$(V, \langle\ ,\ \rangle)$ inner product space.

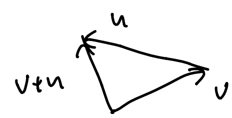
- Def'n

- $\|v\| = \sqrt{\langle v, v\rangle}$ norm ( triangle inequality)

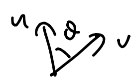
- $\underline{\underline{v \perp u}} \iff \langle v, u\rangle = 0$

  $\Downarrow$

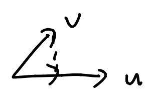
  $\langle v, u\rangle = \|v\|\|u\| \cos\underline{\underline{\theta}} = 0$

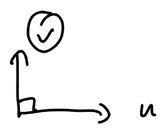



$\|u+v\| \le \|v\| + \|u\|$




v, u


$v \perp u$

- Cauchy - Schwartz.

  $|\langle v, u\rangle| \le \|v\|\|u\|$

- Pythagorean identity

  $\|u+v\|^2 = \|u\|^2 + \|v\|^2$ if $u \perp v$

$(V, \langle \ , \ \rangle)$ finite dim.

(6.23) Def'n. A list of vector $\{e_1, \dots, e_n\}$ is called orthonormal if $\begin{cases} \|e_i\| = 1 \iff \langle e_i, e_i \rangle = 1 \\ e_i \perp e_j \quad i \neq j \iff \langle e_i, e_j \rangle = 0 \quad i \neq j \end{cases}$

( That is, $\langle e_i, e_j \rangle = \begin{cases} 1 & \text{when } i = j \\ 0 & \text{when } i \neq j. \end{cases}$ )

(6.26) If $\{e_1, \dots, e_n\}$ is orthonormal then $\{e_1, \dots, e_n\}$ are linearly independent

pf Suppose $a_1 e_1 + \cdots + a_n e_n = 0$ for some $a_i$

WTS $a_i = 0$

$\langle a_1 e_1 + \cdots + a_n e_n, e_i \rangle = \langle 0, e_i \rangle = 0$

$= a_1 \langle e_1, e_i \rangle + a_2 \langle e_2, e_i \rangle + \cdots + a_n \langle e_n, e_i \rangle$

$= a_i \langle e_i, e_i \rangle = a_i$

Q.

(6.27). An orthonormal basis of $V$ is an orthonormal list of vectors that is also a basis.

Why is orthonormal basis important?

$V$, $\{v_1, \dots, v_n\}$ is a basis

$$V \xrightarrow[\cong]{\mathcal{M}} \mathbb{F}^n \qquad (\mathbb{R}^n \text{ or } \mathbb{C}^n)$$

$$v = a_1 v_1 + \cdots + a_n v_n \longmapsto \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

If $V$ is an inner product space w/ $\langle \cdot, \cdot \rangle$

On $\mathbb{F}^n$, we have the standard inner product $\bullet$

dot product.

$$(V, \langle \cdot, \cdot \rangle) \xrightarrow{\mathcal{M}} (\mathbb{F}^n, \bullet)$$

$\mathcal{M}$ preserves inner product iff $\{v_1, \dots, v_n\}$ is an orthonormal basis.

$$v = a_1 v_1 + \cdots + a_n v_n \longmapsto \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

$$u = b_1 v_1 + \cdots + b_n v_n \longmapsto \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$\langle v, u \rangle = a_1\overline{b_1} + \cdots + a_n\overline{b_n}$$

$$\|v\| = \sqrt{|a_1|^2 + \cdots + |a_n|^2}$$

pf. $\langle v, u \rangle = \underbrace{a_1\overline{b_1} + \cdots + a_n\overline{b_n}}$ $\Longleftrightarrow$ $\{v_1, \cdots, v_n\}$ orthonormal basis

"$\Longleftarrow$" $\langle a_1v_1 + \cdots + a_nv_n \,,\, b_1v_1 + \cdots + b_nv_n \rangle$

$$= \sum_{i,j} a_i\overline{b_j} \langle v_i, v_j \rangle$$

$\{v_1, \cdots, v_n\}$ orthonormal basis

$$= \sum_{i=1}^{n} a_i\overline{b_i} \langle v_i, v_i \rangle = \sum_{i=1}^{n} a_i\overline{b_i}$$

"$\Longrightarrow$" $v_i = 0v_1 + 0v_2 + \cdots + 1 \cdot v_i + \cdots + 0v_n$

$$\longmapsto \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} i\text{th}$$

$$v_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} j\text{th}$$

$\Rightarrow$ $\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$

(6.31) Gram - Schmidt Procedure.

$\{v_1, \cdots, v_m\}$ is a linearly independent vectors of $V$.

$e_1 = \dfrac{v_1}{\|v_1\|}$ For $j = 2, \cdots, m$ define $e_j$ inductively by

$$e_j = \frac{v_j - \langle v_j, e_1 \rangle e_1 - \langle v_j, e_2 \rangle e_2 - \cdots - \langle v_j, e_{j-1} \rangle e_{j-1}}{\| v_j - \langle v_j, e_1 \rangle e_1 - \langle v_j, e_2 \rangle e_2 - \cdots - \langle v_j, e_{j-1} \rangle e_{j-1} \|}$$

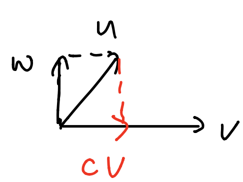
Then $e_1, \cdots, e_m$ is an orthonormal list such that

$$\operatorname{span}\{v_1, \cdots, v_j\} = \operatorname{span}\{e_1, \cdots, e_j\}$$

for any $j = 1, \cdots, m$.

$\left( \Longrightarrow \exists \text{ orthonormal basis for any } (V, \langle \cdot, \cdot \rangle) \right)$

<u>Pf</u> ( Recall : orthogonal decomposition . $v \neq 0$




$c = \dfrac{\langle u, v \rangle}{\|v\|^2} = \dfrac{\langle u, v \rangle}{\langle v, v \rangle}$

$w = u - cv$ $\quad w \perp v$

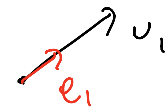
If $\|v\| = 1$, then $c = \langle u, v \rangle$ )

$\{ v_1, \ldots, v_n \}$

• $e_1 = \dfrac{v_1}{\|v_1\|}$

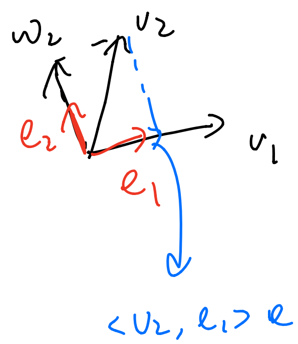
check: $\|e_1\| = \left\| \dfrac{v_1}{\|v_1\|} \right\| = \dfrac{1}{\cancel{\|v_1\|}} \cancel{\|v_1\|} = 1$

$\text{span}\{e_1\} = \text{span}\{v_1\}$




A NUMBER

• $e_2 = \dfrac{v_2 - \langle v_2, e_1 \rangle \cdot e_1}{\| v_2 - \langle v_2, e_1 \rangle \cdot e_1 \|}$ A NUM.




$$\text{span}\{v_1, v_2\} = \text{span}\{e_1, e_2\}$$

$$\| \qquad\qquad \|$$

$$\text{span}\{e_1, v_2\}$$

$e_2 \in \text{span}\{e_1, v_2\}$

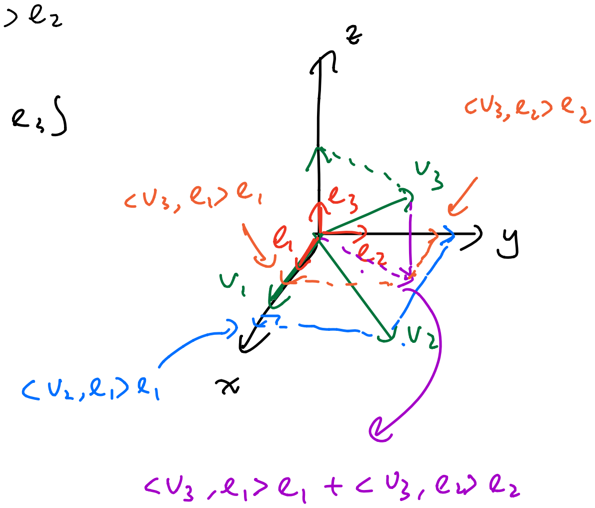
$v_2 \in \text{span}\{e_1, e_2\}$

- $e_3 := \dfrac{v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2}{\| v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2 \|}$

$$\operatorname{span}\{v_1, v_2, v_3\} = \operatorname{span}\{e_1, e_2, e_3\}$$

$$\| \qquad\qquad //$$

$$\operatorname{span}\{e_1, e_2, v_3\}$$




$\square$

Why is $\langle \cdot, \cdot \rangle$ useful?

$\boxed{6C}$ $\langle \cdot, \cdot \rangle$ makes $V/U$ concrete.

$U$ is a subspace of $V$

$$V/U = \{ v + U \mid v \in V \} \cong U^{\perp}$$

(6.45) If $U$ is a subset of $V$, then the orthogonal complement of $U$, denoted by $U^\perp$

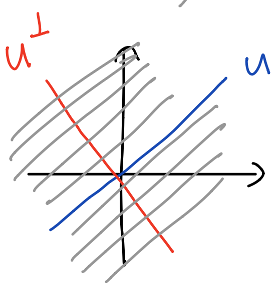
$$U^\perp = \{ v \in V \mid \langle v, u \rangle = 0 \text{ for } \forall u \in U \}$$

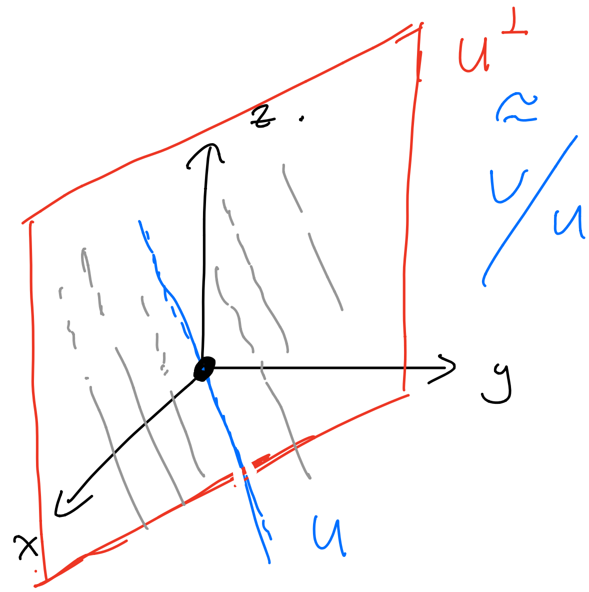
(6.47) If $U$ is a subspace of $V$, then

$$U \oplus U^\perp = V$$

$$\Longrightarrow \quad V/U \cong U^\perp$$

$v + U$



$\boxed{6B}$ $\langle \cdot, \cdot \rangle$ makes $V^* = \mathcal{L}(V, \mathbb{F})$ concrete.

$$V \xrightarrow{\cong} V^* \qquad V \cong V^*$$

$$u \longmapsto \langle \cdot, u \rangle = f_u$$

$f_u(v) = \langle v, u \rangle$ for $\forall v \in V$

check: $f_u \in \mathcal{L}(V, \mathbb{F})$

16.42). (Riesz - Representation Thm)

Let $\varphi \in \mathcal{L}(V, \mathbb{F}) = V^*$, $\exists!$ $u \in V$ such that

$$\varphi = \langle \cdot, u \rangle$$

pf. $\{e_1, \dots, e_n\}$ is an orthonormal basis of $V$.

Let $u = \bar{a}_1 e_1 + \dots + \bar{a}_n e_n$ where $a_i = \varphi(e_i)$

claim. $\varphi = \langle \cdot, u \rangle$

$\langle e_i, u \rangle = \langle e_i, \bar{a}_1 e_1 + \dots + \bar{a}_n e_n \rangle = \bar{\bar{a}}_i \langle e_i, e_i \rangle = a_i = \varphi(e_i)$

D.

Revisit. [6F]

Chap 7.

$A^T = A$

A real matrix.

A is symmetric

A has n

real eigenvalues

(7.29). $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

(Real spectral Thm)