Bayesian Inference for One Parameter Models

The Binomial Model

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MATH 9810

Teenagers and Televisions

In 1998, the New York Times and CBS News polled 1048 randomly selected 13 - 17 year olds to ask them if they had a television in their room.

- $n = 1048$ sampled teenagers
- $y = 692$ had a television in their room
- Inference about θ, the proportion of 13 - 17 year olds who have a television in their room (in 1998)
- What is a reasonable probability model for data?

Binomial Model

- Independent Bernoulli trials $X_i, (i = 1, \ldots, n)$
- "Success" probability $\theta : p(X_i = x_i|\theta) = \theta^{x_i}(1-\theta)^{1-x_i}$
- Y = number of successes = $\sum_{i=1}^{n} X_i$
- $Y|\theta \sim \text{Bin}(n, \theta)$

$$p(Y|\theta) = \binom{n}{y}\theta^y(1-\theta)^{n-y} \text{ for } y = 0, 1, \ldots, n$$

- $E(Y|\theta) = n\theta$, $Var(Y|\theta) = n\theta(1-\theta)$
- R functions: `dbinom`, `pbinom`, `qbinom`, `rbinom`

Classical Inference for θ

Assumptions $\rightarrow Y \mid \theta \sim \text{Bin}(n, \theta)$

- Usual point estimator: observed proportion

$$\hat{\theta} = y/n$$

- $E(\hat{\theta}|\theta) = \theta$, so that $\hat{\theta}$ is an unbiased estimator
- $Var(\hat{\theta}|\theta) = \theta(1-\theta)/n$, so that precision (1/variance) increases for large n and θ near zero or one

Likelihood Function

- Likelihood function: $L(\theta) \propto p(y|\theta)$

$$L(\theta) = \binom{1048}{692} \theta^{692}(1-\theta)^{356} \propto \theta^{692}(1-\theta)^{356}$$

- For FIXED y, look at how probability of the data changes as θ varies over parameter space
- For each value of θ, the likelihood indicates how well that value of θ explains the observed data
- Usually drop terms that do not depend on parameters
- Calculations easier with log likelihood function

$$\log L(\theta) \propto y \log(\theta) + (n-y)\log(1-\theta)$$

Maximum Likelihood Estimate

- What is most likely value of θ for these data?
- Maximum likelihood estimate: value of θ that maximizes the likelihood

Using calculus, we take the derivative of $\log L(\theta)$ and set equal to zero:

$$\begin{aligned} 0 &= \frac{\partial \log L(\theta)}{\partial \theta} = \frac{y}{\theta} - \frac{n-y}{1-\theta} \\ (n-y)\theta &= y(1-\theta) \\ \theta &= y/n \end{aligned}$$

The MLE is $\hat{\theta} = y/n = 692/1048 = .660$.

Functions of Parameters: Odds

- odds: $o(\theta) = \theta/(1-\theta)$
- Likelihood is same under one-to-one transformation
  $p(y \mid o) = p(y \mid \theta(o))$
- MLE of $g(\theta)$ is $g(\hat{\theta})$
- estimated probability that a 13-17 year old will have a TV in their room is 0.660
- estimated odds that a 13-17 year old will have a TV in their room is 1.94 to 1
- estimated odds that a 13-17 year old will not have a TV in their room is $.515$ to 1

Classical Interval Estimates for θ

For large n, central limit theorem:

$$\hat{\theta} = y/n \mathrel{\dot\sim} N(\theta, \theta(1-\theta)/n)$$

This leads to 95% confidence interval for θ,

$$\hat{\theta} \pm 1.96\sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}}$$

$$95\% \text{ CI for } \theta : (0.632, 0.689)$$

Probability that interval covers θ (prior to seeing the data) equals 0.95.

Not same as $p(0.632 < \theta < 0.689) = 0.95$.

Bayesian Inference about θ

Conditional on observed outcome y (and n) the posterior distribution of θ is

$$p(\theta|y) = \frac{p(\theta)p(y|\theta)}{p(y)} \quad 0 < \theta < 1$$

where

$$p(y) = \int_0^1 p(y|\theta)p(\theta)d\theta$$

is the marginal density of data.
Alternatively, we have

$$p(\theta|y) \propto p(\theta)p(y|\theta)$$

subject to normalization to unit integral.

Bayes Inference about θ

- Initial prior uncertainty about θ described by a prior distribution $p(\theta)$.
- Uniform density is a common default choice

$$p(\theta) = 1, \qquad 0 < \theta < 1$$

- Flat density, each point equally weighted
- "uninformative" about true value

Results with Uniform Prior

Posterior $p(\theta|y) \propto \theta^y(1-\theta)^{n-y}(1) = \theta^{y+1-1}(1-\theta)^{n-y+1-1}$

- Recognize that kernel of density is a Beta$(y+1, n-y+1)$

$$p(\theta \mid y) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\,\theta^{a-1}(1-\theta)^{b-1} \qquad \text{for } 0<\theta<1$$

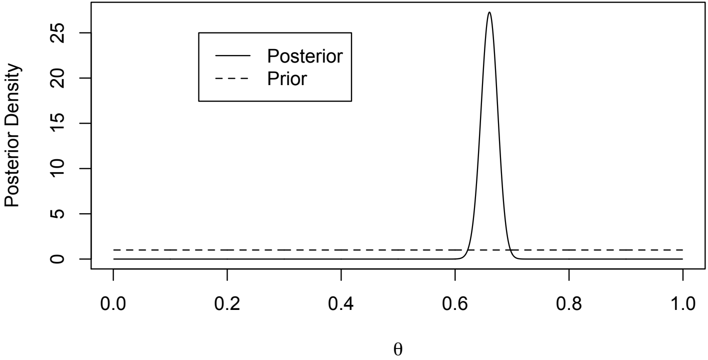
- $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$. When x is integer, $\Gamma(x) = (x-1)!$
- Under uniform prior, posterior distribution for θ is Beta$(693, 357)$
- In R: `dbeta`, `pbeta`, `qbeta`, `rbeta`
    - quantiles (percentiles),

```
> qbeta(c(0.025, 0.5, 0.975), y + 1, n - y + 1);
[1] 0.6310792 0.6601016 0.6883435
```

Posterior Distribution for Television Data

Under uniform prior, posterior distribution for θ is Beta$(693, 357)$

```
theta = seq(0.001, 0.999, by = 0.001)
plot(theta, dbeta(theta,693,357),t="l",lty=1)
lines(theta, dbeta(theta, 1,1), lty=2)
```



Beta Distributions $\theta \sim \text{Beta}(a, b)$

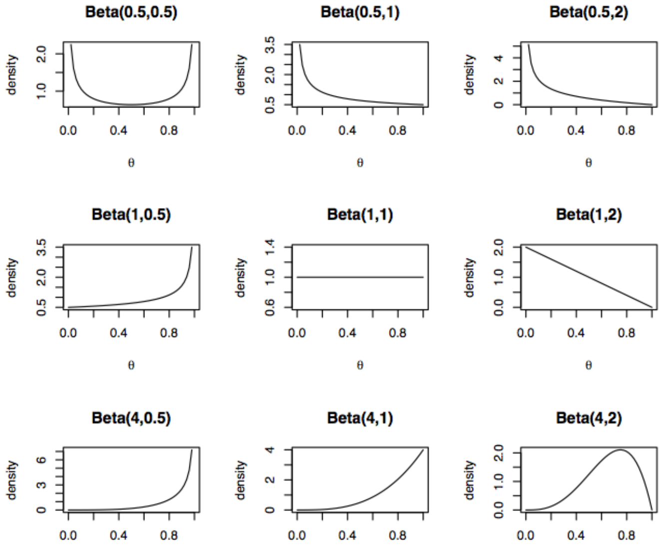
$$p(\theta) = \frac{1}{B(a,b)} \theta^{a-1}(1-\theta)^{b-1} \text{ for } 0 < \theta < 1$$

- $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

$$E(\theta) = \frac{a}{a+b}, \quad Var(\theta) = \frac{ab}{(a+b)^2(a+b+1)}$$

- More concentrated, or "precise", for larger $a+b$
- unique mode at $(a-1)/(a+b-2)$ if $a, b > 1$
- mode at 0 if $a \leq 1$, and/or one at 1 if $b \leq 1$

Shapes of Beta Priors



Beta-Binomial Model

- Prior: $\theta \sim \text{Beta}(a, b) \Rightarrow p(\theta) \propto \theta^{a-1}(1-\theta)^{b-1}$
- Likelihood: $L(\theta) = \theta^{y}(1-\theta)^{n-y}$
- Posterior: $p(\theta \mid Y) \propto L(\theta)p(\theta) \propto \theta^{y+a-1}(1-\theta)^{n-y+b-1}$
- $\theta \mid Y \sim \text{Beta}(y+a, n-y+b)$
- Posterior Mean:

$$\frac{y+a}{n+a+b} = \left(\frac{n}{n+a+b}\right)\left(\frac{y}{n}\right) + \left(\frac{a+b}{n+a+b}\right)\left(\frac{a}{a+b}\right)$$

  weighted average of MLE and prior mean
- a is prior number of "1"s and $a+b$ is prior sample size

Conjugate Prior Distributions

Consider a class of prior distributions, $p(\theta) \in \mathcal{P}$.

We say that the class is conjugate for a sampling model $p(y \mid \theta)$, if $p(\theta) \in \mathcal{P}$ implies that $p(\theta \mid Y) \in \mathcal{P}$ for all $p(\theta) \in \mathcal{P}$ and data y.

- If y has a binomial distribution, then the class of Beta prior distributions is conjugate.
- We will see that sampling models based on exponential families all have conjugate priors.

Posterior Intervals

Credible intervals, or confidence region

$$P(l(y) < \theta < u(y) \mid Y = y) = 0.95$$

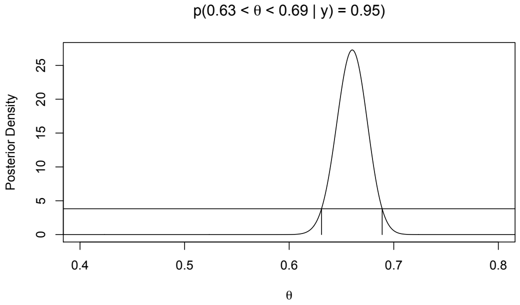
- Quantile-based intervals (equal tails): 95% interval
  $l(y) = 2.5\%$ quantile of $p(\theta \mid y)$
  $u(y) = 97.5\%$ quantile of $p(\theta \mid y)$

```
> qbeta(c(0.025, 0.975), y + 1, n - y + 1);
[1] 0.6310792 0.6883435
```

- We can also compute one-sided quantile-based intervals.

Highest Posterior Density (HPD) intervals

- $P(\theta \in s(y) \mid Y = y) = 0.95$
- For any $\theta_1 \in s(y)$ and $\theta_2 \notin s(y)$, $p(\theta_1 \mid Y = y) > p(\theta_2 \mid Y = y)$.




```
solve.HPD.beta(y, n, h = 0.14, xlim = c(0.4, 0.8))
```

Based on the samples, there is a 95% chance that between 63% and 69% of 13 - 17 year olds who have a television in their room.

Finding an HPD Interval

1. Find mode of posterior $\tilde{\theta}$
2. Construct relative density $(0 < r(\theta) \leq 1)$

$$r(\theta) = \frac{p(\theta \mid Y)}{p(\tilde{\theta} \mid Y)} = \frac{L(\theta \mid Y)p(\theta)}{L(\tilde{\theta} \mid Y)p(\tilde{\theta})}$$

3. Find points of equal density: start with initial height $h \in (0, 1)$ ($h = 0.1$ often is close) and solve for θ_{lh} such that $r(\theta_{lh}) = h$ where $0 < \theta_{lh} < \tilde{\theta}$. Solve for θ_{uh} such that $r(\theta_{uh}) = h$ with $\tilde{\theta} < \theta_{uh} < 1$
4. Calculate $P(\theta_{lh} < \theta < \theta_{uh} \mid Y)$
5. decrease/increase h and repeat Steps 3 – 4 until the probability is approximately 0.95