Engineering Social Learning: Information Design of Time-Locked Sales Campaigns for Online Platforms

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Many online platforms offer time-locked sales campaigns, whereby products are sold at fixed prices for pre-specified lengths of time. Platforms often display *some* information about previous customers' purchase decisions during campaigns. Using a dynamic Bayesian persuasion framework, we study how a revenue-maximizing platform should optimize its information policy for such a setting. We reformulate the platform's problem equivalently by reducing the dimensionality of its message space and *proprietary* history. Specifically, three messages suffice: a *neutral recommendation* that induces a customer to make her purchase decision according to her *private* signal about the product; and a *positive* (resp., *negative*) *recommendation* that induces her to purchase (resp., not purchase) by ignoring her signal. The platform's proprietary history can be represented by the *net purchase position*, a single-dimensional summary statistic that computes the cumulative difference between purchases and non-purchases made by customers having received the neutral recommendation. Subsequently, we establish structural properties of the optimal policy and uncover the platform's fundamental trade-off: long-term information (and revenue) generation versus short-term revenue extraction. Further, we propose and optimize over a class of heuristic policies. The optimal heuristic policy provides only neutral recommendations up to a *cut-off* customer and provides only positive or negative recommendations afterwards, with the recommendation being positive if and only if the net purchase position after the cut-off customer exceeds a *threshold*. This policy is easy-to-implement and numerically shown to perform well. Finally, we demonstrate the generality of our methodology and the robustness of our findings by relaxing some informational assumptions.

Key words: revenue management; dynamic information provision; Bayesian inference; recommendation

1. Introduction

The digital platform economy is bringing radical changes to how we socialize, trade, and exchange information. In contrast to businesses that operated in a traditional economy, online platforms such as Amazon, eBay and Groupon play the role of a market maker by establishing technology-based infrastructures that allow a large number of independent vendors to sell their products and services to a broad range of customers. While these platforms have limited control over tangible instruments such as prices, they can manifest their value by facilitating and controlling the information flow among market participants in real time at no cost, a point we aim to demonstrate in this paper.

To that end, we focus on one particular application innovated by these online platforms, which is known as a *time-locked sales campaign*. This selling mechanism allows third-party vendors to sell their products at a fixed price for a pre-specified length of time, e.g., from a few hours to a few days. As the campaign progresses, the platform displays the time remaining in the campaign. Platforms charge vendors a pre-negotiated commission, i.e., a fraction of the sales.[1] However, customers often face uncertainty about the value of products, deterring them from making the purchase. This uncertainty is particularly significant for new products, products with nuanced features, or products that cater to a niche market. The lack of physical showrooms in the online environment further exacerbates the issue. As such, platforms face the challenge of maximizing sales (i) in the presence of customers' skepticism about the products; (ii) within a limited amount of time; and (iii) without being able to modify the price. To overcome this challenge, platforms may offer prospective customers some information about historical purchase decisions made by previous customers. The format and granularity of information provided vary across different platforms. Motivated by the prevalence and heterogeneity of this practice, we study how a platform should design its information provision strategy for a time-locked sales campaign.

The time-locked sales campaign is best exemplified by Groupon's "Deals of the Day." Customers visiting a deal's web page are presented with some descriptive information about the product (including a verbal introduction, visual demonstration, existing product ratings, warranty terms, and vendor identity) as well as the price, which the vendor typically sets at a discounted level. While this information remains constant for the entire duration of the campaign, Groupon also dynamically updates and displays the time remaining to claim the deal as well as some, possibly vague, information about the up-to-date visits and sales data since the inception of the campaign.[2]

Other platforms adopt different strategies for their campaigns and vary in the granularity of information provided. For example, Amazon's "Deal of the Day" only displays how much time remains to claim the deal. On the other hand, Woot, a daily deals website owned by Amazon, discloses the full time-series sales data.[3] As such, the information provided by Groupon is more granular than that provided by Amazon but is less granular than that provided by Woot.

A fundamental premise behind the platform's strategy is the notion of *social learning* (Bikhchandani et al. 1992). Before making her purchase decision, each customer typically acquires a signal somewhat indicative of the product value (e.g., via word of mouth, product and brand research, or personal experience of using a similar item). While each customer's pre-purchase signal is her

[1] For example, Groupon, eBay and Amazon charge a commission rate of 8-20%, 2-12% and 5-20% respectively.
https://marketplace.groupon.com/support/solutions/articles/5000808521-deal-commission-rates
https://www.ebay.com/help/selling/fees-credits-invoices/selling-fees?id=4364
https://sellercentral.amazon.com/gp/help/external/200336920

[2] We provide in Figure E1 of Appendix E the screen shots of a particular Groupon deal for two time instances.

[3] We provide in Figures E2(a) and E2(b) of Appendix E the screen shots of an Amazon deal and a Woot deal, respectively. Amazon runs another time-locked sales campaign called "Lightning Deal," where customers are shown the percentage of inventory that has been sold. In this paper, we only consider campaigns without inventory constraint.

private information, her purchase decision may reflect the nature of her signal and help other customers revise their valuation of the product. Distinct from the canonical setting where each customer can observe previous customers' actions, however, a customer cannot directly observe others' purchase decisions, which are the platform's *proprietary* information. Thus, to influence upcoming customers' inferences about the product value and, subsequently, their purchase likelihood, the platform can leverage this information advantage and strategically provide some information about previous customers' purchase decisions. In essence, the platform can engineer customers' social learning through the design of its information policy.

In this paper (Section 3), we model the above setting using a dynamic Bayesian persuasion framework (e.g., Kremer et al. 2014), in which the platform designs and commits to a policy that maps its proprietary *history* (i.e., previous customers' purchase decisions and messages received from the platform) to a set of messages (e.g., summary sales statistics, visual signs on the product webpage) dynamically displayed to customers. The platform's problem is then to search for the optimal policy, among all possible ones, to maximize its revenue.

The platform's problem stated above is abstract, because a plethora of messages can be used with possible randomization and dynamic adjustment. To establish an equivalent yet analytically tractable reformulation, we develop (in Section 4) an approach to reduce the dimensionality of the platform's problem, which consists of (i) shrinking the cardinality of the message space to that of a customer's purchase strategies, and (ii) simplifying the representation of the platform's proprietary history through lower-dimensional statistics that summarize the payoff-relevant information. The first reduction allows the platform to restrict its search for the optimal policy within the class of *recommendation policies* that consist of only three messages: a *neutral recommendation* that induces the customer to make the purchase if and only if she receives an optimistic pre-purchase signal; and two *affirmative recommendations*—a *positive recommendation* or a *negative recommendation*—that induce the customer to purchase or not, respectively, regardless of her pre-purchase signal. The credibility of these recommendation messages is warranted by a set of incentive compatibility (IC) constraints for each customer. The second reduction allows the platform to only keep track of a one-dimensional summary statistic, which we term the platform's *net purchase position.* The net purchase position computes the difference between the cumulative numbers of purchases and non-purchases made by customers who have been offered the neutral recommendation. The public belief about the platform's net purchase position is regulated by a martingale constraint. The interaction between the IC constraints and the martingale constraint poses a novel analytical challenge yet reveals important economic insights.

Combining the two reductions mentioned above, we are able to represent the platform's information policy as a mapping, for each customer, from the net purchase position to probability distributions over the three recommendations. Using this reformulation, we solve (in Section 4) for the platform's optimal policy, which turns out to involve a significant level of randomization between affirmative and neutral recommendations at many net purchase positions for majority of customers. Yet, it features two structural patterns: (i) neutral recommendations are predominantly concentrated in the early phase of the campaign, and (ii) between the two affirmative recommendations, positive recommendations are provided at higher net purchase positions than negative recommendations are. These structural properties reveal the fundamental trade-off faced by the platform between long-term information (and, hence, revenue) generation and short-term revenue extraction. In essence, we show that only the neutral recommendation is capable of generating inferential information (about the product value) to the platform, which is beneficial to future revenue extraction. However, offering the neutral recommendation too intensively may restrain the platform's ability to make positive recommendations, resulting in a revenue loss.

For ease of implementation, we develop a heuristic motivated by the above structural properties of the optimal policy. To do so, we restrict (in Section 5) the general recommendation policies to the class of *NA-sequencing* policies. These policies provide each customer t either (i) the neutral recommendation regardless of the platform's net purchase position, or (ii) an affirmative recommendation that can be randomized between the positive or negative recommendation depending on the net purchase position (hence the name NA-sequencing). We find that the optimal NA-sequencing policy front-loads all neutral recommendations and then back-loads the affirmation recommendations. More precisely, this policy provides neutral recommendations to customers arriving up to a *cut-off* customer and provides affirmative recommendations to customers arriving afterwards. The affirmative recommendation is positive if and only if the net purchase position achieved immediately after the cut-off customer exceeds a *threshold*. Such a *cutoff-threshold* structure is entirely consistent with the structural properties of the optimal policy identified above, and resembles the information provision strategy employed by Groupon's "Deals of the Day." We fully characterize both the cut-off customer and the threshold. As demonstrated by an extensive numerical study, the optimal NA-sequencing policy performs relatively well.

Finally, we extend our model and analysis in two separate directions (in Section 6). First, we relax the informational assumption regarding the customers' knowledge about their order of arrival. Then, we incorporate customers' post-purchase signals about the product value to capture settings, in which customers may post product reviews after they purchase and consume the product.[4] We show that the method we developed for the base model (i.e., the reductions of the platform's message space and proprietary history) remains applicable to these two extensions, and that the structure of both the optimal policy and the optimal NA-sequencing policy is preserved.

[4] Such practices are more common for digital goods that can be consumed and reviewed almost instantly after the purchase. We provide in Figure E3 of Appendix E the screen shot of an example on stacksocial.com.

2. Related Literature

Three streams of literature informed and inspired our research: revenue management, social learning, and information design. Here, we briefly review and discuss our contribution to each stream.

The classical revenue management research has focused predominantly on the pricing and inventory instruments as two key levers to maximizing firms' profit (see Talluri and Van Ryzin 2006, Özer and Phillips 2012, for a comprehensive survey). A typical premise therein is that customers know their private types, such as valuation of the product (whereas the seller does not), and hence they have an information advantage over the seller, who can then leverage pricing and inventory-related decisions to screen customers' private valuation (e.g., Courty and Hao 2000, Bergemann et al. 2020). As illustrated in the introduction, online retail platforms may have reversed such an information advantage by collecting massive information about the supply and demand, posing new research challenges and opportunities regarding how to best utilize this proprietary information for revenue management. Drakopoulos et al. (2021) and Lingenbrink and Iyer (2018) are among the first researchers to study the use of information provision as a novel instrument in such settings (see also Johari et al. 2019, for an application in two-sided markets). They demonstrate that obfuscating inventory and demand information can create availability risk and competition among buyers, who will then be induced to make early purchases. Following the same school of thought but in a completely different setting, we study how an online platform can provide information about its historical sales data to induce prospective customers' purchases.

The fundamental linkage between the prior and prospective customers is the notion of social learning (see Chamley 2004, for a comprehensive survey). In retail and service industries, such social learning takes the form of customer reviews, product ratings, and historical sales information. Thus, a burgeoning literature has emerged to examine various implications of social learning on revenue management theory and practice, ranging from pricing policies (Yu et al. 2015, Crapis et al. 2016, Papanastasiou and Savva 2016, Ifrach et al. 2019), control of service rate (Veeraraghavan and Debo 2009), and inventory and product line strategies (Hu et al. 2015) to product design and introduction (Feldman et al. 2018, Araman and Caldentey 2016). This literature mainly focuses on the setting in which both the seller and customers have equal and *full* access to the information generated by the previous customers. Going beyond such a full-disclosure setting, Besbes and Scarsini (2018), Acemoglu et al. (2019) and Garg and Johari (2019) investigate whether and how fast social learning can reveal the true value of a product by using other information provision rules (e.g., summary statistics of the past reviews/ratings). We share with all the authors above the premise that social learning plays an instrumental role in resolving customers' uncertainty about the product or service. Of particular interest to our research are the seminal works by Banerjee

(1992) and Bikhchandani et al. (1992), whose social learning framework forms the building block of our model. However, unlike in these frameworks, customers in our setting cannot directly observe the actions taken by previous customers; rather, their observation is moderated by the platform, which calls for an information design framework.

Methodologically, our modeling framework belongs and contributes to the fast-growing area of research on information design pioneered by Kamenica and Gentzkow (2011); also see Candogan (2020) for a comprehensive survey of its application in operations research. Most recent research, including ours, aims to extend Kamenica and Gentzkow's (2011) static setting to dynamic ones for a variety of application contexts, such as Renault et al. (2017), Ely (2017), Che and Hörner (2018) and Alizamir et al. (2020), to name a few. Unlike our setting with *short-lived* customers, these papers assume that receivers are *long-lived* and can observe the sender's messages previously provided. From this perspective, our work is more closely related to the stream of research initiated by Kremer et al. (2014) and followed by Kleinberg and Slivkins (2017), Papanastasiou et al. (2018), Immorlica et al. (2018) and Immorlica et al. (2019) (see Bimpikis and Papanastasiou 2019, for a review). The central theme in this stream of research is the design of dynamic information policies to incentivize exploration (and to discourage exploitation) in the multi-armed bandit setting, where the sender has proprietary observation of a *post-action* signal (e.g., value of an arm) generated by each receiver's choice. Our model setup forms an almost "dual" setting, where each receiver privately observes a *pre-action* signal and the sender can only observe receivers' actions.[5]

3. Model

In a time-locked sales campaign, an online *platform* (e.g., Groupon) allows a third-party vendor to sell a product to *customers* sequentially visiting the platform over a finite time horizon of length T. The vendor sets a constant price, denoted as p, for the entire selling horizon; thus, the platform has no control over the price. The platform profits from each sale through a pre-specified commission rate, a fraction of the selling price, which we normalize to 1 without loss of generality. The vendor receives the rest of the payments and is responsible for fulfilling the order. Hence, the vendor's revenue is proportional to the platform's revenue. To maximize its revenue, the platform's goal is to generate as many sales as possible. In this paper, we take the platform's perspective and focus on the strategic interaction between the platform and customers.

When selling an innovative or a relatively nuanced product, an online time-locked sales campaign typically faces the challenge of resolving customers' uncertainty about the product's value at the time of purchase (before consumption) primarily due to the lack of a showroom for customers

[5] Immorlica et al. (2019) also consider receivers with private types (e.g., preferences), which, however, are independent of the intrinsic uncertainty (e.g., product value).

to examine or experience the product. For simplicity, the customer enjoys a consumption utility, which we normalize to 1, from a valuable or suitable product; otherwise, the customer enjoys zero consumption utility. We thus model the customer's a priori uncertain utility from consuming the product as a binary random variable $V \in \{0,1\}$. The customer either purchases the product and receives the payoff, $V - p$; or she does not purchase the product and receives zero payoff. To rule out trivial cases, we normalize the price to $p \in (0,1)$ accordingly.

A priori, neither the platform nor the customers know the exact value of V. At the beginning of the selling horizon ($t = 0$), the vendor publishes on the platform some general information about the product (e.g., verbal description, virtual demonstration or even past customers' reviews and testimony), and this information remains constant throughout the selling horizon. Based on this initial public information, the platform and all customers form a common prior expectation $\mathbb{E}[V] = v_0 \in [0,1]$, i.e., $V = 1$ with probability v_0 and $V = 0$ with the complimentary probability $1 - v_0$. In summary, the three exogenous parameters (T, p, v_0) define a time-locked sales campaign and are common knowledge to the platform and all customers.

3.1. Dynamics and information

We divide the selling horizon into discrete time periods indexing the customer's order of arrival. That is, customer $t \in \{1, \ldots, T\}$ arriving in time period t, is the t-th customer visiting the platform. Following the literature (e.g., Banerjee 1992, Bikhchandani et al. 1992, Kremer et al. 2014, Papanastasiou et al. 2018), we start by assuming that customers know their exact order of arrival, i.e., customer t knows that she is the t-th customer. Since the sales campaign is time-locked (i.e., the platform displays the time remaining in the campaign or equivalently the duration of the campaign that has elapsed), this assumption is equivalent to deterministic customer arrivals.[6] In Section 6.1, we relax this informational assumption and show the robustness of our analysis to customers' imprecise knowledge about their order of arrival.

Besides the general information about the product (represented by v_0), customers typically acquire idiosyncratic knowledge and information about the product before making their purchase decisions (e.g., via word of mouth, product research, or experience with similar products). Formally, we represent such knowledge and information acquired by customer t as a pre-purchase binary signal S_t taking symbolic values 1 and -1, which represent an *optimistic* and *pessimistic* assessment, respectively. The signal S_t is customer t's private information, unobservable to the platform and

[6] In practice, many platforms offer to inform customers about their order of arrival by displaying the up-to-date count of visits. Kremer et al. (2014) provide a theoretical justification for such practices: customers would be better off and, hence, have incentives to find their order of arrival; the platform may choose to implement the information policy that assumes customers' knowledge of their order of arrival so as to eliminate such incentives. In our setting, a similar rationale holds, as will be discussed in Section 6.1.

other customers. Nonetheless, the distribution of S_t is public knowledge. Specifically, following Bikhchandani et al. (1992), we model S_t as an identical and independent binary random variable with the following distribution conditional on the value V:

$$\mathbb{P}\left[S_t = 1 \mid V = 1\right] = \mathbb{P}\left[S_t = -1 \mid V = 0\right] = q \in (1/2, 1], \tag{1}$$

where q is referred to as the signal's *precision* and measures how indicative a customer's pre-purchase signal S_t is of the value V.[7] According to Bayes' Rule, an optimistic signal improves the customer's expectation above v_0 (i.e., $\mathbb{E}\left[V \mid S_t = 1\right] = \frac{v_0 q}{v_0 q + (1-v_0)(1-q)} > v_0$) and a pessimistic signal lowers her expectation below v_0 (i.e., $\mathbb{E}\left[V \mid S_t = -1\right] = \frac{v_0(1-q)}{v_0(1-q)+(1-v_0)q} < v_0$). Notably, this information structure is consistent with the social learning literature (e.g., Banerjee 1992, Bikhchandani et al. 1992): if all customers' purchase decisions were publicly observable, social learning emerges because a prospective customer could potentially infer previous customers' private pre-purchase signals (and hence revise her product evaluation) from their purchase decisions.

In practice, many time-locked sales campaigns last for a short time period (e.g., from a few hours to a few days), so customers who have made purchases during the course of campaigns are unable to receive or consume the product in time and provide their product reviews to the platform by the end of campaigns.[8] In Section 6.2, we consider campaigns whereby customers can consume and review products (e.g., digital goods) instantly after the purchase and extend our model to include a public *post-purchase* signal about the product value.

While the customer's private pre-purchase signal is unobservable to the platform, the platform has an information advantage over customers in that the platform observes the purchase decisions of all customers who have visited the platform as well as all information provided to them. Hence, the platform can leverage this advantage to strategically release *some* of its proprietary information to upcoming customers. Formally, let *message* m_t denote the information that the platform provides to customer t. Together with v_0 and S_t, customer t maximizes her expected payoff by deciding whether or not to purchase. We denote customer t's decision as $a_t \in \{1, -1\}$, where $a_t = 1$ and $a_t = -1$ represent a purchase and a non-purchase, respectively. Thus, we say that customer t *follows* her private signal S_t to make the purchase decision if $a_t = S_t$, but it is possible for $a_t \neq S_t$, i.e., she dismisses her private signal when making her purchase decision. Customer t's purchase decision a_t and whatever message m_t she receives are the platform's proprietary information, unobservable to other customers. We denote the platform's *proprietary history* up to customer t as $H_t := \{(m_s, a_s) : s < t\}$, with the convention that $H_1 = \emptyset$, and the space of all possible proprietary histories as $\mathcal{H} := \{H_t : t = 1, 2, \cdots, T\}$.

[7] The signal's precision depends on the quality of the information channel, through which customers acquire their pre-purchase signals, and, hence, is independent of the product value.

[8] Product reviews collected prior to a campaign, if any, remain constant during its selling horizon and have already been captured by the prior expectation v_0.

3.2. The platform's information policy

For generality, we allow the platform to dynamically adjust the information provided to different customers, possibly in a randomized fashion, and do not impose any restriction on the format of such information (e.g., verbal or visual messages are allowed). This allows our framework to capture a wide range of applications including those illustrated in the Introduction (see also Figures E1 and E2). Specifically, the platform designs and commits to[9] an *information policy*, denoted as $(\sigma, \mathcal{M})$, which maps each proprietary up-to-date history $H_t \in \mathcal{H}$ to a probability distribution $\sigma(.|H_t) \in \Delta(\mathcal{M})$ over message space $\mathcal{M}$. The information provided to customer t is then a particular message $m_t \in \mathcal{M}$ drawn according to probability distribution $\sigma(.|H_t)$. For notational simplicity, we sometimes make the message space $\mathcal{M}$ implicit and abbreviate $(\sigma, \mathcal{M})$ as σ.

It is worthwhile to describe two commonly used naïve policies as polar examples of information policies. At one extreme, a *full-disclosure* policy, which resembles the strategy adopted by Woot, reveals to each upcoming customer all the previous customers' purchase decisions; that is, $\mathcal{M} = \mathcal{H}$ and $\sigma(H_t|H_t) = 1$ for all H_t and all t and, hence, $m_t = H_t = (a_1, \cdots, a_{t-1})$ by definition. At the other extreme, a *no-disclosure* policy, which resembles the strategy adopted by Amazon's "Deal of the Day," completely conceals the platform's proprietary history by providing a constant message throughout the campaign; that is, $\mathcal{M}$ consists of a singleton, say, m_0, and $\sigma(m_0|H_t) = 1$. Thus, no inferential information (about the product value) is provided to customers. We study these two naïve policies in Appendix D.

3.3. Customers' and the platform's objectives

The platform provides message $m_t \in \mathcal{M}$ to customer t upon her arrival. Together with the prior expectation v_0 and her private signal S_t, customer t updates her belief about the product value V, and purchases the product if and only if her updated expectation is not smaller than the price:[10]

$$a_t = 1 \text{ if and only if } \mathbb{E}[V \mid S_t, m_t, \sigma] \geq p, \quad \text{for all } t. \tag{2}$$

Under a given policy σ, the actions and beliefs prescribed by (2) constitute a Perfect Bayesian Equilibrium among all customers, who are *short-lived* because their payoffs do not depend on

[9] The platform's commitment means that the platform designs its information policy upfront at the onset of the campaign prior to the realization of its proprietary history, and abides by the policy as its proprietary history unfolds. As articulated in the literature (see Section 2), this commitment assumption is particularly appropriate for online platforms that have reputational concerns and have to automate and computerize the implementation of their information provision for a large number of products and to a widespread audience in real time. Further, the platform is always better off committing whenever possible (Best and Quigley 2020). Indeed, such commitment establishes the meaning of the platform's message, which would otherwise be cheap talk lacking communicative power (Kamenica and Gentzkow 2011).

[10] Throughout the paper, all probabilities or expectations are conditional on the publicly known model primitive parameters $\{T, p, q, v_0\}$, but such dependence is kept implicit for expositional simplicity.

subsequent customers' actions. Following the convention in the literature, we assume that when the customer is indifferent between two actions, she purchases the product.

In anticipation of each customer's response provided in (2), the platform's objective is to maximize its total expected sales over the entire selling horizon, by designing an information policy σ among all policies. That is, the platform's problem can be formulated as

$$\pi^\star := \max_{\sigma} p\, \mathbb{E}\left[\sum_{t=1}^{T} \mathbb{1}[a_t = 1] \,\middle|\, \sigma\right], \quad \text{subject to (2).} \tag{3}$$

In summary, we illustrate the sequence of events in Figure 1. Given the campaign characteristics (T, p, v_0), the platform designs an information policy σ according to (3). Customer t arrives with her private signal S_t, receives the message m_t from the platform, and then makes her purchase decision a_t according to (2). The platform updates its proprietary history to $H_{t+1} = H_t \cup \{m_t, a_t\}$ and time moves on to the next customer $t+1$. The campaign concludes with customer T.

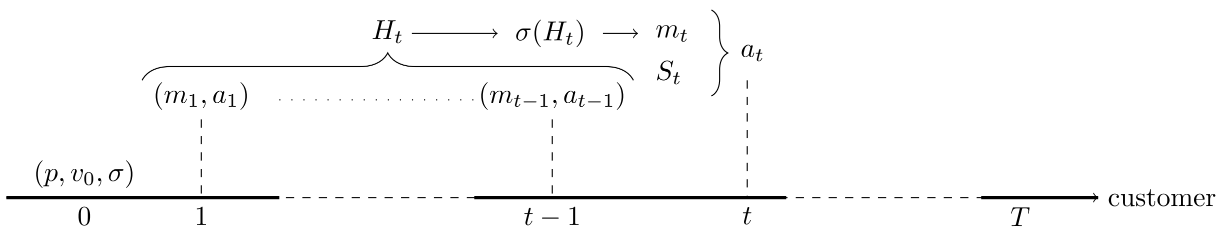


Figure 1 Sequence of events.

4. Reformulation and Solution of the Platform's Problem

In this section, we establish an equivalent, analytically tractable reformulation of the platform's problem (3), which, in its current form, is challenging to analyze due to its full generality. Our methodological innovation here consists in shrinking the dimension of the message space (through *recommendation policies*) and simplifying the representation of the platform's proprietary history (through *net purchase position*). As a result, we identify structural properties of the optimal information policy that reveal important economic insights and trade-offs faced by the platform.

4.1. Recommendation policies and net purchase positions

To reduce the dimension of the message space $\mathcal{M}$, we first characterize customer t's purchase decision specified in (2) in terms of her *interim expectation* $\mathbb{E}[V \mid m_t, \sigma]$. This interim expectation represents customer t's perceived product value based only on the information m_t the platform provides (that is, without conditioning on her private signal S_t).

PROPOSITION 1 **(Customer's purchase decision)**. *Customer t's purchase decision is*

$$a_t = \begin{cases} 1, & \text{if } \mathbb{E}[V \mid m_t, \sigma] \in [v^{**}, 1], \\ S_t, & \text{if } \mathbb{E}[V \mid m_t, \sigma] \in [v^{*}, v^{**}], \\ -1, & \text{if } \mathbb{E}[V \mid m_t, \sigma] \in [0, v^{*}], \end{cases} \tag{4}$$

where thresholds v^* and v^{**} are given by

$$v^* := \frac{p(1-q)}{p(1-q)+(1-p)q} < p < v^{**} := \frac{pq}{pq+(1-p)(1-q)}. \tag{5}$$

Proposition 1 follows from Bayes' rule. When the information provided by the platform does not adequately resolve the customer's uncertainty (i.e., when her interim expectation is between v^* and v^{**}), the customer follows her private signal S_t to make the purchase decision (i.e., $a_t = S_t$). In contrast, when the information provided by the platform leads to a sufficiently conclusive belief (i.e., her interim expectation is either above v^{**} or below v^*), the customer ignores her private signal S_t and makes the purchase decision accordingly: she makes the purchase (resp., no purchase) if her interim expectation is above v^{**} (resp., below v^*).

As a result, the platform's information design problem has a trivial solution when the prior expectation $v_0 \notin [v^*, v^{**})$. In this case, the platform can simply provide a constant message or no information throughout the campaign. Then, all customers make the purchase (resp., no purchase) if $v_0 \geq v^{**}$ (resp., $v_0 < v^{**}$); see Corollary A1 in Appendix A for the formal statement. For this reason, the rest of paper will focus only on the parametric region such that $v_0 \in [v^*, v^{**})$, even though the solution methodology developed below equally applies to the complementary case.

Proposition 1 prescribes three purchasing strategies for the customer as given by (4). As such, we can equate any platform's message to one of the three *incentive compatible recommendations*, symbolically denoted as $\{1, 0, -1\}$: a *positive recommendation* $m_t = 1$ that induces customer t to purchase the product regardless of her private signal ($a_t = 1$), i.e., to form an interim expectation

$$\mathbb{E}[V \mid m_t = 1, \sigma] \geq v^{**}, \quad \text{for all } t; \tag{IC$_1^\sigma$}$$

a *neutral recommendation* $m_t = 0$ that induces customer t to follow her private signal to make the purchase decision ($a_t = S_t$), i.e., to form an interim expectation

$$v^* \leq \mathbb{E}[V \mid m_t = 0, \sigma] \leq v^{**}, \quad \text{for all } t; \tag{IC$_0^\sigma$}$$

and a *negative recommendation* $m_t = -1$ that induces customer t not to purchase the product regardless of her private signal ($a_t = -1$), i.e., to form an interim expectation

$$\mathbb{E}[V \mid m_t = -1, \sigma] \leq v^*, \quad \text{for all } t. \tag{IC$_{-1}^\sigma$}$$

Any incentive compatible information policy $(\sigma, \mathcal{M})$ with $\mathcal{M} := \{1, 0, -1\}$ is called a *recommendation policy*. Because, following either a positive or negative recommendation, a customer makes a definitive purchase decision irrespective of her private signal, we refer to a positive or negative recommendation as an *affirmative recommendation*. The three *incentive compatibility* (IC) constraints (IC_1^σ)-(IC_{-1}^σ) ensure that each recommendation induces a customer's interim expectation that is consistent with the purchase decision intended by the platform. Thus, the IC constraints establish the credibility and meaning of the platform's recommendations.

Our next proposition formally establishes that the platform can restrict the search for its optimal information policy within the class of recommendation policies.

Proposition 2 **(Sufficiency of recommendation policy)**. *For any information policy, there exists a recommendation policy that induces the same purchase decisions from all customers and the same expected revenue for the platform.*

Proposition 2 drastically reduces the dimension of the message space in the platform's information provision policy. In spirit, this result is akin to the Revelation Principle in classical mechanism design problems, which allows the principal to reduce the dimension of mechanism space to that of the agent's private information. In fact, the sufficiency of recommendation policies is known for *static* Bayesian persuasion games (Kamenica and Gentzkow 2011, Proposition 1), whereby one can establish the payoff equivalence of recommendation policies through an argument similar to the Revelation Principle. In our *dynamic* setting, we must additionally show that the reduction to recommendation policies does not diminish the richness of the platform's proprietary history, from which the platform can potentially learn about the product value; see the proof of Proposition 2 (esp., Lemma A1) in Appendix A.

It is worth pointing out that the particular specification of the message space as $\mathcal{M} = \{1, 0, -1\}$ is only for symbolic purposes and notational convenience. Such a recommendation policy can be implemented in practice by using messages in natural language and defining upfront the rule to generate these messages. For instance, message 1 can be framed as encouraging words such as "must buy"; message 0 can be framed as a modest suggestion such as "worth a try"; and message -1 can simply be silence. Furthermore, because customers know their order of arrival, the timing of a message is part of the message: verbally or visually identical messages that are displayed to different customers may be interpreted as different recommendations. Namely, IC constraints (IC_1^σ)-(IC_{-1}^σ) entail different regulations on the recommendation policy for different customers.

Our next step toward obtaining the equivalent formulation for the platform's problem is to identify an efficient representation of the platform's proprietary history, which is a complex object with increasing dimension as the campaign progresses. The following proposition achieves this goal.

PROPOSITION 3 **(Sufficiency of net purchase position)**. *The platform's product valuation conditional on proprietary history H_t generated by recommendation policy σ is*

$$\mathbb{E}\left[V \mid H_t, \sigma\right] = \frac{v_0}{v_0 + (1-v_0)(\frac{1-q}{q})^{N(H_t)}}, \tag{6}$$

where the platform's net purchase position *up to customer t is defined as*[11]

$$N(H_t) := \sum_{(m_s, a_s) \in H_t} a_s \left(1 - |m_s|\right) \in \left\{-(t-1), \cdots, 0, \cdots, t-1\right\}. \tag{7}$$

The significance of Proposition 3 lies in showing that the payoff-relevant information embedded in the platform's proprietary history is succinctly captured by a single-dimensional summary statistic of that history; that is, the net purchase position $N(H_t)$ as defined in (7). The net purchase position computes the difference between the cumulative number of purchases and non-purchases made by customers who have thus far been offered a neutral recommendation. That is, any two proprietary histories H_t and $H'_{t'}$, possibly of different length or even generated by different recommendation policies (say σ and σ'), would induce the same product valuation as long as they yield the same net purchase positions, i.e., $\mathbb{E}\left[V \mid H_t, \sigma\right] = \mathbb{E}\left[V \mid H'_{t'}, \sigma'\right]$, if $N(H_t) = N(H'_{t'})$. Therefore, the payoff-relevant information embedded in H_t is completely summarized in $N(H_t)$. For notational simplicity, we denote the platform's conditional expectation characterized by (6) as

$$v_n := \mathbb{E}\left[V \mid N(H_t) = n, \sigma\right] = \frac{v_0}{v_0 + (1-v_0)(\frac{1-q}{q})^{n}}, \quad \text{for any integer } n. \tag{8}$$

Recall that the platform's recommendations are incentive compatible: upon an affirmative recommendation (positive or negative), the customer finds it optimal to follow the recommendation and makes a purchase decision by dismissing her private signal. Thus, the customer's purchase decision following an affirmative recommendation provides no new information for the platform to update its belief about the product value V. In contrast, upon a neutral recommendation, the customer makes a purchase decision according to her private signal (i.e., $a_t = S_t$). Thus, the customer's purchase decision following a neutral recommendation reveals the customer's private signal, which the platform can use to revise its proprietary belief about the value of the product. Therefore, the platform's product valuation given in (6) or (8) increases in the net purchase position. In summary, affirmative recommendations have only fiscal value but no informational value for the platform, whereas neutral recommendations carry both informational and fiscal values.

[11] We adopt the convention that empty summation equals 0.

4.2. Equivalent reformulation of the platform's problem

Using Propositions 2 and 3, we can now represent a recommendation policy σ for each customer t as a mapping, denoted as r_t, from the integer space of net purchase positions n to the 2-simplex of distributions over the recommendation messages $\mathcal{M}=\{1,0,-1\}$. That is, let $r_t^i(n)=\mathbb{P}[m_t=i \mid N(H_t)=n]$ be the probability that the platform offers recommendation $i\in\{1,0,-1\}$ to customer t given net purchase position n:

$$r_t(n)=\left(r_t^1(n),r_t^0(n),r_t^{-1}(n)\right)\in\mathbb{R}_+^3 \text{ with } \sum_{i\in\{1,0,-1\}} r_t^i(n)=1, \tag{R}$$

for $n=-(T-1),\dots,0,\dots,T-1$, and $t=1,\cdots,T$.

While the platform observes its net purchase position, an arriving customer cannot. Nonetheless, given the recommendation policy $r:=\{r_t:t=1,\cdots,T\}$ committed by the platform, each customer t can form a belief (i.e., a probability distribution) about the net purchase position, which we characterize in the following proposition.

Proposition 4 **(Belief about net purchase position)**. *Given a recommendation policy r, let $z_t(n):=\mathbb{P}[N(H_t)=n \mid r]$ represent customer t's belief that the platform's net purchase position is n. Then, this belief can be obtained recursively by*

$$z_t(n)=\left(1-r_{t-1}^0(n)\right)z_{t-1}(n)+u_{n-1}r_{t-1}^0(n-1)z_{t-1}(n-1)+(1-u_{n+1})r_{t-1}^0(n+1)z_{t-1}(n+1), \tag{N}$$
$$\textit{for } t=2,\cdots,T, \textit{ and } n=-(T-1),\cdots,T-1, \textit{ with } z_t(T)=z_t(-T)=0 \textit{ and } z_1(n)=\mathbb{1}[n=0],$$

where $u_n:=\mathbb{P}[S_t=1 \mid N(H_t)=n]=qv_n+(1-q)(1-v_n)$.

Equation (N) shows how the public belief about the platform's net purchase position, $z:=\{z_t:t=1,\cdots,T\}$, evolves over time. Specifically, the platform reaches net purchase position n in period t from three possible positions faced by the previous customer $t-1$, corresponding to the three terms in (N). First, the current position can remain the same as the previous one at n, if an affirmative recommendation (regardless positive or negative) is provided to customer $t-1$ (because the net purchase position only accounts for purchase decisions under neutral recommendations). For customer $t-1$, the position was n with probability $z_{t-1}(n)$, and the platform provided an affirmative recommendation with probability $r_{t-1}^1(n)+r_{t-1}^{-1}(n)=1-r_{t-1}^0(n)$, leading to the first term in (N). Second, the current position can increase to n from the previous one at $n-1$, if the platform offered a neutral recommendation to customer $t-1$ and the customer made a purchase. For customer $t-1$, the position was $n-1$ with probability $z_{t-1}(n-1)$, the platform provided a neutral recommendation with probability $r_{t-1}^0(n-1)$, and the customer made a purchase (i.e., received an optimistic signal $S_{t-1}=1$) with probability u_{n-1}, leading to the second term in (N).

The third term follows from a similar argument. In essence, the platform's net purchase position evolves according to a (non-homogeneous) random walk, whose transition probability is regulated only by the platform's *neutral* recommendation policy.

Combining the platform's product valuation in Proposition 3 and the public belief about its net purchase position in Proposition 4, customer t makes a Bayesian inference about the product's value V from the platform's recommendation. More precisely, upon receiving recommendation $m_t = i \in \{1, 0, -1\}$, customer t knows that position n is pooled into message $m_t = i$ with probability $r_t^i(n)$ and the platform is at position n with probability $z_t(n)$. Her interim expectation is then $\mathbb{E}[V \mid m_t = i, r] = \frac{\sum_n v_n z_t(n) r_t^i(n)}{\sum_n z_t(n) r_t^i(n)}$. Thus, constraints ($\mathrm{IC}_1^\sigma$)-($\mathrm{IC}_{-1}^\sigma$) can be equivalently expressed as

$$\sum_n (v_n - v^{**}) z_t(n) r_t^1(n) \geq 0, \tag{IC_1^r}$$

$$\sum_n (v_n - v^{**}) z_t(n) r_t^0(n) \leq 0, \quad \sum_n (v_n - v^{*}) z_t(n) r_t^0(n) \geq 0, \quad \text{and} \tag{IC_0^r}$$

$$\sum_n (v_n - v^{*}) z_t(n) r_t^{-1}(n) \leq 0, \quad \text{for all } t. \tag{IC_{-1}^r}$$

The IC constraints (IC_1^r)-(IC_{-1}^r) ensure each customer follows the platform's recommendations to make their purchase decisions. As such, customer t makes a purchase either upon receiving a positive recommendation, which occurs with probability $\sum_n z_t(n) r_t^1(n)$, or upon receiving a neutral recommendation and an optimistic private signal $S_t = 1$, which occurs with probability $\sum_n z_t(n) u_n r_t^0(n)$. Therefore, the platform's expected revenue from customer t is $p\mathbb{E}[\mathbb{1}[a_t = 1] \mid r] = p \sum_n z_t(n) \left[r_t^1(n) + u_n r_t^0(n)\right]$. Taken together, the platform's problem in (3) can be equivalently reformulated as:

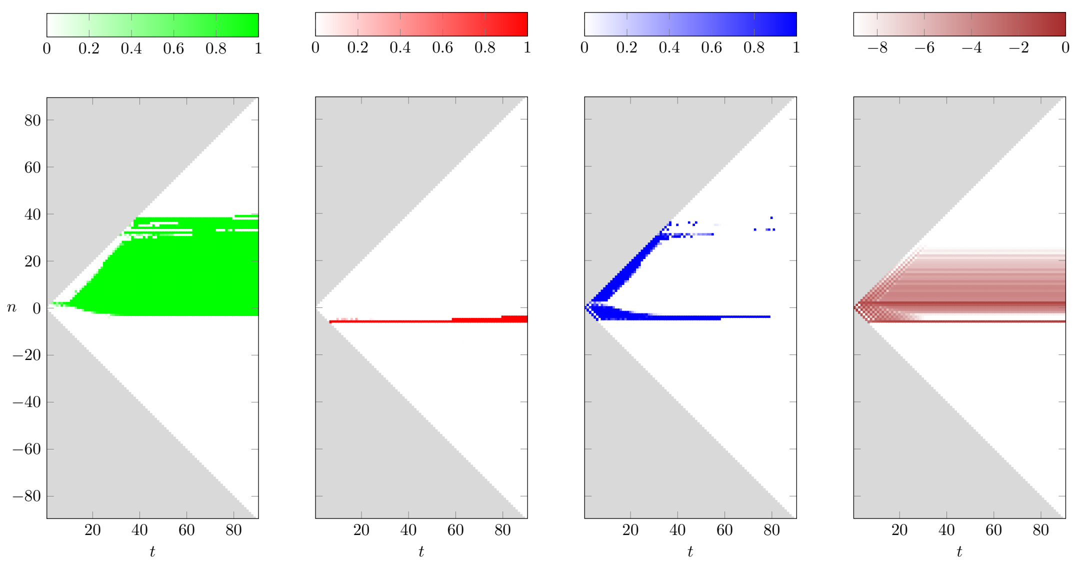
$$\pi^\star := \max_{r,z} \quad p \sum_{t,n} z_t(n) \left[r_t^1(n) + u_n r_t^0(n)\right], \quad \text{subject to (R), (N), } (\mathrm{IC}_1^r), (\mathrm{IC}_0^r), \text{ and } (\mathrm{IC}_{-1}^r), \tag{P}$$

whose solution, denoted as $(r^\star, z^\star)$, then prescribes the platform's optimal recommendation (and, hence, information) policy and the induced public belief about its net purchase position, respectively.

While the platform's problem in formulation (P) significantly reduces the dimensionality of its original formulation in (3), it remains challenging to obtain a complete analytical characterization of the optimal policy. First, formulation (P) is quadratic and, hence, nonlinear in decision variables r and z. To tackle this challenge, we recognize that the decision variables $r_t^i(n)$ and $z_t(n)$ enter the objective and constraints of formulation (P) together as the product $r_t^i(n) z_t(n)$, representing the joint probability that customer t faces net purchase position n *and* is recommended message $i \in \{1, 0, -1\}$. Indeed, as shown by Proposition A1 in Appendix A, treating $r_t^i(n) z_t(n)$ as a single decision variable renders problem (P) a linear program (LP), which can be efficiently solved using existing LP algorithms. As a greater challenge for solving (P), the global belief evolution represented by (N), which connects consecutive customers, interacts with the local constraints (IC_1^r)-(IC_{-1}^r), which are separable across customers. In the next subsection, we leverage the numerical solution of the LP formulation derived from Proposition A1 to identify and establish some structural properties for the platform's problem (P).

4.3. Structural properties of the optimal information policy

For a specific example, Figure 2 illustrates the platform's optimal recommendation policy $r^\star$ and the induced public belief $z^\star$ in heat-map view. The horizontal and vertical coordinates represent customers' order of arrival t and the platform's net purchase position n, respectively. The first three heat maps, Figures 2(a)-2(b), represent the three probabilities $r_t^{i\star}(n) \in [0,1]$ for $i = \pm 1, 0$; the last heat map, Figure 2(d), represents $z_t^\star(n)$ in its logarithmic magnitude.



(a) $r_t^{\star 1}(n)$ (b) $r_t^{\star -1}(n)$ (c) $r_t^{\star 0}(n)$ (d) $\log z_t^\star(n)$

Figure 2 The optimal recommendation policy $r^\star$ and the induced public belief about the net purchase position $z^\star$ for $T = 90, q = .7, p = .7$ and $v_0 = .55 \in [v^*, v^{**}) = [.5, .8448)$. For each subfigure, the color gradient at location (t, n) represents the magnitude of the corresponding quantity, with a darker color gradient corresponding to higher magnitude. The area shaded in gray corresponds to unreachable positions $n \notin (-t, t)$, and $r_t^{\star i}(n)$ $(i = 0, \pm 1)$ is not defined (and is colored white) if $z_t^\star(n) = 0$.

We first examine the pattern of affirmative recommendations in the optimal policy. Figures 2(a) and 2(b) suggest that the positive recommendations are provided to each customer at *higher* net purchase positions than are the negative ones. Indeed, the proposition below analytically establishes this structural property in general.

PROPOSITION 5 **(Structural property of affirmative recommendations)**. *Under an optimal recommendation policy $r^\star$, there exists an integer threshold $n_t^\star \in [-(t-1), t-1]$ for each customer t, such that*

$$r_t^{\star 1}(n) = 1 - r_t^{\star 0}(n) - r_t^{\star -1}(n) = \begin{cases} 1 - r_t^{\star 0}(n), & \text{for } n > n_t^\star, \\ \in [0, 1 - r_t^{\star 0}(n)], & \text{for } n = n_t^\star, \\ 0, & \text{for } n < n_t^\star. \end{cases} \tag{9}$$

Furthermore, $r^\star$ must bind (IC_1^r) for customer t, if $\mathbb{P}[m_t = -1 \mid r^\star] = \sum_n z_t(n) r_t^{\star -1}(n) > 0$.

The optimal affirmative recommendation policy in (9) demonstrates a *threshold* structure: for each customer, the platform should provide the positive (resp., negative) recommendation only at net purchase positions higher (resp., lower) than a threshold position. To understand this result, we make two observations. First, the platform influences the public belief about its net purchase position exclusively through neutral recommendations, which are uniquely capable of generating inferential information about the product value (see Proposition 3). That is, the global constraint (N) in problem (P) depends only on the platform's neutral recommendation policy $\{r_t^0 : t = 1, \cdots, T\}$ (see Proposition 4) and, hence, interacts *only* with (IC_0^r). Thus, given the probability $1 - r_t^{\star 0}(n)$ to be allocated between the two affirmative recommendations at each position n, the platform optimizes its affirmative recommendation policy $r_t^{\pm 1}$ separably for each customer t subject to (IC_1^r) and (IC_{-1}^r). Second, pooling sufficiently high (resp., low) net purchase positions into the positive (resp., negative) recommendation always helps raise (resp., lower) the posterior product valuation, thus strengthening (IC_1^r) (resp., (IC_{-1}^r)). Thus, it is optimal for the platform to allocate the entire probability mass $1 - r_t^{\star 0}(n)$ to the positive recommendation for as many high positions n as possible until (IC_1^r) becomes binding (and to the negative recommendation for the remaining low positions),[12] giving rise to the threshold structure.

We now turn to the pattern of neutral recommendations in the optimal policy. As illustrated by Figure 2(c), the optimal policy calls for more neutral recommendations to earlier customers (as shown by the darker color at a majority of the reachable net purchase positions for smaller t) but utilizes more affirmative recommendations for later customers (as shown by fewer net purchase positions colored for larger t). We recall from Proposition 3 that neutral recommendations enable the platform to learn about the product value V by letting the customer's purchase decision reveal her private signal S_t. As a result, the belief about the platform's net purchase position z_t, which induces a belief about the posterior product valuation $\mathbb{E}[V \mid N(H_t), r]$, becomes significantly dispersed earlier in the selling horizon and then remains relatively constant for the remaining horizon, as shown by Figure 2(d) (as shown by the increasing number of net purchase positions

[12] It is possible that $r_t^{\star 0}(n) = 1$ for all remaining low positions before (IC_1^r) becomes binding, in which case only positive and neutral recommendations are offered to customer t, i.e., $\mathbb{P}[m_t = -1 \mid r^\star] = 0$.

in darker color for small t and almost constant number of positions in darker color for large t). In fact, the following proposition shows that the neutral recommendation induces a *mean-preserving spread* of the belief z_t in the sense of Rothschild and Stiglitz (1970).

Proposition 6 **(Structural property of neutral recommendations, I)**. *Given customer $t-1$'s belief z_{t-1}, let z_t and $\tilde{z}_t$ denote customer t's beliefs generated via (N) by neutral recommendation policy r^0_{t-1} and $\tilde{r}^0_{t-1}$, respectively. Then, $\tilde{z}_t$ is a mean-preserving spread of z_t over $\{v_n : n = 0, \pm 1, \cdots, \pm T\}$ if $\tilde{r}^0_{t-1}(n) \geq r^0_{t-1}(n)$ for all n.*

The mean-preserving spread of the public belief about the platform's net purchase position follows from the evolution constraint (N), and also implies that the platform's posterior product valuation $\mathbb{E}[V \mid N(H_t), r]$ is essentially a martingale. More importantly, Proposition 6 shows that the higher the intensity of neutral recommendation (i.e., higher r^0), the wider the spread of public belief, whose implication is manifested in the next proposition.[13]

Proposition 7 **(Structural property of neutral recommendations, II)**. *The platform's maximal expected revenue from a single customer t,*

$$V(z_t) := \max_{r_t} \; p \sum_n z_t(n) \left[r^1_t(n) + u_n r^0_t(n) \right], \quad \text{subject to (R), } (\mathrm{IC}^r_1), (\mathrm{IC}^r_0), \text{ and } (\mathrm{IC}^r_{-1}), \tag{10}$$

is non-decreasing in the mean-preserving spread of z_t. In particular, for $v_0 \in [v^*, v^{**})$, providing customer t only with the neutral recommendation (i.e., $r^0_t(n) \equiv 1$ for all n) is a feasible solution to (10), which generates an expected revenue of $pu_0 \leq V(z_t)$.

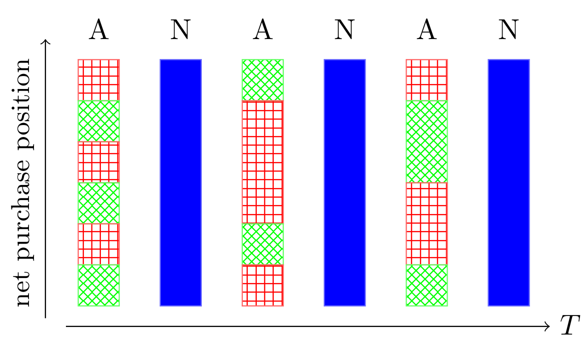
The platform would solve problem (10) if it were to maximize its short-term expected revenue only from customer t alone. Thus, Proposition 7 implies that the platform could potentially boost *future* revenue through a more dispersed public belief about its net purchase position, which, by Proposition 6, can only be achieved by increasing the *current* intensity of providing neutral recommendations. However, offering the neutral recommendation at the highest intensity (i.e., $r^0_t(n) \equiv 1$ for all n) would only generate revenue pu_0, not necessarily the maximal revenue that can be extracted from the current customer. Therefore, these two propositions together highlight the fundamental trade-off faced by the platform between long-term information (and revenue) generation and short-term revenue extraction.

[13] The nature of result and the proof technique of Proposition 7 differ from those in Gentzkow and Kamenica (2016). The latter use the notion of mean-preserving spread to derive the characterization of all Bayes-plausible distributions of the posterior mean, whereas we rank the sender's optimal payoff according to the mean-preserving spread of the prior belief. Gentzkow and Kamenica (2016) restrict their analysis to a one-dimensional continuous state space and use the integral definition of mean-preserving spread and a geometric approach to obtain their characterization, whereas, in a discrete setting, our proof uses the matrix definition of mean-preserving spread and an algebraic approach by directly constructing the defining matrix (see the proof of Proposition 6).

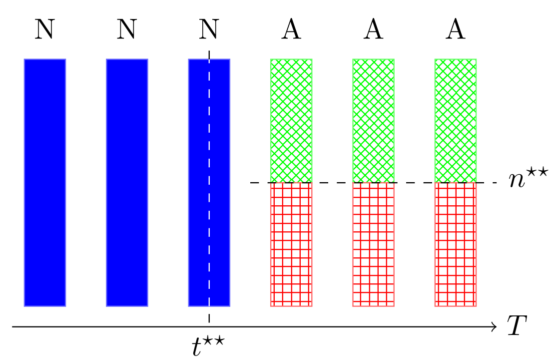
Despite the above structural properties, the optimal policy demonstrated in Figure 2 still involves a significant level of randomization between affirmative and neutral recommendations at many net purchase positions for majority of customers. As such, it does not permit a simple analytical characterization and, hence, can be challenging to prescribe, interpret, and implement in practice. In the next section, we search for a heuristic policy that involves a minimal level of randomization for ease of implementation while still capturing the structural patterns of the optimal policy.

5. NA-Sequencing Policy

In this section, we propose and study a class of recommendation policies, which we term *NA-sequencing policies*. This policy assigns each customer t to be either a *neutral customer*, who is provided exclusively with the neutral recommendation regardless of the platform's net purchase position (i.e., $r_t^0(n) \equiv 1$ for all n), or an *affirmative customer*, who is provided exclusively with an affirmative recommendation that can be randomized between the positive and negative recommendations depending on its net purchase position (i.e., $r_t^1(n) + r_t^{-1}(n) \equiv 1$ and hence $r_t^0(n) \equiv 0$ for all n). That is, the platform sequences the neutral and affirmative recommendations according to the customers' order of arrival, hence the name NA-sequencing policy. Since $r_t^0(\cdot) \in \{0, 1\}$ is only a function of t but not of n, we denote it as r_t^0 for an NA-sequencing policy r. Figure 3(a) illustrates an example NA-sequencing policy, which assigns the even-numbered arrivals to be neutral customers, and the odd-numbered ones to be affirmative customers provided with a mixture of positive and negative recommendations. Clearly, the class of NA-sequencing policies is a subset of the general recommendation policies, which allow full randomization of all three recommendations for each customer t at any net purchase position (i.e., $r_t^1(n) + r_t^0(n) + r_t^{-1}(n) \equiv 1$ for all n).



(a) NA-sequencing policy



(b) Optimal NA-sequencing policy

neutral recommendation positive recommendation negative recommendation

Figure 3 Illustration of a generic NA-sequencing policy and its optimal counterpart.

The main result of this section is to show that the *optimal* policy within the NA-sequencing class takes a simple *cutoff-threshold* structure, as illustrated by Figure 3(b). The optimal NA-sequencing policy provides only neutral recommendations to all customers arriving up to a *cut-off* customer $t^{\star\star}$ and provides only affirmative recommendations afterwards. The affirmative recommendation is positive (resp., negative) if the net purchase position achieved immediately after the cut-off customer exceeds a *threshold* $n^{\star\star}$. Only at the threshold position $n^{\star\star}$ does the platform possibly randomize between the two affirmative recommendations by providing the positive one with probability $x^{\star\star}$. Therefore, the optimal NA-sequencing policy requires the specification of only three policy parameters $(t^{\star\star}, n^{\star\star}, x^{\star\star})$, entails minimal randomization, and is thus simple to implement.[14] For instance, the platform provides no information to the first $t^{\star\star}$ customers. For the remaining $T - t^{\star\star}$ customers, the platform promotes the product *only* if the cumulative purchases made by the first $t^{\star\star}$ customers exceed $(t^{\star\star} + n^{\star\star})/2$, and otherwise remains silent until the end of the campaign.[15] In fact, this mechanism resembles the strategy adopted by Groupon's "Deals of the Day" described in Section 1. To illustrate, Figure E4 in Appendix E records the information provided over the course of the selling campaigns for 15 products on Groupon's "Deals of the Day."

The cutoff-threshold structure of the optimal NA-sequencing policy can be intuited from the structural properties of the optimal policy identified in Section 4.3. Under an NA-sequencing policy, the platform generates inferential information about the product value (at maximal intensity) *only* when it provides the neutral recommendation (only under which, we recall, the customer's private signal is revealed by her purchase decision). In doing so, the platform disperses the belief distribution about its net purchase position in a mean-preserving-spread fashion (see Proposition 6). Intuitively, the platform accumulates more proprietary information and, hence, increases its information advantage. In contrast, the platform generates no information from customers who are offered affirmative recommendations (under which, we recall, the customer's private signal is not revealed by her purchase decision). Thus, any affirmative customer, regardless of how she is offered the two affirmative (positive and negative) recommendations, neither impacts the net purchase position and, hence, nor the *future* revenue. Consequently, the platform can maximize the revenue from that affirmative customer *myopically* (i.e., in isolation from other customers). Importantly, the revenue from an affirmative customer is higher if a larger number of neutral customers were assigned previously, as the belief distribution about the net purchase position would be more dispersed; but the same revenue (pu_0) is generated from any neutral customer

[14] One can, for a small revenue loss, further enforce no randomization at the threshold position, i.e., $x^{\star\star} \in \{0, 1\}$.

[15] In this mechanism, we leverage the customer's order of arrival to differentiate the meaning of the messages: the silence up to the cut-off customer $t^{\star\star}$ is interpreted as the neutral recommendation because it applies regardless of the platform's net purchase position, whereas the silence after the cut-off $t^{\star\star}$ is interpreted as the negative recommendation because it applies only when the cumulative purchases from the first $t^{\star}$ customers fail to exceed $(t^{\star\star} + n^{\star\star})/2$.

regardless of her order of arrival (see Proposition 7). Therefore, it is optimal for the platform to front-load the neutral recommendation and back-load the affirmative recommendation, resulting in the *cutoff* structure. For each affirmative customer, the optimality of providing the positive (resp., negative) recommendation at net purchase positions above (resp., below) a threshold $n^{\star\star}$ is a direct consequence of Proposition 5 (by taking r^0 as zero), which explains the *threshold* structure. As previously alluded to, such a threshold structure allows the platform to maximize the probability of making a positive recommendation (and, hence, maximize the revenue), because it helps strengthen the IC constraints corresponding to both positive and negative recommendations.

In the rest of this section, we formally establish the above-mentioned results. Specifically, we first establish two crucial properties of an NA-sequencing policy (Lemmas 1 and 2). Next, we leverage these properties to fully characterize the optimal NA-sequencing policy (Proposition 8). Finally, we demonstrate its excellent performance numerically and study its welfare implication.

5.1. The optimal NA-sequencing policy

We recall that the main analytical complication of the platform's problem (P) stems from the interaction between the belief evolution (N) and the IC constraints (IC_1^r)-(IC_{-1}^r) imposed for each customer. Since (N) depends on a recommendation policy only through $r_t^0(n)$, an NA-sequencing policy in effect breaks such interaction by making $r_t^0(n)$ independent of net purchase position n. We leverage this property to derive the belief distribution about the net purchase position in closed form, as shown by the following lemma.

Lemma 1 **(Belief about net purchase position under an NA-sequencing policy)**. *Under an NA-sequencing policy r, $z_t(n) = \zeta(\ell_t^r, n)$, where $\ell_t^r := \sum_{s<t} r_s^0$ and*

$$\zeta(s,n) = \binom{s}{\frac{s+n}{2}} \left[v_0 q^{\frac{s+n}{2}} (1-q)^{\frac{s-n}{2}} + (1-v_0)(1-q)^{\frac{s+n}{2}} q^{\frac{s-n}{2}} \right] \mathbb{1}\left[|n| \le s \text{ and } n \equiv s \pmod 2 \right]. \quad (11)$$

Lemma 1 shows that the public belief about the net purchase position z_t depends on NA-sequencing policy r only through ℓ_t^r, a summary statistic calculating the total number of neutral customers assigned up to date. Conditional on the product value $V = 1$ (resp., $V = 0$), a neutral customer makes the purchase if she receives an optimistic signal $S_t = 1$, which occurs with probability q (resp., $1-q$). Subsequently, the number of purchases made by ℓ_t^r neutral customers follows a mixture of two binomial distributions, $\mathrm{Bin}(\ell_t^r, q)$ and $\mathrm{Bin}(\ell_t^r, 1-q)$, with weights $v_0 = \mathbb{P}[V=1]$ and $1 - v_0 = \mathbb{P}[V=0]$, respectively, resulting in (11). Consistent with Proposition 6, a larger number of neutral customers (i.e., larger ℓ_t^r) disperses both binomial distributions and, hence, the belief distribution $\zeta(\ell_t^r, \cdot)$, increasing the platform's information advantage over customers.

Equipped with the belief distribution derived in Lemma 1, we then characterize the optimal policy for an affirmative customer. Notably, we show that the *myopic* revenue maximization for each affirmative customer is also *globally* optimal, because an affirmative customer does not affect the platform's net purchase position (and hence, neither the future revenue). The only relevant IC constraints for an affirmative customer are (IC_1^r) and (IC_{-1}^r), in which we can substitute $\zeta(\ell_t^r, n)$, derived in Lemma 1, for the belief $z_t(n)$. Accordingly, we define[16]

$$(n^{**}(s), x^{**}(s)) := \underset{n\geq -s,\ x\in[0,1]}{\arg\min} \left\{ n : (v_n - v^{**})\,\zeta(s,n)x + \sum_{m>n}(v_m - v^{**})\zeta(s,m) = 0 \right\},\ \text{and} \tag{12}$$

$$(n^{*}(s), x^{*}(s)) := \underset{n\leq s,\ x\in[0,1]}{\arg\max} \left\{ n : (v_n - v^{*})\,\zeta(s,n)(1-x) + \sum_{m<n}(v_m - v^{*})\zeta(s,m) = 0 \right\}. \tag{13}$$

The optimal affirmative policy is then given by the following lemma.

Lemma 2 **(Affirmative customer)**. *Under an NA-sequencing policy r, it is incentive compatible to assign customer t to be an affirmative customer if and only if $\ell_t^r \geq \tau^\circ := \min\{s \geq 0 : n^*(s) > n^{**}(s),\ \text{or}\ n^*(s) = n^{**}(s)\ \text{with}\ x^{**}(s) \geq x^*(s)\}$. The optimal policy for an affirmative customer t is given by*

$$r_t^{\star\star 1}(n) = 1 - r_t^{\star\star -1}(n) = \begin{cases} 1, & \text{for } n > n^{**}(\ell_t^r), \\ x^{**}(\ell_t^r), & \text{for } n = n^{**}(\ell_t^r), \\ 0, & \text{for } n < n^{**}(\ell_t^r), \end{cases} \tag{14}$$

which generates an expected revenue of $pF(\ell_t^r)$ for the platform, where

$$F(s) := x^{**}(s)\zeta(s, n^{**}(s)) + \sum_{n > n^{**}(s)} \zeta(s,n), \tag{15}$$

is a non-decreasing function in s.

Lemma 2 demonstrates that the optimal policy for an affirmative customer has a threshold structure as in (14), under which the expected probability of a positive recommendation (and subsequently a purchase) is given by $F(\ell_t^r)$ as in (15). This threshold structure is consistent with that of the optimal policy shown in Proposition 5. In particular, the threshold position $n^{**}(\ell_t^r)$ and the probability $x^{**}(\ell_t^r)$ of offering the positive recommendation there are, by definition in (12), determined so as to bind (IC_1^r). Similarly, $(n^*(\ell_t^r), x^*(\ell_t^r))$ defined by (13) is determined so as to bind (IC_{-1}^r). Thus, it is incentive compatible for the platform to begin assigning customers to be affirmative ones (i.e., both (IC_1^r) and (IC_{-1}^r) can be satisfied), if and only if the total number of neutral customers assigned thus far, ℓ_t^r, is sufficiently large so that $n^*(\ell_t^r) \geq n^{**}(\ell_t^r)$ with the additional requirement $x^*(\ell_t^r) \leq x^{**}(\ell_t^r)$ when the equality holds. In fact, the following lemma shows that only a small number of neutral customers (i.e., at most three) are needed before assigning customers to be affirmative ones (i.e., to fulfill this condition).[17]

[16] In the proof of Lemma 2, Equations (12) and (13) are shown to be well-defined and unique. In particular, we show that $x^{**}(s) \in [0,1)$; for $s = 0$ and $v_0 = v^*$, (12) generates a unique solution with $n^{**}(0) = 0$ and $x^{**}(0) = 0$, whereas (13) generates a unique $n^*(0) = 0$ but has multiple values of $x^*(0)$ as the solution, in which case we choose $x^*(0) = 0$.

[17] Numerical experiments suggest that $\tau^\circ \leq 3$ is tight, i.e., $\tau^\circ = 3$ for some parameters.

LEMMA 3 **(Minimal neutral customers to enable affirmative customers)**. $\tau^\circ \leq 3$.

Most significantly, Lemma 2 also shows that a larger number of neutral customers sequenced prior to an affirmative customer (and, by Lemma 1, a more dispersed belief distribution) increases the revenue that the platform can extract from that affirmative customer, i.e., $pF(\ell_t^r)$ is non-decreasing in ℓ_t^r. Since a larger value of ℓ_t^r leads to a more dispersed belief distribution $\zeta(\ell_t^r,\cdot)$, Lemma 2 is also consistent with Proposition 7. Together with the fact that neutral customers generate the same revenue pu_0 regardless of their order of arrival (Proposition 7), we obtain our key result:

PROPOSITION 8 **(Optimal NA-sequencing policy)**. *The optimal NA-sequencing policy $r^{\star\star}$ is characterized by a cut-off customer $t^{\star\star}$, a threshold net purchase position $n^{\star\star}$, and a probability $x^{\star\star} \in [0,1]$, such that*

$$r_t^{\star\star 0} = \begin{cases} 1, & \text{if } t \leq t^{\star\star}, \\ 0, & \text{if } t > t^{\star\star}, \end{cases} \quad \text{and} \quad r_t^{\star\star 1}(n) = 1 - r_t^{\star\star 0} - r_t^{\star\star -1}(n) = \begin{cases} 0, & \text{if } t \leq t^{\star\star}, \\ 1, & \text{if } t > t^{\star\star} \text{ and } n > n^{\star\star}, \\ x^{\star\star}, & \text{if } t > t^{\star\star} \text{ and } n = n^{\star\star}, \\ 0, & \text{if } t > t^{\star\star} \text{ and } n < n^{\star\star}. \end{cases} \tag{16}$$

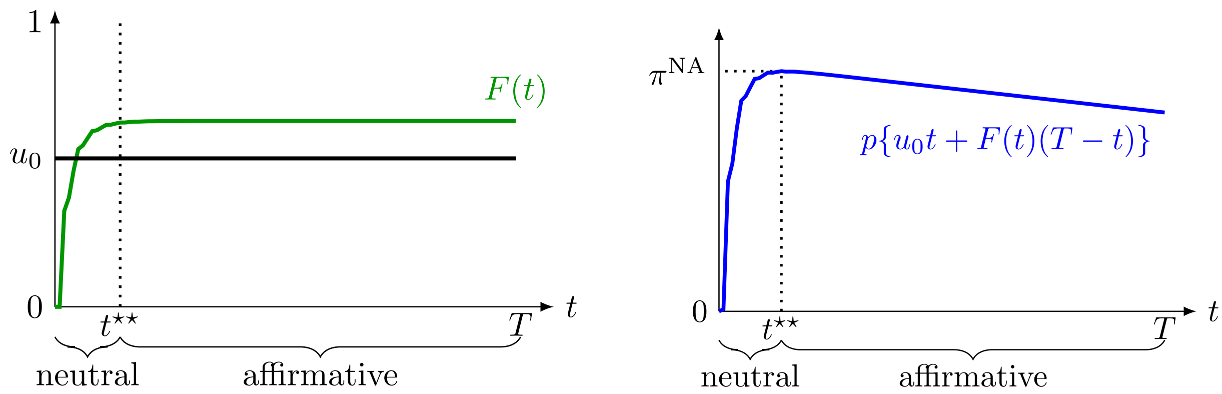
In particular, $t^{\star\star} = T$ if $T < \tau^\circ$, and otherwise $t^{\star\star}$ is given by the solution to

$$\pi^{\mathrm{NA}} := p \max_{t \in [\tau^\circ, T]} u_0 t + F(t)(T-t). \tag{17}$$

Finally, $n^{\star\star} = n^{**}(t^{\star\star})$ and $x^{\star\star} = x^{**}(t^{\star\star})$, where $n^{**}(\cdot)$ and $x^{**}(\cdot)$ is characterized by Lemma 2.

Proposition 8 formally establishes the cutoff-threshold structure illustrated in Figure 3(b) as the optimal NA-sequencing policy, which is fully characterized by three policy parameters $(t^{\star\star}, n^{\star\star}, x^{\star\star})$. Thus, it significantly reduces the computational complexity from a high-dimensional LP in determining the optimal policy (see Proposition A1) to a simple linear search in (17), where the two terms represent the revenues extracted from neutral and affirmative customers, respectively. In other words, the optimal cut-off $t^{\star\star}$ strikes the tradeoff between information accumulation (from neutral customers) and revenue extraction (from affirmative customers).

We illustrate the determination of $t^{\star\star}$ in Figure 4. Since the objective function in (17) is non-decreasing in t for $F(t) \leq u_0$, the platform should keep assigning customer t to be a neutral customer at least until the platform can, if possible, extract higher revenue from an affirmative customer than from a neutral customer (i.e., $F(t^{\star\star}) \geq u_0$). Table E1 in Appendix E shows that, for some parameters (such that v_0 is sufficiently higher than v^*), the optimal NA-sequencing policy only needs to assign less than 20% of all customers to be neutral customers (i.e., $t^{\star\star}/T \leq 20\%$), but for other parameters (such that v_0 is close to v^*), the optimal NA-sequencing policy coincides with the no-disclosure policy and assigns all customers to be neutral ones (i.e., $t^{\star\star} = T$).



(a) $F(t)$. (b) $p\{u_0 t + F(t)(T-t)\}$.

Figure 4 Optimal number of neutral customers $t^{\star\star}$ for $T = 100, q = .7, p = .7$ and $v_0 = .55 \in [v^*, v^{**}) = [.5, .8448)$.

5.2. Performance of optimal NA-sequencing policy

We now demonstrate that the optimal NA-sequencing policy's revenue π^{NA} given by (17) is close to the optimal revenue $\pi^\star$ given by (P).[18] Specifically, we measure the optimality gap by the ratio $(\pi^\star - \pi^{\mathrm{NA}})/\pi^\star$, and compute it by fixing a campaign length $T \in \{50, 100, 500\}$ and varying the price p from 0.1 to 0.9, the prior expectation v_0 from 0.05 to 0.95, and the customer's signal precision q from 0.55 to 0.95, all with a 0.1 increment. Thus, the total number of instances for each value of T is $9 \times 10 \times 5 = 450$, out of which 182 instances satisfy $v_0 \in [v^*, v^{**})$.[19] As shown by Table 1, the optimal NA-sequencing policy is able to garner, on average, 95% of the optimal revenue with a majority of instances garnering more than 90%. In the worst case, at least 75% of the optimal revenue can be captured by the optimal NA-sequencing policy.

Table 1 Summary statistics of the optimality gap between the optimal NA-sequencing policy and optimal recommendation policy $(\pi^\star - \pi^{\mathrm{NA}})/\pi^\star$ across 182 instances of (p, q, v_0) such that $v_0 \in [v^*, v^{**})$.

		$T = 50$	$T = 100$	$T = 500$
Average		5.12%	4.66%	4.11%
Standard deviation		4.45%	4.51%	4.67%
Minimum		0	0	0
Maximum		24.49%	24.85%	25.13%
Number of instances with gap	$\geq 10\%$	25	22	21
	$\geq 15\%$	9	8	8
	$\geq 20\%$	3	3	2

Finally, the following corollary evaluates, as another performance measure of the optimal NA-sequencing policy, the customer's *welfare*, which is defined as her expected payoff under this policy.

[18] In Appendix D, we evaluate how the NA-sequencing policy performs against the full- and no-disclosure benchmarks.

[19] Tables E2 and E3 in Appendix E compute $\pi^\star$ and π^{NA}, respectively. Note that $\pi^{\mathrm{NA}} = \pi^\star$ when $v_0 \notin [v^*, v^{**})$.

COROLLARY 1. *Under the optimal NA-sequencing policy $r^{\star\star}$, all neutral customers obtain the same welfare, which is given by $W_{\le t^{\star\star}} = u_0(v_1 - p)$; and all affirmative customers obtain the same welfare, which is given by $W_{>t^{\star\star}} = (v^{**} - p)F(t^{\star\star})$.*

As demonstrated in Table E4 of Appendix E, it seems that an affirmative customer would be better off than a neutral customer (i.e., $W_{>t^{\star\star}} \ge W_{\le t^{\star\star}}$). This is consistent with concerns expressed in the literature (e.g., Bimpikis and Papanastasiou 2019) about customers' incentives to avoid exploration (which corresponds to being a neutral customer in our setting). Nonetheless, such concern can be resolved by moving the cut-off customer forward to $t^{\star\star} = \max\{t : (v^{**} - p)F(t) \le u_0(v_1 - p)\}$ in our NA-sequencing policy.

6. Extensions

In this section, we extend the model of Section 3 in two directions to broaden the application of our modeling framework and to demonstrate the generality of the methodology developed in Section 4. In Section 6.1, we relax the informational assumption regarding customers' knowledge about their order of arrival; in Section 6.2, we incorporate a post-purchase signal about the product value.

6.1. Imprecise knowledge about the order of arrival

We consider a setting à la Kremer et al. (2014), in which customers do not perfectly know their order of arrival t (whereas the platform does know). Specifically, a (publicly known) sequence of integers $1 = \tau^1 < \tau^2 < \cdots < \tau^k = T + 1$ exists to segment the selling horizon into $k - 1$ *blocks*, and customers only know which block they belong to. A customer arriving in block $b \in \{1, \cdots, k-1\}$ believes that her order of arrival t is uniformly distributed between τ^b and $\tau^{b+1} - 1$, i.e., she forms a belief $g(t \mid b) = \frac{1}{\tau^{b+1} - \tau^b} \mathbb{1}[t \in \{\tau^b, \tau^b + 1, \cdots, \tau^{b+1} - 1\}]$.[20] Notably, when $k = T + 1$, this setting reduces to our base model with customers having precise knowledge about their order of arrival.

Since the platform's observable information remains the same as in the base model, its proprietary history can still be captured by the net purchase position as in (8) and the public belief about its evolution continues to be represented by (N). Furthermore, we can again restrict the analysis to the space of recommendation policies specified by (R). However, each customer, only knowing her arrival in block b, will interpret a message m by integrating her possible order of arrival t over her belief $g(t \mid b)$. More precisely, upon receiving recommendation $m = i \in \{1, 0, -1\}$, all customers in block b infer that their order of arrival is t with probability $g(t \mid b)$ and hence would

[20] The uniform distribution represents the least informative situation in which customers know nothing beyond their blocks. The problem formulation in ($\overline{\mathrm{P}}$) remains valid under the more general assumption that $g(\cdot \mid b)$ has non-overlapping support, i.e., $b' \ne b''$ implies that $g(t \mid b')\, g(t \mid b'') = 0$ for all t.

form their product valuation as $\mathbb{E}[V \mid m=i,b,r] = \frac{\sum_{t,n} v_n z_t(n) r_t^i(n) g(t\,|\,b)}{\sum_{t,n} z_t(n) r_t^i(n) g(t\,|\,b)}$. As such, the IC constraints (IC_1^r)-(IC_{-1}^r) in our base model should be revised to

$$\sum_{t,n}(v_n - v^{**}) z_t(n) r_t^1(n) g(t\,|\,b) \geq 0, \tag{$\overline{\mathrm{IC}}_1^r$}$$

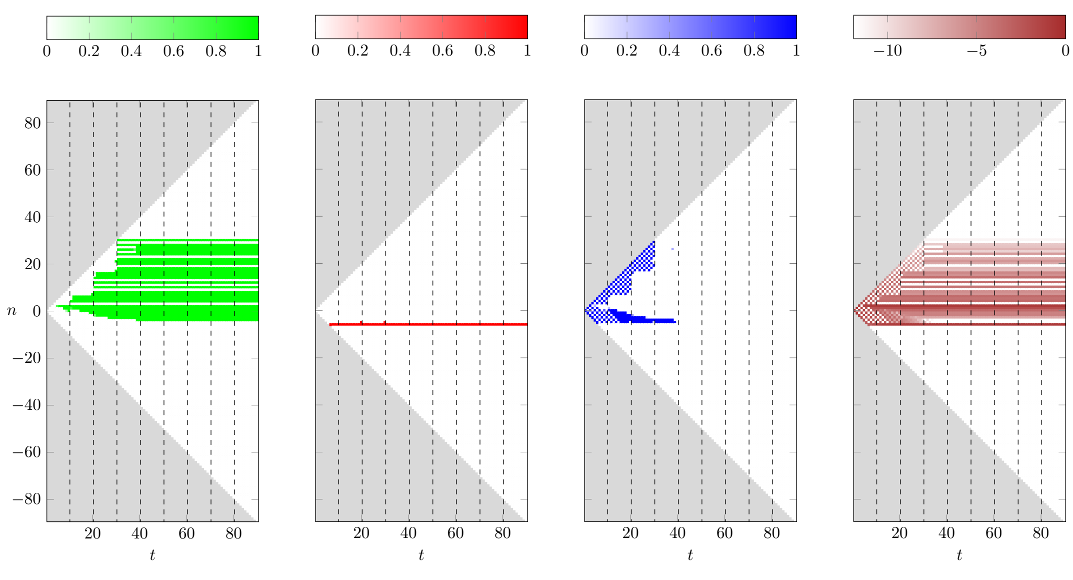
$$\sum_{t,n}(v_n - v^{**}) z_t(n) r_t^0(n) g(t\,|\,b) \leq 0, \quad \sum_{t,n}(v_n - v^{*}) z_t(n) r_t^0(n) g(t\,|\,b) \geq 0, \quad \text{and} \tag{$\overline{\mathrm{IC}}_0^r$}$$

$$\sum_{t,n}(v_n - v^{*}) z_t(n) r_t^{-1}(n) g(t\,|\,b) \leq 0, \quad \text{for all } b. \tag{$\overline{\mathrm{IC}}_{-1}^r$}$$

We note that for a given b, the left-hand side of the constraints above is essentially a weighted sum of the counterparts in constraints (IC_1^r)-(IC_{-1}^r) for all t in block b. In other words, constraints ($\overline{\mathrm{IC}}_1^r$)-($\overline{\mathrm{IC}}_{-1}^r$) are a *relaxation* of (IC_1^r)-(IC_{-1}^r), an observation that has emerged in prior research under different settings (e.g., Kremer et al. 2014, Papanastasiou et al. 2018). Therefore, the platform's information design problem in this setting can be formulated like its problem (P) in the base model, with the constraints (IC_1^r)-(IC_{-1}^r) replaced by their relaxed counterparts ($\overline{\mathrm{IC}}_1^r$)-($\overline{\mathrm{IC}}_{-1}^r$):

$$\overline{\pi}^\star := \max_{r,z} \quad p \sum_{t,n} z_t(n) \left[r_t^1(n) + u_n r_t^0(n) \right], \quad \text{subject to (R), (N), } (\overline{\mathrm{IC}}_1^r), (\overline{\mathrm{IC}}_0^r), \text{ and } (\overline{\mathrm{IC}}_{-1}^r), \tag{$\overline{\mathrm{P}}$}$$

whose solution is denoted as $(\overline{r}^\star, \overline{z}^\star)$.



(a) $\overline{r}_t^{\star 1}(n)$ (b) $\overline{r}_t^{\star -1}(n)$ (c) $\overline{r}_t^{\star 0}(n)$ (d) $\log \overline{z}_t^\star(n)$

Figure 5 The optimal solution $(\overline{r}^\star, \overline{z}^\star)$ to ($\overline{\mathrm{P}}$) for $k=10, T=90, q=.7, p=.7$ and $v_0 = .55 \in [v^*, v^{**}) = [.5, .8448)$ with $\tau^b = 10(b-1)+1$ for $b=1,\cdots,10$ (dashed vertical lines) and $\overline{\pi}^\star = 40.2029 > 40.1294 = \pi^\star$.

Therefore, the platform can extract higher revenue from customers *without* precise knowledge about their order of arrival than from customers *with* such knowledge, i.e., $\overline{\pi}^\star \geq \pi^\star$. As such, customers may have incentives to acquire information about their exact order of arrival (so as to lessen such revenue extraction). Again, we can reformulate ($\overline{\mathrm{P}}$) as an LP and solve it efficiently. Figure 5 illustrates the optimal information policy $(\overline{r}^\star, \overline{z}^\star)$ obtained in this way for the same parameters as in Figure 2. We note that the structural pattern of the optimal policy without customers' precise knowledge about their order of arrival remains similar to that in our base model.

Before concluding the analysis of this relaxed information environment, we restrict our policy space in ($\overline{\mathrm{P}}$) to the NA-sequencing policies specified in Section 5. We make a mild assumption that the first block of arrivals occur up to τ° (i.e., $\tau^2 = \tau^\circ + 1 \leq 4$).

PROPOSITION 9. *The optimal NA-sequencing policy for the base model as characterized in Proposition 8 remains optimal when customers have imprecise knowledge about their order of arrival.*

Intriguingly, Proposition 9 shows that the optimal NA-sequencing policy developed in Section 5 is robust to the informational assumption regarding customers' order of arrival. Intuitively, the optimal NA-sequencing policy has a simple cutoff-threshold structure, whose interpretation does not require customers to know their exact order of arrival other than whether their arrival is before or after the cut-off customer $t^{\star\star}$.

6.2. Public post-purchase signals about product value

We extend our base model to a setting where the delivery and consumption of a product occur instantly following the purchase (e.g., digital goods) so that customers can post reviews that are publicly accessible on the platform.[21] Specifically, we model customer t's review as a *public* trinary signal $\widehat{S}_t \in \{0, \pm 1\}$, where $\widehat{S}_t = 0$ represents no review, and $\widehat{S}_t = 1$ (resp., $\widehat{S}_t = -1$) represents a positive (resp., negative) review. If no purchase is made ($a_t = 0$), we must have $\widehat{S}_t = 0$. Upon a purchase ($a_t = 1$), customer t posts her $\widehat{S}_t \in \{-1, 1\}$ with probability $\beta \in [0,1]$. Note that $\beta = 0$ reduces to our base model. Furthermore, a valuable product $V = 1$ (resp., $V = 0$) would generate $\widehat{S}_t = 1$ (resp., $\widehat{S}_t = -1$) with probability $\widehat{q} \in (1/2, 1]$ (i.e., the *precision* of $\widehat{S}_t$), where we assume that $\widehat{S}_t$ and S_t are independent *conditional on* V.[22] In summary, we have

$$\begin{aligned}
&\mathbb{P}\left[\widehat{S}_t = 0 \,\middle|\, a_t = -1\right] = 1, \quad \mathbb{P}\left[\widehat{S}_t = 0 \,\middle|\, a_t = 1\right] = 1 - \beta, \\
&\mathbb{P}\left[\widehat{S}_t = 1 \,\middle|\, V = 1, a_t = 1\right] = \mathbb{P}\left[\widehat{S}_t = -1 \,\middle|\, V = 0, a_t = 1\right] = \beta \widehat{q}, \quad \text{and} \\
&\mathbb{P}\left[\widehat{S}_t = -1 \,\middle|\, V = 1, a_t = 1\right] = \mathbb{P}\left[\widehat{S}_t = 1 \,\middle|\, V = 0, a_t = 1\right] = \beta(1 - \widehat{q}).
\end{aligned} \tag{18}$$

[21] In practice, product reviews are publicly accessible because they are subject to submitters' verification. Deleting, hiding, or tampering product reviews would disincentivize future submissions and damage the platform's reputation. Nevertheless, our solution method remains equally applicable were the reviews only observable to the platform, in which case the left-hand sides of ($\widehat{\mathrm{IC}}$) would also sum over $\widehat{n}$ and hence the platform's problem ($\widehat{\mathrm{P}}$) is relaxed.

[22] Note, $\widehat{S}_t$ and S_t are not necessarily independent unconditional on V.

Under this setting, both the platform and customer t observe the *public history* $\widehat{H}_t := \{(\widehat{S}_l) : l < t\}$, whereas the platform additionally has proprietary access to history $H_t := \{(m_l, a_l) : l < t\}$ as in our base model. Thus, the platform's information provision policy σ now becomes a mapping from both proprietary and public histories $H_t \cup \widehat{H}_t$ to a probability distribution over the message space $\mathcal{M} = \{1, 0, -1\}$, whose interpretation remains the same as in our base model and maintains its credibility via the following IC constraints: for all $\widehat{H}_t$, [23]

$$\mathbb{E}\left[V \,\middle|\, m_t = 1, \widehat{H}_t, \sigma\right] \geq v^{**}, \;\; v^* \leq \mathbb{E}\left[V \,\middle|\, m_t = 0, \widehat{H}_t, \sigma\right] \leq v^{**}, \text{ and } \mathbb{E}\left[V \,\middle|\, m_t = -1, \widehat{H}_t, \sigma\right] \leq v^*. \quad (19)$$

The key difference between the current setting and our base model lies in expanding the notion of net purchase position to also include net review position as shown in the following proposition.

Proposition 10 **(Sufficiency of net purchase and review positions)**. *The platform's product valuation conditional on history $H_t \cup \widehat{H}_t$ generated by recommendation policy σ is*

$$\mathbb{E}\left[V \,\middle|\, H_t \cup \widehat{H}_t, \sigma\right] = \frac{v_0}{v_0 + (1 - v_0)(\frac{1-q}{q})^{N(H_t)}(\frac{1-\widehat{q}}{\widehat{q}})^{\widehat{N}(\widehat{H}_t)}}, \quad (20)$$

where $N(H_t)$ is the platform's net purchase position defined in (7) and its net review position *up to customer t is defined as:*

$$\widehat{N}(\widehat{H}_t) := \sum_{l<t} \widehat{S}_l \in \{-(t-1), \cdots, 0, \cdots, t-1\}. \quad (21)$$

In contrast with Proposition 3, the addition of public history $\widehat{H}_t$ modifies the platform's product evaluation by an extra term $(\frac{1-\widehat{q}}{\widehat{q}})^{\widehat{N}(\widehat{H}_t)}$ in the denominator of (20). That is, the payoff-relevant information in public history $\widehat{H}_t$ is completely summarized by another single-dimensional statistic $\widehat{N}(\widehat{H}_t)$, the net review position, which is separable from the proprietary history H_t (and, hence, from the net purchase position $N(H_t)$). It essentially computes the difference between the cumulative number of positive and negative reviews from previous customers (no reviews, $\widehat{S}_t = 0$, make zero contribution). As a summary statistics of public history, the net review position is publicly observable by both the platform and customers. For notational simplicity, we denote

$$v_{n,\widehat{n}} := \mathbb{E}\left[V \,\middle|\, N(H_t) = n, \widehat{N}(\widehat{H}_t) = \widehat{n}, \sigma\right] = \frac{v_0}{v_0 + (1 - v_0)(\frac{1-q}{q})^n(\frac{1-\widehat{q}}{\widehat{q}})^{\widehat{n}}}, \text{ for any integers } n, \widehat{n}, \quad (22)$$

together with $u_{n,\widehat{n}} = qv_{n,\widehat{n}} + (1-q)(1-v_{n,\widehat{n}})$, $\widehat{u}_{n,\widehat{n}} = \widehat{q}v_{n,\widehat{n}} + (1-\widehat{q})(1-v_{n,\widehat{n}})$ and $\widetilde{u}_{n,\widehat{n}} = \widehat{q}qv_{n,\widehat{n}} + (1-\widehat{q})(1-q)(1-v_{n,\widehat{n}})$. Accordingly, the platform's policy, again denoted as r, can be

[23] Propositions C1 and C2 in Appendix C generalize Propositions 1 and 2, respectively, and again show the sufficiency of the recommendation policies.

defined as a mapping from each pair of net purchase and review positions $(n,\widehat{n})$ to a probability distribution over recommendation messages $\mathcal{M}$ for each customer t, namely $r_t^i(n,\widehat{n}) := \mathbb{P}\left[m_t = i \,\middle|\, N(H_t)=n, \widehat{N}(\widehat{H}_t)=\widehat{n}\right]$ satisfying

$$r_t(n,\widehat{n}) = \left(r_t^1(n,\widehat{n}), r_t^0(n,\widehat{n}), r_t^{-1}(n,\widehat{n})\right) \in \mathbb{R}_+^3 \text{ with } \sum_{i\in\{1,0,-1\}} r_t^i(n,\widehat{n}) = 1. \tag{$\widehat{\text{R}}$}$$

Following the same step as in the base model (Proposition 4), we now derive the belief evolution about the net purchase and review positions in the next proposition.

PROPOSITION 11 **(Belief about net purchase and review positions)**. *Given a recommendation policy r, let $z_t(n,\widehat{n}) := \mathbb{P}\left[N(H_t)=n, \widehat{N}(\widehat{H}_t)=\widehat{n} \,\middle|\, r\right]$ represent the public belief about the net purchase and review positions. Then, this belief can be obtained recursively by*

$$\begin{aligned} z_t(n,\widehat{n}) = &\left[(1-\beta)r_{t-1}^1(n,\widehat{n}) + r_{t-1}^{-1}(n,\widehat{n})\right] z_{t-1}(n,\widehat{n}) \\ &+(1-\beta)u_{n-1,\widehat{n}}r_{t-1}^0(n-1,\widehat{n})z_{t-1}(n-1,\widehat{n}) + (1-u_{n+1,\widehat{n}})r_{t-1}^0(n+1,\widehat{n})z_{t-1}(n+1,\widehat{n}) \\ &+\beta\widehat{u}_{n,\widehat{n}-1}r_{t-1}^1(n,\widehat{n}-1)z_{t-1}(n,\widehat{n}-1) + \beta(1-\widehat{u}_{n,\widehat{n}+1})r_{t-1}^1(n,\widehat{n}+1)z_{t-1}(n,\widehat{n}+1) \\ &+\beta\widetilde{u}_{n-1,\widehat{n}-1}r_{t-1}^0(n-1,\widehat{n}-1)z_{t-1}(n-1,\widehat{n}-1) \\ &+\beta(u_{n-1,\widehat{n}+1} - \widetilde{u}_{n-1,\widehat{n}+1})r_{t-1}^0(n-1,\widehat{n}+1)z_{t-1}(n-1,\widehat{n}+1), \end{aligned} \tag{$\widehat{\text{N}}$}$$

for $t=2,\cdots,T$, $n=-(T-1),\cdots,T-1$ and $\widehat{n}=-(T-1),\cdots,T-1$, with $z_{t-1}(\pm T,\cdot) = z_{t-1}(\cdot,\pm T) = 0$ and $z_1(n,\widehat{n}) = \mathbb{1}[n=\widehat{n}=0]$.

We derive ($\widehat{\text{N}}$) by enumerating, as represented by each term therein, the probability of reaching the position $(n,\widehat{n})$ from its neighboring positions in the previous period. We note that when $\beta=0$ (i.e., absent public reviews), ($\widehat{\text{N}}$) reduces to (N) in our base model. Equipped with this belief, we can further reformulate the IC constraints in (19) similar to the approach used in our base model:

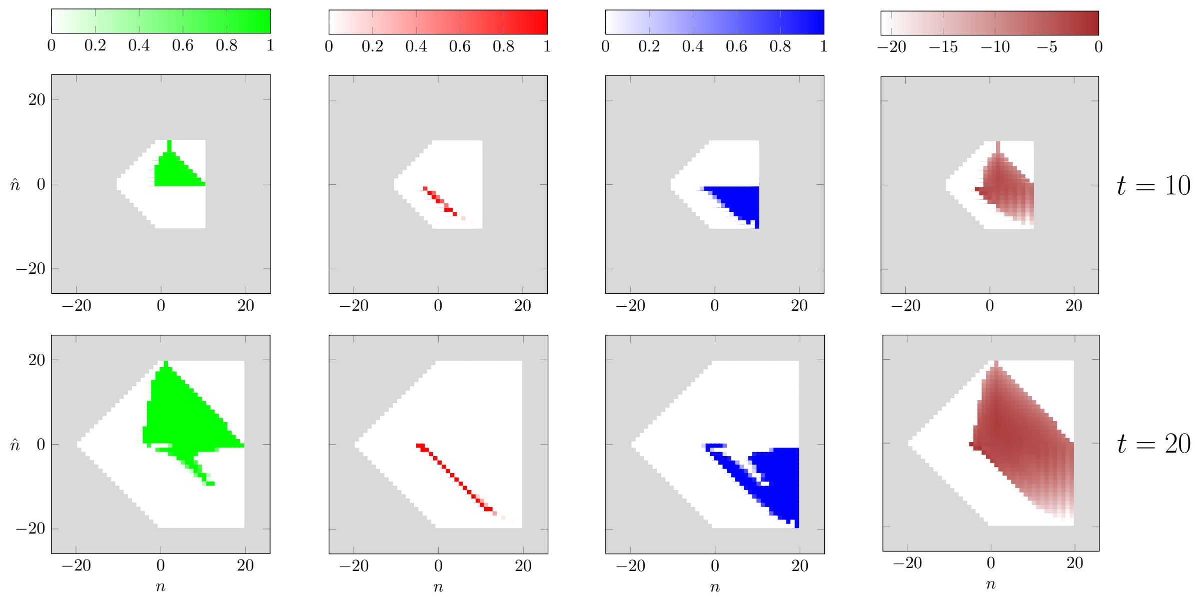
$$\left.\begin{aligned} \textstyle\sum_{n=-(T-1)}^{T-1}[v_{n,\widehat{n}} - v^{**}]z_t(n,\widehat{n})r_t^1(n,\widehat{n}) \geq 0, \\ \textstyle\sum_{n=-(T-1)}^{T-1}[v_{n,\widehat{n}} - v^{**}]z_t(n,\widehat{n})r_t^0(n,\widehat{n}) \leq 0, \\ \textstyle\sum_{n=-(T-1)}^{T-1}[v_{n,\widehat{n}} - v^{*}]z_t(n,\widehat{n})r_t^0(n,\widehat{n}) \geq 0, \\ \textstyle\sum_{n=-(T-1)}^{T-1}[v_{n,\widehat{n}} - v^{*}]z_t(n,\widehat{n})r_t^{-1}(n,\widehat{n}) \leq 0, \end{aligned}\right\} \quad \text{for all } \widehat{n}, t. \tag{$\widehat{\text{IC}}$}$$

Compared to (IC_1^r)-(IC_{-1}^r), the IC constraints in ($\widehat{\text{IC}}$) are separable not only in the order of arrival t but also in the net review position $\widehat{n}$. indeed, each customer t interprets the platform's recommendation in the context of publicly available reviews, so the platform needs to warrant the credibility of its messages for each net review position.

Taken together, the platform's problem in this extended setting is then given by

$$\widehat{\pi}^\star := \max_{r,z} \quad p\sum_{t,n,\widehat{n}} z_t(n,\widehat{n})\left[r_t^1(n,\widehat{n}) + u(n,\widehat{n})r_t^0(n,\widehat{n})\right], \quad \text{subject to } (\widehat{\text{R}}), (\widehat{\text{N}}), \text{ and } (\widehat{\text{IC}}), \tag{$\widehat{\text{P}}$}$$

whose solution is denoted as $(\widehat{r}^\star, \widehat{z}^\star)$. Again, we can further reformulate ($\widehat{\text{P}}$) as an LP by working with the product $z_t(n,\widehat{n})r_t^i(n,\widehat{n})$ and then solving it efficiently.



(a) $\widehat{r}_t^{\star 1}(n,\widehat{n})$ (b) $\widehat{r}_t^{\star -1}(n,\widehat{n})$ (c) $\widehat{r}_t^{\star 0}(n,\widehat{n})$ (d) $\log \widehat{z}_t^{\star}(n,\widehat{n})$

Figure 6 The optimal solution $(\widehat{r}^{\star}, \widehat{z}^{\star})$ to $(\widehat{\mathrm{P}})$ for $t \in \{10, 20\}, T = 24, q = \widehat{q} = p = .7, \beta = .5, v_0 = .55 \in [v^{*}, v^{**}) = [.5, .8448)$. The area shaded in gray corresponds to unreachable $(n, \widehat{n})$ pairs.

Figure 6 illustrates the platform's optimal recommendation policy $(\widehat{r}^{\star}, \widehat{z}^{\star})$ for a specific example. We highlight three observations: (i) The optimal policy's reliance on neutral recommendations appears to diminish over time (i.e., given a net review position $\widehat{n}$, the color gradient in Figure 6(c) reduces along the net purchase position n from the top to the bottom panel), whereas its reliance on affirmative recommendations appears to increase over time (i.e., given $\widehat{n}$, the color gradient in Figures 6(a) and 6(b) increases along the n dimension from the top to the bottom panel). Furthermore, (ii) given time period t and a net review position $\widehat{n}$, positive recommendations are offered at higher net purchase positions than are, if any, negative recommendations (by contrasting the color gradients along n dimension between Figures 6(a) and 6(b)). These two structure patterns are consistent with those of the optimal policy in our base model (see Figure 2). Nuanced in this extended setting, (iii) the optimal policy seems to utilize positive recommendations more intensively at higher (and positive) net review positions and to limit the use of negative recommendations to only negative net review positions (by contrasting the color gradients along the $\widehat{n}$ dimension between Figures 6(a) and 6(b)).

7. Concluding Remarks

In this paper, we study a novel information design problem of an online time-locked sales campaign. In essence, this problem is one of engineering social learning: strategically providing information about historical purchase decisions to influence future customers' product evaluation. As a methodological contribution, we developed a systematic approach to solve dynamic information design problems with a long-term committed sender facing a sequence of short-lived (possibly privately informed) receivers. Our solution method hinges on two reductions: the shrinkage of the sender's message space and the efficient representation of its proprietary history. Using this approach, we characterize the structural properties of the optimal information policy and uncover the fundamental trade-off faced by the sender: long-term information (and revenue) generation versus short-term revenue extraction. As a contribution to the revenue management practice, we demonstrate the power of information provision even in the absence of manipulating the price instrument. Furthermore, we also identify an easy-to-implement, well-performing, information-robust heuristic policy in addition to providing an optimal information policy. Finally, we demonstrate the generality of our solution methodology and the robustness of our findings by relaxing and extending the information assumptions in the base model.

Our analysis also provide general guidelines for online retail platforms to design their information disclosure strategy during sales campaigns. We show that platforms should refrain from making recommendations but focus on observing and collecting consumers' purchase decisions in the early phase of their campaigns. Based on these accumulated information, platforms should offer a small number of assertive recommendations (rather than information of great granularity) in the later phase of their campaigns. The heuristic policy we identified provides an actionable strategy to implement this general principle in practice, and lends itself to a field experiment. One can test and optimize the two key policy parameters, namely the *cut-off* time to stop information accumulation and the *threshold* on up-to-date sales to promote a product. Admittedly, these two policy parameters may vary across different product categories and may depend on campaign lengths and product prices. We can also benchmark the performance of the empirically-optimized heuristic policy with the two naïve information policies (i.e., no-disclosure and full-disclosure policies) widely used in practice. In such a field study, it is important to ensure that the information policy in place is clearly communicated to and understood by customers to ensure meaningful interpretation of platforms' displayed messages.

To conclude our investigation, we comment on three potential directions for future research. First, we solve the platform's problem (P) by further transforming it as an LP. In fact, one could also reformulate it as a high-dimensional deterministic dynamic program (DP) with the public belief about the net purchase position $z_t(\cdot)$, a $(2T+1)$-dimensional vector, as its state variable. The DP reformulation can potentially offer an alternative approach to solve the problem. Second, we employ a parsimonious setup by modeling a customer's private pre-purchase signal S_t as a binary signal. A natural generalization is to model it as consisting of more than two (say k) levels à la Bikhchandani et al. (1992). The solution strategy developed in this paper will still be applicable. Specifically, we can restrict to the class of recommendation policies, albeit with a message space of cardinality

$k+1$, denoted, say, as $\mathcal{M} = \{1, 2, \ldots, k, k+1\}$. Accordingly, the notion of net purchase position will be generalized to a $2(k-1)$-dimensional nonnegative integral vector, which counts the cumulative number of purchases and non-purchases *separately* under each message $j = 2, \ldots, k$ (messages 1 and $k+1$ provide no inferential information about the product value). Thus, a recommendation policy can, for each customer t, be represented as mapping r_t from that vector to the probability distributions over the message space $\mathcal{M}$. Finally, we assume in our current model that customers are short-lived and cannot control their order of arrival. However, in light of Corollary 1 and Table E4, customers *may* have incentives to advance or postpone their arrivals. A rigorous analysis of such strategic behavior would require modeling customers as long-lived agents and studying a dynamic timing game among themselves.

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Electronic Companion

Engineering Social Learning: Information Design of Time-Locked Sales Campaigns for Online Platforms

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Appendix A: Proofs in Section 4

Proof of Proposition 1. Since private signals S_t are mutually independent conditional on V with $\mathbb{P}[S_t=1\mid V=1]=\mathbb{P}[S_t=-1\mid V=0]=q$, the Bayes rule thus immediately implies that

$$\begin{aligned}\mathbb{E}[V\mid S_t=1,m_t,\sigma]&=\mathbb{P}[V=1\mid S_t=1,m_t,\sigma]\\&=\frac{\mathbb{P}[S_t=1\mid V=1]\,\mathbb{P}[V=1\mid m_t,\sigma]}{\mathbb{P}[S_t=1\mid V=1]\,\mathbb{P}[V=1\mid m_t,\sigma]+\mathbb{P}[S_t=1\mid V=0]\,\mathbb{P}[V=0\mid m_t,\sigma]}\\&=\frac{q\mathbb{P}[V=1\mid m_t,\sigma]}{q\mathbb{P}[V=1\mid m_t,\sigma]+(1-q)\mathbb{P}[V=0\mid m_t,\sigma]}>\mathbb{P}[V=1\mid m_t,\sigma]=\mathbb{E}[V\mid m_t,\sigma]\end{aligned}$$

and similarly

$$\begin{aligned}\mathbb{E}[V\mid S_t=-1,m_t,\sigma]&=\mathbb{P}[V=1\mid S_t=-1,m_t,\sigma]\\&=\frac{(1-q)\mathbb{P}[V=1\mid m_t,\sigma]}{(1-q)\mathbb{P}[V=1\mid m_t,\sigma]+q\mathbb{P}[V=0\mid m_t,\sigma]}<\mathbb{P}[V=1\mid m_t,\sigma]=\mathbb{E}[V\mid m_t,\sigma],\end{aligned}$$

where the inequalities follow by $1/2<q\le 1$.

Thus, by (2), $a_t=1$ upon $S_t=-1$ if and only if $\frac{(1-q)\mathbb{P}[V=1\mid m_t,\sigma]}{(1-q)\mathbb{P}[V=1\mid m_t,\sigma]+q\mathbb{P}[V=0\mid m_t,\sigma]}\ge p$, which reduces to $\mathbb{E}[V\mid m_t,\sigma]=\mathbb{P}[V=1\mid m_t,\sigma]\ge\frac{pq}{pq+(1-p)(1-q)}=v^{**}$. Similarly, $a_t=1$ upon $S_t=1$ if and only if $\mathbb{E}[V\mid m_t,\sigma]=\mathbb{P}[V=1\mid m_t,\sigma]<\frac{p(1-q)}{p(1-q)+(1-p)q}=v^*$. Since it is straightforward to verify (5), the above two conditions thus lead to (4). ■

Lemma A1. *For an arbitrary information policy $\left(\tilde{\sigma},\widetilde{\mathcal{M}}\right)$, define a mapping $\varphi:\widetilde{\mathcal{M}}\to\mathcal{M}:=\{1,0,-1\}$ as follows:*

$$\varphi(\widetilde{m}_t):=\begin{cases}1, & \text{if } \mathbb{E}[V\mid\widetilde{m}_t,\tilde{\sigma}]\ge v^{**},\\ 0, & \text{if } v^*\le\mathbb{E}[V\mid\widetilde{m}_t,\tilde{\sigma}]<v^{**},\\ -1, & \text{if } \mathbb{E}[V\mid\widetilde{m}_t,\tilde{\sigma}]\le v^*.\end{cases}\tag{A1}$$

For a platform's proprietary history $\widetilde{H}_t=\{(\widetilde{m}_s,a_s):s<t\}$ under $\left(\tilde{\sigma},\widetilde{\mathcal{M}}\right)$, denote $\varphi(\widetilde{H}_t):=\{(\varphi(\widetilde{m}_s),a_s):s<t\}$ with slight abuse of notation. Then,

$$\mathbb{E}\left[V\,\middle|\,\widetilde{H}'_t,\tilde{\sigma}\right]=\mathbb{E}\left[V\,\middle|\,\widetilde{H}''_t,\tilde{\sigma}\right],\quad \text{if } \varphi(\widetilde{H}'_t)=\varphi(\widetilde{H}''_t).\tag{A2}$$

Furthermore, platform's expectation of the product value evolves as follows

$$\mathbb{E}\left[V\,\middle|\,\widetilde{H}_{t+1},\tilde{\sigma}\right]=\begin{cases}\mathbb{E}\left[V\,\middle|\,\widetilde{H}_t,\tilde{\sigma}\right], & \text{if } \varphi(\widetilde{m}_t)=\pm1 \text{ and } \widetilde{H}_{t+1}=\widetilde{H}_t\cup(\widetilde{m}_t,\varphi(\widetilde{m}_t)),\\ \frac{q\mathbb{P}\left[V=1\,\middle|\,\widetilde{H}_t,\tilde{\sigma}\right]}{q\mathbb{P}\left[V=1\,\middle|\,\widetilde{H}_t,\tilde{\sigma}\right]+(1-q)\mathbb{P}\left[V=0\,\middle|\,\widetilde{H}_t,\tilde{\sigma}\right]}, & \text{if } \varphi(\widetilde{m}_t)=0 \text{ and } \widetilde{H}_{t+1}=\widetilde{H}_t\cup(\widetilde{m}_t,1),\\ \frac{(1-q)\mathbb{P}\left[V=1\,\middle|\,\widetilde{H}_t,\tilde{\sigma}\right]}{(1-q)\mathbb{P}\left[V=1\,\middle|\,\widetilde{H}_t,\tilde{\sigma}\right]+q\mathbb{P}\left[V=0\,\middle|\,\widetilde{H}_t,\tilde{\sigma}\right]}, & \text{if } \varphi(\widetilde{m}_t)=0 \text{ and } \widetilde{H}_{t+1}=\widetilde{H}_t\cup(\widetilde{m}_t,-1).\end{cases}\tag{A3}$$

with $\mathbb{E}[V\mid\tilde{\sigma}]=\mathbb{P}[V=1\mid\tilde{\sigma}]=v_0$.

Proof of Lemma A1. By Proposition 1 and definition of φ in (A1), customer t, who receives $\widetilde{m}_t$ with $\varphi(\widetilde{m}_t) = \pm 1$, makes purchase decision $\widetilde{a}_t = \varphi(\widetilde{m}_t) = \pm 1$, regardless of S_t. Thus, platform does not infer any new information in addition to the inference made from the previous customers purchase decisions and hence its expectation of the product value does not alter, establishing the first line in (A3). Again by Proposition 1 and (A1), customer t, who receives $\widetilde{m}_t$ where $\varphi(\widetilde{m}_t) = 0$, makes purchase decision $\widetilde{a}_t = S_t$. Thus, the second and third lines in (A3) follow from the Bayes rule:

$$\begin{aligned}
\mathbb{E}\left[V \,\middle|\, \widetilde{H}_t \cup (\widetilde{m}_t, \widetilde{a}_t), \varphi(\widetilde{m}_t) = 0, \tilde{\sigma}\right] &= \mathbb{P}\left[V = 1 \,\middle|\, \widetilde{H}_t \cup (\widetilde{m}_t, \widetilde{a}_t), \varphi(\widetilde{m}_t) = 0, \tilde{\sigma}\right] \\
&= \mathbb{P}\left[V = 1 \,\middle|\, \widetilde{H}_t, S_t = \widetilde{a}_t, \tilde{\sigma}\right] \\
&= \frac{\mathbb{P}\left[S_t = \widetilde{a}_t \mid V = 1\right] \mathbb{P}\left[V = 1 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right]}{\mathbb{P}\left[S_t = \widetilde{a}_t \mid V = 1\right] \mathbb{P}\left[V = 1 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right] + \mathbb{P}\left[S_t = \widetilde{a}_t \mid V = 0\right] \mathbb{P}\left[V = 0 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right]},
\end{aligned}$$

where $\mathbb{P}[S_t = 1 \mid V = 1] = \mathbb{P}[S_t = -1 \mid V = 0] = q$ by assumption.

We now demonstrate (A2) by induction. For $t = 1$, since $\widetilde{H}_1 = \emptyset$ and $\mathbb{E}[V \mid \tilde{\sigma}] = v_0$, (A2) holds. Now suppose (A2) holds for t. Let $\widetilde{H}'_{t+1}, \widetilde{H}''_{t+1}$ be two platform's proprietary histories such that $\varphi(\widetilde{H}'_{t+1}) = \varphi(\widetilde{H}''_{t+1})$, which implies that $\varphi(\widetilde{H}'_t) = \varphi(\widetilde{H}''_t)$ and $\varphi(\widetilde{m}'_t) = \varphi(\widetilde{m}''_t)$. Thus, the induction hypothesis immediately implies that $\mathbb{E}\left[V \,\middle|\, \widetilde{H}'_t, \tilde{\sigma}\right] = \mathbb{E}\left[V \,\middle|\, \widetilde{H}''_t, \tilde{\sigma}\right]$ and, further by (A3), we have (A2) holds for $t + 1$. ■

Corollary A1 **(Optimal policy for $v_0 \notin [v^*, v^{**})$).** *For $v_0 \notin [v^*, v^{**})$, any information policy is optimal for the platform. In particular, if $v_0 \geq v^{**}$, all customers make the purchase and the platform's profit is $\pi^\star = pT$; if $v_0 < v^*$, none of the customers make the purchase and the platform's profit is $\pi^\star = 0$.*

Proof of Corollary A1. We prove by induction the following property for $v_0 \notin [v^*, v^{**})$: under any arbitrary information policy σ,

$$\mathbb{E}[V \mid m_t, \sigma] = \mathbb{E}[V \mid H_t, \sigma] = v_0, \tag{A4}$$

for any m_t and H_t such that $\sigma(m_t \mid H_t) > 0$ and $\mathbb{P}[H_t \mid \sigma] > 0$, for all $t \in \{1, \cdots, T\}$.

For $t = 1$, by Bayes rule we have for any m_1

$$\begin{aligned}
\mathbb{E}[V \mid m_1, \sigma] = \mathbb{P}[V = 1 \mid m_1, \sigma] &= \frac{\sum_{H_1} \sigma(m_1 \mid H_1) \mathbb{P}[V = 1 \mid H_1, \sigma] \mathbb{P}[H_1 \mid \sigma]}{\sum_{H_1} \sigma(m_1 \mid H_1) \mathbb{P}[H_1 \mid \sigma]} \\
&= \mathbb{P}[V = 1 \mid H_1, \sigma] = \mathbb{E}[V] = v_0,
\end{aligned}$$

where the third and fourth equalities follow from the fact that $H_1 = \emptyset$. Hence, (A4) holds for $t = 1$. Now suppose (A4) holds for an arbitrary t. Again by Bayes rule, we have for any m_{t+1}

$$\mathbb{E}[V \mid m_{t+1}, \sigma] = \mathbb{P}[V = 1 \mid m_{t+1}, \sigma] = \frac{\sum_{H_{t+1}} \sigma(m_{t+1} \mid H_{t+1}) \mathbb{P}[V = 1 \mid H_{t+1}, \sigma] \mathbb{P}[H_{t+1} \mid \sigma]}{\sum_{H_{t+1}} \sigma(m_{t+1} \mid H_{t+1}) \mathbb{P}[H_{t+1} \mid \sigma]} \tag{A5}$$

Notice for each H_{t+1} with $\mathbb{P}[H_{t+1} \mid \sigma] > 0$, by our induction hypothesis and Proposition 1, that we either have $H_{t+1} = H_t \cup (m_t, 1)$ or $H_{t+1} = H_t \cup (m_t, -1)$ conditional on whether $v_0 \geq v^{**}$ or $v_0 < v^*$, respectively. Consequently, again by our induction hypothesis and (A3) of Lemma A1

$$\mathbb{P}[V = 1 \mid H_{t+1}, \sigma] = \mathbb{E}[V \mid H_{t+1}, \sigma] = \mathbb{E}[V \mid H_t, \sigma] = v_0,$$

for all m_t and H_t that satisfy $\sigma(m_t \mid H_t) > 0$ and $\mathbb{P}[H_t \mid \sigma] > 0$. Hence, (A5) becomes $\mathbb{E}[V \mid m_{t+1}, \sigma] = v_0$ for all m_{t+1} with $\mathbb{P}[m_{t+1} \mid \sigma] > 0$. Therefore, (A4) also holds for $t+1$.

Utilizing (A4) and Proposition 1 it is straightforward to see for all $t \in \{1, \cdots, T\}$, $\mathbb{P}[a_t = 1 \mid \sigma] = 1$ when $v_0 \geq v^{**}$ and $\mathbb{P}[a_t = 1 \mid \sigma] = 0$ when $v_0 < v^*$. This completes the proof. ■

Proof of Proposition 2. For an arbitrary information policy $\left(\tilde{\sigma}, \widetilde{\mathcal{M}}\right)$, we now define a new information policy $(\sigma, \mathcal{M})$ as

$$\sigma(m_t \mid H_t) := \frac{\sum\limits_{\varphi(\widetilde{H}_t) = H_t, \varphi(\widetilde{m}_t) = m_t} \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]}{\sum\limits_{\varphi(\widetilde{H}_t) = H_t} \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]}, \tag{A6}$$

for any $m_t \in \mathcal{M}$ and $H_t = \{(m_s, a_s) : m_s \in \mathcal{M}, a_s \in \{-1, 1\}, s < t\}$.

We then show that the information policy $(\sigma, \mathcal{M})$ defined above satisfies ($\mathrm{IC}_{-1}^{\sigma}$), ($\mathrm{IC}_{0}^{\sigma}$), and ($\mathrm{IC}_{1}^{\sigma}$), and hence is a recommendation policy. By rule of total probability, we first have

$$\begin{aligned}
\mathbb{P}[m_t \mid \sigma] &= \sum_{H_t} \sigma(m_t \mid H_t) \mathbb{P}[H_t \mid \sigma] \\
\text{by (A6)} \quad &= \sum_{H_t} \frac{\sum\limits_{\varphi(\widetilde{H}_t) = H_t, \varphi(\widetilde{m}_t) = m_t} \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]}{\sum\limits_{\varphi(\widetilde{H}_t) = H_t} \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]} \sum_{\varphi(\widetilde{H}_t) = H_t} \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right] \\
&= \sum_{\widetilde{H}_t, \varphi(\widetilde{m}_t) = m_t} \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right] = \sum_{\varphi(\widetilde{m}_t) = m_t} \mathbb{P}[\widetilde{m}_t \mid \tilde{\sigma}],
\end{aligned} \tag{A7}$$

and similarly,

$$\begin{aligned}
\mathbb{P}[m_t, V = 1 \mid \sigma] &= \sum_{H_t} \sigma(m_t \mid H_t) \mathbb{P}[V = 1, H_t \mid \sigma] \\
&= \sum_{H_t} \sigma(m_t \mid H_t) \left\{ \sum_{\varphi(\widetilde{H}_t) = H_t} \mathbb{P}\left[V = 1 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right] \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right] \right\} \\
\text{by (A6)} \quad &= \sum_{H_t} \frac{\sum\limits_{\varphi(\widetilde{H}_t) = H_t, \varphi(\widetilde{m}_t) = m_t} \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]}{\sum\limits_{\varphi(\widetilde{H}_t) = H_t} \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]} \left\{ \sum_{\varphi(\widetilde{H}_t) = H_t} \mathbb{P}\left[V = 1 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right] \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right] \right\} \\
&= \sum_{H_t} \sum_{\varphi(\widetilde{H}_t) = H_t, \varphi(\widetilde{m}_t) = m_t} \mathbb{P}\left[V = 1 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right] \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right] \\
&= \sum_{\varphi(\widetilde{m}_t) = m_t} \sum_{\widetilde{H}_t} \mathbb{P}\left[V = 1 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right] \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right],
\end{aligned}$$

where the fourth equality, utilizing the fact that $V \in \{0, 1\}$ is a binary random variable, follows by (A2). Then, the Bayes rule yields

$$\mathbb{E}[V \mid m_t, \sigma] = \mathbb{P}[V = 1 \mid m_t, \sigma] = \frac{\sum_{\varphi(\widetilde{m}_t) = m_t} \sum\limits_{\widetilde{H}_t} \mathbb{P}\left[V = 1 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right] \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]}{\sum_{\varphi(\widetilde{m}_t) = m_t} \mathbb{P}[\widetilde{m}_t \mid \tilde{\sigma}]}. \tag{A8}$$

On the other hand, the Bayes rule also yields

$$\mathbb{E}[V \mid \widetilde{m}_t, \tilde{\sigma}] = \mathbb{P}[V = 1 \mid \widetilde{m}_t, \tilde{\sigma}] = \frac{\sum\limits_{\widetilde{H}_t} \mathbb{P}\left[V = 1 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right] \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]}{\mathbb{P}[\widetilde{m}_t \mid \tilde{\sigma}]}. \tag{A9}$$

For any $\widetilde{m}_t$ such that $\varphi(\widetilde{m}_t)=0$, (A1) implies that $v^* \le \mathbb{E}[V \mid \widetilde{m}_t, \tilde{\sigma}] < v^{**}$, which, by (A9), is equivalent to

$$v^*\mathbb{P}[\widetilde{m}_t \mid \tilde{\sigma}] \le \sum_{\widetilde{H}_t} \mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right] \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right] < v^{**}\mathbb{P}[\widetilde{m}_t \mid \tilde{\sigma}].$$

Further by (A8), we have $\mathbb{E}[V \mid m_t=0,\sigma] = \mathbb{P}[V=1 \mid m_t=0,\sigma] \in [v^*, v^{**})$, establishing ($\mathrm{IC}_0^\sigma$). By the same token, we can also establish (IC_{-1}^σ) and (IC_1^σ).

Finally, we demonstrate that $(\sigma, \mathcal{M})$ and $\left(\tilde{\sigma}, \widetilde{\mathcal{M}}\right)$ induces the same (ex ante) probability of purchase for each customer and generates the same expected revenue for the platform. By Proposition 1 and (A1), customer t's purchase decision under $\left(\tilde{\sigma}, \widetilde{\mathcal{M}}\right)$ is

$$\tilde{a}_t = \varphi(\widetilde{m}_t) + (1 - |\varphi(\widetilde{m}_t)|)\, S_t. \tag{A10}$$

And customer t's purchase decision under $(\sigma, \mathcal{M})$ is

$$a_t = m_t + (1 - |m_t|)\, S_t. \tag{A11}$$

Hence, by (3), the platform's expected revenue under $\left(\tilde{\sigma}, \widetilde{\mathcal{M}}\right)$ is given by $\frac{pT}{2} + \frac{p}{2}\sum_{t=1}^T \mathbb{E}[\tilde{a}_t \mid \tilde{\sigma}]$, and the platform's expected revenue under $(\sigma, \mathcal{M})$ is given by $\frac{pT}{2} + \frac{p}{2}\sum_{t=1}^T \mathbb{E}[a_t \mid \sigma]$. We now demonstrate these revenues are equal by showing

$$\mathbb{E}[\tilde{a}_t \mid \tilde{\sigma}] = \mathbb{E}[a_t \mid \sigma] \text{ for all } t. \tag{A12}$$

On one hand, by (A10)

$$\begin{aligned}\mathbb{E}[\tilde{a}_t \mid \tilde{\sigma}] &= \mathbb{E}[\varphi(\widetilde{m}_t) + (1-|\varphi(\widetilde{m}_t)|)\, S_t \mid \tilde{\sigma}] \\ &= 1\sum_{\varphi(\widetilde{m}_t)=1} \mathbb{P}[\widetilde{m}_t \mid \tilde{\sigma}] + (-1)\sum_{\varphi(\widetilde{m}_t)=-1} \mathbb{P}[\widetilde{m}_t \mid \tilde{\sigma}] + \sum_{\varphi(\widetilde{m}_t)=0} \mathbb{E}[S_t \mid \widetilde{m}_t, \tilde{\sigma}]\, \mathbb{P}[\widetilde{m}_t \mid \tilde{\sigma}].\end{aligned} \tag{A13}$$

On the other hand, by (A11)

$$\begin{aligned}\mathbb{E}[a_t \mid \sigma] &= \mathbb{E}[m_t + (1-|m_t|)\, S_t \mid \sigma] \\ &= 1\mathbb{P}[m_t = 1 \mid \sigma] + (-1)\mathbb{P}[m_t=-1 \mid \sigma] + \mathbb{E}[S_t \mid m_t = 0, \sigma]\, \mathbb{P}[m_t = 0 \mid \sigma].\end{aligned} \tag{A14}$$

By (A7), the first two terms in (A13) are equal to the first two terms in (A14) respectively, i.e., $\sum_{\varphi(\widetilde{m}_t)=1} \mathbb{P}[\widetilde{m}_t \mid \tilde{\sigma}] = \mathbb{P}[m_t = 1 \mid \sigma]$ and $\sum_{\varphi(\widetilde{m}_t)=-1} \mathbb{P}[\widetilde{m}_t \mid \tilde{\sigma}] = \mathbb{P}[m_t = -1 \mid \sigma]$. Below, we demonstrate the last term in (A13) is also equal to that in (A14), thus completing the proof. To that end, denote $\widetilde{u}_{\widetilde{H}_t} := q\mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right] + (1-q)\mathbb{P}\left[V=0 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right]$ and $u_{H_t} := q\mathbb{P}[V=1 \mid H_t, \sigma] + (1-q)\mathbb{P}[V=0 \mid H_t, \sigma]$. Then, the last term in (A13) can be written as

$$\begin{aligned}\sum_{\varphi(\widetilde{m}_t)=0} \mathbb{E}[S_t \mid \widetilde{m}_t, \tilde{\sigma}]\, \mathbb{P}[\widetilde{m}_t \mid \tilde{\sigma}] &= \sum_{\varphi(\widetilde{m}_t)=0} \sum_{\widetilde{H}_t} \left[1\widetilde{u}_{\widetilde{H}_t} + (-1)(1-\widetilde{u}_{\widetilde{H}_t})\right] \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right] \\ &= \sum_{\varphi(\widetilde{m}_t)=0} \sum_{\widetilde{H}_t} \left[2\widetilde{u}_{\widetilde{H}_t} - 1\right] \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right].\end{aligned}$$

Similarly, the last term in (A14) can be rewritten as

$$\mathbb{E}[S_t \mid m_t = 0, \sigma]\, \mathbb{P}[m_t = 0 \mid \sigma] = \sum_{H_t} \left[1u_{H_t} + (-1)(1-u_{H_t})\right] \sigma(m_t = 0 \mid H_t)\, \mathbb{P}[H_t \mid \sigma]$$

$$
\begin{aligned}
&= \sum_{H_t} \left[2u_{H_t} - 1\right] \sigma\left(m_t = 0 \mid H_t\right) \mathbb{P}\left[H_t \mid \sigma\right] \\
&= \sum_{H_t} \left[2u_{H_t} - 1\right] \frac{\sum\limits_{\varphi(\widetilde{H}_t)=H_t,\varphi(\widetilde{m}_t)=0} \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]}{\sum\limits_{\varphi(\widetilde{H}_t)=H_t} \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]} \mathbb{P}\left[H_t \mid \sigma\right] \\
&= \sum_{H_t} \left[2u_{H_t} - 1\right] \frac{\sum\limits_{\varphi(\widetilde{H}_t)=H_t,\varphi(\widetilde{m}_t)=0} \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]}{\sum\limits_{\varphi(\widetilde{H}_t)=H_t} \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]} \sum_{\varphi(\widetilde{H}_t)=H_t} \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right] \\
&= \sum_{H_t} \sum_{\varphi(\widetilde{H}_t)=H_t,\varphi(\widetilde{m}_t)=0} \left[2u_{H_t} - 1\right] \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right] \\
&= \sum_{\varphi(\widetilde{m}_t)=0} \sum_{\widetilde{H}_t} \left[2\widetilde{u}_{\widetilde{H}_t} - 1\right] \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right] \\
&= \sum_{\varphi(\widetilde{m}_t)=0} \mathbb{E}\left[S_t \mid \widetilde{m}_t, \tilde{\sigma}\right] \mathbb{P}\left[\widetilde{m}_t \mid \tilde{\sigma}\right],
\end{aligned}
$$

where the third equality follows from (A6) and the conversion of u_{H_t} to $\widetilde{u}_{\widetilde{H}_t}$ in the sixth equality follows from (A2) and the fact that $V \in \{0,1\}$ is a binary random variable. ■

Proof of Proposition 3. Under a recommendation policy σ, (A3) implies that the platform's expectation of the product value evolves according to

$$
\mathbb{E}\left[V \mid H_{t+1}, \sigma\right] = \begin{cases} \mathbb{E}\left[V \mid H_t, \sigma\right], & \text{if } H_{t+1} = H_t \cup \pm(1,1), \\ \frac{q\mathbb{P}[V=1 \mid H_t,\sigma]}{q\mathbb{P}[V=1 \mid H_t,\sigma]+(1-q)\mathbb{P}[V=0 \mid H_t,\sigma]}, & \text{if } H_{t+1} = H_t \cup (0,1), \\ \frac{(1-q)\mathbb{P}[V=1 \mid H_t,\sigma]}{(1-q)\mathbb{P}[V=1 \mid H_t,\sigma]+q\mathbb{P}[V=0 \mid H_t,\sigma]}, & \text{if } H_{t+1} = H_t \cup (0,-1), \end{cases} \tag{A15}
$$

with $\mathbb{E}[V \mid \sigma] = \mathbb{P}[V = 1 \mid \sigma] = v_0$. We now prove (6) by induction. For $t = 1$, since $H_1 = \emptyset$ and hence $N(H_1) = 0$, thus (6) follows as $\mathbb{E}[V \mid \sigma] = v_0$. Suppose (6) holds for any arbitrary t (i.e., induction hypothesis). Then, under a recommendation policy σ, we have the following three cases for $t+1$:

- If $H_{t+1} = H_t \cup \pm(1,1)$, (A15) implies that $\mathbb{E}[V \mid H_{t+1}, \sigma] = \mathbb{E}[V \mid H_t, \sigma]$ and hence (6) holds for $t+1$ by induction hypothesis and by noting that $N(H_{t+1}) = N(H_t)$ according to (7).
- If $H_{t+1} = H_t \cup (0,1)$, (A15) and induction hypothesis implies that

$$
\mathbb{E}\left[V \mid H_{t+1}, \sigma\right] = \mathbb{P}\left[V = 1 \mid H_{t+1}, \sigma\right] = \frac{qv_0}{qv_0 + (1-q)(1-v_0)\left(\frac{1-q}{q}\right)^{N(H_t)}} = \frac{v_0}{v_0 + (1-v_0)\left(\frac{1-q}{q}\right)^{N(H_t)+1}},
$$

where the second equality follows by noting that $N(H_{t+1}) = N(H_t) + 1$ according to (7), hence (6) holds for $t+1$.

- If $H_{t+1} = H_t \cup (0,-1)$, (A15) and induction hypothesis implies that

$$
\mathbb{E}\left[V \mid H_{t+1}, \sigma\right] = \mathbb{P}\left[V = 1 \mid H_{t+1}, \sigma\right] = \frac{(1-q)v_0}{(1-q)v_0 + q(1-v_0)\left(\frac{1-q}{q}\right)^{N(H_t)}} = \frac{v_0}{v_0 + (1-v_0)\left(\frac{1-q}{q}\right)^{N(H_t)-1}},
$$

where the second equality follows by noting that $N(H_{t+1}) = N(H_t) - 1$ according to (7), hence (6) holds for $t+1$. ■

Proof of Proposition 4. By the definition of recommendation policy σ, the platform's proprietary history evolves according to

$$\mathbb{P}[H_{t+1} \mid H_t, m_t, \sigma] = \begin{cases} 1, & \text{if } m_t = \pm 1 \text{ and } H_{t+1} = H_t \cup (m_t, m_t), \\ q\mathbb{P}[V=1 \mid H_t, \sigma] + (1-q)\mathbb{P}[V=0 \mid H_t, \sigma], & \text{if } m_t = 0 \text{ and } H_{t+1} = H_t \cup (0,1), \\ (1-q)\mathbb{P}[V=1 \mid H_t, \sigma] + q\mathbb{P}[V=0 \mid H_t, \sigma], & \text{if } m_t = 0 \text{ and } H_{t+1} = H_t \cup (0,-1), \end{cases}$$

which, by the characterization of $\mathbb{E}[V \mid H_t, \sigma]$ (equivalently, $\mathbb{P}[V=1 \mid H_t, \sigma]$ as $V \in \{0,1\}$ is a binary random variable) in (6) and the definition of v_n in (8), translates to

$$\mathbb{P}[N(H_{t+1}) \mid N(H_t) = k, m_t] = \begin{cases} 1, & \text{if } m_t = \pm 1 \text{ and } N(H_{t+1}) = k, \\ qv_k + (1-q)(1-v_k) = u_k, & \text{if } m_t = 0 \text{ and } N(H_{t+1}) = k+1, \\ (1-q)v_k + q(1-v_k) = 1-u_k, & \text{if } m_t = 0 \text{ and } N(H_{t+1}) = k-1, \end{cases} \tag{A16}$$

for any k. Total probability rule implies that

$$\begin{aligned} \mathbb{P}[N(H_{t+1}) = n \mid r] &= \sum_{k=n-1}^{n+1} \mathbb{P}[N(H_{t+1}) = n \mid N(H_t) = k, r]\, \mathbb{P}[N(H_t) = k \mid r] \\ &= \sum_{k=n-1}^{n+1} \sum_{i=-1}^{1} \mathbb{P}[N(H_{t+1}) = n \mid N(H_t) = k, m_t = i]\, \mathbb{P}[m_t = i \mid N(H_t) = k, r]\, z_t(k) \\ &= \sum_{k=n-1}^{n+1} \sum_{i=-1}^{1} \mathbb{P}[N(H_{t+1}) = n \mid N(H_t) = k, m_t = i]\, r_t^i(k) z_t(k), \end{aligned}$$

which, by substituting $\mathbb{P}[N(H_{t+1}) = n \mid N(H_t) = k, m_t = i]$ with (A16), yields (N). ■

PROPOSITION A1 **(LP formulation of (P))**. *The optimal solution to (P) is given by*

$$z_t^\star(n) = \sum_{i \in \{1,0,-1\}} y_t^{i\star}(n), \quad r_t^{i\star}(n) = \frac{y_t^{i\star}(n)}{z_t^\star(n)} \text{ for } i \in \{1,0,-1\} \text{ if } z_t^\star(n) > 0, \tag{A17}$$

and $r_t^\star(n)$ being any vector satisfying (R) if $z_t^\star(n) = 0$, where $y^\star$ is the solution to the following LP

$$\pi^\star = \max_{y \geq 0}\; p \sum_{t=1}^{T} \sum_{n=-(T-1)}^{T-1} y_t^1(n) + u_n y_t^0(n) \tag{A18}$$

$$\begin{aligned} \text{subject to}\quad & \sum_{n=-(T-1)}^{T-1} (v_n - v^{**}) y_t^1(n) \geq 0, \qquad \sum_{n=-(T-1)}^{T-1} (v_n - v^*) y_t^{-1}(n) \leq 0, \\ & \sum_{n=-(T-1)}^{T-1} (v_n - v^{**}) y_t^0(n) \leq 0, \qquad \sum_{n=-(T-1)}^{T-1} (v_n - v^*) y_t^0(n) \geq 0, \quad \textit{for } t = 1, \cdots, T; \\ & \sum_{i \in \{1,0,-1\}} y_t^i(n) = y_{t-1}^1(n) + y_{t-1}^{-1}(n) + u_{n-1} y_{t-1}^0(n-1) + (1-u_{n+1}) y_{t-1}^0(n+1) \\ & \textit{for } t = 2, \cdots, T, \textit{ and } n = -(T-1), \cdots, T-1, \\ & \textit{with } y_{t-1}^0(-T) = y_{t-1}^0(T) = 0 \textit{ and } \sum_{i \in \{1,0,-1\}} y_1^i(n) = \mathbb{1}[n=0]. \end{aligned}$$

Proof of Proposition A1. On one hand, let $z_t^\star(n)$ and $r_t^{i\star}(n)$ for $i \in \{1,0,-1\}$, $n = -T+1, \cdots, T-1$ and $t = 1, \cdots, T$ be the optimal solution to (P). We verify below that $y_t^{i\star}(n) := z_t^\star(n) r_t^{i\star}(n)$ for $i \in \{1,0,-1\}$, $n = -T+1, \cdots, T-1$ and $t = 1, \cdots, T$ is a feasible solution to (A18). First, notice by (R) and (N) that, $y_t^{i\star}(n) = z_t^\star(n) r_t^{i\star}(n) \geq 0$ for $i \in \{1,0,-1\}$, $n = -T+1, \cdots, T-1$ and $t = 1, \cdots, T$. Second, utilizing formulation $y_t^{i\star}(n) = z_t^\star(n) r_t^{i\star}(n)$, we have

$$\sum_{i \in \{1,0,-1\}} y_t^{i\star}(n) = z_t^\star(n) \underbrace{\left[r_t^{1\star}(n) + r_t^{0\star}(n) + r_t^{-1\star}(n)\right]}_{=1 \text{ by (R)}}$$

$$
\begin{aligned}
&= \left[z_t^\star(n) - z_{t-1}^\star(n)\right] + \left[z_{t-1}^\star(n) - z_{t-2}^\star(n)\right] + \cdots + \left[z_2^\star(n) - z_1^\star(n)\right] + z_1^\star(n)\\
\text{by (N)}\quad &= u_{n-1}\sum_{s=1}^{t-1} z_s^\star(n-1) r_s^{0\star}(n-1) + (1-u_{n+1})\sum_{s=1}^{t-1} z_s^\star(n+1) r_s^{0\star}(n+1) - \sum_{s=1}^{t-1} z_s^\star(n) r_s^{0\star}(n) + \mathbb{1}[n=0]\\
&= u_{n-1}\sum_{s=1}^{t-1} y_s^{0\star}(n-1) + (1-u_{n+1})\sum_{s=1}^{t-1} y_s^{0\star}(n+1) - \sum_{s=1}^{t-1} y_s^{0\star}(n) + \mathbb{1}[n=0],
\end{aligned}
$$

for all $n = -T+1, \cdots, T-1$ and $t = 1, \cdots, T$. Remaining set of constraints are also established from utilizing the formulation $y_t^{i\star}(n) = z_t^\star(n) r_t^{i\star}(n)$ and (IC_1^r)-(IC_{-1}^r), such that

$$
\begin{aligned}
\text{by } (\mathrm{IC}_1^r)\quad &= \sum_{n=-T+1}^{T-1} (v_n - v^{**}) y_t^{1\star}(n) = \sum_n (v_n - v^{**}) z_t^\star(n) r_t^{1\star}(n) \geq 0,\\
\text{by } (\mathrm{IC}_0^r)\quad &= \sum_{n=-T+1}^{T-1} (v_n - v^{**}) y_t^{0\star}(n) = \sum_n (v_n - v^{**}) z_t^\star(n) r_t^{0\star}(n) \leq 0,\\
\text{by } (\mathrm{IC}_0^r)\quad &= \sum_{n=-T+1}^{T-1} (v_n - v^{*}) y_t^{0\star}(n) = \sum_n (v_n - v^{*}) z_t^\star(n) r_t^{0\star}(n) \geq 0,\\
\text{by } (\mathrm{IC}_{-1}^r)\quad &= \sum_{n=-T+1}^{T-1} (v_n - v^{*}) y_t^{-1\star}(n) = \sum_n (v_n - v^{*}) z_t^\star(n) r_t^{-1\star}(n) \leq 0.
\end{aligned}
$$

Furthermore, it is straightforward to see that

$$
\begin{aligned}
\pi^\star = p\sum_{t=1}^{T}\sum_n z_t^\star(n)\left[r_t^{1\star}(n) + u_n r_t^{0\star}(n)\right] &= p\sum_{t=1}^{T}\sum_{n=-T+1}^{T-1}\left[y_t^{1\star}(n) + u_n y_t^{0\star}(n)\right]\\
&\leq \text{optimal objective value of (A18).}
\end{aligned}
\tag{A19}
$$

On the other hand, let $y_t^{i\star}(n)$ for $i \in \{1, 0, -1\}$, $n = -T+1, \cdots, T-1$ and $t = 1, \cdots, T$ be the optimal solution to (A18). We can verify below that $z_t^\star(n)$ and $r_t^{i\star}(n)$ for $i \in \{1, 0, -1\}$ defined by (A17) (with $r_t^\star(n)$ being any vector satisfying (R) for $z_t^\star(n) = 0$) is a feasible solution to (P). First, notice that (R) is also satisfied for $z_t^\star(n) > 0$, following from (A17) and the fact that $y_t^{i\star}(n) \geq 0$ for all $i \in \{1, 0, -1\}$, $n = -T+1, \cdots, T-1$ and $t = 1, \cdots, T$,

$$
r_t^{1\star}(n) + r_t^{0\star}(n) + r_t^{-1\star}(n) = \sum_{i\in\{1,0,-1\}} \frac{y_t^{i\star}(n)}{z_t^\star(n)} = \frac{z_t^\star(n)}{z_t^\star(n)} = 1.
$$

Second, by utilizing the fact that $z_1(0) = 1$ and $z_1(n) = 0$ for all $n \neq 0$ and (A17), we have

$$
z_t^\star(n) - z_1^\star(n) = u_{n-1}\sum_{s=1}^{t-1} y_s^{0\star}(n-1) + (1-u_{n+1})\sum_{s=1}^{t-1} y_s^{0\star}(n+1) - \sum_{s=1}^{t-1} y_s^{0\star}(n),
$$

which can equivalently be written as

$$
\begin{aligned}
&\left[z_t^\star(n) - z_{t-1}^\star(n)\right] + \left[z_{t-1}^\star(n) - z_{t-2}^\star(n)\right] + \cdots + \left[z_2^\star(n) - z_1^\star(n)\right] =\\
&\qquad u_{n-1}\sum_{s=1}^{t-1} z_s^\star(n-1) r_s^{0\star}(n-1) + (1-u_{n+1})\sum_{s=1}^{t-1} z_s^\star(n+1) r_s^{0\star}(n+1) - \sum_{s=1}^{t-1} z_s^\star(n) r_s^{0\star}(n),
\end{aligned}
$$

that immediately establishes the feasibility of $z_t^\star(n)$ and $r_t^{i\star}(n)$ for $i \in \{1, 0, -1\}$ defined by (A17), for constraint (N). Again by (A17), we also have

$$
\sum_n (v_n - v^{**}) z_t^\star(n) r_t^{1\star}(n) = \sum_{n=-T+1}^{T-1} (v_n - v^{**}) y_t^{1\star}(n) \geq 0,
$$

$$\sum_n (v_n - v^{**}) z_t^\star(n) r_t^{0\star}(n) = \sum_{n=-T+1}^{T-1} (v_n - v^{**}) y_t^{0\star}(n) \le 0,$$
$$\sum_n (v_n - v^{*}) z_t^\star(n) r_t^{0\star}(n) = \sum_{n=-T+1}^{T-1} (v_n - v^{*}) y_t^{0\star}(n) \ge 0,$$
$$\sum_n (v_n - v^{*}) z_t^\star(n) r_t^{-1\star}(n) = \sum_{n=-T+1}^{T-1} (v_n - v^{*}) y_t^{-1\star}(n) \le 0.$$

establishing the IC constraints (IC_1^r)-(IC_{-1}^r), respectively. Furthermore, it is straightforward to see that optimal objective value of (A18) $= p\sum_{t=1}^{T}\sum_{n=-T+1}^{T-1}\left[y_t^{1\star}(n) + u_n y_t^{0\star}(n)\right] = p\sum_{t=1}^{T}\sum_n z_t^\star(n)\left[r_t^{1\star}(n) + u_n r_t^{0\star}(n)\right] \le \pi^\star$ which, combined with (A19), implies that optimal objective value of (A18) $= \pi^\star$. ■

Proof of Proposition 5. Recall that for an arbitrary recommendation policy r and $t \in \{1, 2, \dots, T\}$, by (N), we have $z_t(n) = 0$ for all $n \notin [-(t-1), t-1]$. We also have, by (N), $\sum_n z_t(n) = 1$.

We first show the existence of $n_t^\star$ for the case where both $\mathbb{P}[m_t = 1 \mid r^\star] = 0$ and $\mathbb{P}[m_t = -1 \mid r^\star] = 0$, which immediately implies $\mathbb{P}[m_t = 0 \mid r^\star] = 1 - \mathbb{P}[m_t = 1 \mid r^\star] - \mathbb{P}[m_t = -1 \mid r^\star] = 1$. Therefore, by (R) and the fact that $\mathbb{P}[m_t = 0 \mid r^\star] = \sum_n z_t(n) r_t^{\star 0}(n) = 1$, we must have $r_t^{\star 0}(n) = 1$ for all n with $z_t(n) > 0$, which directly establishes the existence of $n_t^\star$ given in (9).

We next show the existence of $n_t^\star$ for the case where $\mathbb{P}[m_t = 1 \mid r^\star]$ and $\mathbb{P}[m_t = -1 \mid r^\star]$ are not both equal to 0. If $\mathbb{P}[m_t = 1 \mid r^\star] = \sum_n z_t(n) r_t^{\star 1}(n) = 0$, then it is straightforward to see the existence of $n_t^\star$, with $n_t^\star = t - 1$ and $r_t^{\star 1}(n_t^\star) = 0$. Whereas if $\mathbb{P}[m_t = -1 \mid r^\star] = \sum_n z_t(n) r_t^{\star -1}(n) = 0$, it is again straightforward to see the existence of $n_t^\star$, with $n_t^\star = -(t-1)$ and $r_t^{\star 1}(n_t^\star) = 1 - r_t^{\star 0}(n_t^\star)$.

We lastly show the existence of $n_t^\star$ for $\mathbb{P}[m_t = 1 \mid r^\star]\,\mathbb{P}[m_t = -1 \mid r^\star] > 0$ case. Notice that given $r_t^{\star 0}(\cdot)$ for $t = \{1, \dots, T\}$ (which, by (N), also determines $z_t(\cdot)$ for $t = \{1, \dots, T\}$), platform's problem in (P) takes the following form

$$\begin{aligned} \max_{r^1, r^{-1} \ge 0} \quad & p\sum_{t=1}^{T}\sum_n z_t(n) r_t^1(n) \\ \text{subject to} \quad & r_t^1(n) + r_t^{-1}(n) = 1 - r_t^{\star 0}(n) \text{ for all } (t,n), (\mathrm{IC}_1^r) \text{ and } (\mathrm{IC}_{-1}^r). \end{aligned} \tag{A20}$$

For this case, (9) implies $r^\star$ has $\overline{n} \ge \underline{n}$, for each t and net purchase position pair $(\overline{n}, \underline{n})$ with $z_t(\overline{n}) r_t^{\star 1}(\overline{n}) > 0$ and $z_t(\underline{n}) r_t^{\star -1}(\underline{n}) > 0$. In the remainder of this proof we only focus on the net purchase positions n that can be realized for a particular customer t, i.e. integers n with $z_t(n) > 0$. Suppose in the solution $r^\star$ to problem (A20) there exist net purchase positions $k < l$ with $z_t(k) r_t^{\star 1}(k) > 0$ and $z_t(l) r_t^{\star -1}(l) > 0$. We then define $\tilde{r}$ as follows

$$\begin{aligned} z_t(k)\tilde{r}_t^1(k) = z_t(k) r_t^{\star 1}(k) - \alpha, \quad & z_t(k)\tilde{r}_t^{-1}(k) = z_t(k) r_t^{\star -1}(k) + \alpha, \\ z_t(l)\tilde{r}_t^1(l) = z_t(l) r_t^{\star 1}(l) + \alpha, \quad & z_t(l)\tilde{r}_t^{-1}(l) = z_t(l) r_t^{\star -1}(l) - \alpha, \\ \tilde{r}_t^{-1}(n) = r_t^{\star -1}(n), \quad \tilde{r}_t^1(n) = r_t^{\star -1}(n) \quad \forall n \ne \{k, l\}, \quad & \tilde{r}_{\hat{t}}(n) \equiv r_{\hat{t}}^\star(n) \quad \forall n \text{ and } \hat{t} \ne t, \end{aligned} \tag{A21}$$

where $\alpha = \min\{z_t(k) r_t^{\star 1}(k), z_t(l) r_t^{\star -1}(l)\} > 0$. We now show that $\tilde{r}$ achieves the same objective value while satisfying the constraints listed for problem (A20) for customer t (notice that analyzing for customer t suffices, as $\tilde{r}^0 \equiv r^{\star 0}$, hence both policies induce the same $z_t(\cdot)$ for all t). First, see that

$$\sum_n z_t(n)\tilde{r}_t^1(n) = \sum_{n \ne \{k,l\}} z_t(n)\tilde{r}_t^1(n) + z_t(k)\tilde{r}_t^1(k) + z_t(l)\tilde{r}_t^1(l)$$

$$= \sum_{n\neq\{k,l\}} z_t(n) r_t^{\star 1}(n) + z_t(k) r_t^{\star 1}(k) + z_t(l) r_t^{\star 1}(l) = \sum_n z_t(n) r_t^{\star 1}(n),$$

which illustrates the objective value equivalence between $\tilde{r}$ and $r^\star$. By (A21), it is straightforward to see that $\tilde{r}_t^1(n) + \tilde{r}_t^{-1}(n) = r_t^{\star 1}(n) + r_t^{\star -1}(n) = 1 - r_t^{\star 0}(n) = 1 - \tilde{r}_t^0(n)$ for all n. Next, we show (IC_1^r) and (IC_{-1}^r) are also satisfied by $\tilde{r}$, as

$$\begin{aligned}
\sum_n (v_n - v^{**}) z_t(n) \tilde{r}_t^1(n) &= \sum_{n\neq\{k,l\}} (v_n - v^{**}) z_t(n) \tilde{r}_t^1(n) + (v_k - v^{**}) z_t(k) \tilde{r}_t^1(k) + (v_l - v^{**}) z_t(l) \tilde{r}_t^1(l) \\
&= \sum_{n\neq\{k,l\}} (v_n - v^{**}) z_t(n) r_t^{\star 1}(n) + (v_k - v^{**}) z_t(k) r_t^{\star 1}(k) + (v_l - v^{**}) z_t(l) r_t^{\star 1}(l) + \alpha(v_l - v_k) \\
&= \sum_n (v_n - v^{**}) z_t(n) r_t^{\star 1}(n) + \underbrace{\alpha(v_l - v_k)}_{>0 \text{ by } (8)} > \sum_n (v_n - v^{**}) z_t(n) r_t^{\star 1}(n) \geq 0
\end{aligned}$$

and

$$\begin{aligned}
\sum_n (v_n - v^{*}) z_t(n) \tilde{r}_t^{-1}(n) &= \sum_{n\neq\{k,l\}} (v_n - v^{*}) z_t(n) \tilde{r}_t^{-1}(n) + (v_k - v^{*}) z_t(k) \tilde{r}_t^{-1}(k) + (v_l - v^{*}) z_t(l) \tilde{r}_t^{-1}(l) \\
&= \sum_{n\neq\{k,l\}} (v_n - v^{*}) z_t(n) \tilde{r}_t^{-1}(n) + (v_k - v^{*}) z_t(k) \tilde{r}_t^{-1}(k) + (v_l - v^{*}) z_t(l) \tilde{r}_t^{-1}(l) + \alpha(v_k - v_l) \\
&= \sum_n (v_n - v^{*}) z_t(n) r_t^{\star -1}(n) + \underbrace{\alpha(v_k - v_l)}_{<0 \text{ by } (8)} < \sum_n (v_n - v^{*}) z_t(n) r_t^{\star -1}(n) \leq 0,
\end{aligned}$$

respectively. Repeated iterations of the above procedure establish the existence of such $n_t^\star$ given by (9).

We now prove the second statement of Proposition 5. Suppose in the solution $r^\star$ to problem (A20) we have $\mathbb{P}[m_t = -1 \mid r^\star] = \sum_n z_t(n) r_t^{\star -1}(n) > 0$ (which equivalently means there exists a net purchase position l with $z_t(l) r_t^{\star -1}(l) > 0$) and ($\mathrm{IC}_1^r$) is not binding for customer t. We then define $\tilde{r}_t$ as follows

$$z_t(l)\tilde{r}_t^1(l) = z_t(l) r_t^{\star 1}(l) + \alpha, \quad z_t(l)\tilde{r}_t^{-1}(l) = z_t(l) r_t^{\star -1}(l) - \alpha \quad \text{and} \quad \tilde{r}_t^1(n) = r_t^{\star 1}(n) \quad \forall n \neq l \tag{A22}$$

where α takes positive values with respect to the following three cases. In all of these cases, we show $\tilde{r}_t$ improves the objective value while satisfying the constraints of problem (A20). The first constraint, $r_t^{\star 1}(n) + r_t^{\star -1}(n) = 1 - r_t^{\star 0}(n)$ for all n, is trivially satisfied by (A22), hence its proof is omitted for brevity.

- <u>If $v_l \geq v^{**}$,</u> we have $\alpha = z_t(l) r_t^{\star -1}(l) > 0$. First, see that

$$\begin{aligned}
\sum_n z_t(n) \tilde{r}_t^1(n) &= \sum_{n\neq l} z_t(n) \tilde{r}_t^1(n) + z_t(l) \tilde{r}_t^1(l) \\
&= \sum_{n\neq l} z_t(n) r_t^{\star 1}(n) + z_t(l) r_t^{\star 1}(l) + \alpha > \sum_n z_t(n) r_t^{\star 1}(n)
\end{aligned}$$

which illustrates the objective value improvement by $\tilde{r}_t$ compared to $r_t^\star$. Next, we show (IC_1^r) and (IC_{-1}^r) are also satisfied by $\tilde{r}_t$, as

$$\begin{aligned}
\sum_n (v_n - v^{**}) z_t(n) \tilde{r}_t^1(n) &= \sum_{n\neq l} (v_n - v^{**}) z_t(n) \tilde{r}_t^1(n) + (v_l - v^{**}) z_t(l) \tilde{r}_t^1(l) \\
&= \sum_{n\neq l} (v_n - v^{**}) z_t(n) r_t^{\star 1}(n) + (v_l - v^{**}) z_t(l) r_t^{\star 1}(l) + \underbrace{\alpha(v_l - v^{**})}_{\geq 0} \\
&\geq \sum_n (v_n - v^{**}) z_t(n) r_t^{\star 1}(n) \geq 0
\end{aligned}$$

and

$$\begin{aligned}\sum_n (v_n - v^*) z_t(n) \tilde{r}_t^{-1}(n) &= \sum_{n \neq l} (v_n - v^*) z_t(n) \tilde{r}_t^{-1}(n) + (v_l - v^*) z_t(l) \tilde{r}_t^{-1}(l) \\ &= \sum_{n \neq l} (v_n - v^*) z_t(n) r_t^{\star -1}(n) + (v_l - v^*) z_t(l) r_t^{\star -1}(l) - \underbrace{\alpha(v_l - v^*)}_{<0} \\ &< \sum_n (v_n - v^*) z_t(n) r_t^{\star -1}(n) \leq 0,\end{aligned}$$

respectively.

- If $v_l \in [v^*, v^{**})$, we have $\alpha = \min\left\{ z_t(l) r_t^{\star -1}(l), \frac{\sum_n (v_n - v^{**}) z_t(n) r_t^{\star 1}(n)}{(v^{**} - v_l)} \right\} > 0$. Notice that the second term inside the parenthesis is also strictly positive from our assumption that (IC_1^r) is not binding. First, see that

$$\sum_n z_t(n) \tilde{r}_t^1(n) = \sum_{n \neq l} z_t(n) r_t^{\star 1}(n) + z_t(l) r_t^{\star 1}(l) + \alpha > \sum_n z_t(n) r_t^{\star 1}(n)$$

which illustrates the objective value improvement by $\tilde{r}_t$ compared to $r_t^\star$. Next, we show (IC_1^r) is also satisfied by $\tilde{r}_t$, as

$$\sum_n (v_n - v^{**}) z_t(n) \tilde{r}_t^1(n) = \sum_n (v_n - v^{**}) z_t(n) r_t^{\star 1}(n) + \underbrace{\alpha(v_l - v^{**})}_{<0} \geq 0,$$

where the last inequality is established by the definition of α. Similarly, (IC_{-1}^r) also follows as

$$\sum_n (v_n - v^*) z_t(n) \tilde{r}_t^{-1}(n) = \sum_n (v_n - v^*) z_t(n) r_t^{\star -1}(n) - \underbrace{\alpha(v_l - v^*)}_{\geq 0} \leq \sum_n (v_n - v^*) z_t(n) r_t^{\star -1}(n) \leq 0.$$

- If $v_l < v^*$, we have $\alpha = \min\left\{ z_t(l) r_t^{\star -1}(l), \frac{\sum_n (v_n - v^{**}) z_t(n) r_t^{\star 1}(n)}{(v^{**} - v_l)} \right\} > 0$. This case is assumed to be applied after exhausting all such l with $v_l \geq v^*$ and $z_t(l) r_t^{\star -1}(l) > 0$. First, see that

$$\sum_n z_t(n) \tilde{r}_t^1(n) = \sum_{n \neq l} z_t(n) r_t^{\star 1}(n) + z_t(l) r_t^{\star 1}(l) + \alpha > \sum_n z_t(n) r_t^{\star 1}(n)$$

which illustrates the objective value improvement by $\tilde{r}_t$ compared to $r_t^\star$. Next, we show (IC_1^r) is also satisfied by $\tilde{r}_t$, as

$$\sum_n (v_n - v^{**}) z_t(n) \tilde{r}_t^1(n) = \sum_n (v_n - v^{**}) z_t(n) r_t^{\star 1}(n) + \underbrace{\alpha(v_l - v^{**})}_{<0} \geq 0,$$

where the last inequality is established by the definition of α. Similarly, (IC_{-1}^r) also follows as

$$\begin{aligned}\sum_n (v_n - v^*) z_t(n) \tilde{r}_t^{-1}(n) &= \sum_n (v_n - v^*) z_t(n) r_t^{\star -1}(n) - \underbrace{\alpha(v_l - v^*)}_{<0} \\ &= \underbrace{\sum_{n \neq l} (v_n - v^*) z_t(n) r_t^{\star -1}(n)}_{\substack{\leq 0 \text{ as all such } n \\ \text{with } z_t(l) r_t^{\star -1}(l) > 0 \\ \text{has } v_n < v^*}} + (v_l - v^*) \underbrace{\left[z_t(n) r_t^{\star -1}(n) - \alpha \right]}_{\geq 0} \leq 0.\end{aligned}$$

Combined with the previous cases, this result establishes the second statement of Proposition 5. ■

Proof of Proposition 6. First, notice that

$$v_n = v_{n+1}u_n + v_{n-1}(1-u_n) \quad \forall n. \tag{A23}$$

which is directly established by utilizing (8) and the description of u_n in Proposition 4:

$$\begin{aligned}
v_n =& q v_n + (1-q) v_n \\
=& \underbrace{\frac{qv_0 + (1-q)(1-v_0)(\frac{1-q}{q})^n}{v_0 + (1-v_0)(\frac{1-q}{q})^{n+1}}}_{q} \underbrace{\frac{v_0}{v_0 + (1-v_0)(\frac{1-q}{q})^n}}_{v_n} + \underbrace{\frac{(1-q)v_0 + q(1-v_0)(\frac{1-q}{q})^n}{v_0 + (1-v_0)(\frac{1-q}{q})^{n-1}}}_{1-q} \underbrace{\frac{v_0}{v_0 + (1-v_0)(\frac{1-q}{q})^n}}_{v_n} \\
=& \underbrace{\frac{v_0}{v_0 + (1-v_0)(\frac{1-q}{q})^{n+1}}}_{v_{n+1}} \underbrace{\frac{qv_0 + (1-q)(1-v_0)(\frac{1-q}{q})^n}{v_0 + (1-v_0)(\frac{1-q}{q})^n}}_{u_n} + \underbrace{\frac{v_0}{v_0 + (1-v_0)(\frac{1-q}{q})^{n-1}}}_{v_{n-1}} \underbrace{\frac{(1-q)v_0 + q(1-v_0)(\frac{1-q}{q})^n}{v_0 + (1-v_0)(\frac{1-q}{q})^n}}_{(1-u_n)}.
\end{aligned}$$

As shown by Rothschild and Stiglitz (1970), many equivalent definitions of *mean-preserving spread* exist. We adopt the following one: a belief distribution $\tilde{z}_t$ is said to be a *mean-preserving spread* of belief distribution z_t if there exists a nonnegative $(2T+1)$-square matrix $C = \{c_{mn} : m, n = T, \cdots, 0, \cdots, -T\}$ such that

$$\sum_m c_{mn} = 1, \quad \sum_m v_m c_{mn} = v_n \text{ for all } n, \quad \text{and} \quad \sum_n c_{mn} z_t(n) = \tilde{z}_t(m) \text{ for all } m. \tag{A24}$$

Thus, we prove this Proposition by constructing such a matrix C as follows:

$$c_{mn} := \begin{cases} \alpha_n u_n, & \text{if } m = n+1 \le T, \\ 1-\alpha_n, & \text{if } m = n, \\ \alpha_n(1-u_n), & \text{if } m = n-1 \ge -T, \\ 0, & \text{if otherwise,} \end{cases} \tag{A25}$$

where

$$\alpha_n = \begin{cases} \frac{z_{t-1}(n)\left[\tilde{r}^0_{t-1}(n) - r^0_{t-1}(n)\right]}{z_{t-1}(n+1)(1-u_{n+1})r^0_{t-1}(n+1) + z_{t-1}(n)\left[1 - r^0_{t-1}(n)\right] + z_{t-1}(n-1)u_{n-1}r^0_{t-1}(n-1)} = \frac{z_{t-1}(n)\left[\tilde{r}^0_{t-1}(n) - r^0_{t-1}(n)\right]}{z_t(n)}, & \text{if } z_t(n) > 0, \\ 0, & \text{if otherwise.} \end{cases}$$

We have, by definition, $u_n \in [0,1]$ for all n. Notice that we also have (i) $\alpha_n \ge 0$, as $z_t(n) \ge 0$ for all t, n and the fact that $\tilde{r}^0_{t-1}(n) \ge r^0_{t-1}(n)$ for all n; and (ii) $\alpha_n \le 1$, as $z_t(n) \ge z_{t-1}(n)\left[1 - r^0_{t-1}(n)\right]$ by the definition of α_n and the fact that $\tilde{r}^0_{t-1}(n) \in [r^0_{t-1}(n), 1]$ for all n, hence the nonnegativity of C. It is straightforward to see that the first condition of (A24) is satisfied as, for each n, C matrix given in Equation (A25) has $\sum_m c_{mn} = \alpha_n u_n + 1 - \alpha_n + \alpha_n(1-u_n) = 1$. The second condition of Equation (A24) also trivially follows as, for each n, C matrix given in Equation (A25) has $\sum_m v_m c_{mn} = v_{n+1}\alpha_n u_n + v_n(1-\alpha_n) + v_{n-1}\alpha_n(1-u_n) = v_n\alpha_n + v_n(1-\alpha_n) = v_n$, where the second equality is established by Equation (A23). We lastly show that, for each n, C matrix given in Equation (A25) satisfies the third condition of (A24), as

$$\begin{aligned}
\sum_n c_{mn} z_t(n) =& \alpha_{m+1}(1-u_{m+1}) z_t(m+1) + (1-\alpha_m) z_t(m) + \alpha_{m-1} u_{m-1} z_t(m-1) \\
&= z_{t-1}(m+1)(1-u_{m+1})\left[\tilde{r}^0_{t-1}(m+1) - r^0_{t-1}(m+1)\right] + z_t(m) \\
&\quad - z_{t-1}(m)\left[\tilde{r}^0_{t-1}(m) - r^0_{t-1}(m)\right] + z_{t-1}(m-1)u_{m-1}\left[\tilde{r}^0_{t-1}(m-1) - r^0_{t-1}(m-1)\right] = \tilde{z}_t(m),
\end{aligned}$$

where the second equality follows by the definition of α_m and the last equality follows by the fact that $z_t(m) = \left(1 - r^0_{t-1}(m)\right) z_{t-1}(m) + u_{m-1} r^0_{t-1}(m-1) z_{t-1}(m-1) + (1-u_{m+1}) r^0_{t-1}(m+1) z_{t-1}(m+1)$ given by (N). ■

Proof of Proposition 7. For any feasible policy r_t in (10) under the belief z_t, we prove this proposition by constructing below a recommendation policy $\tilde{r}_t$ that (i) is feasible in (10) under a mean-preserving spread $\tilde{z}_t$ of z_t and (ii) attains the same expected revenue under $\tilde{z}_t$ as that attained by r_t under z_t.

Let us define $\tilde{r}_t$ (for $\tilde{z}_t(m)>0$) as $\tilde{r}_t^i(m) = \frac{\sum_n c_{mn} r_t^i(n) z_t(n)}{\tilde{z}_t(m)}$ for all m and $i \in \{\pm 1, 0\}$. It is then straightforward to see that $\tilde{r}_t$ satisfies (R), as

$$\sum_{i\in\{\pm1,0\}} \tilde{r}_t^i(m) = \frac{\sum_n c_{mn}\left[r_t^1(n)+r_t^0(n)+r_t^{-1}(n)\right] z_t(n)}{\tilde{z}_t(m)} = \frac{\sum_n c_{mn} z_t(n)}{\tilde{z}_t(m)} = \frac{\tilde{z}_t(m)}{\tilde{z}_t(m)} = 1,$$

where third equality follows by (A24). By (A24) and the definition of $\tilde{r}_t$ we have

$$\sum_m \tilde{z}_t(m)\tilde{r}_t^i(m) = \sum_m\sum_n c_{mn} z_t(n) r_t^i(n) = \sum_n z_t(n) r_t^i(n) \underbrace{\sum_m c_{mn}}_{=1} = \sum_m z_t(m) r_t^i(m) \tag{A26}$$

and

$$\sum_m v_m \tilde{z}_t(m)\tilde{r}_t^i(m) = \sum_m\sum_n v_m c_{mn} z_t(n) r_t^i(n) = \sum_n z_t(n) r_t^i(n) \underbrace{\sum_m v_m c_{mn}}_{=v_n} = \sum_m v_m z_t(m) r_t^i(m). \tag{A27}$$

Using (A26) and (A27), we can compute the left hand side of (IC_1^r) under $(\tilde{r}_t, \tilde{z}_t)$ as

$$\begin{aligned}\sum_m (v_m - v^{**})\tilde{z}_t(m)\tilde{r}_t^1(m) &= \sum_m v_m \tilde{z}_t(m)\tilde{r}_t^1(m) - v^{**}\sum_m \tilde{z}_t(m)\tilde{r}_t^1(m)\\ &= \sum_n z_t(n) r_t^1(n) \underbrace{\sum_m v_m c_{mn}}_{=v_n} - v^{**}\sum_n z_t(n) r_t^1(n)\underbrace{\sum_m c_{mn}}_{=1}\\ &= \sum_m v_m z_t(m) r_t^1(m) - v^{**}\sum_m z_t(m) r_t^1(m) = \sum_m (v_m - v^{**}) z_t(m) r_t^1(m) \geq 0;\end{aligned}$$

namely, $(\tilde{r}_t, \tilde{z}_t)$ satisfies (IC_1^r). Similarly, (IC_0^r) and (IC_{-1}^r) are also satisfied by $(\tilde{r}_t, \tilde{z}_t)$.

Again, using (A26) and (A27), we can compute the objective of (10) under $(\tilde{r}_t, \tilde{z}_t)$ as

$$\begin{aligned}p\sum_n \tilde{z}_t(n)\left[\tilde{r}_t^1(n) + u_n \tilde{r}_t^0(n)\right] &= p\left[\sum_n \tilde{z}_t(n)\tilde{r}_t^1(n) + (1-q)\sum_n \tilde{z}_t(n)\tilde{r}_t^0(n) + (2q-1)\sum_n v_n \tilde{z}_t(n)\tilde{r}_t^0(n)\right]\\ &= p\left[\sum_n z_t(n) r_t^1(n) + (1-q)\sum_n z_t(n) r_t^0(n) + (2q-1)\sum_n v_n z_t(n) r_t^0(n)\right]\\ &= p\sum_n z_t(n)\left[r_t^1(n) + u_n r_t^0(n)\right];\end{aligned}$$

namely, $(\tilde{r}_t, \tilde{z}_t)$ achieves the same revenue as (r_t, z_t) does.

For $v_0 \in [v^*, v^{**})$, we note that under $r_t^0(n) \equiv 1$ for all n, (i) (R), (IC_1^r), and (IC_{-1}^r) are automatically satisfied because $r_t^{\pm 1}(n) \equiv 0$ for all n, and, (ii) (IC_0^r) also holds because, by Proposition 6,

$$\sum_n v_n z_t(n) = \sum_n v_n z_1(n) = v_0 \in [v^*, v^{**}). \tag{A28}$$

The corresponding revenue is given by

$$p\sum_n u_n z_t(n) = p\sum_n \left[q v_n + (1-q)(1-v_n)\right] z_t(n) = p\left[q v_0 + (1-q)(1-v_0)\right] = p u_0,$$

where the first and last equalities follow from the definition of u_n and the second one follows from (A28). ■

Appendix B: Proofs in Sections 5

Proof of Lemma 1. We prove this proposition by induction. It is straightforward to see (11) holds for customer $t=1$, since $\ell_1^r=0$ by definition and we have $z_1(0)=1, z_1(n)=0\ \forall n\neq 0$ by (N).

Now, suppose that (11) holds for customer t (i.e., induction hypothesis) and $\ell_t^r=s$. If $r_t^0(n)\equiv 1$ or equivalently $\ell_{t+1}^r=\ell_t^r+1=s+1$, for an arbitrary net purchase position n with $n\equiv s+1 \pmod 2$ and $|n|\leq s+1$, we have

$$z_{t+1}(n)=[\underbrace{qv_{n-1}+(1-q)(1-v_{n-1})}_{u_{n-1}}]z_t(n-1)+[\underbrace{(1-q)v_{n+1}+q(1-v_{n+1})}_{1-u_{n+1}}]z_t(n+1), \tag{B1}$$

by (N) as $r_t^0(n)=1\ \forall n$. For $|n|\leq s-1$, substituting the definition of v_n in (8) into (B1) and using our induction hypothesis, we obtain

$$\begin{aligned}
z_{t+1}(n)=&\frac{v_0q^n+(1-v_0)(1-q)^n}{v_0q^{n-1}+(1-v_0)(1-q)^{n-1}}\binom{s}{\frac{s+n-1}{2}}\left[v_0q^{\frac{s+n-1}{2}}(1-q)^{\frac{s-n+1}{2}}+(1-v_0)(1-q)^{\frac{s+n-1}{2}}q^{\frac{s-n+1}{2}}\right]\\
&+\frac{v_0q^{n+1}(1-q)+(1-v_0)q(1-q)^{n+1}}{v_0q^{n+1}+(1-v_0)(1-q)^{n+1}}\binom{s}{\frac{s+n+1}{2}}\left[v_0q^{\frac{s+n+1}{2}}(1-q)^{\frac{s-n-1}{2}}+(1-v_0)(1-q)^{\frac{s+n+1}{2}}q^{\frac{s-n-1}{2}}\right]\\
=&\left[\binom{s}{\frac{s+n-1}{2}}+\binom{s}{\frac{s+n+1}{2}}\right]\left[v_0q^{\frac{s+n+1}{2}}(1-q)^{\frac{s-n+1}{2}}+(1-v_0)(1-q)^{\frac{s-n+1}{2}}q^{\frac{s+n+1}{2}}\right]\\
=&\binom{s+1}{\frac{s+1+n}{2}}\left[v_0q^{\frac{s+1+n}{2}}(1-q)^{\frac{s+1-n}{2}}+(1-v_0)(1-q)^{\frac{s+1+n}{2}}q^{\frac{s+1-n}{2}}\right]=\zeta(\ell_{t+1}^r,n).
\end{aligned}$$

For $n=s+1$, we have $z_t(n+1)=0$ in (B1) by (N) and hence

$$\begin{aligned}
z_{t+1}(n)=&\frac{v_0q^n+(1-v_0)(1-q)^n}{v_0q^{n-1}+(1-v_0)(1-q)^{n-1}}\left[v_0q^{n-1}+(1-v_0)(1-q)^{n-1}\right]\\
=&[v_0q^n+(1-v_0)(1-q)^n]=\zeta(\ell_{t+1}^r,n).
\end{aligned}$$

For $n=-s-1$, we have $z_t(n-1)=0$ in (B1) by (N) and hence

$$\begin{aligned}
z_{t+1}(n)=&\frac{v_0q^{n+1}(1-q)+(1-v_0)q(1-q)^{n+1}}{v_0q^{n+1}+(1-v_0)(1-q)^{n+1}}\left[v_0(1-q)^{-n-1}+(1-v_0)q^{-n-1}\right]\\
=&v_0q^{n+1}(1-q)+(1-v_0)q(1-q)^{n+1}\left[q^{-n-1}(1-q)^{-n-1}\right]\\
=&[v_0(1-q)^{-n}+(1-v_0)q^{-n}]=\zeta(\ell_{t+1}^r,n).
\end{aligned}$$

Thus, (11) holds for customer $t+1$. If $r_t^0(n)\equiv 0$ or equivalently $\ell_{t+1}^r=\ell_t^r=s$ on the other hand, we have $z_{t+1}(n)=z_t(n)\ \forall_n$ by (N) as $r_t^0(n)=0\ \forall n$. Thus, our induction hypothesis immediately implies that (11) holds again for customer $t+1$. This completes the proof. ■

Lemma B1 **(Properties of $\zeta(\cdot,\cdot)$).** *For any integers $s\geq 0$ and $k\equiv s \pmod 2$, $\zeta(\cdot,\cdot)$ defined in (11) satisfies the following properties,*

$$\sum_n v_n\zeta(s,n)=\sum_n v_n\zeta(s+1,n)=v_0, \tag{B2}$$

$$\sum_{n>k}(v_n-v^{**})\zeta(s,n)=\sum_{n>k+1}(v_n-v^{**})\zeta(s+1,n)+(v_{k+1}-v^{**})\zeta(s,k+2)(1-u_{k+2}), \tag{B3}$$

$$\sum_{n>k-1}(v_n-v^{**})\zeta(s+1,n)=\sum_{n>k}(v_n-v^{**})\zeta(s,n)+(v_{k+1}-v^{**})\zeta(s,k)u_k, \tag{B4}$$

$$\sum_{n<k}(v_n-v^{*})\zeta(s,n)=\sum_{n<k-1}(v_n-v^{*})\zeta(s+1,n)+(v_{k-1}-v^{*})\zeta(s,k-2)u_{k-2}, \tag{B5}$$

$$\sum_{n<k+1}(v_n-v^{*})\zeta(s+1,n)=\sum_{n<k}(v_n-v^{*})\zeta(s,n)+(v_{k-1}-v^{*})\zeta(s,k)(1-u_k). \tag{B6}$$

Proof of Lemma B1. First, notice that by Lemma 1, we can write (B1) as

$$\zeta(s+1,n)=\zeta(s,n-1)u_{n-1}+\zeta(s,n+1)(1-u_{n+1}) \quad \forall_{n,s\geq 0}, \tag{B7}$$

which, in turn, implies

$$\sum_n \zeta(s,n)=\sum_n \zeta(s+1,n)=1. \quad \forall_{s\geq 0} \tag{B8}$$

Where the second equality follows by the fact that $\zeta(0,0)=1$ and $\zeta(0,n)=0\ \forall_{n\neq 0}$. For any $s\geq 0$, by (A23),

$$\begin{aligned}\sum_n v_n\zeta(s,n) &= \sum_n \{v_{n+1}\zeta(s,n)u_n+v_{n-1}\zeta(s,n)(1-u_n)\}\\ &= \sum_n \{v_n\zeta(s,n-1)u_{n-1}+v_n\zeta(s,n+1)(1-u_{n+1})\}\\ \text{by (B7)} &= \sum_n v_n\zeta(s+1,n),\end{aligned}$$

hence establishing the first equality of (B2). The second equality of (B2) follows from the fact that $\zeta(0,0)=1$ and $\zeta(0,n)=0\ \forall n\neq 0$.

Notice that for an arbitrary constant c, by (B2),

$$\sum_n (v_n-c)\zeta(s,n)=\sum_n (v_n-c)\zeta(s+1,n),$$

following from (B8). Subtracting $\sum_{n\leq k}(v_n-c)\zeta(s,n)$ from both sides yields

$$\sum_{n>k}(v_n-c)\zeta(s,n)=\underbrace{\sum_n (v_n-c)\zeta(s+1,n)}_{A}-\underbrace{\sum_{n\leq k}(v_{n+1}-c)\zeta(s,n)u_n+(v_{n-1}-c)\zeta(s,n)(1-u_n)}_{B},$$

where B follows by (A23). Then, re-arranging terms in A and B and utilizing (B7) yield

$$\begin{aligned}A&=\sum_{n>k}(v_n-c)\zeta(s+1,n)+\sum_{n\leq k}(v_n-c)\zeta(s,n+1)(1-u_{n+1})+\sum_{n\leq k}(v_n-c)\zeta(s,n-1)u_{n-1}, \quad \text{and}\\ B&=\sum_{n\leq k}(v_{n+2}-c)\zeta(s,n+1)u_{n+1}+\sum_{n\leq k}(v_n-c)\zeta(s,n+1)(1-u_{n+1}),\end{aligned}$$

which imply

$$\begin{aligned}\sum_{n>k}(v_n-c)\zeta(s,n)&=\sum_{n>k}(v_n-c)\zeta(s+1,n)+\sum_{n\leq k}(v_n-c)\zeta(s,n-1)u_{n-1}-\sum_{n\leq k}(v_{n+2}-c)\zeta(s,n+1)u_{n+1}\\ &=\sum_{n>k}(v_n-c)\zeta(s+1,n)-(v_{k+1}-c)\zeta(s,k)u_k\\ &=\sum_{n>k+1}(v_n-c)\zeta(s+1,n)+(v_{k+1}-c)[\underbrace{\zeta(s+1,k+1)-\zeta(s,k)u_k}_{=\zeta(s,k+2)(1-u_{k+2})\text{ by (B7)}}],\end{aligned}$$

establishing (B3). Then, (B4) follows immediately from (B3) by noting that

$$\begin{aligned}\sum_{n>k}(v_n-c)\zeta(s,n)&=\sum_{n>k-1}(v_n-c)\zeta(s+1,n)-(v_{k+1}-c)\zeta(s+1,k+1)+(v_{k+1}-c)\zeta(s,k+2)(1-u_{k+2})\\ &=\sum_{n>k-1}(v_n-c)\zeta(s+1,n)-(v_{k+1}-c)\zeta(s,k)u_k,\end{aligned}$$

where the last equality again follows from (B7).

To see (B5), we have

$$\begin{aligned}
\sum_{n<k}(v_n-c)\zeta(s,n) &= \sum_{n}(v_n-c)\zeta(s,n)-\sum_{n>k}(v_n-c)\zeta(s,n)-(v_k-c)\zeta(s,k)\\
\text{by (B3)}\quad &= \sum_{n}(v_n-c)\zeta(s+1,n)-\sum_{n>k+1}(v_n-c)\zeta(s+1,n)-(v_{k+1}-c)\zeta(s,k+2)(1-u_{k+2})-(v_k-c)\zeta(s,k)\\
&= \sum_{n<k-1}(v_n-c)\zeta(s+1,n)+(v_{k-1}-c)\underbrace{\zeta(s+1,k-1)}_{=\zeta(s,k-2)u_{k-2}+\zeta(s,k)(1-u_k)\text{ by (B7)}}\\
&\quad+(v_{k+1}-c)[\underbrace{\zeta(s+1,k+1)-(v_{k+1}-c)\zeta(s,k+2)(1-u_{k+2})}_{=\zeta(s,k)u_k\text{ by (B7)}}]-(v_k-c)\zeta(s,k)\\
&= \sum_{n<k-1}(v_n-c)\zeta(s+1,n)+(v_{k-1}-c)\zeta(s,k-2)u_{k-2}\\
&\quad+\zeta(s,k)[\underbrace{(v_{k-1}-c)(1-u_k)+(v_{k+1}-c)u_k-(v_k-c)}_{=0\text{ by (A23)}}]\\
&= \sum_{n<k-1}(v_n-c)\zeta(s+1,n)+(v_{k-1}-c)\zeta(s,k-2)u_{k-2}.
\end{aligned}$$

Finally, we show (B6). By (B5),

$$\begin{aligned}
\sum_{n<k}(v_n-c)\zeta(s,n) &= \sum_{n<k-1}(v_n-c)\zeta(s+1,n)+(v_{k-1}-c)\zeta(s,k-2)u_{k-2}\\
\text{by (B7)} &= \sum_{n<k-1}(v_n-c)\zeta(s+1,n)+(v_{k-1}-c)\zeta(s+1,k-1)-(v_{k-1}-c)\zeta(s,k)(1-u_k)\\
&= \sum_{n<k+1}(v_n-c)\zeta(s+1,n)-(v_{k-1}-c)\zeta(s,k)(1-u_k),
\end{aligned}$$

establishing (B6). ■

LEMMA B2. *If it is incentive compatible to assign customer t as an affirmative customer under an NA-sequencing policy $\hat{r}$ with $\ell_t^{\hat{r}}=s$, then there must uniquely exist $x\in[0,1)$ and $k\equiv s(\mathrm{mod}2)$ with $k\le s$, such that $\sum_n z_t(n)r_t^1(n)=\sum_n z_t(n)\hat{r}_t^1(n)$, $\sum_n(v_n-v^{**})z_t(n)r_t^1(n)\ge 0$ and $\sum_n(v_n-v^{*})z_t(n)r_t^{-1}(n)\le 0$, where r is a threshold affirmative policy defined as*

$$r_t^1(n):=\begin{cases}1, & \text{for } n>k,\\ x\in[0,1), & \text{for } n=k,\\ 0, & \text{for } n<k,\end{cases}\quad \text{and}\quad r_t^{-1}(n):=1-r_t^1(n)\quad \forall n. \tag{B9}$$

Proof of Lemma B2. The proof follows the same argument as that of Proposition 5 by taking $r_t^{\star 0}\equiv 0$ therein. ■

Proof of Lemma 2. We divide the proof into three parts.

Part I. (12) and (13) are well-defined, which consists in showing the existence of $n^{**}(s), x^{**}(s), n^*(s)$ and $x^*(s)$ for any integer $s\ge 0$, as characterized by the following two claims.

CLAIM B1. *$n^{**}(s)\equiv s(mod2), v_{n^{**}(s)}<v^{**}$ and $x^{**}(s)<1$.*

Proof of Claim B1 First, note that for any $n\not\equiv s(mod2)$ that satisfies the equality inside (12), we also have

$$\sum_{m>n-1}(v_m-v^{**})\zeta(s,m)=(v_n-v^{**})\zeta(s,n)x+\sum_{m>n}(v_m-v^{**})\zeta(s,m)=0,$$

as $\zeta(s,n)=0$ for all $n \not\equiv s(mod2)$ by (11). Hence, by (12), $n^{**}(s) \equiv s(mod2)$. Notice also that we must have $x^{**}(s)<1$ if $n^{**}(s)=-s$, as otherwise, by (B2) and the fact that $v_0 \in [v^*, v^{**})$, the equation inside (12) must be violated

$$\sum_{m\geq -s}(v_m - v^{**})\zeta(s,m) = \sum_m (v_m - v^{**})\zeta(s,m) = \sum_m (v_m)\zeta(s,m) - v^{**} = v_0 - v^{**} < 0.$$

Further, for any $n \geq -s+2$ that satisfies the equality inside (12) for $x=1$, we also have

$$\sum_{m>n-2}(v_m - v^{**})\zeta(s,m) = (v_n - v^{**})\zeta(s,n) + \sum_{m>n}(v_m - v^{**})\zeta(s,m) = 0,$$

hence, by (12), (i) $x^{**}(s)<1$.

- When $v_s < v^{**}$, we have $v_m < v^{**}$ for all m with $\zeta(s,m)>0$ by (8). Hence, it is straightforward to verify that the equality inside (12) holds for $n=s, x=0$. For any $n<s$, we have

$$\underbrace{(v_n - v^{**})}_{<0 \text{ as } v_n<v_s<v^{**}} \zeta(s,n)x + \sum_{m>n} \underbrace{(v_m - v^{**})\zeta(s,m)}_{<0 \text{ for } m \text{ with } \zeta(s,m)>0} \leq (v_s - v^{**})\zeta(s,s) < 0,$$

contradicting the equality inside (12). Notice that $v_{n^{**}(s)} < v^{**}$ as $n^{**}(s)=s$, by (8) and the fact that $v_s < v^{**}$.

- When $v_s = v^{**}$, we have $v_m \leq v^{**}$ for all m with $\zeta(s,m)>0$ by (8). Hence, it is straightforward to verify that the equality inside (12) holds for $n=s-2, x=0$. For any $n<s-2$, we have

$$\underbrace{(v_n - v^{**})}_{<0 \text{ as } v_n<v_s=v^{**}} \zeta(s,n)x + \sum_{m>n} \underbrace{(v_m - v^{**})\zeta(s,m)}_{\substack{<0 \text{ for } m<s \text{ with } \zeta(s,m)>0 \\ =0 \text{ for } m=s}} \leq (v_{s-2} - v^{**})\zeta(s,s-2) < 0,$$

contradicting the equality inside (12). Notice that $v_{n^{**}(s)} < v^{**}$ as $n^{**}(s)=s-2$, by (8) and the fact that $v_s = v^{**}$.

- When $v_s > v^{**}$, denote $f_s(n) := \sum_{m>n}(v_m - v^{**})\zeta(s,m)$. It is then straightforward to verify that $f_s(n)$ is positive and non-increasing in $n \leq s-2$ for $v_n \geq v^{**}$ (because $f_s(n) \geq f_s(s-2) = (v_s - v^{**})\zeta(s,s) > 0 = f_s(s)$), and $f_s(n)$ is non-decreasing in n for $v_n \leq v^{**}$ with $f_s(-s-2) = \sum_{m>-s-2}(v_m - v^{**})\zeta(s,m) = v_0 - v^{**} < 0$. Therefore, (ii) there exists a unique $n^{**}(s) \in [-s, s-2]$ satisfying $f_s(n^{**}(s)-2) < 0 \leq f_s(n^{**}(s))$. Accordingly, let

$$x^{**}(s) = \frac{\sum_{m>n^{**}(s)}(v_m - v^{**})\zeta(s,m)}{(v^{**} - v_{n^{**}(s)})\zeta(s,n^{**}(s))} = \frac{f_s(n^{**}(s)) - 0}{f_s(n^{**}(s)) - f_s(n^{**}(s)-2)} \in [0,1).$$

Then, it immediately follows that (iii) $(n^{**}(s), x^{**}(s))$ satisfies the equality inside (12). For any $n<n^{**}(s)$, the equation inside (12) must be violated, because then we must have (iv) $(v_n - v^{**}) < (v_{n^{**}(s)} - v^{**}) < 0$ by (8), as $(v^{**} - v_{n^{**}(s)}) \geq (v^{**} - v_{n^{**}(s)})\zeta(s,n^{**}(s)) = f_s(n^{**}(s)) - f_s(n^{**}(s)-2) > 0$ by (ii). Then, the equation inside (12) can be written as

$$\begin{aligned} 0 = &\underbrace{(v_n - v^{**})\zeta(s,n)}_{\leq 0 \text{ by (iv)}} x + \underbrace{(v_{n^{**}(s)} - v^{**})\zeta(s,n^{**}(s))}_{<0 \text{ by (ii)}} \underbrace{(1-x^{**}(s))}_{>0 \text{ by (i)}} \\ &+ \underbrace{(v_{n^{**}(s)} - v^{**})\zeta(s,n^{**}(s))x^{**}(s) + \sum_{m>n^{**}(s)}(v_m - v^{**})\zeta(s,m)}_{=0 \text{ by (iii)}} + \sum_{m\in(n,n^{**}(s))}(v_m - v^{**})\zeta(s,m), \end{aligned}$$

leading to a contradiction, whereby we use the fact that $v_n < v_m < v_{n^{**}(s)} < v^{**}$ for all (if any) $m \in (n, n^{**}(s))$ by (8). Lastly, notice that $v_{n^{**}(s)} < v^{**}$ as $v_s > v^{**}$ and $\zeta(s,n^{**}(s))>0$ by (ii). $\square$

CLAIM B2. *$n^*(s) \equiv s(mod2), v_{n^*(s)} \geq v^*$ and $x^*(s) \leq 1$, where the second equality holds only when $v_0 = v^*$ and $s = 0$, and the third equality holds only when $v_0 = v^*$.*

Proof of Claim B2 First, note that for any $n \not\equiv s(mod2)$ that satisfies the equality inside (13), we also have

$$\sum_{m<n+1}(v_m - v^*)\zeta(s,m) = (v_n - v^*)\zeta(s,n)(1-x) + \sum_{m<n}(v_m - v^*)\zeta(s,m) = 0,$$

as $\zeta(s,n) = 0$ for all $n \not\equiv s(mod2)$ by (11). Hence, by (13), $n^{**}(s) \equiv s(mod2)$. Next, notice that for any $n \leq s-2$ that satisfies the equality inside (13) for $x = 0$, we also have

$$\sum_{m<n+2}(v_m - v^*)\zeta(s,m) = (v_n - v^*)\zeta(s,n) + \sum_{m<n}(v_m - v^*)\zeta(s,m) = 0,$$

hence, by (13), (v) $x^*(s) > 0$.

- When $v_{-s} > v^*$, we have $v_m > v^*$ for all m with $\zeta(s,m) > 0$ by (8). Hence, it is straightforward to verify that the equality inside (13) holds for $n = -s, x = 1$. For any $n > -s$, we have

$$\underbrace{(v_n - v^*)}_{>0 \text{ as } v_n > v_{-s} > v^*} \underbrace{\zeta(s,n)}_{\substack{>0 \text{ as} \\ n \equiv s(mod2), n \leq s}} (1-x) + \sum_{m<n} \underbrace{(v_m - v^*)\zeta(s,m)}_{>0 \text{ for } m \text{ with } \zeta(s,m)>0} \geq (v_{-s} - v^*)\zeta(s,s) > 0,$$

contradicting the equality inside (13). Notice that $v_{n^*(s)} > v^*$ as $n^*(s) = -s$, by (8) and the fact that $v_{-s} > v^*$.

- When $v_{-s} = v^*$, we have $v_m \geq v^*$ for all m with $\zeta(s,m) > 0$ by (8). Hence, it is straightforward to verify that the equality inside (13) holds for $n = -s+2, x = 0$. For any $n > -s+2$, we have

$$\underbrace{(v_n - v^*)}_{\substack{>0 \text{ as} \\ v_n > v_{-s} = v^*}} \underbrace{\zeta(s,n)}_{\substack{>0 \text{ as} \\ n \equiv s(mod2), n \leq s}} x + \sum_{m<n} \underbrace{(v_m - v^*)\zeta(s,m)}_{\substack{>0 \text{ for } m>-s \text{ with } \zeta(s,m)>0 \\ =0 \text{ for } m=-s}} \geq (v_{-s+2} - v^*)\zeta(s,-s+2) > 0,$$

contradicting the equality inside (13). Notice that $v_{n^*(s)} > v^*$ by (8) and the fact that $v_{-s} = v^*$.

- When $v_{-s} < v^*$, denote $f_s(n) := \sum_{m<n}(v_m - v^*)\zeta(s,m)$. When $v_{-s} < v^*$, it is straightforward to verify that $f_s(n)$ is negative and non-increasing in $n \geq -s+2$ for $v_n \leq v^*$ (because $f_s(n) \leq f_s(-s+2) = (v_{-s} - v^*)\zeta(s,-s) < 0 = f_s(-s)$), and $f_s(n)$ is non-decreasing in n for $v_n \geq v^*$ with $f_s(s+2) = \sum_{m<s+2}(v_m - v^*)\zeta(s,m) = v_0 - v^* \geq 0$. Therefore, (vi) there exists a unique $n^*(s) \in [-s+2, s]$ satisfying $f_s(n^*(s)) \leq 0 < f_s(n^*(s)+2)$. Accordingly, let

$$1 - x^*(s) = \frac{\sum_{m<n^*(s)}(v_m - v^*)\zeta(s,m)}{(v^* - v_{n^*(s)})\zeta(s,n^*(s))} = \frac{f_s(n^*(s)) - 0}{f_s(n^*(s)) - f_s(n^*(s)+2)} \in [0,1).$$

Then, it immediately follows that (vii) $(n^*(s), x^*(s))$ satisfies the equality inside (13). For any $n > n^*(s)$, the equation inside (13) must be violated, because then we must have (viii) $(v_n - v^*) > (v_{n^*(s)} - v^*) > 0$ by (8), as $(v_{n^*(s)} - v^*) \geq (v_{n^*(s)} - v^*)\zeta(s,n^*(s)) = f_s(n^*(s)+2) - f_s(n^*(s)) > 0$ by (vi). Then, the equation inside (13) can be written as

$$\begin{aligned} 0 = {} & \underbrace{(v_n - v^*)}_{>0 \text{ by (viii)}} \underbrace{\zeta(s,n)}_{\substack{>0 \text{ by (N)} \\ \text{as } n \leq s}} (1-x) + \underbrace{(v_{n^*(s)} - v^*)\zeta(s,n^*(s))}_{<0 \text{ by (vi)}} \underbrace{(1 - x^*(s))}_{>0 \text{ by (v)}} \\ & + \underbrace{(v_{n^*(s)} - v^*)\zeta(s,n^*(s))(1 - x^*(s)) + \sum_{m<n^*(s)}(v_m - v^*)\zeta(s,m)}_{=0 \text{ by (vii)}} + \sum_{m \in (n^*(s),n)}(v_m - v^*)\zeta(s,m), \end{aligned}$$

leading to a contradiction, whereby we use the fact that $v^* < v_{n^*(s)} < v_m < v_n$ for all (if any) $m \in (n^*(s), n)$ by (8). Lastly, notice that $v_{n^*(s)} > v^*$ as $v_{-s} < v^*$ and $\zeta(s,n^*(s)) > 0$ by (vi). □

Part II. It is incentive compatible to assign customer t as an affirmative customer if and only if $n^*(s) > n^{**}(s)$ or $n^*(s) = n^{**}(s)$ with $x^{**}(s) \geq x^*(s)$, where $s = \ell_t^r$.

To show the sufficiency, we now demonstrate the following threshold affirmative policy simultaneously satisfies (IC_1^r) and (IC_{-1}^r):

$$r_t^1(n) := \begin{cases} 1, & \text{for } n > n^{**}(s), \\ x^{**}(s) \in [0,1], & \text{for } n = n^{**}(s), \\ 0, & \text{for } n < n^{**}(s), \end{cases} \quad \text{and} \quad r_t^{-1}(n) := 1 - r_t^1(n) \quad \forall n. \tag{B10}$$

Indeed, by (12), we have

$$\sum_n (v_n - v^{**}) z_t(n) r_t^1(n) = \left(v_{n^{**}(s)} - v^{**}\right) \zeta(s, n^{**}(s)) x^{**}(s) + \sum_{n > n^{**}(s)} (v_n - v^{**}) \zeta(s,n) = 0, \tag{B11}$$

establishing (IC_1^r). To establish (IC_{-1}^r), we now show

$$\sum_n (v_n - v^*) z_t(n) r_t^{-1}(n) = \left(v_{n^{**}(s)} - v^*\right) \zeta(s, n^{**}(s))(1 - x^{**}(s)) + \sum_{n < n^{**}(s)} (v_n - v^*) \zeta(s,n) \leq 0. \tag{B12}$$

- For $n^*(s) > n^{**}(s)$, by Claim B2 we have $v_{n^*(s)} \geq v^*$, which leads to $(v_{n^*(s)} - v^*)\zeta(s, n^*(s))(1 - x^*(s)) \geq 0$. Hence, by (13), we have $\sum_{n < n^*(s)} (v_n - v^*)\zeta(s,n) \leq 0$. Then, it is straightforward to see that (B12) holds, such that

$$\begin{aligned} 0 \geq \sum_{n < n^*(s)} (v_n - v^*)\zeta(s,n) \geq & \max\left\{ \sum_{n < n^{**}(s)} (v_n - v^*)\zeta(s,n), \sum_{n \leq n^{**}(s)} (v_n - v^*)\zeta(s,n) \right\} \\ & \geq \left(v_{n^{**}(s)} - v^*\right) \zeta(s, n^{**}(s))(1 - x^{**}(s)) + \sum_{n < n^{**}(s)} (v_n - v^*)\zeta(s,n), \end{aligned}$$

where the second inequality follows by $n^*(s) > n^{**}(s)$ as v_n increases in n by (8), and the third inequality follows from the fact that $x^{**}(s) \in [0,1]$.

- For $n^*(s) = n^{**}(s)$ with $x^{**}(s) \geq x^*(s)$, by Claim B2 we have $v_{n^*(s)} \geq v^*$, which leads to $(v_{n^*(s)} - v^*)\zeta(s, n^*(s))(1 - x^*(s)) \geq 0$. Hence, by (13), we have

$$\begin{aligned} 0 &= \underbrace{\left(v_{n^*(s)} - v^*\right) \zeta(s, n^*(s))(1 - x^*(s))}_{\geq 0} + \sum_{n < n^*(s)} (v_n - v^*)\zeta(s,n) \\ &= \left(v_{n^{**}(s)} - v^*\right) \zeta(s, n^{**}(s))(1 - x^*(s)) + \sum_{n < n^{**}(s)} (v_n - v^*)\zeta(s,n) \\ &\geq \left(v_{n^{**}(s)} - v^*\right) \zeta(s, n^{**}(s))(1 - x^{**}(s)) + \sum_{n < n^{**}(s)} (v_n - v^*)\zeta(s,n), \end{aligned}$$

where the second equality follows by $n^*(s) = n^{**}(s)$, and the third inequality follows from $v_{n^{**}(s)} = v_{n^*(s)} \geq v^*$ by Claim B2 and the fact that $x^{**}(s) \geq x^*(s)$. Thus, (B12) holds.

To show necessity, we note that, if it is incentive compatible to assign customer t (when $\ell_t^r = s$) as an affirmative customer, Lemma B2 implies the existence of $x \in [0,1)$ and $k \equiv s (\mathrm{mod} 2)$ with $k \leq s$, such that r_t specified in (B10) satisfies (IC_1^r),

$$\sum_n (v_n - v^{**}) z_t(n) r_t^1(n) = (v_k - v^{**}) \zeta(s,k) x + \sum_{n > k} (v_n - v^{**}) \zeta(s,n) \geq 0, \tag{B13}$$

and (IC_{-1}^r),

$$\sum_n (v_n - v^*) z_t(n) r_t^{-1}(n) = (v_k - v^*) \zeta(s,k)(1 - x) + \sum_{n < k} (v_n - v^*) \zeta(s,n) \leq 0. \tag{B14}$$

We now claim (B13) and (B14) must imply $n^*(s) > n^{**}(s)$ or $n^*(s) = n^{**}(s)$ with $x^{**}(s) \geq x^*(s)$. To that end, we first establish the following lemma, whose proof is relegated after the proof of Lemma 2.

LEMMA B3. *If it is incentive compatible to assign customer t (when $\ell_t^r = s$) as an affirmative customer, $(n^{**}(s), x^{**}(s))$, $(n^*(s), x^*(s))$ and (x,k) determined by (12), (13) and (B10) of Lemma B2 respectively, satisfy*

$$\text{either} \quad k > n^{**}(s) \quad \text{or} \quad k = n^{**}(s) \text{ with } x \le x^{**}(s), \quad \text{and} \tag{B15}$$

$$\text{either} \quad k < n^{*}(s) \quad \text{or} \quad k = n^{*}(s) \text{ with } x^*(s) \le x. \tag{B16}$$

Lemma B3 immediately implies that $n^*(s) \ge n^{**}(s)$, as $n^*(s) \ge k \ge n^{**}(s)$ by (B15) and (B16). In particular, when $n^*(s) = n^{**}(s) = k$, we have $x^{**}(s) \ge x \ge x^*(s)$.

Part III. If it is incentive compatible to assign customer t as an affirmative customer, then assigning any other customer $t' > t$ as an affirmative customer will also be incentive compatible; and it is optimal for the platform to offer affirmative recommendations according to (B10) and to earn an expected revenue of $pF(s)$, where

$$F(s) = x^{**}(s)\zeta(s, n^{**}(s)) + \sum_{n > n^{**}(s)} \zeta(s,n),$$

is a non-decreasing function in s. In Part II, we show for customer t with $\ell_t^r = s \ge \tau^\circ$, that the threshold affirmative policy given by (B10) simultaneously satisfies (IC_1^r) and (IC_{-1}^r) for $n^*(s) > n^{**}(s)$ or $n^*(s) = n^{**}(s)$ with $x^{**}(s) \ge x^*(s)$. Now, for customer $t' > t$ with $\ell_{t'}^r = s+1$, consider the following policy

$$r_{t'}^1(n) := \begin{cases} 1, & \text{for } n > n^{**}(s)+1, \\ \frac{x^{**}(s)\zeta(s,n^{**}(s))u_{n^{**}(s)} + \zeta(s,n^{**}(s)+2)(1-u_{n^{**}(s)+2})}{\zeta(s+1,n^{**}(s)+1)}, & \text{for } n = n^{**}(s)+1, \\ \frac{x^{**}(s)\zeta(s,n^{**}(s))(1-u_{n^{**}(s)})}{\zeta(s+1,n^{**}(s)-1)}, & \text{for } n = n^{**}(s)-1, \\ 0, & \text{for } n < n^{**}(s)-1, \end{cases} \quad \text{and} \quad r_{t'}^{-1}(n) := 1 - r_{t'}^1(n) \quad \forall n. \tag{B17}$$

Then, we have

$$\begin{aligned} \sum_n (v_n - v^{**})\zeta(s+1,n) r_{t'}^1(n) &= \sum_{n > n^{**}(s)+1} (v_n - v^{**})\zeta(s+1,n) \\ &\quad + (v_{n^{**}(s)+1} - v^{**}) \left[x^{**}(s)\zeta(s,n^{**}(s))u_{n^{**}(s)} + \zeta(s,n^{**}(s)+2)(1-u_{n^{**}(s)+2})\right] \\ &\quad + (v_{n^{**}(s)-1} - v^{**}) \left[x^{**}(s)\zeta(s,n^{**}(s))(1-u_{n^{**}(s)})\right] \\ &= \underbrace{\sum_{n > n^{**}(s)+1} (v_n - v^{**})\zeta(s+1,n) + (v_{n^{**}(s)+1} - v^{**})\zeta(s,n^{**}(s)+2)(1-u_{n^{**}(s)+2})}_{\sum_{n>n^{**}(s)}(v_n - v^{**})\zeta(s,n) \text{ by (B3)}} \\ &\quad + x^{**}(s)\zeta(s,n^{**}(s)) \underbrace{\left[(v_{n^{**}(s)+1} - v^{**})u_{n^{**}(s)} + (v_{n^{**}(s)-1} - v^{**})(1-u_{n^{**}(s)})\right]}_{v_{n^{**}(s)} - v^{**} \text{ by (A23)}} \\ \text{(by (B11))} \quad &= \left(v_{n^{**}(s)} - v^{**}\right)\zeta(s,n^{**}(s))x^{**}(s) + \sum_{n > n^{**}(s)} (v_n - v^{**})\zeta(s,n) = 0. \end{aligned}$$

Therefore, the policy given in (B17) satisfies (IC_1^r). Similarly, we also have

$$\begin{aligned} \sum_n (v_n - v^{*})\zeta(s+1,n) r_{t'}^{-1}(n) &= \sum_{n < n^{**}(s)-1} (v_n - v^{*})\zeta(s+1,n) \\ &\quad + (v_{n^{**}(s)-1} - v^{*}) \left[(1-x^{**}(s))\zeta(s,n^{**}(s))(1-u_{n^{**}(s)}) + \zeta(s,n^{**}(s)-2)u_{n^{**}(s)-2}\right] \\ &\quad + (v_{n^{**}(s)+1} - v^{*}) \left[(1-x^{**}(s))\zeta(s,n^{**}(s))u_{n^{**}(s)}\right] \end{aligned}$$

$$= \underbrace{\sum_{n<n^{**}(s)-1}(v_n - v^*)\zeta(s+1,n) + (v_{n^{**}(s)-1} - v^*)\zeta(s, n^{**}(s)-2)u_{n^{**}(s)-2}}_{\sum_{n<n^{**}(s)}(v_n-v^*)\zeta(s,n) \text{ by (B5)}}$$

$$+ (1-x^{**}(s))\zeta(s,n^{**}(s)) \underbrace{\left[(v_{n^{**}(s)-1} - v^*)(1-u_{n^{**}(s)}) + (v_{n^{**}(s)+1} - v^*)u_{n^{**}(s)}\right]}_{v_{n^{**}(s)}-v^* \text{ by (A23)}}$$

$$\text{(by (B12))} \quad = \left(v_{n^{**}(s)} - v^*\right)\zeta(s,n^{**}(s))(1-x^{**}(s)) + \sum_{n<n^{**}(s)}(v_n - v^*)\zeta(s,n) \leq 0.$$

Therefore, the policy given in (B17) also satisfies (IC^r_{-1}). Hence, in a recursive manner, we show that if it is incentive compatible to assign customer t as an affirmative customer, then assigning any other customer $t' > t$ as an affirmative customer will also be incentive compatible. It is straightforward to see, by Lemma B2 and (12), that offering affirmative recommendations according to (B10) is optimal for the platform. The corresponding expected revenue, that the policy given in (B17) generated from customer t', is as follows

$$\sum_n r^1_{t'}(n)\zeta(s+1,n) = \underbrace{\sum_{n>n^{**}(s)+1}\zeta(s+1,n) + \zeta(s,n^{**}(s)+2)(1-u_{n^{**}(s)+2})}_{=\sum_{n>n^{**}(s)}\zeta(s,n) \text{ by (B7)}} + x^{**}(s)\zeta(s,n^{**}(s)) = F(s),$$

which immediately implies $F(s) \leq F(s+1)$ by Lemma B2. ■

Proof of Lemma B3. First, we establish (B15), by showing

- $\underline{k \geq n^{**}(s)}$. Suppose $k < n^{**}(s)$, then (i) $v_k < v_{n^{**}(s)} < v^{**}$ by (8) (first inequality) and Claim B1 (second inequality). We then have

$$(v_k - v^{**})\zeta(s,k)x + \sum_{m>k}(v_m - v^{**})\zeta(s,m)$$

$$= \underbrace{(v_k - v^{**})\zeta(s,k)}_{\leq 0 \text{ by (i)}} x + \underbrace{(v_{n^{**}(s)} - v^{**})\zeta(s,n^{**}(s))}_{<0 \text{ by Claim } B1} \underbrace{(1-x^{**}(s))}_{>0 \text{ by Claim } B1} + \sum_{m\in(k,n^{**}(s))}(v_m - v^{**})\zeta(s,m)$$

$$+ \underbrace{(v_{n^{**}(s)} - v^{**})\zeta(s,n^{**}(s))x^{**}(s) + \sum_{m>n^{**}(s)}(v_m - v^{**})\zeta(s,m)}_{=0 \text{ by (12)}} < 0,$$

leading to a contradiction with (B13), whereby we use the fact that $v_k < v_m < v_{n^{**}(s)} < v^{**}$ for all (if any) $m \in (k, n^{**}(s))$ by (8).

- $\underline{x \leq x^{**}(s) \text{ when } k = n^{**}(s)}$. Suppose $x > x^{**}(s)$ when $k = n^{**}(s)$, we then have

$$(v_k - v^{**})\zeta(s,k)x + \sum_{m>k}(v_m - v^{**})\zeta(s,m)$$

$$= (v_{n^{**}(s)} - v^{**})\zeta(s,n^{**}(s))x + \sum_{m>n^{**}(s)}(v_m - v^{**})\zeta(s,m)$$

$$= \underbrace{(v_{n^{**}(s)} - v^{**})\zeta(s,n^{**}(s))}_{<0 \text{ by Claim } B1}\underbrace{(x - x^{**}(s))}_{>0} + \underbrace{(v_{n^{**}(s)} - v^{**})\zeta(s,n^{**}(s))x^{**}(s) + \sum_{m>n^{**}(s)}(v_m - v^{**})\zeta(s,m)}_{=0 \text{ by (12)}}$$

$$< 0,$$

leading to a contradiction with (B13).

Next, we establish (B16), by showing

- $\underline{k \le n^*(s)}$. First, note that (ii) $v_{n^*(s)+2} > v^*$ by (8) and Claim B2. Further, we have (iii) $(v_{n^*(s)} - v^*)\zeta(s, n^*(s)) \ge 0$, again by Claim B2. Suppose $k > n^*(s)$, we then have

$$\begin{aligned}(v_k - v^*)\zeta(s,k)(1-x) + \sum_{m<k}(v_m - v^*)\zeta(s,m) = &\underbrace{(v_{n^*(s)} - v^*)\zeta(s,n^*(s))(1-x^*(s)) + \sum_{m<n^*(s)}(v_m - v^*)\zeta(s,m)}_{=0 \text{ by } (13)} \\ &+ \underbrace{(v_{n^*(s)} - v^*)\zeta(s,n^*(s))x^*(s)}_{\ge 0 \text{ by (iii)}} + \underbrace{(v_k - v^*)\zeta(s,k)(1-x)}_{\substack{>0 \text{ by } (8) \text{ and (ii) as } k \ge n^*(s)+2, \\ |k| \le s \text{ by Lemma } B2 \text{ and } x<1}} \\ &+ \sum_{m \in (n^*(s),k)}(v_m - v^*)\zeta(s,m) \\ >&0,\end{aligned}$$

leading to a contradiction with (B14), whereby we use the fact that $v^* < v_{n^*(s)+2} \le v_m$ for all (if any) $m \in (n^*(s), k)$ by (ii).

- $\underline{x^*(s) \le x \text{ when } k = n^*(s)}$. Suppose $x^*(s) > x$ when $k = n^*(s)$. Then, as $x \in [0,1)$, we must have $x^*(s) > 0$, which leads to $v_{n^*(s)} > v^*$ by Claim B2. Therefore, (iv) $(v_{n^*(s)} - v^*)\zeta(s, n^*(s)) > 0$. We then have

$$\begin{aligned}(v_k - v^*)\zeta(s,k)(1-x) + \sum_{m<k}(v_m - v^*)\zeta(s,m) =&(v_{n^*(s)} - v^*)\zeta(s,n^*(s))(1-x) + \sum_{m<n^*(s)}(v_m - v^*)\zeta(s,m) \\ >&\underbrace{(v_{n^*(s)} - v^*)\zeta(s,n^*(s))x^*(s) + \sum_{m<n^*(s)}(v_m - v^*)\zeta(s,m)}_{=0 \text{ by } (13)}\end{aligned}$$

leading to a contradiction with (B14), where the second inequality follows by (iv) and the fact that $x^*(s) > x$. This completes the proof. ■

Proof of Lemma 3. It suffices to show that it is incentive compatible to assign the 4th customer as an affirmative customer after assigning the first 3 customers as neutral ones, i.e., there exists a policy r_4 with $r_4^0(n) \equiv 0$ for all n, which satisfies both (IC_1^r) and (IC_{-1}^r). When $p = v_0$, we have $v^* = v_{-1} < v_0 < v_1 = v^{**}$ by (5) and (8). Thus, (IC_1^r) and (IC_{-1}^r) are straightforwardly satisfied for the 4th customer by an affirmative policy $r_4(\cdot)$ with $r_4^1(3) = r_4^1(1) = 1$ and $r_4^{-1}(-1) = r_4^{-1}(-3) = 1$. In the remainder of this proof, we focus on the case $p < v_0$ (proof for $p > v_0$ follows from a symmetric argument and hence is omitted). In this case, we have $v_{-2} \le v^* < v_{-1} < v_0 \le v^{**} < v_1$, again by (5) and (8), and we make the follow

Claim B3. *When $p < v_0$, there exists a policy r_4 with $r_4^0(n) \equiv 0$ for all n, which satisfies both* (IC_1^r) *and* (IC_{-1}^r)*, if and only if there exists an $x \in [0,1]$ such that*

$$v_3\zeta(3,3) + v_1\zeta(3,1) + v_{-1}\zeta(3,-1)x \ge v^{**}[\zeta(3,3) + \zeta(3,1) + \zeta(3,-1)x] \quad \textit{and} \tag{B18}$$

$$v_{-3}\zeta(3,-3) + v_{-1}\zeta(3,-1)(1-x) \le v^*[\zeta(3,-3) + \zeta(3,-1)(1-x)]. \tag{B19}$$

Sufficiency: It is straightforward to see that, under an affirmative policy $r_4(\cdot)$ with $r_4^1(3) = r_4^1(1) = 1, r_4^1(-1) = x, r_4^1(-3) = 0$ and $r_4^{-1}(n) = 1 - r_4^1(n)$ for all n, (IC_1^r) and (IC_{-1}^r) reduces to (B18) and (B19).

Necessity: Suppose (IC_1^r) and (IC_{-1}^r) hold for the 4th customer. Then, Lemma B2 implies that a threshold policy with $(n^{**}(3), x^{**}(3))$ defined in (12) must also satisfy (IC_1^r) and (IC_{-1}^r). Since $v_{-3} < v^* < v_{-1} <$

$v^{**} < v_1 < v_3$, we must have $n^{**}(3) \in \{-1, -3\}$ by (12). If $n^{**}(3) = -1$, then under threshold policy with $(n^{**}(3), x^{**}(3))$, (IC_1^r) and (IC_{-1}^r) reduce to (B18) (with equality) and (B19), respectively, by taking $x = x^{**}(3)$. If $n^{**}(3) = -3$, then under threshold policy with $(n^{**}(3), x^{**}(3))$, (IC_1^r) reduces to

$$(v_3 - v^{**})\zeta(3,3) + (v_1 - v^{**})\zeta(3,1) + \underbrace{(v_{-1} - v^{**})}_{<0}\zeta(3,-1) + \underbrace{(v_{-3} - v^{**})\zeta(3,-3)x^{**}(3)}_{\leq 0} = 0,$$

which immediately implies (B18) and (B19) by taking $x = 1$, i.e.,

$$(v_3 - v^{**})\zeta(3,3) + (v_1 - v^{**})\zeta(3,1) + (v_{-1} - v^{**})\zeta(3,-1) \geq 0, \quad \text{and} \quad \underbrace{(v_{-3} - v^{*})}_{<0}\zeta(3,-3) < 0. \quad \square$$

Claim B4. *There exists an $x \in [0,1]$ such that* (B18) *and* (B19) *hold, if and only if*

$$\frac{v_0(1-v_0)(2q-1)}{1-u_0}\left[\frac{1+q-q^2}{v^{**}-v_{-1}} + \frac{q-q^2}{v_{-1}-v^{*}}\right] \geq 1. \tag{B20}$$

Proof of Claim B4. We rewrite (B18) and (B19) equivalently as

$$\begin{aligned} \frac{v_3 - v^{**}}{v^{**} - v_{-1}}\zeta(3,3) + \frac{v_1 - v^{**}}{v^{**} - v_{-1}}\zeta(3,1) &\geq \zeta(3,-1)x, \quad \text{and} \\ -\frac{v^{*} - v_{-3}}{v_{-1} - v^{*}}\zeta(3,-3) + \zeta(3,-1) &\leq \zeta(3,-1)x, \quad \text{respectively.} \end{aligned}$$

Hence, there exists an $x \in [0,1]$ such that (B18) and (B19) hold, if and only if

$$\underbrace{\frac{v_3 - v^{**}}{v^{**} - v_{-1}}\zeta(3,3) + \frac{v_1 - v^{**}}{v^{**} - v_{-1}}\zeta(3,1)}_{>0 \text{ (because } v_{-1} < v^{**} < v_1 < v_3)} \geq \underbrace{-\frac{v^{*} - v_{-3}}{v_{-1} - v^{*}}\zeta(3,-3) + \zeta(3,-1)}_{<\zeta(3,-1) \text{ (because } v_{-3} < v^{*})}. \tag{B21}$$

Rearranging terms in (B21) and using (B8) yields

$$\{(v_3 - v_1)\zeta(3,3) + (v_1 - v_{-1})[\zeta(3,3) + \zeta(3,1)]\}\frac{1}{v^{**} - v_{-1}} + (v_{-1} - v_{-3})\zeta(3,-3)\frac{1}{v_{-1} - v^{*}} \geq 1. \tag{B22}$$

Direct application of (11) yields that $\zeta(3,n) = 0$ for all n except the following positions:

$$\zeta(3,3) = u_0 u_1 u_2, \tag{B23}$$

$$\zeta(3,1) = u_0 u_1 (1-u_2) + {u_0}^2(1-u_1) + u_0(1-u_0)u_{-1}, \tag{B24}$$

$$\zeta(3,-1) = (1-u_0)(1-u_{-1})u_{-2} + u_0(1-u_0)(1-u_1) + (1-u_0)^2 u_{-1}, \quad \text{and} \tag{B25}$$

$$\zeta(3,-3) = (1-u_0)(1-u_{-1})(1-u_{-2}). \tag{B26}$$

Further, note that, by (8) and the definition of u_n, we have

$$v_n = \frac{v_{n-1}q}{u_{n-1}} \quad \text{and} \quad v_{-n} = \frac{v_{1-n}(1-q)}{1-u_{1-n}} \quad \text{for all } n. \tag{B27}$$

Utilizing (B27) we derive the following quantities, which are the main components of this proof,

$$v_3 - v_1 = \frac{v_2 - v_1}{u_2} = \frac{v_0(1-v_0)q(1-q)(2q-1)}{{u_0}^2 u_1 u_2}, \tag{B28}$$

$$v_1 - v_{-1} = \frac{v_0 - v_{-1}}{u_0} = \frac{v_0(1-v_0)(2q-1)}{u_0(1-u_0)}, \tag{B29}$$

$$v_{-1} - v_{-3} = \frac{v_{-1} - v_{-2}}{(1-u_{-2})} = \frac{v_0(1-v_0)q(1-q)(2q-1)}{(1-u_0)^2(1-u_{-1})(1-u_{-2})}, \tag{B30}$$

where the first equalities of each line follow directly from (A23).

Utilizing (B23)-(B26) and (B28)-(B30), we have, by straightforward algebraic manipulation,

$$(v_3 - v_1)\zeta(3,3) + (v_1 - v_{-1})\left[\zeta(3,3) + \zeta(3,1)\right] = \frac{v_0(1-v_0)(2q-1)(1+q-q^2)}{1-u_0}, \quad \text{and}$$
$$(v_{-1} - v_{-3})\zeta(3,-3) = \frac{v_0(1-v_0)(2q-1)(q-q^2)}{1-u_0},$$

which renders (B22) equivalent to (B20). $\square$

Combining Claims B3 and B4, we now just need to show (B20) holds. To that end, we note that since $v_{-2} \le v^* < v_{-1}$, there must exists $\lambda^* \in \left(1, \frac{q}{1-q}\right]$ such that

$$v^* = \frac{v_{-1}}{v_{-1} + (1-v_{-1})\lambda^*}, \quad \text{and} \quad v^{**} = \frac{1}{1+\left(\frac{1-q}{q}\right)^2 \frac{1-v^*}{v^*}} = \frac{v_{-1}q^2}{v_{-1}q^2 + (1-v_{-1})(1-q)^2\lambda^*}. \tag{B31}$$

Substituting (B31) into (B20) yields,

$$\begin{aligned}
\text{(B20)} \quad \Leftrightarrow \quad & \frac{v_0(1-v_0)(2q-1)}{(1-u_0)v_{-1}(1-v_{-1})}\left\{\frac{(1+q-q^2)\left[v_{-1}q^2 + (1-v_{-1})(1-q)^2\lambda^*\right]}{q^2 - (1-q)^2\lambda^*} + \frac{(q-q^2)\left[v_{-1} + (1-v_{-1})\lambda^*\right]}{\lambda^* - 1}\right\} \ge 1 \\
\Leftrightarrow \quad & \frac{v_0(1-v_0)(2q-1)}{(1-u_0)v_{-1}(1-v_{-1})}\left\{\frac{q^2(1+q-q^2)}{q^2-(1-q)^2\lambda^*} + \frac{q-q^2}{\lambda^*-1} - (1-v_{-1})\right\} \ge 1 \\
\Leftrightarrow \quad & \frac{q^2(1+q-q^2)}{q^2-(1-q)^2\lambda^*} + \frac{q-q^2}{\lambda^*-1} \ge \frac{q}{2q-1}, \text{ where we use } v_{-1} = \tfrac{v_0(1-q)}{1-u_0} \text{ by (B27).} \\
\Leftrightarrow \quad & q(2q-1)(1+q-q^2)(\lambda^*-1) + (1-q)(2q-1)\left[q^2-(1-q)^2\lambda^*\right] \ge \left[q^2-(1-q)^2\lambda^*\right](\lambda^*-1) \\
\Leftrightarrow \quad & (\lambda^*)^2 - 4q\lambda^* + \frac{q}{1-q} \ge 0 \quad \Leftrightarrow \quad (\lambda^*-2q)^2 + \frac{q}{1-q}(1-2q)^2 \ge 0.
\end{aligned}$$

Thus, (B20) indeed holds, completing the proof. ■

Proof of Proposition 8. If $T < \tau^\circ$, the only feasible NA-sequencing policy is to assign all customers to be neutral ones, and hence (16) holds with $t^{\star\star} = T$. Otherwise ($T \ge \tau^\circ$), Lemma 2, together with Proposition 7 (it is incentive compatible to assign any customer to be a neutral customer and extract the same revenue pu_0 regardless of her order of arrival), immediately implies that the optimal NA-sequencing policy must take the form of equation (16) with $n^{\star\star} = n^{\star\star}(t^{\star\star})$ and $x^{\star\star} = x^{\star\star}(t^{\star\star})$. To determine $t^{\star\star}$, we evaluate and optimize the platform's expected total revenue under the policy in (16), which is exactly given by (17). ■

Proof of Corollary 1. Under the optimal NA-sequencing policy $r^{\star\star}$, we can write customer t's welfare as

$$\begin{aligned}
&\mathbb{E}\left[(V-p)\mathbb{1}\left[a_t = 1\right] \mid r^{\star\star}\right] \\
&\quad = \mathbb{E}\left[V - p \mid m_t = 1, r^{\star\star}\right]\mathbb{P}\left[m_t = 1 \mid r^{\star\star}\right] \\
&\qquad + \mathbb{E}\left[V - p \mid m_t = 0, S_t = 1, r^{\star\star}\right]\mathbb{P}\left[S_t = 1 \mid m_t = 0\right]\mathbb{P}\left[m_t = 0 \mid r^{\star\star}\right] \\
&\quad = \left[\frac{\sum_n v_n z_t(n) r_t^1(n)}{\sum_n z_t(n) r_t^1(n)} - p\right]\sum_n z_t(n) r_t^1(n) \\
&\qquad + \left[\frac{q\mathbb{E}\left[V \mid m_t = 0, r^{\star\star}\right]}{q\mathbb{E}\left[V \mid m_t = 0, r^{\star\star}\right] + (1-q)\left(1 - \mathbb{E}\left[V \mid m_t = 0, r^{\star\star}\right]\right)} - p\right]\left[\frac{\sum_n u_n z_t(n) r_t^0(n)}{\sum_n z_t(n) r_t^0(n)}\right]\sum_n z_t(n) r_t^0(n) \\
&\quad = \sum_n (v_n - p) z_t(n) r_t^1(n) + \sum_n u_n z_t(n) r_t^0(n)\left[\frac{q\sum_n v_n z_t(n) r_t^0(n)}{\sum_n u_n z_t(n) r_t^0(n)} - p\right]
\end{aligned}$$

$$= \underbrace{\sum_n (v_n - v^{**}) z_t(n) r_t^1(n)}_{=0 \text{ by Proposition 8 and (12)}} + (v^{**} - p) \underbrace{\sum_n z_t(n) r_t^1(n)}_{=F(t^{**}) \text{ by Proposition 8 and Lemma 2}} + q \sum_n v_n z_t(n) r_t^0(n) - p \sum_n u_n z_t(n) r_t^0(n).$$

Notice that, by the definition of NA-sequencing policies, for an affirmative customer, the last two terms of the above expression are zero. By the same token, for a neutral customer, the first two terms of the above expression are zero. Furthermore, for the neutral customer case, by (A28), we have $\sum_n v_n z_t(n) r_t^0(n) = \sum_n v_n z_t(n) = v_0$ and $\sum_n u_n z_t(n) r_t^0(n) = \sum_n u_n z_t(n) = u_0$, establishing the Corollary. ■

Appendix C: Proofs in Section 6

Proof of Proposition 9. We first prove the following three parts to show that the qualitative structure of the optimal sequencing policy is of a cutoff structure. Then, since the objective function of the relaxed problem ($\overline{\text{P}}$) is identical to the original problem (P), the optimal number of neutral customers, t^{**}, is again determined by (17).

Part I. It is incentive compatible to assign any customer as a neutral customer, from whom the platform earns an expected revenue of pu_0. Optimal NA-sequencing policy r assigns all customers $t \leq \tau^\circ$ as neutral customers.

When $r_t^0 \equiv 1$ for some t, ($\overline{\text{IC}}_0^r$) constraint for any given b can be written as

$$v^* \sum_t g(t \mid b) \underbrace{\sum_n z_t(n)}_{=1} \leq \sum_t g(t \mid b) \underbrace{\sum_n v_n z_t(n)}_{=v_0 \text{ by (A28)}} < v^{**} \sum_t g(t \mid b) \underbrace{\sum_n z_t(n)}_{=1}.$$

Thus, ($\overline{\text{IC}}_0^r$) is equivalent to $v_0 \in [v^*, v^{**})$ and it is automatically satisfied by our choice of parameters. The revenue extracted from neutral customer t being equal to pu_0 is shown by Proposition 7.

It is immediate to see, by Lemma 2 and Lemma B2, that the platform cannot, incentive compatibly, sequence any customer of block 1 as a neutral customer.

Part II. For each $b \geq 2$ and $t \in [\tau^b, \tau^{b+1} - 1]$, optimal NA-sequencing policy r has the following structure:

$$r_t^0 \equiv \begin{cases} 1, & \textbf{if } t < \bar{t}^b, \\ 0, & \textbf{if } t \geq \bar{t}^b, \end{cases}$$

where $\bar{t}^b \in [\tau^b, \tau^{b+1}]$ is an integer. For each t with $r_t^0 \equiv 0$, the platform uses the following strategy:

$$r_t^1(n) := \begin{cases} 1, & \textbf{for } n > n^{**}(\ell_t^r), \\ x^{**}(\ell_t^r) \in [0,1], & \textbf{for } n = n^{**}(\ell_t^r), \\ 0, & \textbf{for } n < n^{**}(\ell_t^r), \end{cases} \quad \textbf{and} \quad r_t^{-1}(n) := 1 - r_t^1(n) \quad \forall n,$$

where ℓ_t^r is defined as in Lemma 1.

In this proof, we make use of the results in Lemma (B2) and assume hereafter, without loss of generality, that the platform is implying a *threshold affirmative policy* for each of their affirmative customers. In particular, an NA-sequencing policy r has, for each of its t with $r_t^0 \equiv 0$, a $(\hat{n}(\ell_t^r), \hat{x}(\ell_t^r))$ pair, analogous to (k, x) pair in Equation (B9). Existence of $(\hat{n}(\ell_t^r), \hat{x}(\ell_t^r))$ for each $t \geq \tau^b = \tau^\circ + 1$ is granted by Lemma (B2) and Lemma 2.

Suppose for some customer block $b \geq 2$, the platform assigns an affirmative customer t, prior to a neutral customer $t+1$, i.e., $r_t^0 \equiv 0$ and $r_{t+1}^0 \equiv 1$ for $t \in [\tau^b, \tau^{b+1}-2]$, which is a certainty in a contradicting case to the first statement of Part II. We now define another NA-sequencing policy $\tilde{r}$ that swaps the orders of affirmative customer t and neutral customer $t+1$ while keeping the rest of the policy the same as r, i.e., $\tilde{r}_t^0 \equiv 1, \tilde{r}_{t+1}^0 \equiv 0$ and $\tilde{r}_{\hat{t}} \equiv r_{\hat{t}}$ for all $\hat{t} \neq \{t, t+1\}$. Notice that, by Part I, ($\overline{\text{IC}}_0^r$) of customer block b continues to hold for $\tilde{r}$ and the revenue extracted from the neutral customers is the same with r. We next show that ($\overline{\text{IC}}_1^r$) and ($\overline{\text{IC}}_{-1}^r$) also continue to hold under $\tilde{r}$, but the platform extracts more revenue from the affirmative customers than under r. Let's say, $\tilde{r}_{t+1}$ is as follows (for $\ell_{t+1}^{\tilde{r}} = s+1$, or equivalently, $\ell_t^r = s$):

$$\tilde{r}_{t+1}^1(n) := \begin{cases} 1, & \text{for } n > \hat{n}(s)+1, \\ \frac{\hat{x}(s)\zeta(s,\hat{n}(s))u_{\hat{n}(s)} + \zeta(s,\hat{n}(s)+2)(1-u_{\hat{n}(s)+2})}{\zeta(s+1,\hat{n}(s)+1)}, & \text{for } n = \hat{n}(s)+1, \\ \frac{\hat{x}(s)\zeta(s,\hat{n}(s))(1-u_{\hat{n}(s)})}{\zeta(s+1,\hat{n}(s)-1)}, & \text{for } n = \hat{n}(s)-1, \\ 0, & \text{for } n < \hat{n}(s)-1, \end{cases} \quad \text{and} \quad \tilde{r}_{t+1}^{-1}(n) := 1 - \tilde{r}_{t+1}^1(n) \quad \forall n. \tag{C1}$$

Then, we have

$$\begin{aligned} \sum_n (v_n - v^{**})\zeta(s+1,n)\tilde{r}_{t+1}^1(n) &= \sum_{n > \hat{n}(s)+1} (v_n - v^{**})\zeta(s+1,n) \\ &\quad + (v_{\hat{n}(s)+1} - v^{**})\left[\hat{x}(s)\zeta(s,\hat{n}(s))u_{\hat{n}(s)} + \zeta(s,\hat{n}(s)+2)(1-u_{\hat{n}(s)+2})\right] \\ &\quad + (v_{\hat{n}(s)-1} - v^{**})\left[\hat{x}(s)\zeta(s,\hat{n}(s))(1-u_{\hat{n}(s)})\right] \\ &= \underbrace{\sum_{n > \hat{n}(s)+1} (v_n - v^{**})\zeta(s+1,n) + (v_{\hat{n}(s)+1} - v^{**})\zeta(s,\hat{n}(s)+2)(1-u_{\hat{n}(s)+2})}_{\sum_{n > \hat{n}(s)}(v_n - v^{**})\zeta(s,n) \text{ by (B3)}} \\ &\quad + \hat{x}(s)\zeta(s,\hat{n}(s)) \underbrace{\left[(v_{\hat{n}(s)+1} - v^{**})u_{\hat{n}(s)} + (v_{\hat{n}(s)-1} - v^{**})(1-u_{\hat{n}(s)})\right]}_{v_{\hat{n}(s)} - v^{**} \text{ by (A23)}} \\ &= \sum_n (v_n - v^{**})\zeta(s,n) r_t^1(n), \end{aligned}$$

which immediately implies $g(t+1 \mid b)\sum_n (v_n - v^{**})\zeta(s+1,n)\tilde{r}_{t+1}^1(n) = g(t \mid b)\sum_n (v_n - v^{**})\zeta(s,n) r_t^1(n)$. Notice also that $\ell_{t'}^{\tilde{r}} = \ell_{t'}^r$ for all $t' > t+1$. Therefore, by Lemma 1, $\tilde{r}$ satisfies ($\overline{\text{IC}}_1^r$). Similarly,

$$\begin{aligned} \sum_n (v_n - v^{*})\zeta(s+1,n)\tilde{r}_{t+1}^{-1}(n) &= \sum_{n < \hat{n}(s)-1} (v_n - v^{*})\zeta(s+1,n) \\ &\quad + (v_{\hat{n}(s)-1} - v^{*})\left[(1-\hat{x}(s))\zeta(s,\hat{n}(s))(1-u_{\hat{n}(s)}) + \zeta(s,\hat{n}(s)-2)u_{\hat{n}(s)-2}\right] \\ &\quad + (v_{\hat{n}(s)+1} - v^{*})\left[(1-\hat{x}(s))\zeta(s,\hat{n}(s))u_{\hat{n}(s)}\right] \\ &= \underbrace{\sum_{n < \hat{n}(s)-1} (v_n - v^{*})\zeta(s+1,n) + (v_{\hat{n}(s)-1} - v^{*})\zeta(s,\hat{n}(s)-2)u_{\hat{n}(s)-2}}_{\sum_{n < \hat{n}(s)}(v_n - v^{*})\zeta(s,n) \text{ by (B5)}} \\ &\quad + (1-\hat{x}(s))\zeta(s,\hat{n}(s)) \underbrace{\left[(v_{\hat{n}(s)-1} - v^{*})(1-u_{\hat{n}(s)}) + (v_{\hat{n}(s)+1} - v^{*})u_{\hat{n}(s)}\right]}_{v_{\hat{n}(s)} - v^{*} \text{ by (A23)}} \\ &= \sum_n (v_n - v^{*})\zeta(s+1,n) r_t^{-1}(n), \end{aligned}$$

which immediately implies $g\,(t+1\,|\,b)\sum_n(v_n - v^*)\zeta(s+1,n)\tilde{r}^{-1}_{t+1}(n) = g\,(t\,|\,b)\sum_n(v_n - v^*)\zeta(s,n)r^{-1}_t(n)$. Therefore, by Lemma 1, $\tilde{r}$ also satisfies ($\overline{\text{IC}}^r_{-1}$). Lastly, the expected revenue that $\tilde{r}_{t+1}$ generates from customer $t+1$ is equal to

$$\sum_n \tilde{r}_{t+1}(n)\zeta(s+1,n) = \underbrace{\sum_{n>\hat{n}(s)+1}\zeta(s+1,n)+\zeta(s,\hat{n}(s)+2)(1-u_{\hat{n}(s)+2})}_{=\sum_{n>\hat{n}(s)}\zeta(s,n) \text{ by (B7)}}+\hat{x}(s)\zeta(s,\hat{n}(s)) = \sum_n r_t(n)\zeta(s,n),$$

which, combining with the fact that $\ell^{\tilde{r}}_{t'} = \ell^r_{t'}$ for all $t' > t+1$, establishes that $\tilde{r}$ achieves a higher expected revenue for the platform than r, while satisfying ($\overline{\text{IC}}^r_1$), ($\overline{\text{IC}}^r_0$), and ($\overline{\text{IC}}^r_{-1}$), contradicting the optimality of r. Hence, in a recursive manner, we can show that the optimal NA-sequencing policy is of the form given by the first statement of Part II.

By Lemma 1, structure of $g\,(\cdot\,|\,b)$ and the first statement of Part II, for a given b, we can write ($\overline{\text{IC}}^r_1$) and ($\overline{\text{IC}}^r_{-1}$) as, respectively:

$$g\left(\bar{t}^b\,\middle|\,b\right)\sum_n(v_n - v^{**})\zeta(\ell^r_{\bar{t}^b},n)\sum_{t\in[\bar{t}^b,\tau^{b+1}-1]} r^1_t(n) \geq 0,$$
$$g\left(\bar{t}^b\,\middle|\,b\right)\sum_n(v_n - v^{*})\zeta(\ell^r_{\bar{t}^b},n)\sum_{t\in[\bar{t}^b,\tau^{b+1}-1]} r^{-1}_t(n) \leq 0.$$

Then, by Lemma 2, it is straightforward to see that the threshold affirmative policy $\left(n^{**}(\ell^r_{\bar{t}^b}),x^{**}(\ell^r_{\bar{t}^b})\right)$ is optimal for each affirmative customer $t\in[\bar{t}^b,\tau^{b+1}-1]$ and this generates, per affirmative customer, an expected revenue of $pF(\ell^r_{\bar{t}^b})$ for the platform.

Part III. For each $b\in\{2,\ldots,k-1\}$, **optimal NA-sequencing policy** r **has** $r^0_{\tau^b-1}\geq r^0_{\tau^b}$.

Suppose for some $b\geq 2$, r has $r^0_{\tau^b-1}\equiv 0$ and $r^0_{\tau^b}\equiv 1$, which is the only contradicting case to the statement of Part III. Let's say τ^b-1 equals to t. We now define another NA-sequencing policy $\tilde{r}$ that swaps the orders of affirmative customer t and neutral customer $t+1$ while keeping the rest of the policy the same as r, i.e., $\tilde{r}^0_t\equiv 1, \tilde{r}^0_{t+1}\equiv 0$ and $\tilde{r}_{\hat{t}}\equiv r_{\hat{t}}$ for all $\hat{t}\notin\{t,t+1\}$. Again, by Part I, ($\overline{\text{IC}}^r_0$) of customer blocks $b-1$ and b continue to hold for $\tilde{r}$ and the revenue extracted from the neutral customers is the same with r. Notice that, by Part II, the individual contributions of r^1_t and $\tilde{r}^1_{t+1}$ to the left hand sides of ($\overline{\text{IC}}^r_1$) constraints of customer blocks $b-1$ and b are, respectively:

$$g\,(t\,|\,b-1)\left\{\left(v_{n^{**}(\ell^r_t)}-v^{**}\right)\zeta(\ell^r_t,n^{**}(\ell^r_t))x^{**}(\ell^r_t)+\sum_{n>n^{**}(\ell^r_t)}(v_n-v^{**})\zeta(\ell^r_t,n)\right\}=0,$$
$$g\,(t+1\,|\,b)\left\{\left(v_{n^{**}(\ell^{\tilde{r}}_{t+1})}-v^{**}\right)\zeta(\ell^{\tilde{r}}_{t+1},n^{**}(\ell^{\tilde{r}}_{t+1}))x^{**}(\ell^{\tilde{r}}_{t+1})+\sum_{n>n^{**}(\ell^{\tilde{r}}_{t+1})}(v_n-v^{**})\zeta(\ell^{\tilde{r}}_{t+1},n)\right\}=0.$$

Therefore, ($\overline{\text{IC}}^r_1$) also continues to hold under $\tilde{r}$. Similarly, ($\overline{\text{IC}}^r_{-1}$) also holds under $\tilde{r}$ as, by Equation (B12), each affirmative period's contribution to the left hand side of ($\overline{\text{IC}}^r_{-1}$) is negative. Lastly, by Lemma 2, the expected revenue that $\tilde{r}_{t+1}$ generates from customer $t+1$ is greater or equal than what r generates from customer t, as $\ell^{\tilde{r}}_{t+1}=\ell^r_t+1$. As $\ell^{\tilde{r}}_{t'}=\ell^r_{t'}$ for all $t'>t+1$, we have established that $\tilde{r}$ achieves a higher expected revenue for the platform than r, while satisfying ($\overline{\text{IC}}^r_1$), ($\overline{\text{IC}}^r_0$), and ($\overline{\text{IC}}^r_{-1}$), contradicting the optimality of r. Recursive application of Parts II and III establishes the Proposition. ■

PROPOSITION C1 **(Customer's purchase decision)**. *For any public history $\widehat{H}_t$, customer t's purchase decision is*

$$a_t = \begin{cases} 1, & \text{if } \mathbb{E}\left[V \,\middle|\, m_t, \widehat{H}_t, \sigma\right] \in [v^{**}, 1], \\ S_t, & \text{if } \mathbb{E}\left[V \,\middle|\, m_t, \widehat{H}_t, \sigma\right] \in [v^{*}, v^{**}], \\ -1, & \text{if } \mathbb{E}\left[V \,\middle|\, m_t, \widehat{H}_t, \sigma\right] \in [0, v^{*}], \end{cases} \tag{C2}$$

where thresholds v^* and v^{**} are given by

$$v^* := \frac{p(1-q)}{p(1-q)+(1-p)q} < p < v^{**} := \frac{pq}{pq+(1-p)(1-q)}. \tag{C3}$$

Proof of Proposition C1. Since private signals S_t are mutually independent conditional on V with $\mathbb{P}[S_t = 1 \mid V = 1] = \mathbb{P}[S_t = -1 \mid V = 0] = q$, the Bayes rule thus immediately implies that

$$\begin{aligned}\mathbb{E}\left[V \,\middle|\, S_t = 1, m_t, \widehat{H}_t, \sigma\right] &= \mathbb{P}\left[V = 1 \,\middle|\, S_t = 1, m_t, \widehat{H}_t, \sigma\right] \\ &= \frac{\mathbb{P}[S_t = 1 \mid V = 1]\,\mathbb{P}\left[V = 1 \,\middle|\, m_t, \widehat{H}_t, \sigma\right]}{\mathbb{P}[S_t = 1 \mid V = 1]\,\mathbb{P}\left[V = 1 \,\middle|\, m_t, \widehat{H}_t, \sigma\right] + \mathbb{P}[S_t = 1 \mid V = 0]\,\mathbb{P}\left[V = 0 \,\middle|\, m_t, \widehat{H}_t, \sigma\right]} \\ &= \frac{q\mathbb{P}\left[V = 1 \,\middle|\, m_t, \widehat{H}_t, \sigma\right]}{q\mathbb{P}\left[V = 1 \,\middle|\, m_t, \widehat{H}_t, \sigma\right] + (1-q)\mathbb{P}\left[V = 0 \,\middle|\, m_t, \widehat{H}_t, \sigma\right]} > \mathbb{P}\left[V = 1 \,\middle|\, m_t, \widehat{H}_t, \sigma\right] = \mathbb{E}\left[V \,\middle|\, m_t, \widehat{H}_t, \sigma\right]\end{aligned}$$

and similarly

$$\begin{aligned}\mathbb{E}\left[V \,\middle|\, S_t = -1, m_t, \widehat{H}_t, \sigma\right] &= \mathbb{P}\left[V = 1 \,\middle|\, S_t = -1, m_t, \widehat{H}_t, \sigma\right] \\ &= \frac{(1-q)\mathbb{P}\left[V = 1 \,\middle|\, m_t, \widehat{H}_t, \sigma\right]}{(1-q)\mathbb{P}\left[V = 1 \,\middle|\, m_t, \widehat{H}_t, \sigma\right] + q\mathbb{P}\left[V = 0 \,\middle|\, m_t, \widehat{H}_t, \sigma\right]} < \mathbb{P}\left[V = 1 \,\middle|\, m_t, \widehat{H}_t, \sigma\right] = \mathbb{E}\left[V \,\middle|\, m_t, \widehat{H}_t, \sigma\right],\end{aligned}$$

where the inequalities follow by $q \in (1/2, 1]$.

Thus, by Equation (2), $a_t = 1$ upon $S_t = -1$ if and only if $\frac{(1-q)\mathbb{P}\left[V=1 \,\middle|\, m_t, \widehat{H}_t, \sigma\right]}{(1-q)\mathbb{P}\left[V=1 \,\middle|\, m_t, \widehat{H}_t, \sigma\right] + q\mathbb{P}\left[V=0 \,\middle|\, m_t, \widehat{H}_t, \sigma\right]} \geq p$, which reduces to $\mathbb{E}\left[V \,\middle|\, m_t, \widehat{H}_t, \sigma\right] = \mathbb{P}\left[V = 1 \,\middle|\, m_t, \widehat{H}_t, \sigma\right] \geq \frac{pq}{pq+(1-p)(1-q)} = v^{**}$. Similarly, $a_t = 1$ upon $S_t = 1$ if and only if $\mathbb{E}\left[V \,\middle|\, m_t, \widehat{H}_t, \sigma\right] = \mathbb{P}\left[V = 1 \,\middle|\, m_t, \widehat{H}_t, \sigma\right] < \frac{p(1-q)}{p(1-q)+(1-p)q} = v^*$. Since it is straightforward to verify (C3), the above two conditions thus lead to (C2). ■

LEMMA C1. *For an arbitrary information provision policy $\left(\tilde{\sigma}, \widetilde{\mathcal{M}}\right)$, define a mapping $\varphi : \widetilde{\mathcal{M}} \to \mathcal{M} := \{1, 0, -1\}$ as follows:*

$$\varphi(\widetilde{m}_t) := \begin{cases} 1, & \text{if } \mathbb{E}\left[V \,\middle|\, \widetilde{m}_t, \widehat{H}_t, \tilde{\sigma}\right] \geq v^{**}, \\ 0, & \text{if } v^* \leq \mathbb{E}\left[V \,\middle|\, \widetilde{m}_t, \widehat{H}_t, \tilde{\sigma}\right] < v^{**}, \\ -1, & \text{if } \mathbb{E}\left[V \,\middle|\, \widetilde{m}_t, \widehat{H}_t, \tilde{\sigma}\right] \leq v^{*}. \end{cases} \tag{C4}$$

For a platform's proprietary history $\widetilde{H}_t = \{(\widetilde{m}_s, a_s) : s < t\}$ under $\left(\tilde{\sigma}, \widetilde{\mathcal{M}}\right)$, denote $\varphi(\widetilde{H}_t) := \{(\varphi(\widetilde{m}_s), a_s) : s < t\}$ with slight abuse of notation. Then, for any public history $\widehat{H}_t$,

$$\mathbb{E}\left[V \,\middle|\, \widetilde{H}'_t, \widehat{H}_t, \tilde{\sigma}\right] = \mathbb{E}\left[V \,\middle|\, \widetilde{H}''_t, \widehat{H}_t, \tilde{\sigma}\right], \quad \text{if } \varphi(\widetilde{H}'_t) = \varphi(\widetilde{H}''_t). \tag{C5}$$

Furthermore, platform's expectation of the product value evolves according to $\mathbb{E}[V \mid \tilde{\sigma}] = \mathbb{P}[V=1 \mid \tilde{\sigma}] = v(0,0) = v_0$ *and*

$$\mathbb{E}\left[V \,\middle|\, \widetilde{H}_{t+1}, \widehat{H}_{t+1}, \tilde{\sigma}\right] = \begin{cases} \mathbb{E}\left[V \,\middle|\, \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right], & \text{if } \varphi(\widetilde{m}_t)=\pm 1, \widetilde{H}_{t+1}=\widetilde{H}_t \cup (\widetilde{m}_t, \varphi(\widetilde{m}_t)), \widehat{H}_{t+1}=\widehat{H}_t \cup (0), \\ \frac{q\mathbb{P}\left[V=1 \mid \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right]}{q\mathbb{P}\left[V=1 \mid \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right] + (1-q)\mathbb{P}\left[V=0 \mid \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right]}, & \text{if } \varphi(\widetilde{m}_t)=0, \widetilde{H}_{t+1}=\widetilde{H}_t \cup (\widetilde{m}_t, 1), \widehat{H}_{t+1}=\widehat{H}_t \cup (0), \\ \frac{(1-q)\mathbb{P}\left[V=1 \mid \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right]}{(1-q)\mathbb{P}\left[V=1 \mid \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right] + q\mathbb{P}\left[V=0 \mid \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right]}, & \text{if } \varphi(\widetilde{m}_t)=0, \widetilde{H}_{t+1}=\widetilde{H}_t \cup (\widetilde{m}_t, -1), \widehat{H}_{t+1}=\widehat{H}_t \cup (0), \\ \frac{\widehat{q}\mathbb{P}\left[V=1 \mid \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right]}{\widehat{q}\mathbb{P}\left[V=1 \mid \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right] + (1-\widehat{q})\mathbb{P}\left[V=0 \mid \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right]}, & \text{if } \varphi(\widetilde{m}_t)=1, \widetilde{H}_{t+1}=\widetilde{H}_t \cup (\widetilde{m}_t, 1), \widehat{H}_{t+1}=\widehat{H}_t \cup (1), \\ \frac{(1-\widehat{q})\mathbb{P}\left[V=1 \mid \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right]}{(1-\widehat{q})\mathbb{P}\left[V=1 \mid \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right] + \widehat{q}\mathbb{P}\left[V=0 \mid \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right]}, & \text{if } \varphi(\widetilde{m}_t)=1, \widetilde{H}_{t+1}=\widetilde{H}_t \cup (\widetilde{m}_t, 1), \widehat{H}_{t+1}=\widehat{H}_t \cup (-1), \\ \frac{\widehat{q}q\mathbb{P}\left[V=1 \mid \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right]}{\widehat{q}q\mathbb{P}\left[V=1 \mid \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right] + (1-\widehat{q})(1-q)\mathbb{P}\left[V=0 \mid \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right]}, & \text{if } \varphi(\widetilde{m}_t)=0, \widetilde{H}_{t+1}=\widetilde{H}_t \cup (\widetilde{m}_t, 1), \widehat{H}_{t+1}=\widehat{H}_t \cup (1), \\ \frac{(1-\widehat{q})q\mathbb{P}\left[V=1 \mid \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right]}{(1-\widehat{q})q\mathbb{P}\left[V=1 \mid \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right] + \widehat{q}(1-q)\mathbb{P}\left[V=0 \mid \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right]}, & \text{if } \varphi(\widetilde{m}_t)=0, \widetilde{H}_{t+1}=\widetilde{H}_t \cup (\widetilde{m}_t, 1), \widehat{H}_{t+1}=\widehat{H}_t \cup (-1). \end{cases} \tag{C6}$$

Proof of Lemma C1. By Proposition C1 and definition of φ in (C4), customer t, who receives $\widetilde{m}_t$ with $\varphi(\widetilde{m}_t) = \pm 1$, makes purchase decision $\widetilde{a}_t = \varphi(\widetilde{m}_t) = \pm 1$, regardless of S_t. If this customer also does not leave a review or a rating on the platform, $\widehat{S}_t = 0$, then platform does not infer any new information in addition to the inference made from the previous customers interactions and hence its expectation of the product value does not alter, establishing the first line in (C6). Again by Proposition C1 and (C4), customer t, who receives $\widetilde{m}_t$ where $\varphi(\widetilde{m}_t) = 0$, makes purchase decision $\widetilde{a}_t = S_t$, that is followed by $\widehat{S}_t \in \{0, \pm 1\}$. Thus, the remaining lines in (C6) follow from the Bayes rule:

$$\begin{aligned} &\mathbb{E}\left[V \,\middle|\, \widetilde{H}_t \cup (\widetilde{m}_t, \widetilde{a}_t), \widehat{H}_t \cup (\widehat{S}_t), \tilde{\sigma}\right] \\ &= \mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t \cup (\widetilde{m}_t, \widetilde{a}_t), \widehat{H}_t \cup (\widehat{S}_t), \tilde{\sigma}\right] = \mathbb{P}\left[V=1 \,\middle|\, \widetilde{a}_t, \widehat{S}_t, \widetilde{m}_t, \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right] \\ &= \frac{\mathbb{P}\left[\widetilde{a}_t, \widehat{S}_t \,\middle|\, \widetilde{m}_t, V=1\right] \mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right]}{\mathbb{P}\left[\widetilde{a}_t, \widehat{S}_t \,\middle|\, \widetilde{m}_t, V=1\right] \mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right] + \mathbb{P}\left[\widetilde{a}_t, \widehat{S}_t \,\middle|\, \widetilde{m}_t, V=0\right] \mathbb{P}\left[V=0 \,\middle|\, \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right]} \\ &= \frac{\mathbb{P}\left[\widehat{S}_t \,\middle|\, \widetilde{a}_t, V=1\right] \mathbb{P}\left[\widetilde{a}_t \mid \widetilde{m}_t, V=1\right] \mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right]}{\mathbb{P}\left[\widehat{S}_t \,\middle|\, \widetilde{a}_t, V=1\right] \mathbb{P}\left[\widetilde{a}_t \mid \widetilde{m}_t, V=1\right] \mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right] + \mathbb{P}\left[\widehat{S}_t \,\middle|\, \widetilde{a}_t, V=0\right] \mathbb{P}\left[\widetilde{a}_t \mid \widetilde{m}_t, V=0\right] \mathbb{P}\left[V=0 \,\middle|\, \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right]}, \end{aligned}$$

where $\mathbb{P}[\widetilde{a}_t = 1 \mid \varphi(\widetilde{m}_t) = 0, V=1] = \mathbb{P}[S_t = 1 \mid V=1] = q$ (similarly, $\mathbb{P}[\widetilde{a}_t = 1 \mid \varphi(\widetilde{m}_t) = 0, V=0] = \mathbb{P}[S_t = 1 \mid V=0] = 1 - q$) and $\mathbb{P}[\widetilde{a}_t = -1 \mid \varphi(\widetilde{m}_t) = 0, V=0] = \mathbb{P}[S_t = -1 \mid V=0] = q$ (similarly, $\mathbb{P}[\widetilde{a}_t = -1 \mid \varphi(\widetilde{m}_t) = 0, V=1] = \mathbb{P}[S_t = -1 \mid V=1] = 1 - q$) by assumption and $\mathbb{P}\left[\widehat{S}_t \,\middle|\, \widetilde{a}_t, V\right]$ is given by (18).

We now demonstrate (C5) by induction. For $t=1$, since $\widetilde{H}_1 = \widehat{H}_1 = \emptyset$ and $\mathbb{E}[V \mid \tilde{\sigma}] = v_{0,0} = v_0$, (C5) holds. Now suppose (C5) holds for t. Let $\widetilde{H}'_{t+1}, \widetilde{H}''_{t+1}$ be two platform's proprietary histories such that $\varphi(\widetilde{H}'_{t+1}) = \varphi(\widetilde{H}''_{t+1})$, which implies that $\varphi(\widetilde{H}'_t) = \varphi(\widetilde{H}''_t)$ and $\varphi(\widetilde{m}'_t) = \varphi(\widetilde{m}''_t)$. Thus, the induction hypothesis immediately implies that $\mathbb{E}\left[V \,\middle|\, \widetilde{H}'_t, \widehat{H}_t, \tilde{\sigma}\right] = \mathbb{E}\left[V \,\middle|\, \widetilde{H}''_t, \widehat{H}_t, \tilde{\sigma}\right]$ and, further by (C6), we have (C5) holds for $t+1$. ■

PROPOSITION C2 **(Sufficiency of recommendation policy)**. *For any information provision policy, there exists a recommendation policy that induces the same purchase decisions from all customers and the same expected revenue for the platform.*

Proof of Proposition C2. For an arbitrary information provision policy $\left(\tilde{\sigma}, \widetilde{\mathcal{M}}\right)$, we now define a new information provision policy $(\sigma, \mathcal{M})$ as

$$\sigma\left(m_t \,\middle|\, H_t, \widehat{H}_t\right) := \frac{\sum_{\varphi(\widetilde{H}_t)=H_t, \varphi(\widetilde{m}_t)=m_t} \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t, \widehat{H}_t\right) \mathbb{P}\left[\widetilde{H}_t, \widehat{H}_t \,\middle|\, \tilde{\sigma}\right]}{\sum_{\varphi(\widetilde{H}_t)=H_t} \mathbb{P}\left[\widetilde{H}_t, \widehat{H}_t \,\middle|\, \tilde{\sigma}\right]}, \tag{C7}$$

for any $m_t \in \mathcal{M}, H_t = \{(m_l, a_l) : m_l \in \mathcal{M}, a_l \in \{-1, 1\}, l < t\}$ and $\widehat{H}_t := \{(\widehat{S}_l) : l < t\}$.

We then show that the information provision policy $(\sigma, \mathcal{M})$ defined above satisfies (19), and hence is a recommendation policy. By rule of total probability, we first have

$$\begin{aligned}
\mathbb{P}[m_t \mid \sigma] &= \sum_{H_t, \widehat{H}_t} \sigma\left(m_t \,\middle|\, H_t, \widehat{H}_t\right) \mathbb{P}\left[H_t, \widehat{H}_t \,\middle|\, \sigma\right] \\
\text{by (C7)} \quad &= \sum_{H_t, \widehat{H}_t} \frac{\sum_{\varphi(\widetilde{H}_t)=H_t, \varphi(\widetilde{m}_t)=m_t} \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t, \widehat{H}_t\right) \mathbb{P}\left[\widetilde{H}_t, \widehat{H}_t \,\middle|\, \tilde{\sigma}\right]}{\sum_{\varphi(\widetilde{H}_t)=H_t} \mathbb{P}\left[\widetilde{H}_t, \widehat{H}_t \,\middle|\, \tilde{\sigma}\right]} \sum_{\varphi(\widetilde{H}_t)=H_t} \mathbb{P}\left[\widetilde{H}_t, \widehat{H}_t \,\middle|\, \tilde{\sigma}\right] \\
&= \sum_{\widetilde{H}_t, \varphi(\widetilde{m}_t)=m_t, \widehat{H}_t} \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t, \widehat{H}_t\right) \mathbb{P}\left[\widetilde{H}_t, \widehat{H}_t \,\middle|\, \tilde{\sigma}\right] = \sum_{\varphi(\widetilde{m}_t)=m_t} \mathbb{P}\left[\widetilde{m}_t \mid \tilde{\sigma}\right],
\end{aligned} \tag{C8}$$

and similarly,

$$\begin{aligned}
\mathbb{P}[m_t, V=1 \mid \sigma] &= \sum_{H_t, \widehat{H}_t} \sigma\left(m_t \,\middle|\, H_t, \widehat{H}_t\right) \mathbb{P}\left[V=1, H_t, \widehat{H}_t \,\middle|\, \sigma\right] \\
&= \sum_{H_t, \widehat{H}_t} \sigma\left(m_t \,\middle|\, H_t, \widehat{H}_t\right) \left\{ \sum_{\varphi(\widetilde{H}_t)=H_t} \mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right] \mathbb{P}\left[\widetilde{H}_t, \widehat{H}_t \,\middle|\, \tilde{\sigma}\right] \right\} \\
\text{by (C7)} \quad &= \sum_{H_t, \widehat{H}_t} \frac{\sum_{\varphi(\widetilde{H}_t)=H_t, \varphi(\widetilde{m}_t)=m_t} \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t, \widehat{H}_t\right) \mathbb{P}\left[\widetilde{H}_t, \widehat{H}_t \,\middle|\, \tilde{\sigma}\right]}{\sum_{\varphi(\widetilde{H}_t)=H_t} \mathbb{P}\left[\widetilde{H}_t, \widehat{H}_t \,\middle|\, \tilde{\sigma}\right]} \left\{ \sum_{\varphi(\widetilde{H}_t)=H_t} \mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right] \mathbb{P}\left[\widetilde{H}_t, \widehat{H}_t \,\middle|\, \tilde{\sigma}\right] \right\} \\
&= \sum_{H_t, \widehat{H}_t} \sum_{\varphi(\widetilde{H}_t)=H_t, \varphi(\widetilde{m}_t)=m_t} \mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right] \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t, \widehat{H}_t\right) \mathbb{P}\left[\widetilde{H}_t, \widehat{H}_t \,\middle|\, \tilde{\sigma}\right] \\
&= \sum_{\widetilde{H}_t, \varphi(\widetilde{m}_t)=m_t, \widehat{H}_t} \mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right] \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t, \widehat{H}_t\right) \mathbb{P}\left[\widetilde{H}_t, \widehat{H}_t \,\middle|\, \tilde{\sigma}\right],
\end{aligned}$$

where the fourth equality, utilizing the fact that $V \in \{0, 1\}$ is a binary random variable, follows by (C5). Then, the Bayes rule yields

$$\mathbb{E}\left[V \,\middle|\, m_t, \widehat{H}_t, \sigma\right] = \mathbb{P}\left[V=1 \,\middle|\, m_t, \widehat{H}_t, \sigma\right] = \frac{\sum_{\widetilde{H}_t, \varphi(\widetilde{m}_t)=m_t} \mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right] \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t, \widehat{H}_t\right) \mathbb{P}\left[\widetilde{H}_t, \widehat{H}_t \,\middle|\, \tilde{\sigma}\right]}{\sum_{\varphi(\widetilde{m}_t)=m_t} \mathbb{P}\left[\widetilde{m}_t, \widehat{H}_t \,\middle|\, \tilde{\sigma}\right]}. \tag{C9}$$

On the other hand, the Bayes rule also yields

$$\mathbb{E}\left[V \,\middle|\, \widetilde{m}_t, \widehat{H}_t, \tilde{\sigma}\right] = \mathbb{P}\left[V=1 \,\middle|\, \widetilde{m}_t, \widehat{H}_t, \tilde{\sigma}\right] = \frac{\sum_{\widetilde{H}_t} \mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right] \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t, \widehat{H}_t\right) \mathbb{P}\left[\widetilde{H}_t, \widehat{H}_t \,\middle|\, \tilde{\sigma}\right]}{\mathbb{P}\left[\widetilde{m}_t, \widehat{H}_t \,\middle|\, \tilde{\sigma}\right]}. \tag{C10}$$

For any $\widetilde{m}_t$ such that $\varphi(\widetilde{m}_t)=0$, (C4) implies that $v^* \le \mathbb{E}\left[V \,\middle|\, \widetilde{m}_t, \widehat{H}_t, \tilde{\sigma}\right] < v^{**}$, which, by (C10), is equivalent to

$$v^*\mathbb{P}\left[\widetilde{m}_t, \widehat{H}_t \,\middle|\, \tilde{\sigma}\right] \le \sum_{\widetilde{H}_t} \mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right] \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t, \widehat{H}_t\right) \mathbb{P}\left[\widetilde{H}_t, \widehat{H}_t \,\middle|\, \tilde{\sigma}\right] < v^{**}\mathbb{P}\left[\widetilde{m}_t, \widehat{H}_t \,\middle|\, \tilde{\sigma}\right].$$

Further by (C9), we have $\mathbb{E}\left[V \,\middle|\, m_t=0, \widehat{H}_t, \sigma\right] = \mathbb{P}\left[V=1 \,\middle|\, m_t=0, \widehat{H}_t, \sigma\right] \in [v^*, v^{**})$, establishing the second inequality (19). By the same token, we can also establish the first and the last inequalities of (19), which ensures the incentive compatibility of $(\sigma, \mathcal{M})$.

Finally, we demonstrate that $(\sigma, \mathcal{M})$ and $\left(\tilde{\sigma}, \widetilde{\mathcal{M}}\right)$ induces the same (ex ante) probability of purchase for each customer and generates the same expected revenue for the platform. By Proposition C1 and (C4), customer t's purchase decision under $\left(\tilde{\sigma}, \widetilde{\mathcal{M}}\right)$ is

$$\tilde{a}_t = \varphi(\widetilde{m}_t) + (1 - |\varphi(\widetilde{m}_t)|)\, S_t. \tag{C11}$$

And customer t's purchase decision under $(\sigma, \mathcal{M})$ is

$$a_t = m_t + (1 - |m_t|)\, S_t. \tag{C12}$$

Hence, by Equation (3), the platform's expected revenue under $\left(\tilde{\sigma}, \widetilde{\mathcal{M}}\right)$ is given by $\frac{pT}{2} + \frac{p}{2}\sum_{t=1}^{T} \mathbb{E}\left[\tilde{a}_t \mid \tilde{\sigma}\right]$, and the platform's expected revenue under $(\sigma, \mathcal{M})$ is given by $\frac{pT}{2} + \frac{p}{2}\sum_{t=1}^{T} \mathbb{E}\left[a_t \mid \sigma\right]$. We now demonstrate these revenues are equal by showing

$$\mathbb{E}\left[\tilde{a}_t \mid \tilde{\sigma}\right] = \mathbb{E}\left[a_t \mid \sigma\right] \text{ for all } t. \tag{C13}$$

On one hand, by (C11)

$$\begin{aligned}
\mathbb{E}\left[\tilde{a}_t \mid \tilde{\sigma}\right] &= \mathbb{E}\left[\varphi(\widetilde{m}_t) + (1 - |\varphi(\widetilde{m}_t)|)\, S_t \mid \tilde{\sigma}\right] \\
&= 1 \sum_{\varphi(\widetilde{m}_t)=1} \mathbb{P}\left[\widetilde{m}_t \mid \tilde{\sigma}\right] + (-1) \sum_{\varphi(\widetilde{m}_t)=-1} \mathbb{P}\left[\widetilde{m}_t \mid \tilde{\sigma}\right] + \sum_{\varphi(\widetilde{m}_t)=0} \mathbb{E}\left[S_t \mid \widetilde{m}_t, \tilde{\sigma}\right] \mathbb{P}\left[\widetilde{m}_t \mid \tilde{\sigma}\right].
\end{aligned} \tag{C14}$$

On the other hand, by (C12)

$$\begin{aligned}
\mathbb{E}\left[a_t \mid \sigma\right] &= \mathbb{E}\left[m_t + (1 - |m_t|)\, S_t \mid \sigma\right] \\
&= 1\mathbb{P}\left[m_t = 1 \mid \sigma\right] + (-1)\mathbb{P}\left[m_t = -1 \mid \sigma\right] + \mathbb{E}\left[S_t \mid m_t = 0, \sigma\right] \mathbb{P}\left[m_t = 0 \mid \sigma\right].
\end{aligned} \tag{C15}$$

By (C8), the first two terms in (C14) are equal to the first two terms in (C15) respectively, i.e., $\sum_{\varphi(\widetilde{m}_t)=1} \mathbb{P}\left[\widetilde{m}_t \mid \tilde{\sigma}\right] = \mathbb{P}\left[m_t = 1 \mid \sigma\right]$ and $\sum_{\varphi(\widetilde{m}_t)=-1} \mathbb{P}\left[\widetilde{m}_t \mid \tilde{\sigma}\right] = \mathbb{P}\left[m_t = -1 \mid \sigma\right]$. Below, we demonstrate the last term in (C14) is also equal to that in (C15), thus completing the proof. To that end, denote $u'_{\widetilde{H}_t, \widehat{H}_t} = q\mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right] + (1-q)\mathbb{P}\left[V=0 \,\middle|\, \widetilde{H}_t, \widehat{H}_t, \tilde{\sigma}\right]$ and $u_{H_t, \widehat{H}_t} = q\mathbb{P}\left[V=1 \,\middle|\, H_t, \widehat{H}_t, \sigma\right] + (1 - q)\mathbb{P}\left[V=0 \,\middle|\, H_t, \widehat{H}_t, \sigma\right]$. Then, the last term in (C14) can be written as

$$\begin{aligned}
\sum_{\varphi(\widetilde{m}_t)=0} \mathbb{E}\left[S_t \mid \widetilde{m}_t, \tilde{\sigma}\right] \mathbb{P}\left[\widetilde{m}_t \mid \tilde{\sigma}\right] &= \sum_{\widetilde{H}_t, \varphi(\widetilde{m}_t)=0, \widehat{H}_t} \left[1u'_{\widetilde{H}_t, \widehat{H}_t} + (-1)(1 - u'_{\widetilde{H}_t, \widehat{H}_t})\right] \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t, \widehat{H}_t\right) \mathbb{P}\left[\widetilde{H}_t, \widehat{H}_t \,\middle|\, \tilde{\sigma}\right] \\
&= \sum_{\widetilde{H}_t, \varphi(\widetilde{m}_t)=0, \widehat{H}_t} \left[2u'_{\widetilde{H}_t, \widehat{H}_t} - 1\right] \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t, \widehat{H}_t\right) \mathbb{P}\left[\widetilde{H}_t, \widehat{H}_t \,\middle|\, \tilde{\sigma}\right].
\end{aligned}$$

Similarly, the last term in (C15) can be rewritten as

$$\begin{aligned}
\mathbb{E}\left[S_t \mid m_t=0,\sigma\right]\mathbb{P}\left[m_t=0 \mid \sigma\right] &= \sum_{H_t,\widehat{H}_t}\left[1u_{H_t,\widehat{H}_t}+(-1)(1-u_{H_t,\widehat{H}_t})\right]\sigma\left(m_t=0 \,\middle|\, H_t,\widehat{H}_t\right)\mathbb{P}\left[H_t,\widehat{H}_t \,\middle|\, \sigma\right]\\
&= \sum_{H_t,\widehat{H}_t}\left[2u_{H_t,\widehat{H}_t}-1\right]\sigma\left(m_t=0 \,\middle|\, H_t,\widehat{H}_t\right)\mathbb{P}\left[H_t,\widehat{H}_t \,\middle|\, \sigma\right]\\
&= \sum_{H_t,\widehat{H}_t}\left[2u_{H_t,\widehat{H}_t}-1\right]\frac{\sum_{\varphi(\widetilde{H}_t)=H_t,\varphi(\widetilde{m}_t)=0}\tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t,\widehat{H}_t\right)\mathbb{P}\left[\widetilde{H}_t,\widehat{H}_t \,\middle|\, \tilde{\sigma}\right]}{\sum_{\varphi(\widetilde{H}_t)=H_t}\mathbb{P}\left[\widetilde{H}_t,\widehat{H}_t \,\middle|\, \tilde{\sigma}\right]}\mathbb{P}\left[H_t,\widehat{H}_t \,\middle|\, \sigma\right]\\
&= \sum_{H_t,\widehat{H}_t}\left[2u_{H_t,\widehat{H}_t}-1\right]\frac{\sum_{\varphi(\widetilde{H}_t)=H_t,\varphi(\widetilde{m}_t)=0}\tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t,\widehat{H}_t\right)\mathbb{P}\left[\widetilde{H}_t,\widehat{H}_t \,\middle|\, \tilde{\sigma}\right]}{\sum_{\varphi(\widetilde{H}_t)=H_t}\mathbb{P}\left[\widetilde{H}_t,\widehat{H}_t \,\middle|\, \tilde{\sigma}\right]}\sum_{\varphi(\widetilde{H}_t)=H_t}\mathbb{P}\left[\widetilde{H}_t,\widehat{H}_t \,\middle|\, \tilde{\sigma}\right]\\
&= \sum_{\widetilde{H}_t,\varphi(\widetilde{m}_t)=0,\widehat{H}_t}\left[2{u'}_{\widetilde{H}_t,\widehat{H}_t}-1\right]\tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t,\widehat{H}_t\right)\mathbb{P}\left[\widetilde{H}_t,\widehat{H}_t \,\middle|\, \tilde{\sigma}\right]\\
&= \sum_{\varphi(\widetilde{m}_t)=0}\mathbb{E}\left[S_t \mid \widetilde{m}_t,\tilde{\sigma}\right]\mathbb{P}\left[\widetilde{m}_t \mid \tilde{\sigma}\right],
\end{aligned}$$

where the third equality follows from (C7) and the conversion of $u_{H_t,\widehat{H}_t}$ to ${u'}_{\widetilde{H}_t,\widehat{H}_t}$ in the sixth equality follows from (C5) and the fact that $V \in \{0,1\}$ is a binary random variable. ■

Proof of Proposition 10. Under a recommendation policy r, (C6) implies that the platform's expectation of the product value evolves according to

$$\mathbb{E}\left[V \,\middle|\, H_{t+1},\widehat{H}_{t+1},r\right] = \begin{cases}
\mathbb{E}\left[V \,\middle|\, H_t,\widehat{H}_t,r\right], & \text{if } H_{t+1}=H_t\cup\pm(1,1),\widehat{H}_{t+1}=\widehat{H}_t\cup(0),\\
\frac{q\mathbb{P}\left[V=1 \,\middle|\, H_t,\widehat{H}_t,r\right]}{q\mathbb{P}\left[V=1 \,\middle|\, H_t,\widehat{H}_t,r\right]+(1-q)\mathbb{P}\left[V=0 \,\middle|\, H_t,\widehat{H}_t,r\right]}, & \text{if } H_{t+1}=H_t\cup(0,1),\widehat{H}_{t+1}=\widehat{H}_t\cup(0),\\
\frac{(1-q)\mathbb{P}\left[V=1 \,\middle|\, H_t,\widehat{H}_t,r\right]}{(1-q)\mathbb{P}\left[V=1 \,\middle|\, H_t,\widehat{H}_t,r\right]+q\mathbb{P}\left[V=0 \,\middle|\, H_t,\widehat{H}_t,r\right]}, & \text{if } H_{t+1}=H_t\cup(0,-1),\widehat{H}_{t+1}=\widehat{H}_t\cup(0),\\
\frac{\widehat{q}q\mathbb{P}\left[V=1 \,\middle|\, H_t,\widehat{H}_t,r\right]}{\widehat{q}q\mathbb{P}\left[V=1 \,\middle|\, H_t,\widehat{H}_t,r\right]+(1-\widehat{q})(1-q)\mathbb{P}\left[V=0 \,\middle|\, H_t,\widehat{H}_t,r\right]}, & \text{if } H_{t+1}=H_t\cup(0,1),\widehat{H}_{t+1}=\widehat{H}_t\cup(1),\\
\frac{(1-\widehat{q})q\mathbb{P}\left[V=1 \,\middle|\, H_t,\widehat{H}_t,r\right]}{(1-\widehat{q})q\mathbb{P}\left[V=1 \,\middle|\, H_t,\widehat{H}_t,r\right]+\widehat{q}(1-q)\mathbb{P}\left[V=0 \,\middle|\, H_t,\widehat{H}_t,r\right]}, & \text{if } H_{t+1}=H_t\cup(0,1),\widehat{H}_{t+1}=\widehat{H}_t\cup(-1),\\
\frac{\widehat{q}\mathbb{P}\left[V=1 \,\middle|\, H_t,\widehat{H}_t,r\right]}{\widehat{q}\mathbb{P}\left[V=1 \,\middle|\, H_t,\widehat{H}_t,r\right]+(1-\widehat{q})\mathbb{P}\left[V=0 \,\middle|\, H_t,\widehat{H}_t,r\right]}, & \text{if } H_{t+1}=H_t\cup(1,1),\widehat{H}_{t+1}=\widehat{H}_t\cup(1),\\
\frac{(1-\widehat{q})\mathbb{P}\left[V=1 \,\middle|\, H_t,\widehat{H}_t,r\right]}{(1-\widehat{q})\mathbb{P}\left[V=1 \,\middle|\, H_t,\widehat{H}_t,r\right]+\widehat{q}\mathbb{P}\left[V=0 \,\middle|\, H_t,\widehat{H}_t,r\right]}, & \text{if } H_{t+1}=H_t\cup(1,1),\widehat{H}_{t+1}=\widehat{H}_t\cup(-1).
\end{cases} \tag{C16}$$

with $\mathbb{E}[V \mid r]=\mathbb{P}[V=1 \mid r]=v_{0,0}=v_0$. We now prove (20) by induction. For $t=1$, since $H_1=\widehat{H}_1=\emptyset$ and hence $N(H_1)=\widehat{N}(\widehat{H}_1)=0$, thus (20) follows as $\mathbb{E}[V \mid r]=v_{0,0}=v_0$. Suppose (20) holds for any arbitrary t (i.e., induction hypothesis). Then, under a recommendation policy r, we have the following seven cases for $t+1$:

- If $H_{t+1}=H_t\cup\pm(1,1),\widehat{H}_{t+1}=\widehat{H}_t\cup(0)$, (C16) implies that $\mathbb{E}\left[V \,\middle|\, H_{t+1},\widehat{H}_{t+1},\sigma\right]=\mathbb{E}\left[V \,\middle|\, H_t,\widehat{H}_t,r\right]$, hence (20) holds for $t+1$ by the induction hypothesis and by noting that $N(H_{t+1})=N(H_t)$ and $\widehat{N}(\widehat{H}_{t+1})=\widehat{N}(\widehat{H}_t)$, according to Equations (7) and (21), respectively.
- If $H_{t+1}=H_t\cup(0,1),\widehat{H}_{t+1}=\widehat{H}_t\cup(0)$, (C16) and the induction hypothesis implies that

$$\mathbb{E}\left[V \,\middle|\, H_{t+1},\widehat{H}_{t+1},r\right]=\frac{qv_0}{qv_0+(1-q)(1-v_0)\left(\frac{1-q}{q}\right)^{N(H_t)}\left(\frac{1-\widehat{q}}{\widehat{q}}\right)^{\widehat{N}(\widehat{H}_t)}}=\frac{v_0}{v_0+(1-v_0)\left(\frac{1-q}{q}\right)^{N(H_t)+1}\left(\frac{1-\widehat{q}}{\widehat{q}}\right)^{\widehat{N}(\widehat{H}_t)}}.$$

As $N(H_{t+1}) = N(H_t) + 1$ and $\widehat{N}(\widehat{H}_{t+1}) = \widehat{N}(\widehat{H}_t)$, (20) holds for $t+1$.

- If $H_{t+1} = H_t \cup (0,-1)\,, \widehat{H}_{t+1} = \widehat{H}_t \cup (0)$, (C16) and the induction hypothesis implies that

$$\mathbb{E}\left[V \,\middle|\, H_{t+1}, \widehat{H}_{t+1}, r\right] = \frac{(1-q)v_0}{(1-q)v_0 + q(1-v_0)\left(\frac{1-q}{q}\right)^{N(H_t)}\left(\frac{1-\widehat{q}}{\widehat{q}}\right)^{\widehat{N}(\widehat{H}_t)}} = \frac{v_0}{v_0 + (1-v_0)\left(\frac{1-q}{q}\right)^{N(H_t)-1}\left(\frac{1-\widehat{q}}{\widehat{q}}\right)^{\widehat{N}(\widehat{H}_t)}}.$$

As $N(H_{t+1}) = N(H_t) - 1$ and $\widehat{N}(\widehat{H}_{t+1}) = \widehat{N}(\widehat{H}_t)$, (20) holds for $t+1$.

- If $H_{t+1} = H_t \cup (0,1)\,, \widehat{H}_{t+1} = \widehat{H}_t \cup (1)$, (C16) and the induction hypothesis implies that

$$\mathbb{E}\left[V \,\middle|\, H_{t+1}, \widehat{H}_{t+1}, r\right] = \frac{\widehat{q}qv_0}{\widehat{q}qv_0 + (1-\widehat{q})(1-q)(1-v_0)\left(\frac{1-q}{q}\right)^{N(H_t)}\left(\frac{1-\widehat{q}}{\widehat{q}}\right)^{\widehat{N}(\widehat{H}_t)}} = \frac{v_0}{v_0 + (1-v_0)\left(\frac{1-q}{q}\right)^{N(H_t)+1}\left(\frac{1-\widehat{q}}{\widehat{q}}\right)^{\widehat{N}(\widehat{H}_t)+1}}.$$

As $N(H_{t+1}) = N(H_t) + 1$ and $\widehat{N}(\widehat{H}_{t+1}) = \widehat{N}(\widehat{H}_t) + 1$, (20) holds for $t+1$.

- If $H_{t+1} = H_t \cup (0,1)\,, \widehat{H}_{t+1} = \widehat{H}_t \cup (-1)$, (C16) and the induction hypothesis implies that

$$\mathbb{E}\left[V \,\middle|\, H_{t+1}, \widehat{H}_{t+1}, r\right] = \frac{(1-\widehat{q})qv_0}{(1-\widehat{q})qv_0 + \widehat{q}(1-q)(1-v_0)\left(\frac{1-q}{q}\right)^{N(H_t)}\left(\frac{1-\widehat{q}}{\widehat{q}}\right)^{\widehat{N}(\widehat{H}_t)}} = \frac{v_0}{v_0 + (1-v_0)\left(\frac{1-q}{q}\right)^{N(H_t)+1}\left(\frac{1-\widehat{q}}{\widehat{q}}\right)^{\widehat{N}(\widehat{H}_t)-1}}.$$

As $N(H_{t+1}) = N(H_t) + 1$ and $\widehat{N}(\widehat{H}_{t+1}) = \widehat{N}(\widehat{H}_t) - 1$, (20) holds for $t+1$.

- If $H_{t+1} = H_t \cup (1,1)\,, \widehat{H}_{t+1} = \widehat{H}_t \cup (1)$, (C16) and the induction hypothesis implies that

$$\mathbb{E}\left[V \,\middle|\, H_{t+1}, \widehat{H}_{t+1}, r\right] = \frac{\widehat{q}v_0}{\widehat{q}v_0 + (1-\widehat{q})(1-v_0)\left(\frac{1-q}{q}\right)^{N(H_t)}\left(\frac{1-\widehat{q}}{\widehat{q}}\right)^{\widehat{N}(\widehat{H}_t)}} = \frac{v_0}{v_0 + (1-v_0)\left(\frac{1-q}{q}\right)^{N(H_t)}\left(\frac{1-\widehat{q}}{\widehat{q}}\right)^{\widehat{N}(\widehat{H}_t)+1}}.$$

As $N(H_{t+1}) = N(H_t)$ and $\widehat{N}(\widehat{H}_{t+1}) = \widehat{N}(\widehat{H}_t) + 1$, (20) holds for $t+1$.

- If $H_{t+1} = H_t \cup (1,1)\,, \widehat{H}_{t+1} = \widehat{H}_t \cup (-1)$, (C16) and the induction hypothesis implies that

$$\mathbb{E}\left[V \,\middle|\, H_{t+1}, \widehat{H}_{t+1}, r\right] = \frac{(1-\widehat{q})v_0}{(1-\widehat{q})v_0 + \widehat{q}(1-v_0)\left(\frac{1-q}{q}\right)^{N(H_t)}\left(\frac{1-\widehat{q}}{\widehat{q}}\right)^{\widehat{N}(\widehat{H}_t)}} = \frac{v_0}{v_0 + (1-v_0)\left(\frac{1-q}{q}\right)^{N(H_t)}\left(\frac{1-\widehat{q}}{\widehat{q}}\right)^{\widehat{N}(\widehat{H}_t)-1}}.$$

As $N(H_{t+1}) = N(H_t)$ and $\widehat{N}(\widehat{H}_{t+1}) = \widehat{N}(\widehat{H}_t) - 1$, (20) holds for $t+1$. ■

Proof of Proposition 11. By the definition of recommendation policy r, the platform's proprietary history can be calculated recursively as

$$\begin{aligned}
&\mathbb{P}\left[H_{t+1}, \widehat{H}_{t+1} \,\middle|\, H_t, \widehat{H}_t, r\right] \\
&= \mathbb{P}\left[H_{t+1}, \widehat{H}_{t+1}, V=1 \,\middle|\, H_t, \widehat{H}_t, r\right] + \mathbb{P}\left[H_{t+1}, \widehat{H}_{t+1}, V=0 \,\middle|\, H_t, \widehat{H}_t, r\right] \\
&= \mathbb{P}\left[a_t, \widehat{S}_t \,\middle|\, m_t, V=1\right]\mathbb{P}\left[V=1 \,\middle|\, H_t, \widehat{H}_t, r\right] + \mathbb{P}\left[a_t, \widehat{S}_t \,\middle|\, m_t, V=0\right]\mathbb{P}\left[V=0 \,\middle|\, H_t, \widehat{H}_t, r\right] \\
&= \mathbb{P}\left[\widehat{S}_t \,\middle|\, a_t, V=1\right]\mathbb{P}\left[a_t \mid m_t, V=1\right]\mathbb{P}\left[V=1 \,\middle|\, H_t, \widehat{H}_t, r\right] + \mathbb{P}\left[\widehat{S}_t \,\middle|\, a_t, V=0\right]\mathbb{P}\left[a_t \mid m_t, V=0\right]\mathbb{P}\left[V=0 \,\middle|\, H_t, \widehat{H}_t, r\right],
\end{aligned}$$

and it evolves according to

$$= \begin{cases}
1-\beta, & \text{if } H_{t+1} = H_t \cup (1,1)\,, \widehat{H}_{t+1} = \widehat{H}_t \cup (0), \\
1, & \text{if } H_{t+1} = H_t \cup (-1,-1)\,, \widehat{H}_{t+1} = \widehat{H}_t \cup (0), \\
(1-\beta)\left\{q\mathbb{P}\left[V=1 \,\middle|\, H_t, \widehat{H}_t, r\right] + (1-q)\mathbb{P}\left[V=0 \,\middle|\, H_t, \widehat{H}_t, r\right]\right\}, & \text{if } H_{t+1} = H_t \cup (0,1)\,, \widehat{H}_{t+1} = \widehat{H}_t \cup (0), \\
(1-q)\mathbb{P}\left[V=1 \,\middle|\, H_t, \widehat{H}_t, r\right] + q\mathbb{P}\left[V=0 \,\middle|\, H_t, \widehat{H}_t, r\right], & \text{if } H_{t+1} = H_t \cup (0,-1)\,, \widehat{H}_{t+1} = \widehat{H}_t \cup (0), \\
\beta\left\{\widehat{q}\mathbb{P}\left[V=1 \,\middle|\, H_t, \widehat{H}_t, r\right] + (1-\widehat{q})\mathbb{P}\left[V=0 \,\middle|\, H_t, \widehat{H}_t, r\right]\right\}, & \text{if } H_{t+1} = H_t \cup (1,1)\,, \widehat{H}_{t+1} = \widehat{H}_t \cup (1), \\
\beta\left\{(1-\widehat{q})\mathbb{P}\left[V=1 \,\middle|\, H_t, \widehat{H}_t, r\right] + \widehat{q}\mathbb{P}\left[V=0 \,\middle|\, H_t, \widehat{H}_t, r\right]\right\}, & \text{if } H_{t+1} = H_t \cup (1,1)\,, \widehat{H}_{t+1} = \widehat{H}_t \cup (-1), \\
\beta\left\{\widehat{q}q\mathbb{P}\left[V=1 \,\middle|\, H_t, \widehat{H}_t, r\right] + (1-\widehat{q})(1-q)\mathbb{P}\left[V=0 \,\middle|\, H_t, \widehat{H}_t, r\right]\right\}, & \text{if } H_{t+1} = H_t \cup (0,1)\,, \widehat{H}_{t+1} = \widehat{H}_t \cup (1), \\
\beta\left\{(1-\widehat{q})q\mathbb{P}\left[V=1 \,\middle|\, H_t, \widehat{H}_t, r\right] + \widehat{q}(1-q)\mathbb{P}\left[V=0 \,\middle|\, H_t, \widehat{H}_t, r\right]\right\}, & \text{if } H_{t+1} = H_t \cup (0,1)\,, \widehat{H}_{t+1} = \widehat{H}_t \cup (-1).
\end{cases}$$

By the characterization of $\mathbb{E}\left[V \,\middle|\, H_t, \widehat{H}_t, r\right]$ (equivalently, $\mathbb{P}[V=1 \mid H_t, r]$ as $V \in \{0,1\}$ is a binary random variable) in (20) and (22), this evolution can be represented as

$$\mathbb{P}\left[N(H_{t+1}), \widehat{N}(\widehat{H}_{t+1}) \,\middle|\, N(H_t)=k, \widehat{N}(\widehat{H}_t)=l\right] \tag{C17}$$
$$= \begin{cases} 1-\beta, & \text{if } m_t=1, N(H_{t+1})=k, \widehat{N}(\widehat{H}_{t+1})=l, \\ 1, & \text{if } m_t=-1, N(H_{t+1})=k, \widehat{N}(\widehat{H}_{t+1})=l, \\ (1-\beta)\{qv_{k,l}+(1-q)[1-v_{k,l}]\}=(1-\beta)u_{k,l}, & \text{if } m_t=0, N(H_{t+1})=k+1, \widehat{N}(\widehat{H}_{t+1})=l, \\ (1-q)v_{k,l}+q[1-v_{k,l}]=1-u_{k,l}, & \text{if } m_t=0, N(H_{t+1})=k-1, \widehat{N}(\widehat{H}_{t+1})=l, \\ \beta\{\widehat{q}v_{k,l}+(1-\widehat{q})[1-v_{k,l}]\}=\beta\widehat{u}_{k,l}, & \text{if } m_t=1, N(H_{t+1})=k, \widehat{N}(\widehat{H}_{t+1})=l+1, \\ \beta\{(1-\widehat{q})v_{k,l}+\widehat{q}[1-v_{k,l}]\}=\beta(1-\widehat{u}_{k,l}), & \text{if } m_t=1, N(H_{t+1})=k, \widehat{N}(\widehat{H}_{t+1})=l-1, \\ \beta\{\widehat{q}qv_{k,l}+(1-\widehat{q})(1-q)[1-v_{k,l}]\}=\beta\widetilde{u}_{k,l}, & \text{if } m_t=0, N(H_{t+1})=k+1, \widehat{N}(\widehat{H}_{t+1})=l+1, \\ \beta\{(1-\widehat{q})qv_{k,l}+\widehat{q}(1-q)[1-v_{k,l}]\}=\beta(1-\widetilde{u}_{k,l}), & \text{if } m_t=0, N(H_{t+1})=k+1, \widehat{N}(\widehat{H}_{t+1})=l-1, \end{cases}$$

for any k and l. For any given t, total probability rule implies that

$$\begin{aligned} &\mathbb{P}\left[N(H_{t+1})=n, \widehat{N}(\widehat{H}_{t+1})=\widehat{n} \,\middle|\, r\right] \\ &= \sum_{k=n-1}^{n+1}\sum_{l=\widehat{n}-1}^{\widehat{n}+1} \mathbb{P}\left[N(H_{t+1})=n, \widehat{N}(\widehat{H}_{t+1})=\widehat{n} \,\middle|\, N(H_t)=k, \widehat{N}(\widehat{H}_t)=l, r\right] \mathbb{P}\left[N(H_t)=k, \widehat{N}(\widehat{H}_t)=l \,\middle|\, r\right] \\ &= \sum_{k=n-1}^{n+1}\sum_{l=\widehat{n}-1}^{\widehat{n}+1}\sum_{i=-1}^{1} \mathbb{P}\left[N(H_{t+1})=n, \widehat{N}(\widehat{H}_{t+1})=\widehat{n} \,\middle|\, N(H_t)=k, \widehat{N}(\widehat{H}_t)=l, m_t=i\right] \mathbb{P}\left[m_t=i \,\middle|\, N(H_t)=k, \widehat{N}(\widehat{H}_t)=l, r\right] z_t(k,l) \\ &= \sum_{k=n-1}^{n+1}\sum_{l=\widehat{n}-1}^{\widehat{n}+1}\sum_{i=-1}^{1} \mathbb{P}\left[N(H_{t+1})=n, \widehat{N}(\widehat{H}_{t+1})=\widehat{n} \,\middle|\, N(H_t)=k, \widehat{N}(\widehat{H}_t)=l, m_t=i\right] r_t^i(k,l) z_t(k,l), \end{aligned}$$

which, by (C17), yields ($\widehat{\mathrm{N}}$). ■

Appendix D: Full-Disclosure and No-Disclosure Policies

Two naïve policies widely used in practice are the no-disclosure (e.g., Amazon) and full-disclosure (e.g., Woot) policies, as defined in Section 3. They represent two polar modes of information provision. In this appendix, we first characterize the platform's revenues under the two benchmark policies, and then quantify how much additional revenue the platform earns under the optimal NA-sequencing policy relative to them.

Under the no-disclosure policy, the platform withholds any information about its proprietary history from upcoming customers, who thus make their purchase decisions purely based on the prior expectation $v_0 \in [v^*, v^{**})$. By Proposition 1, therefore, all customers simply follow their private signals to make their purchase decisions, resulting in an expected revenue equal to that under an NA-sequencing policy that assigns all customers as a neutral customer. Thus, the platform's expected revenue under the no-disclosure policy is $\pi^{\mathrm{ND}} := pu_0T$ (see Proposition 7).

In contrast, the full-disclosure policy equalizes the information between the platform and customers, a setting explored by Bikhchandani et al. (1992). In this case, each customer observes the platform's net purchase position and bases her purchase decision on the belief identified in Proposition 3. Once a customer's posterior expectation exceeds v^{**} (resp., falls below v^*), *positive* (resp., *negative*) *information cascade* occurs: all the subsequent customers will make the purchase (resp., not make the purchase) irrespective of their private signals, and hence their purchase decisions will generate no new information about the product value.

Prior to the information cascade, customers make their purchase decision according to their private signals. The following proposition fully characterizes the platform's expected revenue π^{FD} under the full-disclosure policy.

PROPOSITION D1 **(Full-disclosure policy)**. *The platform's expected revenue under the full-disclosure policy is given by*

$$\pi^{FD} = p \begin{cases} \dfrac{u_0(Tu_1+1-u_0)-[u_0(1-u_1)]^{\frac{T}{2}}\left[T+Tu_0^2(u_1-1)+2u_0u_1(u_0-1)\right]}{1-u_0(1-u_1)} \\ \quad + \dfrac{2u_0(1-u_0)\left[(1-u_1)\left[u_0-[u_0(1-u_1)]^{\frac{T+2}{2}}\right]-u_1\right]}{[1-u_0(1-u_1)]^2}, & \text{for } v_0 \in [v^*, p) \text{ and even } T, \\ \dfrac{u_0(T+u_0-1)+u_0[(1-u_0)u_{-1}]^{\frac{T}{2}}[1-u_0(T+1)-(T+2)(1-u_0)(1-u_{-1})]}{1-(1-u_0)u_{-1}} \\ \quad + \dfrac{2u_0(1-u_0)\left[u_{-1}\left[[(1-u_0)u_{-1}]^{\frac{T+2}{2}}-1+u_0\right]+1-u_{-1}\right]}{[1-(1-u_0)u_{-1}]^2}, & \text{for } v_0 \in [p, v^{**}) \text{ and even } T, \\ \dfrac{u_0(Tu_1+1-u_0)-[u_0(1-u_1)]^{\frac{T+1}{2}}\left[(T+2)u_0(1-u_0)+[u_0(T+1)-1]\frac{u_1}{1-u_1}\right]}{1-u_0(1-u_1)} \\ \quad + \dfrac{2u_0(1-u_0)\left[1-[u_0(1-u_1)]^{\frac{T+1}{2}}\right][u_0(1-u_1)-u_1]}{[1-u_0(1-u_1)]^2}, & \text{for } v_0 \in [v^*, p) \text{ and odd } T, \\ \dfrac{u_0(T+u_0-1)+u_0[(1-u_0)u_{-1}]^{\frac{T+1}{2}}\left[1-u_0(T+2)-(T+1)\frac{1-u_{-1}}{u_{-1}}\right]}{1-(1-u_0)u_{-1}} \\ \quad + \dfrac{2u_0(1-u_0)\left[1-[(1-u_0)u_{-1}]^{\frac{T+1}{2}}\right][1-u_{-1}-(1-u_0)u_{-1}]}{[1-(1-u_0)u_{-1}]^2}, & \text{for } v_0 \in [p, v^{**}) \text{ and odd } T. \end{cases} \tag{D1}$$

In particular, π^{FD} can have discontinuity in v_0 or p only at $v_0 = p$.

Notably π^{FD}, as a function of two main product characteristics, prior expectation v_0 and price p, is discontinuous at $v_0 = p$, as the occurrence of information cascade follows different patterns for $v_0 \geq p$ versus for $v_0 < p$.[24] The former (resp., latter) case represents a product with a promising prospect and a reasonable price (resp., a less promising and pricier product) that a customer would purchase (resp., not purchase) purely based on the publicly available information about the product without referring to her private signal nor the platform's message.

To demonstrate the value of information design, we now benchmark the revenue under the optimal NA-sequencing policy, π^{NA}, characterized in Equation (17) against those of the no-disclosure and full-disclosure policies.The measurements we adopt are the percentage increase of π^{NA} relative to the revenue under the benchmark policies, namely $(\pi^{\mathrm{NA}} - \pi^{\mathrm{ND}})/\pi^{\mathrm{ND}}$ and $(\pi^{\mathrm{NA}} - \pi^{\mathrm{FD}})/\pi^{\mathrm{FD}}$. In Figure D1, we compute and plot these two ratios by varying the price p and the prior expectation v_0 respectively while fixing all the other parameters.

As can be seen from Figure D1, the optimal NA-sequencing policy always outperforms the two naïve policies and can increase the revenue under those two policies significantly (from 20% to 80% for most of the parametric instances). Specifically, relative to no disclosure, the optimal NA-sequencing policy manifests its value in information *provision* and this value becomes higher for products with lower prices or more promising prospects (i.e., $(\pi^{\mathrm{NA}} - \pi^{\mathrm{ND}})/\pi^{\mathrm{ND}}$ is decreasing in p and increasing in v_0, as shown by Figures D1(a) and D1(b), respectively). Relative to full disclosure, the optimal NA-sequencing policy manifests its value

[24] As shown by (D7) and (D8), for $v_0 \geq p$, the positive (resp., negative) cascade *can* occur after odd (resp., even) number of customers starting from the second (resp., third) customer, whereas for $v_0 < p$ the positive (resp., negative) cascade *can* occur after even (resp., odd) number of customers starting from the third (resp., second) customer.

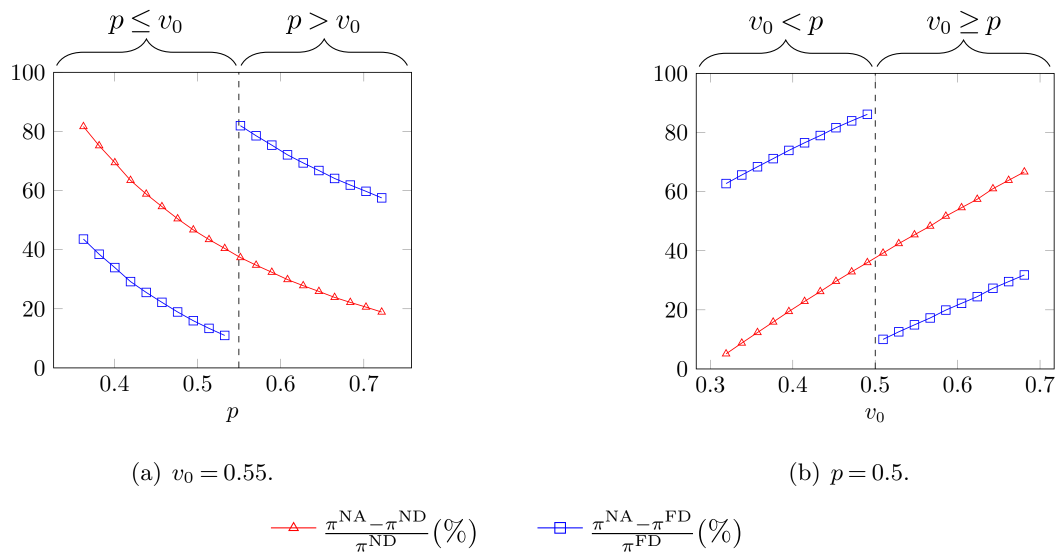


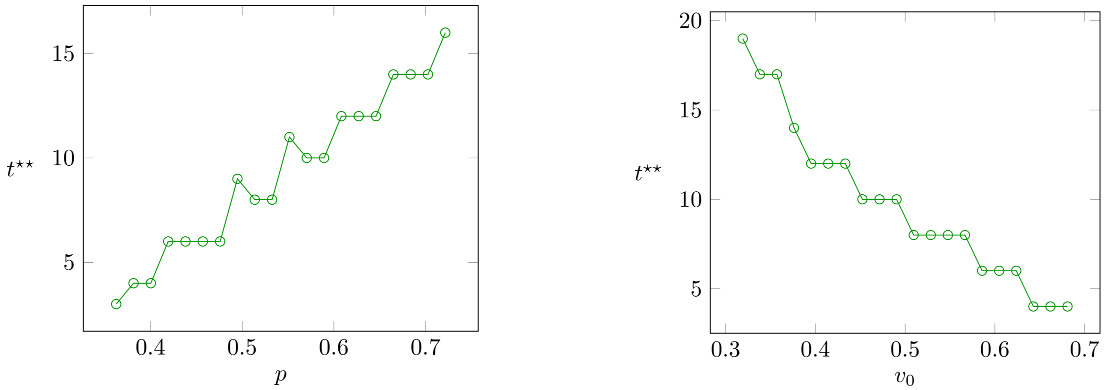
Figure D1 Relative revenue performance of optimal NA-sequencing policy π^{NA} against no-disclosure policy π^{ND} and full-disclosure policy π^{FD} (for $q = .7$ and $T = 100$), plotted for the range of $v_0 \in [v^*, v^{**})$.

in information *obfuscation* and this value $((\pi^{\mathrm{NA}} - \pi^{\mathrm{FD}})/\pi^{\mathrm{FD}})$ demonstrates a discontinuity in the product's price p or prior prospect v_0 at $p = v_0$, as can be seen from Figures D1(a) and D1(b), respectively. Such a discontinuity emerges from the discontinuity of π^{FD} as pointed out above. Thus, Figure D1 suggests that the optimal NA-sequencing policy can bring a higher revenue improvement over the full-disclosure policy for $p > v_0$ than for $p \leq v_0$ (i.e., $(\pi^{\mathrm{NA}} - \pi^{\mathrm{FD}})/\pi^{\mathrm{FD}}$ is higher for $p > v_0$ than for $p \leq v_0$).

Furthermore, by comparing the two benchmark policies, Figure D1 reveals that the full-disclosure policy generates a higher revenue than the no-disclosure policy for $p \leq v_0$, as the optimal NA-sequencing policy yields a lower revenue improvement over the full-disclosure policy than over the no-disclosure policy (i.e., $(\pi^{\mathrm{NA}} - \pi^{\mathrm{ND}})/\pi^{\mathrm{ND}} > (\pi^{\mathrm{NA}} - \pi^{\mathrm{FD}})/\pi^{\mathrm{FD}}$ implies that $\pi^{\mathrm{FD}} > \pi^{\mathrm{ND}}$). It is the other way around for $p > v_0$. In other words, if we regard the set of all information policies as a continuum spectrum ranging from the no-disclosure to full-disclosure policies, then the optimal NA-sequencing policy seems to be closer to the full-disclosure policy (resp., the no-disclosure policy) for products with more (resp., less) promising prospects and/or lower prices (resp., higher prices).

Indeed, as indicated by Figure D2, the number of neutral customers $t^{\star\star}$ assigned by the optimal NA-sequencing policy is in general increasing in p and decreasing in v_0:[25] by assigning a larger number of neutral customers, the platform discloses less of its proprietary information and hence makes the NA-sequencing policy closer to the no-disclosure policy. In essence, the optimal NA-sequencing policy acts to optimize the occurrence of information cascade to $t^{\star\star}$ (rather than the first time that the customer's posterior expectation falls outside of $[v^*, v^{**})$ under the full-disclosure policy) and maximizes the probability of a positive information cascade to $F(t^{\star\star})$. The practical implication of our observations above is evident. For products with less promising prospect (e.g., with mediocre existing ratings) or pricier products, platforms should withhold the

[25] Local non-monotonicity in $t^{\star\star}$ is mainly caused by the discreteness in $t^{\star\star}$.



(a) $v_0 = 0.55$. (b) $p = 0.5$.

Figure D2 Optimal number of neutral customers $t^{\star\star}$, for $T = 100$ and $q = .7$.

sales history for longer time and accrue more convincing evidence (by letting customers make their purchase decisions based on their own pre-purchase signals) before making an affirmative recommendation. However, platforms do not need to wait too long to provide affirmative recommendations for highly rated or heavily discounted products.

Proof of Proposition D1. We first establish the following two claims D1 and D2.

CLAIM D1. *The full-disclosure policy induces the same purchase decisions from all customers and the same expected revenue for the platform as the following recommendation policy:*

$$(r_t^1(n), r_t^0(n), r_t^{-1}(n)) = \begin{cases} (1,0,0), & \textit{if } n \geq n^+, \\ (0,1,0), & \textit{if } n \in (n^-, n^+), \\ (0,0,1), & \textit{if } n \leq n^-, \end{cases} \qquad \textit{for all } t \in \{1,2,...,T\}, \tag{D2}$$

where $n^+ := \min_{n\in\mathbb{Z}} \{n : v_n \geq v^{**}\}$ *and* $n^- := \max_{n\in\mathbb{Z}} \{n : v_n < v^*\}$. *Let* $N(H_t)$ *be the net purchase position up to time* t *under the recommendation policy in* (D2), *and denote* $\tau^+ := \min\{t = 1, \cdots, T : N(H_{t+1}) \geq n^+\}$ *and* $\tau^- := \min\{t = 1, \cdots, T : N(H_{t+1}) \leq n^-\}$. *Then, the platform's expected revenue under the full-disclosure policy is given by*

$$\pi^{FD} := p \sum_{t=1}^{T} \left\{ u_0 t \left(\mathbb{P}[\tau^+ = t] + \mathbb{P}[\tau^- = t] \right) + (T-t)\mathbb{P}[\tau^+ = t] \right\}. \tag{D3}$$

Proof of Claim D1. Let $\left(\tilde{\sigma}, \widetilde{\mathcal{H}}\right)$ denote the full-disclosure policy, whereby the message space $\widetilde{\mathcal{H}}$ consists of the platform's proprietary history $\widetilde{H}_t$. Define mapping $\varphi : \widetilde{\mathcal{H}} \to \mathcal{M} = \{1, 0, -1\}$ as

$$\varphi(\widetilde{H}_t) = \begin{cases} 1, & \text{if } \mathbb{E}[V|\widetilde{H}_t, \tilde{\sigma}] \geq v^{**}, \\ 0, & \text{if } \mathbb{E}[V|\widetilde{H}_t, \tilde{\sigma}] \in [v^*, v^{**}), \\ -1, & \text{if } \mathbb{E}[V|\widetilde{H}_t, \tilde{\sigma}] < v^*, \end{cases} \tag{D4}$$

and denote $\gamma(\widetilde{H}_t) := \{(\varphi(\widetilde{H}_s), a_s) : s < t\}$. Further define the recommendation policy $(\sigma, \mathcal{M})$ to be

$$\sigma(m \mid H_t) = \frac{\sum_{\gamma(\widetilde{H}_t)=H_t, \varphi(\widetilde{H}_t)=m} \tilde{\sigma}\left(\widetilde{H}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]}{\sum_{\gamma(\widetilde{H}_t)=H_t} \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]} = \frac{\sum_{\gamma(\widetilde{H}_t)=H_t, \varphi(\widetilde{H}_t)=m} \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]}{\sum_{\gamma(\widetilde{H}_t)=H_t} \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]}, \tag{D5}$$

for any $m \in \mathcal{M}$ and $H_t = \{(m_s, a_s) : m_s \in \mathcal{M}, a_s \in \{-1,1\}, s < t\}$. Then, (D4) immediately implies

$$(\sigma(m=1 \mid H_t), \sigma(m=0 \mid H_t), \sigma(m=-1 \mid H_t)) = \begin{cases} (1,0,0), \text{ if } \mathbb{E}[V|\gamma(\widetilde{H}_t) = H_t, \tilde{\sigma}] \geq v^{**}, \\ (0,1,0), \text{ if } \mathbb{E}[V|\gamma(\widetilde{H}_t) = H_t, \tilde{\sigma}] \in [v^*, v^{**}), \\ (0,0,1), \text{ if } \mathbb{E}[V|\gamma(\widetilde{H}_t) = H_t, \tilde{\sigma}] < v^*. \end{cases}$$

Following similar argument as in the proof of Proposition 2, we have full-disclosure policy $\tilde{\sigma}$ and σ induce the same purchase decisions from all customers and the same expected revenue for the platform. By (A8) and (A9), the above equation can also be rewritten as

$$(\sigma(m=1 \mid H_t), \sigma(m=0 \mid H_t), \sigma(m=-1 \mid H_t)) = \begin{cases} (1,0,0), \text{ if } \mathbb{E}[V|H_t, \sigma] \geq v^{**}, \\ (0,1,0), \text{ if } \mathbb{E}[V|H_t, \sigma] \in [v^*, v^{**}), \\ (0,0,1), \text{ if } \mathbb{E}[V|H_t, \sigma] < v^*, \end{cases}$$

which, by utilizing the definitions of n^-, n^+ and Proposition 3, can be expressed as

$$(\sigma(m=1 \mid H_t), \sigma(m=0 \mid H_t), \sigma(m=-1 \mid H_t)) = \begin{cases} (1,0,0), \text{ if } N(H_t) \geq n^+, \\ (0,1,0), \text{ if } N(H_t) \in (n^-, n^+), \\ (0,0,1), \text{ if } N(H_t) \leq n^-, \end{cases}$$

and hence, establishing the equivalence between recommendation policies σ and r given in (D2).

By (4), customer t's purchase decision under the recommendation policy r given in (D2) follows

$$a_t = \begin{cases} 1, & \text{if } N(H_t) \geq n^+, \\ S_t, & \text{if } N(H_t) \in (n^-, n^+), \\ -1, & \text{if } N(H_t) \leq n^-, \end{cases} \tag{D6}$$

Thus, the net purchase position $N(H_t) \geq n^+$ (resp., $N(H_t) \leq n^-$) remains unchanged and generates a sale with probability 1 (resp., 0) after $t > \tau^+$ (resp., $t > \tau^-$). Then, utilizing (3), we can denote the expected revenue of a recommendation policy r given in (D2) as follows

$$\begin{aligned} & p\sum_{t=1}^{T} \mathbb{P}\left[\tau^+ \wedge \tau^- = t\right] \left\{\mathbb{P}[a_t = 1 \mid m_t = 0, r]t + \mathbb{1}[\tau^+ = t](T-t)\right\} \\ &= p\sum_{t=1}^{T} \left\{\left(\mathbb{P}[\tau^+ = t] + \mathbb{P}[\tau^- = t]\right) t \sum_n \underbrace{\mathbb{P}[S_t = 1|N(H_t) = n]}_{=qv_n + (1-q)(1-v_n)} \underbrace{\mathbb{P}[N(H_t) = n|r]}_{z_t(n)} + \mathbb{P}[\tau^+ = t](T-t)\right\} \\ &= p\sum_{t=1}^{T} \left\{\left(\mathbb{P}[\tau^+ = t] + \mathbb{P}[\tau^- = t]\right) t[q \underbrace{\sum_n v_n z_t(n)}_{=v_0 \text{ by (A28)}} + (1-q) \underbrace{\sum_n v_n - v_n z_t(n))}_{=1-v_0 \text{ by (A28) and (B8)}}] + \mathbb{P}[\tau^+ = t](T-t)\right\}. \end{aligned}$$

Thus, (D3) follows from $u_0 = qv_0 + (1-q)(1-v_0)$ definition given in Proposition 4. □

CLAIM D2. *The probabilities of positive and negative cascade occurring right after time $t \geq 1$ are given by*

$$\mathbb{P}\left[\tau^+ = t\right] = \begin{cases} [(1-u_0)(u_{-1})]^{\frac{t-1}{2}} u_0 \mathbb{1}[t \equiv 1(mod 2)], \ \textit{for } v_0 \in [p, v^{**}), \\ (u_0)^{\frac{t}{2}}(1-u_1)^{\frac{t}{2}-1} u_1 \mathbb{1}[t \equiv 0(mod 2)], \ \textit{for } v_0 \in [v^*, p), \end{cases} \quad \textit{and} \tag{D7}$$

$$\mathbb{P}\left[\tau^- = t\right] = \begin{cases} (1-u_0)^{\frac{t}{2}}(u_{-1})^{\frac{t}{2}-1}(1-u_{-1}) \mathbb{1}[t \equiv 0(mod 2)], \ \textit{for } v_0 \in [p, v^{**}), \\ [(u_0)(1-u_1)]^{\frac{t-1}{2}}(1-u_0)\mathbb{1}[t \equiv 1(mod 2)], \ \textit{for } v_0 \in [v^*, p), \end{cases} \quad \textit{respectively.} \tag{D8}$$

Proof of Claim D2. We now derive (D7) and (D8) for the case of $v_0 \in [p, v^{**})$. The case of $v_0 \in [v^*, p)$ can be derived in a similar fashion. For $v_0 \in [p, v^{**})$, in order for the net purchase position $N(H_t) = n$ to stay within (n^-, n^+) (i.e. $n \in (n^-, n^+)$), the first customer must not make the purchase (must receive a pessimistic signal, which occurs with probability $(1-u_0)$ as $N(H_1)=0$) and the net purchase position gets updated to $N(H_2)=-1$; the second customer must make the purchase (must receive an optimistic signal, which occurs with probability u_{-1}) and the net purchase position gets updated to $N(H_3)=0$; the third customer must not make the purchase (must receive a pessimistic signal, which occurs with probability $(1-u_0)$) and the net purchase position gets updated to $N(H_4)=-1$. By induction, the net purchase position must alternate between 0 and -1 in order for the net purchase position $N(H_t)=n$ to stay within (n^-,n^+), i.e. $N(H_t)=0$ for odd t and $N(H_t)=-1$ for even t. Therefore, in order for τ^+ to be equal to t, a customer whose order of arrival is an odd number must make the purchase - must receive an optimistic signal (which occurs with probability u_0 as $N(H_t)$ would be 0). Similarly, in order for τ^- to be equal to t, a customer whose order of arrival is an even number must not make the purchase - must receive a pessimistic signal (which occurs with probability $1-u_{-1}$ as $N(H_t)$ would be -1). So $\mathbb{P}[\tau^+=t]$ and $\mathbb{P}[\tau^-=t]$ follows as in (D7) and (D8). $\square$

We now establish (D1). We only demonstrate the case of $v_0\in[p,v^{**})$ and odd T; all other cases follow similar argument. Re-writing (D3) we have

$$\pi^{\mathrm{FD}}=p\left[\sum_{t=1}^{T}[T+t(u_0-1)]\mathbb{P}[\tau^+=t]+\sum_{t=1}^{T}u_0t\mathbb{P}[\tau^-=t]\right]. \tag{D9}$$

Utilizing (D7) we can represent the first term of (D9) as

$$=[T+(u_0-1)]u_0+[T+3(u_0-1)]u_0(1-u_0)(u_{-1})+\cdots+[T+(T-1)(u_0-1)]u_0(1-u_0)^{\frac{T-1}{2}}(u_{-1})^{\frac{T-1}{2}}.$$

Next, we obtain the following representation by multiplying the expression with $1-(1-u_0)u_{-1}$

$$\begin{aligned}
=&[T+(u_0-1)]u_0+[T+3(u_0-1)]u_0(1-u_0)(u_{-1})+\cdots+[T+(T-1)(u_0-1)]u_0(1-u_0)^{\frac{T-1}{2}}(u_{-1})^{\frac{T-1}{2}}\\
&-\left[[T+(u_0-1)]u_0(1-u_0)(u_{-1})+[T+3(u_0-1)]u_0(1-u_0)^2(u_{-1})^2+\cdots+[T+(T-1)(u_0-1)]u_0(1-u_0)^{\frac{T+1}{2}}(u_{-1})^{\frac{T+1}{2}}\right]\\
=&[T+(u_0-1)]u_0+2(u_0-1)u_0\left[(1-u_0)(u_{-1})+(1-u_0)^2(u_{-1})^2+\cdots+(1-u_0)^{\frac{T-1}{2}}(u_{-1})^{\frac{T-1}{2}}\right]\\
&-[T+(T-1)(u_0-1)]u_0(1-u_0)^{\frac{T+1}{2}}(u_{-1})^{\frac{T+1}{2}}\\
=&[T+(u_0-1)]u_0+2(u_0-1)u_0\left[(1-u_0)(u_{-1})+(1-u_0)^2(u_{-1})^2+\cdots+(1-u_0)^{\frac{T-1}{2}}(u_{-1})^{\frac{T-1}{2}}+(1-u_0)^{\frac{T+1}{2}}(u_{-1})^{\frac{T+1}{2}}\right]\\
&-[T+(T+1)(u_0-1)]u_0(1-u_0)^{\frac{T+1}{2}}(u_{-1})^{\frac{T+1}{2}}\\
=&[T+(u_0-1)]u_0+2(u_0-1)u_0(1-u_0)(u_{-1})\left[1+(1-u_0)(u_{-1})+\cdots+(1-u_0)^{\frac{T-1}{2}}(u_{-1})^{\frac{T-1}{2}}\right]\\
&-[T+(T+1)(u_0-1)]u_0(1-u_0)^{\frac{T+1}{2}}(u_{-1})^{\frac{T+1}{2}}\\
=&[T+(u_0-1)]u_0+\frac{2(u_0-1)u_0\left[1-[(1-u_0)(u_{-1})]^{\frac{T+1}{2}}\right]}{1-(1-u_0)u_{-1}}-[T+(T+1)(u_0-1)]u_0(1-u_0)^{\frac{T+1}{2}}(u_{-1})^{\frac{T+1}{2}},
\end{aligned}$$

where the last equality follows from the sum of geometric series formula. Then, we divide the expression back again by $1-(1-u_0)u_{-1}$ to get

$$=\frac{[T+(u_0-1)]u_0-[T+(T+1)(u_0-1)]u_0(1-u_0)^{\frac{T+1}{2}}(u_{-1})^{\frac{T+1}{2}}}{1-(1-u_0)u_{-1}}+\frac{2(u_0-1)u_0\left[1-[(1-u_0)(u_{-1})]^{\frac{T+1}{2}}\right]}{[1-(1-u_0)u_{-1}]^2}.$$

By similar analysis, second term of (D9) can be represented as

$$=-\frac{(T+1)u_0(1-u_{-1})(u_{-1})^{-1}\left[(1-u_0)u_{-1}\right]^{\frac{T+1}{2}}}{1-(1-u_0)u_{-1}}+\frac{2u_0(1-u_{-1})(1-u_0)\left[1-[(1-u_0)(u_{-1})]^{\frac{T+1}{2}}\right]}{[1-(1-u_0)u_{-1}]^2}.$$

Combining the above-derived representations of first and second term of (D9) and a few simple steps of algebra, we reach at (D1). ■

Appendix E: Additional Figures and Detailed Numerical Results

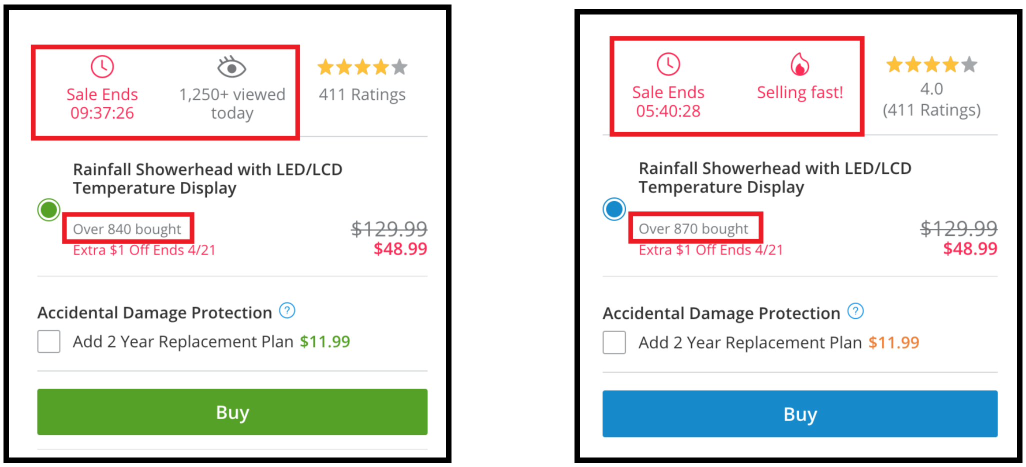
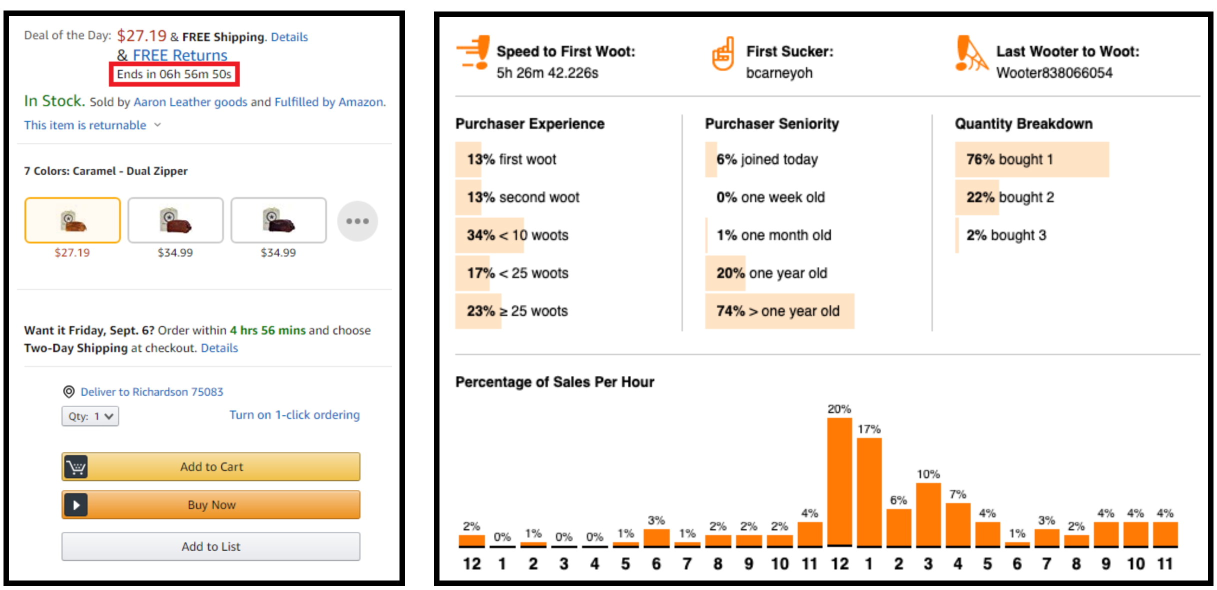


Figure E1 Key information provided for a time-locked sales campaign on Groupon.com.



(a) Amazon.com (b) Woot.com

Figure E2 Key information provided for a time-locked sales campaign on Amazon.com and Woot.com.

Table E1 $t^{\star\star}/T$ for $T = 100, p = 0.5, v_0 \in \{0.025, 0.05, \cdots, 0.975\}$ and $q \in \{0.525, 0.55, \cdots, 0.975\}$.

$p = 0.5$	q																			
v_0	0.525	0.55	0.575	0.6	0.625	0.65	0.675	0.7	0.725	0.75	0.775	0.8	0.825	0.85	0.875	0.9	0.925	0.95	0.975	
0.025																			1.00	
0.05																		1.00	1.00	
0.075																	1.00	1.00	1.00	
0.1																1.00	1.00	1.00	1.00	
0.125															1.00	1.00	1.00	1.00	1.00	
0.15														1.00	1.00	1.00	1.00	1.00	1.00	
0.175													1.00	1.00	1.00	1.00	1.00	1.00	1.00	
0.2				$\{v_0 < v^*\}$								1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	$\{t^{\star\star} = T\}$
0.225											1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	
0.25										1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	
0.275									1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	
0.3								0.22	0.20	0.82	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	
0.325							0.21	0.19	0.17	0.15	0.13	0.10	0.10	0.10	0.08	0.08	0.08	1.00	1.00	
0.35						0.21	0.19	0.17	0.15	0.12	0.12	0.10	0.10	0.08	0.08	0.07	0.06	0.05	0.05	
0.375					0.23	0.19	0.17	0.14	0.12	0.12	0.10	0.10	0.07	0.08	0.07	0.07	0.05	0.05	0.04	
0.4				0.25	0.21	0.19	0.14	0.12	0.12	0.10	0.10	0.09	0.08	0.07	0.07	0.06	0.05	0.05	0.04	
0.425			0.25	0.21	0.19	0.17	0.15	0.12	0.10	0.10	0.10	0.08	0.08	0.07	0.07	0.05	0.05	0.05	0.04	
0.45		0.27	0.21	0.19	0.17	0.15	0.12	0.10	0.10	0.10	0.08	0.08	0.07	0.07	0.05	0.05	0.05	0.04	0.04	
0.475	0.27	0.21	0.17	0.17	0.15	0.13	0.10	0.10	0.10	0.08	0.08	0.08	0.07	0.05	0.05	0.05	0.05	0.04	0.04	
0.5	0.18	0.18	0.15	0.13	0.13	0.13	0.10	0.11	0.08	0.08	0.08	0.07	0.05	0.05	0.05	0.05	0.05	0.04	0.04	
0.525		0.09	0.11	0.11	0.11	0.11	0.08	0.08	0.08	0.08	0.08	0.05	0.05	0.05	0.05	0.05	0.05	0.04	0.04	
0.55			0.07	0.09	0.09	0.09	0.09	0.08	0.08	0.08	0.07	0.05	0.05	0.05	0.05	0.05	0.05	0.04	0.04	
0.575				0.04	0.07	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.05	0.05	0.05	0.04	0.04	0.04	
0.6					0.04	0.07	0.07	0.06	0.06	0.06	0.06	0.06	0.05	0.05	0.05	0.05	0.04	0.04	0.04	$\{t^{\star\star} < T\}$
0.625						0.04	0.04	0.06	0.06	0.06	0.06	0.06	0.05	0.05	0.05	0.05	0.04	0.04	0.04	
0.65							0.04	0.04	0.04	0.06	0.06	0.05	0.05	0.05	0.05	0.05	0.04	0.04	0.04	
0.675								0.04	0.04	0.04	0.04	0.03	0.03	0.05	0.05	0.04	0.04	0.04	0.02	
0.7									0.04	0.04	0.04	0.04	0.03	0.03	0.03	0.03	0.04	0.04	0.02	
0.725										0.03	0.04	0.04	0.04	0.03	0.03	0.03	0.03	0.03	0.03	
0.75											0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	
0.775												0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	
0.8				$\{v_0 \geq v^{**}\}$									0.03	0.03	0.03	0.03	0.03	0.03	0.03	
0.825														0.03	0.03	0.03	0.03	0.03	0.03	
0.85															0.03	0.03	0.03	0.03	0.03	
0.875																0.03	0.03	0.03	0.03	
0.9																	0.03	0.03	0.03	
0.925																		0.02	0.02	
0.95																			0.02	
0.975																				

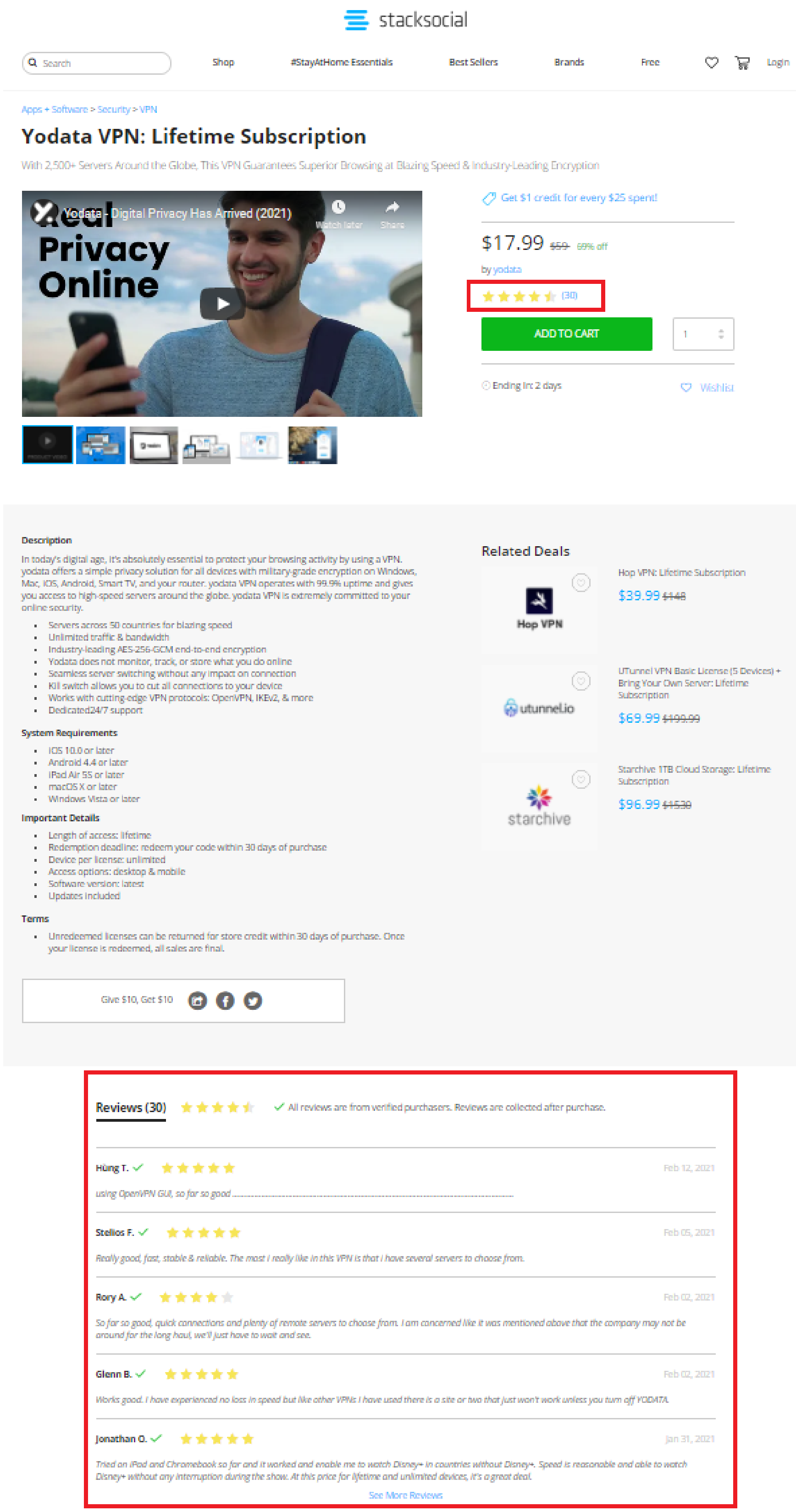


Figure E3 Key information provided for a time-locked sales campaign on stacksocial.com.

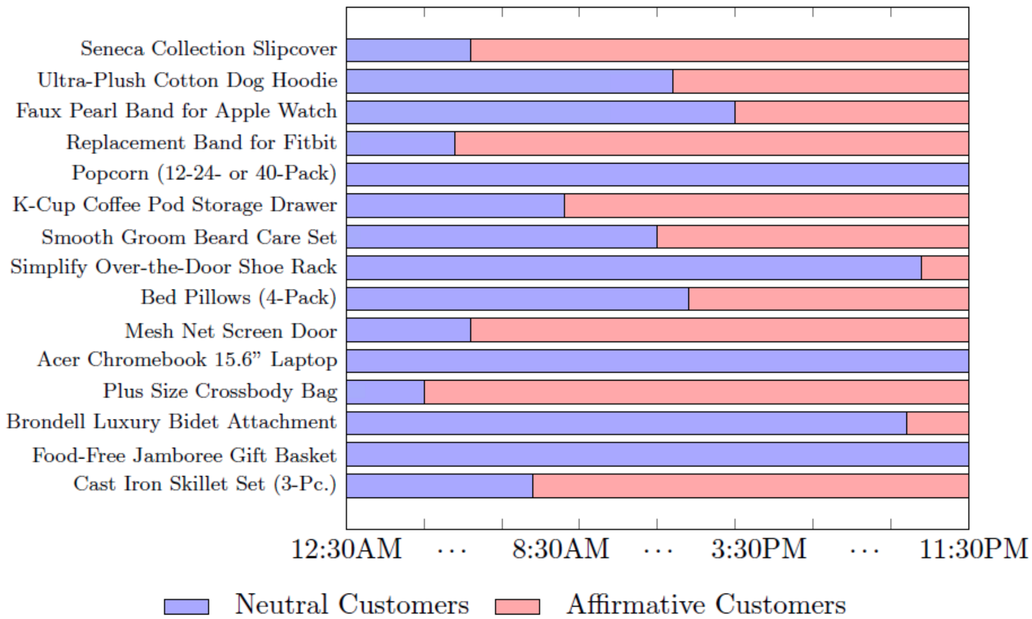


Figure E4 Groupon daily deals on 4/8/19. Data collected in 30 minute intervals for 15 products. Groupon revealed no information or showed up-to-date count of visits to the product webpage during blue time stamps, and then switch to highlight the message "Selling Fast!" during red time stamps.

Table E2 $\pi^{\star}$ data used in Table 1.

$T = 50$

q	v_0	p = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$q = 0.55$	0.05	0	0	0	0	0	0	0	0	0
	0.15	5	0	0	0	0	0	0	0	0
	0.25	5	10	0	0	0	0	0	0	0
	0.35	5	10	15	0	0	0	0	0	0
	0.45	5	10	15	20	16.155	0	0	0	0
	0.55	5	10	15	20	25	0	0	0	0
	0.65	5	10	15	20	25	30	0	0	0
	0.75	5	10	15	20	25	30	35	0	0
	0.85	5	10	15	20	25	30	35	40	0
	0.95	5	10	15	20	25	30	35	40	45
$q = 0.65$	0.05	0	0	0	0	0	0	0	0	0
	0.15	4.366	4.792	0	0	0	0	0	0	0
	0.25	5	7.842	8.297	0	0	0	0	0	0
	0.35	5	10	11.694	12.271	12.732	0	0	0	0
	0.45	5	10	15	15.995	16.653	17.133	0	0	0
	0.55	5	10	15	19.687	20.730	21.425	0	0	0
	0.65	5	10	15	20	25	25.919	26.573	0	0
	0.75	5	10	15	20	25	30	31.590	31.998	0
	0.85	5	10	15	20	25	30	35	37.839	37.304
	0.95	5	10	15	20	25	30	35	40	45
$q = 0.75$	0.05	1.575	0	0	0	0	0	0	0	0
	0.15	3.255	4.319	5.261	0	0	0	0	0	0
	0.25	5	6.330	7.554	8.625	9.375	0	0	0	0
	0.35	5	8.320	9.914	11.275	12.425	13.087	0	0	0
	0.45	5	10	12.234	14.012	15.475	16.613	17.750	0	0
	0.55	5	10	14.566	16.690	18.631	20.138	21.702	0	0
	0.65	5	10	15	19.382	21.712	23.788	25.656	27.545	0
	0.75	5	10	15	20	25	27.348	29.757	31.919	33.732
	0.85	5	10	15	20	25	30	33.766	36.486	38.734
	0.95	5	10	15	20	25	30	35	40	43.994
$q = 0.85$	0.05	1.158	1.910	0	0	0	0	0	0	0
	0.15	2.285	3.386	4.462	5.472	0	0	0	0	0
	0.25	3.408	4.894	6.311	7.702	8.999	10.133	0	0	0
	0.35	4.532	6.384	8.206	9.933	11.624	13.172	14.437	0	0
	0.45	5	7.876	10.069	12.221	14.248	16.212	17.936	19.059	0
	0.55	5	9.370	11.934	14.466	16.943	19.252	21.435	23.127	0
	0.65	5	10	13.802	16.712	19.582	22.375	24.934	27.194	29.565
	0.75	5	10	15	18.963	22.223	25.431	28.530	31.262	34.134
	0.85	5	10	15	20	25	28.491	32.046	35.440	38.707
	0.95	5	10	15	20	25	30	35	39.535	43.419
$q = 0.95$	0.05	0.582	1.040	1.494	1.940	0	0	0	0	0
	0.15	1.281	2.184	3.090	3.989	4.878	5.750	6.585	0	0
	0.25	1.978	3.340	4.686	6.038	7.382	8.709	10.005	11.222	0
	0.35	2.675	4.488	6.299	8.087	9.885	11.669	13.425	15.113	16.526
	0.45	3.372	5.637	7.900	10.158	12.388	14.628	16.845	19.004	20.931
	0.55	4.069	6.786	9.501	12.212	14.918	17.588	20.266	22.896	25.336
	0.65	4.767	7.935	11.102	14.266	17.426	20.579	23.686	26.787	29.740
	0.75	5	9.084	12.703	16.320	19.934	23.543	27.142	30.678	34.145
	0.85	5	10	14.304	18.374	22.443	26.508	30.567	34.610	38.550
	0.95	5	10	15	20	25	29.473	33.993	38.507	43.000

$T = 100$

q	v_0	p = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$q = 0.55$	0.05	0	0	0	0	0	0	0	0	0
	0.15	10	0	0	0	0	0	0	0	0
	0.25	10	20	0	0	0	0	0	0	0
	0.35	10	20	30	0	0	0	0	0	0
	0.45	10	20	30	40	34.721	0	0	0	0
	0.55	10	20	30	40	50	0	0	0	0
	0.65	10	20	30	40	50	60	0	0	0
	0.75	10	20	30	40	50	60	70	0	0
	0.85	10	20	30	40	50	60	70	80	0
	0.95	10	20	30	40	50	60	70	80	90
$q = 0.65$	0.05	0	0	0	0	0	0	0	0	0
	0.15	8.784	9.619	0	0	0	0	0	0	0
	0.25	10	15.765	16.743	0	0	0	0	0	0
	0.35	10	20	23.533	24.873	26.042	0	0	0	0
	0.45	10	20	30	32.238	33.862	35.230	0	0	0
	0.55	10	20	30	39.571	41.833	43.681	0	0	0
	0.65	10	20	30	40	50	52.341	54.315	0	0
	0.75	10	20	30	40	50	60	63.799	65.650	0
	0.85	10	20	30	40	50	60	70	76.324	77.147
	0.95	10	20	30	40	50	60	70	80	90
$q = 0.75$	0.05	3.158	0	0	0	0	0	0	0	0
	0.15	6.547	8.682	10.537	0	0	0	0	0	0
	0.25	10	12.717	15.187	17.297	18.750	0	0	0	0
	0.35	10	16.731	19.905	22.666	24.925	26.189	0	0	0
	0.45	10	20	24.583	28.121	31.100	33.323	35.726	0	0
	0.55	10	20	29.272	33.518	37.381	40.458	43.677	0	0
	0.65	10	20	30	38.929	43.587	47.718	51.641	55.638	0
	0.75	10	20	30	40	50	54.887	59.749	64.382	68.576
	0.85	10	20	30	40	50	60	67.764	73.300	78.295
	0.95	10	20	30	40	50	60	70	80	88.314
$q = 0.85$	0.05	2.326	3.823	0	0	0	0	0	0	0
	0.15	4.593	6.807	8.950	10.959	0	0	0	0	0
	0.25	6.854	9.823	12.683	15.454	18.035	20.281	0	0	0
	0.35	9.118	12.822	16.460	19.950	23.320	26.400	28.899	0	0
	0.45	10	15.821	20.206	24.504	28.605	32.519	35.941	38.137	0
	0.55	10	18.824	23.953	29.014	33.961	38.638	42.983	46.325	0
	0.65	10	20	27.703	33.525	39.260	44.840	50.025	54.514	59.361
	0.75	10	20	30	38.041	44.562	50.974	57.164	62.702	68.527
	0.85	10	20	30	40	50	57.113	64.223	71.000	77.702
	0.95	10	20	30	40	50	60	70	79.216	87.004
$q = 0.95$	0.05	1.168	2.084	2.990	3.882	0	0	0	0	0
	0.15	2.570	4.382	6.193	7.990	9.767	11.507	13.174	0	0
	0.25	3.969	6.692	9.395	12.099	14.784	17.437	20.026	22.453	0
	0.35	5.368	8.994	12.614	16.208	19.802	23.368	26.878	30.247	33.057
	0.45	6.767	11.297	15.822	20.338	24.819	29.298	33.729	38.042	41.881
	0.55	8.166	13.600	19.029	24.451	29.862	35.228	40.581	45.837	50.705
	0.65	9.566	15.903	22.236	28.565	34.884	41.189	47.433	53.632	59.529
	0.75	10	18.206	25.444	32.678	39.906	47.124	54.320	61.427	68.353
	0.85	10	20	28.652	36.792	44.929	53.059	61.177	69.262	77.177
	0.95	10	20	30	40	50	58.995	68.034	77.062	86.047

$T = 500$

q	v_0	p = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$q = 0.55$	0.05	0	0	0	0	0	0	0	0	0
	0.15	50	0	0	0	0	0	0	0	0
	0.25	50	100	0	0	0	0	0	0	0
	0.35	50	100	150	0	0	0	0	0	0
	0.45	50	100	150	200	193.144	0	0	0	0
	0.55	50	100	150	200	250	0	0	0	0
	0.65	50	100	150	200	250	300	0	0	0
	0.75	50	100	150	200	250	300	350	0	0
	0.85	50	100	150	200	250	300	350	400	0
	0.95	50	100	150	200	250	300	350	400	450
$q = 0.65$	0.05	0	0	0	0	0	0	0	0	0
	0.15	44.132	48.233	0	0	0	0	0	0	0
	0.25	50	79.151	84.405	0	0	0	0	0	0
	0.35	50	100	118.287	126.000	133.428	0	0	0	0
	0.45	50	100	150	162.339	172.116	181.460	0	0	0
	0.55	50	100	150	198.646	210.959	222.718	0	0	0
	0.65	50	100	150	200	250	264.180	277.773	0	0
	0.75	50	100	150	200	250	300	322.050	337.107	0
	0.85	50	100	150	200	250	300	350	384.738	400.006
	0.95	50	100	150	200	250	300	350	400	450
$q = 0.75$	0.05	15.825	0	0	0	0	0	0	0	0
	0.15	32.880	43.588	52.752	0	0	0	0	0	0
	0.25	50	63.811	76.259	86.672	0	0	0	0	0
	0.35	50	84.012	99.834	113.791	124.925	131.002	0	0	0
	0.45	50	100	123.368	140.996	156.100	167.011	179.662	0	0
	0.55	50	100	146.914	168.143	187.381	203.020	219.634	0	0
	0.65	50	100	150	195.304	218.587	239.155	259.607	280.827	0
	0.75	50	100	150	200	250	275.200	299.731	324.285	348.162
	0.85	50	100	150	200	250	300	339.760	367.928	395.429
	0.95	50	100	150	200	250	300	350	400	442.955
$q = 0.85$	0.05	11.669	19.124	0	0	0	0	0	0	0
	0.15	23.050	34.174	44.859	54.855	0	0	0	0	0
	0.25	34.426	49.256	63.652	77.472	90.321	101.468	0	0	0
	0.35	45.804	64.321	82.490	100.089	116.891	132.219	144.599	0	0
	0.45	50	79.387	101.297	122.764	143.462	162.970	179.984	190.762	0
	0.55	50	94.455	120.106	145.396	170.103	193.721	215.369	231.915	0
	0.65	50	100	138.918	168.029	196.689	224.556	250.754	273.067	297.901
	0.75	50	100	150	190.666	223.277	255.322	286.235	314.220	343.785
	0.85	50	100	150	200	250	286.094	321.637	355.482	389.682
	0.95	50	100	150	200	250	300	350	396.662	435.706
$q = 0.95$	0.05	5.860	10.436	14.963	19.416	0	0	0	0	0
	0.15	12.878	21.966	31.017	40.001	48.878	57.570	65.889	0	0
	0.25	19.895	33.508	47.070	60.585	74.007	87.264	100.192	112.300	0
	0.35	26.911	45.042	63.140	81.170	99.135	116.958	134.496	151.322	165.301
	0.45	33.928	56.577	79.198	101.776	124.263	146.652	168.800	190.345	209.481
	0.55	40.944	68.112	95.256	122.365	149.418	176.346	203.103	229.367	253.660
	0.65	47.961	79.646	111.314	142.954	174.551	206.071	237.407	268.390	297.840
	0.75	50	91.182	127.372	163.544	199.684	235.770	271.746	307.413	342.020
	0.85	50	100	143.431	184.134	224.818	265.469	306.054	346.476	386.199
	0.95	50	100	150	200	250	295.169	340.363	385.504	430.425

Table E3 π^{NA} data used in Table 1.

$T = 50$

q	v_0	p = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$q = 0.55$	0.05	0	0	0	0	0	0	0	0	0
	0.15	5	0	0	0	0	0	0	0	0
	0.25	5	10	0	0	0	0	0	0	0
	0.35	5	10	15	0	0	0	0	0	0
	0.45	5	10	15	20	15.561	0	0	0	0
	0.55	5	10	15	20	25	0	0	0	0
	0.65	5	10	15	20	25	30	0	0	0
	0.75	5	10	15	20	25	30	35	0	0
	0.85	5	10	15	20	25	30	35	40	0
	0.95	5	10	15	20	25	30	35	40	45
$q = 0.65$	0.05	0	0	0	0	0	0	0	0	0
	0.15	4.139	4.433	0	0	0	0	0	0	0
	0.25	5	7.307	7.779	0	0	0	0	0	0
	0.35	5	10	10.938	11.562	12.536	0	0	0	0
	0.45	5	10	15	14.983	15.743	16.748	0	0	0
	0.55	5	10	15	19.402	19.495	20.331	0	0	0
	0.65	5	10	15	20	25	24.517	25.317	0	0
	0.75	5	10	15	20	25	30	30.169	30.533	0
	0.85	5	10	15	20	25	30	35	36.292	35.885
	0.95	5	10	15	20	25	30	35	40	45
$q = 0.75$	0.05	1.375	0	0	0	0	0	0	0	0
	0.15	2.836	3.441	4.875	0	0	0	0	0	0
	0.25	5	5.557	6.450	7.500	9.375	0	0	0	0
	0.35	5	7.849	8.921	10.111	11.394	12.800	0	0	0
	0.45	5	10	11.552	12.926	14.409	15.967	17.592	0	0
	0.55	5	10	14.309	15.911	17.576	19.341	21.172	0	0
	0.65	5	10	15	19.014	20.935	22.871	24.905	26.904	0
	0.75	5	10	15	20	25	26.648	28.821	31.093	33.240
	0.85	5	10	15	20	25	30	33.145	35.505	37.720
	0.95	5	10	15	20	25	30	35	40	43.246
$q = 0.85$	0.05	0.925	1.850	0	0	0	0	0	0	0
	0.15	1.876	2.557	3.825	5.100	0	0	0	0	0
	0.25	3.131	4.156	5.232	6.500	8.125	9.750	0	0	0
	0.35	4.407	5.823	7.250	8.716	10.226	11.850	13.825	0	0
	0.45	5	7.478	9.282	11.154	13.009	14.915	16.815	18.750	0
	0.55	5	9.182	11.400	13.586	15.868	18.107	20.409	22.684	0
	0.65	5	10	13.478	16.137	18.706	21.392	24.011	26.707	29.348
	0.75	5	10	15	18.632	21.694	24.652	27.726	30.715	33.760
	0.85	5	10	15	20	25	28.082	31.550	34.874	38.209
	0.95	5	10	15	20	25	30	35	39.219	42.922
$q = 0.95$	0.05	0.475	0.950	1.425	1.900	0	0	0	0	0
	0.15	1.091	1.850	2.775	3.700	4.625	5.550	6.475	0	0
	0.25	1.819	3.003	4.202	5.500	6.875	8.250	9.625	11.000	0
	0.35	2.544	4.203	5.859	7.530	9.202	10.950	12.775	14.600	16.425
	0.45	3.269	5.400	7.531	9.659	11.800	13.942	16.084	18.232	20.475
	0.55	3.995	6.596	9.199	11.788	14.401	17.014	19.627	22.241	24.849
	0.65	4.723	7.795	10.868	13.945	17.015	20.087	23.172	26.256	29.319
	0.75	5	8.997	12.540	16.084	19.638	23.173	26.717	30.272	33.827
	0.85	5	10	14.221	18.233	22.247	26.275	30.281	34.290	38.315
	0.95	5	10	15	20	25	29.380	33.858	38.358	42.810

$T = 100$

q	v_0	p = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$q = 0.55$	0.05	0	0	0	0	0	0	0	0	0
	0.15	10	0	0	0	0	0	0	0	0
	0.25	10	20	0	0	0	0	0	0	0
	0.35	10	20	30	0	0	0	0	0	0
	0.45	10	20	30	40	33.732	0	0	0	0
	0.55	10	20	30	40	50	0	0	0	0
	0.65	10	20	30	40	50	60	0	0	0
	0.75	10	20	30	40	50	60	70	0	0
	0.85	10	20	30	40	50	60	70	80	0
	0.95	10	20	30	40	50	60	70	80	90
$q = 0.65$	0.05	0	0	0	0	0	0	0	0	0
	0.15	8.466	9.094	0	0	0	0	0	0	0
	0.25	10	15.030	16.066	0	0	0	0	0	0
	0.35	10	20	22.554	23.952	25.793	0	0	0	0
	0.45	10	20	30	30.977	32.691	34.769	0	0	0
	0.55	10	20	30	39.183	40.307	42.264	0	0	0
	0.65	10	20	30	40	50	50.604	52.631	0	0
	0.75	10	20	30	40	50	60	61.825	63.737	0
	0.85	10	20	30	40	50	60	70	74.364	75.375
	0.95	10	20	30	40	50	60	70	80	90
$q = 0.75$	0.05	2.750	0	0	0	0	0	0	0	0
	0.15	5.813	6.930	9.750	0	0	0	0	0	0
	0.25	10	11.335	13.088	15.000	18.750	0	0	0	0
	0.35	10	15.965	18.141	20.544	23.032	25.630	0	0	0
	0.45	10	20	23.386	26.282	29.296	32.388	35.550	0	0
	0.55	10	20	28.885	32.191	35.726	39.344	43.030	0	0
	0.65	10	20	30	38.416	42.346	46.477	50.684	54.937	0
	0.75	10	20	30	40	50	53.970	58.539	63.313	67.940
	0.85	10	20	30	40	50	60	67.015	72.169	77.103
	0.95	10	20	30	40	50	60	70	80	87.416
$q = 0.85$	0.05	1.850	3.700	0	0	0	0	0	0	0
	0.15	3.810	5.116	7.650	10.200	0	0	0	0	0
	0.25	6.358	8.413	10.521	13.000	16.250	19.500	0	0	0
	0.35	8.916	11.765	14.648	17.559	20.511	23.700	27.650	0	0
	0.45	10	15.137	18.780	22.514	26.225	29.986	33.748	37.540	0
	0.55	10	18.523	23.015	27.467	32.013	36.519	41.091	45.637	0
	0.65	10	20	27.213	32.533	37.788	43.144	48.441	53.822	59.146
	0.75	10	20	30	37.571	43.696	49.747	55.908	61.989	68.071
	0.85	10	20	30	40	50	56.529	63.449	70.311	77.160
	0.95	10	20	30	40	50	60	70	78.860	86.395
$q = 0.95$	0.05	0.950	1.900	2.850	3.800	0	0	0	0	0
	0.15	2.196	3.700	5.550	7.400	9.250	11.100	12.950	0	0
	0.25	3.657	6.029	8.412	11.000	13.750	16.500	19.250	22.000	0
	0.35	5.117	8.430	11.752	15.080	18.412	21.900	25.550	29.200	32.850
	0.45	6.579	10.840	15.094	19.366	23.638	27.914	32.188	36.468	40.950
	0.55	8.043	13.242	18.459	23.654	28.870	34.086	39.306	44.519	49.742
	0.65	9.511	15.650	21.807	27.970	34.104	40.265	46.425	52.589	58.737
	0.75	10	18.067	25.156	32.264	39.372	46.446	53.550	60.654	67.758
	0.85	10	20	28.524	36.559	44.613	52.667	60.681	68.726	76.772
	0.95	10	20	30	40	50	58.856	67.855	76.854	85.801

$T = 500$

q	v_0	p = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$q = 0.55$	0.05	0	0	0	0	0	0	0	0	0
	0.15	50	0	0	0	0	0	0	0	0
	0.25	50	100	0	0	0	0	0	0	0
	0.35	50	100	150	0	0	0	0	0	0
	0.45	50	100	150	200	190.209	0	0	0	0
	0.55	50	100	150	200	250	0	0	0	0
	0.65	50	100	150	200	250	300	0	0	0
	0.75	50	100	150	200	250	300	350	0	0
	0.85	50	100	150	200	250	300	350	400	0
	0.95	50	100	150	200	250	300	350	400	450
$q = 0.65$	0.05	0	0	0	0	0	0	0	0	0
	0.15	43.329	46.799	0	0	0	0	0	0	0
	0.25	50	77.688	83.414	0	0	0	0	0	0
	0.35	50	100	116.670	124.568	133.046	0	0	0	0
	0.45	50	100	150	160.298	170.247	180.782	0	0	0
	0.55	50	100	150	197.848	208.496	220.440	0	0	0
	0.65	50	100	150	200	250	261.284	275.130	0	0
	0.75	50	100	150	200	250	300	318.710	334.151	0
	0.85	50	100	150	200	250	300	350	381.282	397.157
	0.95	50	100	150	200	250	300	350	400	450
$q = 0.75$	0.05	13.750	0	0	0	0	0	0	0	0
	0.15	29.738	34.913	48.750	0	0	0	0	0	0
	0.25	50	57.892	66.353	75.000	93.750	0	0	0	0
	0.35	50	81.072	92.685	104.407	116.287	128.291	0	0	0
	0.45	50	100	119.198	134.107	149.083	164.220	179.431	0	0
	0.55	50	100	146.034	163.979	182.160	200.377	218.766	0	0
	0.65	50	100	150	194.187	215.415	236.846	258.284	279.907	0
	0.75	50	100	150	200	250	273.535	298.177	322.794	347.457
	0.85	50	100	150	200	250	300	338.698	366.206	393.875
	0.95	50	100	150	200	250	300	350	400	441.374
$q = 0.85$	0.05	9.250	18.500	0	0	0	0	0	0	0
	0.15	19.325	25.586	38.250	51.000	0	0	0	0	0
	0.25	32.210	42.509	52.863	65.000	81.250	97.500	0	0	0
	0.35	45.105	59.449	73.906	88.359	102.848	118.500	138.250	0	0
	0.45	50	76.491	94.960	113.504	132.069	150.641	169.258	187.888	0
	0.55	50	93.511	116.125	138.691	161.325	183.992	206.644	229.362	0
	0.65	50	100	137.241	163.984	190.641	217.384	244.129	270.854	297.603
	0.75	50	100	150	189.231	220.074	250.815	281.689	312.477	343.268
	0.85	50	100	150	200	250	284.417	319.226	354.185	389.033
	0.95	50	100	150	200	250	300	350	396.137	434.943
$q = 0.95$	0.05	4.750	9.500	14.250	19.000	0	0	0	0	0
	0.15	11.035	18.500	27.750	37.000	46.250	55.500	64.750	0	0
	0.25	18.391	30.235	42.096	55.000	68.750	82.500	96.250	110.000	0
	0.35	25.742	42.313	58.904	75.499	92.094	109.500	127.750	146.000	164.250
	0.45	33.093	54.413	75.713	97.042	118.371	139.700	161.027	182.363	204.750
	0.55	40.444	66.499	92.555	118.585	144.649	170.712	196.776	222.825	248.894
	0.65	47.814	78.586	109.376	140.166	170.926	201.724	232.522	263.319	294.091
	0.75	50	90.675	126.199	161.723	197.248	232.737	268.268	303.800	339.300
	0.85	50	100	143.026	183.284	223.542	263.800	304.017	344.281	384.546
	0.95	50	100	150	200	250	294.845	339.832	384.820	429.765

Table E4 $W_{\le t^{\star\star}} - W_{>t^{\star\star}}$ under the optimal NA-sequencing policy $r^{\star\star}$ for $T = 100, p \in \{0.1, 0.2, \cdots, 0.9\}, v_0 \in \{0.05, 0.15, \cdots, 0.95\}$ and $q \in \{0.65, 0.75, 0.85, 0.95\}$.

$q = 0.65$		p								
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
v_0	0.05	0	0	0	0	0	0	0	0	0
	0.15	-0.0035	-0.0362	0	0	0	0	0	0	0
	0.25	0	-0.0129	-0.0449	0	0	0	0	0	0
	0.35	0	0	-0.0208	-0.0501	-0.0798	0	0	0	0
	0.45	0	0	0	-0.0245	-0.0525	-0.0806	0	0	0
	0.55	0	0	0	0	-0.0252	-0.0517	0	0	0
	0.65	0	0	0	0	0	-0.0231	-0.0477	0	0
	0.75	0	0	0	0	0	0	-0.0178	-0.0403	0
	0.85	0	0	0	0	0	0	0	-0.0091	-0.0304
	0.95	0	0	0	0	0	0	0	0	0
$q = 0.75$		p								
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
v_0	0.05	-0.0313	0	0	0	0	0	0	0	0
	0.15	-0.0097	-0.0324	-0.0703	0	0	0	0	0	0
	0.25	0	-0.0198	-0.0413	-0.0625	-0.0938	0	0	0	0
	0.35	0	-0.0085	-0.0278	-0.0471	-0.0664	-0.0858	0	0	0
	0.45	0	0	-0.0121	-0.0319	-0.0496	-0.0672	-0.0848	0	0
	0.55	0	0	-0.0001	-0.0169	-0.0328	-0.0487	-0.0647	0	0
	0.65	0	0	0	-0.0018	-0.0161	-0.0303	-0.0446	-0.0589	0
	0.75	0	0	0	0	0	-0.0113	-0.0246	-0.0372	-0.0499
	0.85	0	0	0	0	0	0	-0.0046	-0.0153	-0.0264
	0.95	0	0	0	0	0	0	0	0	-0.0031
$q = 0.85$		p								
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
v_0	0.05	-0.029	-0.0659	0	0	0	0	0	0	0
	0.15	-0.009	-0.0223	-0.0531	-0.0741	0	0	0	0	0
	0.25	-0.0048	-0.0169	-0.029	-0.0445	-0.0638	-0.0783	0	0	0
	0.35	-0.0002	-0.011	-0.0224	-0.0333	-0.044	-0.0559	-0.0697	0	0
	0.45	0	-0.0063	-0.0161	-0.0254	-0.0351	-0.0446	-0.0541	-0.0636	0
	0.55	0	-0.0003	-0.0092	-0.0179	-0.026	-0.0345	-0.0427	-0.051	0
	0.65	0	0	-0.0032	-0.0098	-0.0173	-0.0242	-0.0315	-0.0384	-0.0455
	0.75	0	0	0	-0.0027	-0.008	-0.0143	-0.0199	-0.0259	-0.0317
	0.85	0	0	0	0	0	-0.004	-0.0083	-0.0133	-0.0179
	0.95	0	0	0	0	0	0	0	-0.0008	-0.0042
$q = 0.95$		p								
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
v_0	0.05	-0.017	-0.031	-0.0371	-0.0405	0	0	0	0	0
	0.15	-0.0039	-0.0103	-0.0223	-0.029	-0.0333	-0.0362	-0.0384	0	0
	0.25	-0.0027	-0.0069	-0.0108	-0.0174	-0.0237	-0.0282	-0.0314	-0.0339	0
	0.35	-0.0018	-0.0051	-0.009	-0.0124	-0.0158	-0.0201	-0.0244	-0.0278	-0.0304
	0.45	-0.001	-0.0045	-0.0073	-0.0102	-0.0131	-0.016	-0.0189	-0.0218	-0.0249
	0.55	-0.0004	-0.0033	-0.0057	-0.0081	-0.0104	-0.0128	-0.0153	-0.0177	-0.0201
	0.65	0	-0.0013	-0.004	-0.0059	-0.0078	-0.0097	-0.0117	-0.0137	-0.0156
	0.75	0	-0.0003	-0.0023	-0.0038	-0.0052	-0.0066	-0.0081	-0.0095	-0.011
	0.85	0	0	-0.0002	-0.0016	-0.0026	-0.0036	-0.0045	-0.0055	-0.0065
	0.95	0	0	0	0	0	-0.0005	-0.001	-0.0015	-0.002

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