Recall. [8B]

(8.21) V is a complex vector space. $T \in \mathcal{L}(V)$

$\lambda_1, \dots, \lambda_m$ are distinct eigenvalues

a) $V = G(T,\lambda_1) \oplus \cdots \oplus G(T,\lambda_m)$

b) Each $G(T,\lambda_i)$ is inv. under T.

c) $(T-\lambda_i I)|_{G(T,\lambda_i)}$ is nilpotent.

$$\mathcal{M}(T) = \begin{pmatrix} \boxed{A_1} & & & \\ & \boxed{A_2} & & \\ & & \boxed{\ } & \\ & & & \ddots & \\ & & & & \boxed{A_m} \end{pmatrix}$$

UPPER TRI.

Block Diagonal.

$A_i = \mathcal{M}(T|_{G(T,\lambda_i)})$

size $= \dim(G(T,\lambda_i))$

(8.29). V is a complex vector space. $T \in \mathcal{L}(V)$

λ_i distinct. $\exists$ a basis of V such that

$$\mathcal{M}(T) = \begin{pmatrix} \boxed{A_1} & & & \\ & \boxed{A_2} & & \\ & & \ddots & \\ & & & \boxed{A_m} \end{pmatrix} \quad \&$$

$$A_i = \begin{pmatrix} \lambda_i & & * \\ & \ddots & \\ 0 & & \lambda_i \end{pmatrix}_{d_i \times d_i}$$

$d_i = \dim G(T,\lambda_i)$ ...
is called multiplicity of λ_i

pf.

A_i represents $T|_{G(T,\lambda_i)}$

$\because$ $(T-\lambda_i I)|_{G(T,\lambda_i)}$ is nilpotent.

$\exists$ a basis of $G(T,\lambda_i)$ call $\{v_1^i, v_2^i, \dots, v_{d_i}^i\}$

$$\mathcal{M}\left(T-\lambda_i I|_{G(T,\lambda_i)}\right) = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \qquad (8.19)$$

$$\shortparallel$$

$$\mathcal{M}(T|_{G(T,\lambda_i)}) - \mathcal{M}(\lambda_i I)$$

$$= \mathcal{M}(T|_{G(T,\lambda_i)}) - \lambda_i \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$$= \mathcal{M}(T|_{G(T,\lambda_i)}) - \begin{pmatrix} \lambda_i & & 0 \\ & \ddots & \\ 0 & & \lambda_i \end{pmatrix}$$

$$\mathcal{M}(T|_{G(T,\lambda_i)}) = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} + \begin{pmatrix} \lambda_i & & 0 \\ & \ddots & \\ 0 & & \lambda_i \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_i & & * \\ & \ddots & \\ 0 & & \lambda_i \end{pmatrix}$$

$\square$

•

$$\mathcal{M}(T) = \begin{pmatrix} \overset{G(\lambda_1, T)}{\boxed{\begin{matrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_1 \end{matrix}}} & & 0 \\ & \overset{G(\lambda_2, T)}{\boxed{\begin{matrix} \lambda_2 & & * \\ & \ddots & \\ 0 & & \lambda_2 \end{matrix}}} & \\ 0 & & \ddots \;\; \underset{G(\lambda_n, T)}{\boxed{\lambda_n}} \end{pmatrix}$$

Jordan Block

$$\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$

Jordan normal form (8.59, 8.60)

V is a vector space over $\mathbb{C}$. $\underline{T} \in \mathcal{L}(V)$.

$\exists$ A BASIS OF V, SUCH THAT

$$\mathcal{M}(T) = \begin{pmatrix} \boxed{\boxed{J_1} \quad \boxed{J_2}} \; A_1 & & & \\ & \ddots & & \\ & & & \boxed{\boxed{J_p}} \; A_m \end{pmatrix}$$

where $J_i = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$

$\leftarrow$ Jordan Block.

(8.55)' If $N \in \mathcal{L}(V)$ is nilpotent,

$\exists$ a basis of V, such that

$$\mathcal{M}(N) = \begin{pmatrix} 0 & a_1 & & & 0 \\ & 0 & a_2 & & \\ & & 0 & \ddots & \\ & & & \ddots & a_3 \\ 0 & & & & 0 \end{pmatrix}$$

$\boxed{a_i \in \{0 \text{ ~~or~~ } 1\}}$

(8.55) $N \in \mathcal{L}(V)$ is nilpotent. $\exists\ v_1, \cdots, v_n \in V$

& $\exists\ m_1, m_2, \cdots, m_n \geq 0$ such that

a) $\{ v_1, Nv_1, N^2 v_1, \cdots, N^{m_1} v_1, \cdots\cdots,$

$v_n, Nv_n, N^2 v_n, \cdots N^{m_n} v_n \}$ is a basis

of V.

b) $N^{m_1+1} v_1 = 0, \cdots\cdots N^{m_n+1} v_n = 0$

Ex. $N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} : \mathbb{C}^3 \to \mathbb{C}^3.$

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$Nv_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

$$N^2 v_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \qquad m_1 = 2$$

$$\underline{N^3 v_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \vec{0} \qquad \underline{\underline{m_1 + 1 = 3}}$$

$$\left\{ \underset{\substack{\uparrow \\ w_1}}{N^2 v_1},\ \underset{\substack{\uparrow \\ w_2}}{N v_1},\ \underset{\substack{\uparrow \\ w_3}}{v_1} \right\} = \left\{ \underline{\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \right\}$$

$M(N)$ WRT $\{w_1, w_2, w_3\}$

$$N(w_1) = N(N^2 v_1) = N^3 v_1 = 0$$

$$N(w_2) = N(N v_1) = N^2 v_1 = w_1$$

$$=$$

$$N(w_3) = N(v_1) = w_2$$

$$M(N) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

EXAMPLE 2

$$\begin{pmatrix} \boxed{\begin{matrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{matrix}} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} \boxed{0} & 0 & 0 \\ 0 & \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} \end{matrix} \end{pmatrix} = N \qquad 6\times 6$$

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$\boxed{\text{as in } (8.55)}$

$$v_1, \quad N(v_1) = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad N^2(v_1) = \begin{pmatrix} 6 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \underline{N^3(v_1)} = 0$$

$m_1 = 2$

$$v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad N v_2 = 0 \cdot v_2 = 0. \qquad m_2 = 0$$

$$v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad N v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \underline{\underline{N^2 v_3 = 0}} \qquad m_3 = 1.$$

$\{ N^2(v_1), N(v_1), v_1, v_2, N v_3, v_3 \}$ IS A BASIS.

FIND $M(N)$ W.R.T.

$$N\left(N^2(v_1)\right) = N^3 v_1 = 0$$

$$N\left(N(v_1)\right) = \underline{\underline{N^2 v_1}} \quad , \quad N(v_1) = \underline{\underline{N(v_1)}}$$

$$N(v_2) = 0$$

$$N\left(N(v_3)\right) = N^2 v_3 = 0 \qquad N(v_3) = \underline{\underline{N(v_3)}}$$

$$M(N) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Jordan normal form (8.59, 8.60)

V is a vector space over $\mathbb{C}$. $T \in \mathcal{L}(V)$.

$\exists$ A BASIS OF V, SUCH THAT

$$\mathcal{M}(T) = \begin{pmatrix} \underbrace{\begin{matrix} J_1 & \\ & J_2 \end{matrix}}_{A_1} & & \\ & \ddots & \\ & & \underbrace{J_p}_{A_m} \end{pmatrix}$$

where $$\underbrace{J_i = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}}_{\text{Jordan Block.}}$$ λ is an eigenvalue.

Pf. $V = G(\lambda_1, T) \oplus G(\lambda_2, T) \oplus \cdots \oplus G(\lambda_m, T)$

WHERE $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues

We know

$$(T - \lambda_i I)\big|_{G(\lambda_i, T)}$$ is nilpotent.

By (8.55), for each $G(\lambda_i, T)$ $\exists$

$v_1^i, \dots, v_{n_i}^i$ such that

$$\{ N^* v_1^i, N^{*-1} v_1^i, \dots, v_1^i, \dots\dots, N^* v_{n_i}^i, \dots, v_{n_i}^i \}$$

is a basis &

$$\mathcal{M}\left(T - \lambda_i I \big|_{G(\lambda_i, T)} \right) = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix}_{6\times 6}$$

nil.

$$\text{“} = \text{”} \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & 0 \\ & & 0 & 1 & \\ & & & 0 & 1 \\ & 0 & & & 0 \end{pmatrix}$$

$$\text{“} = \text{”} \begin{pmatrix} 0 & & & & 0 \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ 0 & & & & 0 \end{pmatrix}$$

$$\mathcal{M}\left(T \big|_{G(\lambda_i, T)} \right) - \lambda_i \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} =$$

$$\mathcal{M}\left(T|_{G(\lambda_i,T)}\right) =$$

$$\text{“=”} \begin{pmatrix} \lambda_i & 1 & & & & \\ & \lambda_i & 1 & & & \\ & & \lambda_i & & & \\ & & & \lambda_i & & \\ & & & & \lambda_i & 1 \\ & & & & & \lambda_i \end{pmatrix}_{6\times 6} \quad U_2',\ U_3'$$

$$\text{“=”} \begin{pmatrix} \lambda_i & 1 & & & 0 \\ & \lambda_i & 1 & & \\ & & \lambda_i & 1 & \\ & & & \lambda_i & 1 \\ 0 & & & & \lambda_i \end{pmatrix}_{5\times 5} \quad U_1'$$

$$\text{“=”} \begin{pmatrix} \lambda_i & & & & 0 \\ & \ddots & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \lambda_i \end{pmatrix} \quad \begin{matrix} U_1' \\ U_2' \\ \vdots \\ U_3' \end{matrix}$$