Warning Against Recurring Risks: An Information Design Approach

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The World Health Organization seeks effective ways to alert its member states about global pandemics. Motivated by this challenge, we study a public agency's problem of designing warning policies to mitigate potential disasters that occur with advance notice. The agency privately receives early information about recurring harmful events and issues warnings to induce an uninformed stakeholder to take preemptive actions. The agency's decision to issue a warning critically depends on its reputation, which we define as the stakeholder's belief regarding the accuracy of the agency's information. The agency faces then a trade-off between eliciting a proper response today and maintaining its reputation in order to elicit responses to future events.

We formulate this problem as a dynamic Bayesian persuasion game, which we solve in closed form. We find that the agency sometimes strategically misrepresents its advance information about a current threat in order to cultivate its future reputation. When its reputation is sufficiently low, the agency *downplays* the risk and actually downplays more as its reputation improves. By contrast, when its reputation is high, the agency sometimes *exaggerates* the threat and exaggerates more as its reputation deteriorates. Only when its reputation is moderate does the agency send warning messages that fully disclose its private information.

Our study suggests a plausible and novel rationale for some of the false alarms or omissions observed in practice. We further test the robustness of our findings to imperfect advance information, disasters without advance notice, and heterogeneous receivers.

Key words: Information Design, Bayesian Persuasion Game, Dynamic Programming, Statistical Decision, Global Health, Disaster Management

1. Introduction

When public agencies receive early signals about a potential disaster, they issue warnings for the public or other stakeholders to take early but costly mitigation actions. When the adverse event does not materialize, however, these false alarms affect the agencies' reputation and hence their ability to mobilize timely responses to future threats. Thus, when sounding an alarm, the agencies need to weigh their current ability to elicit immediate actions against the efficacy of their future warning messages.

The challenge of balancing this fundamental trade-off is perhaps best exemplified by the World Health Organization (WHO)'s decisions to proclaim a so-called "Public Health Emergency of International Concern (PHEIC)".[1] WHO has the sole authority to declare a PHEIC in anticipation of a global infectious disease outbreak, and to request that member states take costly measures to contain the epidemic at its onset (World Health Organization 2005). However, PHEIC declarations, or the lack thereof, have been regularly criticized for misrepresenting such risks, sometimes discrediting the agency's reputation and casting doubts on the quality of its decisions.

For example, WHO was accused of exaggerating the risk of a relatively mild outbreak by declaring a PHEIC for the 2009 H1N1 epidemic (Flynn 2010).[2] WHO's reputation was further damaged by its failure to declare a PHEIC early enough for the 2014 West Africa Ebola outbreak and, this time, for having downplayed the risk of an international crisis (Sengupta 2015). This loss of reputation reduces WHO's ability to effectively elicit containment actions from its member states and affects its future willingness to declare PHEICs (Moon et al. 2015). In fact, a panel of experts in global health (World Health Organization 2015) concluded that WHO's failure in tackling the 2014 Ebola outbreak partly stemms from the reputation loss that followed the 2009 H1N1 epidemic.[3]

WHO is not alone in facing this challenge. The U.S. Food and Drug Administration (FDA)[4] appears to have downplayed the cardiovascular risks associated with anti-inflammator Rofecoxib. Although FDA was aware of these risks, the agency failed to issue any advisory warning until the manufacturer voluntarily withdrew the drug from the market (Topol 2004). By contrast, FDA exaggerated the risk associated with certain antidepressants in 2004 (Friedman 2014). Similarly, the U.S. National Weather Service appears to have downplayed the risk associated with the flood of the Red River in North Dakota and Minnesota in 1997 (Pielke 1999), but exaggerated the risk associated with hurricanes Bertha and Fran in 1996 (Dow and Cutter 1998).

Our research provides a novel rationale for misrepresentations such as these. We elucidate how to resolve the trade-offs between eliciting a proper response today and maintaining its reputation in order to elicit responses to future adverse events. In doing so, we uncover the underlying mechanisms by which the need to regulate the agency's reputation result in downplaying or exaggerating the

[1] The international regulation defines a "public health emergency of international concern" as an extraordinary event which is determined: (i) to constitute a public health risk to other states through the international spread of disease and (ii) to potentially require a coordinated international response (see World Health Organization 2005).

[2] A review committee, however, argued that that the WHO declaration of a PHEIC was appropriate, see World Health Organization (2011) and Fineberg (2014).

[3] Specifically, the panel states in its report (World Health Organization 2015, page 13) that WHO did not declare a PHEIC at the onset of the Ebola outbreak in part because "WHO had been previously criticized for declaring a PHEIC for pandemic influenza H1N1."

[4] Part of the FDA's mandate is to release warnings and even recall products that are deemed too dangerous; see Code of Federal Regulations (CFR) Title 21 §310.305, 314.80, 314.98, 600.80, 1271.350 and Section 760 of the Food Drug and Cosmetic Act (FDCA).

current risk. To that end, we model the problem of designing a warning policy in an information design framework (e.g., Kamenica and Gentzkow 2011), which we solve in closed form.

More specifically, we capture the dynamic interaction between an agency (the sender) and the public (the receiver) in the face of possible harmful events with a two-period Bayesian persuasion game. In this set-up, a harmful event occurs at the end of each period according to a known prior probability, in which case both parties incur a loss. Before the event fully materializes, the receiver can take costly mitigation actions that reduces the losses of both parties. The sender has a stronger incentive for the receiver to act than does the receiver himself. This is the case, for example, when the mitigation benefits a third party for whom the sender is also responsible, or when the sender does not internalize the mitigation costs.

The sender cannot force the receiver to act, but the sender may observe an advance signal indicating the event and warn the receiver accordingly. In this paper, we mostly consider disasters that always occur with advance notice, i.e., the sender always observes an advance signal when the event ultimately materializes. (Nonetheless, we relax this assumption in Section 6.2.) Global pandemics, for instance, always emerge first as local outbreaks. However, the quality of these signals may be imperfect, in that the sender may observe an advance signal in the absence of an upcoming threat. Thus, in our context, the sender's *reputation* corresponds to the receiver's belief (i.e., subjective probability) that the sender has access to accurate signals.

Whether or not the sender observes a signal at the beginning of a period is her private information. However, the sender may misrepresent this private information to induce the receiver to take proper actions. More formally, the sender issues a warning according to a pre-specified probability, which depends on her current and historical information. The receiver then decides whether to act based on the sender's current reputation and warning messages. At the end of the period, both parties observe whether the event actually materializes, and the receiver adjusts the sender's reputation using Bayes' rule based on whether the receiver was correctly alerted.

In this setting, a *full-disclosure* policy corresponds to issuing a warning with probability one in the presence of an advance signal, and with probability zero in its absence. By contrast, the sender *misrepresents* her private signal when she warns the receiver with a probability that is strictly between zero and one. The more she raises (resp. reduces) the warning probability in the absence (resp. presence) of an advance signal, the more the sender misrepresents this information.

In the global health context, for instance, harmful events are global pandemics. The receiver is a member state, which can implement costly measures to contain a pandemic, when the epidemic emerges in another country. These measures include border controls, mass vaccinations, quarantine, etc. The sender is WHO, who observes a signal from independent institutions or local government agencies (e.g., the Global Outbreak Alert and Response Network[5] or National IHR Focal Points of member states, as defined by the World Health Organization (2005)) when a local epidemic outbreak requires the receiver to take containment actions. According to the international regulations (World Health Organization 2005, page 13), however, WHO "shall not make this information generally available," and thus the presence or absence of this signal is confidential to the organization. WHO then determines whether or not to warn the international community with a PHEIC declaration. WHO's incentives, however, are not fully aligned with those of the member states. Indeed, WHO does not bear all mitigation costs and seeks to protect the international community, while a state solely focuses on protecting its own population and incurs the full cost of its mitigation actions.

To tackle the sender's problem, we first establish that her reputation fully captures her ability to influence the receiver's action in each period. As a result, the sender's probability of issuing a warning in a given period depends only on her current reputation and on whether or not she observes an advance signal. In making these warning decisions, therefore, the sender needs to trade off the effect her warnings have on the receiver's action in a given period against the impact the warnings have on the sender's reputation in the next period.

We first study the single-period version (which also corresponds to the last period) of our problem, in which the future is irrelevant. In this case, we show that full disclosure of the sender's private information is optimal, provided that her reputation is above a critical threshold. Otherwise, the receiver always ignores the sender's warning, and never acts. In other words, the mere misalignment of incentives between the sender and receiver does not necessarily suffice to justify misrepresenting the sender's private information. This finding contrasts with the existing literature on static Bayesian persuasion games.

We then fully characterize the optimal policy of our two-period problem. In this case, full disclosure is also optimal in the first period when the harmful event is highly likely. Otherwise, our results indicate that the sender may be better off by misrepresenting her private information. In other words, the interplay between the misaligned incentives and the need to cultivate reputation gives rise to strategic misrepresentations of the sender's information.

Specifically, when reputation is low (i.e., below a critical threshold), the sender never releases a warning in the absence of an advance signal, as in full disclosure. However, the sender keeps refraining from warning the receiver (i.e., the warning probability is less than one), even when she observes an advance signal about a threat. In this sense, the sender *downplays* the risk. Further, and perhaps surprisingly, the higher the sender's reputation goes, the more she downplays the risk.

[5] See `http://www.who.int/ihr/alert_and_response/outbreak-network/en/`.

When her reputation is right below the critical threshold, she stops warning the receiver altogether, regardless of her advance information.

For higher levels of reputation (i.e., above the previous critical threshold), full disclosure becomes optimal, unless these reputation levels are very high (i.e., above a second threshold) and the sender sufficiently values the future. In this case, the sender warns the receiver in the presence of an advance signal, as in full disclosure. However, she also warns the receiver with positive probability even in the signal's absence. That is, the sender *exaggerates* the threat. Further, a sender with a higher reputation exaggerates the risk less.

We also uncover novel intuitions for these information distortions (more details are offered in Section 5). In short, when the sender's initial reputation is weak, she lacks the ability to elicit an action from the receiver in the current period. As such, she can only hope to improve her reputation so as to acquire this capability in the next period. To that end, the sender misrepresents her private information by refraining from issuing warnings and hence limiting the risk of false alarms.

By contrast, when the sender's reputation is sufficiently high, she is capable of mobilizing the receiver in the current period. The sender then needs to protect her reputation so as to retain this capability in the subsequent period. To that end, the sender exaggerates the threat, which makes her warnings more prevalent in general. The receiver thus attributes the instance of a false alarm more to the sender's excessive propensity to issue warnings than to the poor quality of the sender's information. This limits the resulting loss of reputation.

Finally, we examine the robustness of our findings by extending our analysis in various directions (see Section 6). Specifically, we allow for signals with more general error rate configurations, and explore the corresponding optimal warning policies. We also discuss the case of multiple heterogeneous receivers. Despite the more complex structure of the corresponding policies, the main insights from our base model, and in particular the incentives to misrepresent the risk, persist in these settings.

2. Literature Review

The application context of our paper speaks to the management science and operations research literature that studies the containment of contagious disease outbreaks (e.g., Sun et al. 2009, Wang et al. 2009, and references therein). In a broader sense, we contribute to the longstanding literature on public health emergency response (Green and Kolesar 2004, Jacobson et al. 2012) as well as to the growing literature on disaster management (see Gupta et al. 2016, for an excellent review).

The focus in the above-mentioned research corresponds to the receiver's mitigation actions in our set-up, and the goal therein is to improve the effectiveness of those actions through better physical deployment of medical resources. Because those studies focus primarily on operational and implementation level decisions, they implicitly assume that the stakeholders have sufficient information to trigger timely containment measures in case of an outbreak. Our work aims to fill in this gap by examining the flow of information that is critical to those triggering mechanisms. From this perspective, our research contributes to the emerging literature on public information dissemination in crisis management (e.g., Eftekhar et al. 2017, Yoo et al. 2016).

Closely related to our paper is work by Pinker (2007), who studies the joint use of physical defensive resources as well as public warnings to minimize the damage caused by terrorists' attacks. Bakshi and Pinker (2018) built on this set-up to explicitly model the strategic behaviors of the terrorists. Their key insight is that effective warnings may in fact deter the terrorist attack rather than alert and prepare the public.

Both Pinker (2007) and Bakshi and Pinker (2018) account for "warning fatigue" by assuming a predetermined cost function that penalizes the sender for raising false alarms. This is in the spirit of Paté-Cornell (1986), one of the earliest studies that offers prescriptive guidelines to design public warning systems. Indeed, Paté-Cornell (1986) formalizes the notion of warning fatigues as a warning response rate, which increases with the quality of previous alarms. By contrast, we account for the effect of false alarms by fully endogenizing the receiver's strategic actions. In our set-up, the public responds to false alarms through Bayesian updating, which enables us to endogenize the notion of reputation. Managing this reputation then becomes central to the sender's problem.

Further, Bakshi and Pinker (2018) focus on strategic terrorist attacks, while we study epidemics and natural disasters. Thus, the chance of a disaster is endogenous in their set-up, while the occurrence of harmful events is exogenous in ours. (See Zhuang and Bier (2007) for discussion of the fundamental difference between terrorism-induced crises and other disasters). But this difference means that in Bakshi and Pinker's (2018) set-up, the sender cannot commit to a warning policy, while in our set-up the sender has commitment powers. As a result, they consider a dynamic zero-sum game framework, while we face an information design problem, which we cast in a dynamic persuasion game framework.

More generally, our paper is related to the literature on reputation management (see, e.g., Corona and Randhawa 2017 and Dai and Singh 2019 for recent works on reputation in management science). Of particular interest for our paper is the stream of research initiated by Holmström (1999) that uses signal-jamming models. In this framework, a manager seeks to influence her reputation in the market overtime. Reputation corresponds to the market's belief about the manager's type (her competence), which the market updates dynamically. As in our set-up, the manager is a priori uninformed about her own type, which persists over time. In Holmström (1999), however, the manager manipulates her reputation by exerting unobservable effort, which thus corresponds to a moral hazard context. By contrast, the sender influences the receiver's learning process through information provision in our set-up.

The notion of *organizational reputation* has also attracted significant research efforts in political science (Carpenter and Krause 2012, and the references therein). The wealth of empirical evidences accumulated by this field establishes that public agencies sometimes distort their welfare-maximizing decisions so as to protect their reputations. Examples include FDA's strategic delay in drug approvals (Carpenter 2004), FDA's reluctance to issue public announcement and warnings (Maor 2011), emulations and biases of economic and fiscal forecasts among agencies (Krause and Corder 2007). Our research shows that, in the face of recurrent risks, maximizing social welfare may actually require these distortions. We further make predictions that can be empirically tested.

From the methodological perspective, our modeling framework is based on the Bayesian persuasion game recently introduced by Kamenica and Gentzkow (2011) and belongs to the rapidly-evolving area of information design (Bergemann and Morris 2019). Researchers in this area recognize and establish the importance and relevance of commitment power possessed by the information designer (the sender in our model). This powerful framework has since been extended to various dynamic settings (e.g., Ely 2017), and applied to different operations management problems such as online platform management (Papanastasiou et al. 2018, Drakopoulos et al. 2018, Lingenbrink and Iyer 2018) and queuing control (Lingenbrink and Iyer Forthcoming). In all these papers, the sender is the unique source of information for the receiver. In our set-up, however, the receiver also observes the actual realizations of events. This enables learning about the sender through Bayesian updating, which our notion of reputation captures. To the best of our knowledge, ours is the first attempt to account for such learning effects in a Bayesian persuasion framework. In the presence of these effects, the sender needs to make the trade-off between eliciting an action in the current period and managing its reputation for future elicitation, leading to novel insights.

3. Model Description and Problem Formulation

To design effective warning strategies, we model the interaction between a sender (hereafter "she") and a receiver (hereafter "he") as a Bayesian persuasion game that unfolds over two time periods, $t = 1, 2$. In a global health context, the sender is WHO and the receiver is a member state that can take early mitigation actions to contain a pandemic.

3.1. Harmful Events and Their Costs

In each period t, both players face the prospect of a harmful event. This event corresponds, for instance, to an international pandemic such as the 2009 H1N1 pandemic or 2014-2016 West Africa Ebola epidemic. The event occurs at the end of the period according to random variables X_t with realizations $x_t \in \{0, 1\}$, where the event happens if $x_t = 1$, and does not happen otherwise. Events

X_1 and X_2 are independent and identically distributed according to a Bernoulli distribution with parameter π. Thus, $\pi = \mathbb{P}[X_t = 1]$ represents the prior belief of both players about the occurrence of a harmful event in each period $t = 1, 2$. We also denote $\overline{\pi} \equiv 1 - \pi = \mathbb{P}[X_t = 0]$.

If an event occurs in period t (i.e., $x_t = 1$), both the sender and the receiver suffer losses that we denote as ℓ_s and ℓ_r, respectively. Depending on the context, these losses account for casualties, economic costs, ex-post remediation efforts (e.g., rescue efforts, quarantine, etc.), negative externalities imposed on third parties, etc. The sender's loss, for instance, may be larger than that of the receiver ($\ell_s \geq \ell_r$) when the event affects a third party for which the sender is also accountable, or when the sender is a social planner.

When WHO is the sender, for instance, the receiver is a member state, who has the means to take mitigation actions to control a local outbreak that may start in another part of the world. WHO's objective is to minimize the losses of all its member states, including those who do not have the resources to avert a pandemic.[6] If the receiver does not take pre-emptive action and the local outbreak becomes a global pandemic (i.e., the adverse event occurs), the disease may spread to the receiver as well as other populations. WHO then internalizes the losses of all affected countries, and hence suffers more than the receiver, who only incurs his own losses.

To reduce these losses, the receiver can take costly mitigation actions before the event occurs (i.e., before X_t realizes). This corresponds to border restrictions or mass vaccination at the onset of an epidemic outbreak, evacuation in the case of extreme weather conditions, etc. We denote by $a_t \in \{0, 1\}$ the receiver's decision such that the receiver does not act when $a_t = 0$ and acts otherwise. In particular, when $a_t = 1$, the receiver takes mitigation actions at cost κ_r, which reduce both losses ℓ_i by amount δ_i for $i = s, r$. When the sender is a central planner and accounts for all or part of the receiver's costs, this also imposes cost κ_s on the sender.

Table 1 The Cost Structure

(a) Receiver's Cost $c_r(a_t, x_t)$.

a_t \ x_t	0	1
0	0	ℓ_r
1	κ_r	$\kappa_r + \ell_r - \delta_r$

(b) Sender's Cost $c_s(a_t, x_t)$.

a_t \ x_t	0	1
0	0	ℓ_s
1	κ_s	$\kappa_s + \ell_s - \delta_s$

Thus, the event's realization x_t and the receiver's action a_t fully determine the costs incurred at the end of the period by the sender and the receiver, which we denote as $c_s(a_t, x_t)$ and $c_r(a_t, x_t)$,

[6] The constitution of World Health Organization (1946) starts with the following principle: "The enjoyment of the highest attainable standard of health is one of the fundamental rights of every human being without distinction of race, religion, political belief, economic or social condition."

respectively. Table 1 summarizes these costs for all possible scenarios. In particular, when $(a_t, x_t) = (1,1)$, the receiver takes early mitigation actions and the harmful event actually occurs. Player $i = s, r$ suffers then a net loss of $\ell_i - \delta_i + \kappa_i$. By contrast, when $(a_t, x_t) = (1,0)$, both players only bear their respective mitigation costs κ_i without incurring any other losses since the harmful event does not materialize.

To rule out trivial cases, we assume that the mitigation actions are cost-effective for both the sender and the receiver, i.e., $\delta_s > \kappa_s$ and $\delta_r > \kappa_r$. Nonetheless, the marginal benefit from the mitigation action is higher for the sender than for the receiver, i.e., $\delta_s/\kappa_s > \delta_r/\kappa_r > 1$. This is the case, for instance, when the sender does not internalize the receiver's mitigation cost or when the receiver's mitigation effort generates additional positive externalities for the sender.

3.2. Early Signals

While the receiver has the resources to mitigate the effects of a disaster, the sender has proprietary access to early signals about this event. In this paper, we mostly consider disasters that occur with advance notice, such as global pandemics and hurricanes, which always emerge first as local outbreaks and tropical storms, respectively. (We relax this assumption in Section 6.2.) In our set-up, therefore, events that ultimately materialize (i.e. when $x_t = 1$) are always preceded by early signals that the sender privately observes at the beginning of the period. Conversely, the absence of these signals must imply that an event will not occur (i.e., $x_t = 0$). More formally, we define signal Z_t as a binary random variable with realizations $z_t \in \{0,1\}$, such that $z_t = 1$ when the sender observes the advance signal and $z_t = 0$ when she does not. The value of z_t, i.e., whether an advance signal is present or absent at the beginning of period t, is the sender's private information. In particular, $z_t = 0$ must suggest $x_t = 0$.

Depending on the quality of the information she has access to, however, the sender may pick up signals that wrongly indicate a danger so that $z_t = 1$ does not necessarily imply $x_t = 1$. Specifically, we represent the signals' quality as type $\theta \in \{H, L\}$: *high-quality* signals $(\theta = H)$ always correctly predict the occurrence of the event, while *low-quality* signals $(\theta = L)$ are completely uninformative, i.e., the event's probability remains equal to the prior π if the sender receives a signal of type $\theta = L$. This means that the signal is always present when $\theta = L$. (We extend our result in Section 6.1 to the case where the signal is not entirely uninformative and can be absent when $\theta = L$.)

The signal's type is unknown a priori to both parties[7] and $p_0 \equiv \mathbb{P}[\theta = H]$ represents the common prior belief at time $t = 0$ that the sender has access to high-quality signals (and hence $1 - p_0 = \mathbb{P}[\theta = L]$). As time unfolds and new information is revealed, both parties update their belief about

[7] Here, the signals' quality type is assumed to persist across the periods. This is the case when the sender relies on the same infrastructure to receive the signals, as WHO does with the Global Outbreak Alert and Response Network or National IHR Focal Points of member states, as defined by World Health Organization (2005).

the quality of the signal. The sender, however, privately observes the signals and thus better infer over time the signal's true type, giving her an informational advantage about the signal type. This situation gives rise to a Bayesian persuasion game, whereby the sender can control and obfuscate her information to influence the receiver's learning process and ultimately his actions.[8]

3.3. Warning Policies

Based on her private information at the beginning of each period, the sender may decide to warn the receiver about a potential danger. In its most general form, the sender's warning policy specifies, for each period, a mapping from the sender's historical information and her current advance signal to a probability of issuing a warning. Such a warning probability can be broadly thought of as an abstraction of the severity of warning messages (e.g., the frequency of sounding an alarm).

Formally, let $\mathbf{h}_{s,t-1}$ represent the history of the game at time t, as observed by the sender, which includes the entire trajectory of all random variable realizations (and her private signals in particular) as well as all players' decisions until the end of period $t-1$. We define a warning decision at time t as $\boldsymbol{\omega}_t(\cdot) = (\omega_t^0(\cdot), \omega_t^1(\cdot))$, where $\omega_t^z(\mathbf{h}_{s,t-1}) \in [0,1]$ for $z \in \{0,1\}$, such that the sender issues a warning with probability $\omega_t^1(\mathbf{h}_{s,t-1})$ in the presence of an advance signal and with probability $\omega_t^0(\mathbf{h}_{s,t-1})$ otherwise. Later in this section, we introduce a notion of "reputation," which is sufficient to capture the sender's history.

The warning decisions $\omega_t^1(\mathbf{h}_{s,t-1}) = 1$ and $\omega_t^0(\mathbf{h}_{s,t-1}) = 0$ correspond to a *full-disclosure* policy in period t, since the sender's warning perfectly reveals her private signal in that period. Deviation from full disclosure consists in misrepresenting this information with a positive probability, i.e., by having $\omega_t^1(\mathbf{h}_{s,t-1}) < 1$ or $\omega_t^0(\mathbf{h}_{s,t-1}) > 0$. The closer the sender sets probabilities $\omega_t^0(\mathbf{h}_{s,t-1})$ and $\omega_t^1(\mathbf{h}_{s,t-1})$ to one another, the more she obfuscates or, equivalently, the less she discloses her private information. In the extreme, when these two probabilities are equal, $\omega_t^1(\mathbf{h}_{s,t-1}) = \omega_t^0(\mathbf{h}_{s,t-1})$, the warning message does not carry any meaningful information about her private signal.

In our set-up, the misrepresentation $\omega_t^0(\mathbf{h}_{s,t-1}) > 0$ implies that the sender *exaggerates* the risk. Indeed, define D_t as the binary random variable that indicates whether the receiver is warned in period t. Let warning message $d_t \in \{0,1\}$ denote the realization of D_t. Then, $d_t = 1$ if the sender ends up issuing a warning and $d_t = 0$ otherwise. When $\omega_t^0(\mathbf{h}_{s,t-1}) > 0$, the sender may warn the receiver ($d_t = 1$) even though she has received an early signal ($z_t = 0$). This means that $D_t > Z_t$ (according to the first-order stochastic dominance) and thus the message that is sent to the receiver

[8] This framework is reasonable to model WHO, which was founded by different countries (member states) "for the purpose of co-operation among themselves and with others to promote and protect the health of all peoples" (World Health Organization 1946). As such, the organization plays the role of a "coordinating authority on international health work," and thus extensively relies on expertise and infrastructures independent of the agency. While the nature of these information sources is public knowledge a priori, the exact signals it generate are not public.

is "stronger" than the actual signal. Similarely, the sender *downplays* the risk when $\omega_t^1(\mathbf{h}_{s,t-1}) < 1$, which implies $D_t < Z_t$.

A warning policy Ω maps all possible histories to four probability distributions, two per period. Formally, define $\Omega \equiv [\boldsymbol{\omega}_1(\cdot), \boldsymbol{\omega}_2(\cdot))]$, such that the sender warns the receiver in period 1 and 2 with probabilities $\boldsymbol{\omega}_1(\mathbf{h}_{s,0})$ and $\boldsymbol{\omega}_2(\mathbf{h}_{s,1})$, respectively, and where $\mathbf{h}_{s,0} = \emptyset$ and $\mathbf{h}_{s,1} = \{d_1, x_1, z_1\}$. Warning policy Ω is announced to the receiver at the beginning of time horizon. The sender's problem is to design and commit to a warning policy Ω at time zero so as to minimize her total discounted cost over the two periods, the formulation of which is to be provided in Section 3.6.

Subsequently, upon receiving warning d_t, the receiver decides whether or not to take mitigation actions. The receiver makes this choice based on the sender's warning message and on the history of the game he has observed thus far. Specifically, define $\mathbf{h}_{r,t-1}$ as the receiver's history at the beginning of period t, where $\mathbf{h}_{r,0} = \emptyset$ and $\mathbf{h}_{r,1} = \{d_1, x_1\}$. The receiver only partially observes the sender's history, i.e. $\mathbf{h}_{r,t-1} \subset \mathbf{h}_{s,t-1}$, since signal z_t is the sender's private information. Overall, given warning d_t, history $\mathbf{h}_{r,t-1}$, and warning policy Ω, the receiver chooses action $a_t(d_t, \mathbf{h}_{r,t-1}|\Omega)$ in period t, which is characterized in Section 3.5.

3.4. Sender's Reputation

As time unfolds, the receiver contrasts the warning messages he received with the actual realization of events and thus assesses the quality of the sender's advance signals. This assessment at the beginning of period t corresponds to the receiver's posterior probability $p_{t-1} \equiv \mathbb{P}[\theta = H \mid \mathbf{h}_{r,t-1}]$, $t = 1, 2$ (recall that $p_0 = \mathbb{P}[\theta = H]$). Hereafter, we refer to p_{t-1} as the sender's *reputation* at the beginning of period t. The next lemma establishes the central role that reputation plays in designing warning policies.

LEMMA 1 **(Sufficiency of Reputation)**. *It is without loss of generality to restrict our analysis to the class of policies such that, for any t, $\boldsymbol{\omega}_t(\mathbf{h}_{s,t-1}) = \boldsymbol{\omega}_t(\mathbf{h}'_{s,t-1})$ and $a_t(d_t, \mathbf{h}_{r,t-1}|\Omega) = a_t(d_t, \mathbf{h}'_{r,t-1}|\Omega)$ for any two histories $\mathbf{h}_{r,t-1} \subset \mathbf{h}_{s,t-1}$ and $\mathbf{h}'_{r,t-1} \subset \mathbf{h}'_{s,t-1}$ that share the same reputation $p_{t-1} = \mathbb{P}[\theta = H \mid \mathbf{h}_{r,t-1}] = \mathbb{P}[\theta = H \mid \mathbf{h}'_{r,t-1}]$.*

To obtain this lemma, we essentially demonstrate (see online appendix C.2) that the sender's reputation fully captures all payoff-relevant information embedded in both players' histories. Thus, both players' decisions can be recast as functions of p_t instead of $\mathbf{h}_{r,t}$ and $\mathbf{h}_{s,t}$, without loss of generality. In particular, according to Lemma 1, the sender's warning policy $\omega_t^z(\mathbf{h}_{s,t-1}) = \mathbb{P}[D_t = 1 \mid Z_t = z, \mathbf{h}_{s,t-1}]$ can be redefined, with slight abuse of notation, as $\omega_t^z(p_{t-1}) = \mathbb{P}[D_t = 1 \mid Z_t = z, p_{t-1}]$, i.e., $\omega_t^z(\cdot)$ now maps $[0,1] \mapsto [0,1]$. Therefore, the lemma implies that the sender's decisions only depend on her current reputation, p_{t-1}. We refer in the following to a warning policy as $\Omega = \left[\boldsymbol{\omega}_1(\cdot), \boldsymbol{\omega}_2(\cdot)\right]$ with $\boldsymbol{\omega}_t(\cdot) = (\omega_t^0(\cdot), \omega_t^1(\cdot))$ mapping $[0,1] \mapsto [0,1]^2$ for each period

t. Similarly, we can redefine the receiver's decision in each period t as $a_t(d_t, p_{t-1} \mid \Omega)$, which is a mapping from $\{0,1\} \times [0,1]$ to $\{0,1\}$ for any given warning policy Ω.

Notably, the notion of reputation defined above is a dynamic measurement of the sender's competency (i.e., perceived quality of her advance signal), whose evolution not only depends on the sender's historical performance but also the sender's warning policy. More specifically, given policy Ω, the receiver updates reputation p_{t-1} at the end of each period by contrasting warning message d_t with realization x_t. Define updated reputation $p_t^{d,x} \equiv \mathbb{P}[\theta = H \mid D_t = d, X_t = x, p_{t-1}, \Omega]$ as the posterior probability of p_{t-1}. The following result characterizes reputation $p_t^{d,x}$ using Bayes' rule.

LEMMA 2. *Given realizations $D_t = d$ and $X_t = x$ with $(d,x) \in \{0,1\}^2$, reputation p_{t-1} is updated at the end of period t to $p_t^{d,x}$ according to*

$$p_t^{1,1} = p_t^{0,1} = p_{t-1}, \tag{1}$$

$$p_t^{1,0} = \frac{p_{t-1} w_t^0}{p_{t-1} w_t^0 + (1 - p_{t-1}) w_t^1}, \tag{2}$$

$$p_t^{0,0} = \frac{p_{t-1} \overline{w}_t^0}{p_{t-1} \overline{w}_t^0 + (1 - p_{t-1}) \overline{w}_t^1}, \tag{3}$$

where warning probabilities $w_t^1 = \omega_t^1(p_{t-1})$, $w_t^0 = \omega_t^0(p_{t-1})$, and $\overline{w}_t^z = 1 - w_t^z$ for $z \in \{0,1\}$. In particular, $p_t^{0,0} \geq p_{t-1} \geq p_t^{1,0}$ if and only if $w_t^1 \geq w_t^0$.

To understand Lemma 2, we only consider in the following warning decisions such that $\omega_t^1(\cdot) \geq \omega_t^0(\cdot)$. We show in Section 3.5 that this restriction is without loss of generality. In this case, Lemma 2 indicates that the sender's reputation for the next period is damaged only if she sends a *false alarm* in the current period (2), i.e., she issues a warning ($d_t = 1$) but the event does not materialize ($x_t = 0$). On the other hand, her reputation improves only if she remains correctly silent (3), i.e., no warning ($d_t = 0$) is issued and the event does not occur ($x_t = 0$). Otherwise, her reputation is not affected (1). (Section 6.2 studies settings where the receiver also updates the sender's reputation from an event's occurrence, with $p_t^{1,1} \geq p_{t-1} \geq p_t^{0,1}$.)

Indeed, we consider events that always occur with advance notice. Thus, when the event actually happens ($x_t = 1$), the receiver knows for sure that the sender has obtained a signal (i.e., $z_t = 1$) regardless of type θ and that warning decision d_t could only be drawn from probability w_t^1. The value of d_t, therefore, does not provide any new information about quality type θ. By contrast, when the event does not happen ($x_t = 0$), the relative weight of the warning probabilities w_t^0 and w_t^1 indicates the likelihood of a warning ($d_t = 1$) being triggered by the sender's advance signal, and hence the value of d_t allows the receiver to make an inference about type θ, as long as $w_t^0 \neq w_t^1$.

More specifically, updated reputation $p_t^{1,0}$ and $p_t^{0,0}$ do not significantly diverge from p_{t-1} when ratios w_t^1/w_t^0 and thus $\overline{w}_t^1/\overline{w}_t^0$ are close to one (see equations (2)-(3)). By contrast, an increase in ratio $w_t^1/w_t^0 \geq 1$, and thus a decrease in ratio $\overline{w}_t^1/\overline{w}_t^0 \leq 1$, moves $p_t^{1,0}$ and $p_t^{0,0}$ away from p_{t-1} in opposite directions, where $p_t^{1,0}$ decreases and $p_t^{0,0}$ increases.

In short, Lemma 2 indicates how the sender's decision affects her future reputation. She can be conservative and set the two warning probabilities, w_t^1 and w_t^0, close to each other so as to maintain her reputation around its current level. This deviation from full disclosure obfuscates her private information, and thus prevents the receiver from learning about the quality of the sender's signal. Alternatively, she can set these probabilities apart (i.e., leaning towards a full-disclosure policy) and, thus, better reveal her private information, which makes more drastic adjustments in her reputation. However, this is riskier, as the direction of these adjustments depends on the presence of an advance signal and the realization of the event, both of which are out of her control. In particular, full disclosure induces *perfect learning* with $p_1^{1,0} = 0$ and $p_1^{0,0} = 1$, i.e., the sender completely loses her reputation in the case of a false alarm but gains the highest reputation otherwise.

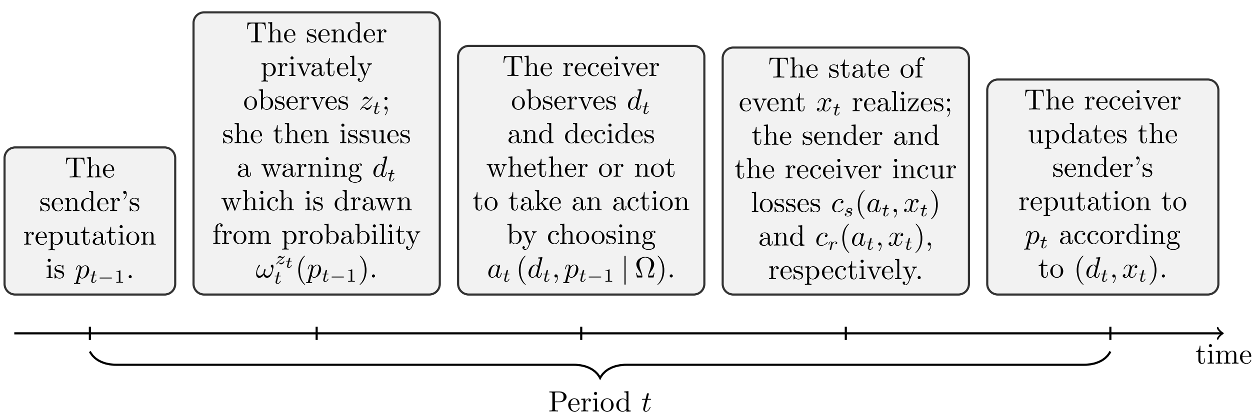


Figure 1 Sequence of events during period t

Finally, Figure 1 depicts the sequence of events that unfold in each period. First, the sender observes an advance signal or the lack thereof. She next warns the receiver according to a probability that depends on both this private information and her current reputation. Based on this warning message and the sender's reputation, the receiver decides whether or not to act. Finally, the true state of the disaster is revealed (i.e., it occurs or fails to occur), costs are incurred (according to Table 1) and the receiver updates the sender's reputation (following Lemma 2).

3.5. The Receiver's Best Response

For any given warning policy Ω, the receiver seeks to minimize his total expected discounted cost by choosing his optimal actions in periods $t = 1, 2$. This problem is separable in time and can be solved independently for each period. Thus, the receiver's optimal action in period

t depends on warning probabilities $\boldsymbol{\omega}_t(\cdot) = (\omega_t^0(\cdot), \omega_t^1(\cdot))$, warning message d_t, and reputation p_{t-1} (Lemma 1), and is defined, with slight abuse of notation, as $a^\star(d_t, p_{t-1}, \boldsymbol{\omega}_t(p_{t-1})) \equiv \arg\min_{a\in\{0,1\}} \mathbb{E}\left[c_r(a, X_t) \mid d_t, p_{t-1}, \boldsymbol{\omega}_t(p_{t-1})\right]$.[9] The following result characterizes this optimal choice.

LEMMA 3. *Given warning probabilities* $\boldsymbol{w} = (w^0, w^1) \in [0,1]^2$, *warning message* $d \in \{0,1\}$ *and reputation* $p \in [0,1]$, *the optimal decision of the receiver is equal to,*

$$a^\star(d, p, \boldsymbol{w}) = \begin{cases} d, & \text{if} \quad (p^* - p)\, w^1 + p w^0 \le \min\{0, p^*\} \\ 0, & \text{if} \quad 0 < (p^* - p)\, w^1 + p w^0 < p^* \\ 1, & \text{if} \quad p^* < (p^* - p)\, w^1 + p w^0 < 0 \\ (1-d), & \text{if} \quad (p^* - p)\, w^1 + p w^0 \ge \max\{0, p^*\}, \end{cases} \tag{4}$$

where reputation threshold $p^* \equiv 1 - \eta_r$ *with* $\eta_r \equiv \pi/\overline{\pi}\,(\delta_r/\kappa_r - 1) > 0$.

Lemma 3 allows us, without loss of generality, to restrict feasible decisions $\boldsymbol{\omega}_t(\cdot) : [0,1] \mapsto [0,1]^2$ to mappings $\boldsymbol{\omega}_t(\cdot) : [0,1] \mapsto \mathcal{W}$, where $\mathcal{W} \equiv \{(w^0, w^1) \in [0,1]^2 : w^0 \le w^1\}$. To see this, note that when $w_t^0 > w_t^1$, the meaning of $d = 1$ and $d = 0$ can be switched ex-ante to "not warning" and "warning" the receiver, respectively.[10] Lemma 3 then implies that

$$a^\star(d, p, \boldsymbol{w}) = \begin{cases} d, & \text{if} \quad w^0/w^1 \le (p - p^*)/p \le 1 \text{ or } \overline{w}^0/\overline{w}^1 \ge (p - p^*)/p \ge 1, \\ 0, & \text{if} \quad (p - p^*)/p \le w^0/w^1 \le 1, \\ 1, & \text{if} \quad 1 \le \overline{w}^0/\overline{w}^1 \le (p - p^*)/p, \end{cases} \qquad \text{for } \boldsymbol{w} \in \mathcal{W}. \tag{5}$$

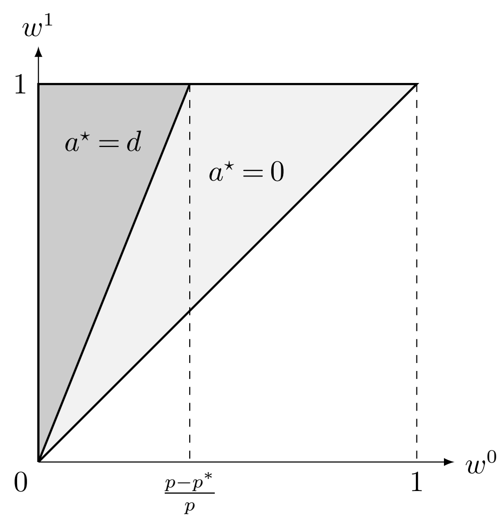


Figure 2 Illustration of the partition of $\mathcal{W}$ given reputation p for $p^* > 0$.

Figure 2 depicts feasible set $\mathcal{W}$ when reputation $p \ge p^* \ge 0$. Set $\mathcal{W}$ corresponds to the shaded area (above the diagonal of the unit square), which is split into two non-intersecting triangles. In

[9] As a convention, when multiple best responses exist, the receiver chooses the one that minimizes the sender's cost.

[10] Indeed, the final case in (4) corresponds to the first one under the mirror policy $\left(1 - \omega_t^0(\cdot), 1 - \omega_t^1(\cdot)\right)$.

the upper triangle, the sender's warning decision is close to full disclosure (i.e., w^0 is close to zero and w^1 is close to one so that $w^0/w^1 \le (p-p^*)/p$), and the receiver takes action according to the warning message ($a^\star = d$). By contrast, in the lower triangle, the sender's warning message is not very indicative of the sender's private signal information (i.e., w^0/w^1 is close to one), in which case the receiver ignores the warning message and always chooses not to act ($a^\star = 0$). In particular, when the sender's reputation becomes lower than p^*, the upper triangle disappears and the receiver prefers never to act ($a^\star = 0$).

Threshold p^* plays a key role in our analysis, and only depends on the receiver's cost structure. It is positive if and only if $\pi < \kappa_r/\delta_r$. When $p^* \le 0$ (i.e., the event is sufficiently likely to occur with $\pi \ge \kappa_r/\delta_r$), the receiver has enough incentive to take mitigation action, in which case the sender's warning message acts to prevent overreaction from the receiver. Similar to the case with $p^* \ge 0$, the receiver will follow the warning message ($a^\star = d$) only when the sender's warning decision is close to full disclose (i.e., $\overline{w}^0/\overline{w}^1 \ge (p - p_r^*)/p$). Otherwise, the receiver will always act ($a^\star = 1$).

Overall, the sender needs to disclose enough of her private information (i.e., set w^0 and w^1 apart), so that the receiver acts according to her warning message ($a^\star = d$). Otherwise, the warning message is incapable of influencing the receiver when the sender obfuscates her private information too much (i.e., when w^0 and w^1 are close to each other). In fact, the lower the reputation, the more disclosure is needed to influence the receiver, as $(p-p^*)/p$ is increasing (resp. decreasing) in p when $p^* \ge 0$ (resp. $p^* < 0$). If reputation is too low ($p < p^*$), inducing an action is never feasible. In this sense, threshold $p^* \ge 0$ is the minimum reputation level required to influence the receiver.

This points to a fundamental trade-off for the sender. To influence the receiver in the current period, she needs to sufficiently disclose her private information. But, as discussed in Section 3.3, this also increases the risk of damaging her reputation, and hence her ability to influence the receiver in the next period.

3.6. Sender's Problem

In anticipation of the receiver's response specified in Lemma 3, the sender's objective is to design and commit to a warning policy $\Omega = [\boldsymbol{\omega}_1(\cdot), \boldsymbol{\omega}_2(\cdot)]$, that minimizes her total expected discounted cost, with discount factor $\rho \in [0,1]$,

$$\min_{\Omega} \left\{ \sum_{t=1}^{2} \rho^{t-1} \mathbb{E}\left[c_s\left(a^\star\left(D_t, p_{t-1}, \boldsymbol{\omega}_t(p_{t-1})\right), X_t\right) \mid \Omega, p_0 \right] \right\}. \tag{6}$$

In our set-up, the sender has full commitment power in the sense that she can assure the receiver that she will be true to her word. The sender does not know what the future signals will be, but designs and announces at time $t = 0$ a *dynamic* policy that determines how she will warn the receiver as her private signals and the events unfold overtime, i.e., in all possible future states of the world. Commitment power simply means that the sender always abides by this announced policy.[11] As established in Lemma 1, the sender's commitment power allows her to search for the optimal policy within the set of dynamic policies that are only contingent on the *public* history, whose payoff-relevant information is in turn summarized by the sender's *reputation*.

Commitment power typically holds when the sender's policy is set by law or when the sender's credibility is severely damaged if she deviates from her announced policy (see, for instance, Kamenica and Gentzkow 2011, Che and Hörner 2017). In the case of WHO, for instance, the warning policy is set by international regulations (World Health Organization 2005). In addition, WHO makes a number of important health recommendations for non-communicable diseases.[12] Losing its credibility by deviating from its own pandemic warning policy could significantly reduce the impact of WHO's announcements concerning other health issues.

Further, when the sender has commitment power, she is always better off using it (see Fudenberg and Tirole (1991), Ch.3 for a general discussion and Best and Quigley (2017) for the case of persuasion games). Indeed, without commitment power, the sender's messages become cheap talk, a form of friction that increases her expected costs. In fact, our findings provide a meaningful lower-bound benchmark for the sender's expected costs in the non-commitment case.

Finally, we note that the sender's decision $\boldsymbol{\omega}_t(\cdot)$ depends on the public history via reputation p_{t-1}, and is thus dynamic. In particular, the sender's problem (6) has the following dynamic programming formulation:

$$J_2(p_1) \equiv \min_{\boldsymbol{w}_2 \in \mathcal{W}} \mathbb{E}\left[c_s\left(a^\star\left(D_2, p_1, \boldsymbol{w}_2\right), X_2\right) \mid p_1\right], \tag{7}$$

$$J_1(p_0) \equiv \min_{\boldsymbol{w}_1 \in \mathcal{W}} \mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) + \rho J_2(p_1) \mid p_0\right], \tag{8}$$

where $J_t(.)$ represents the optimal cost function at the beginning of period t and p_1 stochastically evolves from p_0 according to Lemma 2.

In the following sections, we solve dynamic program (7)-(8) via backward induction. We start with (7), which corresponds to a single-period problem. We then move to the first-period problem (8), the solution of which yields the optimal warning policy of the original problem (6).

4. The Single Period Problem

The second period in our setting corresponds to a single-period problem, in which the sender's future reputation is irrelevant. The sender only seeks to mobilize the receiver's action in the present

[11] Only in this way can the sender's messages have a uniquely-defined meaning for the receiver to interpret. In the case of WHO, for instance, commitment power means that WHO can convince its member states that it will follow its own regulation, which specifies in advance under which circumstances the organization will raise (or not) an alarm against a pandemic of international concern (World Health Organization 2005).

[12] See http://www.who.int/topics/en/.

period. To be consistent with the notation of Problem (7), we denote the optimal warning policy and cost function of the single period problem as $\boldsymbol{\omega}_2^\star(\cdot)$ and $J_2(\cdot)$, respectively.

Proposition 1. *Given reputation p at the beginning of a single-period problem, the optimal warning policy is given as follows, where p^* is defined in Lemma 3.*

- *If $p \geq p^*$, full disclosure policy $\boldsymbol{\omega}_2^\star(p) = (0,1)$ is optimal, and the receiver always acts upon receiving a warning, i.e., $a^\star(d, p, \boldsymbol{\omega}_2^\star(p)) = d$.*
- *If $p < p^*$, any policy is optimal and the receiver never acts, i.e., $a^\star(d, p, \boldsymbol{\omega}_2(p)) = 0$ for any feasible policy $\boldsymbol{\omega}_2(p)$.*

Furthermore,

$$J_2(p) = \begin{cases} \pi\ell_s - \overline{\pi}\kappa_s(p + \eta_s - 1) & \text{if } p \geq p^* \\ \pi\ell_s & \text{if } p < p^*, \end{cases} \tag{9}$$

where $\eta_s \equiv \pi/\overline{\pi}\,(\delta_s/\kappa_s - 1) > 0$.

Since any policy is optimal when $p < p^*$, we assume by convention hereafter that the sender follows the full-disclosure policy in this case as well. Hence, we have the optimal warning decisions $\boldsymbol{\omega}_2^\star(p) = (0,1)$ for all $p \in [0,1]$ in the single-period problem. Indeed, the sender aims to elicit the correct action from the receiver only for this particular period. The absence of a signal indicates no upcoming event and the sender wants the receiver not to overreact so that unnecessary mitigation costs can be avoided. On the other hand, the presence of a signal strengthens the sender's belief about the risk and her incentive to induce an action from the receiver. However, the receiver has no incentive to do so a priori unless the sender presents adequately strong evidence. Therefore, in both cases, the sender has an incentive to perfectly reveal her private information, making full disclosure the best policy in the single-period setting.

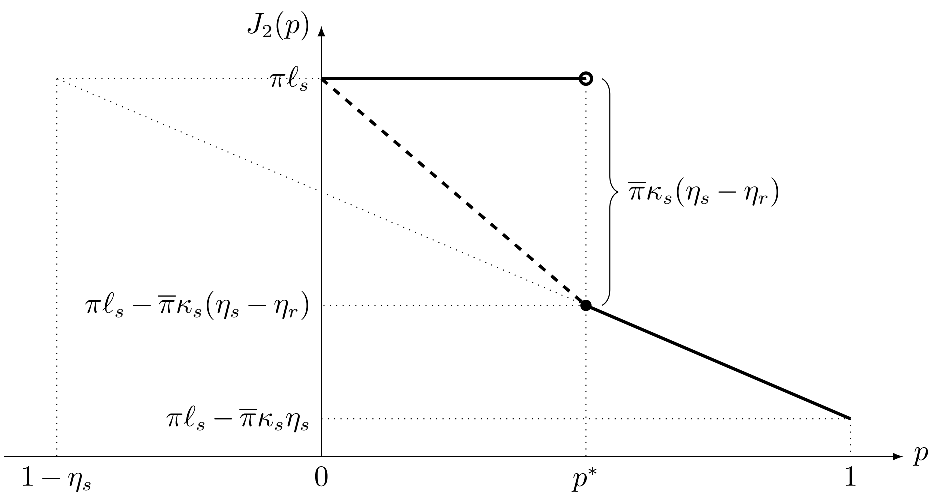


Figure 3 **Illustration of $J_2(p)$ for $p^* > 0$ and $\eta_s > 1$.**

Proposition 1 implies that if reputation is high enough, the sender possesses persuasive power and the receiver fully complies with the sender's warning recommendation. (This is particularly true for all p_0 when $p^* \leq 0$.) By contrast, if reputation is low, the receiver ignores the sender and refrains from acting even when the sender warns. More specifically, when the sender's initial reputation is low (i.e., $p < p^*$), revealing the presence of an advance signal (by issuing a warning) does increase the receiver's risk assessment, but not to the extent of eliciting an action. In this case, the receiver becomes completely inactive and the sender's expected cost is independent of her reputation, as Figure 3 illustrates. When reputation is high enough (i.e., $p \geq p^* > 0$), however, a warning message raises the receiver's risk assessment sufficiently high to elicit an action. In this case, the private signal of a sender with higher reputation is perceived to be more accurate and thus lowers her expected losses, as shown in Figure 3. When reputation is right at the threshold ($p = p^* > 0$) the sender's expected cost $J_2(p)$ is discontinuous as illustrated in Figure 3. The size of this drop, given by $\overline{\pi}\kappa_s(\eta_s - \eta_r) = \pi\kappa_s\left[\delta_s/\kappa_s - \delta_r/\kappa_r\right] \geq 0$, captures the difference in the marginal benefits of mitigation between the sender and the receiver. (When $p^* \leq 0$, only the upper branch of (9) is active and $J_2(p)$ is linear without discontinuity.)

In particular, when $0 < \kappa_s/\delta_s \leq \pi \leq \kappa_r/\delta_r < 1$, the incentives of the sender and the receiver are misaligned a priori. In this case, indeed, the receiver is reluctant to act a priori, i.e., $\pi\ell_r \leq \overline{\pi}\kappa_r + \pi(\kappa_r + \ell_r - \delta_r)$, whereas the sender wants the receiver to act a priori, i.e., $\pi\ell_s \geq \overline{\pi}\kappa_s + \pi(\kappa_s + \ell_s - \delta_s)$. In contrast with the extant literature on static Bayesian persuasion games (e.g., Kamenica and Gentzkow 2011), Proposition 1 then implies that the mere misalignment of incentives between the sender and receiver does not necessarily justify misrepresenting the sender's private information, since full disclosure is optimal for the single-period problem.

5. Optimal Warning Policies and Misrepresentation of Risk

We characterize in this section the optimal policy for the first-period problem (8), which constitutes our main result. In contrast to the single-period problem, the sender now needs to account for the impact of her current warning decision on her future reputation, p_1, which in turn determines her ability to elicit an action in the next period. As discussed in the previous section, this ability is effective in the second period if and only if her future reputation is higher than threshold p^*. To achieve this, however, the sender sometimes needs to compromise her ability to elicit an action in the current period.

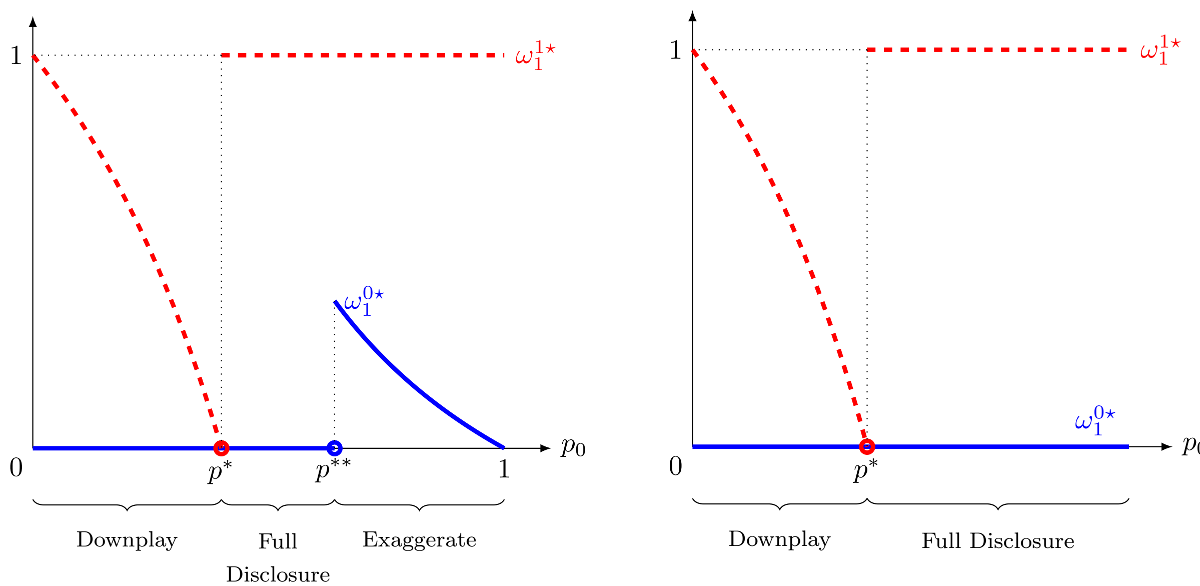
5.1. Characterization of the Optimal Policy

In problem (8), cost-to-go $J_2(\cdot)$ is determined by Proposition 1 and the terminal state p_1 evolves from the sender's prior reputation, p_0, according to Lemma 2. The next theorem constitutes our main result and provides the full characterization of the sender's optimal warning policy in the first-period (where p^* and η_r are defined in Lemma 3 and η_s is defined Proposition 1.)

Theorem 1. *If $\pi < \kappa_s/\delta_s$ or $\pi > \kappa_r/\delta_r$, the optimal warning policy in the first period is given by $\boldsymbol{\omega}_1^\star(p_0) = (0,1)$ for all $p_0 \in [0,1]$. Otherwise (i.e., $\kappa_s/\delta_s \le \pi \le \kappa_r/\delta_r$), the optimal warning policy in the first period is given by*

$$\boldsymbol{\omega}_1^\star(p_0) = \begin{cases} \left(0, \frac{p^* - p_0}{p^*(1-p_0)}\right), & \text{if } 0 \le p_0 < p^*, \\ (0,1), & \text{if } p^* \le p_0 < p^{**}, \\ \left(\frac{p^*}{(1-p^*)} \frac{(1-p_0)}{p_0}, 1\right), & \text{if } p^{**} \le p_0 \le 1, \end{cases} \tag{10}$$

with $p^{**} \equiv (2-p^*)p^* > p^* > 0$ for $\rho \ge \widehat{\rho}$ and $p^{**} \equiv 1$ otherwise, where $\widehat{\rho} \equiv p^*/[\overline{\pi}\eta_r(\eta_s - 1)]$.



(a) $\rho \ge \widehat{\rho}$. (b) $\rho < \widehat{\rho}$.

Figure 4 The optimal warning policy in the first period for $\pi \in (\kappa_s/\delta_s, \kappa_r/\delta_r)$.

When the event is either highly likely or unlikely, the sender's and receiver's incentives are aligned a priori: both of them prefer either to take mitigation action (in the case of $\pi > \kappa_r/\delta_r$), or not to act (in the case of $\pi < \kappa_s/\delta_s$). In this case, Theorem 1 shows that full disclosure remains optimal for the first period. By contrast, when the occurrence of the event becomes uncertain and the sender's and receiver's incentives are no longer aligned a priori, Theorem 1 demonstrates that the optimal warning policy may deviate from full disclosure in two distinctive directions. Specifically, if reputation p_0 is low (i.e., $0 \le p_0 < p^*$), the sender refrains from warning the receiver with positive probability in the presence of an advance signal, but remains silent in its absence, as illustrated by both Figures 4(a) and 4(b). In other words, the sender sometimes *downplays* the risk of a harmful event even though her private information indicates otherwise. Further, $\omega_1^{1\star}(p_0)$ decreases with p_0.

This means, perhaps surprisingly, that a sender with a higher reputation is more likely to downplay the risk by withholding her private information.

If reputation p_0 is high enough (i.e., $p_0 \geq p^{**} > p^* \geq 0$) and the sender sufficiently cares about the future (i.e., $\rho \geq \widehat{\rho}$), the sender always warns the receiver in the presence of a signal, but persists in warning him with positive probability even in the absence of a signal. (See Figure 4(a).) That is, the sender sometimes *exaggerates* the threat even when she knows with certainty that no event is on the horizon. In this case, the probability of exaggerating the threat, $\omega_1^{0\star}(p_0)$, decreases with reputation p_0, and hence the deviation away from full-disclosure actually diminishes when the sender's reputation goes higher.

Otherwise, when reputation p_0 takes intermediate values (i.e., $0 \leq p^* \leq p_0 < p^{**}$), full disclosure is optimal (Figure 4(a)). In particular, when the future becomes less important (i.e., $\rho < \widehat{\rho}$), full disclosure remains optimal for reputation p_0 up to $p^{**} = 1$ (Figure 4(b)).

It is worth noting that the optimal warning policy depends on the sender's and the receiver's cost parameters only through η_s and η_r, respectively. Varying the cost parameters while keeping η_s and η_r fixed does not change the optimal warning policy. We leverage this observation in Section 6.3 when we explore the case of multiple heterogeneous receivers. (The dependence of the optimal warning policy on probability π is examined in Appendix A; see Proposition A.1 and Figure A.1).

Finally, we describe below the receiver's response to the sender's optimal policy from Lemma 3.

COROLLARY 1. *In response to the sender's optimal warning policy, the receiver acts on a warning in the first period if and only if reputation p_0 exceeds p^*, i.e., $a_1^\star(d, p_0, \boldsymbol{\omega}_1^\star(p_0)) = d$ for $p_0 \geq p^*$ and $a_1^\star(d, p_0, \boldsymbol{\omega}_1^\star(p_0)) = 0$ for $p_0 < p^*$. In particular, for $\pi > \kappa_r/\delta_r$, the receiver always acts on a warning, i.e., $a_1^\star(d, p_0, \boldsymbol{\omega}_1^\star(p_0)) \equiv d$ for all $p_0 \in [0,1]$.*

In light of (5), Corollary 1 states that, whenever possible (i.e., $p \geq p^*$), the sender keeps the two warning probabilities $w_1^{1\star}$ and $w_1^{0\star}$ distinctive enough to elicit an action in the first period.

5.2. Optimal Posterior Reputation

We can alternatively interpret Theorem 1 in terms of the sender's posterior reputation. Indeed, for any Bayesian persuasion game (e.g., Kamenica and Gentzkow 2011), there is a one-to-one correspondence between the information provision decisions (i.e., the warning probabilities herein) and the receiver's posterior belief (i.e., the posterior reputation herein). In fact, a powerful solution approach recently developed for dynamic persuasion games (e.g., Ely 2017, Renault et al. 2017) leverages this change of decision variables. In those settings, it suffices to require the sender's choice of the receiver's posterior belief to be consistent in that the receiver's expected posterior belief given the prior remains unchanged, i.e., the belief evolution satisfies the martingale property. It then implies that the convex envelope of the sender's objective function determines the optimal posterior belief and, thus, her optimal information provision decisions.

In our set-up, Lemma 2 implies that the evolution of the sender's reputation indeed satisfies the martingale property $p_0 = \mathbb{E}[p_1 \mid p_0]$, or, more explicitly,

$$p_0 = \underbrace{\pi}_{\mathbb{P}[X_1=1]} p_0 + \underbrace{(1-\pi)\left[p_0 w_1^0 + (1-p_0) w_1^1\right]}_{\mathbb{P}[D_1=1, X_1=0 \mid p_0, \Omega]} p_1^{1,0} + \underbrace{(1-\pi)\left[p_0 \overline{w}_1^0 + (1-p_0)\overline{w}_1^1\right]}_{\mathbb{P}[D_1=0, X_1=0 \mid p_0, \Omega]} p_1^{0,0}. \tag{11}$$

However, in our context, the need to account for the learning effect (resulting from contrasting the sender's warning decisions with the exogenous event realizations) limits the space of plausible martingale beliefs, preventing us from reformulating (8) in terms of posterior reputation purely subject to the martingale constraint. In particular, the transition law governing the reputation evolution (c.f., (2)-(3)) is not exogenously given as in Ely (2017) and Renault et al. (2017) but, rather, depends on the warning decisions endogenously. Nonetheless, the optimal warning policy can still be expressed in terms of posterior reputations, as stated below.

Corollary 2. *The optimal warning policy induces the sender's posterior reputation at the end of the first period to be* $\left(p_1^{1,0}, p_1^{0,0}\right) = (0,1)$ *for* $\pi < \kappa_s/\delta_s$ *or* $\pi > \kappa_r/\delta_r$*, and*

$$\left(p_1^{1,0}, p_1^{0,0}\right) = \begin{cases} (0, p^*), & \text{if } 0 \le p_0 < p^*, \\ (0,1), & \text{if } p^* \le p_0 < p^{**}, \\ (p^*, 1), & \text{if } p^{**} \le p_0 \le 1, \end{cases} \tag{12}$$

for $\kappa_s/\delta_s \le \pi \le \kappa_r/\delta_r$*, where* p^{**} *is defined in Theorem 1.*

For a highly likely event with $\pi > \kappa_r/\delta_r$, threshold $p^* < 0$ and, hence, the second period's expected cost $J_2(p_1)$ reduces to a fully linear function over $p_1 \in [0,1]$. As such, $\mathbb{E}[J_2(p_1) \mid p_0] = J_2(\mathbb{E}[p_1 \mid p_0]) = J_2(p_0)$, independent of the first period's warning decision. Thus, the optimal warning decision is purely driven by the first-period problem and, thus, must be full disclosure according to Proposition 1, inducing perfect learning (i.e., $p_1^{1,0} = 0$ and $p_1^{0,0} = 1$). For the remaining discussion, we focus on the case $\pi \le \kappa_r/\delta_r$ (i.e., $p^* \ge 0$).

When reputation $p_0 < p^*$, the receiver cannot be influenced in the first period (see Lemma 3) and, thus, the sender only minimizes her future expected cost $\mathbb{E}[J_2(p_1) \mid p_0]$ subject to the martingale constraint (11). As suggested by the convex envelope of $J_2(p_1)$, it is optimal for the sender to set $p_1^{1,0} = 0$ and to boost her reputation $p_1^{0,0}$ at least above p^*. Indeed, for $\kappa_s/\delta_s \le \pi \le \kappa_r/\delta_r$, the dashed line in Figure 3 is steeper than the downward sloping solid line, suggesting that the the convex envelope of $J_2(p_1)$ is achieved by setting $p_1^{0,0} = p^*$. Otherwise, for $\pi < \kappa_s/\delta_s$, the sender boosts $p_1^{0,0}$ all the way to 1 through full disclosure.

When $p_0 \ge p^* > 0$, the sender is capable of eliciting an action in the first period, and thus faces a trade-off between maximizing such capability in the current period and maintaining it for the future. Indeed, the former incentive favors warning policies closer to full disclosure (i.e., at least $w_1^1/w_1^0 \geq p_0/(p_0 - p^*)$ according to (5)), which leads to a more drastic split of p_0 into $p_1^{1,0}$ and $p_1^{0,0}$ in (11); the latter incentive instead calls for a more conservative split so as to keep $p_1^{1,0}$ above p^*.

For $0 \leq p^* \leq p_0 < p^{**}$, it is impossible to achieve both objectives. Since the sender's reputation drops only in the event of a false alarm, the former incentive thus dominates and full disclosure is optimal, creating the widest split of p_0 and inducing perfect learning (i.e., $p_1^{1,0} = 0$ and $p_1^{0,0} = 1$).

For $p_0 \geq p^{**} > p^* \geq 0$, the sender is able to maintain $p_1^{1,0} \geq p^*$ while simultaneously eliciting an action in the first period. As long as $p_1^{1,0} \geq p^*$, the linearity of $J_2(p_0)$ in $p_0 \geq p^*$ (see Figure 3) implies that $\mathbb{E}[J_2(p_1) \mid p_0] = J_2(\mathbb{E}[p_1 \mid p_0]) = J_2(p_0)$ is independent of the first period's warning decision. It is then optimal to induce the widest possible split of p_0 by setting $p_1^{1,0} = p^*$ and $p_1^{0,0} = 1$, enacting a warning policy closest to full disclosure. Indeed, only when the event is uncertain (i.e., $\pi \in (\kappa_s/\delta_s, \kappa_r/\delta_r)$) and the future becomes important (i.e., $\rho \geq \widehat{\rho}$) does the sender's warning carry significant information value in the next period. Thus, the sender prefers to set $p_1^{1,0} = p^* \geq 0$ and $p_1^{0,0} = 1$ rather than adopt full disclosure (see Figure 4(a)). Otherwise (i.e., for $\pi < \kappa_s/\delta_s$ or $\rho < \widehat{\rho}$), the cost increment the sender incurs by deviating from full disclosure in the first period exceeds the benefit she derives from maintaining her reputation in the future, and full disclosure is preferred (see Figure 4(b)).

5.3. Rationale for Misrepresenting the Risk

Next, we provide intuitions for the deviations from full disclosure identified in Theorem 1.

Why downplay? When the sender's initial reputation is too low to elicit an action in the first period (i.e., $p_0 < p^*$, see Proposition 1), the sender only seeks to elicit an action in the next period by boosting her future reputation to a level that is high enough (i.e., above threshold p^*). The only scenario that improves reputation is when the sender remains correctly silent (see Lemma 2). Under the full-disclosure policy (with $w_1^0 = 0$ and $w_1^1 = 1$), this is achieved only in the absence of an advance signal, in which case reputation becomes perfect at the end of the period (with $p_1^{0,0} = 1$). By contrast, in the presence of such a signal, the sender always warns the receiver under full disclosure and her reputation can only drop, incentivizing her to deviate from full disclosure. By sometimes downplaying this risk ($w_1^1 < 1$), she retains the possibility of remaining correctly silent even in the presence of such a signal.

In essence, when obtaining an advance signal, the sender mimics, with a positive probability, the decision she would have made in its absence. Thus, when the event does not materialize, she can still pretend that her private signal is of high quality. (When the event does occur, her reputation is intact, as shown in Lemma 2.) Fully aware of this manipulation, the receiver will no longer update the sender's reputation all the way to perfection ($p_1^{0,0} < 1$), even if the sender remains correctly silent. Nonetheless, the receiver still updates the sender's reputation upward to a level high enough to elicit an action (i.e., above p^*). In this sense, the sender trades off the magnitude of a reputation boost with the likelihood of inducing this boost. For a sender with higher reputation (i.e., with higher value of p_0), the latter benefit dominates the former, leading to more downplaying (i.e., w_1^1 takes smaller values).

Overall, this deviation from full disclosure is akin to the bait-and-switch strategy first identified by Rayo and Segal (2010) for online advertisement problems. In our context, when the event does not ultimately occur, the "bait" is the absence of an advance signal. Indeed, it would induce a receiver's optimistic belief about the sender under full disclosure. The "switch" is the presence of such a signal as it would lead to a receiver's pessimistic belief under full disclosure. But downplaying the risk overturns this pessimistic belief by pooling the switch and bait together into the same warning decision.

Why exaggerate? By contrast, when the sender's initial reputation is sufficiently high (i.e., $p_0 \geq p^{**} > p^* \geq 0$), she can elicit an action in the current period, but also in the next one as long as she maintains her reputation at a sufficiently high level (i.e., above p^*). However, her reputation always drops following a false alarm (see Lemma 2). Therefore, if she cares enough about the future (i.e., $\rho \geq \widehat{\rho}$), the sender seeks to mitigate the damage to her future reputation caused by a false alarm. This is impossible under full disclosure, as a false alarm would completely discredit the sender (with $p_1^{1,0} = 0$). The sender thus has an incentive to deviate from full disclosure.

Perhaps surprisingly, the sender deviates from full disclosure by sometimes warning the receiver even in the absence of an advanced signal (i.e. $w_1^0 > 0$), thereby exacerbating the likelihood of a false alarm and thus inducing the receiver to over-react in the first period. However, knowing that the sender is exaggerating, the receiver now rationalizes a false alarm with two alternative explanations. In addition to the possibly low quality of the sender's private information, the false alarm can also be attributed to the sender's propensity of emitting excessive warning messages. As such, the updated reputation following a false alarm drops, but is no longer fully ruined (i.e., $p_0 > p_1^{1,0} \geq 0$). In particular, when her initial reputation is large enough ($p_0 \geq p^{**}$), exaggerating indeed allows the sender to maintain her reputation at a sufficiently high level (i.e., above p^*) and thus to retain the ability of eliciting an action in the future, even after a false alarm. In fact, the more the sender exaggerates the risk, the less the receiver lays the blame for a false alarm on the quality of the sender's private information, and thus the less reputation drops. As a result, for a sender with lower reputation (i.e., with lower value of p_0), more exaggeration (i.e., larger w_1^0) is required to maintain her ability to elicit an action in the next period.

The exaggeration strategy that may generate unwarranted warnings shares some resemblance with the "spamming" tactic suggested by Che and Hörner (2017) for Internet recommender systems, whereby the online platform may over-recommend a product that has yet to be found worthy of said recommendation. However, the goal of such a spamming tactic is to facilitate social learning by incentivizing exploration of unexplored products, whereas the exaggeration strategy in our context does the opposite and aims to hamper the receiver's learning about the sender by strategically misrepresenting her private information.

6. Extensions

This section presents three extensions of our base model by relaxing some of its fundamental assumptions, and investigates their impacts on the sender's incentives to misrepresent the risk.

6.1. Imperfect and Informative Signals

In our base model, the presence of an advance signal either provides perfect information or is entirely uninformative depending on its quality; that is, $1 = \mathbb{P}[X_t = 1 \mid Z_t = 1, \theta = H] \geq \mathbb{P}[X_t = 1 \mid Z_t = 1, \theta = L] = \pi$. In this subsection, we relax this assumption and explore imperfect but informative advance signals such that $1 \geq \mathbb{P}[X_t = 1 \mid Z_t = 1, \theta = H] \geq \mathbb{P}[X_t = 1 \mid Z_t = 1, \theta = L] \geq \pi$.

Specifically, we assume that the signal of quality $\theta \in \{H, L\}$ has a false positive rate $\bar{\beta}_\theta \equiv 1 - \beta_\theta$ such that $\beta_\theta \equiv \mathbb{P}[Z_t = 0 \mid X_t = 0,\ \theta]$ and $1 \geq \bar{\beta}_L \geq \bar{\beta}_H \geq 0$. We denote $\beta \equiv \beta_H - \beta_L \geq 0$ and continue to assume that the events must occur with advance notice, i.e., $P[Z_t = 1 \mid X_t = 1, \theta] = 1$ for $\theta \in \{H, L\}$. In particular, our base model corresponds to $\beta_H = \bar{\beta}_L = \beta = 1$.

To focus on the most interesting case, we assume further that the sender always prefers to act when a signal is present regardless of its quality ($a_t = Z_t$), whereas the receiver prefers to act only in the presence of a high quality signal ($a_t = 1$ only if $Z_t = 1$ and $\theta = H$). As shown by Lemma C.1 in Appendix C, this corresponds to $\bar{\beta}_H \leq \eta_r \leq \bar{\beta}_L \leq \eta_s$, which is consistent with our general assumption that $\delta_s/\kappa_s > \delta_r/\kappa_r > 1$. The next result characterizes the optimal warning policy for this setting, where p^* generalizes its counterpart in Lemma 3 for the base model.

THEOREM 2. *For the second period, full disclosure policy $\boldsymbol{\omega}_2^\star(p_1) = (0, 1)$ is optimal. For the first period, thresholds $\hat{p}_1, \hat{p}_2, \hat{p}_3$ and $p^* \equiv (\bar{\beta}_L - \eta_r)/\beta$ exist such that $\hat{p}_1 \leq p^* \leq \hat{p}_2 \leq \hat{p}_3 \leq 1$, and*

$$\boldsymbol{\omega}_1^\star(p_0) = \begin{cases} (0,1), & \text{if } 0 \leq p_0 < \hat{p}_1, \\ \left(0, \frac{p^* - p_0}{p^*(\beta_L - \beta p_0) - \bar{\beta}_H p_0}\right), & \text{if } \hat{p}_1 \leq p_0 < p^*, \\ (0,1), & \text{if } p^* \leq p_0 < \hat{p}_2, \\ \left(\frac{(\bar{\beta}_L - \beta p_0)p^* - \bar{\beta}_H p_0}{\beta_H p_0 - (\beta_L + \beta p_0)p^*}, 1\right), & \text{if } \hat{p}_2 \leq p_0 < \hat{p}_3, \\ (0,1), & \text{if } \hat{p}_3 \leq p_0 \leq 1. \end{cases} \tag{13}$$

(The closed-form expressions of $\hat{p}_1$, $\hat{p}_2$, and $\hat{p}_3$ are given in the proof of Theorem 2 in Appendix D.)

In essence, Theorem 2 states that the structure of the optimal warning policy, and in particular the need to downplay and exaggerate the risk in the first period, persists when the signals are imperfect but informative, as long as p_0 does not take too extreme values (i.e., $\hat{p}_1 \leq p_0 \leq \hat{p}_3$). If p_0

is too low ($p_0 < \hat{p}_1$), the imperfect quality of the signal makes it impossible for the sender to boost her next period's reputation to a level high enough (updated reputation p_1 is always less than p^* for any warning policy). In other words, the receiver will always ignore the sender in both periods and any warning policy is optimal in this case. Per convention (see Section 4), the optimal policy corresponds then to full disclosure. Conversely, if p_0 is sufficiently high ($p_0 \geq \hat{p}_3$), the reputation can never drop to below the critical threshold p^* for any warning policy. Thus, maintaining a high level of reputation in the next period does not require misrepresenting the risk, while full disclosure minimizes the costs of the current period.

6.2. Events without Advance Notice

Thus far, we only considered events that always occur with advance notice; that is, the absence of a signal perfectly predicts the nonoccurrence of the event. As a consequence, the sender's reputation remains unchanged when an event occurs (i.e., $p_1^{1,1} = p_1^{0,1} = p_0$, see Lemma 2). We now relax this assumption, and consider events that may occur even without advance notice. In this case, reputation will also improve or deteriorate depending on whether or not the sender correctly warns against an event that eventually occurs (i.e., $p_1^{1,1} \geq p_0 \geq p_1^{0,1}$).

To that end, we extend our base model by allowing the low-quality signal to be informative, with false negative rate $\bar{\alpha}_L \equiv \mathbb{P}[Z_t = 0 \mid X_t = 1, \theta = L]$. The false positive rate $\bar{\beta}_L$ is general (see Section 6.1) such that $\bar{\alpha}_L + \bar{\beta}_L \leq 1$, which ensures that the presence of a signal is more indicative of the event than its absence. We further assume that the false positive rate dominates the false negative one, i.e., $\bar{\beta}_L \geq \bar{\alpha}_L$, which generalizes our base model (where $\bar{\beta}_L = 1 > 0 = \bar{\alpha}_L$).[13] For simplicity, we focus on the case where both parties prefer not to act a priori and, as in the previous section, the sender always prefers to act upon observing a signal, whereas the receiver prefers to act in the presence of a high quality signal. This corresponds to $0 \leq \eta_r \leq \bar{\beta}_L/\alpha_L \leq \eta_s \leq 1$ with $\alpha_L \equiv 1 - \bar{\alpha}_L$; see Lemma C.1 of Appendix C.

The next result characterizes the optimal warning policy for this setting, where p^* generalizes its counterpart in Lemma 3 for the base model.

Theorem 3. *For the second period, full disclosure policy $\boldsymbol{\omega}_2^\star(p_1) = (0,1)$ is optimal. For the first period, thresholds $\hat{p}_1, \cdots, \hat{p}_6$ and $p^* \equiv (\bar{\beta}_L - \alpha_L\eta_r)/(\bar{\beta}_L + \bar{\alpha}_L\eta_r)$ exist such that $0 \leq \hat{p}_1 \leq \hat{p}_2 \leq \hat{p}_3 \leq$*

[13] Our solution procedure equally applies to the case where $\bar{\alpha}_L \leq \bar{\beta}_L$.

$p^* \leq \hat{p}_4 \leq \hat{p}_5 \leq \hat{p}_6 \leq 1$, *and*

$$\boldsymbol{\omega}_1^{\star}(p_0) = \begin{cases} (0,1), & \text{if } \ 0 \leq p_0 \leq \hat{p}_1 \\ (0, w_1^{1\star}), \quad w_1^{1\star} < 1, & \text{if } \ \hat{p}_1 < p_0 < \hat{p}_2 \\ (w_1^{0\star}, 1), \quad w_1^{0\star} \geq 0 \text{ with ``='' at } p_0 = \hat{p}_2, & \text{if } \ \hat{p}_2 \leq p_0 < \hat{p}_3 \\ (0,1), & \text{if } \ \hat{p}_3 \leq p_0 \leq \hat{p}_4 \\ (0, w_1^{1\star}), \quad w_1^{1\star} < 1, & \text{if } \ \hat{p}_4 < p_0 \leq \hat{p}_5 \\ (w_1^{0\star}, 1), \quad w_1^{0\star} > 0, & \text{if } \ \hat{p}_5 < p_0 \leq \hat{p}_6 \\ (w_1^{0\star}, w_1^{1\star}), \ w_1^{0\star} > 0, \ w_1^{1\star} \leq 1, & \text{if } \ \hat{p}_6 < p_0 \leq 1. \end{cases} \tag{14}$$

(The proof of Theorem 3 in online appendix D provides closed-form expressions for the optimal warning probabilities and thresholds $\hat{p}_i$, $i = 1, \cdots, 6$.)

Theorem 3 reveals that the sender continues to have incentives to misrepresent the risk in this more general case, albeit in a more intricate way. In short, when reputation is very low ($p_0 \leq \hat{p}_1$), any policy is optimal and the sender might as well remain truthful (as in Theorem 2). As reputation improves but remains low ($p_0 \leq \hat{p}_3$), the sender seeks to boost her reputation by first downplaying and then exaggerating the risk. As in the base model, full disclosure remains optimal for intermediate reputation ($\hat{p}_3 < p_0 < \hat{p}_4$). When reputation reaches higher levels ($p_0 > \hat{p}_4$), the sender protects her reputation by first downplaying, then exaggerating and finally both downplaying and exaggerating the risk.

As in the base model, downplaying the risk for $p_0 < p^*$ acts to boost reputation by remaining correctly silent ($p_1^{0,0} = p^*$), while exaggerating the risk for $p_0 > p^*$ aims to mitigate a reputation damage due to a false alarm ($p_1^{1,0} = p^*$). In this more general setting, however, reputation also increases when the sender correctly warns against an event ($p^{1,1} > p_0$). By exaggerating the risk for $p_0 < p^*$, the sender can boost her reputation ($p_1^{1,1} = p^*$) through a correct warning. Similarly, reputation decreases ($p^{0,1} < p_0$) when the sender misses an event without warning, but downplaying the risk for $p_0 > p^*$ limits such reputation damage ($p_1^{0,1} = p^*$). In fact, when the reputation level is very high ($p_0 > \hat{p}_6$), the sender can leverage both misrepresentations of the risk to protect her reputation ($p_1^{0,1} = p_1^{1,0} = p^*$).

Hence, the sender may misrepresent the risk so as to improve or protect her reputation and thus to ensure that the future reputation level reaches critical value p^*, which is just enough to influence the receiver. In this sense, the fundamental insights of Corollary 2 and Section 5.3, extend to this more general case.

Nonetheless, the sender's misrepresentations now aim to manage four different posterior reputations (as opposed to two in the base model), resulting in multiple tradeoffs and the seven possible intervals on reputation p_0 prescribed in Theorem 3. As a consequence, the optimal policy may require two additional types of risk misrepresentations, which consist in exaggerating and downplaying the risk when reputation is high and low, respectively. In particular, the sender can only influence the receiver when reputation is high enough ($p_0 > p^*$). In this case, downplaying the risk means that the sender purposefully dissuades the receiver from acting, even upon receiving a signal of high perceived quality ($p_0 > \hat{p}_4 > p^*$). The following proposition offers sufficient conditions under which such extreme misrepresentations of the risk are not optimal.

PROPOSITION 2. *If $\beta_L/\bar{\alpha}_L \geq \pi/\overline{\pi}$ and $\alpha_L/\bar{\beta}_L \leq \frac{\eta_r^{-1}+\rho\pi\eta_s^{-1}}{1+\rho\pi}$, the thresholds in Theorem 3 satisfy*

$$0 \leq \hat{p}_1 \leq \hat{p}_2 = \hat{p}_3 \leq p^* \leq \hat{p}_4 = \hat{p}_5 \leq \hat{p}_6 = 1, \tag{15}$$

and thus $w_1^{0\star}(p_0)$=0 for all $p_0 \in [0, p^*)$ and $w_1^{1\star}(p_0)$=1 for all $p_0 \in (p^*, 1]$.

The proposition's first condition on $\beta_L/\bar{\alpha}_L$ is equivalent to state that a true negative is more likely than a false negative. Since $\alpha_L/\bar{\beta}_L \geq 1$, the second condition on $\alpha_L/\bar{\beta}_L$ further requires that the low-quality signal's likelihood ratio[14] is not too different from one. Under these conditions, the proposition states then that the sender should never exaggerate and downplay the risk when reputation is low ($p_0 < p^*$) and high ($p_0 \geq p^*$), respectively. In other words, we retrieve the overall structure of the optimal policy depicted in Theorem 2, for which downplaying is optimal only for low reputation ($p_0 < p^*$), while exaggeration is optimal only for high reputation ($p_0 > p^*$).

6.3. Heterogeneous Receivers

Finally, we explore the robustness of our main results when the sender faces multiple heterogeneous receivers. If the sender is able to use a separate communication channel to interact with each receiver privately and receivers cannot communicate among themselves, Theorem 1 and Proposition 1 directly apply to each receiver and the sender can tailor different warning policies accordingly. Our main results then hold. In this section, we explore the more challenging case, where the sender uses a unique public communication channel and hence commits to a unique warning policy simultaneously applying to all receivers (which is arguably true for WHO). Formally, $\boldsymbol{\omega}_t^\star(\cdot)$ denotes this public optimal warning probability in period $t = 1, 2$.

Our base model can give rise to several sources of heterogeneity. Receivers may differ in their own cost parameters (ℓ_r, δ_r and κ_r) but also in the effects of their actions on the sender's costs (through δ_s and κ_s). For simplicity, we consider two different receivers, indexed by $j \in \{1, 2\}$, and explore each source of heterogeneity in isolation.

Consider first two receivers, who only differ in their effects on the sender's cost ℓ_s^j, δ_s^j and κ_s^j (with a slight abuse of notation), and define η_s^j and $\widehat{\rho}^j$ as in Proposition 1 and Theorem 1, respectively. The receivers are identical otherwise and, in particular, share the same cost parameters ℓ_r, δ_r and

[14] The likelihood ratio $\alpha_\theta/\bar{\beta}_\theta$ measures the informativeness of the signal. A non-informative signal corresponds to a ratio equal to one (e.g., the low-quality signal in our base model), whereas the ratio for a perfect signal equals infinity (e.g., the high-quality signal in our base model).

κ_r (and hence same parameters η_r and p^* as defined in Lemma 3) with $\kappa_r/\delta_r > (\kappa_s^1+\kappa_s^2)/(\delta_s^1+\delta_s^2)$. The following proposition provides sufficient conditions for $\boldsymbol{\omega}_t^\star(\cdot)$ $(t=1,2)$ to retain the optimal structure in our base model.

PROPOSITION 3. *In the setting described above, $\boldsymbol{\omega}_2^\star(p_1)=(0,1)$ for all $p_1\in[0,1]$. If $\pi>\kappa_r/\delta_r$ or $\pi<(\kappa_s^1+\kappa_s^2)/(\delta_s^1+\delta_s^2)$, then $\boldsymbol{\omega}_1^\star(p_0)=(0,1)$ for all $p_0\in[0,1]$. If $(\kappa_s^1+\kappa_s^2)/(\delta_s^1+\delta_s^2)\le\pi\le\kappa_r/\delta_r$, then $\boldsymbol{\omega}_1^\star(p_0)$ is given by (10) with $\delta_s\equiv\delta_s^1+\delta_s^2$, $\kappa_s\equiv\kappa_s^1+\kappa_s^2$, and hence $\eta_s\equiv\pi/\overline{\pi}\,(\delta_s/\kappa_s-1)$.*

The first part of Proposition 3 extends Proposition 1 and demonstrates the optimality of the full-disclosure policy in the second period. The next two statements show that a sender, who faces receivers heterogeneous in their effects on the sender's cost, determines her optimal warning policy as if she is facing a *single* receiver, whose effect on the sender's cost is simply the aggregate of the two (i.e., with $\delta_s\equiv\delta_s^1+\delta_s^2$ and $\kappa_s\equiv\kappa_s^1+\kappa_s^2$). In other words, the structure of the optimal warning policy identified in Theorem 1 continues to hold, i.e., the sender downplays the risk when her reputation is below p^* and may exaggerate the risk when her reputation is above p^{**}. Specifically, recall from Theorem 1 that parameters δ_s and κ_s only affect the optimal warning probability given in (10) through p^{**} by determining whether or not $\rho\ge\widehat{\rho}\equiv p^*/[\overline{\pi}\eta_r(\eta_s-1)]$ (i.e., whether or not exaggerating the risk is optimal for high values of p_0).

We now turn to the second source of heterogeneity. Consider then two receivers, who only differ in parameters ℓ_r^j, δ_r^j and κ_r^j (with a slight abuse of notation), and define η_r^j and p^{j*} as in Lemma 3, and $\widehat{\rho}^j$ and p^{j**} as in Theorem 1. The receivers are identical otherwise, with same parameters δ_s and κ_s. Without loss of generality, assume $\kappa_r^2/\delta_r^2>\kappa_r^1/\delta_r^1(>\kappa_s/\delta_s)$ and thus $\eta_r^1>\eta_r^2$, which further implies that $p^{1*}<p^{2*}$, $\widehat{\rho}^1<\widehat{\rho}^2$ and $p^{1**}<p^{2**}$. The next proposition partially characterizes the optimal warning policy, and shows that misrepresenting the risk can still be optimal in this set-up.

PROPOSITION 4. *In the setting described above, $\boldsymbol{\omega}_2^\star(p_1)=(0,1)$ for all $p_1\in[0,1]$. If $\pi>\kappa_r^2/\delta_r^2$ or $\pi<\kappa_s/\delta_s$, then $\boldsymbol{\omega}_1^\star(p_0)=(0,1)$ for all $p_0\in[0,1]$. If $\kappa_s/\delta_s\le\pi\le\kappa_r^1/\delta_r^1$, then*

(i) $\boldsymbol{\omega}_1^\star(p_0)=(0,\omega_1^{1\star}(p_0))$ *with* $\omega_1^{1\star}(p_0)<1$ *if* $p_0<p^{1*}$,

(ii) $\boldsymbol{\omega}_1^\star(p_0)=(0,1)$ *if* $p^{2*}\le p_0<p^{2**}$, *and*

(iii) $\boldsymbol{\omega}_1^\star(p_0)=(\omega_1^{0\star}(p_0),1)$ *with* $\omega_1^{0\star}(p_0)>0$ *if* $p_0\ge p^{2**}$.

Hence, downplaying and exaggerating the risk remain optimal in this set-up when reputation is sufficiently low ($p_0<p^{1*}$) and high ($p_0\ge p^{2**}$), respectively. For these reputation levels, the sender has incentives to misrepresent the risk to each receiver in the same direction, and thus does not need to discriminate between the receivers. Similarly, full-disclosure remains optimal if the sender has no incentive to misrepresent the risk to each receiver (i.e. if $p^{2*}\le p_0<p^{1**}$). For other intermediate levels of reputation, the sender may have incentives to misrepresent the risk in different directions, thus weakening the magnitude of the misrepresentations we identified in our base model.

7. Concluding Remarks

In this paper, we study a dynamic setting in which an informed agency alerts an uninformed party to take preemptive actions against potential disasters. Our findings reveal that maintaining the ability to elicit such actions in the future offers a rationale for warning messages to downplay or exaggerate the risk. The effectiveness of the agency's warnings in eliciting an action critically depends on the agency's reputation. When reputation is low, the agency downplays the threat in the hope of boosting its reputation. When reputation is high, the agency exaggerates the threat in order to limit a possible loss of reputation.

For disasters that always occur with advance notice no further form of risk misrepresentations is required. However, if disasters can also occur without early signals, the sender may further exaggerate and downplay the risk when her reputation is low and high, respectively. The rationale for these additional misrepresentations is the same: the sender seeks to boost or protect her reputation to maintain the effectiveness of her future warnings.

It is worth noting that WHO has been accused for exaggerating or downplaying the risk of global pandemics (Flynn 2010, Sengupta 2015, World Health Organization 2015). Many reasons have been advanced for these risk misrepresentations, such as conflicts of interest and budgeting issues (de Véricourt 2017, and references therein). Our work suggests that the management of an agency's reputation to maximize the effectiveness of its warnings over time can induce similar behaviors. This provides an alternative rationale, which has never been explored.

This rationale is at least consistent with several PHEICs declared by WHO over the years. Indeed, WHO's success in tackling the 2003 SARS outbreak led to the revision of the International Health Regulations (World Health Organization 2005), which made WHO the key coordination agency to collect information about and prevent public health risks. Given this high level of reputation, our model predicts that the agency should exaggerate the signal it receives. This is exactly what WHO was accused of doing after it declared its first PHEIC for the 2009 flu pandemic. This, in turn, seemed to have damaged WHO's reputation. Our model predicts that the agency should then downplay the risk, which is, again, what the agency was accused of doing when it refrained from declaring a PHEIC for the 2014 Ebola outbreak.

In the end, our stylized model can only yield qualitative results. Yet, our work articulates a new and general hypothesis for further empirical investigations. If this appears challenging in the context of WHO, because of the extreme low frequency of adverse events, other settings such as FDA (Carpenter 2006, Maor 2011) or NWS (Ripberger et al. 2015) may be more promising. A possible approach could leverage the behavioral literature, which suggests that past failures are reasonable proxy for losses in reputation (Bolton and Katok 2018). The study could then determine whether the probability of issuing a new warning increases (resp. decreases) following an agency's failure (resp. a success) to alert against previous hazards. The strength of the alert or delays in the agency's decisions may further constitute relevant and more practical alternatives to the warning probability (Carpenter 2004). In this sense, our findings also contribute to the growing literature on organizational reputation in the public sector (Carpenter and Krause 2012).

Likewise, our findings provide a theoretical benchmark for future behavioral experimental studies. In their recent work, Bolton and Katok (2018) explore how *receivers* react to different ways of communicating warning messages. Similar experimental studies could examine the *sender*'s warning behavior and, in particular, the different information distortions we uncover in this paper.

Finally, we have restricted the analysis to a two-period model to keep the analysis tractable, even though agencies such as WHO typically face multiple potential threats over time. Having more periods can potentially lead to more complex dynamics and definitely constitutes a promising future research direction. Nevertheless, our two-period set-up is still appropriate for two reasons. First, it is already rich enough to identify the key trade-off between eliciting an action today and cultivating a high level of future reputation. Second, it is a reasonable approximation for agencies whose primary focus is on short- and mid-term threats. We conjecture that the qualitative nature of some of our results will remain unaffected for longer time horizons when the discount factor is not too high.

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Electronic Companion

Warning Against Recurring Risks: An Information Design Approach

Appendix A: Effect of the Event's Probability π

In the context of our base model, the following result examines how the prior probability of an event π affects the optimal warning policy.

PROPOSITION A.1. *For any $0 \le p_0 \le 1$, there exist two thresholds $\breve{\pi}(p_0), \hat{\pi}(p_0) \in [\kappa_s/\delta_s, \kappa_r/\delta_r]$ such that $\breve{\pi}(p_0) \le \hat{\pi}(p_0)$, and it is optimal for the sender to (i) downplay the risk if $\pi \in [\kappa_s/\delta_s, \breve{\pi}(p_0))$ with $w_1^{1\star} < 1$ decreasing in π, (ii) exaggerate the threat if $\pi \in [\hat{\pi}(p_0), \kappa_r/\delta_r)$ with $w_1^{0\star} > 0$ decreasing in π, and (iii) fully disclose otherwise.*

Furthermore, threshold $\breve{\pi}(p_0)$ strictly decreases in p_0 with $\breve{\pi}(0) = \kappa_r/\delta_r$ and $\breve{\pi}(p_0) = \kappa_s/\delta_s$ for $p_0 \ge \breve{p}$ where $\breve{p} \equiv (\delta_s\kappa_r - \kappa_s\delta_r) / (\delta_s\kappa_r - \kappa_s\kappa_r) < 1$; and threshold $\hat{\pi}(p_0)$ first strictly decreases in p_0 with $\hat{\pi}(0) = \kappa_r/\delta_r$ and then remains equal to a constant $\pi^\dagger \in (\kappa_s/\delta_s, \kappa_r/\delta_r)$.

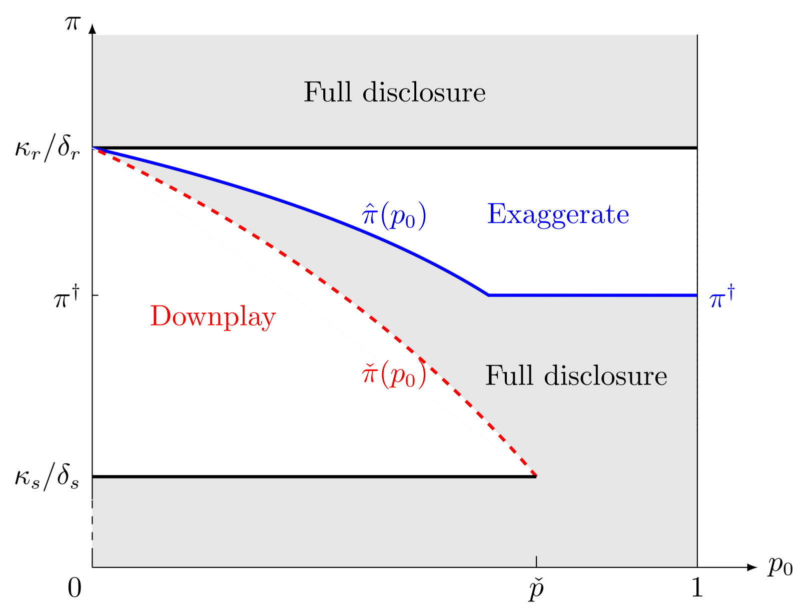

Figure A.1 Optimal warning probabilities for the first period in the (p_0, π) space

In effect, Proposition A.1 partitions the two dimensional (p_0, π) parameter space into four separate regions, as illustrated by Figure A.1. The first region corresponds to events that are highly likely to occur (i.e., $\pi \ge \kappa_r/\delta_r$), in which case full disclosure is optimal.

In the second region, when the event is relatively unlikely (i.e., $\kappa_s/\delta_s < \pi < \breve{\pi}(p_0)$) and the reputation is below $\breve{p}$, the sender downplays the risk with positive probability. Recall that downplaying the risk boosts the sender's reputation to p^* when no event occurs. Therefore, lower reputation p_0 offers stronger incentives for the sender to downplay (i.e., larger $\breve{\pi}(p_0)$ and hence wider range of π), as does a less likely event (i.e., smaller π implies smaller $w_1^{1\star}$ for fixed p_0).

In the third region, when the event is relatively likely (i.e., $\hat{\pi}(p_0) \le \pi < \kappa_r/\delta_r$), the sender exaggerates the threat with positive probability. Indeed, the purpose of exaggeration is to maintain reputation above p^*. This is easier to achieve for a sender with a higher reputation (i.e., higher p_0), who thus exaggerates for wider range of events (i.e., smaller $\hat{\pi}(p_0)$). Meanwhile, since threshold p^* decreases in π, a more likely event necessitates less exaggeration (i.e., $w_1^{0\star}$ decreases in π).

In the last remaining area, full disclosure is again optimal. In particular, when the event is uncertain (i.e., $\kappa_s/\delta_s \le \pi \le \kappa_r/\delta_r$), the full-disclosure interval on π shrinks as reputation deteriorates (i.e., smaller p_0). That is, a sender with a lower reputation tends to misrepresent her information more.

Taken together, exaggerating the threat is optimal when reputation and the event's likelihood are both relatively high, while downplaying the risk becomes optimal when both take relatively low values. Notably, these two distortions never take place at the same time, i.e., the downplay and exaggeration regions do not overlap in Figure A.1.

Proof of Proposition A.1. Consider $p_0 \in [0,1]$. First, note that $p^* = (\kappa_r - \pi\delta_r)/(\bar{\pi}\kappa_r)$ decreases as π increases over the $[\kappa_s/\delta_s, \kappa_r/\delta_r]$ interval. In particular, $p^* = 0$ when $\pi = \kappa_r/\delta_r$ and $p^* = \check{p}$ when $\pi = \kappa_s/\delta_s$, where $\check{p}$ is defined as, $\check{p} = (\delta_s\kappa_r - \kappa_s\delta_r)/(\delta_s\kappa_r - \kappa_s\kappa_r)$. Thus, for $p_0 < \check{p}$, there always exists a value of $\pi \in [\kappa_s/\delta_s, \kappa_r/\delta_r]$ for which $p^* = p_0$. We denote this value by $\check{\pi}(p_0)$, and note that $\check{\pi}(p_0)$ is decreasing in p_0. Recall from Theorem 1 that downplaying the risk is optimal when $0 \le p_0 < p^*$, and $\kappa_s/\delta_s < \pi < \kappa_r/\delta_r$. It follows that when π gets smaller than $\check{\pi}(p_0)$, the value of p^* exceeds p_0, and downplaying becomes optimal. For $p_0 \ge \check{p}$, on the other hand, p^* always remains below p_0 regardless of the value of π. In this case, downplaying is never optimal. For these values of p_0, we let $\check{\pi}(p_0) = \kappa_s/\delta_s$.

The proof for $\hat{\pi}(p_0)$ is similar. Specifically, it is straightforward to show that both quantities $\widehat{\rho}$ and $p^*(2-p^*)$ decrease as π increases over the $[\kappa_s/\delta_s, \kappa_r/\delta_r]$ interval. Let $\pi^\dagger$ represent the value of π (if it exists) for which $\widehat{\rho} = \rho$. Similarly, let $\pi^\ddagger(p_0)$ denote the value of π for which $p^*(2-p^*) = p_0$, and note that $\pi^\ddagger(p_0)$ is decreasing in p_0. Then, $\pi \ge \pi^\dagger$ induces the value of $\widehat{\rho}$ to fall below ρ, and $\pi \ge \pi^\ddagger(p_0)$ induces the value of $p^*(2-p^*)$ to fall below p_0. Recall from Theorem 1 that exaggerating the threat is optimal when $\rho \ge \widehat{\rho}$ and $p_0 \ge p^*(2-p^*)$. Therefore, setting $\hat{\pi}(p_0) = \max\{\pi^\dagger, \pi^\ddagger(p_0)\}$ completes the proof in this case.

Further, $\hat{\pi}(p_0) \ge \check{\pi}(p_0)$ for any given p_0 follows from $p^*(2-p^*) \ge p^*$, since the value of π that induces $p^*(2-p^*) = p_0$ should be bigger than the value of π that induces $p^* = p_0$. The monotonicity of $w_1^{1\star}$ when downplaying is optimal, and $w_1^{0\star}$ when exaggerating is optimal can be immediately verified noting that $(p^*-p_0)/[p^*(1-p_0)]$ and $p^*(1-p_0)/[p_0(1-p^*)]$ are both increasing in p^* (and hence decreasing in π). $\square$

Appendix B: Proofs for Results in Sections 4 and 5

Proof of Proposition 1. First assume $\pi \le \kappa_r/\delta_r$, which in turn implies that $p^* = (\kappa_r - \pi\delta_r)/(\bar{\pi}\kappa_r) \ge 0$. For the single period problem with the sender's initial reputation p, her ex ante expected cost as in (7) becomes,

$$\mathbb{E}\left[c_s\left(a^\star\left(D,p,\boldsymbol{w}\right),X\right)\mid \boldsymbol{w},p\right] = \begin{cases} \mathbb{E}\left[c_s\left(D,X\right)\mid \boldsymbol{w},p\right], & \text{if } \frac{w^1}{w^0} \ge \frac{p}{p-p^*} \text{ and } p \ge p^*, \\ \mathbb{E}\left[c_s\left(0,X\right)\mid \boldsymbol{w},p\right], & \text{if } \frac{w^1}{w^0} < \frac{p}{p-p^*} \text{ or } p < p^*, \end{cases}$$

where the receiver's optimal action is plugged in from (5). Then following (C.17)-(C.20), we have

$$\begin{cases} \mathbb{E}\left[c_s\left(D,X\right)\mid \boldsymbol{w},p\right], & \text{if } \frac{w^1}{w^0} \ge \frac{p}{p-p^*} \text{ and } p \ge p^*, \\ \mathbb{E}\left[c_s\left(0,X\right)\mid \boldsymbol{w},p\right], & \text{if } \frac{w^1}{w^0} < \frac{p}{p-p^*} \text{ or } p < p^*, \end{cases}$$

$$
\begin{aligned}
&= \begin{cases} \pi w^1(\kappa_s + \ell_s - \delta_s) + \pi(1-w^1)\ell_s + \overline{\pi}\kappa_s\left[p w^0 + (1-p)w^1\right], & \text{if } \frac{w^1}{w^0} \geq \frac{p}{p-p^*} \text{ and } p \geq p^*, \\ \pi \ell_s, & \text{if } \frac{w^1}{w^0} < \frac{p}{p-p^*} \text{ or } p < p^*, \end{cases} \\
&= \begin{cases} \pi \ell_s + \overline{\pi}\kappa_s\left[p w^0 - (p+\eta_s-1)w^1\right], & \text{if } \frac{w^1}{w^0} \geq \frac{p}{p-p^*} \text{ and } p \geq p^*, \\ \pi \ell_s, & \text{if } \frac{w^1}{w^0} < \frac{p}{p-p^*} \text{ or } p < p^*. \end{cases}
\end{aligned} \tag{B.1}
$$

This implies that when $p < p^*$, the sender's cost is constant and independent of her warning policy. Hence, any feasible warning policy $\boldsymbol{\omega}(.)$ is optimal. When $p \geq p^*$, on the other hand, the sender's cost is linear in both w^0 and w^1 with positive and negative coefficients, respectively. To see this, note that

$$
\begin{aligned}
p^* + \eta_s - 1 = \frac{\pi}{\overline{\pi}}\left[\frac{\delta_s}{\kappa_s} - \frac{\delta_r}{\kappa_r}\right] &\Rightarrow p + \eta_s - 1 \geq \frac{\pi}{\overline{\pi}}\left[\frac{\delta_s}{\kappa_s} - \frac{\delta_r}{\kappa_r}\right] \text{ for all } p \geq p^* \\
&\Rightarrow p + \eta_s - 1 > 0 \text{ for all } p \geq p^*\,,
\end{aligned} \tag{B.2}
$$

where the last inequality is due to $\delta_s/\kappa_s > \delta_r/\kappa_r$. Thus, it is optimal for the sender to set the value of w^0 (resp. w^1) as low (resp. high) as possible, i.e., the full disclosure policy $\boldsymbol{\omega}^\star(p) = (0,1)$ is optimal. Subsequently, the sender's optimal cost function follows immediately by plugging in the optimal warning policy $\boldsymbol{\omega}^\star(p) = (0,1)$ in (B.1).

When $\pi > \kappa_r/\delta_r$, it follows that $p^* < 0$. In this case, we always have $p \geq p^*$, which gives us

$$
\begin{aligned}
\mathbb{E}\left[c_s\left(a^\star\left(D,p,\boldsymbol{w}\right),X\right) \mid \boldsymbol{w},p\right] &= \begin{cases} \mathbb{E}\left[c_s\left(D,X\right) \mid \boldsymbol{w},p\right], & \text{if } \frac{\overline{w}^1}{\overline{w}^0} \leq \frac{p}{p-p^*}, \\ \mathbb{E}\left[c_s\left(1,X\right) \mid \boldsymbol{w},p\right], & \text{if } \frac{\overline{w}^1}{\overline{w}^0} > \frac{p}{p-p^*}, \end{cases} \\
&= \begin{cases} \pi\ell_s - \overline{\pi}\kappa_s\left[(p+\eta_s-1)w^1 - p w^0\right], & \text{if } \frac{\overline{w}^1}{\overline{w}^0} \leq \frac{p}{p-p^*}, \\ \pi\ell_s - \overline{\pi}\kappa_s(\eta_s - 1), & \text{if } \frac{\overline{w}^1}{\overline{w}^0} > \frac{p}{p-p^*}. \end{cases}
\end{aligned} \tag{B.3}
$$

Note that the full disclosure policy satisfies the $\overline{w}^1/\overline{w}^0 \leq p/(p-p^*)$ requirement and leads to payoff $\pi\ell_s - \overline{\pi}\kappa_s(p+\eta_s-1)$, which is always smaller than $\pi\ell_s - \overline{\pi}\kappa_s(\eta_s-1)$. Hence, the full disclosure policy is optimal for any $p \in [0,1]$ in this case. □

Proof of Theorem 1. First assume $\pi < \kappa_r/\delta_r$, which in turn implies that $p^* = (\kappa_r - \pi\delta_r)/(\overline{\pi}\kappa_r) \geq 0$. To proceed with the proof, we consider two different cases corresponding to the initial value of reputation p_0, and solve each case separately.

- **Case 1:** $p_0 \in [0, p^*)$**.**

  When $p_0 < p^*$, Lemma 3 implies that $a^\star(d_1, p_0, \boldsymbol{w}) = 0$ for all $(w^0, w^1) \in [0,1]^2$ and $d_1 \in \{0,1\}$. In this case, the sender's warning policy in the first period has no influence on the receiver's action in that period, and $\mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \boldsymbol{\omega}_1(p_0)\right), X_1\right) \mid \Omega, p_0\right] = \pi\ell_s$. Therefore, the sender's warning policy only serves to influence her future reputation p_1. Specifically, we have

$$
\begin{aligned}
J_1(p_0) =& \pi\ell_s + \rho \min_{\boldsymbol{w}_1 \in \mathcal{W}} \mathbb{E}\left[J_2(p_1) \mid \boldsymbol{w}_1, p_0\right] \\
=& \pi\ell_s + \rho \min_{\boldsymbol{w}_1 \in \mathcal{W}} \sum_{(d,x)\in\{0,1\}^2} \Pr\left[p_1 = p_1^{d,x} \,\middle|\, \boldsymbol{w}_1, p_0\right] J_2\left(p_1^{d,x}\right) \\
=& \pi\ell_s + \rho\pi J_2(p_0) + \rho\overline{\pi} \min_{\boldsymbol{w}_1 \in \mathcal{W}} \Big\{ \left(p_0 w_1^0 + (1-p_0)w_1^1\right) J_2(p_1^{1,0}) \\
&+ \left(1 - p_0 w_1^0 - (1-p_0)w_1^1\right) J_2(p_1^{0,0}) \Big\}\,,
\end{aligned} \tag{B.4}
$$

  where $\Pr\left[p_1 = p_1^{d,x} \,\middle|\, \boldsymbol{w}_1, p_0\right]$ follows from (C.17)-(C.20), and $p_1^{d,x}$ is as defined in Lemma 2.

Now, we can substitute for $J_2(.)$ from (9) to find the optimal values for w_1^0 and w_1^1. We already know that $p_1^{1,0} < p^*$ since $p_1^{1,0} \le p_0$ and $p_0 < p^*$. If w_1^0 and w_1^1 are such that $p_1^{0,0}$ is also below p^*, then (B.4) reduces to $J_1(p_0) = \pi\ell_s + \rho\pi^2\ell_s + \rho\pi\bar{\pi}\ell_s = (1+\rho)\pi\ell_s$. On the other hand, if w_1^0 and w_1^1 induce $p_1^{0,0} \ge p^*$, then

$$
\begin{aligned}
J_1(p_0) =& \pi\ell_s + \rho\pi^2\ell_s + \rho\bar{\pi} \min_{\boldsymbol{w}_1 \in \mathcal{W}} \Bigg\{ \left(p_0 w_1^0 + (1-p_0) w_1^1\right) \pi\ell_s \\
&\qquad + \left(1 - p_0 w_1^0 - (1-p_0) w_1^1\right) \left[\pi\ell_s - \bar{\pi}\kappa_s \left(\frac{p_0(1-w_1^0)}{1 - p_0 w_1^0 - (1-p_0) w_1^1} + \eta_s - 1\right)\right]\Bigg\} \\
=& \pi\ell_s + \rho\pi^2\ell_s + \rho\bar{\pi}\pi\ell_s - \rho\bar{\pi}^2\kappa_s \min_{\boldsymbol{w}_1 \in \mathcal{W}} \left\{ \left(1 - p_0 w_1^0 - (1-p_0) w_1^1\right) \left(\frac{p_0(1-w_1^0)}{1 - p_0 w_1^0 - (1-p_0) w_1^1} + \eta_s - 1\right)\right\} \\
=& (1+\rho)\pi\ell_s - \rho\bar{\pi}^2\kappa_s \min_{\boldsymbol{w}_1 \in \mathcal{W}} \left\{ p_0(1-w_1^0) + \left(1 - p_0 w_1^0 - (1-p_0) w_1^1\right)(\eta_s - 1)\right\} \\
=& (1+\rho)\pi\ell_s - \rho\bar{\pi}^2\kappa_s(p_0 + \eta_s - 1) + \rho\bar{\pi}^2\kappa_s \min_{\boldsymbol{w}_1 \in \mathcal{W}} \left\{ p_0\eta_s w_1^0 + (1-p_0)(\eta_s - 1) w_1^1 \right\}, \qquad \text{(B.5)}
\end{aligned}
$$

which is always bounded from above by $(1+\rho)\pi\ell_s$ (e.g., by setting $w_1^0 = w_1^1 = 1$). Therefore, at optimality we must have $p_1^{0,0} \ge p^*$, or equivalently, $p^*\left(1 - p_0 w_1^0 - (1-p_0) w_1^1\right) \le p_0(1-w_1^0)$. Putting altogether, we have the following minimization problem,

$$
\begin{aligned}
&\min_{\boldsymbol{w}_1 \in \mathcal{W}} \left\{ p_0\eta_s w_1^0 + (1-p_0)(\eta_s - 1) w_1^1 \right\} \\
&\text{subject to: } p^*\left(1 - p_0 w_1^0 - (1-p_0) w_1^1\right) \le p_0(1-w_1^0)\,.
\end{aligned}
$$

When $\eta_s < 1$ (i.e., when $\pi < \kappa_s/\delta_s$), the coefficient of w_1^1 in the objective function is negative. Given that the coefficient of w_1^0 is always positive (noting that $\eta_s \ge 0$), the minimum is achieved when $w_1^0 = 0$ and $w_1^1 = 1$, which also satisfies the constraint. This yields the optimality of the full disclosure policy.

When $\eta_s \ge 1$, on the other hand, the objective is minimized by $w_1^0 = w_1^1 = 0$, which violates the constraint. Thus, the constraint must be binding. Calculating the value of w_1^1 in terms of w_1^0 from the equality constraint and substituting it in the objective (after some simple algebra) produces a positive coefficient for w_1^0. This implies that the optimal value of w_1^0 must be zero. Plugging this back into the constraint gives us the final solution $\boldsymbol{\omega}_1^\star(p_0) = \left(0, \frac{p^*-p_0}{p^*(1-p_0)}\right)$, as given in (10).[15]

Substituting this optimal policy in (D.4) yield the optimal cost function

$$
\begin{aligned}
J_1(p_0) =& \pi\ell_s + \rho\pi J_2(p_0) + \rho\bar{\pi}\left(p_0 w_1^{0\star} + (1-p_0) w_1^{1\star}\right) J_2(0) + \rho\bar{\pi}\left(1 - p_0 w_1^{0\star} - (1-p_0) w_1^{1\star}\right) J_2(p^*) \\
=& \pi\ell_s + \rho\pi^2\ell_s + \rho\bar{\pi}\left(\frac{p^*-p_0}{p^*}\right)\pi\ell_s + \rho\bar{\pi}\left(1 - \frac{p^*-p_0}{p^*}\right)\left[\pi\ell_s - \bar{\pi}\kappa_s(p^* + \eta_s - 1)\right] \\
=& (1+\rho)\pi\ell_s - \rho\bar{\pi}^2\kappa_s\left(\frac{p^*+\eta_s-1}{p^*}\right) p_0\,.
\end{aligned}
$$

- **Case 2:** $p_0 \in [p^*, 1]$**.**

[15] An alternative, and perhaps shorter proof follows by noting that $\left(p_0 w_1^0 + (1-p_0) w_1^1\right) p_1^{1,0} + \left(1 - p_0 w_1^0 - (1-p_0) w_1^1\right) p_1^{0,0} = p_0$. Thus, we must have the minimum value in (B.4) be achieved by the convex envelope of $J_2(\cdot)$ on $[0, p^*]$. When $\eta_s \ge 1$ as in Figure 3, this immediately implies that the optimal $\boldsymbol{w}_1$ must induce $p_1^{1,0} = 0$ and $p_1^{0,0} = p^*$, leading to $w_1^0 = 0$, and $w_1^1 = (p^* - p_0)/(p^*(1-p_0))$. When $\eta_s < 1$, the convex envelope becomes a straight line, and the optimal $\boldsymbol{w}_1$ must induce $p_1^{1,0} = 0$ and $p_1^{0,0} = 1$, leading to $w_1^0 = 0$, and $w_1^1 = 1$.

When $p_0 \geq p^*$, it is possible for the sender to induce an action in the first period. Hence, she has to balance the current incentive to persuade the receiver to act and the future incentive to maintain a high reputation for the second period.

As our solution strategy, we solve two subproblems depending on whether the next period's reputation p_1 remains above p^* or not. Then, we compare the optimal costs of these two subproblems, the smaller of which determines the overall optimal cost and the corresponding optimal warning policy for this case.

To this end, for any given $p_0 \geq p^*$, define sets $\mathcal{W}^{\geq}(p_0)$ and $\mathcal{W}^{<}(p_0)$ as,

$$\mathcal{W}^{\geq}(p_0) = \left\{(w^0, w^1) \in \mathcal{W} : p_1^{1,0} \geq p^*\right\} = \left\{(w^0, w^1) \in [0,1]^2 : 1 \leq \frac{w^1}{w^0} \leq \frac{p_0(1-p^*)}{p^*(1-p_0)}\right\},$$
$$\mathcal{W}^{<}(p_0) = \left\{(w^0, w^1) \in \mathcal{W} : p_1^{1,0} < p^*\right\} = \left\{(w^0, w^1) \in [0,1]^2 : \frac{p_0(1-p^*)}{p^*(1-p_0)} < \frac{w^1}{w^0}\right\}.$$

Notice that $\mathcal{W}^{\geq}(p_0)$ and $\mathcal{W}^{<}(p_0)$ partition the feasible set $\mathcal{W}$ and hence, the optimal solution to the sender's problem must belong to one of these two sets. Let $J_1^{\geq}(.)$ and $J_1^{<}(.)$ represent the sender's optimal cost function at the beginning of the horizon (as a function of initial reputation p_0) when her warning policy *in the first period* is constrained to be in $\mathcal{W}^{\geq}(p_0)$ and $\mathcal{W}^{<}(p_0)$, respectively. More specifically, it follows from (8) that,

$$J_1^{\geq}(p_0) \equiv \min_{\boldsymbol{w}_1 \in \mathcal{W}^{\geq}(p_0)} \mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) + \rho J_2(p_1) \mid \boldsymbol{w}_1, p_0\right], \tag{B.6}$$
$$J_1^{<}(p_0) \equiv \min_{\boldsymbol{w}_1 \in \mathcal{W}^{<}(p_0)} \mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) + \rho J_2(p_1) \mid \boldsymbol{w}_1, p_0\right], \tag{B.7}$$

where $J_1(p_0) = \min\left\{J_1^{\geq}(p_0), J_1^{<}(p_0)\right\}$.

Subproblem I. First, consider $J_1^{\geq}(p_0)$. By definition, any $\boldsymbol{w}_1 \in \mathcal{W}^{\geq}(p_0)$ induces $p_1 \in [p^*, 1]$. Further, we know from (9) that $J_2(p_1)$ is linear over this interval. Hence, the sender's cost in the second period becomes independent of her warning policy since

$$\mathbb{E}\left[J_2(p_1) \mid \boldsymbol{w}_1, p_0\right] = J_2\left(\mathbb{E}\left[p_1 \mid \boldsymbol{w}_1, p_0\right]\right) = J_2(p_0) = \pi\ell_s - \bar{\pi}\kappa_s(p_0 + \eta_s - 1).$$

This implies that we only need to minimize the sender's cost in the first period, given by $\mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) \mid \boldsymbol{w}_1, p_0\right]$, over $\mathcal{W}^{\geq}(p_0)$. For this, we denote $p^\diamond = (2-p^*)p^*$, and consider two different ranges for p_0:

(i) If $p^* \leq p_0 < p^\diamond$: In this case, we have

$$\begin{aligned}
\boldsymbol{w}_1 \in \mathcal{W}^{\geq}(p_0) \Rightarrow & 1 \leq \frac{w_1^1}{w_1^0} \leq \frac{p_0(1-p^*)}{p^*(1-p_0)} \xRightarrow{p^* \leq p_0 < p^\diamond} 1 \leq \frac{w_1^1}{w_1^0} < \frac{p_0}{p_0 - p^*} \\
\Rightarrow & 0 < (p^* - p_0)w_1^1 + p_0 w_1^0 < p^* \xRightarrow{\text{Lemma 3}} a^\star(d_1, p_0, \boldsymbol{w}_1) = 0 \\
\Rightarrow & \mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) \mid \boldsymbol{w}_1, p_0\right] = \pi\ell_s \\
\Rightarrow & J_1^{\geq}(p_0) = \pi\ell_s + \rho\left(\pi\ell_s - \bar{\pi}\kappa_s(p_0 + \eta_s - 1)\right).
\end{aligned} \tag{B.8}$$

Thus, the sender's cost does not depend on the warning policy, and any feasible policy is optimal.

(ii) If $p_0 \geq p^\diamond$: In this case, the receiver's action in the first period can be triggered even when the policy is restricted to be in $\mathcal{W}^{\geq}(p_0)$. That is,

$$\begin{aligned}
p_0 \geq p^\diamond \Rightarrow & \frac{p_0(1-p^*)}{p^*(1-p_0)} \geq \frac{p_0}{p_0-p^*} \\
\Rightarrow & \exists \boldsymbol{w}_1 \in \mathcal{W}^{\geq}(p_0): \frac{w_1^1}{w_0^1} \geq \frac{p_0}{p_0-p^*} \\
\overset{(B.1)}{\Longrightarrow} & \mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) \mid \boldsymbol{w}_1, p_0\right] = \pi \ell_s + \bar{\pi}\kappa_s\left[p_0 w_1^0 - (p_0+\eta_s-1)w_1^1\right] .
\end{aligned}$$

To minimize this cost, since the coefficient of w_1^0 (resp. w_1^1) is positive (resp. negative by (B.2)), it is optimal to set the value of w_1^0 (resp. w_1^1) as low (resp. high) as possible while still satisfying $\boldsymbol{w}_1 \in \mathcal{W}^{\geq}(p_0)$. This implies that the solution is on the boundary of set $\mathcal{W}^{\geq}(p_0)$ so that $w_1^1/w_1^0 = p_0(1-p^*)/p^*(1-p_0)$.

Substituting for w_1^0 based on this ratio then gives us

$$\mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) \mid \boldsymbol{w}_1, p_0\right] = \pi \ell_s + \bar{\pi}\kappa_s\left[\frac{p^*(1-p_0)}{(1-p^*)} - (p_0+\eta_s-1)\right] w_1^1 .$$

Consider the coefficient of w_1^1. Using (B.2), we have

$$\frac{p^*(1-p_0)}{(1-p^*)} - (p_0+\eta_s-1) < \frac{p^*(1-p_0)}{(1-p^*)} - p_0 + p^* \leq \frac{p^*(1-p^\diamond)}{(1-p^*)} - p_0 + p^* = p^*(1-p^*) - p_0 + p^* = p^\diamond - p_0 \leq 0 .$$

Since the coefficient of w_1^1 is negative, the cost is minimized by setting $w_1^1 = 1$. Taken altogether,

$$\begin{aligned}
\boldsymbol{\omega}_1^{\star\geq}(p_0) &= \left(\frac{p^*(1-p_0)}{p_0(1-p^*)}, 1\right) \in \mathcal{W}^{\geq}(p_0) , \\
J_1^{\geq}(p_0) &= \pi \ell_s + \bar{\pi}\kappa_s\left[\frac{p^*(1-p_0)}{(1-p^*)} - (p_0+\eta_s-1)\right] + \rho\left(\pi \ell_s - \bar{\pi}\kappa_s(p_0+\eta_s-1)\right) . \qquad \text{(B.9)}
\end{aligned}$$

Subproblem II. Next, consider $J_1^{<}(p_0)$. We have

$$\begin{aligned}
\mathbb{E}\left[J_2(p_1) \mid \boldsymbol{w}_1, p_0\right] =& \sum_{(d,x)\in\{0,1\}^2} \Pr\left[p_1 = p_1^{d,x} \,\middle|\, \boldsymbol{w}_1, p_0\right] J_2\left(p_1^{d,x}\right) \\
=& \pi J_2(p_0) + \bar{\pi}\left(p_0 w_1^0 + (1-p_0)w_1^1\right) J_2(p_1^{1,0}) + \bar{\pi}\left(1 - p_0 w_1^0 - (1-p_0)w_1^1\right) J_2(p_1^{0,0}) ,
\end{aligned}$$

where $\Pr\left[p_1 = p_1^{d,x} \,\middle|\, \boldsymbol{w}_1, p_0\right]$ follows from (C.17)-(C.20), and $p_1^{d,x}$ is as defined in Lemma 2.

Then, noting that $p_1^{1,0} < p^*$ (by definition of $\mathcal{W}^{<}(p_0)$), and using (9), the above expression reduces to

$$\begin{aligned}
& \pi\left[\pi \ell_s - \bar{\pi}\kappa_s(p_0+\eta_s-1)\right] + \bar{\pi}\left(p_0 w_1^0 + (1-p_0)w_1^1\right)\pi \ell_s \\
& \qquad + \bar{\pi}\left(1 - p_0 w_1^0 - (1-p_0)w_1^1\right)\left[\pi \ell_s - \bar{\pi}\kappa_s\left(\frac{p_0(1-w_1^0)}{p_0(1-w_1^0)+(1-p_0)(1-w_1^1)} + \eta_s - 1\right)\right] \\
=& \pi \ell_s - \pi\bar{\pi}\kappa_s(p_0+\eta_s-1) - \bar{\pi}^2\kappa_s\left[p_0+\eta_s-1-p_0\eta_s w_1^0 - (1-p_0)(\eta_s-1)w_1^1\right] .
\end{aligned}$$

Therefore, since $p_0 \geq p^*$, (B.1) gives us,

$$\begin{aligned}
& \mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) + \rho J_2(p_1) \mid \boldsymbol{w}_1, p_0\right] \\
& \quad = \rho\left[\pi \ell_s - \pi\bar{\pi}\kappa_s(p_0+\eta_s-1) - \bar{\pi}^2\kappa_s\left[p_0+\eta_s-1-p_0\eta_s w_1^0 - (1-p_0)(\eta_s-1)w_1^1\right]\right] \\
& \quad + \begin{cases} \pi \ell_s + \bar{\pi}\kappa_s\left[p_0 w_1^0 - (p_0+\eta_s-1)w_1^1\right], & \text{if } \frac{w_1^1}{w_1^0} \geq \frac{p_0}{p_0-p^*} \\ \pi \ell_s, & \text{if } \frac{w_1^1}{w_1^0} < \frac{p_0}{p_0-p^*} \end{cases}
\end{aligned}$$

$$
\begin{aligned}
&= (1+\rho)\pi\ell_s - \rho\pi\bar{\pi}\kappa_s(p_0+\eta_s-1) - \rho\bar{\pi}^2\kappa_s\,(p_0+\eta_s-1) \\
&+ \begin{cases} \bar{\pi}\kappa_s \left\{ \underbrace{[1+\rho\bar{\pi}\eta_s]\,p_0}_{\phi_1} w_1^0 - \underbrace{[p_0+\eta_s-1-\rho\bar{\pi}(1-p_0)(\eta_s-1)]}_{\phi_2} w_1^1 \right\}, & \text{if } \frac{w_1^1}{w_1^0} \geq \frac{p_0}{p_0-p^*} \\ \rho\bar{\pi}^2\kappa_s \left[\underbrace{p_0\eta_s}_{\phi_3} w_1^0 + \underbrace{(1-p_0)(\eta_s-1)}_{\phi_4} w_1^1 \right], & \text{if } \frac{w_1^1}{w_1^0} < \frac{p_0}{p_0-p^*}\,. \end{cases}
\end{aligned}
$$

Now, we consider the coefficients of w_1^0 and w_1^1 in the above expressions, and investigate their signs:

$$\eta_s > 0 \Rightarrow \phi_1 > 0 \text{ and } \phi_3 > 0\,.$$

$$p_0+\eta_s-1-\rho\bar{\pi}(1-p_0)(\eta_s-1) = p_0 + (\eta_s-1)\underbrace{[1-\rho\bar{\pi}(1-p_0)]}_{\leq 1} > 0 \Rightarrow \phi_2 > 0\,,$$

where the second to last inequality is due to (B.2). This implies that the term in the upper branch is minimized by $(w_1^0, w_1^1) = (0,1)$, which also satisfies the requirement that $w_1^1/w_1^0 \geq p_0/(p_0-p^*)$. Substituting $(w_1^0, w_1^1) = (0,1)$ reduces the term to,

$$
\begin{aligned}
&\bar{\pi}\kappa_s \left\{ [1+\rho\bar{\pi}\eta_s]\,p_0 w_1^0 - [p_0+\eta_s-1-\rho\bar{\pi}(1-p_0)(\eta_s-1)]\,w_1^1 \right\} \\
=& -\bar{\pi}\kappa_s\,[p_0+\eta_s-1-\rho\bar{\pi}(1-p_0)(\eta_s-1)] \\
=& -\bar{\pi}\kappa_s(p_0+\eta_s-1) + \rho\bar{\pi}^2\kappa_s(1-p_0)(\eta_s-1)\,. \qquad \text{(B.10)}
\end{aligned}
$$

For the term in the lower branch, we note that the sign of ϕ_4 is the same as $\eta_s - 1$. If $\eta_s \geq 1$ and hence $\phi_4 \geq 0$, the term in the lower branch is minimized by $(w_1^0, w_1^1) = (0,0)$ (ignoring the requirement that $w_1^1/w_1^0 < p_0/(p_0-p^*)$), and equals zero. When $\eta_s < 1$, on the other hand, the term is minimized by $(w_1^0, w_1^1) = (0,1)$ (ignoring the requirement that $w_1^1/w_1^0 < p_0/(p_0-p^*)$), and equals $\rho\bar{\pi}^2\kappa_s(1-p_0)(\eta_s-1)$. Following (B.10), in both cases ($\eta_s \geq 1$ and $\eta_s < 1$), the term in the upper branch is the smaller term, and the cost is therefore minimized by $(w_1^0, w_1^1) = (0,1)$. Thus, $\boldsymbol{\omega}_1^{\star<}(p_0) = (0,1) \in \mathcal{W}^<(p_0)$. Substituting this policy in the cost function derived above yields,

$$
\begin{aligned}
J_1^<(p_0) =& (1+\rho)\pi\ell_s - \rho\pi\bar{\pi}\kappa_s(p_0+\eta_s-1) - \rho\bar{\pi}^2\kappa_s\,(p_0+\eta_s-1) - \bar{\pi}\kappa_s\,[p_0+\eta_s-1-\rho\bar{\pi}(1-p_0)(\eta_s-1)] \\
=& (1+\rho)\pi\ell_s - \bar{\pi}\kappa_s\,[(1+\rho)(p_0+\eta_s-1) - \rho\bar{\pi}(1-p_0)(\eta_s-1)]\,. \qquad \text{(B.11)}
\end{aligned}
$$

Finally, putting the two subproblems together, we obtain the optimal cost function and its corresponding optimal policy for $p_0 \geq p^*$ by comparing $J_1^{\geq}(p_0)$ and $J_1^<(p_0)$ from (B.8), (B.9), and (B.11). In particular,

$$
J_1^{\geq}(p_0) - J_1^<(p_0) = \begin{cases} \bar{\pi}\kappa_s\,[(1-\rho+\rho\pi)(\eta_s-1) + [1+\rho\bar{\pi}(\eta_s-1)]\,p_0]\,, & \text{if } p^* \leq p_0 < p^\diamond, \\ \bar{\pi}\kappa_s(1-p_0)\left[\frac{p^*}{1-p^*} - \rho\bar{\pi}(\eta_s-1)\right], & \text{if } p_0 \geq p^\diamond. \end{cases}
$$

The result immediately follows by simple algebra. That is, for $p^* \leq p_0 < p^\diamond$, we have

$$
\begin{aligned}
J_1^{\geq}(p_0) - J_1^<(p_0) &= \bar{\pi}\kappa_s\,[(1-\rho+\rho\pi)(\eta_s-1) + [1+\rho\bar{\pi}(\eta_s-1)]\,p_0] \\
&= \bar{\pi}\kappa_s\left[\underbrace{(1-\rho+\rho\pi+\rho\bar{\pi}p_0)}_{\leq 1}(\eta_s-1) + p_0 \right] > 0\,,
\end{aligned}
$$

where the last inequality is due to (B.2). Therefore, the optimal policy for both $\eta_s < 1$ and $\eta_s \geq 1$ is given by the full disclosure policy in this case.

For $p_0 \geq p^\diamond$, on the other hand,

$$J_1^{\geq}(p_0) - J_1^{<}(p_0) = \bar{\pi}\kappa_s(1-p_0)\left[\frac{p^*}{1-p^*} - \rho\bar{\pi}(\eta_s - 1)\right],$$

which is non-negative if $\eta_s < 1$, thus suggesting the optimality of the full disclosure policy. If $\eta_s \geq 1$, however, the term is non-negative only if $\rho < p^*/[(1-p^*)\bar{\pi}(\eta_s-1)]$. Thus,

$$\boldsymbol{\omega}_1^\star(p_0) = \begin{cases} (0,1), & \text{if } p^* \leq p_0 < p^\diamond, \\ \left(\frac{p^*}{1-p^*}\frac{1-p_0}{p_0}, 1\right), & \text{if } p_0 \geq p^\diamond \text{ and } \eta_s \geq 1 \text{ and } \rho \geq \frac{p^*}{(1-p^*)\bar{\pi}(\eta_s-1)}, \\ (0,1), & \text{if otherwise}. \end{cases} \tag{B.12}$$

Similarly, the optimal cost function becomes,

$$J_1(p_0) = \begin{cases} (1+\rho)\pi\ell_s - \overline{\pi}\kappa_s\left[(1+\rho)(p_0+\eta_s-1) - \rho\overline{\pi}(1-p_0)(\eta_s-1)\right], & \text{if } p^* \leq p_0 < p^\diamond, \\ (1+\rho)\pi\ell_s - \overline{\pi}\kappa_s\left[(1+\rho)(p_0+\eta_s-1) - \frac{p^*(1-p_0)}{(1-p^*)}\right], & \text{if } p_0 \geq p^\diamond \text{ and } \eta_s \geq 1 \text{ and } \rho \geq \frac{p^*}{(1-p^*)\bar{\pi}(\eta_s-1)}, \\ (1+\rho)\pi\ell_s - \overline{\pi}\kappa_s\left[(1+\rho)(p_0+\eta_s-1) - \rho\overline{\pi}(1-p_0)(\eta_s-1)\right], & \text{if otherwise}. \end{cases} \tag{B.13}$$

Denoting $p^{**} = p^\diamond$ for $\rho \geq p^*/[(1-p^*)\bar{\pi}(\eta_s-1)]$ and $p^{**} = 1$ otherwise, together with the result of Case 1, completes the proof for $\pi \leq \kappa_r/\delta_r$.

When $\pi > \kappa_r/\delta_r$, we have $p^* = (\kappa_r - \pi\delta_r)/(\overline{\pi}\kappa_r) < 0$. In this case, by Proposition 1, the full-disclosure policy is always optimal in the single period problem, and the receiver's optimal action always coincides with the sender's warning message, i.e., $a^\star(d,p,\boldsymbol{w}^\star) = d$ for all $p \in [0,1]$. It follows that

$$\begin{aligned} \mathbb{E}\left[c_s\left(a^\star\left(D,p,\boldsymbol{w}^\star\right),X\right) \mid \boldsymbol{w}^\star, p\right] &= \mathbb{E}\left[c_s\left(D,X\right) \mid \boldsymbol{w}^\star, p\right] \\ &= \pi(\kappa_s + \ell_s - \delta_s) + \bar{\pi}\kappa_s(1-p). \end{aligned}$$

Therefore, the same argument applies to the second period in the two period problem, and $J_2(p) = \pi(\kappa_s + \ell_s - \delta_s) + \bar{\pi}\kappa_s(1-p)$. Since $J_2(.)$ is continuous and linear, $\mathbb{E}\left[J_2(p_1) \mid \boldsymbol{w}_1, p_0\right] = J_2\left(\mathbb{E}\left[p_1 \mid \boldsymbol{w}_1, p_0\right]\right) = J_2(p_0)$. Substituting this in (8) reduces the two period problem to a single period one. By the same argument, we conclude that full-disclosure is also optimal in the first period. Thus, $\boldsymbol{\omega}_1^\star(p_0) = (0,1)$, and

$$\begin{aligned} J_1(p_0) = \pi(\kappa_s + \ell_s - \delta_s) + \bar{\pi}\kappa_s(1-p_0) + \rho J_2(p_0) &= (1+\rho)\left[\pi(\kappa_s + \ell_s - \delta_s) + \bar{\pi}\kappa_s(1-p)\right] \\ &= (1+\rho)\left[\pi\ell_s - \overline{\pi}\kappa_s(\eta_s - 1 + p_0)\right]. \end{aligned}$$

Taking all cases together completes the proof of Theorem 1. □

Proof of Corollary 1. The proof follows by substituting the optimal policies characterized in Theorem 1 into Lemma 3. □

Proof of Corollary 2. The posterior reputation in (12) immediately follows by substituting the optimal warning policy in each case into (2) and (3). □

Appendix C: Results for General Signal Quality and Proofs in Section 3

This appendix establishes results for general quality structure of early signals. Specifically, we denote the error rates of Z_t as

$$\alpha_\theta := \mathbb{P}[Z_t = 1 \mid X_t = 1, \theta], \quad \text{and} \quad \beta_\theta := \mathbb{P}[Z_t = 0 \mid X_t = 0, \theta], \quad \text{for } \theta \in \{H, L\}. \tag{C.1}$$

Thus, $\bar{\alpha}_\theta := 1 - \alpha_\theta$ and $\bar{\beta}_\theta := 1 - \beta_\theta$ respectively represent the false negative and false positive rates for type θ. The following assumptions apply for both the base model in Section 3 and extensions in Section 6, and hence are imposed throughout this appendix.

A1. $0 \le \alpha_L \le \alpha_H \le 1$ and $0 \le \beta_L \le \beta_H \le 1$, where the two inequalities cannot both hold as equalities. We denote $\alpha := \alpha_H - \alpha_L \ge 0$ and $\beta := \beta_H - \beta_L \ge 0$ and hence $\alpha + \beta > 0$.

A2. $\alpha_H + \beta_H \ge \alpha_L + \beta_L \ge 1$, which ensures the presence of a signal to be more indicative of the event than its absence (i.e., $\mathbb{P}[Z_t = 1 \mid X_t = 1, \theta] \ge \mathbb{P}[Z_t = 1 \mid X_t = 0, \theta]$ for $\theta \in \{H, L\}$).

LEMMA C.1. *The model parameters satisfy the following properties:*

1. *Prior to the realization of Z_t, player $i \in \{s, r\}$ prefers $a_t = 1$ if and only if $\eta_i := \frac{\pi(\delta_i - \kappa_i)}{\bar{\pi}\kappa_i} \ge 1$.*
2. $0 \le \bar{\beta}_H/\alpha_H \le \bar{\beta}_L/\alpha_L \le 1 \le \beta_L/\bar{\alpha}_L \le \beta_H/\bar{\alpha}_H$ *and hence*

$$\underbrace{\mathbb{P}[X_t = 1 \mid Z_t = 1, H]}_{\pi\alpha_H/(\pi\alpha_H + \bar{\pi}\bar{\beta}_H)} > \underbrace{\mathbb{P}[X_t = 1 \mid Z_t = 1, L]}_{\pi\alpha_L/(\pi\alpha_L + \bar{\pi}\bar{\beta}_L)} \ge \pi \ge \underbrace{\mathbb{P}[X_t = 1 \mid Z_t = 0, L]}_{\pi\bar{\alpha}_L/(\pi\bar{\alpha}_L + \bar{\pi}\beta_L)} > \underbrace{\mathbb{P}[X_t = 1 \mid Z_t = 0, H]}_{\pi\bar{\alpha}_H/(\pi\bar{\alpha}_H + \bar{\pi}\beta_H)}. \tag{C.2}$$

3. *If the quality type θ is publicly known, then player $i \in \{s, r\}$ prefers $a_t = 1$ upon $Z_t = 1$ if and only if $\eta_i \ge \bar{\beta}_\theta/\alpha_\theta$, while player $i \in \{s, r\}$ prefers $a_t = 1$ upon $Z_t = 0$ if and only if $\eta_i \ge \beta_\theta/\bar{\alpha}_\theta$.*
4. *For $i \in \{s, r\}$, $p_i^* := \frac{\bar{\beta}_L - \alpha_L\eta_i}{\beta + \alpha\eta_i}$ and $q_i^* := \frac{-\beta_L + \bar{\alpha}_L\eta_i}{\beta + \alpha\eta_i}$ are monotonically decreasing and increasing in $\eta_i \ge 0$, respectively, and satisfy Table C.1. In particular, $p_i^* - q_i^* = (1 - \eta_i)/(\beta + \alpha\eta_i) \ge 0$ if and only if $\eta_i \le 1$.*

Table C.1 **Properties of p_i^* and q_i^* for $i \in \{s, r\}$.**

$\eta_i \in$	$[0, \bar{\beta}_H/\alpha_H]$	$[\bar{\beta}_H/\alpha_H, \bar{\beta}_L/\alpha_L]$	$[\bar{\beta}_L/\alpha_L, \beta_L/\bar{\alpha}_L]$	$[\beta_L/\bar{\alpha}_L, \beta_H/\bar{\alpha}_H]$	$[\beta_H/\bar{\alpha}_H, \infty)$
p_i^*	≥ 1	$\in [0, 1]$	≤ 0		
q_i^*	≤ 0			$\in [0, 1]$	≥ 1

5. *$\eta_s > \eta_r$, $p_s^* < p_r^*$ and $q_s^* > q_r^*$.*
6. *$p_r^* q_s^* < p_s^* q_r^*$.*

Proof of Lemma C.1. We organize the proof according to the results.

1. Prior to the realization of Z_t, player i prefers $a_t = 1$ if and only if $\mathbb{E}[c_i(1, X_t)] \le \mathbb{E}[c_i(0, X_t)]$, which is, according to Table 1, equivalent to

$$\bar{\pi}\kappa_i + \pi(\kappa_i + \ell_i - \delta_i) \le \pi\ell_i \quad \Leftrightarrow \quad \eta_i \ge 1.$$

2. Straightforward application of Bayes Rule yields

$$\mathbb{P}[X_t=1 \mid Z_t=1,\theta]=\frac{\pi\alpha_\theta}{\pi\alpha_\theta+\overline{\pi}\bar{\beta}_\theta}, \quad \text{and} \quad \mathbb{P}[X_t=1 \mid Z_t=0,\theta]=\frac{\pi\bar{\alpha}_\theta}{\pi\bar{\alpha}_\theta+\overline{\pi}\beta_\theta}, \quad \text{for } \theta\in\{H,L\}. \tag{C.3}$$

   By straightforward verification, this property and hence (C.2) follow from the fact that $\alpha_H+\beta_H \geq \alpha_L+\beta_L \geq 1$.

3. Knowing θ, player i prefers $a_t=1$ upon $Z_t=1$ if and only if $\mathbb{E}[c_i(1,X_t) \mid Z_t=1,\theta] \leq \mathbb{E}[c_i(0,X_t) \mid Z_t=1,\theta]$, which is, according to (C.3), equivalent to

$$\overline{\pi}\bar{\beta}_\theta\kappa_i+\pi\alpha_\theta(\kappa_i+\ell_i-\delta_i)\leq \pi\alpha_\theta\ell_i \quad \Leftrightarrow \quad \eta_i\geq \bar{\beta}_\theta/\alpha_\theta.$$

   Following similar argument, player i knowing θ prefers $a_t=1$ upon $Z_t=0$ if and only if $\eta_i\geq\beta_\theta/\bar{\alpha}_\theta$.

4. This follows directly from the definition of p_i^* and q_i^* and the second property.

5. By definition, assumption $\delta_s/\kappa_s>\delta_r/\kappa_r>1$ immediately implies that $\eta_s>\eta_r$, from which $p_s^*<p_r^*$ and $q_s^*>q_r^*$ follow from the last property.

6. By the second property, direct calculation reveals that

$$p_s^*q_r^*-p_r^*q_s^*=\frac{(\bar{\beta}_L-\alpha_L\eta_s)(\bar{\alpha}_L\eta_r-\beta_L)-(\bar{\beta}_L-\alpha_L\eta_r)(\bar{\alpha}_L\eta_s-\beta_L)}{(\beta+\alpha\eta_s)(\beta+\alpha\eta_r)}=\frac{(\alpha_L\beta_L-\bar{\alpha}_L\bar{\beta}_L)(\eta_s-\eta_r)}{(\beta+\alpha\eta_s)(\beta+\alpha\eta_r)}>0,$$

   where the positivity follows from the second and fourth properties. □

Property 3 in Lemma C.1 implies that

A3. $\bar{\beta}_H/\alpha_H\leq\eta_r\leq\bar{\beta}_L/\alpha_L\leq\eta_s\leq\beta_L/\bar{\alpha}_L$, if the sender always prefers $a_t=Z_t$ regardless of θ, whereas the receiver prefers $a_t=1$ only if $Z_t=1$ and $\theta=H$.

C.1. Game Formulation

This appendix documents the game formulation in the original form of *histories* defined in Section 3.3, and then presents its reduction to the *reputation*-based formulation of Section 3.4. Subsequently, for the general quality structure of early signals, we prove Lemma 1 and obtain the following two Lemmas as the generalization of Lemmas 2 and 3, respectively.

LEMMA C.2 **(Updated Reputation)**. *Given realizations $D_t=d$ and $X_t=x$ with $(d,x)\in\{0,1\}^2$, credibility p_{t-1} is updated at the end of period t to $p_t^{d,x}$ according to*

$$p_t^{1,1}=\frac{\alpha_H w_t^1+\bar{\alpha}_H w_t^0}{(\alpha_L+\alpha p_{t-1})w_t^1+(\bar{\alpha}_L-\alpha p_{t-1})w_t^0}p_{t-1}, \tag{C.4}$$

$$p_t^{0,1}=\frac{\alpha_H \overline{w}_t^1+\bar{\alpha}_H \overline{w}_t^0}{(\alpha_L+\alpha p_{t-1})\overline{w}_t^1+(\bar{\alpha}_L-\alpha p_{t-1})\overline{w}_t^0}p_{t-1}, \tag{C.5}$$

$$p_t^{1,0}=\frac{\bar{\beta}_H w_t^1+\beta_H w_t^0}{(\bar{\beta}_L-\beta p_{t-1})w_t^1+(\beta_L+\beta p_{t-1})w_t^0}p_{t-1}, \tag{C.6}$$

$$p_t^{0,0}=\frac{\bar{\beta}_H \overline{w}_t^1+\beta_H \overline{w}_t^0}{(\bar{\beta}_L-\beta p_{t-1})\overline{w}_t^1+(\beta_L+\beta p_{t-1})\overline{w}_t^0}p_{t-1}, \tag{C.7}$$

where warning probabilities $w_t^1=\omega_t^1(p_{t-1})$, $w_t^0=\omega_t^0(p_{t-1})$, and $\overline{w}_t^z=1-w_t^z$ for $z\in\{0,1\}$. In particular, $p_t^{1,1},p_t^{0,0}\geq p_{t-1}$ and $p_t^{0,1},p_t^{1,0}\leq p_{t-1}$ if and only if $w_t^1\geq w_t^0$.

LEMMA C.3 **(Receiver's Best Response).** *Given warning probabilities $\boldsymbol{w} = (w^0, w^1) \in [0,1]^2$, warning message $d \in \{0,1\}$ and reputation $p \in [0,1]$, the optimal decision of the receiver is given by,*

$$\text{for } \eta_r \le 1, \quad a^\star(d,p,\boldsymbol{w}) = \begin{cases} d, & \text{if } \; (p_r^* - p)\,w^1 + (p - q_r^*)\,w^0 \le 0 \\ 0, & \text{if } \; 0 \le (p_r^* - p)\,w^1 + (p - q_r^*)\,w^0 \le p_r^* - q_r^* \\ (1-d), & \text{if } \; (p_r^* - p)\,w^1 + (p - q_r^*)\,w^0 \ge p_r^* - q_r^*\,, \end{cases} \tag{C.8}$$

$$\text{and for } \eta_r > 1, \quad a^\star(d,p,\boldsymbol{w}) = \begin{cases} d, & \text{if } \; (p_r^* - p)\,w^1 + (p - q_r^*)\,w^0 \le p_r^* - q_r^* \\ 1, & \text{if } \; p_r^* - q_r^* \le (p_r^* - p)\,w^1 + (p - q_r^*)\,w^0 \le 0 \\ (1-d), & \text{if } \; (p_r^* - p)\,w^1 + (p - q_r^*)\,w^0 \ge 0. \end{cases} \tag{C.9}$$

Problem Formulation. We first describe the receiver's best response. We note that the receiver's action in each period has no impact on his costs in other periods and also does not influence the sender's reputation trajectory throughout the game. Thus, given the sender's warning policy Ω and the receiver's history $\mathbf{h}_{r,t-1}$, the receiver's best action in period t upon receiving warning d_t is given by

$$a^\star(d_t, \mathbf{h}_{r,t-1} \mid \Omega) := \underset{a \in \{0,1\}}{\arg\min}\, \mathbb{E}\left[c_r(a, X_t) \mid d_t, \mathbf{h}_{r,t-1}, \Omega\right]. \tag{C.10}$$

Subsequently, the sender's problem is to minimize her ex ante expected total discounted cost:

$$\min_{\Omega} \mathbb{E}\left[\sum_{t=1}^{2} \rho^{t-1} c_s\left(a^\star(D_t, \mathbf{h}_{r,t-1} \mid \Omega), X_t\right) \,\middle|\, \Omega, p_0\right], \tag{C.11}$$

where we slightly abuse the notation and denote $\mathbf{h}_{r,1} = \{D_1, X_1\}$ rather than the realized outcome $\{d_1, x_1\}$ from the first period.

For any warning policy $\Omega = [\boldsymbol{\omega}_1(\cdot), \boldsymbol{\omega}_2(\cdot))]$, we define the *interim* warning probabilities as

$$\widehat{\omega}_t^z(\mathbf{h}_{r,t-1}) \equiv \mathbb{E}\left[\omega_t^z(\mathbf{h}_{s,t-1}) \mid Z_t = z, \mathbf{h}_{r,t-1}, \Omega\right] = \mathbb{P}\left[D_t = 1 \mid Z_t = z, \mathbf{h}_{r,t-1}, \Omega\right], \text{ for } z \in \{0,1\} \text{ and } t = 1,2. \tag{C.12}$$

Correspondingly, we define a warning policy $\widehat{\Omega} = [\widehat{\boldsymbol{\omega}}_1(\cdot), \widehat{\boldsymbol{\omega}}_2(\cdot))]$, which depends on the receiver's history $\{\mathbf{h}_{r,t-1} : t = 1,2\}$. In the following expressions, we omit the dependence of $\widehat{\omega}_t^z(\mathbf{h}_{r,t-1})$ on $\mathbf{h}_{r,t-1}$ and simply write it as $\widehat{\omega}_t^z$ without the risk of confusion.

Calculating Conditional Probabilities. By (C.12) and (C.1), we recognize that, for $\theta \in \{H, L\}$,

$$\mathbb{P}\left[D_t = 1 \mid X_t = 1, \theta, \mathbf{h}_{r,t-1}, \Omega\right] = \widehat{\omega}_t^1 \mathbb{P}\left[Z_t = 1 \mid X_t = 1, \theta\right] + \widehat{\omega}_t^0 \mathbb{P}\left[Z_t = 0 \mid X_t = 1, \theta\right] = \alpha_\theta \widehat{\omega}_t^1 + \bar{\alpha}_\theta \widehat{\omega}_t^0, \tag{C.13}$$

$$\mathbb{P}\left[D_t = 0 \mid X_t = 1, \theta, \mathbf{h}_{r,t-1}, \Omega\right] = 1 - \mathbb{P}\left[D_t = 1 \mid X_t = 1, \theta, \mathbf{h}_{r,t-1}, \Omega\right] = \alpha_\theta(1 - \widehat{\omega}_t^1) + \bar{\alpha}_\theta(1 - \widehat{\omega}_t^0), \tag{C.14}$$

$$\mathbb{P}\left[D_t = 1 \mid X_t = 0, \theta, \mathbf{h}_{r,t-1}, \Omega\right] = \widehat{\omega}_t^1 \mathbb{P}\left[Z_t = 1 \mid X_t = 0, \theta\right] + \widehat{\omega}_t^0 \mathbb{P}\left[Z_t = 0 \mid X_t = 0, \theta\right] = \bar{\beta}_\theta \widehat{\omega}_t^1 + \beta_\theta \widehat{\omega}_t^0, \tag{C.15}$$

$$\mathbb{P}\left[D_t = 0 \mid X_t = 0, \theta, \mathbf{h}_{r,t-1}, \Omega\right] = 1 - \mathbb{P}\left[D_t = 1 \mid X_t = 0, \theta, \mathbf{h}_{r,t-1}, \Omega\right] = \bar{\beta}_\theta(1 - \widehat{\omega}_t^1) + \beta_\theta(1 - \widehat{\omega}_t^0). \tag{C.16}$$

Recalling (C.1) and $p_{t-1} \equiv \mathbb{P}\left[\theta = H \mid \mathbf{h}_{r,t-1}, \Omega\right]$, we thus have

$$\begin{aligned}
\mathbb{P}\left[D_t = 1 \mid X_t = 1, \mathbf{h}_{r,t-1}, \Omega\right] &= p_{t-1}\left(\alpha_H \widehat{\omega}_t^1 + \bar{\alpha}_H \widehat{\omega}_t^0\right) + (1 - p_{t-1})\left(\alpha_L \widehat{\omega}_t^1 + \bar{\alpha}_L \widehat{\omega}_t^0\right) \\
&= (\alpha_L + \alpha p_{t-1})\,\widehat{\omega}_t^1 + (\bar{\alpha}_L - \alpha p_{t-1})\,\widehat{\omega}_t^0, \\
\mathbb{P}\left[D_t = 0 \mid X_t = 1, \mathbf{h}_{r,t-1}, \Omega\right] &= 1 - \mathbb{P}\left[D_t = 1 \mid X_t = 1, \mathbf{h}_{r,t-1}, \Omega\right] = (\alpha_L + \alpha p_{t-1})\,(1 - \widehat{\omega}_t^1) + (\bar{\alpha}_L - \alpha p_{t-1})\,(1 - \widehat{\omega}_t^0), \\
\mathbb{P}\left[D_t = 1 \mid X_t = 0, \mathbf{h}_{r,t-1}, \Omega\right] &= p_{t-1}\left(\bar{\beta}_H \widehat{\omega}_t^1 + \beta_H \widehat{\omega}_t^0\right) + (1 - p_{t-1})\left(\bar{\beta}_L \widehat{\omega}_t^1 + \beta_L \widehat{\omega}_t^0\right) \\
&= \left(\bar{\beta}_L - \beta p_{t-1}\right)\widehat{\omega}_t^1 + (\beta_L + \beta p_{t-1})\,\widehat{\omega}_t^0,
\end{aligned}$$

$$\mathbb{P}[D_t=0 \mid X_t=0, \mathbf{h}_{r,t-1}, \Omega] = 1-\mathbb{P}[D_t=1 \mid X_t=0, \mathbf{h}_{r,t-1}, \Omega] = \left(\bar{\beta}_L - \beta p_{t-1}\right)(1-\widehat{\omega}_t^1) + (\beta_L+\beta p_{t-1})(1-\widehat{\omega}_t^0),$$

which immediately imply

$$\mathbb{P}[D_t=1, X_t=1 \mid \mathbf{h}_{r,t-1}, \Omega] = \pi\left[(\alpha_L+\alpha p_{t-1})\widehat{\omega}_t^1 + (\bar{\alpha}_L - \alpha p_{t-1})\widehat{\omega}_t^0\right] \tag{C.17}$$

$$\mathbb{P}[D_t=0, X_t=1 \mid \mathbf{h}_{r,t-1}, \Omega] = \pi\left[(\alpha_L+\alpha p_{t-1})(1-\widehat{\omega}_t^1) + (\bar{\alpha}_L - \alpha p_{t-1})(1-\widehat{\omega}_t^0)\right], \tag{C.18}$$

$$\mathbb{P}[D_t=1, X_t=0 \mid \mathbf{h}_{r,t-1}, \Omega] = \bar{\pi}\left[\left(\bar{\beta}_L - \beta p_{t-1}\right)\widehat{\omega}_t^1 + (\beta_L+\beta p_{t-1})\widehat{\omega}_t^0\right], \tag{C.19}$$

$$\mathbb{P}[D_t=0, X_t=0 \mid \mathbf{h}_{r,t-1}, \Omega] = \bar{\pi}\left[\left(\bar{\beta}_L - \beta p_{t-1}\right)(1-\widehat{\omega}_t^1) + (\beta_L+\beta p_{t-1})(1-\widehat{\omega}_t^0)\right]. \tag{C.20}$$

Proof of Lemma C.2. By Bayes' Rule, the odds ratio is updated according to

$$\begin{aligned}
\frac{p_t^{d,x}}{1-p_t^{d,x}} &= \frac{\mathbb{P}[\theta=H \mid D_t=d, X_t=x, \mathbf{h}_{r,t-1}, \Omega]}{\mathbb{P}[\theta=L \mid D_t=d, X_t=x, \mathbf{h}_{r,t-1}, \Omega]} \\
&= \frac{\mathbb{P}[D_t=d, X_t=x \mid H, \mathbf{h}_{r,t-1}, \Omega]}{\mathbb{P}[D_t=d, X_t=x \mid L, \mathbf{h}_{r,t-1}, \Omega]} \frac{\mathbb{P}[\theta=H \mid \mathbf{h}_{r,t-1}, \Omega]}{\mathbb{P}[\theta=L \mid \mathbf{h}_{r,t-1}, \Omega]} \\
&= \frac{\mathbb{P}[D_t=d \mid X_t=x, H, \mathbf{h}_{r,t-1}, \Omega]\,\mathbb{P}[X_t=x]\,\mathbb{P}[\theta=H \mid \mathbf{h}_{r,t-1}, \Omega]}{\mathbb{P}[D_t=d \mid X_t=x, L, \mathbf{h}_{r,t-1}, \Omega]\,\mathbb{P}[X_t=x]\,\mathbb{P}[\theta=L \mid \mathbf{h}_{r,t-1}, \Omega]} \\
&= \frac{\mathbb{P}[D_t=d \mid X_t=x, H, \mathbf{h}_{r,t-1}, \Omega]}{\mathbb{P}[D_t=d \mid X_t=x, L, \mathbf{h}_{r,t-1}, \Omega]} \frac{p_{t-1}}{1-p_{t-1}}.
\end{aligned}$$

Solving for $p_t^{d,x}$ from the above equation yields

$$p_t^{d,x} = \frac{p_{t-1}\mathbb{P}[D_t=d \mid X_t=x, H, \mathbf{h}_{r,t-1}, \Omega]}{p_{t-1}\mathbb{P}[D_t=d \mid X_t=x, H, \mathbf{h}_{r,t-1}, \Omega] + (1-p_{t-1})\mathbb{P}[D_t=d \mid X_t=x, L, \mathbf{h}_{r,t-1}, \Omega]}. \tag{C.21}$$

By substituting (C.13)-(C.16) into (C.21), we obtain

$$p_t^{1,1} = \frac{p_{t-1}\left(\alpha_H\widehat{\omega}_t^1 + \bar{\alpha}_H\widehat{\omega}_t^0\right)}{p_{t-1}\left(\alpha_H\widehat{\omega}_t^1 + \bar{\alpha}_H\widehat{\omega}_t^0\right) + (1-p_{t-1})\left(\alpha_L\widehat{\omega}_t^1 + \bar{\alpha}_L\widehat{\omega}_t^0\right)} = \frac{\alpha_H\widehat{\omega}_t^1 + \bar{\alpha}_H\widehat{\omega}_t^0}{(\alpha_L+\alpha p_{t-1})\widehat{\omega}_t^1 + (\bar{\alpha}_L-\alpha p_{t-1})\widehat{\omega}_t^0} p_{t-1}, \tag{C.22}$$

$$p_t^{0,1} = \frac{\alpha_H(1-\widehat{\omega}_t^1) + \bar{\alpha}_H(1-\widehat{\omega}_t^0)}{(\alpha_L+\alpha p_{t-1})(1-\widehat{\omega}_t^1) + (\bar{\alpha}_L-\alpha p_{t-1})(1-\widehat{\omega}_t^0)} p_{t-1}, \tag{C.23}$$

$$p_t^{1,0} = \frac{p_{t-1}\left(\bar{\beta}_H\widehat{\omega}_t^1 + \beta_H\widehat{\omega}_t^0\right)}{p_{t-1}\left(\bar{\beta}_H\widehat{\omega}_t^1 + \beta_H\widehat{\omega}_t^0\right) + (1-p_{t-1})\left(\bar{\beta}_L\widehat{\omega}_t^1 + \beta_L\widehat{\omega}_t^0\right)} = \frac{\bar{\beta}_H\widehat{\omega}_t^1 + \beta_H\widehat{\omega}_t^0}{\left(\bar{\beta}_L-\beta p_{t-1}\right)\widehat{\omega}_t^1 + (\beta_L+\beta p_{t-1})\widehat{\omega}_t^0} p_{t-1}, \tag{C.24}$$

$$p_t^{0,0} = \frac{\bar{\beta}_H(1-\widehat{\omega}_t^1) + \beta_H(1-\widehat{\omega}_t^0)}{\left(\bar{\beta}_L-\beta p_{t-1}\right)(1-\widehat{\omega}_t^1) + (\beta_L+\beta p_{t-1})(1-\widehat{\omega}_t^0)} p_{t-1}. \tag{C.25}$$

Therefore, (C.4)-(C.7) follow immediately from (C.22)-(C.25) once we show that the warning probabilities only depend on p_{t-1} (rather than $\mathbf{h}_{s,t-1}$). This is established in the proof of Lemma 1 below.

Finally, by rearranging terms, we note that $p_t^{1,1} \geq p_{t-1}$ is equivalent to $\alpha(1-p_{t-1})(\widehat{\omega}_t^1 - \widehat{\omega}_t^0) \geq 0$, which holds if and only if $\widehat{\omega}_t^1 \geq \widehat{\omega}_t^0$. Similarly, we can establish the equivalence of $p_t^{0,0} \geq p_{t-1}$ and $p_t^{0,1}, p_t^{1,0} \leq p_{t-1}$ to $\widehat{\omega}_t^1 \geq \widehat{\omega}_t^0$. □

Proof of Lemma C.3. First, note that

$$\mathbb{E}[c_r(a, X_t) \mid D_t, \mathbf{h}_{r,t-1}, \Omega] = c_r(a,1)\,\mathbb{P}[X_t=1 \mid D_t, \mathbf{h}_{r,t-1}, \Omega] + c_r(a,0)\,\mathbb{P}[X_t=0 \mid D_t, \mathbf{h}_{r,t-1}, \Omega].$$

Then, the receiver's best action $a^\star(1, \mathbf{h}_{r,t-1} \mid \Omega) = 1$ if and only if

$$\underbrace{\{c_r(1,1) - c_r(0,1)\}}_{\kappa_r - \delta_r}\mathbb{P}[X_t=1 \mid D_t=1, \mathbf{h}_{r,t-1}, \Omega] + \underbrace{\{c_r(1,0) - c_r(0,0)\}}_{\kappa_r}\mathbb{P}[X_t=0 \mid D_t=1, \mathbf{h}_{r,t-1}, \Omega] \leq 0,$$

or equivalently,

$$\frac{\mathbb{P}\left[X_t = 0 \mid D_t = 1, \mathbf{h}_{r,t-1}, \Omega\right]}{\mathbb{P}\left[X_t = 1 \mid D_t = 1, \mathbf{h}_{r,t-1}, \Omega\right]} \le \frac{\delta_r - \kappa_r}{\kappa_r}. \tag{C.26}$$

On the other hand, (C.17) and (C.19) imply that

$$\frac{\mathbb{P}\left[X_t = 0 \mid D_t = 1, \mathbf{h}_{r,t-1}, \Omega\right]}{\mathbb{P}\left[X_t = 1 \mid D_t = 1, \mathbf{h}_{r,t-1}, \Omega\right]} = \frac{\mathbb{P}\left[D_t = 1, X_t = 0 \mid \mathbf{h}_{r,t-1}, \Omega\right]}{\mathbb{P}\left[D_t = 1, X_t = 1 \mid \mathbf{h}_{r,t-1}, \Omega\right]} = \frac{\bar{\pi}}{\pi} \times \frac{\left(\bar{\beta}_L - \beta p_{t-1}\right)\widehat{\omega}_t^1 + \left(\beta_L + \beta p_{t-1}\right)\widehat{\omega}_t^0}{\left(\alpha_L + \alpha p_{t-1}\right)\widehat{\omega}_t^1 + \left(\bar{\alpha}_L - \alpha p_{t-1}\right)\widehat{\omega}_t^0},$$

which, together with (C.26), suggests that

$$\begin{aligned}
a^\star\left(1, \mathbf{h}_{r,t-1} \mid \Omega\right) = 1 \quad &\Leftrightarrow \quad \frac{\left(\bar{\beta}_L - \beta p_{t-1}\right)\widehat{\omega}_t^1 + \left(\beta_L + \beta p_{t-1}\right)\widehat{\omega}_t^0}{\left(\alpha_L + \alpha p_{t-1}\right)\widehat{\omega}_t^1 + \left(\bar{\alpha}_L - \alpha p_{t-1}\right)\widehat{\omega}_t^0} \le \frac{\pi(\delta_r - \kappa_r)}{\bar{\pi}\kappa_r} = \eta_r \\
&\Leftrightarrow \quad \left(\underbrace{\frac{\bar{\beta}_L - \alpha_L \eta_r}{\beta + \alpha\eta_r}}_{p_r^*} - p_{t-1}\right)\widehat{\omega}_t^1 + \left(p_{t-1} - \underbrace{\frac{-\beta_L + \bar{\alpha}_L\eta_r}{\beta + \alpha\eta_r}}_{q_r^*}\right)\widehat{\omega}_t^0 \le 0
\end{aligned} \tag{C.27}$$

Following similar argument and using (C.18) and (C.20), we also have

$$a^\star\left(0, \mathbf{h}_{r,t-1} \mid \Omega\right) = 1 \quad \Leftrightarrow \quad \left(p_r^* - p_{t-1}\right)\widehat{\omega}_t^1 + \left(p_{t-1} - q_r^*\right)\widehat{\omega}_t^0 \ge p_r^* - q_r^*. \tag{C.28}$$

Therefore, given sender's warning policy $\widehat{\Omega}$, the receiver's best response at period t depends on $\widehat{\boldsymbol{\omega}}_t(\mathbf{h}_{r,t-1})$ and his history $\mathbf{h}_{r,t-1}$, where the latter dependence is only through p_{t-1}. Hence, we can denote the receiver's best response as $a^\star\left(d_t, p_{t-1}, \widehat{\boldsymbol{\omega}}_t(\mathbf{h}_{r,t-1})\right)$. The proof is then complete once we show that the warning probabilities only depend on p_{t-1} (rather than $\mathbf{h}_{s,t-1}$). This is done in Lemma 1 below. In particular, (C.8) and (C.9) follow from (C.27) and (C.28) by recognizing that $p_r^* - q_r^* = (1 - \eta_r)/(\beta + \alpha\eta_r) \ge 0$ if and only if $\eta_r \le 1$ (Lemma C.1). $\square$

C.2. Proofs for Results in Section 3

Proof of Lemma 1. Using the result of Lemma C.3 above, we can rewrite the sender's ex ante expected total discounted cost as

$$\begin{aligned}
&\mathbb{E}\left[\sum_{t=1}^{2} \rho^{t-1} c_s\left(a^\star\left(D_t, \mathbf{h}_{r,t-1} \mid \Omega\right), X_t\right) \,\middle|\, \Omega, p_0\right] \\
=&\mathbb{E}\left[\sum_{t=1}^{2} \rho^{t-1} c_s\left(a^\star\left(D_t, p_{t-1}, \widehat{\boldsymbol{\omega}}_t(\mathbf{h}_{r,t-1})\right), X_t\right) \,\middle|\, \Omega, p_0\right] \\
=&\mathbb{E}\left[\sum_{t=1}^{2} \rho^{t-1} \mathbb{E}\left[c_s\left(a^\star\left(D_t, p_{t-1}, \widehat{\boldsymbol{\omega}}_t(\mathbf{h}_{r,t-1})\right), X_t\right) \,\middle|\, \mathbf{h}_{r,t-1}, \widehat{\Omega}, p_0\right] \,\middle|\, \Omega, p_0\right].
\end{aligned} \tag{C.29}$$

Now, notice that the joint probability distribution of (D_t, X_t) conditional on $\left\{\mathbf{h}_{r,t-1}, \widehat{\Omega}, p_0\right\}$ is given by (C.17)-(C.20), which depends on $\mathbf{h}_{r,t-1}$ only through p_{t-1}, and $\widehat{\boldsymbol{\omega}}_t(\mathbf{h}_{r,t-1})$ for $t = 1, 2$. Therefore, $\mathbb{E}\left[c_s\left(a^\star\left(D_t, p_{t-1}, \widehat{\boldsymbol{\omega}}_t(\mathbf{h}_{r,t-1})\right), X_t\right) \,\middle|\, \mathbf{h}_{r,t-1}, \widehat{\Omega}, p_0\right]$ is a function of only p_{t-1} and $\widehat{\boldsymbol{\omega}}_t(\mathbf{h}_{r,t-1})$. This allows us to simplify the sender's cost as,

$$\begin{aligned}
&\mathbb{E}\left[\left(\mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \widehat{\boldsymbol{\omega}}_1(\mathbf{h}_{r,0})\right), X_1\right) \mid \widehat{\boldsymbol{\omega}}_1(\mathbf{h}_{r,0}), p_0\right] + \rho\mathbb{E}\left[c_s\left(a^\star\left(D_2, p_1, \widehat{\boldsymbol{\omega}}_2(\mathbf{h}_{r,1})\right), X_2\right) \mid \widehat{\boldsymbol{\omega}}_2(\mathbf{h}_{r,1}), p_1\right]\right) \,\middle|\, \Omega, p_0\right] \\
=&\mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \widehat{\boldsymbol{\omega}}_1(\mathbf{h}_{r,0})\right), X_1\right) \mid \widehat{\boldsymbol{\omega}}_1(\mathbf{h}_{r,0}), p_0\right] + \rho\mathbb{E}\left[\left(\mathbb{E}\left[c_s\left(a^\star\left(D_2, p_1, \widehat{\boldsymbol{\omega}}_2(\mathbf{h}_{r,1})\right), X_2\right) \mid \widehat{\boldsymbol{\omega}}_2(\mathbf{h}_{r,1}), p_1\right]\right) \,\middle|\, \Omega, p_0\right].
\end{aligned}$$

Moreover, p_t is a stochastic process evolving according to (C.22)-(C.25), and only depends on p_{t-1} and $\widehat{\boldsymbol{\omega}}_t(\mathbf{h}_{r,t-1})$, which further reduces the sender's cost to,

$$\mathbb{E}\left[c_s\left(a^\star\left(D_1,p_0,\widehat{\boldsymbol{\omega}}_1(\mathbf{h}_{r,0})\right),X_1\right)\mid\widehat{\boldsymbol{\omega}}_1(\mathbf{h}_{r,0}),p_0\right]+\rho\mathbb{E}\left[\left(\mathbb{E}\left[c_s\left(a^\star\left(D_2,p_1,\widehat{\boldsymbol{\omega}}_2(\mathbf{h}_{r,1})\right),X_2\right)\mid\widehat{\boldsymbol{\omega}}_2(\mathbf{h}_{r,1}),p_1\right]\right)\,\middle|\,\widehat{\boldsymbol{\omega}}_1(\mathbf{h}_{r,0}),p_0\right].$$

Subsequently, the sender's interim warning probabilities $\widehat{\boldsymbol{\omega}}_t(\mathbf{h}_{r,t-1})$ (and hence her ex ante warning probabilities $\boldsymbol{\omega}_t(\mathbf{h}_{s,t-1})$) can be expressed only as functions of p_{t-1} without loss of generality. $\square$

Proof of Lemmas 2 and 3. These two lemmas are a special case of Lemmas C.2 and C.3, respectively, by taking $\alpha_L=\alpha_H=\beta_H=1$ and $\beta_L=0$ and hence $p^*=p_r^*=1-\eta_r$ and $q_r^*=0$ implied from the base model. $\square$

C.3. The Second-Period Policy

LEMMA C.4. *For $\eta_r\le 1$, the sender's expected cost in a single period is given by*

$$\mathbb{E}\left[c_s\left(a^\star\left(D,p,\boldsymbol{w}\right),X\right)\mid\boldsymbol{w},p\right]=\begin{cases}\pi\ell_s+\overline{\pi}\kappa_s\left(\beta+\alpha\eta_s\right)\left[\left(p_s^*-p\right)w^1+\left(p-q_s^*\right)w^0\right], & \text{if } \frac{w^0}{w^1}\le\frac{p-p_r^*}{p-q_r^*}\\ \pi\ell_s, & \text{if } \frac{w^0}{w^1}\ge\frac{p-p_r^*}{p-q_r^*},\end{cases}\tag{C.30}$$

Proof of Lemma C.4. For $\eta_r\le 1$ and $w^0\le w^1\le 1$, we must have $\left(p_r^*-p\right)w^1+\left(p-q_r^*\right)w^0\le p_r^*-q_r^*$ by Lemma C.1. Thus, (C.8) implies the following two cases, from which (C.30) follows.

- If $\frac{w^0}{w^1}\le\frac{p-p_r^*}{p-q_r^*}\le 1$, then $a^\star\left(D,p,\boldsymbol{w}\right)=D$ and (C.17)-(C.20), together with Table 1, imply

$$\begin{aligned}
&\mathbb{E}\left[c_s\left(a^\star\left(D,p,\boldsymbol{w}\right),X\right)\mid\boldsymbol{w},p\right]=\mathbb{E}\left[c_s\left(D,X\right)\mid\boldsymbol{w},p\right]\\
&=\underbrace{c_s(1,1)}_{\kappa_s+\ell_s-\delta_s}\pi\left[\left(\alpha_L+\alpha p_{t-1}\right)w^1+\left(\bar{\alpha}_L-\alpha p_{t-1}\right)w^0\right]+\underbrace{c_s(0,1)}_{\ell_s}\pi\left[\left(\alpha_L+\alpha p_{t-1}\right)\overline{w}^1+\left(\bar{\alpha}_L-\alpha p_{t-1}\right)\overline{w}^0\right]\\
&\quad+\underbrace{c_s(1,0)}_{\kappa_s}\overline{\pi}\left[\left(\bar{\beta}_L-\beta p_{t-1}\right)w^1+\left(\beta_L+\beta p_{t-1}\right)w^0\right]+\underbrace{c_s(0,0)}_{0}\overline{\pi}\left[\left(\bar{\beta}_L-\beta p_{t-1}\right)\overline{w}^1+\left(\beta_L+\beta p_{t-1}\right)\overline{w}^0\right]\\
&=\pi\ell_s+\pi(\kappa_s-\delta_s)\left[\left(\alpha_L+\alpha p_{t-1}\right)w^1+\left(\bar{\alpha}_L-\alpha p_{t-1}\right)w^0\right]+\overline{\pi}\kappa_s\left[\left(\bar{\beta}_L-\beta p_{t-1}\right)w^1+\left(\beta_L+\beta p_{t-1}\right)w^0\right]\\
&=\pi\ell_s+\overline{\pi}\kappa_s\left(\beta+\alpha\eta_s\right)\left[\left(\frac{\bar{\beta}_L-\alpha_L\eta_s}{\beta+\alpha\eta_s}-p\right)w^1+\left(p-\frac{-\beta_L+\bar{\alpha}_L\eta_s}{\beta+\alpha\eta_s}\right)w^0\right],\\
&=\pi\ell_s+\overline{\pi}\kappa_s\left(\beta+\alpha\eta_s\right)\left[\left(p_s^*-p\right)w^1+\left(p-q_s^*\right)w^0\right].
\end{aligned}$$

- If $1\ge\frac{w^0}{w^1}\ge\frac{p-p_r^*}{p-q_r^*}$, then $a^\star\left(D,p,\boldsymbol{w}\right)\equiv 0$, then, according to Table 1,

$$\mathbb{E}\left[c_s\left(a^\star\left(D,p,\boldsymbol{w}\right),X\right)\mid\boldsymbol{w},p\right]=\mathbb{E}\left[c_s\left(0,X\right)\mid\boldsymbol{w},p\right]=\pi\underbrace{c_s(0,1)}_{\ell_s}+\overline{\pi}\underbrace{c_s(0,0)}_{0}=\pi\ell_s.\quad\square$$

PROPOSITION C.1. *Suppose that Assumption A3 holds. Given reputation p at the beginning of a single-period problem, the optimal warning policy is given as follows.*

- *for $p<p_r^*$, any policy is optimal and the receiver never acts, i.e., $a^\star\left(d,p,\boldsymbol{\omega}_2^\star(p)\right)=0$ for any feasible policy $\boldsymbol{\omega}_2^\star(p)$;*
- *for $p\ge p_r^*$, full disclosure $\boldsymbol{\omega}_2^\star(p)=(w_2^{0\star},w_2^{1\star})=(0,1)$ is optimal and the receiver always acts upon receiving a warning, i.e., $a^\star\left(d,p,\boldsymbol{\omega}_2^\star(p)\right)=d$.*

Furthermore,

$$J_2(p)=\begin{cases}\pi\ell_s, & \text{if } p<p_r^*,\\ \pi\ell_s+\overline{\pi}\kappa_s\left(\beta+\alpha\eta_s\right)\left(p_s^*-p\right), & \text{if } p\ge p_r^*,\end{cases}\tag{C.31}$$

with a unique discontinuity $J_2(p_r^*-)-J_2(p_r^*+)=\overline{\pi}\kappa_s\left(\beta+\alpha\eta_s\right)\left(p_r^*-p_s^*\right)>0$.

Proof of Proposition C.1. By Lemma C.1, we have $\eta_r \le 1$, $p_s^* \le 0$ and $p_r^* \ge 0 \ge q_s^* > q_r^*$ by Lemma C.1. Thus, the sender's single-period problem is to choose $\boldsymbol{w}$ to minimize her expected cost in (C.30).

- For $p < p_r^*$, we must have $w^0/w^1 \ge 0 > (p-p_r^*)/(p-q_r^*)$, which implies that $a^\star(D,p,\boldsymbol{w}) \equiv 0$ by (C.8) and $\mathbb{E}[c_s(a^\star(D,p,\boldsymbol{w}),X) \mid \boldsymbol{w},p] \equiv \pi\ell_s$ for any $\boldsymbol{w}$ by (C.30), i.e., any policy is optimal. Thus, we obtain the first result.

- For $p \ge p_r^* \ge 0$, we have, by (C.30), for all $w^0/w^1 \le (p-p_r^*)/(p-q_r^*) \le 1$,

$$\begin{aligned}\mathbb{E}[c_s(a^\star(D,p,\boldsymbol{w}),X) \mid \boldsymbol{w},p] =& \pi\ell_s + \bar{\pi}\kappa_s(\beta+\alpha\eta_s)\left[\underbrace{(p_s^*-p)}_{\le 0}w^1 + \underbrace{(p-q_s^*)}_{\ge 0}w^0\right] \\ \ge& \pi\ell_s + \bar{\pi}\kappa_s(\beta+\alpha\eta_s)(p_s^*-p),\end{aligned} \tag{C.32}$$

  where "=" holds with $(w^1,w^0)=(1,0)$ satisfying $w^0/w^1 \le (p-p_r^*)/(p-q_r^*)$. Since

$$\pi\ell_s + \bar{\pi}\kappa_s(\beta+\alpha\eta_s)(p_s^*-p) \le \pi\ell_s = \mathbb{E}[c_s(a^\star(D,p,\boldsymbol{w}),X) \mid \boldsymbol{w},p], \quad \text{for } w^0/w^1 \ge (p-p_r^*)/(p-q_r^*),$$

  the optimal policy is indeed $(w_2^{0\star}, w_2^{1\star}) = (0,1)$ with $\pi\ell_s + \bar{\pi}\kappa_s(\beta+\alpha\eta_s)(p_s^*-p)$ as the sender's optimal expected cost, leading to the second result. By (C.8), we also have $a^\star(d,p,\boldsymbol{\omega}_2^\star(p)) = d$. $\square$

LEMMA C.5. *For any $p_0 \in [0,1]$, define*

$$\begin{aligned} A_r(p_0) &:= (\alpha_H - \alpha p_r^*)p_0 - \alpha_L p_r^*, \qquad & \bar{A}_r(p_0) &:= (\bar{\alpha}_H + \alpha p_r^*)p_0 - \bar{\alpha}_L p_r^* = p_0 - p_r^* - A_r(p_0);\\ B_r(p_0) &:= (\beta_H - \beta p_r^*)p_0 - \beta_L p_r^*, & \bar{B}_r(p_0) &:= (\bar{\beta}_H + \beta p_r^*)p_0 - \bar{\beta}_L p_r^* = p_0 - p_r^* - B_r(p_0).\end{aligned} \tag{C.33}$$

Then, for $p_r^* \in [0,1]$, the posterior probability $p_1^{d,x}$ satisfies the following properties:

$$p_1^{1,1}\begin{cases} < p_r^* \text{ for all } w_1^1 \ge w_1^0, & \text{if } p_0 < \alpha_L p_r^*/(\alpha_H - \alpha p_r^*),\\ \ge p_r^* \text{ only for } w_1^1/w_1^0 \ge -\bar{A}_r(p_0)/A_r(p_0), & \text{if } \alpha_L p_r^*/(\alpha_H-\alpha p_r^*) \le p_0 < p_r^*,\\ \ge p_r^* \text{ for all } w_1^1 \ge w_1^0, & \text{if } p_0 \ge p_r^*;\end{cases} \tag{C.34}$$

$$p_1^{0,0}\begin{cases} < p_r^* \text{ for all } w_1^1 \ge w_1^0, & \text{if } p_0 < \beta_L p_r^*/(\beta_H - \beta p_r^*),\\ \ge p_r^* \text{ only for } \overline{w}_1^0/\overline{w}_1^1 \ge -\bar{B}_r(p_0)/B_r(p_0), & \text{if } \beta_L p_r^*/(\beta_H-\beta p_r^*) \le p_0 < p_r^*,\\ \ge p_r^* \text{ for all } w_1^1 \ge w_1^0, & \text{if } p_0 \ge p_r^*;\end{cases} \tag{C.35}$$

$$p_1^{0,1}\begin{cases} < p_r^* \text{ for all } w_1^1 \ge w_1^0, & \text{if } p_0 < p_r^*,\\ \ge p_r^* \text{ only for } \overline{w}_1^0/\overline{w}_1^1 \le -A_r(p_0)/\bar{A}_r(p_0), & \text{if } p_r^* \le p_0 < \bar{\alpha}_L p_r^*/(\bar{\alpha}_H + \alpha p_r^*),\\ \ge p_r^* \text{ for all } w_1^1 \ge w_1^0, & \text{if } p_0 \ge \bar{\alpha}_L p_r^*/(\bar{\alpha}_H + \alpha p_r^*);\end{cases} \tag{C.36}$$

$$p_1^{1,0}\begin{cases} < p_r^* \text{ for all } w_1^1 \ge w_1^0, & \text{if } p_0 < p_r^*,\\ \ge p_r^* \text{ only for } w_1^1/w_1^0 \le -B_r(p_0)/\bar{B}_r(p_0), & \text{if } p_r^* \le p_0 < \bar{\beta}_L p_r^*/(\bar{\beta}_H + \beta p_r^*),\\ \ge p_r^* \text{ for all } w_1^1 \ge w_1^0, & \text{if } p_0 \ge \bar{\beta}_L p_r^*/(\bar{\beta}_H + \beta p_r^*),\end{cases} \tag{C.37}$$

where the inequalities $p_1^{d,x} \ge p_r^*$ hold with "=" when the corresponding inequalities regarding (w_1^0, w_1^1) also hold with "=".

In particular, when $\{\alpha_L p_r^*/(\alpha_H - \alpha p_r^*), \beta_L p_r^*/(\beta_H - \beta p_r^*)\} \le p_0 < p_r^*$ (i.e., $A_r(p_0) \ge 0$, $B_r(p_0) \ge 0$, $\bar{A}_r(p_0) \le p_0 - p_r^* < 0$ and $\bar{B}_r(p_0) \le p_0 - p_r^* < 0$), then $p_1^{1,1} = p_1^{0,0} = p_r^*$ for $(w_1^0, w_1^1) = (w_\dagger^0, w_\dagger^1)$, where

$$\frac{w_\dagger^1}{w_\dagger^0} = -\frac{\bar{A}_r(p_0)}{A_r(p_0)} \quad \text{and} \quad \frac{\overline{w}_\dagger^0}{\overline{w}_\dagger^1} = -\frac{\bar{B}_r(p_0)}{B_r(p_0)} \quad \Leftrightarrow \quad w_\dagger^0 = \frac{A_r(p_0)}{A_r(p_0) - \bar{B}_r(p_0)} \quad \text{and} \quad w_\dagger^1 = \frac{\bar{A}_r(p_0)}{\bar{A}_r(p_0) - B_r(p_0)}. \tag{C.38}$$

Similarly, when $p_r^* \le p_0 < \{\bar{\alpha}_L p_r^*/(\bar{\alpha}_H + \alpha p_r^*), \bar{\beta}_L p_r^*/(\bar{\beta}_H + \beta p_r^*)\}$ (i.e., $A_r(p_0) > p_0 - p_r^*$, $B_r(p_0) > p_0 - p_r^*$, $\bar{A}_r(p_0) < 0$ and $\bar{B}_r(p_0) < 0$), then $p_1^{0,1} = p_1^{1,0} = p_r^*$ for $(w_1^0, w_1^1) = (w_\ddagger^0, w_\ddagger^1)$, where

$$\frac{w_\ddagger^1}{w_\ddagger^0} = -\frac{B_r(p_0)}{\bar{B}_r(p_0)} \quad \text{and} \quad \frac{\overline{w}_\ddagger^0}{\overline{w}_\ddagger^1} = -\frac{A_r(p_0)}{\bar{A}_r(p_0)} \quad \Leftrightarrow \quad w_\ddagger^0 = \frac{\bar{B}_r(p_0)}{\bar{B}_r(p_0) - A_r(p_0)} \quad \text{and} \quad w_\ddagger^1 = \frac{B_r(p_0)}{B_r(p_0) - \bar{A}_r(p_0)}. \tag{C.39}$$

Moreover, we have

$$\frac{\alpha_L p_r^*}{\alpha_H - \alpha p_r^*} \le \frac{\beta_L p_r^*}{\beta_H - \beta p_r^*} \quad \text{if and only if} \quad \alpha\beta_L - \beta\alpha_L \ge 0; \tag{C.40}$$

$$\frac{\bar{\alpha}_L p_r^*}{\bar{\alpha}_H + \alpha p_r^*} \le \frac{\bar{\beta}_L p_r^*}{\bar{\beta}_H + \beta p_r^*} \quad \text{if and only if} \quad \alpha\beta_L - \beta\alpha_L \ge \alpha - \beta. \tag{C.41}$$

Proof of Lemma C.5. We only demonstrate (C.34). All other properties follow from similar argument.

- If $p_0 < \alpha_L p_r^* / (\alpha_H - \alpha p_r^*)$, rearranging terms yields $\alpha_H p_0 / (\alpha_L + \alpha p_0) < p_r^*$. On the other hand, it is straightforward to see that

$$p_1^{1,1} = \frac{\alpha_H w_1^1 + \bar{\alpha}_H w_1^0}{(\alpha_L + \alpha p_0)\, w_1^1 + (\bar{\alpha}_L - \alpha p_0)\, w_1^0} p_0 \le \frac{\alpha_H p_0}{\alpha_L + \alpha p_0},$$

  where "=" holds for $(w_1^1, w_1^0) = (1, 0)$. As such, we must have $p_1^{1,1} < p_r^*$.

- If $\alpha_L p_r^* / (\alpha_H - \alpha p_r^*) \le p_0 < p_r^*$, it is straightforward to verify that $-\bar{A}_r(p_0) > A_r(p_0) \ge 0$. Thus, by rearranging terms

$$p_1^{1,1} = \frac{\alpha_H w_1^1 + \bar{\alpha}_H w_1^0}{(\alpha_L + \alpha p_0)\, w_1^1 + (\bar{\alpha}_L - \alpha p_0)\, w_1^0} p_0 \ge p_r^* \quad \Leftrightarrow \quad \frac{w_1^1}{w_1^0} \ge \frac{-\bar{A}_r(p_0)}{A_r(p_0)}.$$

- Finally, if $p_0 \ge p_r^*$, then $p_1^{1,1} \ge p_0 \ge p_r^*$.

To see (C.40), we note, by rearranging terms, that

$$\frac{\alpha_L p_r^*}{\alpha_H - \alpha p_r^*} \le \frac{\beta_L p_r^*}{\beta_H - \beta p_r^*} \quad \Leftrightarrow \quad (\alpha\beta_L - \beta\alpha_L)\,(1 - p_r^*) \ge 0.$$

Similarly, we can obtain (C.41) . □

Lemma C.6. *For any $p_0 \in [0,1]$, define*

$$\begin{aligned} A_s(p_0) &:= (\alpha_H - \alpha p_s^*)\, p_0 - \alpha_L p_s^* \ge 0, & \bar{A}_s(p_0) &:= (\bar{\alpha}_H + \alpha p_s^*)\, p_0 - \bar{\alpha}_L p_s^* = p_0 - p_s^* - A_s(p_0) \ge 0; \\ B_s(p_0) &:= (\beta_H - \beta p_s^*)\, p_0 - \beta_L p_s^* \ge 0, & \bar{B}_s(p_0) &:= (\bar{\beta}_H + \beta p_s^*)\, p_0 - \bar{\beta}_L p_s^* = p_0 - p_s^* - B_s(p_0) \ge 0. \end{aligned} \tag{C.42}$$

Under Assumption A3 and for any given $\boldsymbol{w}_1 = (w_1^0, w_1^1)$,

- *if $\{p_1^{0,1}, p_1^{1,0}\} \le p_0 \le \{p_1^{0,0}, p_1^{1,1}\} < p_r^*$, then*

$$\mathbb{E}\left[J_2(p_1) \mid \boldsymbol{w}_1, p_0\right] = \pi \ell_s; \tag{C.43}$$

- *if $\{p_1^{0,1}, p_1^{1,0}\} \le p_0 \le p_1^{0,0} < p_r^* \le p_1^{1,1}$, then*

$$\mathbb{E}\left[J_2(p_1) \mid \boldsymbol{w}_1, p_0\right] = \pi \ell_s - \overline{\pi} \kappa_s (\beta + \alpha\eta_s) \cdot \pi \left[A_s(p_0) w_1^1 + \bar{A}_s(p_0) w_1^0\right]; \tag{C.44}$$

- *if $\{p_1^{0,1}, p_1^{1,0}\} \le p_0 \le p_1^{1,1} < p_r^* \le p_1^{0,0}$, then*

$$\begin{aligned} \mathbb{E}\left[J_2(p_1) \mid \boldsymbol{w}_1, p_0\right] &= \pi \ell_s - \overline{\pi} \kappa_s (\beta + \alpha\eta_s) \cdot \overline{\pi} \left[\bar{B}_s(p_0) \overline{w}_1^1 + B_s(p_0) \overline{w}_1^0\right] \\ &= \pi \ell_s + \overline{\pi} \kappa_s (\beta + \alpha\eta_s) \cdot \overline{\pi} \left[(p_s^* - p_0) + \bar{B}_s(p_0) w_1^1 + B_s(p_0) w_1^0\right]; \end{aligned} \tag{C.45}$$

- *if $\{p_1^{0,1}, p_1^{1,0}\} < p_r^* \le \{p_1^{0,0}, p_1^{1,1}\}$, then*

$$\begin{aligned} \mathbb{E}\left[J_2(p_1) \mid \boldsymbol{w}_1, p_0\right] &= \pi \ell_s - \overline{\pi} \kappa_s (\beta + \alpha\eta_s) \left\{\pi \left[A_s(p_0) w_1^1 + \bar{A}_s(p_0) w_1^0\right] + \overline{\pi} \left[\bar{B}_s(p_0) \overline{w}_1^1 + B_s(p_0) \overline{w}_1^0\right]\right\} \\ &= \pi \ell_s + \overline{\pi} \kappa_s (\beta + \alpha\eta_s) \left\{\overline{\pi} (p_s^* - p_0) + \left[\overline{\pi} \bar{B}_s(p_0) - \pi A_s(p_0)\right] w_1^1 + \left[\overline{\pi} B_s(p_0) - \pi \bar{A}_s(p_0)\right] w_1^0\right\}; \end{aligned} \tag{C.46}$$

- *if* $p_1^{1,0} < p_r^* \le p_1^{0,1} \le p_0 \le \{p_1^{0,0}, p_1^{1,1}\}$, *then*

$$\mathbb{E}\left[J_2(p_1) \mid \boldsymbol{w}_1, p_0\right] = \pi\ell_s + \overline{\pi}\kappa_s(\beta+\alpha\eta_s)\left[p_s^* - p_0 + \overline{\pi}\bar{B}_s(p_0)w_1^1 + \overline{\pi}B_s(p_0)w_1^0\right]; \tag{C.47}$$

- *if* $p_1^{0,1} < p_r^* \le p_1^{1,0} \le p_0 \le \{p_1^{0,0}, p_1^{1,1}\}$, *then*

$$\mathbb{E}\left[J_2(p_1) \mid \boldsymbol{w}_1, p_0\right] = \pi\ell_s + \overline{\pi}\kappa_s(\beta+\alpha\eta_s)\left[\overline{\pi}(p_s^* - p_0) - \pi A_s(p_0)w_1^1 - \pi\bar{A}_s(p_0)w_1^0\right]; \tag{C.48}$$

- *if* $p_r^* \le \{p_1^{0,1}, p_1^{1,0}\} \le p_0 \le \{p_1^{0,0}, p_1^{1,1}\}$, *then*

$$\mathbb{E}\left[J_2(p_1) \mid \boldsymbol{w}_1, p_0\right] = \pi\ell_s + \overline{\pi}\kappa_s(\beta+\alpha\eta_s)(p_s^* - p_0). \tag{C.49}$$

Proof of Lemma C.6. Following from (C.31), we have

- if $\{p_1^{0,1}, p_1^{1,0}\} \le p_0 \le \{p_1^{0,0}, p_1^{1,1}\} < p_r^*$, then $J_2(p_1) \equiv \pi\ell_s$ and hence we obtain (C.43);
- if $\{p_1^{0,1}, p_1^{1,0}\} \le p_0 \le p_1^{0,0} < p_r^* \le p_1^{1,1}$, then $J_2\left(p_1^{0,1}\right) = J_2\left(p_1^{1,0}\right) = J_2\left(p_1^{0,0}\right) \equiv \pi\ell_s$ and $J_2\left(p_1^{1,1}\right) = \pi\ell_s + \overline{\pi}\kappa_s(\beta+\alpha\eta_s)\left(p_s^* - p_1^{1,1}\right)$, and hence (C.44) is implied by

$$\begin{aligned}\mathbb{E}\left[J_2(p_1) \mid \boldsymbol{w}_1, p_0\right] =& \pi\ell_s + \overline{\pi}\kappa_s(\beta+\alpha\eta_s)\left(p_s^* - p_1^{1,1}\right)\mathbb{P}\left[D_1 = 1, X_1 = 1 \mid \boldsymbol{w}_1, p_0\right] \\ =& \pi\ell_s + \overline{\pi}\kappa_s(\beta+\alpha\eta_s)\pi\left[p_s^*(\alpha_L + \alpha p_0)w_1^1 + p_s^*(\bar{\alpha}_L - \alpha p_0)w_1^0 - \left(\alpha_H w_1^1 + \bar{\alpha}_H w_1^0\right)p_0\right],\end{aligned}$$

  where the second equality follows from by (C.17) and (C.4);
- if $\{p_1^{0,1}, p_1^{1,0}\} \le p_0 \le p_1^{1,1} < p_r^* \le p_1^{0,0}$, then $J_2\left(p_1^{0,1}\right) = J_2\left(p_1^{1,0}\right) = J_2\left(p_1^{1,1}\right) \equiv \pi\ell_s$ and $J_2\left(p_1^{0,0}\right) = \pi\ell_s + \overline{\pi}\kappa_s(\beta+\alpha\eta_s)\left(p_s^* - p_1^{0,0}\right)$, and hence

$$\mathbb{E}\left[J_2(p_1) \mid \boldsymbol{w}_1, p_0\right] = \pi\ell_s + \overline{\pi}\kappa_s(\beta+\alpha\eta_s)\left(p_s^* - p_1^{0,0}\right)\mathbb{P}\left[D_1 = 0, X_1 = 0 \mid \boldsymbol{w}_1, p_0\right],$$

  which implies (C.45) by (C.20) and (C.7);
- if $\{p_1^{0,1}, p_1^{1,0}\} < p_r^* \le \{p_1^{0,0}, p_1^{1,1}\}$, then $J_2\left(p_1^{0,1}\right) = J_2\left(p_1^{1,0}\right) \equiv \pi\ell_s$ and $J_2\left(p_1^{d,x}\right) = \pi\ell_s + \overline{\pi}\kappa_s(\beta+\alpha\eta_s)\left(p_s^* - p_1^{d,x}\right)$ for $(d,x) \in \{(0,0),(1,1)\}$, and hence

$$\begin{aligned}\mathbb{E}\left[J_2(p_1) \mid \boldsymbol{w}_1, p_0\right] = \pi\ell_s + \overline{\pi}\kappa_s(\beta+\alpha\eta_s)\Big\{&\left(p_s^* - p_1^{0,0}\right)\mathbb{P}\left[D_1 = 0, X_1 = 0 \mid \boldsymbol{w}_1, p_0\right] \\ &+ \left(p_s^* - p_1^{1,1}\right)\mathbb{P}\left[D_1 = 1, X_1 = 1 \mid \boldsymbol{w}_1, p_0\right]\Big\},\end{aligned}$$

  from which (C.46) follows by similar argument as above;
- if $p_1^{1,0} < p_r^* \le p_1^{0,1} \le p_0 \le \{p_1^{0,0}, p_1^{1,1}\}$, then $J_2\left(p_1^{1,0}\right) \equiv \pi\ell_s$ and $J_2\left(p_1^{d,x}\right) = \pi\ell_s + \overline{\pi}\kappa_s(\beta+\alpha\eta_s)\left(p_s^* - p_1^{d,x}\right)$ for $(d,x) \neq (1,0)$, and hence using the fact that $\mathbb{E}\left[p_1 \mid \boldsymbol{w}_1, p_0\right] = p_0$, we have

$$\mathbb{E}\left[J_2(p_1) \mid \boldsymbol{w}_1, p_0\right] = \pi\ell_s + \overline{\pi}\kappa_s(\beta+\alpha\eta_s)(p_s^* - p_0) - \overline{\pi}\kappa_s(\beta+\alpha\eta_s)\left(p_s^* - p_1^{1,0}\right)\mathbb{P}\left[D_1 = 1, X_1 = 0 \mid \boldsymbol{w}_1, p_0\right],$$

  from which (C.47) follows by similar argument as above;
- if $p_1^{1,0} < p_r^* \le p_1^{0,1} \le p_0 \le \{p_1^{0,0}, p_1^{1,1}\}$, then $J_2\left(p_1^{0,1}\right) \equiv \pi\ell_s$ and $J_2\left(p_1^{d,x}\right) = \pi\ell_s + \overline{\pi}\kappa_s(\beta+\alpha\eta_s)\left(p_s^* - p_1^{d,x}\right)$ for $(d,x) \neq (0,1)$, and hence using the fact that $\mathbb{E}\left[p_1 \mid \boldsymbol{w}_1, p_0\right] = p_0$, we have

$$\mathbb{E}\left[J_2(p_1) \mid \boldsymbol{w}_1, p_0\right] = \pi\ell_s + \overline{\pi}\kappa_s(\beta+\alpha\eta_s)(p_s^* - p_0) - \overline{\pi}\kappa_s(\beta+\alpha\eta_s)\left(p_s^* - p_1^{0,1}\right)\mathbb{P}\left[D_1 = 0, X_1 = 1 \mid \boldsymbol{w}_1, p_0\right],$$

  from which (C.48) follows by similar argument as above;

- if $p_r^* \le \{p_1^{0,1}, p_1^{1,0}\} \le p_0 \le \{p_1^{0,0}, p_1^{1,1}\}$, then $J_2\left(p_1^{d,x}\right) = \pi\ell_s + \overline{\pi}\kappa_s(\beta + \alpha\eta_s)\left(p_s^* - p_1^{d,x}\right)$ for all (d,x), and hence (C.49) follows from the fact that $\mathbb{E}[p_1 \mid \boldsymbol{w}_1, p_0] = p_0$. □

LEMMA C.7. *Under Assumption A3,*

$$\overline{\pi}\bar{B}_s(p_0) - \pi A_s(p_0) \begin{cases} \ge 0 \text{ for all } p_0, & \text{if } \pi/\overline{\pi} \le \bar{\beta}_H/\alpha_H, \\ \ge 0 \text{ only for } p_0 \le \frac{(\overline{\pi}\bar{\beta}_L - \pi\alpha_L)p_s^*}{\overline{\pi}\bar{\beta}_H - \pi\alpha_H + (\overline{\pi}\beta + \pi\alpha)p_s^*} \in [0,1], & \text{if } \bar{\beta}_H/\alpha_H < \pi/\overline{\pi} \le \bar{\beta}_L/\alpha_L, \\ \le 0 \text{ for all } p_0, & \text{if } \pi/\overline{\pi} \ge \bar{\beta}_L/\alpha_L; \end{cases} \tag{C.50}$$

$$\overline{\pi}B_s(p_0) - \pi\bar{A}_s(p_0) \begin{cases} \ge 0 \text{ for all } p_0, & \text{if } \pi/\overline{\pi} \le \beta_L/\bar{\alpha}_L, \\ \ge 0 \text{ only for } p_0 \ge \frac{(\overline{\pi}\beta_L - \pi\bar{\alpha}_L)p_s^*}{\overline{\pi}\beta_H - \pi\bar{\alpha}_H - (\overline{\pi}\beta + \pi\alpha)p_s^*} \in [0,1], & \text{if } \beta_L/\bar{\alpha}_L < \pi/\overline{\pi} \le \beta_H/\bar{\alpha}_H, \\ \le 0 \text{ for all } p_0, & \text{if } \pi/\overline{\pi} \ge \beta_H/\bar{\alpha}_H; \end{cases} \tag{C.51}$$

Proof of Lemma C.7. By definition (C.42), we have

$$\overline{\pi}\bar{B}_s(p_0) - \pi A_s(p_0) = \left[\overline{\pi}\bar{\beta}_H - \pi\alpha_H + (\overline{\pi}\beta + \pi\alpha)p_s^*\right]p_0 - \left(\overline{\pi}\bar{\beta}_L - \pi\alpha_L\right)p_s^*, \quad \text{and} \tag{C.52}$$

$$\overline{\pi}B_s(p_0) - \pi\bar{A}_s(p_0) = \left[\overline{\pi}\beta_H - \pi\bar{\alpha}_H - (\overline{\pi}\beta + \pi\alpha)p_s^*\right]p_0 - \left(\overline{\pi}\beta_L - \pi\bar{\alpha}_L\right)p_s^*. \tag{C.53}$$

Therefore,

- If $\pi/\overline{\pi} \le \bar{\beta}_H/\alpha_H$ (i.e., $\overline{\pi}\bar{\beta}_H - \pi\alpha_H \ge 0$), then (C.52) implies that

  $$\overline{\pi}\bar{B}_s(p_0) - \pi A_s(p_0) \ge \left(\overline{\pi}\bar{\beta}_H - \pi\alpha_H\right)p_0 + (\overline{\pi}\beta + \pi\alpha)p_s^* - \left(\overline{\pi}\bar{\beta}_L - \pi\alpha_L\right)p_s^* = \left(\overline{\pi}\bar{\beta}_H - \pi\alpha_H\right)(p_0 - p_s^*) \ge 0,$$

  where the inequalities follow from $p_s^* \le 0$ and $p_0 \in [0,1]$. Thus, we obtain the first line in (C.50).

- If $\bar{\beta}_H/\alpha_H < \pi/\overline{\pi} \le \bar{\beta}_L/\alpha_L$ (i.e., $\overline{\pi}\bar{\beta}_H - \pi\alpha_H < 0$ and $\overline{\pi}\bar{\beta}_L - \pi\alpha_L > 0$), then it is straightforward to verify that $\overline{\pi}\bar{\beta}_H - \pi\alpha_H + (\overline{\pi}\beta + \pi\alpha)p_s^* \le \left(\overline{\pi}\bar{\beta}_L - \pi\alpha_L\right)p_s^* \le 0$. Therefore, (C.52) immediately implies the second line in (C.50).

- If $\pi/\overline{\pi} \ge \bar{\beta}_L/\alpha_L$, then $\overline{\pi}\bar{\beta}_H - \pi\alpha_H \le 0$ and $\overline{\pi}\bar{\beta}_L - \pi\alpha_L \le 0$ and (C.52) implies that the third line in (C.50) because of $p_s^* \le 0$.

Following similar argument, we also obtain (C.51). □

Appendix D: Proofs for Results in Section 6

LEMMA D.1. *For $p_r^* \in [0,1]$, there exist a unique $p^\diamond \in \left(p_r^*, \frac{\bar{\beta}_L p_r^*}{\beta_H + \beta p_r^*}\right)$ such that*

$$(p - q_r^*)\bar{B}_r(p) + (p - p_r^*)B_r(p) \le (\textit{resp. } \ge) 0, \quad \textit{for } p_0 \in [p_r^*, p^\diamond] \ (\textit{resp. } p_0 \in [p^\diamond, 1]). \tag{D.1}$$

In particular, if $\beta_H = 1$ (and hence $\beta = \bar{\beta}_L$), then

$$p^\diamond = p_r^* - \frac{\beta}{2}p_r^*(p_r^* - q_r^*) + \sqrt{\frac{\beta^2}{4}(p_r^*)^2(p_r^* - q_r^*)^2 + \beta p_r^*(p_r^* - q_r^*)(1 - p_r^*)}; \tag{D.2}$$

Proof of Lemma D.1. For $p_0 \in \left[p_r^*, \frac{\bar{\beta}_L p_r^*}{\beta_H + \beta p_r^*}\right)$, we first verify that (D.1) indeed has a unique solution $p^\diamond > p_r^*$ and hence $p^\diamond$ is well-defined. To that end, we note, by (C.33), that the quadratic function

$$(p - q_r^*)\bar{B}_r(p) + (p - p_r^*)B_r(p) = (p - q_r^*)(p - p_r^*) - (p_r^* - q_r^*)B_r(p)$$

is convex (i.e., open upward) and takes negative value at $p = p_r^*$ and positive value at $p = \frac{\bar{\beta}_L p_r^*}{\beta_H + \beta p_r^*}$, because $B_r(p) \ge 0$ for $p \ge p_r^*$ and $\bar{B}_r(p) \le (=) 0$ for $p \le (=) \frac{\bar{\beta}_L p_r^*}{\beta_H + \beta p_r^*}$. Thus, (D.1) is established.

When $\beta_H = 1$, straightforward calculation reveals that

$$(p - q_r^*)\bar{B}_r(p) + (p - p_r^*)B_r(p) = (p - p_r^*)^2 - \beta p_r^*(p_r^* - q_r^*)(1 - p),$$

whose root above p_r^* can indeed be verified as given by (D.2). □

Proof of Theorem 2. To start the proof, note that in the context of Appendix C, this theorem corresponds to having $\alpha_L = \alpha_H = 1$. As such, Assumption A3 reduces to $\bar{\beta}_H \leq \eta_r \leq \bar{\beta}_L \leq \eta_s$, which is equivalent to the setting of Section 6.1. Now, the optimality of the full disclosure policy for the second period follows immediately from Proposition C.1. To derive the optimal warning policy for the first period, we follow an approach similar to the proof of Theorem 1 by considering two different cases corresponding to the initial value of reputation p_0.

- **Case 1:** $p_0 \in [0, p_r^*)$.

  When $p_0 < p_r^* = (\bar{\beta}_L - \eta_r)/\beta$, Lemma C.3 implies that $a^\star(d_1, p_0, \boldsymbol{w}) = 0$ for all $(w^0, w^1) \in [0,1]^2$ and $d_1 \in \{0,1\}$. In this case, the sender's warning policy in the first period has no influence on the receiver's action in that period, and $\mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \boldsymbol{\omega}_1(p_0)\right), X_1\right) \mid \Omega, p_0\right] = \pi \ell_s$. Therefore, the sender's warning policy only serves to influence her future reputation p_1. Specifically, we have

$$
\begin{aligned}
J_1(p_0) =& \pi \ell_s + \rho \min_{\boldsymbol{w}_1 \in \mathcal{W}} \mathbb{E}\left[J_2(p_1) \mid \boldsymbol{w}_1, p_0\right] \\
=& \pi \ell_s + \rho \min_{\boldsymbol{w}_1 \in \mathcal{W}} \sum_{(d,x) \in \{0,1\}^2} \mathbb{P}\left[p_1 = p_1^{d,x} \,\middle|\, \boldsymbol{w}_1, p_0\right] J_2\left(p_1^{d,x}\right) \\
=& \pi \ell_s + \rho \pi J_2(p_0) + \rho \bar{\pi} \min_{\boldsymbol{w}_1 \in \mathcal{W}} \left\{ \left((\beta_L + \beta p_0) w_1^0 + (\bar{\beta}_L - \beta p_0) w_1^1\right) J_2(p_1^{1,0}) \right. \\
& \left. + \left(1 - (\beta_L + \beta p_0) w_1^0 - (\bar{\beta}_L - \beta p_0) w_1^1\right) J_2(p_1^{0,0}) \right\} , \qquad \text{(D.3)}
\end{aligned}
$$

  where $\mathbb{P}\left[p_1 = p_1^{d,x} \,\middle|\, \boldsymbol{w}_1, p_0\right]$ follows from (C.17)-(C.20), and $p_1^{d,x}$ is as defined in Lemma C.2.

  Now, we can substitute for $J_2(.)$ from (C.31) to find the optimal values for w_1^0 and w_1^1. We already know that $p_1^{1,0} < p_r^*$ since $p_1^{1,0} \leq p_0$ and $p_0 < p_r^*$. If w_1^0 and w_1^1 are such that $p_1^{0,0}$ is also below p_r^*, then (D.3) reduces to $J_1(p_0) = \pi \ell_s + \rho \pi^2 \ell_s + \rho \pi \bar{\pi} \ell_s = (1+\rho)\pi \ell_s$. On the other hand, if w_1^0 and w_1^1 induce $p_1^{0,0} \geq p_r^*$, then

$$
\begin{aligned}
J_1(p_0) =& \pi \ell_s + \rho \pi^2 \ell_s + \rho \bar{\pi} \min_{\boldsymbol{w}_1 \in \mathcal{W}} \left\{ \left((\beta_L + \beta p_0) w_1^0 + (\bar{\beta}_L - \beta p_0) w_1^1\right) \pi \ell_s \right. \\
& \left. + \left(1 - (\beta_L + \beta p_0) w_1^0 - (\bar{\beta}_L - \beta p_0) w_1^1\right) \left[\pi \ell_s - \bar{\pi} \kappa_s \left(\frac{\left(\beta_H (1 - w_1^0) + \bar{\beta}_H (1 - w_1^1)\right) \beta p_0}{(\bar{\beta}_L - \beta p_0)(1 - w_1^1) + (\beta_L + \beta p_0)(1 - w_1^0)} + \eta_s - \bar{\beta}_L \right) \right] \right\} \\
=& (1+\rho)\pi \ell_s - \rho \bar{\pi}^2 \kappa_s (\beta p_0 + \eta_s - \bar{\beta}_L) \\
& + \rho \bar{\pi}^2 \kappa_s \min_{\boldsymbol{w}_1 \in \mathcal{W}} \left\{ \left[\beta_H \beta p_0 + (\eta_s - \bar{\beta}_L)(\beta_L + \beta p_0)\right] w_1^0 + \left[\bar{\beta}_H \beta p_0 + (\eta_s - \bar{\beta}_L)(\bar{\beta}_L - \beta p_0)\right] w_1^1 \right\} , \\
& \qquad \text{(D.4)}
\end{aligned}
$$

  which is always bounded from above by $(1+\rho)\pi \ell_s$ (e.g., by setting $w_1^0 = w_1^1 = 1$). Therefore, the sender always prefers to set w_1^0 and w_1^1 so that they induce $p_1^{0,0} \geq p_r^*$, or equivalently, $p_r^* \left(1 - (\beta_L + \beta p_0) w_1^0 - (\bar{\beta}_L - \beta p_0) w_1^1\right) \leq \left(\beta_H (1 - w_1^0) + \bar{\beta}_H (1 - w_1^1)\right) p_0$. For sufficiently small values of p_0, however, this cannot be achieved and $p_1^{0,0} < p_r^*$ for all $(w_1^0, w_1^1) \in \mathcal{W}$. More precisely,

$$
\begin{aligned}
& \max_{\boldsymbol{w}_1 \in \mathcal{W}} \left\{ \frac{\left(\beta_H (1 - w_1^0) + \bar{\beta}_H (1 - w_1^1)\right) p_0}{(\bar{\beta}_L - \beta p_0)(1 - w_1^1) + (\beta_L + \beta p_0)(1 - w_1^0)} \right\} = \frac{\beta_H p_0}{\beta_L + \beta p_0} \\
& \Rightarrow p_1^{0,0} < p_r^* \text{ for all } (w_1^0, w_1^1) \in \mathcal{W} \text{ if } p_0 < \frac{\beta_L p_r^*}{\beta_H - \beta p_r^*} .
\end{aligned}
$$

  This implies that any warning policy is optimal when $p_0 < \beta_L p_r^* / (\beta_H - \beta p_r^*)$ since the sender's cost in both periods is independent of the warning policy, and $J_1(p_0) = (1+\rho)\pi \ell_s$.

For $p_0 \geq \beta_L p_r^* / (\beta_H - \beta p_r^*)$, on the other hand, there always exists a $(w_1^0, w_1^1) \in \mathcal{W}$ that induces $p_1^{0,0} \geq p_r^*$. Putting altogether, we have the following minimization problem for $p_0 \geq \beta_L p_r^* / (\beta_H - \beta p_r^*)$,

$$\min_{\boldsymbol{w}_1 \in \mathcal{W}} \left\{ \left[\beta_H \beta p_0 + (\eta_s - \bar{\beta}_L)(\beta_L + \beta p_0)\right] w_1^0 + \left[\bar{\beta}_H \beta p_0 + (\eta_s - \bar{\beta}_L)(\bar{\beta}_L - \beta p_0)\right] w_1^1 \right\}$$
$$\text{subject to: } \left[\beta_H \beta p_0 + (\eta_r - \bar{\beta}_L)(\beta_L + \beta p_0)\right] w_1^0 + \left[\bar{\beta}_H \beta p_0 + (\eta_r - \bar{\beta}_L)(\bar{\beta}_L - \beta p_0)\right] w_1^1 \leq \beta p_0 + \eta_r - \bar{\beta}_L \,.$$

Since we assume $\eta_s \geq \bar{\beta}_L$, the coefficient of both w_1^0 and w_1^1 in the objective function is non-negative. Hence, the objective is minimized by $w_1^0 = w_1^1 = 0$, which violates the constraint (note that the right hand side of the constraint is negative given that $p_0 < p_r^*$). This implies that the constraint must be binding. Next, we note that the coefficients of w_1^1 and w_1^0 in the constraint add up to $(\beta p_0 + \eta_r - \bar{\beta}_L)$, which is negative. Therefore, at least one of these two coefficients must be negative. Then, calculating the value of w_1^1 in terms of w_1^0 from the equality constraint and substituting it in the objective (after some simple algebra) produces a positive coefficient for w_1^0. It follows that the optimal value of w_1^0 must be zero. Plugging this back into the constraint gives us the final solution $\boldsymbol{\omega}_1^\star(p_0) = \left(0, \frac{p^* - p_0}{p^*(\bar{\beta}_L - \beta p_0) - \bar{\beta}_H p_0}\right)$, as given in (13).

- **Case 2:** $p_0 \in [p_r^*, 1]$**.**

When $p_0 \geq p_r^*$, it is possible for the sender to induce an action in the first period. Hence, she has to balance the current incentive to persuade the receiver to act and the future incentive to maintain a high reputation for the second period.

As our solution strategy, we solve two subproblems depending on whether the next period's reputation p_1 remains above p_r^* or not. Then, we compare the optimal costs of these two subproblems, the smaller of which determines the overall optimal cost and the corresponding optimal warning policy for this case.

To this end, for any given $p_0 \geq p_r^*$, define sets $\mathcal{W}^{\geq}(p_0)$ and $\mathcal{W}^{<}(p_0)$ as,

$$\begin{aligned}
\mathcal{W}^{\geq}(p_0) &= \left\{(w^0, w^1) \in \mathcal{W} : p_1^{1,0} \geq p_r^*\right\} \\
&= \left\{(w^0, w^1) \in \mathcal{W} : \left[\bar{\beta}_H p_0 - (\bar{\beta}_L - \beta p_0) p_r^*\right] w_1^1 + \left[\beta_H p_0 - (\beta_L + \beta p_0) p_r^*\right] w_1^0 \geq 0\right\} \,, \\
\mathcal{W}^{<}(p_0) &= \left\{(w^0, w^1) \in \mathcal{W} : p_1^{1,0} < p_r^*\right\} \\
&= \left\{(w^0, w^1) \in \mathcal{W} : \left[\bar{\beta}_H p_0 - (\bar{\beta}_L - \beta p_0) p_r^*\right] w_1^1 + \left[\beta_H p_0 - (\beta_L + \beta p_0) p_r^*\right] w_1^0 < 0\right\} \,.
\end{aligned}$$

Notice that $\mathcal{W}^{\geq}(p_0)$ and $\mathcal{W}^{<}(p_0)$ partition the feasible set $\mathcal{W}$ and hence, the optimal solution to the sender's problem must belong to one of these two sets. Let $J_1^{\geq}(.)$ and $J_1^{<}(.)$ represent the sender's optimal cost function at the beginning of the horizon (as a function of initial reputation p_0) when her warning policy *in the first period* is constrained to be in $\mathcal{W}^{\geq}(p_0)$ and $\mathcal{W}^{<}(p_0)$, respectively. More specifically, it follows from (8) that,

$$J_1^{\geq}(p_0) \equiv \min_{\boldsymbol{w}_1 \in \mathcal{W}^{\geq}(p_0)} \mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) + \rho J_2(p_1) \,|\, \boldsymbol{w}_1, p_0\right] \,, \tag{D.5}$$

$$J_1^{<}(p_0) \equiv \min_{\boldsymbol{w}_1 \in \mathcal{W}^{<}(p_0)} \mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) + \rho J_2(p_1) \,|\, \boldsymbol{w}_1, p_0\right] \,, \tag{D.6}$$

where $J_1(p_0) = \min\left\{J_1^{\geq}(p_0), J_1^{<}(p_0)\right\}$.

Subproblem I. First, consider $J_1^{\geq}(p_0)$. By definition, any $\boldsymbol{w}_1 \in \mathcal{W}^{\geq}(p_0)$ induces $p_1 \in [p_r^*, 1]$. Further, we know from (C.31) that $J_2(p_1)$ is linear over this interval. Hence, the sender's cost in the second period becomes independent of her warning policy since

$$\mathbb{E}\left[J_2(p_1) \mid \boldsymbol{w}_1, p_0\right] = J_2\left(\mathbb{E}\left[p_1 \mid \boldsymbol{w}_1, p_0\right]\right) = J_2(p_0) = \pi\ell_s - \bar{\pi}\kappa_s(\beta p_0 + \eta_s - \bar{\beta}_L) .$$

This implies that we only need to minimize the sender's cost in the first period, given by $\mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) \mid \boldsymbol{w}_1, p_0\right]$, over $\mathcal{W}^{\geq}(p_0)$. For this, we denote $p^\dagger = \bar{\beta}_L p_r^*/(\bar{\beta}_H + \beta p_r^*) \geq p_r^*$. By applying Lemma D.1 to the parametric value $\alpha_L = \alpha_H = 1$ (and hence $q_r^* = -\beta_L/\beta$), there exists a unique $p^\diamond \in (p_r^*, p^\dagger)$ such that

$$\frac{\beta_H p_0 - (\beta_L + \beta p_0)p_r^*}{(\bar{\beta}_L - \beta p_0)p_r^* - \bar{\beta}_H p_0} \leq (\text{resp. } \geq) \frac{\beta p_0 + \beta_L}{\beta(p_0 - p_r^*)}, \quad \text{for } p_0 \in [p_r^*, p^\diamond] \text{ (resp. } p_0 \in [p^\diamond, 1]).$$

We thus consider three different ranges for p_0:

(i) For $p_r^* \leq p_0 < p^\diamond < p^\dagger$: From $p_0 < p^\dagger$, we have $\bar{\beta}_H p_0 - (\bar{\beta}_L - \beta p_0)p_r^* < 0$. Thus,

$$\begin{aligned}
\boldsymbol{w}_1 \in \mathcal{W}^{\geq}(p_0) \Rightarrow & 1 \leq \frac{w_1^1}{w_1^0} \leq \frac{\beta_H p_0 - (\beta_L + \beta p_0)p_r^*}{(\bar{\beta}_L - \beta p_0)p_r^* - \bar{\beta}_H p_0} \xRightarrow{p_r^* \leq p_0 < p^\diamond} 1 \leq \frac{w_1^1}{w_1^0} < \frac{\beta p_0 + \beta_L}{\beta(p_0 - p_r^*)} \\
\Rightarrow & 0 < \beta(p_r^* - p_0)w_1^1 + (\beta p_0 + \beta_L)w_1^0 < 1 - \eta_r \xRightarrow{\text{Lemma C.3}} a^\star(d_1, p_0, \boldsymbol{w}_1) = 0 \\
\Rightarrow & \mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) \mid \boldsymbol{w}_1, p_0\right] = \pi\ell_s \\
\Rightarrow & J_1^{\geq}(p_0) = \pi\ell_s + \rho\left(\pi\ell_s - \bar{\pi}\kappa_s(\beta p_0 + \eta_s - \bar{\beta}_L)\right) .
\end{aligned} \tag{D.7}$$

Thus, the sender's cost does not depend on the warning policy, and any feasible policy is optimal.

(ii) For $p^\diamond \leq p_0 < p^\dagger$:In this case, the receiver's action in the first period can be triggered even when the policy is restricted to be in $\mathcal{W}^{\geq}(p_0)$. That is,

$$\begin{aligned}
p_0 \geq p^\diamond \Rightarrow & \frac{\beta_H p_0 - (\beta_L + \beta p_0)p_r^*}{(\bar{\beta}_L - \beta p_0)p_r^* - \bar{\beta}_H p_0} \geq \frac{\beta p_0 + \beta_L}{\beta(p_0 - p_r^*)} \\
\Rightarrow & \exists \boldsymbol{w}_1 \in \mathcal{W}^{\geq}(p_0) : \frac{w_1^1}{w_0^1} \geq \frac{\beta p_0 + \beta_L}{\beta(p_0 - p_r^*)} \\
\xRightarrow{\text{(C.32)}} & \mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) \mid \boldsymbol{w}_1, p_0\right] = \pi\ell_s + \bar{\pi}\kappa_s\left[(\beta p_0 + \beta_L)w_1^0 - (\beta p_0 + \eta_s - \bar{\beta}_L)w_1^1\right] .
\end{aligned}$$

To minimize this cost, since the coefficient of w_1^0 (resp. w_1^1) is positive (resp. negative), it is optimal to set the value of w_1^0 (resp. w_1^1) as low (resp. high) as possible while still satisfying $\boldsymbol{w}_1 \in \mathcal{W}^{\geq}(p_0)$. This implies that the solution is on the boundary of set $\mathcal{W}^{\geq}(p_0)$ so that $w_1^1/w_1^0 = \left[\beta_H p_0 - (\beta_L + \beta p_0)p_r^*\right] / \left[(\bar{\beta}_L - \beta p_0)p_r^* - \bar{\beta}_H p_0\right]$. Substituting for w_1^0 based on this ratio then gives us

$$\mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) \mid \boldsymbol{w}_1, p_0\right] = \pi\ell_s + \bar{\pi}\kappa_s\left[\frac{(\beta p_0 + \beta_L)\left[(\bar{\beta}_L - \beta p_0)p_r^* - \bar{\beta}_H p_0\right]}{\beta_H p_0 - (\beta_L + \beta p_0)p_r^*} - (\beta p_0 + \eta_s - \bar{\beta}_L)\right] w_1^1 .$$

Consider the coefficient of w_1^1. We have,

$$p_0 \geq p^\diamond \Rightarrow \frac{\beta_H p_0 - (\beta_L + \beta p_0)p_r^*}{(\bar{\beta}_L - \beta p_0)p_r^* - \bar{\beta}_H p_0} \geq \frac{\beta p_0 + \beta_L}{\beta(p_0 - p_r^*)} \xRightarrow{\eta_s \geq \eta_r} \frac{\beta_H p_0 - (\beta_L + \beta p_0)p_r^*}{(\bar{\beta}_L - \beta p_0)p_r^* - \bar{\beta}_H p_0} \geq \frac{\beta p_0 + \beta_L}{\beta(p_0 - p_s^*)} = \frac{\beta p_0 + \beta_L}{\beta p_0 + \eta_s - \bar{\beta}_L}.$$

Thus, the coefficient of w_1^1 is negative and hence, the cost is minimized by setting $w_1^1=1$. Taken altogether,

$$\begin{aligned}\boldsymbol{\omega}_1^{\star\geq}(p_0) &= \left(\frac{(\bar\beta_L-\beta p_0)p_r^*-\bar\beta_H p_0}{\beta_H p_0-(\beta_L+\beta p_0)p_r^*},1\right)\in\mathcal{W}^{\geq}(p_0)\,,\\ J_1^{\geq}(p_0) &= \pi\ell_s+\bar\pi\kappa_s\left[\frac{(\beta p_0+\beta_L)\left[(\bar\beta_L-\beta p_0)p_r^*-\bar\beta_H p_0\right]}{\beta_H p_0-(\beta_L+\beta p_0)p_r^*}-(\beta p_0+\eta_s-\bar\beta_L)\right]+\rho\left(\pi\ell_s-\bar\pi\kappa_s(\beta p_0+\eta_s-\bar\beta_L)\right)\,.\end{aligned}\tag{D.8}$$

(iii) For $p_0\geq p^\dagger$: In this case, $p_0\geq p^\dagger$ implies that $\bar\beta_H p_0-(\bar\beta_L-\beta p_0)p_r^*\geq 0$. Further, since $p_0\geq p_r^*$ gives us $\beta_H p_0-(\beta_L+\beta p_0)p_r^*\geq 0$, every $(w^0,w^1)\in\mathcal{W}$ belongs to $\mathcal{W}^{\geq}(p_0)$, i.e., $\mathcal{W}=\mathcal{W}^{\geq}(p_0)$. Then, similar to Case (ii) above, there always exists a $\boldsymbol{w}_1\in\mathcal{W}^{\geq}(p_0)$ that induces the receiver to act. Now, since $\mathcal{W}=\mathcal{W}^{\geq}(p_0)$, the sender prefers to adopt the full disclosure policy, that is, to set w_1^0 (resp. w_1^1) as low (resp. high) as possible. Subsequently, we have

$$\begin{aligned}\boldsymbol{\omega}_1^{\star\geq}(p_0) &= (0,1)\in\mathcal{W}^{\geq}(p_0)\,,\\ J_1^{\geq}(p_0) &= \pi\ell_s-\bar\pi\kappa_s(\beta p_0+\eta_s-\bar\beta_L)+\rho\left(\pi\ell_s-\bar\pi\kappa_s(\beta p_0+\eta_s-\bar\beta_L)\right)\,.\end{aligned}\tag{D.9}$$

Subproblem II. Next, consider $J_1^{<}(p_0)$. We have

$$\begin{aligned}\mathbb{E}\left[J_2(p_1)\,|\,\boldsymbol{w}_1,p_0\right] &= \sum_{(d,x)\in\{0,1\}^2}\Pr\left[p_1=p_1^{d,x}\,\middle|\,\boldsymbol{w}_1,p_0\right]J_2\left(p_1^{d,x}\right)\\ &= \pi J_2(p_0)+\bar\pi\left((\beta_L+\beta p_0)w_1^0+(\bar\beta_L-\beta p_0)w_1^1\right)J_2(p_1^{1,0})\\ &\quad+\bar\pi\left(1-(\beta_L+\beta p_0)w_1^0-(\bar\beta_L-\beta p_0)w_1^1\right)J_2(p_1^{0,0})\,,\end{aligned}$$

where $\Pr\left[p_1=p_1^{d,x}\,\middle|\,\boldsymbol{w}_1,p_0\right]$ follows from (C.17)-(C.20), and $p_1^{d,x}$ is as defined in Lemma C.2.

Then, noting that $p_1^{1,0}<p_r^*$ (by definition of $\mathcal{W}^{<}(p_0)$), and using (C.31), the above expression reduces to

$$\begin{aligned}&\pi\left[\pi\ell_s+\bar\pi\kappa_s(\bar\beta_L-\eta_s-\beta p_0)\right]+\bar\pi\left((\beta_L+\beta p_0)w_1^0+(\bar\beta_L-\beta p_0)w_1^1\right)\pi\ell_s\\ &\quad+\bar\pi\left(1-(\beta_L+\beta p_0)w_1^0-(\bar\beta_L-\beta p_0)w_1^1\right)\left[\pi\ell_s+\bar\pi\kappa_s\left(\bar\beta_L-\eta_s-\frac{\beta p_0\left[\beta_H(1-w_1^0)+\bar\beta_H(1-w_1^1)\right]}{(\beta_L+\beta p_0)(1-w_1^0)+(\bar\beta_L-\beta p_0)(1-w_1^1)}\right)\right]\\ &=\pi\ell_s+\pi\bar\pi\kappa_s(\bar\beta_L-\eta_s-\beta p_0)+\bar\pi^2\kappa_s\left[\left(1-(\beta_L+\beta p_0)w_1^0-(\bar\beta_L-\beta p_0)w_1^1\right)(\bar\beta_L-\eta_s)-\beta p_0\left[\beta_H(1-w_1^0)+\bar\beta_H(1-w_1^1)\right]\right]\,.\end{aligned}$$

Therefore, since $p_0\geq p_r^*$, (C.32) gives us,

$$\begin{aligned}&\mathbb{E}\left[c_s\left(a^\star(D_1,p_0,\boldsymbol{w}_1),X_1\right)+\rho J_2(p_1)\,|\,\boldsymbol{w}_1,p_0\right]\\ &=\rho\left(\pi\ell_s+\pi\bar\pi\kappa_s(\bar\beta_L-\eta_s-\beta p_0)+\bar\pi^2\kappa_s\left[\left(1-(\beta_L+\beta p_0)w_1^0-(\bar\beta_L-\beta p_0)w_1^1\right)(\bar\beta_L-\eta_s)-\beta p_0\left[\beta_H(1-w_1^0)+\bar\beta_H(1-w_1^1)\right]\right]\right)\\ &+\begin{cases}\pi\ell_s+\bar\pi\kappa_s\left[(\beta p_0+\beta_L)w_1^0+(\bar\beta_L-\eta_s-\beta p_0)w_1^1\right], & \text{if } \frac{w_1^0}{w_1^1}\leq\frac{\beta p_0+\eta_r-\bar\beta_L}{\beta p_0+\beta_L}\\ \pi\ell_s, & \text{if } \frac{w_1^0}{w_1^1}>\frac{\beta p_0+\eta_r-\bar\beta_L}{\beta p_0+\beta_L}\end{cases}\\ &=(1+\rho)\pi\ell_s+\rho\pi\bar\pi\kappa_s(\bar\beta_L-\eta_s-\beta p_0)+\rho\bar\pi^2\kappa_s(\bar\beta_L-\eta_s-\beta p_0)\\ &+\begin{cases}\bar\pi\kappa_s\left\{\underbrace{\left(\rho\bar\pi\beta_H\beta p_0+\left[1+\rho\bar\pi(\eta_s-\bar\beta_L)\right](\beta p_0+\beta_L)\right)}_{\phi_1}w_1^0+\underbrace{\left(\rho\bar\pi\bar\beta_H\beta p_0+\left[1+\rho\bar\pi(\eta_s-\bar\beta_L)\right](\bar\beta_L-\beta p_0)-\eta_s\right)}_{\phi_2}w_1^1\right\}, & \text{if } \frac{w_1^0}{w_1^1}\leq\frac{\beta p_0+\eta_r-\bar\beta_L}{\beta p_0+\beta_L}\\ \rho\bar\pi^2\kappa_s\left[\underbrace{\left(\beta_H\beta p_0+(\beta_L+\beta p_0)(\eta_s-\bar\beta_L)\right)}_{\phi_3}w_1^0+\underbrace{\left(\bar\beta_H\beta p_0+(\bar\beta_L-\beta p_0)(\eta_s-\bar\beta_L)\right)}_{\phi_4}w_1^1\right], & \text{if } \frac{w_1^0}{w_1^1}>\frac{\beta p_0+\eta_r-\bar\beta_L}{\beta p_0+\beta_L}\,.\end{cases}\end{aligned}$$

Now, we consider the coefficients of w_1^0 and w_1^1 in the above expressions, and investigate their signs:

$$\eta_s \geq \bar{\beta}_L \Rightarrow \phi_1 > 0 \text{ and } \phi_3 > 0\,.$$

$$\rho\bar{\pi}(\bar{\beta}_L - \beta p) < 1 \Rightarrow \left[\rho\bar{\pi}(\bar{\beta}_L - \beta p) - 1\right](\eta_s - \bar{\beta}_L) < 0 \Rightarrow \phi_2 < 0\,,$$

where the first inequality above is by assumption. This implies that the term in the upper branch is minimized by $(w_1^0, w_1^1) = (0,1)$, which also satisfies the requirement that $w_1^0/w_1^1 \leq (\beta p_0 + \eta_r - \bar{\beta}_L)/(\beta p_0 + \beta_L)$. Substituting $(w_1^0, w_1^1) = (0,1)$ reduces the term to,

$$\bar{\pi}\kappa_s\left\{\left(\rho\bar{\pi}\beta_H\beta p_0 + \left[1+\rho\bar{\pi}(\eta_s - \bar{\beta}_L)\right](\beta p_0 + \beta_L)\right)w_1^0 + \left(\rho\bar{\pi}\bar{\beta}_H\beta p_0 + \left[1+\rho\bar{\pi}(\eta_s - \bar{\beta}_L)\right](\bar{\beta}_L - \beta p_0) - \eta_s\right)w_1^1\right\} =$$
$$\bar{\pi}\kappa_s\left(\rho\bar{\pi}\bar{\beta}_H\beta p_0 + \left[1+\rho\bar{\pi}(\eta_s - \bar{\beta}_L)\right](\bar{\beta}_L - \beta p_0) - \eta_s\right). \tag{D.10}$$

For the term in the lower branch, we note that

$$\bar{\beta}_L - \eta_s - \beta p_0 \leq 0 \Rightarrow \phi_2 \leq \phi_4\,.$$

Given that $\phi_3 > 0$, it immediately follows that for any (w_1^0, w_1^1), the lower branch is dominated by (D.10). Thus, $\boldsymbol{\omega}_1^{\star<}(p_0) = (0,1) \in \mathcal{W}^<(p_0)$. Substituting this policy in the cost function derived above yields,

$$\begin{aligned} J_1^<(p_0) =& (1+\rho)\pi\ell_s + \rho\pi\bar{\pi}\kappa_s(\bar{\beta}_L - \eta_s - \beta p_0) + \rho\bar{\pi}^2\kappa_s(\bar{\beta}_L - \eta_s - \beta p_0) + \bar{\pi}\kappa_s\left(\rho\bar{\pi}\bar{\beta}_H\beta p_0 + \left[1+\rho\bar{\pi}(\eta_s - \bar{\beta}_L)\right](\bar{\beta}_L - \beta p_0) - \eta_s\right) \\ =& (1+\rho)\pi\ell_s + \bar{\pi}\kappa_s\left[(1+\rho)(\bar{\beta}_L - \eta_s - \beta p_0) + \rho\bar{\pi}\left(\bar{\beta}_H\beta p_0 + (\eta_s - \bar{\beta}_L)(\bar{\beta}_L - \beta p_0)\right)\right]\,. \end{aligned} \tag{D.11}$$

Finally, putting the two subproblems together, we obtain the optimal cost function and its corresponding optimal policy for $p_0 \geq p^*$ by comparing $J_1^{\geq}(p_0)$ and $J_1^<(p_0)$ from (D.7) – (D.9), and (D.11). In particular,

$$J_1^{\geq}(p_0) - J_1^<(p_0) = \begin{cases} \bar{\pi}\kappa_s\left[\beta p_0 + \eta_s - \bar{\beta}_L - \rho\bar{\pi}\left(\bar{\beta}_H\beta p_0 + (\eta_s - \bar{\beta}_L)(\bar{\beta}_L - \beta p_0)\right)\right], & \text{if } p_r^* \leq p_0 < p^\diamond, \\ \bar{\pi}\kappa_s\left[\frac{(\beta p_0 + \beta_L)\left[(\bar{\beta}_L - \beta p_0)p_r^* - \bar{\beta}_H p_0\right]}{\beta_H p_0 - (\beta_L + \beta p_0)p_r^*} - \rho\bar{\pi}\left(\bar{\beta}_H\beta p_0 + (\eta_s - \bar{\beta}_L)(\bar{\beta}_L - \beta p_0)\right)\right], & \text{if } p^\diamond \leq p_0 < p^\dagger \\ -\rho\bar{\pi}^2\kappa_s\left(\bar{\beta}_H\beta p_0 + (\eta_s - \bar{\beta}_L)(\bar{\beta}_L - \beta p_0)\right), & \text{if } p_0 \geq p^\dagger\,. \end{cases}$$

The result immediately follows by simple algebra. That is, for $p_r^* \leq p_0 < p^\diamond$, we have

$$J_1^{\geq}(p_0) - J_1^<(p_0) = \bar{\pi}\kappa_s\left[(\eta_s - \bar{\beta}_L)\left(1 - \rho\bar{\pi}(\bar{\beta}_L - \beta p_0)\right) + \beta p_0(1 - \rho\bar{\pi}\bar{\beta}_H)\right] > 0\,.$$

Therefore, the optimal policy is given by the full disclosure policy in this case.

For $p^\diamond \leq p_0 < p^\dagger$, on the other hand,

$$p_0 < p^\dagger \Rightarrow (\bar{\beta}_L - \beta p_0)p_r^* - \bar{\beta}_H p_0 > 0\,.$$

This implies that

$$J_1^{\geq}(p_0) - J_1^<(p_0) \leq 0 \quad \Leftrightarrow \quad \rho \geq \frac{(\beta p_0 + \beta_L)\left[(\bar{\beta}_L - \beta p_0)p_r^* - \bar{\beta}_H p_0\right]}{\bar{\pi}\left[\beta_H p_0 - (\beta_L + \beta p_0)p_r^*\right]\left[\bar{\beta}_H\beta p_0 + (\eta_s - \bar{\beta}_L)(\bar{\beta}_L - \beta p_0)\right]}\,.$$

For $p_0 \geq p^\dagger$, we have $J_1^{\geq}(p_0) - J_1^<(p_0) \leq 0$ and the optimality of the full disclosure policy is obvious. Thus,

$$\boldsymbol{\omega}_1^\star(p_0) = \begin{cases} (0,1), & \text{if } p_r^* \leq p_0 < p^\diamond, \\ \left(\frac{(\bar{\beta}_L - \beta p_0)p_r^* - \bar{\beta}_H p_0}{\beta_H p_0 - (\beta_L + \beta p_0)p_r^*}, 1\right), & \text{if } p^\diamond \leq p_0 < p^\dagger \text{ and } \rho \geq \frac{(\beta p_0 + \beta_L)\left[(\bar{\beta}_L - \beta p_0)p_r^* - \bar{\beta}_H p_0\right]}{\bar{\pi}\left[\beta_H p_0 - (\beta_L + \beta p_0)p_r^*\right]\left[\bar{\beta}_H\beta p_0 + (\eta_s - \bar{\beta}_L)(\bar{\beta}_L - \beta p_0)\right]}\,, \\ (0,1), & \text{otherwise}\,. \end{cases} \tag{D.12}$$

Combining Cases 1 and 2 completes the proof for $\pi \le \kappa_r/\delta_r$, and the thresholds specified in Theorem 2 are given by $p^* = p_r^*$, $\hat{p}_1 = \beta_L p_r^*/(\beta_H - \beta p_r^*)$, $\hat{p}_3 = p^\dagger$, and

$$\hat{p}_2 = \begin{cases} p^\diamond, & \text{if } \rho \ge \frac{(\beta p_0+\beta_L)\left[(\bar\beta_L-\beta p_0)p_r^*-\bar\beta_H p_0\right]}{\bar\pi[\beta_H p_0-(\beta_L+\beta p_0)p_r^*]\left[\bar\beta_H\beta p_0+(\eta_s-\bar\beta_L)(\bar\beta_L-\beta p_0)\right]}\,, \\ p^\dagger, & \text{otherwise}\,. \quad \square \end{cases} \tag{D.13}$$

In the following theorem, we let $p^\diamond$ be identified by Lemma D.1 and define

$$\Theta_f(p_0) := p_s^* - p_0 + \rho\pi\bar{A}_s(p_0) + \rho\bar{\pi}\bar{B}_s(p_0), \tag{D.14}$$

$$\Theta_d(p_0) := \left[p_s^* - p_0 + \rho\bar{\pi}\bar{B}_s(p_0)\right]\frac{p_0 - p_r^*}{A_r(p_0)}, \tag{D.15}$$

$$\Theta_e(p_0) := p_s^* - p_0 - \frac{\bar{B}_r(p_0)}{B_r(p_0)}(p_0 - q_s^*) + \rho\pi\bar{A}_s(p_0)\left[1 + \frac{\bar{B}_r(p_0)}{B_r(p_0)}\right], \tag{D.16}$$

$$\Theta_{de}(p_0) := \frac{\bar{B}_r(p_0)}{\bar{B}_r(p_0) - A_r(p_0)}\left[p_0 - q_s^* - \frac{B_r(p_0)}{\bar{B}_r(p_0)}(p_s^* - p_0)\right]. \tag{D.17}$$

THEOREM D.1. *Suppose that Assumption A3 holds with $\eta_s \le 1$ and $\alpha\beta_L - \beta\alpha_L \le \alpha - \beta \le 0$.*

- *For $p_0 \in \left[0, \frac{\beta_L p_r^*}{\beta_H - \beta p_r^*}\right)$, any policy is optimal.*
- *For $p_0 \in \left[\frac{\beta_L p_r^*}{\beta_H - \beta p_r^*}, \frac{\alpha_L p_r^*}{\alpha_H - \alpha p_r^*}\right)$, the optimal policy is to boost $p_1^{0,0} = p_r^*$ (while $p_1^{1,1} < p_r^*$) by setting*

$$w_1^{0\star} = 0 \quad \text{and} \quad w_1^{1\star} = \frac{p_r^* - p_0}{\bar\beta_L p_r^* - (\bar\beta_H + \beta p_r^*)\,p_0} \in (0,1], \tag{D.18}$$

  where $w_1^{1\star} = 1$ if and only if $p_0 = \frac{\beta_L p_r^*}{\beta_H - \beta p_r^*}$.
- *For $p_0 \in \left[\frac{\alpha_L p_r^*}{\alpha_H - \alpha p_r^*}, p_r^*\right)$,*
  1. *if $\bar\pi\bar{B}_s(p_0) - \pi A_s(p_0) > 0$ and $\bar\pi B_s(p_0) - \pi\bar{A}_s(p_0) \ge 0$, then the optimal policy is to boost $p_1^{0,0} = p_r^*$ and $p_1^{1,1} > p_r^*$ by setting (D.18);*
  2. *if $\bar\pi\bar{B}_s(p_0) - \pi A_s(p_0) \le 0$ and $\bar\pi B_s(p_0) - \pi\bar{A}_s(p_0) < 0$, then the optimal policy is to boost $p_1^{1,1} = p_r^*$ and $p_1^{0,0} > p_r^*$ by setting*

$$w_1^{0\star} = \frac{(\alpha_H - \alpha p_r^*)\,p_0 - \alpha_L p_r^*}{\bar\alpha_L p_r^* - (\bar\alpha_H + \alpha p_r^*)\,p_0} \in [0,1) \quad \text{and} \quad w_1^{1\star} = 1, \tag{D.19}$$

     where $w_1^{0\star} = 0$ if and only if $p_0 = \frac{\alpha_L p_r^*}{\alpha_H - \alpha p_r^*}$;
  3. *otherwise (i.e., $\bar\pi\bar{B}_s(p_0) - \pi A_s(p_0) \le 0$ and $\bar\pi B_s(p_0) - \pi\bar{A}_s(p_0) \ge 0$), full disclosure $(w_1^{0\star}, w_1^{1\star}) = (0,1)$ is optimal with $\{p_1^{1,1}, p_1^{0,0}\} \ge p_r^* > \{p_1^{0,1}, p_1^{1,0}\}$.*
- *For $p_0 \in [p_r^*, p^\diamond)$, full disclosure $(w_1^{0\star}, w_1^{1\star}) = (0,1)$ is optimal if $\Theta_f(p_0) \le \Theta_d(p_0)$; otherwise, the optimal policy is to keep $p_1^{0,1} = p_r^* > p_1^{1,0}$ by setting*

$$w_1^{0\star} = 0, \quad \text{and} \quad w_1^{1\star} = \frac{p_0 - p_r^*}{A_r(p_0)} \in [0,1). \tag{D.20}$$

- *For $p_0 \in \left[p^\diamond, \frac{\bar\beta_L p_r^*}{\bar\beta_H + \beta p_r^*}\right)$, the optimal policy is given by*

  — full disclosure $(w_1^{0\star}, w_1^{1\star}) = (0,1)$ with $\{p_1^{1,1}, p_1^{0,0}\} \ge p_r^* > \{p_1^{0,1}, p_1^{1,0}\}$, if $\Theta_f(p_0) \le \{\Theta_d(p_0), \Theta_e(p_0), \Theta_{de}(p_0)\}$;

  — setting (D.20) to keep $p_1^{0,1} = p_r^* > p_1^{1,0}$, if $\Theta_d(p_0) \le \{\Theta_f(p_0), \Theta_e(p_0), \Theta_{de}(p_0)\}$;

— *setting the following probability to keep* $p_1^{1,0} = p_r^* > p_1^{0,1}$

$$w_1^{0\star} = -\frac{\bar{B}_r(p_0)}{B_r(p_0)} \in (0,1], \quad \textit{and} \quad w_1^{1\star} = 1, \tag{D.21}$$

if $\Theta_e(p_0) \leq \{\Theta_f(p_0), \Theta_d(p_0), \Theta_{de}(p_0)\}$;

— *setting the following probability to keep* $p_1^{1,0} = p_1^{0,1} = p_r^*$

$$w_1^{0\star} = \frac{\bar{B}_r(p_0)}{\bar{B}_r(p_0) - A_r(p_0)} \in (0,1] \quad \textit{and} \quad w_1^{1\star} = \frac{B_r(p_0)}{B_r(p_0) - \bar{A}_r(p_0)} \in (0,1], \tag{D.22}$$

if $\Theta_{de}(p_0) \leq \{\Theta_f(p_0), \Theta_d(p_0), \Theta_e(p_0)\}$.

- *For* $p_0 \in \left[\frac{\bar{\beta}_L p_r^*}{\bar{\beta}_H + \beta p_r^*}, \frac{\bar{\alpha}_L p_r^*}{\bar{\alpha}_H + \alpha p_r^*}\right)$, *full disclosure* $(w_1^{0\star}, w_1^{1\star}) = (0,1)$ *is optimal with* $\{p_1^{1,1}, p_1^{0,0}, p_1^{1,0}\} \geq p_r^* > p_1^{0,1}$ *if* $p_s^* - p_0 + \rho\pi\bar{A}_s(p_0) \leq (p_s^* - p_0)(p_0 - p_r^*)/A_r(p_0)$; *otherwise, the optimal policy is to keep* $p_1^{1,0} \geq p_1^{0,1} = p_r^*$ *by setting* (D.20).
- *For* $p_0 \in \left[\frac{\bar{\alpha}_L p_r^*}{\bar{\alpha}_H + \alpha p_r^*}, 1\right)$, *full disclosure* $(w_1^{0\star}, w_1^{1\star}) = (0,1)$ *is optimal with* $\{p_1^{1,1}, p_1^{0,0}, p_1^{1,0}, p_1^{0,1}\} \geq p_r^*$.

Subsequently, the receiver takes no action regardless of the sender's warning decisions for $p_0 \in [0, p_r^*)$, *whereas the receiver acts upon a warning, i.e.,* $a_1^\star(d, p_0, \boldsymbol{\omega}_1^\star(p_0)) = d$, *for* $p_0 \in [p_r^*, 1]$.

Proof of Theorem D.1. To identify the sender's optimal warning policy in the first period, we need to solve (8). We first note that when Assumption A3 holds with $\eta_s \leq 1$, we have $q_r^* < q_s^* \leq p_s^* \leq 0 \leq p_r^* \leq 1$. By (C.40) and (C.41), $\alpha\beta_L - \beta\alpha_L \leq \alpha - \beta \leq 0$ implies

$$0 \leq \frac{\beta_L p_r^*}{\beta_H - \beta p_r^*} \leq \frac{\alpha_L p_r^*}{\alpha_H - \alpha p_r^*} \leq p_r^* \leq \frac{\bar{\beta}_L p_r^*}{\bar{\beta}_H + \beta p_r^*} \leq \frac{\bar{\alpha}_L p_r^*}{\bar{\alpha}_H + \alpha p_r^*} \leq 1, \tag{D.23}$$

which divides our analysis into the following cases:

- For $p_0 \in \left[0, \frac{\beta_L p_r^*}{\beta_H - \beta p_r^*}\right)$, (C.34)-(C.37) imply that $\left\{p_1^{0,1}, p_1^{1,0}\right\} \leq p_0 \leq \left\{p_1^{0,0}, p_1^{1,1}\right\} < p_r^*$, which then, by (C.30) and (C.43), implies that $\mathbb{E}[c_s(a^\star(D_1, p_0, \boldsymbol{w}_1), X_1) \mid \boldsymbol{w}_1, p_0] = \mathbb{E}[J_2(p_1) \mid \boldsymbol{w}_1, p_0] \equiv \pi\ell_s$. Hence, any policy is optimal and

$$J_1(p_0) = \mathbb{E}[c_s(a^\star(D_1, p_0, \boldsymbol{w}_1), X_1) + \rho J_2(p_1) \mid \boldsymbol{w}_1, p_0] = (1+\rho)\pi\ell_s.$$

  Because of $p_0 < p_r^*$, similar argument as in the proof of Proposition C.1 suggests that $a^\star(D_1, p_0, \boldsymbol{w}_1) \equiv 0$.
- For $p_0 \in \left[\frac{\beta_L p_r^*}{\beta_H - \beta p_r^*}, \frac{\alpha_L p_r^*}{\alpha_H - \alpha p_r^*}\right)$, (C.34), (C.36) and (C.37) imply that $\left\{p_1^{0,1}, p_1^{1,0}\right\} \leq p_0 \leq p_1^{1,1} < p_r^*$ and hence $\mathbb{E}[c_s(a^\star(D_1, p_0, \boldsymbol{w}_1), X_1) \mid \boldsymbol{w}_1, p_0] \equiv \pi\ell_s$ by (C.30). In addition, by (C.35), we have the following two cases to examine:

  — If $1 \leq \overline{w}_1^0/\overline{w}_1^1 < -\bar{B}_r(p_0)/B_r(p_0)$, then $p_1^{0,0} < p_r^*$ and (C.43) implies that

$$\mathbb{E}[c_s(a^\star(D_1, p_0, \boldsymbol{w}_1), X_1) + \rho J_2(p_1) \mid \boldsymbol{w}_1, p_0] \equiv (1+\rho)\pi\ell_s, \tag{D.24}$$

  which will be shown below to be dominated.

  — If $\overline{w}_1^0/\overline{w}_1^1 \geq -\bar{B}_r(p_0)/B_r(p_0) > 1$, then $p_1^{0,0} \geq p_r^*$ and (C.45) implies that

$$\begin{aligned}
&\mathbb{E}[c_s(a^\star(D_1, p_0, \boldsymbol{w}_1), X_1) + \rho J_2(p_1) \mid \boldsymbol{w}_1, p_0] \\
=&(1+\rho)\pi\ell_s + \rho\overline{\pi}^2\kappa_s(\beta + \alpha\eta_s)\left[(p_s^* - p_0) + \bar{B}_s(p_0)w_1^1 + B_s(p_0)w_1^0\right] \\
=&(1+\rho)\pi\ell_s - \rho\overline{\pi}^2\kappa_s(\beta + \alpha\eta_s)\left[\bar{B}_s(p_0)\overline{w}_1^1 + B_s(p_0)\overline{w}_1^0\right] \leq (1+\rho)\pi\ell_s,
\end{aligned} \tag{D.25}$$

where the coefficients $\bar{B}_s(p_0) \geq 0$ and $B_s(p_0) \geq 0$ according to (C.42). Hence, to minimize the sender's expected cost, we take the smallest possible w_1^0 and w_1^1 satisfying $\overline{w}_1^0/\overline{w}_1^1 \geq -\bar{B}_r(p_0)/B_r(p_0)$, which binds the inequality (i.e., $p_1^{0,0} = p_r^*$) and yields $w_1^{0\star} = 0$ and $w_1^{1\star} = 1 + B_r(p_0)/\bar{B}_r(p_0) = (p_0 - p_r^*)/\bar{B}_r(p_0)$, leading to (D.18). It is then straightforward to verify that $w_1^{1\star} = 1$ if and only if $B_r(p_0) = 0$, i.e., $p_0 = \frac{\beta_L p_r^*}{\beta_H - \beta p_r^*}$. Because of $p_0 < p_r^*$, similar argument as in the proof of Proposition C.1 suggests that $a^\star(D_1, p_0, \boldsymbol{w}_1) \equiv 0$.

- For $p_0 \in \left[\frac{\alpha_L p_r^*}{\alpha_H - \alpha p_r^*}, p_r^*\right)$, (C.30) implies that $\mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) \mid \boldsymbol{w}_1, p_0\right] \equiv \pi \ell_s$. Again, because of $p_0 < p_r^*$, similar argument as in the proof of Proposition C.1 suggests that $a^\star(D_1, p_0, \boldsymbol{w}_1) \equiv 0$. In addition, by (C.34) and (C.35), we have the following four cases to examine:
  - —If $1 \leq w_1^1/w_1^0 < -\bar{A}_r(p_0)/A_r(p_0)$ and $1 \leq \overline{w}_1^0/\overline{w}_1^1 < -\bar{B}_r(p_0)/B_r(p_0)$, then $\{p_1^{0,0}, p_1^{1,1}\} < p_r^*$ and we again obtain (D.24), which is again dominated by the next case.
  - —If $1 \leq w_1^1/w_1^0 < -\bar{A}_r(p_0)/A_r(p_0)$ and $\overline{w}_1^0/\overline{w}_1^1 \geq -\bar{B}_r(p_0)/B_r(p_0) > 1$, then $p_1^{0,0} \geq p_r^* > p_1^{1,1}$ and we again obtain (D.25), which is minimized (but not achievable) at $(w_\dagger^0, w_\dagger^1)$ defined in (C.38), the minimal possible (w_1^0, w_1^1) satisfying $w_1^1/w_1^0 < -\bar{A}_r(p_0)/A_r(p_0)$ and $\overline{w}_1^0/\overline{w}_1^1 \geq -\bar{B}_r(p_0)/B_r(p_0)$, because $\bar{B}_s(p_0) \geq 0$ and $B_s(p_0) \geq 0$ according to (C.42). Thus, the sender's infimum expected cost in this case is given by

$$\inf J_1\left(p_1^{0,0} \geq p_r^* > p_1^{1,1}\right) = (1+\rho)\pi\ell_s - \rho\overline{\pi}^2\kappa_s\left(\beta + \alpha\eta_s\right)\left[\bar{B}_s(p_0)\overline{w}_\dagger^1 + B_s(p_0)\overline{w}_\dagger^0\right] < (1+\rho)\pi\ell_s. \tag{D.26}$$

  - —If $w_1^1/w_1^0 \geq -\bar{A}_r(p_0)/A_r(p_0) > 1$ and $1 \leq \overline{w}_1^0/\overline{w}_1^1 < -\bar{B}_r(p_0)/B_r(p_0)$, then $p_1^{1,1} \geq p_r^* > p_1^{0,0}$ and (C.44) implies that

$$\begin{aligned}&\mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) + \rho J_2(p_1) \mid \boldsymbol{w}_1, p_0\right]\\ =&(1+\rho)\pi\ell_s - \pi\overline{\pi}\kappa_s\left(\beta + \alpha\eta_s\right)\left[A_s(p_0)w_1^1 + \bar{A}_s(p_0)w_1^0\right] \leq (1+\rho)\pi\ell_s,\end{aligned}$$

where the coefficients $A_s(p_0) \geq 0$ and $\bar{A}_s(p_0) \geq 0$ according to (C.42). Thus, the sender's expected cost in this case is minimized (but not achievable) again at $(w_\dagger^0, w_\dagger^1)$ defined in (C.38), the maximal possible (w_1^0, w_1^1) satisfying $w_1^1/w_1^0 \geq -\bar{A}_r(p_0)/A_r(p_0)$ and $\overline{w}_1^0/\overline{w}_1^1 < -\bar{B}_r(p_0)/B_r(p_0)$. Thus, the sender's infimum expected cost in this case is given by

$$\inf J_1\left(p_1^{1,1} \geq p_r^* > p_1^{0,0}\right) = (1+\rho)\pi\ell_s - \pi\overline{\pi}\kappa_s\left(\beta + \alpha\eta_s\right)\left[A_s(p_0)w_\dagger^1 + \bar{A}_s(p_0)w_\dagger^0\right]. \tag{D.27}$$

  - —If both $w_1^1/w_1^0 \geq -\bar{A}_r(p_0)/A_r(p_0) > 1$ and $\overline{w}_1^0/\overline{w}_1^1 \geq -\bar{B}_r(p_0)/B_r(p_0) > 1$, then $\{p_1^{1,1}, p_1^{0,0}\} \geq p_r^* > p_0$ and (C.46) implies that

$$\begin{aligned}&\mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) + \rho J_2(p_1) \mid \boldsymbol{w}_1, p_0\right]\\ =&(1+\rho)\pi\ell_s - \overline{\pi}\kappa_s\left(\beta + \alpha\eta_s\right)\left\{\pi\left[A_s(p_0)w_1^1 + \bar{A}_s(p_0)w_1^0\right] + \overline{\pi}\left[\bar{B}_s(p_0)\overline{w}_1^1 + B_s(p_0)\overline{w}_1^0\right]\right\},\\ =&(1+\rho)\pi\ell_s + \overline{\pi}\kappa_s\left(\beta + \alpha\eta_s\right)\left\{\overline{\pi}\left(p_s^* - p_0\right) + \left[\overline{\pi}\bar{B}_s(p_0) - \pi A_s(p_0)\right]w_1^1 + \left[\overline{\pi}B_s(p_0) - \pi\bar{A}_s(p_0)\right]w_1^0\right\},\end{aligned} \tag{D.28}$$

the minimum of which over $w_1^1/w_1^0 \geq -\bar{A}_r(p_0)/A_r(p_0) > 1$ and $\overline{w}_1^0/\overline{w}_1^1 \geq -\bar{B}_r(p_0)/B_r(p_0) > 1$ must obviously be lower than $\inf J_1\left(p_1^{0,0} \geq p_r^* > p_1^{1,1}\right)$ and $\inf J_1\left(p_1^{1,1} \geq p_r^* > p_1^{0,0}\right)$, respectively defined in

(D.26) and (D.27). Therefore, it suffices to search for the optimal policy within this case. Note that, by Lemmas C.7 and C.1, we will never encounter situation where $\overline{\pi}\bar{B}_s(p_0) - \pi A_s(p_0) > 0$ and $\overline{\pi} B_s(p_0) - \pi \bar{A}_s(p_0) < 0$. Thus, we just need to discuss the following three cases:

1. If $\overline{\pi}\bar{B}_s(p_0) - \pi A_s(p_0) > 0$ and $\overline{\pi} B_s(p_0) - \pi \bar{A}_s(p_0) \geq 0$, then the minimum of (D.28) is achieved at the smallest possible (w_1^0, w_1^1) satisfying $w_1^1/w_1^0 \geq -\bar{A}_r(p_0)/A_r(p_0) > 1$ and $\overline{w}_1^0/\overline{w}_1^1 \geq -\bar{B}_r(p_0)/B_r(p_0) > 1$, which is given by (D.18).
2. If $\overline{\pi}\bar{B}_s(p_0) - \pi A_s(p_0) \leq 0$ and $\overline{\pi} B_s(p_0) - \pi \bar{A}_s(p_0) < 0$, then the minimum of (D.28) is achieved at the largest possible (w_1^0, w_1^1) satisfying $w_1^1/w_1^0 \geq -\bar{A}_r(p_0)/A_r(p_0) > 1$ and $\overline{w}_1^0/\overline{w}_1^1 \geq -\bar{B}_r(p_0)/B_r(p_0) > 1$, which is given by (D.19).
3. If $\overline{\pi}\bar{B}_s(p_0) - \pi A_s(p_0) \leq 0$ and $\overline{\pi} B_s(p_0) - \pi \bar{A}_s(p_0) \geq 0$, then the minimum of (D.28) is achieved at the largest possible w_1^1 but the smallest possible w_1^0 satisfying $w_1^1/w_1^0 \geq -\bar{A}_r(p_0)/A_r(p_0) > 1$ and $\overline{w}_1^0/\overline{w}_1^1 \geq -\bar{B}_r(p_0)/B_r(p_0) > 1$, which is given by $(w_1^0, w_1^1) = (0, 1)$.

- For $p_0 \in [p_r^*, p^\diamond)$, we have $\frac{p_0 - q_r^*}{p_0 - p_r^*} > -B_r(p_0)/\bar{B}_r(p_0)$, which leads to the following cases to examine:

1. If $1 \leq \overline{w}_1^0/\overline{w}_1^1 \leq -A_r(p_0)/\bar{A}_r(p_0)$ and $1 \leq w_1^1/w_1^0 \leq -B_r(p_0)/\bar{B}_r(p_0) < \frac{p_0 - q_r^*}{p_0 - p_r^*}$, then (C.34)-(C.37) suggest $p_0 \geq \{p_1^{1,0}, p_1^{0,1}\} \geq p_r^*$. Thus, (C.30) and (C.49) imply that

$$\mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) + \rho J_2(p_1) \mid \boldsymbol{w}_1, p_0\right] \equiv (1+\rho)\pi \ell_s + \rho \overline{\pi} \kappa_s \left(\beta + \alpha \eta_s\right)\left(p_s^* - p_0\right) := \Gamma, \quad \text{(D.29)}$$

which will be shown to be dominated (see (D.31)).

2. If $\overline{w}_1^0/\overline{w}_1^1 > -A_r(p_0)/\bar{A}_r(p_0)$ and $1 \leq w_1^1/w_1^0 \leq -B_r(p_0)/\bar{B}_r(p_0) < \frac{p_0 - q_r^*}{p_0 - p_r^*}$, then (C.34)-(C.37) suggest $p_0 \geq p_1^{1,0} \geq p_r^* > p_1^{0,1}$. Thus, (C.30) and (C.48) imply that

$$\begin{aligned}
&\mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) + \rho J_2(p_1) \mid \boldsymbol{w}_1, p_0\right] \\
=&(1+\rho)\pi \ell_s + \rho \overline{\pi} \kappa_s \left(\beta + \alpha \eta_s\right)\left[\overline{\pi}(p_s^* - p_0) - \pi A_s(p_0) w_1^1 - \pi \bar{A}_s(p_0) w_1^0\right] \\
\geq&(1+\rho)\pi \ell_s + \rho \overline{\pi} \kappa_s \left(\beta + \alpha \eta_s\right)\left[\overline{\pi}(p_s^* - p_0) - \pi\left(A_s(p_0) + \bar{A}_s(p_0)\right)\right] = \Gamma,
\end{aligned}$$

which is thus dominated by the previous case.

3. If $\overline{w}_1^0/\overline{w}_1^1 > -A_r(p_0)/\bar{A}_r(p_0)$ and $1 \leq -B_r(p_0)/\bar{B}_r(p_0) < w_1^1/w_1^0 < \frac{p_0 - q_r^*}{p_0 - p_r^*}$, then (C.34)-(C.37) suggest $p_0 \geq p_r^* > \{p_1^{1,0}, p_1^{0,1}\}$. Similar argument as in the previous case shows that this case is also dominated.
4. If $\overline{w}_1^0/\overline{w}_1^1 \leq -A_r(p_0)/\bar{A}_r(p_0)$ and $1 \leq -B_r(p_0)/\bar{B}_r(p_0) < w_1^1/w_1^0 < \frac{p_0 - q_r^*}{p_0 - p_r^*}$, then (C.34)-(C.37) suggest $p_0 \geq p_1^{0,1} \geq p_r^* > p_1^{1,0}$. Similar argument as in the previous case shows that this case is again dominated.
5. If $\overline{w}_1^0/\overline{w}_1^1 \leq -A_r(p_0)/\bar{A}_r(p_0)$ and $1 \leq -B_r(p_0)/\bar{B}_r(p_0) < \frac{p_0 - q_r^*}{p_0 - p_r^*} \leq w_1^1/w_1^0$, then (C.34)-(C.37) suggest $p_0 \geq p_1^{0,1} \geq p_r^* > p_1^{1,0}$. Thus, (C.30) and (C.47) imply that

$$\begin{aligned}
&\mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) + \rho J_2(p_1) \mid \boldsymbol{w}_1, p_0\right] \\
=&\pi \ell_s + \overline{\pi} \kappa_s \left(\beta + \alpha \eta_s\right)\left[\left(p_s^* - p_0\right) w_1^1 + \left(p_0 - q_s^*\right) w_1^0\right] \\
&+ \rho \pi \ell_s + \rho \overline{\pi} \kappa_s \left(\beta + \alpha \eta_s\right)\left[p_s^* - p_0 + \overline{\pi} \bar{B}_s(p_0) w_1^1 + \overline{\pi} B_s(p_0) w_1^0\right]
\end{aligned}$$

$$=\Gamma+\overline{\pi}\kappa_s\left(\beta+\alpha\eta_s\right)\left\{\underbrace{\left[p_s^*-p_0+\rho\overline{\pi}\bar{B}_s(p_0)\right]}_{<0}w_1^1+\underbrace{\left[p_0-q_s^*+\rho\overline{\pi}B_s(p_0)\right]}_{>0}w_1^0\right\},\tag{D.30}$$

where the signs of the coefficients for w_1^1 and w_1^0 follow from (C.42). Therefore, (D.30) is minimized along $\overline{w}_1^0/\overline{w}_1^1=-A_r(p_0)/\bar{A}_r(p_0)\geq 1$ (i.e., $p_1^{0,1}=p_r^*$), which yields

$$\begin{aligned}&\left[p_s^*-p_0+\rho\overline{\pi}\bar{B}_s(p_0)\right]w_1^1+\left[p_0-q_s^*+\rho\overline{\pi}B_s(p_0)\right]w_1^0\\=&\left\{\left[p_s^*-p_0+\rho\overline{\pi}\bar{B}_s(p_0)\right]-\frac{A_r(p_0)}{\bar{A}_r(p_0)}\left[p_0-q_s^*+\rho\overline{\pi}B_s(p_0)\right]\right\}w_1^1+\text{constant independent of }(w_1^0,w_1^1)\end{aligned}$$

Because $-A_r(p_0)/\bar{A}_r(p_0)\geq 1$, we have

$$\left[p_s^*-p_0+\rho\overline{\pi}\bar{B}_s(p_0)\right]-\frac{A_r(p_0)}{\bar{A}_r(p_0)}\left[p_0-q_s^*+\rho\overline{\pi}B_s(p_0)\right]\geq p_s^*-q_s^*+\rho\overline{\pi}\underbrace{\left[\bar{B}_s(p_0)+B_s(p_0)\right]}_{p_0-p_s^*}>0.$$

Hence, (D.30) is minimized at the smallest possible w_1^1 satisfying $\overline{w}_1^0/\overline{w}_1^1=-A_r(p_0)/\bar{A}_r(p_0)$, namely (D.20). The corresponding sender's cost from (D.30) is given by

$$J_1\left(p_1^{0,1}=p_r^*>p_1^{1,0}\right)=\Gamma+\overline{\pi}\kappa_s\left(\beta+\alpha\eta_s\right)\Theta_d(p_0)<\Gamma,\tag{D.31}$$

where we note $\Theta_d(p_0)$ is of the same sign as $p_s^*-p_0+\rho\overline{\pi}\bar{B}_s(p_0)<0$.

6 If $\overline{w}_1^0/\overline{w}_1^1>-A_r(p_0)/\bar{A}_r(p_0)$ and $1\leq -B_r(p_0)/\bar{B}_r(p_0)<\frac{p_0-q_r^*}{p_0-p_r^*}\leq w_1^1/w_1^0$, then (C.34)-(C.37) suggest $\{p_1^{0,1},p_1^{1,0}\}<p_r^*\leq p_0$. Thus, (C.30) and (C.46) imply that

$$\begin{aligned}&\mathbb{E}\left[c_s\left(a^\star\left(D_1,p_0,\boldsymbol{w}_1\right),X_1\right)+\rho J_2(p_1)\,|\,\boldsymbol{w}_1,p_0\right]\\=&\pi\ell_s+\overline{\pi}\kappa_s\left(\beta+\alpha\eta_s\right)\left[\left(p_s^*-p_0\right)w_1^1+\left(p_0-q_s^*\right)w_1^0\right]\\&+\rho\pi\ell_s+\rho\overline{\pi}\kappa_s\left(\beta+\alpha\eta_s\right)\left\{\overline{\pi}\left(p_s^*-p_0\right)+\left[\overline{\pi}\bar{B}_s(p_0)-\pi A_s(p_0)\right]w_1^1+\left[\overline{\pi}B_s(p_0)-\pi\bar{A}_s(p_0)\right]w_1^0\right\}\\=&\Gamma+\overline{\pi}\kappa_s\left(\beta+\alpha\eta_s\right)\left\{-\rho\pi\left(p_s^*-p_0\right)+\underbrace{\left[p_s^*-p_0+\rho\overline{\pi}\bar{B}_s(p_0)-\rho\pi A_s(p_0)\right]}_{<0}w_1^1\right.\\&\left.+\underbrace{\left[p_0-q_s^*+\rho\overline{\pi}B_s(p_0)-\rho\pi\bar{A}_s(p_0)\right]}_{>0}w_1^0\right\},\end{aligned}\tag{D.32}$$

where we note that, because $0\leq\bar{A}_s(p_0)\leq p_0-p_s^*$ and $\rho\pi\leq 1$,

$$p_0-q_s^*+\rho\overline{\pi}B_s(p_0)-\rho\pi\bar{A}_s(p_0)\geq p_0-q_s^*+\rho\overline{\pi}B_s(p_0)-(p_0-p_s^*)=p_s^*-q_s^*+\rho\overline{\pi}B_s(p_0)>0.$$

Hence, (D.32) is minimized by full disclosure $(w_1^0,w_1^1)=(0,1)$ to achieve

$$\begin{aligned}J_1\left(p_r^*>\{p_1^{0,1},p_1^{1,0}\}\right)=&\Gamma+\overline{\pi}\kappa_s\left(\beta+\alpha\eta_s\right)\left[\left(1-\rho\pi\right)\left(p_s^*-p_0\right)+\rho\overline{\pi}\bar{B}_s(p_0)-\rho\pi A_s(p_0)\right]\\=&\Gamma+\overline{\pi}\kappa_s\left(\beta+\alpha\eta_s\right)\Theta_f(p_0).\end{aligned}\tag{D.33}$$

To determine the sender's optimal policy for $p_0\in[p_r^*,p^\diamond)$, we now compare $J_1\left(p_1^{0,1}=p_r^*>p_1^{1,0}\right)$ and $J_1\left(p_r^*>\{p_1^{0,1},p_1^{1,0}\}\right)$ respectively given by (D.31) and (D.33), leading to the result.

- For $p_0\in\left[p^\diamond,\frac{\bar{\beta}_Lp_r^*}{\bar{\beta}_H+\beta p_r^*}\right)$, we have $1\leq\frac{p_0-q_r^*}{p_0-p_r^*}\leq -B_r(p_0)/\bar{B}_r(p_0)$, which leads to the following cases to examine:

1. If $1 \le \overline{w}_1^0/\overline{w}_1^1 \le -A_r(p_0)/\bar{A}_r(p_0)$ and $1 \le w_1^1/w_1^0 \le \frac{p_0-q_r^*}{p_0-p_r^*} \le -B_r(p_0)/\bar{B}_r(p_0)$, then (C.34)-(C.37) suggest $p_0 \ge \{p_1^{1,0}, p_1^{0,1}\} \ge p_r^*$. We thus obtain the same sender's cost Γ as in (D.29), which will be shown below to be dominated.

2. If $\overline{w}_1^0/\overline{w}_1^1 > -A_r(p_0)/\bar{A}_r(p_0) \ge 1$ and $1 \le w_1^1/w_1^0 \le \frac{p_0-q_r^*}{p_0-p_r^*} \le -B_r(p_0)/\bar{B}_r(p_0)$, then (C.34)-(C.37) suggest $p_0 \ge p_1^{1,0} \ge p_r^* > p_1^{0,1}$ and (C.30) suggests that $\mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) \mid \boldsymbol{w}_1, p_0\right] \equiv \pi \ell_s$. Hence, the sender's cost must be higher than Γ, and this case is again dominated.

3. If $\overline{w}_1^0/\overline{w}_1^1 > -A_r(p_0)/\bar{A}_r(p_0) \ge 1$ and $\frac{p_0-q_r^*}{p_0-p_r^*} \le w_1^1/w_1^0 \le -B_r(p_0)/\bar{B}_r(p_0)$, then (C.34)-(C.37) suggest $p_0 \ge p_1^{1,0} \ge p_r^* > p_1^{0,1}$. Hence, (C.30) and (C.48) suggest that

$$
\begin{aligned}
&\mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) + \rho J_2\left(p_1\right) \mid \boldsymbol{w}_1, p_0\right] \\
=&\pi \ell_s + \overline{\pi} \kappa_s\left(\beta + \alpha \eta_s\right)\left[\left(p_s^* - p_0\right) w_1^1 + \left(p_0 - q_s^*\right) w_1^0\right] \\
&+ \rho \pi \ell_s + \rho \overline{\pi} \kappa_s\left(\beta + \alpha \eta_s\right)\left[\overline{\pi}(p_s^* - p_0) - \pi A_s(p_0) w_1^1 - \pi \bar{A}_s(p_0) w_1^0\right] \\
=&\Gamma + \overline{\pi} \kappa_s\left(\beta + \alpha \eta_s\right)\left\{\rho \pi(p_0 - p_s^*) + \underbrace{\left[p_s^* - p_0 - \rho \pi A_s(p_0)\right]}_{\le p_r^* - p_0 < 0} w_1^1 + \underbrace{\left[p_0 - q_s^* - \rho \pi \bar{A}_s(p_0)\right]}_{> p_0 - p_s^* - \bar{A}_s(p_0) = A_s(p_0) \ge 0} w_1^0\right\},
\end{aligned}
\tag{D.34}
$$

where the signs of the coefficients for w_1^1 and w_1^0 follow from (C.42) and the fact that $q_s^* < p_s^* \le 0 < p_r^* \le p_0$. Hence, (D.34) is minimized along $w_1^1/w_1^0 = -B_r(p_0)/\bar{B}_r(p_0) \ge 1$ (i.e., $p_1^{1,0} = p_r^*$), yielding

$$
\begin{aligned}
&\left[p_s^* - p_0 - \rho \pi A_s(p_0)\right] w_1^1 + \left[p_0 - q_s^* - \rho \pi \bar{A}_s(p_0)\right] w_1^0 \\
=&\underbrace{\left\{\left[p_0 - q_s^* - \rho \pi \bar{A}_s(p_0)\right] - B_r(p_0)/\bar{B}_r(p_0)\left[p_s^* - p_0 - \rho \pi A_s(p_0)\right]\right\}}_{<0} w_1^0,
\end{aligned}
$$

where the negativity of the coefficient of w_1^0 follows from the fact that $A_s(p_0) \ge 0$ and $\bar{A}_s(p_0) \ge 0$ and the observation that

$$
p_0 - q_s^* - \frac{B_r(p_0)}{\bar{B}_r(p_0)}\left(p_s^* - p_0\right) = \left(p_0 - p_s^*\right)\left[\frac{p_0 - q_s^*}{p_0 - p_s^*} + \frac{B_r(p_0)}{\bar{B}_r(p_0)}\right] < \left(p_0 - p_s^*\right)\left[\frac{p_0 - q_r^*}{p_0 - p_r^*} + \frac{B_r(p_0)}{\bar{B}_r(p_0)}\right] \le 0.
$$

Therefore, (D.34) must be minimized by the largest w_1^0 satisfying $w_1^1/w_1^0 = -B_r(p_0)/\bar{B}_r(p_0) \ge 1$, namely (D.21). The corresponding sender's cost from (D.34) is given by

$$
J_1\left(p_1^{1,0} = p_r^* > p_1^{0,1}\right) = \Gamma + \overline{\pi} \kappa_s\left(\beta + \alpha \eta_s\right) \Theta_e(p_0). \tag{D.35}
$$

4. If $1 \le \overline{w}_1^0/\overline{w}_1^1 \le -A_r(p_0)/\bar{A}_r(p_0)$ and $\frac{p_0-q_r^*}{p_0-p_r^*} \le w_1^1/w_1^0 \le -B_r(p_0)/\bar{B}_r(p_0)$, then (C.34)-(C.37) suggest $p_0 \ge \left\{p_1^{1,0}, p_1^{0,1}\right\} \ge p_r^*$. Hence, (C.30) and (C.49) suggest that

$$
\begin{aligned}
&\mathbb{E}\left[c_s\left(a^\star\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) + \rho J_2\left(p_1\right) \mid \boldsymbol{w}_1, p_0\right] \\
=&(1+\rho) \pi \ell_s + \overline{\pi} \kappa_s\left(\beta + \alpha \eta_s\right)\left[\left(p_s^* - p_0\right) w_1^1 + \left(p_0 - q_s^*\right) w_1^0 + \rho(p_s^* - p_0)\right] \\
=&\Gamma + \overline{\pi} \kappa_s\left(\beta + \alpha \eta_s\right)\left[\underbrace{\left(p_s^* - p_0\right)}_{<0} w_1^1 + \underbrace{\left(p_0 - q_s^*\right)}_{>0} w_1^0\right].
\end{aligned}
\tag{D.36}
$$

Therefore, (D.36) is minimized either along $\overline{w}_1^0/\overline{w}_1^1 = -A_r(p_0)/\bar{A}_r(p_0) \ge 1$ (i.e., $p_1^{0,1} = p_r^*$) or along along $w_1^1/w_1^0 = -B_r(p_0)/\bar{B}_r(p_0) \ge 1$ (i.e., $p_1^{1,0} = p_r^*$). The former case yields

$$
\left(p_s^* - p_0\right) w_1^1 + \left(p_0 - q_s^*\right) w_1^0 = \underbrace{\left\{p_s^* - p_0 - \frac{A_r(p_0)}{\bar{A}_r(p_0)}\left(p_0 - q_s^*\right)\right\}}_{\ge p_s^* - q_s^* > 0} w_1^1 + \text{constant independent of } (w_1^0, w_1^1);
$$

the latter case yields

$$(p_s^* - p_0)\, w_1^1 + (p_0 - q_s^*)\, w_1^0 = \underbrace{\left\{ -\frac{B_r(p_0)}{\bar{B}_r(p_0)}\,(p_s^* - p_0) + (p_0 - q_s^*) \right\}}_{=(p_0-p_s^*)\left[\frac{p_0-q_s^*}{p_0-p_s^*}+\frac{B_r(p_0)}{\bar{B}_r(p_0)}\right]<(p_0-p_s^*)\left[\frac{p_0-q_r^*}{p_0-p_r^*}+\frac{B_r(p_0)}{\bar{B}_r(p_0)}\right]\le 0} \qquad w_1^0.$$

Thus, (D.36) is minimized by the intersection between $\overline{w}_1^0/\overline{w}_1^1 = -A_r(p_0)/\bar{A}_r(p_0) \geq 1$ and $w_1^1/w_1^0 = -B_r(p_0)/\bar{B}_r(p_0) \geq 1$, which is $(w_\ddagger^0, w_\ddagger^1)$ given by (C.39), or equivalently, (D.22). The corresponding sender's cost from (D.36) is thus given by

$$\begin{aligned} J_1\left(p_1^{1,0} = p_1^{0,1} = p_r^*\right) =& \Gamma + \overline{\pi}\kappa_s\,(\beta + \alpha\eta_s)\left[(p_s^* - p_0)\, w_\ddagger^1 + (p_0 - q_s^*)\, w_\ddagger^0\right] \\ =& \Gamma + \overline{\pi}\kappa_s\,(\beta + \alpha\eta_s)\,\Theta_{de}(p_0) < \Gamma, \end{aligned} \tag{D.37}$$

where we note that $\Theta_{de}(p_0)$ is of the same sign as $p_0 - q_s^* - \frac{B_r(p_0)}{\bar{B}_r(p_0)}\,(p_s^* - p_0) < 0$.

5 If $1 \leq \overline{w}_1^0/\overline{w}_1^1 \leq -A_r(p_0)/\bar{A}_r(p_0)$ and $w_1^1/w_1^0 > -B_r(p_0)/\bar{B}_r(p_0) \geq \frac{p_0-q_r^*}{p_0-p_r^*}$, then following the same analysis as in case 5 for $p_0 \in [p_r^*, p^\diamond)$, we obtain the optimal policy given by (D.20), which implies that $p_1^{0,1} = p_r^* > p_1^{1,0}$, and the corresponding sender's cost given by $J_1\left(p_1^{0,1} = p_r^* > p_1^{1,0}\right) < \Gamma$ in (D.31).

6 If $\overline{w}_1^0/\overline{w}_1^1 > -A_r(p_0)/\bar{A}_r(p_0) \geq 1$ and $w_1^1/w_1^0 > -B_r(p_0)/\bar{B}_r(p_0) \geq \frac{p_0-q_r^*}{p_0-p_r^*}$, then following the same analysis as in case 6 for $p_0 \in [p_r^*, p^\diamond)$, we obtain the full disclosure $(w_1^0, w_1^1) = (0,1)$ as the optimal policy, and the corresponding sender's cost given by $J_1\left(p_r^* > \{p_1^{0,1}, p_1^{1,0}\}\right)$ in (D.33).

To determine the sender's optimal policy for $p_0 \in \left[p^\diamond, \frac{\bar{\beta}_L p_r^*}{\bar{\beta}_H + \beta p_r^*}\right)$, we need to compare $J_1\left(p_1^{1,0} = p_r^* > p_1^{0,1}\right)$ in (D.35), $J_1\left(p_1^{1,0} = p_1^{0,1} = p_r^*\right)$ in (D.37), $J_1\left(p_1^{0,1} = p_r^* > p_1^{1,0}\right)$ in (D.31), and $J_1\left(p_r^* > \{p_1^{0,1}, p_1^{1,0}\}\right)$ in (D.33), leading to the result.

- For $p_0 \in \left[\frac{\bar{\beta}_L p_r^*}{\bar{\beta}_H + \beta p_r^*}, \frac{\bar{\alpha}_L p_r^*}{\bar{\alpha}_H + \alpha p_r^*}\right)$, (C.37) implies that $p_1^{1,0} \geq p_r^*$, which leads to the following case to examine:

1 If $1 \leq \overline{w}_1^0/\overline{w}_1^1 \leq -A_r(p_0)/\bar{A}_r(p_0)$ and $1 \leq w_1^1/w_1^0 \leq \frac{p_0-q_r^*}{p_0-p_r^*}$, then similar to case 1 for $p_0 \in \left[p^\diamond, \frac{\bar{\beta}_L p_r^*}{\bar{\beta}_H + \beta p_r^*}\right)$, the sender's cost is given by Γ in (D.29), which will be shown below to be dominated.

2 If $\overline{w}_1^0/\overline{w}_1^1 > -A_r(p_0)/\bar{A}_r(p_0) \geq 1$ and $1 \leq w_1^1/w_1^0 \leq \frac{p_0-q_r^*}{p_0-p_r^*}$, then similar to case 2 for $p_0 \in \left[p^\diamond, \frac{\bar{\beta}_L p_r^*}{\bar{\beta}_H + \beta p_r^*}\right)$, the sender's cost must be higher than Γ, and this case is again dominated.

3 If $1 \leq \overline{w}_1^0/\overline{w}_1^1 \leq -A_r(p_0)/\bar{A}_r(p_0)$ and $w_1^1/w_1^0 \geq \frac{p_0-q_r^*}{p_0-p_r^*} \geq 1$, then following similar analysis in case 4 for $p_0 \in \left[p^\diamond, \frac{\bar{\beta}_L p_r^*}{\bar{\beta}_H + \beta p_r^*}\right)$, we have the sender's cost given by (D.36), which is then minimized at the smallest w_1^1 satisfying $\overline{w}_1^0/\overline{w}_1^1 = -A_r(p_0)/\bar{A}_r(p_0) \geq 1$ (i.e., $p_1^{0,1} = p_r^*$), i.e., the optimal policy is given by (D.20), and the corresponding sender's cost is given by

$$J_1\left(p_1^{1,0} \geq p_1^{0,1} = p_r^*\right) = \Gamma + \overline{\pi}\kappa_s\,(\beta + \alpha\eta_s)\,(p_s^* - p_0)\,\frac{p_0 - p_r^*}{A_r(p_0)} < \Gamma. \tag{D.38}$$

4 If $\overline{w}_1^0/\overline{w}_1^1 > -A_r(p_0)/\bar{A}_r(p_0) \geq 1$ and $w_1^1/w_1^0 \geq \frac{p_0-q_r^*}{p_0-p_r^*} \geq 1$, then (C.34)-(C.37) suggest $p_0 \geq p_1^{1,0} \geq p_r^* > p_1^{0,1}$. Thus, following similar analysis in case 3 for $p_0 \in \left[p^\diamond, \frac{\bar{\beta}_L p_r^*}{\bar{\beta}_H + \beta p_r^*}\right)$, we obtain the sender's cost given by (D.34), which implies that the optimal policy is the full disclosure $(w_1^0, w_1^1) = (0,1)$ and the sender's corresponding cost is given by

$$\begin{aligned} J_1\left(p_0 \geq p_1^{1,0} \geq p_r^* > p_1^{0,1}\right) =& \Gamma + \overline{\pi}\kappa_s\,(\beta + \alpha\eta_s)\left[\rho\pi(p_0 - p_s^*) + p_s^* - p_0 - \rho\pi A_s(p_0)\right] \\ =& \Gamma + \overline{\pi}\kappa_s\,(\beta + \alpha\eta_s)\left[p_s^* - p_0 + \rho\pi\bar{A}_s(p_0)\right] < \Gamma. \end{aligned} \tag{D.39}$$

To determine the sender's optimal policy for $p_0 \in \left[\frac{\bar{\beta}_L p_r^*}{\bar{\beta}_H + \beta p_r^*}, \frac{\bar{\alpha}_L p_r^*}{\bar{\alpha}_H + \alpha p_r^*}\right)$, we need to compare $J_1\left(p_1^{1,0} \geq p_1^{0,1}\right)$ in (D.38), and $J_1\left(p_0 \geq p_1^{1,0} \geq p_r^* > p_1^{0,1}\right)$ in (D.39), leading to the result.

- For $p_0 \in \left[\frac{\bar{\alpha}_L p_r^*}{\bar{\alpha}_H + \alpha p_r^*}, 1\right)$, then (C.34)-(C.37) suggest $p_0 \geq \{p_1^{1,0}, p_1^{0,1}\} \geq p_r^*$. Thus, (C.49) implies that $\mathbb{E}\left[J_2(p_1) \,|\, \boldsymbol{w}_1, p_0\right] = \pi\ell_s + \bar{\pi}\kappa_s\left(\beta + \alpha\eta_s\right)\left(p_s^* - p_0\right)$ is independent of the warning policy $\boldsymbol{w}_1$ and hence the optimal policy is the full disclosure $w_1^{0\star} = 0$ and $w_1^{1\star} = 1$ as determined by the first-period one as in Proposition 1. The corresponding sender's cost is given by

$$J_1\left(p_0 \geq \{p_1^{1,0}, p_1^{0,1}\} \geq p_r^*\right) = \Gamma + \bar{\pi}\kappa_s\left(\beta + \alpha\eta_s\right)\left(p_s^* - p_0\right). \quad \square \tag{D.40}$$

Lemma D.2. *Suppose $\alpha_H = \beta_H = 1$, $\alpha \leq \beta$ and $0 \leq \eta_r \leq \bar{\beta}_L/\alpha_L \leq \eta_s \leq 1$. Let*

$$p^d := \max\{p_0 \in [p_r^*, 1] : \psi(p_0) \geq 0\} > p_r^*, \quad \textit{with} \quad \psi(p_0) := p_r^*\left[p_0 - p_s^* - \rho\bar{\pi}\bar{B}_s(p_0)\right] + \rho\pi p_s^* A_r(p_0), \tag{D.41}$$

$$p^e := \max\{p_0 \in [p_r^*, 1] : \phi(p_0) \geq 0\} > p_r^*, \quad \textit{with} \quad \phi(p_0) := p_r^*\left[p_0 - q_s^* - \rho\pi\bar{A}_s(p_0)\right] + \rho\bar{\pi} p_s^* B_r(p_0). \tag{D.42}$$

Then, we have

1. *$\Theta_f(p_0) \leq \{\Theta_d(p_0), \Theta_e(p_0), \Theta_{de}(p_0)\}$ for $p_r^* \leq p_0 \leq \min\{p^d, p^e\}$;*
2. *If $p^d < p^e$, then $\Theta_d(p_0) < \Theta_f(p_0) \leq \Theta_e(p_0)$ and $\Theta_d(p_0) \leq \Theta_{de}(p_0)$ for $p^d < p_0 \leq p^e$;*
3. *If $p^d < p^e < 1$, then there exists a threshold $p^{de} \in [p^e, 1)$ such that $\Theta_d(p_0) \leq \Theta_{de}(p_0) \leq \Theta_e(p_0) \leq \Theta_f(p_0)$ for $p^e < p_0 \leq p^{de}$ and $\{\Theta_{de}(p_0), \Theta_e(p_0)\} \leq \Theta_d(p_0) \leq \Theta_f(p_0)$ for $p^{de} < p_0 \leq 1$;*
4. *If $p^e < p^d$, then $\Theta_e(p_0) < \Theta_f(p_0) \leq \Theta_d(p_0)$ and $\Theta_e(p_0) \leq \Theta_{de}(p_0)$ for $p^e < p_0 \leq p^d$;*
5. *If $p^e \leq p^d < 1$, then there exists a threshold $p^{de} \in [p^d, 1]$ such that $\Theta_e(p_0) \leq \Theta_{de}(p_0) \leq \Theta_d(p_0) \leq \Theta_f(p_0)$ for $p^d < p_0 \leq p^{de}$ and $\Theta_{de}(p_0) \leq \{\Theta_d(p_0), \Theta_e(p_0)\} \leq \Theta_f(p_0)$ for $p^{de} < p_0 \leq 1$.*

Proof of Lemma D.2. When $\alpha_H = \beta_H = 1$, direct verification gives that

$$A_r(p_0) = p_0 - p_r^* + \alpha p_r^*(1 - p_0) \geq 0, \quad \bar{A}_r(p_0) = -\alpha p_r^*(1 - p_0) \leq 0, \tag{D.43}$$

$$B_r(p_0) = p_0 - p_r^* + \beta p_r^*(1 - p_0) \geq 0, \quad \bar{B}_r(p_0) = -\beta p_r^*(1 - p_0) \leq 0, \tag{D.44}$$

$$A_s(p_0) = p_0 - p_s^* + \alpha p_s^*(1 - p_0) \geq 0, \quad \bar{A}_s(p_0) = -\alpha p_s^*(1 - p_0) \geq 0, \tag{D.45}$$

$$B_s(p_0) = p_0 - p_s^* + \beta p_s^*(1 - p_0) \geq 0, \quad \bar{B}_s(p_0) = -\beta p_s^*(1 - p_0) \geq 0. \tag{D.46}$$

Direct calculation reveals that

$$\begin{aligned}
\Theta_d(p_0) - \Theta_f(p_0) &= \left[p_s^* - p_0 + \rho\bar{\pi}\bar{B}_s(p_0)\right]\frac{p_0 - p_r^*}{A_r(p_0)} - \left[p_s^* - p_0 + \rho\pi\bar{A}_s(p_0) + \rho\bar{\pi}\bar{B}_s(p_0)\right] \\
&= -\frac{\bar{A}_r(p_0)}{A_r(p_0)}\left[p_0 - p_s^* - \rho\bar{\pi}\bar{B}_s(p_0)\right] - \rho\pi\bar{A}_s(p_0) \\
&= \frac{1}{A_r(p_0)}\left\{-\bar{A}_r(p_0)\left[p_0 - p_s^* - \rho\bar{\pi}\bar{B}_s(p_0)\right] - \rho\pi A_r(p_0)\bar{A}_s(p_0)\right\} = \frac{\alpha(1 - p_0)}{A_r(p_0)}\psi(p_0).
\end{aligned} \tag{D.47}$$

Straightforward calculation yields that

$$\psi(p_0) = \left[p_r^* + \rho\pi p_s^* - \rho\left(\bar{\pi}\beta + \pi\alpha\right)p_r^* p_s^*\right]p_0 - p_r^* p_s^*\left[1 - \rho\left(\bar{\pi}\beta - \pi\bar{\alpha}\right)\right], \tag{D.48}$$

which is linear in p_0 and satisfies

$$\psi(p_r^*) = p_r^*\left[p_r^* - p_s^* + \rho p_s^*(1 - p_r^*)\left(\pi\alpha + \bar{\pi}\beta\right)\right] \geq p_r^*\left[p_r^* - p_s^* + p_s^*(1 - p_r^*)\right] = (p_r^*)^2(1 - p_s^*) > 0, \quad \text{and}$$

$$\psi(1) = p_r^*(1-p_s^*) + \rho\pi p_s^*(1-p_r^*). \tag{D.49}$$

Thus, we must have

Claim 1. $p^d > p_r^*$ such that $\psi(p_0) \geq 0$ for all $p_0 \in [p_r^*, p^d]$ and $\psi(p_0) < 0$ for all $p_0 \in (p^d, 1]$.

Similarly, we have

$$\begin{aligned}
\Theta_e(p_0) - \Theta_f(p_0) =& p_s^* - p_0 - \frac{\bar{B}_r(p_0)}{B_r(p_0)}(p_0 - q_s^*) + \rho\pi\bar{A}_s(p_0)\left[1 + \frac{\bar{B}_r(p_0)}{B_r(p_0)}\right] - \left[p_s^* - p_0 + \rho\pi\bar{A}_s(p_0) + \rho\bar{\pi}\bar{B}_s(p_0)\right] \\
=& -\frac{\bar{B}_r(p_0)}{B_r(p_0)}\left[p_0 - q_s^* - \rho\pi\bar{A}_s(p_0)\right] - \rho\bar{\pi}\bar{B}_s(p_0) \\
=& \frac{1}{B_r(p_0)}\left\{-\bar{B}_r(p_0)\left[p_0 - q_s^* - \rho\pi\bar{A}_s(p_0)\right] - \rho\bar{\pi}\bar{B}_s(p_0)B_r(p_0)\right\} = \frac{\beta(1-p_0)}{B_r(p_0)}\phi(p_0).
\end{aligned} \tag{D.50}$$

Again, straightforward calculation yields that

$$\phi(p_0) = \left[p_r^* + \rho\bar{\pi}p_s^* - \rho\left(\bar{\pi}\beta + \pi\alpha\right)p_r^*p_s^*\right]p_0 - p_r^*\left[q_s^* - \rho\left(\pi\alpha - \bar{\pi}\bar{\beta}\right)p_s^*\right] \tag{D.51}$$

which is linear in p_0 and satisfies

$$\begin{aligned}
\phi(p_r^*) =& p_r^*\left[p_r^* - q_s^* + \rho p_s^*(1-p_r^*)\left(\pi\alpha + \bar{\pi}\beta\right)\right] && \text{(D.52)} \\
\geq& p_r^*\left[p_r^* - p_s^* + \rho p_s^*(1-p_r^*)\left(\pi\alpha + \bar{\pi}\beta\right)\right] = \psi(p_r^*) > 0, \text{ (because } p_s^* \geq q_s^*) \quad \text{and} \\
\phi(1) =& p_r^*(1-q_s^*) + \rho\bar{\pi}p_s^*(1-p_r^*). && \text{(D.53)}
\end{aligned}$$

Thus, we must have

Claim 2. $p^e > p_r^*$ such that $\phi(p_0) \geq 0$ for all $p_0 \in [p_r^*, p^e]$ and $\phi(p_0) < 0$ for all $p_0 \in (p^e, 1]$.

We further note that

$$\phi(p_0) - \psi(p_0) = p_r^*(p_s^* - q_s^*) + \rho(\bar{\pi} - \pi)p_s^*\left(p_0 - p_r^*\right). \tag{D.54}$$

Next, we compute

$$\begin{aligned}
\Theta_{de}(p_0) - \Theta_f(p_0) =& \frac{\bar{B}_r(p_0)}{\bar{B}_r(p_0) - A_r(p_0)}\left[p_0 - q_s^* - \frac{B_r(p_0)}{\bar{B}_r(p_0)}(p_s^* - p_0)\right] - \left[p_s^* - p_0 + \rho\pi\bar{A}_s(p_0) + \rho\bar{\pi}\bar{B}_s(p_0)\right] \\
=& \frac{1}{A_r(p_0) - \bar{B}_r(p_0)}\Big\{\underbrace{-\bar{B}_r(p_0)\left[p_0 - q_s^* - \rho\pi\bar{A}_s(p_0)\right] - \rho\bar{\pi}\bar{B}_s(p_0)B_r(p_0)}_{(1-p_0)\beta\phi(p_0)} \\
& \underbrace{-\bar{A}_r(p_0)\left[p_0 - p_s^* - \rho\bar{\pi}\bar{B}_s(p_0)\right] - \rho\pi A_r(p_0)\bar{A}_s(p_0)}_{(1-p_0)\alpha\psi(p_0)}\Big\} \\
=& \frac{1-p_0}{A_r(p_0) - \bar{B}_r(p_0)}\left[\alpha\psi(p_0) + \beta\phi(p_0)\right],
\end{aligned} \tag{D.55}$$

which, together with (D.47) and (D.50), further implies that

$$\begin{aligned}
\Theta_{de}(p_0) - \Theta_d(p_0) =& \frac{1-p_0}{A_r(p_0) - \bar{B}_r(p_0)}\left[\alpha\psi(p_0) + \beta\phi(p_0)\right] - \frac{\alpha(1-p_0)}{A_r(p_0)}\psi(p_0) \\
=& \frac{1-p_0}{A_r(p_0)\left[A_r(p_0) - \bar{B}_r(p_0)\right]}\left\{\beta A_r(p_0)\phi(p_0) + \alpha\bar{B}_r(p_0)\psi(p_0)\right\} \\
=& \frac{\beta(1-p_0)}{A_r(p_0)\left[A_r(p_0) - \bar{B}_r(p_0)\right]}\left\{A_r(p_0)\phi(p_0) + \bar{A}_r(p_0)\psi(p_0)\right\}, \quad \text{and} && \text{(D.56)} \\
\Theta_{de}(p_0) - \Theta_e(p_0) =& \frac{1-p_0}{A_r(p_0) - \bar{B}_r(p_0)}\left[\alpha\psi(p_0) + \beta\phi(p_0)\right] - \frac{\beta(1-p_0)}{B_r(p_0)}\phi(p_0)
\end{aligned}$$

$$=\frac{1-p_0}{B_r(p_0)\left[A_r(p_0)-\bar{B}_r(p_0)\right]}\left\{\beta\bar{A}_r(p_0)\phi(p_0)+\alpha B_r(p_0)\psi(p_0)\right\}$$
$$=\frac{\alpha(1-p_0)}{B_r(p_0)\left[A_r(p_0)-\bar{B}_r(p_0)\right]}\left\{\bar{B}_r(p_0)\phi(p_0)+B_r(p_0)\psi(p_0)\right\}. \tag{D.57}$$

- If $p_r^* \le p_0 \le \min\{p^d, p^e\}$, then $\{\psi(p_0), \phi(p_0)\} \ge 0$ by Claim 1 and 2. Hence, Result 1 immediately follows from (D.47), (D.50) and (D.55).
- If $p^d < p_0 \le p^e$, then $\psi(p_0) < 0 \le \phi(p_0)$. Hence, (D.47) and (D.50) imply that $\Theta_d(p_0) < \Theta_f(p_0) \le \Theta_e(p_0)$; and (D.56), together with (D.43), implies that $\Theta_d(p_0) \le \Theta_{de}(p_0)$. Thus, Result 2 is obtained.
- If $p^e < p_0 \le p^d$, then $\phi(p_0) < 0 \le \psi(p_0)$. Hence, (D.47) and (D.50) imply that $\Theta_e(p_0) < \Theta_f(p_0) \le \Theta_d(p_0)$; and (D.57), together with (D.44), implies that $\Theta_e(p_0) \le \Theta_{de}(p_0)$. Thus, Result 4 is obtained.
- If $\max\{p^d, p^e\} < p_0 \le 1$, then $\{\psi(p_0), \phi(p_0)\} < 0$ and hence, by (D.47), (D.50) and (D.55), $\{\Theta_d(p_0), \Theta_e(p_0), \Theta_{de}(p_0)\} \le \Theta_f(p_0)$. In particular, we have $\psi(1) < 0$ and $\phi(1) < 0$. We make two claims:

  <u>Claim 3</u>. $A_r(p_0)\phi(p_0) + \bar{A}_r(p_0)\psi(p_0)$ is a concave quadratic function in p_0.

  Proof of Claim 3: Direct calculation reveals that

$$\begin{aligned}
A_r(p_0)\phi(p_0)+\bar{A}_r(p_0)\psi(p_0) =&(p_0-p_r^*)\phi(p_0)+\bar{A}_r(p_0)\left[\psi(p_0)-\phi(p_0)\right]\\
\text{(by (D.42) and (D.54))}\quad =&(p_0-p_r^*)\left\{\left[p_r^*+\rho\overline{\pi}p_s^*-\rho p_r^*p_s^*\left(\overline{\pi}\beta+\pi\alpha\right)\right]p_0-p_r^*q_s^*+\rho p_r^*p_s^*\left(\pi\alpha-\overline{\pi}\bar{\beta}\right)\right\}\\
&+\alpha p_r^*(1-p_0)\left[\rho(\overline{\pi}-\pi)p_s^*(p_0-p_r^*)+p_r^*(p_s^*-q_s^*)\right]\\
=&\left\{p_r^*+\rho\overline{\pi}p_s^*\left[1-(\alpha+\beta)p_r^*\right]\right\}p_0^2\\
&-p_r^*\left\{p_r^*+q_s^*+\alpha p_r^*(p_s^*-q_s^*)+\rho\overline{\pi}p_s^*\left[\bar{\alpha}+\bar{\beta}-(\alpha+\beta)p_r^*\right]\right\}p_0\\
&+(p_r^*)^2\left[\alpha p_s^*+\bar{\alpha}q_s^*+\rho\overline{\pi}(1-\alpha-\beta)p_s^*\right].
\end{aligned}$$

  We note that the coefficient of p_0^2 is

$$p_r^*+\rho\overline{\pi}p_s^*\left[1-(\alpha+\beta)p_r^*\right]<p_r^*(1-q_s^*)+\rho\overline{\pi}p_s^*(1-p_r^*)=\phi(1)<0,$$

  immediately yielding Claim 3.

  <u>Claim 4</u>. $\bar{B}_r(p_0)\phi(p_0) + B_r(p_0)\psi(p_0)$ is a concave quadratic function in p_0.

  Proof of Claim 4: Direct calculation reveals that

$$\begin{aligned}
\bar{B}_r(p_0)\phi(p_0)+B_r(p_0)\psi(p_0) =&\bar{B}_r(p_0)\left[\phi(p_0)-\psi(p_0)\right]+(p_0-p_r^*)\psi(p_0)\\
\text{(by (D.41) and (D.54))}\quad =&\beta p_r^*(p_0-1)\left[\rho(\overline{\pi}-\pi)p_s^*(p_0-p_r^*)+p_r^*(p_s^*-q_s^*)\right]\\
&+(p_0-p_r^*)\left\{\left[p_r^*+\rho\pi p_s^*-\rho p_r^*p_s^*\left(\overline{\pi}\beta+\pi\alpha\right)\right]p_0-p_r^*p_s^*+\rho p_r^*p_s^*\left(\overline{\pi}\beta-\pi\bar{\alpha}\right)\right\}\\
=&\left\{p_r^*+\rho\pi p_s^*\left[1-(\alpha+\beta)p_r^*\right]\right\}p_0^2\\
&-p_r^*\left\{p_r^*+p_s^*-\beta p_r^*(p_s^*-q_s^*)+\rho\pi p_s^*\left[\bar{\alpha}+\bar{\beta}-(\alpha+\beta)p_r^*\right]\right\}p_0\\
&+(p_r^*)^2\left[\bar{\beta}p_s^*+\beta q_s^*+\rho\pi(1-\alpha-\beta)p_s^*\right].
\end{aligned}$$

  We note that the coefficient of p_0^2 is

$$p_r^*+\rho\pi p_s^*\left[1-(\alpha+\beta)p_r^*\right]<p_r^*(1-p_s^*)+\rho\pi p_s^*(1-p_r^*)=\psi(1)<0,$$

  immediately yielding Claim 4.

  $\boxed{\text{If } p^d < p^e \le p_0 \le 1}$, then Result 3 follows by considering two cases below:

1. $\psi(p_0) \le \phi(p_0) \le 0$ for $p_0 \in [p^e, 1]$. By (D.56), $\Theta_{de}(p_0) - \Theta_d(p_0) \propto A_r(p_0)\phi(p_0) + \bar{A}_r(p_0)\psi(p_0)$. Since
$$A_r(p^e)\phi(p^e) + \bar{A}_r(p^e)\psi(p^e) = \bar{A}_r(p^e)\psi(p^e) \ge 0, \quad \text{and}$$
$$A_r(1)\phi(1) + \bar{A}_r(1)\psi(1) = (1 - p_r^*)\phi(1) < 0,$$
Claim 3 above implies that there must exists $p^{de} \in [p^e, 1)$ such that $A_r(p_0)\phi(p_0) + \bar{A}_r(p_0)\psi(p_0) \ge (<)0$ for $p_0 \le (>)p^{de}$. Therefore, $\Theta_d(p_0) \le \Theta_{de}(p_0)$ for $p_0 \in [p^e, p^{de}]$ and $\Theta_{de}(p_0) < \Theta_d(p_0)$ for $p_0 \in (p^{de}, 1]$. On the other hand, by (D.57), $\Theta_{de}(p_0) - \Theta_e(p_0) \propto \bar{B}_r(p_0)\phi(p_0) + B_r(p_0)\psi(p_0) \le \left[\bar{B}_r(p_0) + B_r(p_0)\right]\psi(p_0) = (p_0 - p_r^*)\psi(p_0) \le 0$; that is, $\Theta_{de}(p_0) \le \Theta_e(p_0)$ for $p_0 \in [p^e, 1]$.
2. There exists $p^\flat \in (p^e, 1)$ such that $\psi(p_0) \le \phi(p_0) \le 0$ for $p_0 \in [p^e, p^\flat]$ and $\phi(p_0) \le \psi(p_0) \le 0$ for $p_0 \in [p^\flat, 1]$. Now that
$$A_r(p^e)\phi(p^e) + \bar{A}_r(p^e)\psi(p^e) = \bar{A}_r(p^e)\psi(p^e) \ge 0, \quad \text{and}$$
$$A_r(p^\flat)\phi(p^\flat) + \bar{A}_r(p^\flat)\psi(p^\flat) = (p^\flat - p_r^*)\phi(p^\flat) = (p^\flat - p_r^*)\psi(p^\flat) \le 0,$$
Claim 3 above implies that there must exists $p^{de} \in [p^e, p^\flat]$ such that $A_r(p_0)\phi(p_0) + \bar{A}_r(p_0)\psi(p_0) \ge (\le)0$ for $p_0 \le (\ge)p^{de}$. Therefore, $\Theta_d(p_0) \le \Theta_{de}(p_0)$ for $p_0 \in [p^e, p^{de}]$ and $\Theta_{de}(p_0) < \Theta_d(p_0)$ for $p_0 \in [p^{de}, 1]$. On the other hand, similar to the previous case, we have $\Theta_{de}(p_0) \le \Theta_e(p_0)$ for $p_0 \in [p^e, p^\flat]$, which implies that $\Theta_d(p_0) \le \Theta_{de}(p_0) \le \Theta_e(p_0)$ for $p_0 \in [p^e, p^{de}]$. Finally, we compute, by (D.47) and (D.50), that
$$\begin{aligned}\Theta_d(p_0) - \Theta_e(p_0) =& \frac{\alpha(1-p_0)}{A_r(p_0)}\psi(p_0) - \frac{\beta(1-p_0)}{B_r(p_0)}\phi(p_0) \\ =& \frac{1-p_0}{A_r(p_0)B_r(p_0)}\left[\alpha B_r(p_0)\psi(p_0) - \beta A_r(p_0)\phi(p_0)\right],\end{aligned} \tag{D.58}$$
which is nonnegative for $p_0 \in [p^\flat, 1]$, because $\phi(p_0) \le \psi(p_0) \le 0$ and $\beta A_r(p_0) - \alpha B_r(p_0) = (\beta - \alpha)(p_0 - p_r^*) \ge 0$ by noting that $\beta \ge \alpha$.

$\boxed{\text{If } p^e \le p^d \le p_0 \le 1}$, then we must have $\phi(p_0) \le \psi(p_0) \le 0$ for $p_0 \in [p^d, 1]$, because $\phi(p_r^*) > \psi(p_r^*)$ while $\phi(p^d) \le \psi(p^d) = 0$. By (D.56), $\Theta_{de}(p_0) - \Theta_d(p_0) \propto A_r(p_0)\phi(p_0) + \bar{A}_r(p_0)\psi(p_0) \le \left[A_r(p_0) + \bar{A}_r(p_0)\right]\psi(p_0) = (p_0 - p_r^*)\psi(p_0) \le 0$. By (D.57), $\Theta_{de}(p_0) - \Theta_e(p_0) \propto \bar{B}_r(p_0)\phi(p_0) + B_r(p_0)\psi(p_0)$. Since
$$\bar{B}_r(p^d)\phi(p^d) + B_r(p^d)\psi(p^d) = \bar{B}_r(p^d)\phi(p^d) \ge 0, \quad \text{and}$$
$$\bar{B}_r(1)\phi(1) + B_r(1)\psi(1) = (1 - p_r^*)\psi(1) \le 0,$$
Claim 4 above implies that there must exists $p^{de} \in [p^d, 1]$ such that $\bar{B}_r(p_0)\phi(p_0) + B_r(p_0)\psi(p_0) \ge (\le)0$ for $p_0 \le (\ge)p^{de}$. Therefore, $\Theta_e(p_0) \le \Theta_{de}(p_0)$ for $p_0 \in [p^d, p^{de}]$ and $\Theta_{de}(p_0) \le \Theta_e(p_0)$ for $p_0 \in [p^{de}, 1]$, leading to Result 5. $\square$

Proof of Theorem 3. The second-period optimal policy in Theorem 3 follows directly from Proposition C.1. For the first-period optimal policy, we apply Theorem D.1 to the parametric values $\alpha_H = \beta_H = 1$ and $\beta = \bar{\beta}_L \ge \bar{\alpha}_L = \alpha$, which immediately also imply that $\alpha\beta_L - \beta\alpha_L = \alpha - \beta \ge 0$. In this case, (D.23) reduces to
$$0 \le \underbrace{\frac{\beta_L p_r^*}{\beta_H - \beta p_r^*}}_{\frac{\beta_L p_r^*}{1-\bar{\beta}_L p_r^*}} \le \underbrace{\frac{\alpha_L p_r^*}{\alpha_H - \alpha p_r^*}}_{\frac{\alpha_L p_r^*}{1-\bar{\alpha}_L p_r^*}} \le p_r^* \le \frac{\bar{\beta}_L p_r^*}{\bar{\beta}_H + \beta p_r^*} = \frac{\bar{\alpha}_L p_r^*}{\bar{\alpha}_H + \alpha p_r^*} = 1. \tag{D.59}$$

We will show that the thresholds specified in (14) are given by

$$\hat{p}_1 = \frac{\beta_L p_r^*}{1-\bar{\beta}_L p_r^*}, \tag{D.60}$$

$$\hat{p}_2 = \frac{\alpha_L p_r^*}{1-\bar{\alpha}_L p_r^*} \vee \left(\frac{\left(\overline{\pi}\bar{\beta}_L - \pi\alpha_L\right) p_s^*}{\left(\overline{\pi}\bar{\beta}_L + \pi\bar{\alpha}_L\right) p_s^* - \pi} \wedge p_r^* \right) \geq \hat{p}_1, \tag{D.61}$$

$$\hat{p}_3 = \hat{p}_2 \vee \left(\frac{\left(\overline{\pi}\beta_L - \pi\bar{\alpha}_L\right) p_s^*}{\overline{\pi} - \left(\overline{\pi}\bar{\beta}_L + \pi\bar{\alpha}_L\right) p_s^*} \wedge p_r^* \right) \geq \hat{p}_2, \tag{D.62}$$

$$\hat{p}_4 = \begin{cases} p^d > p_r^* \geq \hat{p}_3, & \text{if } p^d < p^e, \\ p^e \vee (p^d \wedge p^\diamond) > p_r^* \geq \hat{p}_3, & \text{if } p^d \geq p^e, \end{cases} \tag{D.63}$$

$$\hat{p}_5 = \begin{cases} 1 > p^d = \hat{p}_4, & \text{if } p^d < p^e = 1, \\ p^{de} \vee p^\diamond > p^d = \hat{p}_4, & \text{if } p^d < p^e < 1, \\ p^\diamond > p^d = \hat{p}_4, & \text{if } p^e \leq p^d < p^\diamond, \\ p^e \vee p^\diamond = \hat{p}_4, & \text{if } p^e \vee p^\diamond \leq p^d, \end{cases} \tag{D.64}$$

$$\hat{p}_6 = \begin{cases} 1 \geq p^e \vee p^\diamond = \hat{p}_5, & \text{if } p^e \leq p^d = 1, \\ p^{de} > p^\diamond \geq \hat{p}_5, & \text{if } p^e \leq p^d < 1 \text{ and } p^\diamond < p^{de}, \\ p^\diamond = \hat{p}_5, & \text{if } p^e \leq p^d < 1 \text{ and } p^\diamond \geq p^{de}, \\ \hat{p}_5, & \text{if } p^d < p^e, \end{cases} \tag{D.65}$$

and $p^* = p_r^*$, where $p^\diamond$ is given by (D.2), and $\{p^d, p^e, p^{de}\}$ are given by Lemma D.2.

- By Theorem D.1, any policy is optimal for $p_0 \in \left[0, \frac{\beta_L p_r^*}{1-\bar{\beta}_L p_r^*}\right)$. By convention, we can take it as full disclosure policy $\boldsymbol{\omega}_1^\star(p_0) = (0,1)$. This leads to (D.60) by also noticing that $w_1^{1\star} = 1$ in (D.18) if and only if $p_0 = \frac{\beta_L p_r^*}{1-\bar{\beta}_L p_r^*}$.

- By Theorem D.1, the optimal policy takes the form $(w_1^{0\star} = 0, w_1^{1\star} < 1)$ as specified by (D.18), for $p_0 \in \left(\hat{p}_1, \frac{\alpha_L p_r^*}{\alpha_H - \alpha p_r^*}\right)$, and for $p_0 \in \left[\frac{\alpha_L p_r^*}{\alpha_H - \alpha p_r^*}, p_r^*\right)$ that satisfies $\overline{\pi}\bar{B}_s(p_0) - \pi A_s(p_0) > 0$ and $\overline{\pi} B_s(p_0) - \pi \bar{A}_s(p_0) \geq 0$. By (C.50) and (C.51), $\overline{\pi}\bar{B}_s(p_0) - \pi A_s(p_0) > 0$ and $\overline{\pi} B_s(p_0) - \pi \bar{A}_s(p_0) \geq 0$ both hold if and only if $\pi/\overline{\pi} < \bar{\beta}_L/\alpha_L$ and $p_0 < \frac{\left(\overline{\pi}\bar{\beta}_L - \pi\alpha_L\right)p_s^*}{\left(\overline{\pi}\bar{\beta}_L + \pi\bar{\alpha}_L\right)p_s^* - \pi}$. This leads to (D.61) by noticing that $p_s^* \leq 0$.

- By Theorem D.1, the optimal policy takes the form $(w_1^{0\star} \geq 0, w_1^{1\star} = 1)$ as specified by (D.19), for $p_0 \in \left[\frac{\alpha_L p_r^*}{\alpha_H - \alpha p_r^*}, p_r^*\right)$ that satisfies $\overline{\pi}\bar{B}_s(p_0) - \pi A_s(p_0) \leq 0$ and $\overline{\pi} B_s(p_0) - \pi \bar{A}_s(p_0) < 0$. By (C.50) and (C.51), $\overline{\pi}\bar{B}_s(p_0) - \pi A_s(p_0) \leq 0$ and $\overline{\pi} B_s(p_0) - \pi \bar{A}_s(p_0) < 0$ both hold if and only if $\pi/\overline{\pi} > \beta_L/\bar{\alpha}_L$ and $p_0 < \frac{\left(\overline{\pi}\beta_L - \pi\bar{\alpha}_L\right)p_s^*}{\overline{\pi} - \left(\overline{\pi}\bar{\beta}_L + \pi\bar{\alpha}_L\right)p_s^*}$. Since when $\pi/\overline{\pi} > \beta_L/\bar{\alpha}_L \geq \bar{\beta}_L/\alpha_L$, we must have $\hat{p}_2 = \frac{\alpha_L p_r^*}{\alpha_H - \alpha p_r^*}$. This leads to (D.62) by noticing that $p_s^* \leq 0$.

- By (C.50) and (C.51), for $p_0 \in [\hat{p}_3, p_r^*)$, we must have $\overline{\pi}\bar{B}_s(p_0) - \pi A_s(p_0) \leq 0$ and $\overline{\pi} B_s(p_0) - \pi \bar{A}_s(p_0) \geq 0$, which, by Theorem D.1, further implies that the optimal policy is given by full disclosure $(w_1^{0\star}, w_1^{1\star}) = (0,1)$. Furthermore, by Lemma D.2 and Theorem D.1, the full disclosure $(w_1^{0\star}, w_1^{1\star}) = (0,1)$ is also optimal for $p_0 \in [p_r^*, \min\{p^d, p^e\}]$ and for $p_0 \in [p^e, p^d] \cap [p_r^*, p^\diamond]$, thus leading to (D.63).

- By Lemma D.2 and Theorem D.1, the optimal policy takes the form $(w_1^{0\star} = 0, w_1^{1\star} < 1)$ as specified by (D.20) (i) for $p^e \leq p^d < p_0 \leq p^\diamond$, or (ii) for $p^d < p_0 \leq p^e = 1$, or (iii) for $p_0 \in (p^d, p^{de} \vee p^\diamond]$ if $p^d < p^e < 1$. This leads to (D.64).

- By Lemma D.2 and Theorem D.1, the optimal policy takes the form $(w_1^{0\star} > 0, w_1^{1\star} = 1)$ as specified by (D.21) (i) for $p^e \vee p^\diamond < p_0 \leq p^d = 1$, or (ii) for $p^e \vee p^\diamond < p_0 \leq p^{de}$ if $p^e \leq p^d < 1$.

- By Lemma D.2 and Theorem D.1, the optimal policy takes the form $(w_1^{0\star} > 0, w_1^{1\star} \leq 1)$ as specified by (D.21) or (D.22) for $p^{de} \vee p^{\diamond} < p_0 \leq 1$, which is equivalent to $\hat{p}_6 < p_0 \leq 1$. $\square$

Proof of Proposition 2. If $\beta_L/\bar{\alpha}_L \geq \pi/\bar{\pi}$, we immediately have $\frac{(\bar{\pi}\beta_L - \pi\bar{\alpha}_L)p_s^*}{\bar{\pi} - (\bar{\pi}\bar{\beta}_L + \pi\bar{\alpha}_L)p_s^*} \leq 0$. Hence, (D.62) implies that $\hat{p}_2 = \hat{p}_3$. The other condition $\alpha_L/\bar{\beta}_L \leq \frac{\eta_r^{-1} + \rho\pi\eta_s^{-1}}{1+\rho\pi}$ is equivalent to $p^d = 1$, because, by (D.41) and (D.49), we have

$$
\begin{aligned}
\psi(1) =& p_r^*(1-p_s^*) + \rho\pi p_s^*(1-p_r^*) = (1-p_s^*)(1-p_r^*)\left[\frac{p_r^*}{1-p_r^*} + \rho\pi\frac{p_s^*}{1-p_s^*}\right] \\
=& (1-p_s^*)(1-p_r^*)\left[\bar{\beta}_L\eta_r^{-1} - \alpha_L + \rho\pi\left(\bar{\beta}_L\eta_s^{-1} - \alpha_L\right)\right] \geq 0 \quad \Leftrightarrow \quad \alpha_L/\bar{\beta}_L \leq \frac{\eta_r^{-1} + \rho\pi\eta_s^{-1}}{1+\rho\pi}.
\end{aligned}
$$

Thus, (D.64) and (D.65) immediately implies that $\hat{p}_4 = \hat{p}_5$ and $\hat{p}_6 = 1$, respectively. $\square$

Proof of Proposition 3. Let $c_s^j(a,x)$ represent the sender's cost corresponding to receiver j (as defined in Table 1(b)), and $J_t^j(p)$ denote the sender's cost-to-go in period t when she is only facing receiver j. Further, note that since cost parameters ℓ_r, δ_r and κ_r remain the same across the two receivers, they share the same p^* and best response function $a^\star(d,p,\boldsymbol{w})$.

First, we have the sender's cost in the second period:

$$
\begin{aligned}
J_2(p_1) =& \min_{\boldsymbol{w}_2\in\mathcal{W}} \mathbb{E}\left[c_s^1\left(a^\star\left(D_2,p_1,\boldsymbol{w}_2\right),X_2\right) + c_s^2\left(a^\star\left(D_2,p_1,\boldsymbol{w}_2\right),X_2\right) \,\middle|\, p_1\right] \\
\geq& \min_{\boldsymbol{w}_2\in\mathcal{W}} \mathbb{E}\left[c_s^1\left(a^\star\left(D_2,p_1,\boldsymbol{w}_2\right),X_2\right) \,\middle|\, p_1\right] + \min_{\boldsymbol{w}_2\in\mathcal{W}} \mathbb{E}\left[c_s^2\left(a^\star\left(D_2,p_1,\boldsymbol{w}_2\right),X_2\right) \,\middle|\, p_1\right] \\
=& J_2^1(p_1) + J_2^2(p_1),
\end{aligned}
$$

where the last equality is achieved at $\boldsymbol{w}_2 = (0,1)$ for any p_1 according to Proposition 1. Since plugging in $\boldsymbol{w}_2 = (0,1)$ in the first minimization problem results in the above inequality to hold as an equality, this policy must also be the optimal policy solving the first minimization problem and achieving $J_2(p_1)$.

Thus, $J_2^j(p)$ is obtained by generalizing (9) as,

$$
J_2^j(p) = \begin{cases} \pi\ell_s^j - [\bar{\pi}\kappa_s^j p + \pi(\delta_s^j - \kappa_s^j) - \bar{\pi}\kappa_s^j] & \text{if } p \geq p^*, \\ \pi\ell_s^j & \text{if } p < p^*, \end{cases} \tag{D.66}
$$

and subsequently we have,

$$
J_2(p) = \begin{cases} \pi\ell_s - \bar{\pi}\kappa_s(p + \eta_s - 1) & \text{if } p \geq p^* \\ \pi\ell_s & \text{if } p < p^*, \end{cases} \tag{D.67}
$$

where $\ell_s \equiv \ell_s^1 + \ell_s^2$, $\delta_s \equiv \delta_s^1 + \delta_s^2$, $\kappa_s \equiv \kappa_s^1 + \kappa_s^2$, and $\eta_s \equiv \pi/\bar{\pi}\,(\delta_s/\kappa_s - 1)$.

Next, we consider the per-period cost for the sender when she is facing the two receivers simultaneously. For $\pi \leq \kappa_r/\delta_r$, which in turn implies that $p^* = (\kappa_r - \pi\delta_r)/(\bar{\pi}\kappa_r) \geq 0$, (B.1) leads to

$$
\mathbb{E}\left[c_s^j\left(a^\star\left(D,p,\boldsymbol{w}\right),X\right) \,\middle|\, \boldsymbol{w},p\right] = \begin{cases} \pi\ell_s^j + \bar{\pi}\kappa_s^j\left[pw^0 - (p+\eta_s^j - 1)w^1\right], & \text{if } \frac{w^1}{w^0} \geq \frac{p}{p-p^*} \text{ and } p \geq p^*, \\ \pi\ell_s^j, & \text{if } \frac{w^1}{w^0} < \frac{p}{p-p^*} \text{ or } p < p^*, \end{cases}
$$

which further gives us

$$
\mathbb{E}\left[c_s^1\left(a^\star\left(D,p,\boldsymbol{w}\right),X\right) + c_s^2\left(a^\star\left(D,p,\boldsymbol{w}\right),X\right) \,\middle|\, \boldsymbol{w},p\right] = \begin{cases} \pi\ell_s + \bar{\pi}\kappa_s\left[pw^0 - (p+\eta_s - 1)w^1\right], & \text{if } \frac{w^1}{w^0} \geq \frac{p}{p-p^*} \text{ and } p \geq p^*, \\ \pi\ell_s, & \text{if } \frac{w^1}{w^0} < \frac{p}{p-p^*} \text{ or } p < p^*. \end{cases} \tag{D.68}
$$

Similarly, for $\pi > \kappa_r/\delta_r$ and hence $p^* < 0$, (B.3) implies that

$$
\mathbb{E}\left[c_s^j\left(a^\star\left(D,p,\boldsymbol{w}\right),X\right) \,\middle|\, \boldsymbol{w},p\right] = \begin{cases} \pi\ell_s^j - \bar{\pi}\kappa_s^j\left[(p+\eta_s^j - 1)w^1 - pw^0\right], & \text{if } \frac{\overline{w}^1}{\overline{w}^0} \leq \frac{p}{p-p^*}, \\ \pi\ell_s^j - \bar{\pi}\kappa_s^j(\eta_s^j - 1), & \text{if } \frac{\overline{w}^1}{\overline{w}^0} > \frac{p}{p-p^*}, \end{cases}
$$

which leads to

$$\mathbb{E}\left[c_s^1\left(a^\star\left(D,p,\boldsymbol{w}\right),X\right)+c_s^2\left(a^\star\left(D,p,\boldsymbol{w}\right),X\right)\,\middle|\,\boldsymbol{w},p\right]=\begin{cases}\pi\ell_s-\overline{\pi}\kappa_s\left[(p+\eta_s-1)w^1-pw^0\right], & \text{if } \frac{\overline{w}^1}{\overline{w}^0}\le\frac{p}{p-p^*},\\ \pi\ell_s-\overline{\pi}\kappa_s(\eta_s-1), & \text{if } \frac{\overline{w}^1}{\overline{w}^0}>\frac{p}{p-p^*}.\end{cases} \tag{D.69}$$

Putting all together, we conclude that both the cost-to-go function $J_2(p)$ in (D.67) and the period cost functions in (D.68) and (D.69) are exactly the cost-to-go function and period cost functions, respectively, when the sender faces a single receiver whose cost parameters are given by $\delta_s\equiv\delta_s^1+\delta_s^2$, $\kappa_s\equiv\kappa_s^1+\kappa_s^2$, and $\eta_s\equiv\pi/\overline{\pi}\left(\delta_s/\kappa_s-1\right)$. Thus, the result follows from Theorem 1. $\square$

Proof of Proposition 4. Let $a^{\star j}(d,p,\boldsymbol{w})$ represent receiver j's optimal action, and $J_t^j(.)$ denote the sender's cost-to-go in period t when she is only facing receiver j. First, we have the sender's cost in the second period:

$$\begin{aligned}J_2(p_1)&=\min_{\boldsymbol{w}_2\in\mathcal{W}}\mathbb{E}\left[c_s\left(a^{\star1}\left(D_2,p_1,\boldsymbol{w}_2\right),X_2\right)+c_s\left(a^{\star2}\left(D_2,p_1,\boldsymbol{w}_2\right),X_2\right)\,\middle|\,p_1\right]\\&\ge\min_{\boldsymbol{w}_2\in\mathcal{W}}\mathbb{E}\left[c_s\left(a^{\star1}\left(D_2,p_1,\boldsymbol{w}_2\right),X_2\right)\,\middle|\,p_1\right]+\min_{\boldsymbol{w}_2\in\mathcal{W}}\mathbb{E}\left[c_s\left(a^{\star2}\left(D_2,p_1,\boldsymbol{w}_2\right),X_2\right)\,\middle|\,p_1\right]\\&=J_2^1(p_1)+J_2^2(p_1),\end{aligned}$$

where the last equality is achieved at $\boldsymbol{w}_2=(0,1)$ for any p_1 according to Proposition 1. Subsequently, $\boldsymbol{w}_2=(0,1)$ must also be the optimal policy solving the first minimization problem and achieving $J_2(p_1)$. Therefore, the inequality above must be equality.

Noting that

$$\begin{aligned}J_1(p_0)&=\min_{\boldsymbol{w}_1\in\mathcal{W}}\mathbb{E}\left[c_s\left(a^{\star1}\left(D_1,p_0,\boldsymbol{w}_1\right),X_1\right)+c_s\left(a^{\star2}\left(D_1,p_0,\boldsymbol{w}_1\right),X_1\right)+\rho J_2(p_1)\,\middle|\,p_0\right]\\&\ge\min_{\boldsymbol{w}_1\in\mathcal{W}}\mathbb{E}\left[c_s\left(a^{\star1}\left(D_1,p_0,\boldsymbol{w}_1\right),X_1\right)+\rho J_2^1(p_1)\,\middle|\,p_0\right]+\min_{\boldsymbol{w}_1\in\mathcal{W}}\mathbb{E}\left[c_s\left(a^{\star2}\left(D_1,p_0,\boldsymbol{w}_1\right),X_1\right)+\rho J_2^2(p_1)\,\middle|\,p_0\right]\\&=J_1^1(p_0)+J_1^2(p_0),\end{aligned}$$

where, by Theorem 1, the last equality is achieved at $\boldsymbol{w}_1=(0,1)$ for any p_0 when $\pi>\kappa_r^2/\delta_r^2>\kappa_r^1/\delta_r^1>\kappa_s/\delta_s$ or $\pi<\kappa_s/\delta_s$. Thus, $\boldsymbol{w}_1=(0,1)$ must also be the optimal policy solving the first minimization problem and achieving $J_1(p_0)$. Therefore, the inequality above must be equality. Using the exact same argument also proves that the full disclosure policy is optimal when $\kappa_s/\delta_s\le\pi\le\kappa_r^1/\delta_r^1$ and $p^{2*}\le p_0<p^{1**}$.

The proof for the remaining cases follows steps very similar to the proof of Theorem 1, except the cost-to-go function in the second period in now given by

$$J_2(p)=\begin{cases}2\pi\ell_s, & \text{if } p<p_1^*\\ 2\pi\ell_s-\overline{\pi}\kappa_s(p+\eta_s-1), & \text{if } p_1^*\le p<p_2^*\\ 2\pi\ell_s-2\overline{\pi}\kappa_s(p+\eta_s-1), & \text{if } p\ge p_2^*.\end{cases} \tag{D.70}$$

Subsequently, when $p_0<p_1^*$, the sender's warning message in the first period cannot trigger an action from either of the two receivers and only serves to improve the sender's reputation for the second period. Thus,

$$\begin{aligned}J_1(p_0)&=2\pi\ell_s+\rho\min_{\boldsymbol{w}_1\in\mathcal{W}}\mathbb{E}\left[J_2(p_1)\,|\,\boldsymbol{w}_1,p_0\right]\\&=2\pi\ell_s+\rho\min_{\boldsymbol{w}_1\in\mathcal{W}}\sum_{(d,x)\in\{0,1\}^2}\Pr\left[p_1=p_1^{d,x}\,\middle|\,\boldsymbol{w}_1,p_0\right]J_2\left(p_1^{d,x}\right)\\&=2\pi\ell_s+\rho\pi J_2(p_0)+\rho\overline{\pi}\min_{\boldsymbol{w}_1\in\mathcal{W}}\left\{\left(p_0w_1^0+(1-p_0)w_1^1\right)J_2(p_1^{1,0})\right.\\&\qquad\left.+\left(1-p_0w_1^0-(1-p_0)w_1^1\right)J_2(p_1^{0,0})\right\},\end{aligned} \tag{D.71}$$

where $\Pr\left[p_1 = p_1^{d,x} \,\middle|\, \boldsymbol{w}_1, p_0\right]$ follows from (C.17)-(C.20), and $p_1^{d,x}$ is as defined in Lemma 2. To plug in the value of $J_2(.)$ from (D.70), we already know $p_1^{1,0} \le p_0 < p_1^*$, and hence $J_2(p_1^{1,0}) = 2\pi\ell_s$. Depending on the value of $p_1^{0,0}$, there are three different cases to consider: (i) $p_1^{0,0} < p_1^*$, (ii) $p_1^* \le p_1^{0,0} < p_2^*$, and (iii) $p_1^{0,0} \ge p_2^*$.

For the first case, $p_1^{0,0} < p_1^*$ implies $J_2(p_1^{0,0}) = 2\pi\ell_s$, and hence, (D.71) gives us $J_1(p_0) = 2\pi\ell_s + \rho(2\pi\ell_s) = 2(1+\rho)\pi\ell_s$.

For the second case, we have

$$\begin{aligned} J_1(p_0) =& 2\pi\ell_s + 2\rho\pi^2\ell_s + \rho\bar{\pi} \min_{\boldsymbol{w}_1 \in \mathcal{W}} \Bigg\{ \left(p_0 w_1^0 + (1-p_0) w_1^1\right) 2\pi\ell_s \\ &+ \left(1 - p_0 w_1^0 - (1-p_0) w_1^1\right) \left[2\pi\ell_s - \bar{\pi}\kappa_s \left(\frac{p_0(1-w_1^0)}{1 - p_0 w_1^0 - (1-p_0) w_1^1} + \eta_s - 1\right)\right]\Bigg\} \\ =& 2(1+\rho)\pi\ell_s - \rho\bar{\pi}^2\kappa_s(p_0 + \eta_s - 1) + \rho\bar{\pi}^2\kappa_s \min_{\boldsymbol{w}_1 \in \mathcal{W}} \left\{p_0\eta_s w_1^0 + (1-p_0)(\eta_s - 1) w_1^1\right\} . \end{aligned} \tag{D.72}$$

Thus, we have the following minimization problem,

$$\begin{aligned} &\min_{\boldsymbol{w}_1 \in \mathcal{W}} \left\{p_0\eta_s w_1^0 + (1-p_0)(\eta_s - 1) w_1^1\right\} \\ \text{subject to: } & p_1^* \left(1 - p_0 w_1^0 - (1-p_0) w_1^1\right) \le p_0(1 - w_1^0) \\ & p_0(1-w_1^0) < p_2^* \left(1 - p_0 w_1^0 - (1-p_0) w_1^1\right) . \end{aligned}$$

From the proof of Theorem 1, we already know that $(w_1^0, w_1^1) = \left(0, \frac{p_1^* - p_0}{p_1^*(1-p_0)}\right)$ is the optimal solution to the above minimization problem in the absence of the second constraint. But since this solution is feasible to the second constraint, it must remain optimal even when the second constraint is added.

The analysis of the third case is similar, and leads to the following minimization problem,

$$\begin{aligned} &\min_{\boldsymbol{w}_1 \in \mathcal{W}} \left\{p_0\eta_s w_1^0 + (1-p_0)(\eta_s - 1) w_1^1\right\} \\ \text{subject to: } & p_2^* \left(1 - p_0 w_1^0 - (1-p_0) w_1^1\right) \le p_0(1 - w_1^0) , \end{aligned}$$

for which $(w_1^0, w_1^1) = \left(0, \frac{p_2^* - p_0}{p_2^*(1-p_0)}\right)$ is optimal.

The optimal policy for $p_0 < p_1^*$ is then obtained by comparing the optimal solutions to the above three cases. First, (D.72) is always bounded from above by $2(1+\rho)\pi\ell_s$ (e.g., by setting $w_1^0 = w_1^1 = 1$). Therefore, the first case is always dominated by the second case. The proof is then complete by noting that the optimal solutions to both the second and the third case induce $w_1^{0\star} = 0$ and $w_1^{1\star} < 1$.

For $p_0 \ge p^{2*}$, it is guaranteed that $p_1^{0,0} \ge p^{2*}$ and hence, $J_2(p_1^{0,0}) = 2\pi\ell_s - 2\overline{\pi}\kappa_s(p + \eta_s - 1)$. We then need to consider different possible values for $p_1^{1,0}$, which in turn, determines $J_2(p_1^{1,0})$ according to (D.70). To that end, we define three sets

$$\begin{aligned} \mathcal{W}^1(p_0) &= \left\{(w^0, w^1) \in \mathcal{W} : p_1^{1,0} \ge p_2^*\right\} , \\ \mathcal{W}^2(p_0) &= \left\{(w^0, w^1) \in \mathcal{W} : p_1^* \le p_1^{1,0} < p_2^*\right\} , \\ \mathcal{W}^3(p_0) &= \left\{(w^0, w^1) \in \mathcal{W} : p_1^{1,0} < p_1^*\right\} , \end{aligned}$$

which together partition the feasible set $\mathcal{W}$, and denote their corresponding optimal objective values by $J_1^1(p_0)$, $J_1^2(p_0)$, and $J_1^3(p_0)$, respectively.

- **Subproblem I:** When $\boldsymbol{w} \in \mathcal{W}^1(p_0)$, we have $p_1^{0,0}, p_1^{1,0} \in [p_2^*, 1]$. Then, (D.70) implies that the sender's cost in the second period becomes independent of her warning policy and

$$\mathbb{E}\left[J_2(p_1) \,|\, \boldsymbol{w}_1, p_0\right] = J_2\left(\mathbb{E}\left[p_1 \,|\, \boldsymbol{w}_1, p_0\right]\right) = J_2(p_0) = 2\pi\ell_s - 2\bar{\pi}\kappa_s(p_0 + \eta_s - 1)\,.$$

This implies that we only need to minimize the sender's cost in the first period, given by $\mathbb{E}\left[c_s\left(a^{\star 1}\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) + c_s\left(a^{\star 2}\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) \,|\, \boldsymbol{w}_1, p_0\right]$, over $\mathcal{W}^1(p_0)$. For this, we denote $p^{\diamond 1} = (2 - p_1^*)p_1^*$ and $p^{\diamond 2} = (2 - p_2^*)p_2^*$, and consider three different ranges for p_0:

(i) If $p_2^* \le p_0 < p^{\diamond 1}$: Neither of the receivers can be induced to act and we have,

$$\begin{aligned}&\mathbb{E}\left[c_s\left(a^{\star 1}\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) \,\middle|\, \boldsymbol{w}_1, p_0\right] = \mathbb{E}\left[c_s\left(a^{\star 2}\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) \,\middle|\, \boldsymbol{w}_1, p_0\right] = \pi\ell_s \\ \Rightarrow\, &J_1^1(p_0) = 2\pi\ell_s + 2\rho\left(\pi\ell_s - \bar{\pi}\kappa_s(p_0 + \eta_s - 1)\right)\,. \qquad \text{(D.73)}\end{aligned}$$

Thus, the sender's cost does not depend on her warning policy, and any feasible policy is optimal.

(ii) If $p^{\diamond 1} \le p_0 < p^{\diamond 2}$: The second receiver cannot be induced to act but there exists a feasible policy in $\mathcal{W}^1(p_0)$ that can trigger the first receiver's action. Hence,

$$\begin{aligned}&\mathbb{E}\left[c_s\left(a^{\star 2}\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) \,\middle|\, \boldsymbol{w}_1, p_0\right] = \pi\ell_s \\ &\mathbb{E}\left[c_s\left(a^{\star 1}\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) \,\middle|\, \boldsymbol{w}_1, p_0\right] = \pi\ell_s + \bar{\pi}\kappa_s\left[p_0 w_1^0 - (p_0 + \eta_s - 1)w_1^1\right].\end{aligned}$$

To minimize this cost, since the coefficient of w_1^0 (resp. w_1^1) is positive (resp. negative), it is optimal to set the value of w_1^0 (resp. w_1^1) as low (resp. high) as possible while still satisfying $\boldsymbol{w}_1 \in \mathcal{W}^1(p_0)$. This implies that the solution is on the boundary of set $\mathcal{W}^1(p_0)$ so that $w_1^1/w_1^0 = p_0(1-p_2^*)/p_2^*(1-p_0)$. Substituting for w_1^0 based on this ratio then gives us

$$\mathbb{E}\left[c_s\left(a^{\star 1}\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) \,\middle|\, \boldsymbol{w}_1, p_0\right] = \pi\ell_s + \bar{\pi}\kappa_s\left[\frac{p_2^*(1-p_0)}{(1-p_2^*)} - (p_0 + \eta_s - 1)\right] w_1^1\,,$$

in which the coefficient of w_1^1 is negative. As a result, the cost is minimized by setting $w_1^1 = 1$, leading to

$$J_1^1(p_0) = 2\pi\ell_s + \bar{\pi}\kappa_s\left[\frac{p_2^*(1-p_0)}{(1-p_2^*)} - (p_0 + \eta_s - 1)\right] + 2\rho\left(\pi\ell_s - \bar{\pi}\kappa_s(p_0 + \eta_s - 1)\right)\,. \qquad \text{(D.74)}$$

(iii) If $p_0 \ge p^{\diamond 2}$: There exists a feasible policy in $\mathcal{W}^1(p_0)$ that can trigger both receivers' actions. Thus, we have

$$\begin{aligned}&\mathbb{E}\left[c_s\left(a^{\star 1}\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) \,\middle|\, \boldsymbol{w}_1, p_0\right] = \mathbb{E}\left[c_s\left(a^{\star 2}\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) \,\middle|\, \boldsymbol{w}_1, p_0\right] \qquad \text{(D.75)} \\ &= \pi\ell_s + \bar{\pi}\kappa_s\left[p_0 w_1^0 - (p_0 + \eta_s - 1)w_1^1\right].\end{aligned}$$

Exactly similar to Case (ii) above, the total cost is minimized by setting the value of w_1^0 (resp. w_1^1) as low (resp. high) as possible while still satisfying $\boldsymbol{w}_1 \in \mathcal{W}^1(p_0)$. This leads to

$$J_1^1(p_0) = 2\pi\ell_s + 2\bar{\pi}\kappa_s\left[\frac{p_2^*(1-p_0)}{(1-p_2^*)} - (p_0 + \eta_s - 1)\right] + 2\rho\left(\pi\ell_s - \bar{\pi}\kappa_s(p_0 + \eta_s - 1)\right)\,. \qquad \text{(D.76)}$$

- **Subproblem II:** When $\boldsymbol{w} \in \mathcal{W}^2(p_0)$, we have

$$\begin{aligned}\mathbb{E}\left[J_2(p_1) \,|\, \boldsymbol{w}_1, p_0\right] = &\sum_{(d,x)\in\{0,1\}^2} \Pr\left[p_1 = p_1^{d,x} \,\middle|\, \boldsymbol{w}_1, p_0\right] J_2\left(p_1^{d,x}\right) \\ =&\pi J_2(p_0) + \bar{\pi}\left(p_0 w_1^0 + (1-p_0)w_1^1\right) J_2(p_1^{1,0}) + \bar{\pi}\left(1 - p_0 w_1^0 - (1-p_0)w_1^1\right) J_2(p_1^{0,0}),\end{aligned}$$

where $\Pr\left[p_1 = p_1^{d,x} \,\middle|\, \boldsymbol{w}_1, p_0\right]$ follows from (C.17)-(C.20), and $p_1^{d,x}$ is as defined in Lemma 2. Then, we can use (D.70) to plug in the values of $J_2(p_1^{1,0})$ and $J_2(p_1^{0,0})$, and get

$$\begin{aligned}&\pi\left[2\pi\ell_s - 2\bar{\pi}\kappa_s(p_0+\eta_s-1)\right] + \bar{\pi}\left(p_0 w_1^0 + (1-p_0)w_1^1\right)\left[2\pi\ell_s - \bar{\pi}\kappa_s\left(\frac{p_0 w_1^0}{p_0 w_1^0 + (1-p_0)w_1^1} + \eta_s - 1\right)\right] \\ &\quad + \bar{\pi}\left(1 - p_0 w_1^0 - (1-p_0)w_1^1\right)\left[2\pi\ell_s - 2\bar{\pi}\kappa_s\left(\frac{p_0(1-w_1^0)}{p_0(1-w_1^0) + (1-p_0)(1-w_1^1)} + \eta_s - 1\right)\right] \\ =&2\pi\ell_s - 2\pi\bar{\pi}\kappa_s(p_0+\eta_s-1) - \bar{\pi}^2\kappa_s\left[2p_0 + 2\eta_s - 2 - p_0\eta_s w_1^0 - (1-p_0)(\eta_s-1)w_1^1\right].\end{aligned}$$

Therefore, since $p_0 \geq p_2^*$, we get

$$\begin{aligned}&\mathbb{E}\left[c_s\left(a^{\star 1}\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) + c_s\left(a^{\star 2}\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) + \rho J_2(p_1) \,\middle|\, \boldsymbol{w}_1, p_0\right] \\ &\quad = \rho\left[2\pi\ell_s - 2\pi\bar{\pi}\kappa_s(p_0+\eta_s-1) - \bar{\pi}^2\kappa_s\left[2p_0 + 2\eta_s - 2 - p_0\eta_s w_1^0 - (1-p_0)(\eta_s-1)w_1^1\right]\right] \\ &\quad + \begin{cases} 2\pi\ell_s + 2\bar{\pi}\kappa_s\left[p_0 w_1^0 - (p_0+\eta_s-1)w_1^1\right], & \text{if } \frac{w_1^1}{w_1^0} \geq \frac{p_0}{p_0 - p_2^*} \\ 2\pi\ell_s + \bar{\pi}\kappa_s\left[p_0 w_1^0 - (p_0+\eta_s-1)w_1^1\right], & \text{if } \frac{p_0}{p_0-p_1^*} \leq \frac{w_1^1}{w_1^0} < \frac{p_0}{p_0-p_2^*} \\ 2\pi\ell_s, & \text{if } \frac{w_1^1}{w_1^0} < \frac{p_0}{p_0-p_1^*} \end{cases} \\ &\quad = 2(1+\rho)\pi\ell_s - 2\rho\pi\bar{\pi}\kappa_s(p_0+\eta_s-1) - 2\rho\bar{\pi}^2\kappa_s\left(p_0+\eta_s-1\right) \\ &\quad + \begin{cases} \bar{\pi}\kappa_s\left\{\underbrace{\left[2+\rho\bar{\pi}\eta_s\right]p_0}_{\phi_1} w_1^0 - \underbrace{\left[2p_0 + 2\eta_s - 2 - \rho\bar{\pi}(1-p_0)(\eta_s-1)\right]}_{\phi_2} w_1^1\right\}, & \text{if } \frac{w_1^1}{w_1^0} \geq \frac{p_0}{p_0-p_2^*} \\ \bar{\pi}\kappa_s\left\{\underbrace{\left[1+\rho\bar{\pi}\eta_s\right]p_0}_{\phi_3} w_1^0 - \underbrace{\left[p_0 + \eta_s - 1 - \rho\bar{\pi}(1-p_0)(\eta_s-1)\right]}_{\phi_4} w_1^1\right\}, & \text{if } \frac{p_0}{p_0-p_1^*} \leq \frac{w_1^1}{w_1^0} < \frac{p_0}{p_0-p_2^*} \\ \rho\bar{\pi}^2\kappa_s\left[\underbrace{p_0\eta_s}_{\phi_5} w_1^0 + \underbrace{(1-p_0)(\eta_s-1)}_{\phi_6} w_1^1\right], & \text{if } \frac{w_1^1}{w_1^0} < \frac{p_0}{p_0-p_1^*}. \end{cases}\end{aligned}$$

Now, exactly similar to the proof of Theorem 1 and using (B.2), we can show that $\phi_2 > \phi_4 > 0$. Moreover, it is obvious that $\phi_1 > \phi_3 > \phi_5 > 0$. Thus, the upper branch is minimized by $(w_1^0, w_1^1) = (0,1)$, which also satisfies the requirement $w_1^1/w_1^0 \geq p_0/(p_0 - p_2^*)$, and leads to $-\bar{\pi}\kappa_s\phi_2$. Similarly, the middle branch is also minimized by $(w_1^0, w_1^1) = (0,1)$ (irrespective of the requirement $p_0/(p_0 - p_1^*) \leq w_1^1/w_1^0 < p_0/(p_0 - p_2^*)$), and leads to $-\bar{\pi}\kappa_s\phi_4$. It follows that the upper branch always dominates the middle branch. The same argument also applies to the lower branch (that is, in both cases $\eta_s < 1$ and $\eta_s \geq 1$, it leads to an outcome that is dominated by $-\bar{\pi}\kappa_s\phi_2$). We therefore conclude that

$$\begin{aligned}J_1^2(p_0) =&2(1+\rho)\pi\ell_s - 2\rho\pi\bar{\pi}\kappa_s(p_0+\eta_s-1) - 2\rho\bar{\pi}^2\kappa_s\left(p_0+\eta_s-1\right) - \bar{\pi}\kappa_s\left[2p_0 + 2\eta_s - 2 - \rho\bar{\pi}(1-p_0)(\eta_s-1)\right] \\ =&\, 2(1+\rho)\pi\ell_s - \bar{\pi}\kappa_s\left[2(1+\rho)(p_0+\eta_s-1) - \rho\bar{\pi}(1-p_0)(\eta_s-1)\right]. \end{aligned} \tag{D.77}$$

- **Subproblem III:** When $\boldsymbol{w} \in \mathcal{W}^3(p_0)$, the analysis follows steps similar to the case $\boldsymbol{w} \in \mathcal{W}^2(p_0)$ above, and leads to

$$\mathbb{E}\left[J_2(p_1) \,|\, \boldsymbol{w}_1, p_0\right] = 2\pi\ell_s - 2\pi\bar{\pi}\kappa_s(p_0+\eta_s-1) - 2\bar{\pi}^2\kappa_s\left[p_0 + \eta_s - 1 - p_0\eta_s w_1^0 - (1-p_0)(\eta_s-1)w_1^1\right],$$

and hence,

$$
\begin{aligned}
&\mathbb{E}\left[c_s\left(a^{\star 1}\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) + c_s\left(a^{\star 2}\left(D_1, p_0, \boldsymbol{w}_1\right), X_1\right) + \rho J_2(p_1) \,\middle|\, \boldsymbol{w}_1, p_0\right] \\
&\quad = 2(1+\rho)\pi\ell_s - 2\rho\pi\bar{\pi}\kappa_s(p_0+\eta_s-1) - 2\rho\bar{\pi}^2\kappa_s\left(p_0+\eta_s-1\right) \\
&\quad + \begin{cases}
\bar{\pi}\kappa_s\left\{\underbrace{2\left[1+\rho\bar{\pi}\eta_s\right]p_0}_{\phi_1} w_1^0 - \underbrace{2\left[p_0+\eta_s-1-\rho\bar{\pi}(1-p_0)(\eta_s-1)\right]}_{\phi_2} w_1^1\right\}, & \text{if } \frac{w_1^1}{w_1^0} \geq \frac{p_0}{p_0-p_2^*} \\
\bar{\pi}\kappa_s\left\{\underbrace{\left[1+2\rho\bar{\pi}\eta_s\right]p_0}_{\phi_3} w_1^0 - \underbrace{\left[p_0+\eta_s-1-2\rho\bar{\pi}(1-p_0)(\eta_s-1)\right]}_{\phi_4} w_1^1\right\}, & \text{if } \frac{p_0}{p_0-p_1^*} \leq \frac{w_1^1}{w_1^0} < \frac{p_0}{p_0-p_2^*} \\
2\rho\bar{\pi}^2\kappa_s\left[\underbrace{p_0\eta_s}_{\phi_5} w_1^0 + \underbrace{(1-p_0)(\eta_s-1)}_{\phi_6} w_1^1\right], & \text{if } \frac{w_1^1}{w_1^0} < \frac{p_0}{p_0-p_1^*}\,.
\end{cases}
\end{aligned}
$$

We note that $\phi_1 > \phi_3 > \phi_5 > 0$, and $\phi_2 > 0$ and $\phi_2 > \phi_4$. Thus, the upper branch is always minimized by $(w_1^0, w_1^1) = (0,1)$, which also satisfies the requirement $w_1^1/w_1^0 \geq p_0/(p_0 - p_2^*)$, and leads to $-\bar{\pi}\kappa_s\phi_2$. On the other hand, depending on whether $\phi_4 \geq 0$ or $\phi_4 < 0$, the middle branch is minimized by $(w_1^0, w_1^1) = (0,1)$ or $(w_1^0, w_1^1) = (0,1)$ (irrespective of the requirement $p_0/(p_0 - p_1^*) \leq w_1^1/w_1^0 < p_0/(p_0 - p_2^*)$), and leads to $-\bar{\pi}\kappa_s\phi_4$ or zero, respectively. It follows that the upper branch always dominates the middle branch. The same argument also applies to the lower branch, thereby guaranteeing the optimality of the upper branch. We therefore conclude that

$$
\begin{aligned}
J_1^3(p_0) =& 2(1+\rho)\pi\ell_s - 2\rho\pi\bar{\pi}\kappa_s(p_0+\eta_s-1) - 2\rho\bar{\pi}^2\kappa_s\left(p_0+\eta_s-1\right) - 2\bar{\pi}\kappa_s\left[p_0+\eta_s-1-\rho\bar{\pi}(1-p_0)(\eta_s-1)\right] \\
=& 2(1+\rho)\pi\ell_s - 2\bar{\pi}\kappa_s\left[(1+\rho)(p_0+\eta_s-1) - \rho\bar{\pi}(1-p_0)(\eta_s-1)\right]\,. \qquad \text{(D.78)}
\end{aligned}
$$

Finally, putting the three subproblems together, we obtain the optimal cost function and its corresponding optimal policy for $p_0 \geq p_2^*$ by comparing $J_1^1(p_0)$, $J_1^2(p_0)$ and $J_1^3(p_0)$. In particular, using simple algebra, the following conclusions can be made about the optimal warning policy in the first period:

(i) If $p_2^* \leq p_0 < p^{\diamond 1}$:

$$J_1^3(p_0) \leq J_1^1(p_0) \Rightarrow \min\{J_1^1(p_0), J_1^2(p_0), J_1^3(p_0)\} = J_1^2(p_0) \text{ or } J_1^3(p_0) \Rightarrow \boldsymbol{w}_1^\star = (0,1).$$

(ii) If $p_0 \geq p^{\diamond 1}$ and $\eta_s < 1$:

$$\eta_s < 1 \Rightarrow J_1^3(p_0) \leq J_1^2(p_0) \text{ and } J_1^2(p_0) < J_1^1(p_0) \Rightarrow \boldsymbol{w}_1^\star = (0,1).$$

(iii) If $p^{\diamond 1} \leq p_0 < p^{\diamond 2}$ and $\eta_s \geq 1$:

$$\eta_s \geq 1 \Rightarrow J_1^2(p_0) \leq J_1^3(p_0) \text{ and } J_1^2(p_0) < J_1^1(p_0) \Rightarrow \boldsymbol{w}_1^\star = (0,1).$$

(iv) If $p_0 \geq p^{\diamond 2}$ and $\eta_s \geq 1$:

$$\eta_s \geq 1 \Rightarrow J_1^2(p_0) \leq J_1^3(p_0) \text{ and } \begin{cases} J_1^2(p_0) < J_1^1(p_0), \text{ if } \rho < 2p_2^*/(\bar{\pi}(1-p_2^*)(\eta_s-1)) & \Rightarrow \boldsymbol{w}_1^\star = (0,1), \\ J_1^1(p_0) \leq J_1^2(p_0), \text{ if } \rho \geq 2p_2^*/(\bar{\pi}(1-p_2^*)(\eta_s-1)) & \Rightarrow \boldsymbol{w}_1^\star = (w_1^{0\star} > 0, w_1^{1\star} = 1)\,. \end{cases} \qquad \square$$