Chapter 1

Combinatorial Analysis

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Outline

The basic rule of counting

- ☞ Suppose an experiment consists outcome 1 and outcome 2. If there are n_1 possibilities for outcome 1, and n_2 possibilities outcome 2, then together there are

$$n_1 \times n_2$$

possibilities for the experiment.

Example: a team of one boy and one girl is to be made form a group of 5 girls and 2 boys. How many different teams teams are there?
(Team A is different from Team B if at least one player is different.)

$$\begin{array}{ccccc} G_1B_1 & G_2B_1 & G_3B_1 & G_4B_1 & G_5B_1 \\ G_1B_2 & G_2B_2 & G_3B_2 & G_4B_2 & G_5B_2 \end{array}$$

$$5 \times 2 = 10$$

The generalized rule of counting

☞ Suppose an experiment consists r different outcomes, with the i-th outcome having n_i possibilities, then together there are

$$n_1 \times n_2 \times \cdots \times n_r = \prod_{i=1}^{r} n_i$$

possibilities for the experiment.

Example: how many different license plates?

____	____	____	____	____	____
letter	letter	letter	number	number	number

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17,576,000$$

Permutations

Example: how many different arrangements of the letters a, b, c?

$$\underline{\quad 3 \quad} \times \underline{\quad 2 \quad} \times \underline{\quad 1 \quad} = 3! = 6$$

- Each of these arrangements is a *permutation*.
- The order matters!
- Number of permutations of n different objects

$$n \times (n-1) \times \cdots 2 \times 1 = n!$$

Question

Number of permutations of the letters in the word "Clemson"?

Question

Number of permutations of the letters in the word "Facebook"?

(a) $7!$

(b) $8!$

(c) $8!/2$

Notice that there are two "o" in the word "Facebook".

$$\begin{array}{cccccccc} \mathsf{c} & \mathsf{e} & \mathsf{F} & \mathsf{a} & \mathsf{b} & \mathsf{o}_1 & \mathsf{o}_2 & \mathsf{k} \\ \mathsf{c} & \mathsf{e} & \mathsf{F} & \mathsf{a} & \mathsf{b} & \mathsf{o}_2 & \mathsf{o}_1 & \mathsf{k} \end{array}$$

The same words! Different orders between the two "o" do not matter.

$$\frac{\#(\text{permutations if all different})}{\#(\text{permutations among the 2 ``o''})} = \frac{8!}{2!}$$

☞ Among n objects, if n_1 are alike, n_2 are alike, ..., n_r are alike, then there are

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

different permutations.

Question

Number of permutations of the letters in the word "google"?

(a) $6!/2$

(b) $5!/2$

(c) $6!/4$

(d) $6!$

Recap

The basic rule of counting

- r different outcomes; the i-th outcome having n_i possibilities, then the number of possibilities is

$$\prod_{i=1}^{r} n_i$$

Permutations

- Number of permutations of n different objects is $n!$.
- Number of permutations of n objects, if n_1 are alike, n_2 are alike, . . . , n_r are alike, is

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

Example: (a) In how many ways can 4 married couples line up?
(b) What if couples must stand together?

Part (a): 8 different people

$$8! = 40320$$

Part (b): there are $4! = 24$ ways that the couples can be arranged, and each couple can be arranged in $2! = 2$ ways. So the answer is

$$(4!) \times (2!) \times (2!) \times (2!) \times (2!) = 384.$$

Question

How many permutations of 2 numbers among $\{1, 2, 3, 4, 5, 6\}$ are there?

(a) $6 \times 6 = 36$

(b) $2^6 = 64$

(c) $6 \times 5 = 30$

(d) $6!$

$$\overline{6} \times \overline{5} = 6!/4!$$

☞ If we have n items and want to select r of them,

$$\#(\text{permutations}) = n \times (n-1) \times \cdots \times (n-r+1) = \frac{n!}{(n-r)!}$$

<u>What if the order doesn't matter? E.g. handshakes.</u>

Combinations: order doesn't matter!

When order matters, there are $r!$ different orderings of the r items selected.

- If we have n items and want to select r of them,

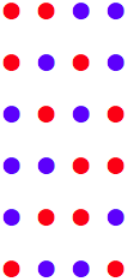
$$\#(\text{combinations}) = \frac{n \times (n-1) \times \cdots \times (n-r+1)}{r!} = \frac{n!}{(n-r)!r!}$$

- Define *choose*

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

- The number $\binom{n}{r}$ is pronounced as n choose r, it is the number of ways to pick r objects from a set of n distinct objects.
- $0 \le r \le n$, otherwise 0
- Recall $0! = 1$

For example, the number of ways to arrange two red marbles and two blue marbles is



$$\binom{4}{2} = \frac{4!}{2! \times 2!} = \frac{24}{2 \times 2} = 6.$$

Example: how many handshakes take place between a group of 6 people if everyone need to shake hands with everyone else?

Same questions: how many combinations of 2 numbers among $\{1,2,3,4,5,6\}$ are there?

{1,2} {1,3} {1,4} {1,5} {1,6}
{2,3} {2,4} {2,5} {2,6}
{3,4} {3,5} {3,6}
{4,5} {4,6}
{5,6}

$$\binom{6}{2} = \frac{6!}{4!2!} = \frac{6 \times 5}{2 \times 1} = 15$$

Example: Poker hand. A standard poker deck has 52 cards, in four suits (clubs, diamonds, hearts, spades) of thirteen cards each (2, 3, ..., 10, Jack, Queen, King, Ace).

Question

How many poker hands (5 cards) can be dealt from a deck of 52 cards?

Combinations, choose 5 out of 52 different cards.

$$\binom{52}{5} = \frac{52!}{47!5!} = \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = 2,598,960$$

How many distinct hands of "four of a kind" (four of the five cards are of the same rank)?

$\binom{13}{1}$	$\times$	$\binom{4}{4}$	$\times$	$\binom{48}{1}$
the same rank		suits for the same rank		the rest 1 card

Example: among 4 married couples, we want to select a group of 3 people that is not allowed to contain a married couple. How many choices are there?

Number of choices if the group can contain married couple(s):

$$N_1 = \binom{8}{3} = \frac{8!}{3! \times 5!} = 56$$

Number of choices if the group contains married couple(s)?
Then it can only contain one couple.

$$N_2 = \binom{4}{1} \times \binom{6}{1} = 24$$

The number of choices that the group does not have a couple:

$$N_1 - N_2 = 32$$

Properties of combinations $\binom{n}{r} = \frac{n!}{(n-r)!r!}$

1. $$\binom{n}{1} = n \qquad\qquad \binom{n}{n} = 1$$

2. $$\binom{n}{r} = \binom{n}{n-r}$$

3. $$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}, \quad 1 \le r \le n$$

Binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Proof: 1) mathematical induction or 2) combinatorial consideration.
Think about license plates that are formed by n digits, where each digit can be a letter or a number. $a = 26, b = 10$.
Total number of distinct plates:

$$N = (a+b)^n = N_0 + N_1 + \cdots + N_n,$$

where N_k is the number of distinct plates that contains exactly k number of letters.

$$N_k = \binom{n}{k} \times a^k \times b^{n-k}$$

Question

How many subsets are there of the set $\{1, 2, \ldots, n\}$?

For each $0 \le k \le n$, there are $\binom{n}{k}$ different subsets of size k. Then

$$\#(\text{subsets}) = \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1+1)^n$$

Use the Binomial Theorem to simplify

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1+1)^n = 2^n$$

Any other way to solve this question?

Multinomial coefficients

Example: a police department of 10 officers wants to have 5 of the officers patrol streets, 2 doing paperwork, and 3 at the donut shop, how many ways can this be done?

$$\binom{10}{5}\binom{5}{2}\binom{3}{3} = \frac{10!}{5!(10-5)!} \cdot \frac{5!}{3!(5-3)!} = \frac{10!}{5!3!2!}$$

☞ *Multinomial coefficient*: a set of n distinct items is to be divided into r distinct groups of respective sizes $n_1, \ldots, n_r$, where $n_1 + n_2 + \cdots + n_r = n$. Number of possible divisions is

$$\binom{n}{n_1, n_2, \ldots, n_r} \overset{\text{def}}{=} \frac{n!}{n_1! n_2! \cdots n_r!}$$

- $$\binom{n}{n_1, n_2, \ldots, n_r} = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n_r}{n_r}$$
- When $r = 2$, becomes binomial coefficient (choose function)

$$\binom{n}{n_1, n_2} = \binom{n}{n_1}$$

Note that $n_1 + n_2 = n$

- Multinomial Theorem

$$(a_1 + a_2 + \cdots + a_r)^n = \sum_{n_1+\cdots+n_r=n} \binom{n}{n_1, n_2, \ldots, n_r} a_1^{n_1} a_2^{n_2} \cdots a_r^{n_r}$$

- The Binomial theorem is a special case when $r = 2$.