① V is real vector space $\dim V = n$

$T \in \mathcal{L}(V)$

⊛ Complexification. (9.2), (9.5)

V is a vector space over $\mathbb{R}$, with basis $\{v_1, \dots, v_n\}$.

$$V = \{ a_1 v_1 + \cdots + a_n v_n \mid a_i \in \mathbb{R} \}$$

The complexification of V, denoted by $V_{\mathbb{C}}$

$$V_{\mathbb{C}} = \{ a_1 v_1 + \cdots + a_n v_n \mid a_i \in \mathbb{C} \}$$

e.g. $V = \mathbb{R}^2 = \text{span}\{ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \}$

$$V_{\mathbb{C}} = \underbrace{\text{span}\{ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \}} \text{ over } \mathbb{C}.$$

$$= \{ z_1 e_1 + z_2 e_2 \mid z_i \in \mathbb{C} \}$$

$$= \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} : z_i \in \mathbb{C} \right\} \cong \mathbb{C}^2$$

dim of $V_{\mathbb{C}}$ as a complex vector space $= 2$

dim of $V_{\mathbb{C}}$ as a real vector space $= 4$

$V_{\mathbb{C}} = \text{span}\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, i\begin{pmatrix} 1 \\ 0 \end{pmatrix}, i\begin{pmatrix} 0 \\ 1 \end{pmatrix} \}$ as real vector space

$$\begin{pmatrix} 1+i \\ i-2 \end{pmatrix} \text{ in } V_{\mathbb{C}}$$

$$= (1+i)\begin{pmatrix}1\\0\end{pmatrix} + (i-2)\begin{pmatrix}0\\1\end{pmatrix}$$ $V_{\mathbb{C}}$ as complex v. s.

$$= 1\cdot\begin{pmatrix}1\\0\end{pmatrix} + i\begin{pmatrix}1\\0\end{pmatrix} + i\begin{pmatrix}0\\1\end{pmatrix} - 2\begin{pmatrix}0\\1\end{pmatrix}$$ $V_{\mathbb{C}}$ as real v. s.

* $T \in \mathcal{L}(V) : V \to V$ real vector space.

$\{v_1, \dots, v_n\}$ is a basis of V

$T(v_1) = \boxed{w_1 = a_{11}v_1 + a_{21}v_2 + \cdots + a_{n1}v_n}$

$T(v_2) = w_2$

$\vdots$

$T(v_n) = w_n$

$T(v) = T(a_1v_1 + \cdots + a_nv_n) = a_1 T(v_1) + \cdots + a_n T(v_n)$

$T_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$ $\quad V_{\mathbb{C}} = \operatorname{span}\{v_1, \dots, v_n\}$

$v_1 \longmapsto w_1 = a_{11}v_1 + \cdots + a_{n1}v_n.$

With respect to $\underline{\{v_1, \dots, v_n\}}$

$$M(T) = M(T_{\mathbb{C}})$$

$$M(T) = \begin{pmatrix} \boxed{\begin{matrix} a_{11} \\ \vdots \\ a_{n1} \end{matrix}} & \cdots \end{pmatrix} = M(T_{\mathbb{C}})$$

$G(T, \lambda_1) = \operatorname{span}\{e_1\}$ $\dim = 1$ $\mathbb{R}$

e.g., $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & -1 & 1 \end{pmatrix} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ w.r.t e_1, e_2, e_3

$$\underline{\hphantom{xxxxxxxxxxxxxxxxxxxx}}$$

$*$ $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & -1 & 1 \end{pmatrix} : \mathbb{C}^3 \longrightarrow \mathbb{C}^3.$ w.r.t e_1, e_2, e_3

$$P_{char}(x) := \left(\prod_{i=1}^{m} (x - \lambda_i)^{d_i} \right.$$

$$[10.B] = \det\left(-\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & -1 & 1 \end{pmatrix} + xI\right) = \det\begin{pmatrix} x-1 & 0 & -1 \\ 0 & x-1 & -4 \\ 0 & 1 & x-1 \end{pmatrix}$$

$$= (x-1)\left(\underline{(x-1)^2 + 4} \right) = 0$$

$\lambda_1 = 1 \qquad \lambda_2 = 1+2i \qquad \lambda_3 = 1-2i = \overline{\lambda_2}$

$v_1 = e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -i \\ -4i \\ 2 \end{pmatrix} \qquad v_3 = \begin{pmatrix} i \\ 4i \\ 2 \end{pmatrix} = \overline{v_2}$

$A_{\mathbb{C}}$: $\boxed{A_{\mathbb{C}}(v_1) = \lambda_1 v_1 = v_1}$

$$\boxed{\begin{aligned} A_{\mathbb{C}}(v_2) &= \lambda_2 v_2 \\ A_{\mathbb{C}}(v_3) &= \lambda_3 v_3 \end{aligned}}$$

Jordan normal form

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix}$$

A : $A v_1 = \lambda_1 v_1 = v_1$

v_1, $\lambda_1 = 1$

$\uparrow \qquad \uparrow$

e.v. $\quad$ e.value.

$$v_2 = \begin{pmatrix} -i \\ -4i \\ 2 \end{pmatrix} \qquad v_3 = \begin{pmatrix} i \\ 4i \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -i \\ -4i \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \qquad = i u_1 + u_2.$$

$$= -i \underbrace{\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}}_{u_1} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}}_{u_2} = -i u_1 + u_2$$

$\text{span}_{\mathbb{C}}\{v_2, v_3\} = \text{span}_{\mathbb{C}}\{u_1, u_2\}$ invariant

under $A_{\mathbb{C}}$.

<u>claim</u> $\text{span}_{\mathbb{R}}\{u_1, u_2\}$ is invariant A.

w.r.t. $\{v_1, u_1, u_2\}$

$\mathbb{R}^3 = \langle v_1 \rangle \oplus \langle u_1, u_2 \rangle$

$\lambda = a + bi$

eigenvalue of $A_{\mathbb{C}}$

$$M(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{pmatrix}$$

$v_1 \quad A(u_1) \quad A(u_2)$

$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ for A.

$$A_{\mathbb{C}}(v_2) = \lambda_2 v_2 = \overset{\lambda_2}{(1+2i)}\overset{v_2}{(-u_1 i + u_2)}$$
$$= (u_2 + 2u_1) + (-u_1 + 2u_2)i$$

$\|$

$$A_{\mathbb{C}}(-u_1 i + u_2) = -A_{\mathbb{C}}(u_1)i + A_{\mathbb{C}}(u_2)$$
$$= A(u_2) - A(u_1)i$$

$$\left.\begin{aligned} A(u_1) &= u_1 - 2u_2 \\ A(u_2) &= 2u_1 + u_2 \end{aligned}\right\} \text{ in } \operatorname{span}\{u_1, u_2\}$$

In chap. 10 If V is a real vector space. & $T \in \mathcal{L}(V)$, then we will work w/ $V_{\mathbb{C}}$ & $T_{\mathbb{C}} \in \mathcal{L}(V_{\mathbb{C}})$

[10.A] Trace.

- "change of bases"

$T \in \mathcal{L}(V)$

$\{u_1, \dots, u_n\}$ & $\{v_1, \dots, v_n\}$ are both bases of V.

$$\mathcal{M}(T, (u_1, \dots, u_n), (v_1, \dots, v_n))$$

$$\begin{tikzcd}
(V, u_1, \dots, u_n) \arrow[r, "T"] \arrow[d, phantom, "\cong"] & (V, v_1, \dots, v_n) \arrow[d, phantom, "\cong"] \\
\mathbb{F}^n \arrow[r, "\mathcal{M}(T)"] & \mathbb{F}^n
\end{tikzcd}$$

$$\mathcal{M}(T) = \begin{pmatrix} s_{11} & & \\ \vdots & \cdots & \\ s_{n1} & & \end{pmatrix}$$

$T(u_1) = s_{11} v_1 + s_{21} v_2 + \cdots + s_{n1} v_n$

Defn Two $n \times n$ matrices A & B are similar if $\exists$ an invertible matrix S such that $A = S^{-1} B S$ (A is a conjugate of B)

Claim: $T \in \mathcal{L}(V)$

$\{u_1, \dots, u_n\}$ & $\{v_1, \dots, v_n\}$ are bases.

$\mathcal{M}(T, (u_1, \dots, u_n))$ is similar to

$\mathcal{M}(T, (v_1, \dots, v_n))$

pf.

$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\qquad A = \begin{pmatrix} a_{11} & \cdots \\ a_{21} & \\ \vdots & \\ a_{n1} & \end{pmatrix}$ $\qquad \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}$

$$
\begin{tikzcd}
(V, u_1, \cdots, u_n) \arrow[r, "T"] & (V, u_1, \cdots, u_n) \\
F^n \arrow[u, phantom, "\shortparallel"] \arrow[r, "{\mathcal{M}(T, u_1, \cdots, u_n) = A}"] \arrow[d, "\cancel{I = \mathrm{id}}\ S"'] & F^n \arrow[u, phantom, "\shortparallel"] \arrow[d, "{S = \mathcal{M}(\mathrm{id}, u_1, \cdots, u_n, v_1, \cdots, v_n)}"] \\
(V, v_1, \cdots, v_n) \arrow[r, "T"] & (V, v_1, \cdots, v_n) \\
F^n \arrow[u, phantom, "\shortparallel"] \arrow[r, "{\mathcal{M}(T, v_1, \cdots, v_n) = B}"] & F^n \arrow[u, phantom, "\shortparallel"]
\end{tikzcd}
$$

$\begin{pmatrix} s_{11} \\ \vdots \\ s_{n1} \end{pmatrix}$

$$\boxed{SA = BS} \qquad A = S^{-1}BS$$

$$S \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} = B \cdot \begin{pmatrix} s_{11} \\ \vdots \\ s_{n1} \end{pmatrix}$$

= the first col of SA & BS

$$S = \begin{pmatrix} s_{11} & s_{12} & \\ s_{21} & \vdots & \\ \vdots & \vdots & \cdots \\ s_{n1} & s_{n2} & \end{pmatrix} :$$ is invertible $\Leftarrow$

$\{u_i\}$ & $\{v_i\}$ are BASES

$$u_1 = s_{11} v_1 + s_{21} v_2 + \cdots + s_{n1} v_n$$

$$u_2 = s_{12} v_1 + \cdots\cdots\cdots + s_{n2} v_n$$

D.