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$$\mathcal{L}(V, W) \xrightarrow{\mathcal{M}} \mathbb{F}^{m \times n}$$

TSO

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$$\dim(V) = \dim(\text{Range } T) + \dim(\text{ker}(T))$$

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$$T: V \rightarrow W$$

$$T \text{ is}$$

invertible

$$T \text{ is Iso.}$$

$$\Updownarrow$$

$$\det(M_T) \neq 0.$$

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$$T, U \text{ is invariant subspace.}$$

$$T_u: V/U \rightarrow V/U.$$

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$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix}$$

$$A = \begin{pmatrix} -1 & 4 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{pmatrix} \quad \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 3 \\ \lambda_3 = 4 \end{matrix}$$

$$\det(zI - A) = \det \begin{pmatrix} z+1 & -4 & -3 \\ 0 & z-3 & -4 \\ 0 & 0 & z-4 \end{pmatrix}$$

$$= (z+1)(z-3)(z-4)$$

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5C,8 $T \in \mathcal{L}(V)$ (complex).

5C Eigen vector space

$$E(T, \lambda_i) = \{ v \in V \mid Tv = \lambda_i v \}$$

$$= \{ v \in V \mid (T - \lambda_i) v = 0 \}$$

8 $G(T, \lambda_i) = \{ v \in V \mid (T - \lambda_i)^j v = 0 \text{ for some } j \}$

$$E(T, \lambda_i) \subseteq G(T, \lambda_i)$$

$G(T, 2) = \text{span}\{u_1, u_2\}$

$\lambda_1 = 2 \quad \lambda_2 = 5$

u_1, u_2, u_3, u_4

$$\begin{pmatrix} \boxed{2} & \boxed{1} & & \\ & \boxed{2} & & \\ & & & \boxed{2} \\ & & & \boxed{5} \end{pmatrix}$$

$E(T, 2) = \text{span}\{u_1, u_2\}$

n_1

$G(T, 2)$

$E(T, 5) = \text{span}\{u_4\}$

$G(T, 5)$

$$\boxed{8} \quad V = \underbrace{G(T, \lambda_1)}_{U1} \oplus \dots \oplus \underbrace{G(T, \lambda_m)}_{U1}$$

$$V \neq E(T, \lambda_1) \oplus \dots \oplus E(T, \lambda_m)$$

$$\boxed{5c} \quad \text{If } V = E(T, \lambda_1) \oplus \dots \oplus E(T, \lambda_m)$$

then T is diagonalizable

(the Jordan normal form of T
is diagonal)

$$\Leftrightarrow G(T, \lambda_i) = E(T, \lambda_i) \quad \forall \lambda_i$$

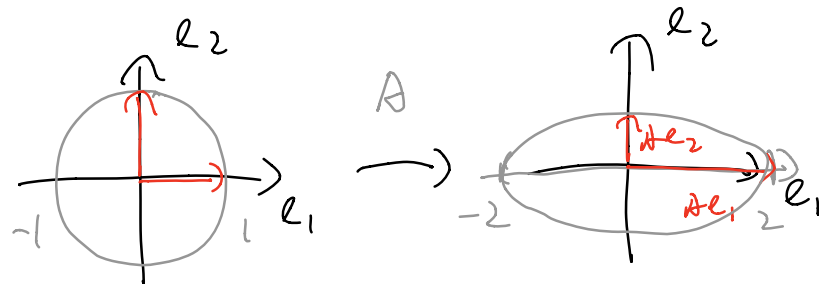
$$\begin{pmatrix} 2 & \textcircled{1} & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \textcircled{1} \\ 2 \\ 0 \end{pmatrix}$$

$$G(T, 2) \neq E(T, 2)$$

u_2 is not an eigen vector.

$$\begin{pmatrix} \lambda_1 & \text{---} \\ & \lambda_1 \end{pmatrix}$$

Ex. $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$



Ex $\begin{pmatrix} 2 & 1 \\ 0 & 2^{-1} \end{pmatrix} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$\lambda_1 = 2 \quad v_1$

$\lambda_2 = 2^{-1} \quad v_2$

$\begin{pmatrix} 2 & 0 \\ 0 & 2^{-1} \end{pmatrix}$ wrt v_1, v_2

