18A1 Generalized eigenvector & Generalized eigenspace.

$$G(\lambda, T) = \{ v \in V \mid (T-\lambda I)^j (v) = 0 \text{ FOR some } j \}$$

Eigenvalue $\quad = \ker((T-\lambda I)^n) \qquad n = \dim(V)$

#1 $\boxed{8A}$ $T \in \mathcal{L}(\mathbb{C}^2) \qquad T(w, z) = (z, 0)$

Find all generalized eigenvectors of T.

Pf WRT the standard basis $e_1 = (1, 0)$ $e_2 = (0, 1)$

$$\mathcal{M}(T) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Eigenvalue $\lambda = 0 \qquad n = \dim(\mathbb{C}^2) = 2$

$$G(0, T) = \ker\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 \right) = \ker\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) = \mathbb{C}^2.$$

nilponent $\boxed{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow$ All non-zero vectors are generalized eigenvector.

(8.16) An operator N in $\mathcal{L}(V)$ is called nilpotent if $N^k = 0$ for some $k \in \mathbb{Z}^+$. (Def'n)

(8.18) If $N \in \mathcal{L}(V)$ is a nilponent operator. then $N^{\dim(V)} = 0$.

PR $\quad G(0, N) = \{ v \in V \mid N^j(v) = 0 \text{ for some } j \}$

$\because$ N is nilpotent

$\therefore$ $\exists\, k$ s.t. $N^k = 0$ by definition.

$\Rightarrow$ $G(0, N) = V$

$\Rightarrow$ $G(0, N) = \ker(N^{\dim U})$ $\Bigg\}$ $\Rightarrow$

$\ker(N^{\dim U}) = V \Rightarrow N^{\dim V} = 0$

$\square$

(8.19). N is a nilpotent operator. then $\exists$ a basis of V such that w.r.t. that basis

$$M(N) = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \qquad n = \dim(U)$$

PF.

$$\begin{array}{ccccccccc} \text{Null}(N) & \subseteq & \text{Null}(N^2) & \subseteq & \text{Null}(N^3) & \subseteq \cdots & \subseteq & \text{Null}(N^n) & = \cdots \\ \| & & \| & & & & & \| & \\ \text{span}\{\underline{v_1}, v_2\} & & \text{span}\{v_1, v_2, v_3, v_4\} & & \cdots\cdots & & & \text{span}\{v_1, v_2, v_3, v_4, v_5\} & \end{array}$$

with $\text{Null}(N^n) = V$.

Claim. w.r.t. $\{v_1, v_2, \ldots, v_5\}$ a basis of V.

$M(N)$ in the desired form.

$N(v_1) = 0$ $\qquad N(v_2) = 0$

$N(v_3) \in \text{Null}(N) \Leftarrow N(N(v_3)) = N^2(v_3) = 0$

$\Leftarrow v_3 \in \text{Null}(N^2)$ $\qquad N(v_4) \in \text{Null}(N)$

$N(v_3) = a_1 v_1 + a_2 v_2$ $\qquad N(v_4) = * v_1 + * v_2$

$v_5 \in Null(N^3) \qquad N(v_5) \in Null(N^2) = span\{v_1, v_2, v_3, v_4\}$

$N^2(N(v_5)) = N^3(v_5) = 0$

$N(v_5) = * v_1 + * v_2 + * v_3 + * v_4$

$$M(N) = \begin{pmatrix} 0 & 0 & a_1 & * & * \\ 0 & 0 & a_2 & * & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{5\times 5}$$

2.

8B1.

(8.21) Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \dots, \lambda_m$ be distinct eigenvalues of T. Then

a) $V = G(\lambda_1, T) \oplus G(\lambda_2, T) \oplus \cdots \oplus G(\lambda_m, T)$ $\square$

b) Each $G(\lambda_i, T)$ is invariant under T. ✓

c) $(T - \lambda_i I)\big|_{G(\lambda_i, T)}$ is nilpotent, over $G(\lambda_i, T)$ ✓

(8.20) $T \in \mathcal{L}(V)$ $\quad$ $p(x)$ is a polynomial. Then

Null$(p(T))$ & Range$(p(T))$ ARE BOTH INVARIANT under T.

Remark: FOR us $p(x) = (x - \lambda_i)^n$ $\quad$ $p(T) = (T - \lambda I)^n$

Pf $\quad v \in \text{Null}(p(T))$ $\quad$ WTS $\quad T(v) \in \text{Null}(p(T))$

$$p(T) \cdot (T(v)) = T \cdot p(T)(v) = T(0) = 0$$

$v \in \text{Range}(p(T))$ $\quad$ WTS $\quad T(v) \in \text{Range}(p(T))$

$\exists\, w \in V$ s.t.

$$v = p(T)(w) \Rightarrow T(v) = T \cdot p(T)(w)$$
$$= p(T)(T(w))$$
$$\in \text{Range}(p(T))$$

$\square$.

(8.21) Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \cdots, \lambda_m$ be distinct eigenvalues of T. Then

a) $V = G(\lambda_1, T) \oplus G(\lambda_2, T) \oplus \cdots \oplus G(\lambda_m, T)$?

b) Each $G(\lambda_i, T)$ is invariant under T. ✓

c) $(T - \lambda_i I)|_{G(\lambda_i, T)}$ is nilpotent over $G(\lambda_i, T)$ ✓

pf b). $G(\lambda_i, T) = \mathrm{Null}\left((T - \lambda_i I)^n\right)$

$= \mathrm{Null}(p(T))$ w/ $p(x) = (x - \lambda_i)^n$

is invariant under T by (8.20).

c) Claim that

$$(T - \lambda_i I)^n \big|_{G(\lambda_i, T)} = 0$$

$\Leftarrow$ $G(\lambda_i, T) = \mathrm{Null}\left((T - \lambda_i I)^n\right)$

$\Rightarrow$ $T - \lambda_i I \big|_{G(\lambda_i, T)}$ is nilpotent over $G(\lambda_i, T)$

?

a). Remark We know in general

$$G(\lambda_1, T) \oplus G(\lambda_2, T) \oplus \cdots \oplus G(\lambda_m, T) \neq V$$

① BASE CASE.

$n=1$. $T \in \mathcal{L}(V) \Rightarrow T(v) = \lambda v$ FOR some λ. (HW)

$V = G(\lambda, T)$ (DONE).

② Assume a) holds for V w/ $\dim(V) = n$

WTS a) holds for V of $\dim(V) = n+1$.

$\because$ V is a vector space over complex number,

$\exists$ an eigenvector v corresponding to an eigenvalue λ_1 of T.

$G(\lambda_1, T) \neq \{0\} \Rightarrow \dim(G(\lambda_1, T)) > 0$.

Note that

$G(\lambda_1, T) = \text{null}((T - \lambda_1 I)^{n+1})$ $\quad$ $\dim V = n+1$

DEFINE $U = \text{Range}((T - \lambda_1 I)^{n+1})$

Claim: $V = G(\lambda_1, T) \oplus U$. $\quad$ $\boxed{\text{IN our case } S = (T - \lambda_1 I)}$

inv. $\quad$ inv.

LEM (8.5) $S \in \mathcal{L}(V)$ $n = \dim V$

$V = \text{null}(S^n) \oplus \text{Range}(S^n)$.

PR. BY the fundamental theorem of linear op.

$\dim V = \dim \ker(S^n) + \dim \text{range}(S^n)$

e.g. $S = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{R}^2)$

$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$

$\text{Null}(S) = \text{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\shortparallel$

$\text{Range}(S) = \text{span}\left\{\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$

$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

It suffices to show that

$\ker(S^n) \cap \text{range}(S^n) = \{0\}$

Suppose that $v \in \ker(S^n) \cap \text{range}(S^n)$ WTS $v = 0$.

$\because v \in \text{range}(S^n)$ $\exists\, w \in V$ s.t.

$\boxed{S^n(w) = v}$

$v \in \ker(S^n)$

$\Rightarrow \quad S^n(v) = S^n\big(S^n(w)\big) = S^{2n}(w) = 0$

$w \in \ker(S^{2n}) = \text{Null}(S^{2n})$

$\text{Null}(S) \subseteq \text{Null}(S^2) \subseteq \text{Null}(S^3) \subseteq \cdots \subseteq \text{Null}(S^n) = \text{Null}(S^{n+1})$

$= \cdots$

$\Rightarrow \quad \text{Null}(S^{2n}) = \text{Null}(S^n)$

$w \in \text{Null}(S^n)$

$\Rightarrow \quad v = S^n(w) = 0$

$V = \underline{G(\lambda_1, T)} \oplus \underline{U}$ $\quad \dim(U) < n+1$

& U is invariant under T

Apply the induction hypothesis to $T|_U \in \mathcal{L}(U)$

$U = \underline{G(\lambda_2, T|_U) \oplus G(\lambda_3, T|_U) \oplus \cdots \oplus G(\lambda_m, T|_U)}.$

$G(\lambda_2, T) \oplus G(\lambda_3, T) \oplus \cdots \oplus G(\lambda_m, T)$

⊛ compare $G(\lambda_2, T|_U)$ & $G(\lambda_2, T)$.

① $\underline{G(\lambda_2, T|_U) \subseteq G(\lambda_2, T)}$

$G(\lambda_2, T|_U) = \{ v \in U \mid \exists j \quad (\lambda_2 I - T)^j(v) = 0 \}$

$G(\lambda_2, T) = \{ v \in V \mid \exists j \text{ w/ } (\lambda_2 I - T)^j(v) = 0 \}$

② WTS $G(\lambda_2, T) \subseteq G(\lambda_2, T|_U).$

$\underline{v \in G(\lambda_2, T)}$ w.T.S $v \in G(\lambda_2, T|_U)$

$v \in V$

$\because V = G(\lambda_1, T) \oplus G(\lambda_2, T|_U) \oplus \cdots \oplus G(\lambda_m, T|_U)$

UNIQUE.

$\therefore \exists^{\vee} v_1 \in G(\lambda_1, T), \quad v_2 \in G(\lambda_2, T|_U) \cdots \quad v_m \in G(\lambda_m, T|_U)$

$\cap$ $G(\lambda_2, T)$

s.t. $v = v_1 + v_2 + \cdots + v_m$

$$v_1 + (v_2 - v) + \cdots + v_m = 0 \qquad (*)$$

$v_1 \in G(\lambda_1, T)$, $\quad (v_2 - v) \in G(\lambda_2, T)$, $\quad v_m \in G(\lambda_m, T)$

$\Rightarrow$ $v_1, \quad v_2 - v, \quad \ldots, \quad v_m$ are linearly dependent.

$\Rightarrow$ $v_1 = 0, \quad v_2 - v = 0, \quad \ldots, \quad v_m = 0$

$\Uparrow$

$\because$ IF $v_i \neq 0$ $(i \neq 2)$ THEN v_i IS A G.E.V CORRESPONDING to λ_i.

$\Rightarrow$ $v = v_2 \in G(\lambda_2, T|_U)$. $\quad$ D.

(8.21) Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ be distinct eigenvalues of T. Then

a) $V = G(\lambda_1, T) \oplus G(\lambda_2, T) \oplus \cdots \oplus G(\lambda_m, T)$

b) Each $G(\lambda_i, T)$ is invariant under T.

c) $(T - \lambda_i I)\big|_{G(\lambda_i, T)}$ is nilpotent, over $G(\lambda_i, T)$

(8.23) Suppose V is a vector space over $\mathbb{C}$. Given $T \in \mathcal{L}(V)$ $\exists$ a basis of V that consists of generalized eigen vectors of T.

(5.41). pf. $V = G_1(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$

$\{v_1, \ldots, v_k\} \subset G_1(\lambda_1, T)$, $\quad \{w_1, \ldots, w_\ell\} \subset G(\lambda_m, T)$

(8.25). $T \in \mathcal{L}(\mathbb{C}^3)$

$T(z_1, z_2, z_3) = (6z_1 + 3z_2 + 4z_3,\ 6z_2 + 2z_3,\ 7z_3)$

WRT the <u>standard basis</u>

$$\mathcal{M}(T) = \begin{pmatrix} 6 & 3 & 4 \\ 0 & 6 & 2 \\ 0 & 0 & 7 \end{pmatrix}$$

$\lambda_1 = 6, \quad \lambda_2 = 7$

$$G(6, T) = \text{Null}\left((6I - T)^3 \right)$$

$$= \text{Null}\left(\begin{pmatrix} 0 & 3 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}^3 \right) = \text{span}\{(1,0,0), (0,1,0)\}$$

$$G(7, T) = \text{Null}\left((T - 7I)^3 \right)$$

$$= \text{Null}\left(\begin{pmatrix} -1 & 3 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}^3 \right)$$

$$= \text{Null}\begin{pmatrix} -1 & 9 & -8 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 9 & -8 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \begin{cases} -x + 9y - 8z = 0 \\ -y + 2z = 0 \Rightarrow y = 2z \end{cases}$$

$\Rightarrow -x + 9(2z) - 8z = 0$

$\Rightarrow$ LET $z = 1$

$x = 10 \qquad y = 2.$

$(x, y, z) = (10, 2, 1).$

$G(7, T) = \text{span}\{(10, 2, 1)\}.$

$$\mathbb{C}^3 = \overset{U_1}{G(6,T)} \oplus \overset{U_2}{G(7,T)} \leftarrow \dim = 1$$

$$= \operatorname{span}\{\underset{v_1}{(1,0,0)}, \underset{v_2}{(0,1,0)}, \underset{v_3}{(10,2,1)}\} \quad (8.23)$$

INU[illegible]

$T(u)$

?

$G(6,T)$

IR $u \in G(6,T)$

$$T(v_1) = (6,0,0) = 6 \cdot v_1$$

$$T(v_2) = (3,6,0) = 3v_1 + 6v_2$$

$$T(v_3) = 7v_3$$

$$T|_{\operatorname{span}\{v_1,v_2\}} = \begin{pmatrix} 6 & 3 \\ 0 & 6 \end{pmatrix}$$

$$\mathcal{M}(T) = \begin{pmatrix} \boxed{\begin{matrix} 6 & 3 \\ 0 & 6 \end{matrix}} & 0 \\ \begin{matrix} 0 & 0 \end{matrix} & \boxed{7} \end{pmatrix} \leftarrow T|_{\operatorname{span}\{v_3\}} = (7)$$

$$T(z_1, z_2, z_3) = (6z_1 + 3z_2 + 4z_3,\ 6z_2 + 2z_3,\ 7z_3)$$

$$T = \begin{pmatrix} 6 & 3 & 4 \\ 0 & 6 & 2 \\ 0 & 0 & 7 \end{pmatrix}$$

(8.27) Block diagonal matrix.

$$\begin{pmatrix} \boxed{A_1} & & & 0 \\ & \boxed{A_2} & & \\ & & \ddots & \\ 0 & & & \boxed{A_m} \end{pmatrix}$$

eg

$$\begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 2 & 3 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 2 & 2 & 5 & 0 \\ 0 & 0 & 0 & 0 & \boxed{2} \end{pmatrix} \quad 5\times5$$

3×3

Why is Block diagonal matrix good?

Given $T \in \mathcal{L}(V)$

$V = U_1 \oplus U_2 \oplus U_3$ s.t. U_i are invariant under T

$$\begin{array}{ccccccc} V & = & U_1 & \oplus & U_2 & \oplus & U_3 \\ \downarrow T & & \downarrow T|_{U_1} & & \downarrow T|_{U_2} & & \downarrow T|_{U_3} \\ V & = & U_1 & \oplus & U_2 & \oplus & U_3 \end{array}$$

A basis $V = \{ \underbrace{v_1, v_2, \dots, v_k}_{U_1}, \underbrace{u_1, \dots, u_l}_{U_2}, \underbrace{w_1, \dots, w_m}_{U_3} \}$

$A_1 = \mathcal{M}(T|_{U_1})$

$$\mathcal{M}(T) = \begin{array}{c} \left(\begin{array}{c|c|c} \boxed{\begin{matrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{matrix}} & 0 & \\ \hline \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \boxed{\begin{matrix} * & * & * \\ * & A_2 & * \\ * & * & * \end{matrix}} & \\ \hline \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & 0 & \boxed{A_3} \end{array} \right) \end{array} \begin{array}{ll} v_1 & \\ \vdots & k \text{ row} \\ v_k & \\ u_1 & \\ \vdots & l \text{ row} \\ u_l & \\ w_1 & \\ \vdots & m \text{ row} \\ w_m & \end{array}$$

$A_2 = \mathcal{M}(T|_{U_2})$

$\mathcal{M}(T|_{U_3})$

$T(v_1) \in U_1 = \operatorname{span}\{v_1, \dots, v_k\}$

$= * v_1 + * v_2 + \cdots + * v_k$

$T(u_1) \in U_2 = \operatorname{span}\{u_1, \dots, u_l\}$

$= * u_1 + \cdots + * u_l$

e.g.,

$$\begin{array}{c|c} e_1 & \\ e_2 & \\ e_3 & \\ e_4 & \end{array}\begin{pmatrix} \boxed{\begin{matrix} 1 & 2 \\ 3 & 5 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} 1 & 2 \\ 0 & 1 \end{matrix}} \end{pmatrix} = \underline{T} : \mathbb{R}^4 \to \mathbb{R}^4 !$$

$$\mathbb{R}^4 = \overset{U_1}{\underbrace{\text{span}\{e_1, e_2\}}_{\text{inv. under } T}} \oplus \overset{U_2}{\underbrace{\text{span}\{e_3, e_4\}}_{\text{inv. under } T}}$$

$$T = T|_{\underline{U_1}} \oplus T|_{\underline{U_2}}$$

$$T|_{U_1} : \underset{U_1}{\mathbb{R}^2} \to \underset{U_1}{\mathbb{R}^2} \qquad T|_{U_1} = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$$

$$T|_{U_2} : \underset{U_2}{\mathbb{R}^2} \to \underset{U_2}{\mathbb{R}^2} \qquad T|_{U_2} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$