text{female}) = 39\%$.
- $P(\text{math major}) = 42\%$.
- $P(\text{female and math major}) = 24\%$.

Find the probability that a student is either female or a math major.

1. Conditions: event $A = \{\text{female}\}$, $B = \{\text{math major}\}$,

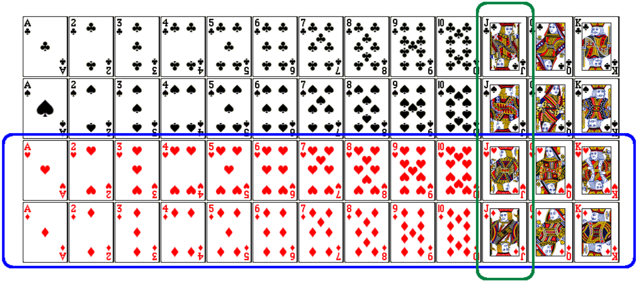
$$P(A) = 0.39, P(B) = 0.42, P(A \cap B) = 0.24$$

2. Question: find $P(A \cup B)$.
3. Formula: inclusion-exclusion

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.39 + 0.42 - 0.24 = 0.57$$

Question

What is the probability of drawing a jack or a red card from a well shuffled full deck?



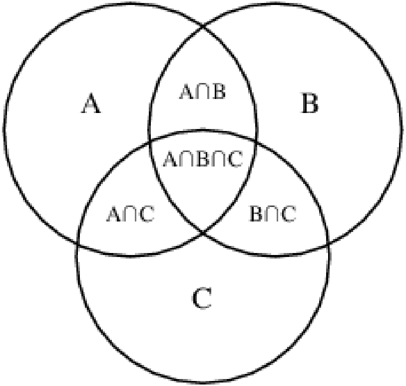
$$P(jack\ or\ red) = P(jack) + P(red) - P(jack\ and\ red)$$
$$= \frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{28}{52}$$

Figure from *http://www.milefoot.com/math/discrete/counting/cardfreq.htm* .

Propositions

☞ Inclusion-Exclusion: three events A, B, C (not necessarily disjoint)

$$\begin{aligned} P(A \cup B \cup C) =& P(A) + P(B) + P(C) - P(A \cap B) \\ & - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \end{aligned}$$



Question

Suppose that for a randomly selected student in a probability class,

- $P(\text{female}) = 39\%$, $P(\text{math major}) = 42\%$, $P(\text{vegetarian}) = 3\%$.
- $P(\text{female and math major}) = 24\%$, $P(\text{female and vegetarian}) = 3\%$, $P(\text{math major and vegetarian}) = 0\%$
- $P(\text{female and math major and vegetarian}) = 0\%$

Find the probability that a student is either female, a math major or a vegetarian.

1. Conditions: Event $A = \{\text{female}\}$, $B = \{\text{math major}\}$, $C = \{\text{vegetarian}\}$,
2. Question: find $P(A \cup B \cup C)$.
3. Formula: inclusion-exclusion

$$\begin{aligned} P(A \cup B \cup C) =& P(A) + P(B) + P(C) - P(A \cap B) \\ & - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \\ & = 0.39 + 0.42 + 0.03 - 0.24 - 0.03 - 0 + 0 = 0.57 \end{aligned}$$

Recap

Three axioms of probability P

1. $0 \leq P(E) \leq 1$
2. $P(S) = 1$
3. $P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$, if $E_i \cap E_j = \emptyset$ for $i \neq j$

Propositions of probability

- $P(A^c) = 1 - P(A)$
- $P(B \cap A^c) = P(B) - P(A)$, if $A \subseteq B$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cup B \cup C)$

Sample spaces with equally likely outcomes

- Suppose a sample space has N equally likely outcomes $\{1\}, \ldots, \{N\}$, then

$$S = \{1\} \cup \ldots \cup \{N\}$$

Disjointness gives

$$1 = P(S) = P(\{1\}) + \ldots + P(\{N\}) = NP(\{i\}),$$

so for each $1 \leq i \leq N$,

$$P(\{i\}) = \frac{1}{N}$$

- For event E in a sample space S with equally likely outcomes,

$$P(E) = \frac{\#(E)}{\#(S)}$$

Notation:

Cardinality - $\#(E) =$ number of elements in set E

Probability of rolling an even number with a six sided die?

$$E = \{2, 4, 6\} \text{ and } S = \{1, 2, 3, 4, 5, 6\}$$

$$P(E) = 3/6 = 1/2$$

A couple has two kids, what is the probability that they are not both girls?

$$E = \{BB, GB, BG\} \text{ and } S = \{BB, GG, GB, BG\}$$

$$P(E) = 3/4$$

Question

Two fair four-sided dice are rolled. Two events:

$$A = \{\text{sum of two rolls is 5}\}$$

$$B = \{\text{minimum roll is 2}\}$$

1. Compute $P(A)$ and $P(B)$
2. Compute $P(A \cup B)$

Hint: sample space

1,1	1,2	1,3	1,4
2,1	2,2	2,3	2,4
3,1	3,2	3,3	3,4
4,1	4,2	4,3	4,4

1. $P(A) = \frac{4}{16}$, $P(B) = \frac{5}{16}$
2. $$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{4}{16} + \frac{5}{16} - \frac{2}{16} = \frac{7}{16}$$

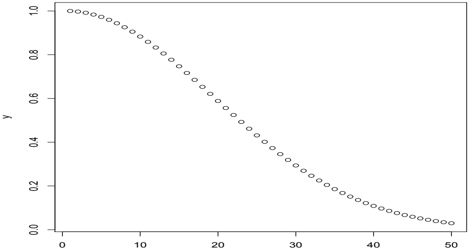
Example: birthday problem

Ignoring leap years, and assuming birthdays are equally likely to be any day of the year, what is the chance of no tie in birthdays among n students?

$$\#(\text{birthdays of } n \text{ people}) = 365^n$$

$$\#(\text{no match}) = 365 \times 364 \times \cdots \times (365 - n + 1)$$

$$P(\text{no match}) = \frac{365!}{(365-n)!\,365^n}$$



Birthday problem (cont.)

Question

For this class ($n = 34$), $P(\text{no match}) = 20.5\%$. What is the chance of at least a tie in birthdays among 34 students?

$$P(\text{at least a tie}) = 1 - P(\text{no tie}) = 0.795$$

Example: randomly pair 4 keys {a, b, c, d} with 3 locks {a, b, c}. Compute $P(\text{at least one match})$.

Let A denote the event that lock a and key a matches. Similarly, B, C.

- $P(A) = \frac{3\times2}{4\times3\times2}$
- $P(A\cap B) = \frac{2}{4\times3\times2}$
- $P(A\cap B\cap C) = \frac{1}{4\times3\times2}$
- $P(\text{at least one match}) = P(A\cup B\cup C) =$

$$P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)$$
$$=\frac{6}{24}\times3-\frac{2}{24}\times3+\frac{1}{24}=\frac{13}{24}$$