Recall

V is a finite dim vector space. $T \in \mathcal{L}(V)$

$\mathcal{M}(T, v_1, \cdots, v_n) = B$, $\mathcal{M}(T, u_1, \cdots, u_n) = A$

A and B are similar.

$$A = S^{-1} B S$$

$$S = \mathcal{M}(Id_V, (u_1, \cdots, u_n), v_1, \cdots, v_n)$$

$u_1 = S_{11} v_1 + \cdots + S_{n1} v_1$

$\cdots$

$u_n = S_{1n} v_1 + \cdots + S_{nn} v_n$

$$\begin{array}{ccc} u_1 & & u_n \\ \begin{pmatrix} S_{11} \\ \vdots \\ S_{n1} \end{pmatrix} & \cdots & \begin{pmatrix} S_{1n} \\ \vdots \\ S_{nn} \end{pmatrix} \end{array}$$

Trace of a matrix.

$$A_{n\times n} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{22} \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}_{n\times n}$$

$$tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}.$$

(10.14). If A, B are both $n\times n$ matrices

$$\boxed{tr(AB) = tr(BA)}$$ $(AB \neq BA)$

Pf. $tr(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} (a_{i1} b_{1i} + a_{i2} b_{2i} + \cdots + a_{in} b_{ni})$

$$\begin{array}{c} \\ i\text{th} \end{array}\begin{pmatrix} & & \\ a_{i1} & \cdots & a_{in} \\ & & \end{pmatrix}\begin{array}{c} i\text{th} \\ \begin{pmatrix} & b_{1i} & \\ & \vdots & \\ & b_{ni} & \end{pmatrix}\end{array}$$

$A \qquad B$

$$= \sum_{i=1}^{n}\left(\sum_{k=1}^{n} a_{ik}b_{ki}\right)$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ki}a_{ik}$$

$$= \sum_{k=1}^{n}\sum_{i=1}^{n} b_{ki}a_{ik}$$

$$= \sum_{k=1}^{n} (BA)_{kk} = \underline{\underline{tr(BA)}}$$

(10.15) $T \in \mathcal{L}(V)$ $\{v_1, \cdots, v_n\}$ & $\{u_1, \cdots, u_n\}$ are two basis of V.

$$tr(\underbrace{\mathcal{M}(T, v_1, \cdots, v_n)}_{= A}) = tr(\underbrace{\mathcal{M}(T, u_1, \cdots, u_n)}_{= B})$$

Define $tr(\mathcal{M}(T))$ is the trace of T $\underline{\underline{tr(T)}}$

Pf

$A = S^{-1}BS$

$$\underline{\underline{tr(A)}} = tr(S^{-1}BS) = tr(BS \cdot S^{-1}) = \underline{\underline{tr(B)}}$$

$\square$

Remark $T \in \mathcal{L}(V)$. If $T: V \to V$ where V is a real vector space. Then we will consider $T_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$ in the following definition

"Def'n" $\quad \operatorname{tr}(T) = \sum_{i=1}^{m} d_i \cdot \lambda_i$

where $\lambda_1, \dots, \lambda_m$ are distinction eigenvalues

& $d_i = \dim G(\lambda_i, T)$ multiplicity of λ_i.

pf. $\quad T \in \mathcal{L}(V) \qquad V$ is complex vector space.

$\exists$ a basis $\{v_1, \dots, v_n\}$ such that

$$\mathcal{M}(T, v_1, \dots, v_n) = \begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_1 \end{matrix}} & & & \\ & \boxed{\begin{matrix} \lambda_2 & & \\ & \ddots & \\ & & \lambda_2 \end{matrix}} & & \\ & & \ddots & \\ & & & \lambda_m \end{pmatrix} \begin{matrix} d_1 \\ \\ d_2. \\ \\ \end{matrix}$$

$$\operatorname{tr}(\mathcal{M}(T, v_1, \dots, v_n)) = \sum_{i=1}^{m} d_i \lambda_i$$

$\square$.

Ex. $\quad T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3.$

- $\mathcal{M}(T, e_1, e_2, e_3) = \overset{A}{\overset{\parallel}{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & -1 & 1 \end{pmatrix}}} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3.$

$$\operatorname{tr}(A) = 1+1+1 = 3. = \operatorname{tr}(T)$$

- $T_{\mathbb{C}} : \mathbb{C}^3 \longrightarrow \mathbb{C}^3. \qquad \operatorname{tr}(T) = \lambda_1 + \lambda_2 + \lambda_3 = 3$

$$\begin{cases} \lambda_1 = 1 & \longrightarrow \quad v_1 = e_1 \\ \lambda_2 = 1+2i & \longrightarrow \quad v_2 = \begin{pmatrix} -i \\ -4i \\ 2 \end{pmatrix} \\ \lambda_3 = 1-2i & \end{cases}$$

$$v_3 = \begin{pmatrix} i \\ 4i \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & \\ & 1+2i & \\ & & 1-2i \end{pmatrix} = S^{-1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & -1 & 1 \end{pmatrix} S$$

$$\begin{pmatrix} 1 & & \\ & 1+2i & \\ & & 1-2i \end{pmatrix} = \mathcal{M}(T_{\mathbb{C}}, e_1, v_2, v_3)$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & -1 & 1 \end{pmatrix} = \mathcal{M}(T_{\mathbb{C}}, e_1\ e_2\ e_3)$$

$$S = \begin{array}{c} \begin{pmatrix} 1 & -i & i \\ 0 & -4i & 4i \\ 0 & 2 & 2 \end{pmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ e_1 & v_2 & v_3 \end{array} \end{array}$$

$e_1 = e_1$

$$v_2 = \begin{pmatrix} -i \\ -4i \\ 2 \end{pmatrix} = -i\, e_1 + (-4i) e_2 + 2 e_3$$

$$v_3 = \begin{pmatrix} i \\ 4i \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -i & i \\ 0 & -4i & 4i \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}} & \begin{matrix} \\ 1+2i \\ \\ \end{matrix} & \begin{matrix} \\ \\ 1-2i \end{matrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}} & \begin{matrix} -i \\ -4i \\ 2 \end{matrix} & \begin{matrix} i \\ 4i \\ 2 \end{matrix} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -i & i \\ 0 & -4i & 4i \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \lambda_1 e_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\boxed{10\,B}$ Determinant.

- Determinant of a matrix.

(10.27). • A permutation of $(1, \dots, n)$ is a list $(m_1, \dots, m_n)$ that contains each of the number $1, \dots, n$ once.

- $\text{Perm } n$ = the set of all permutation of $(1, \dots, n)$.

$|\text{Perm } n| = n!$

- sign of a permutation is

$+$ IR the natural order has been changed even # of time.

$-$ Otherwise.

e.g. $n=2$: $\{\underset{+}{(1,2)}, \underset{-}{(2,1)}\}$ = perm 2.

$n=3$:

$(1,2,3)\ +$ $\quad$ $(1,3,2)\ -$

$(2,1,3)\ -$ $\rightarrow$ $(2,3,1)\ +$

$(3,2,1)\ -$ $\rightarrow$ $(3,1,2)\ +$

(10.33). Det (A)

Suppose that A is an $n\times n$ matrix.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

Then

$$\det A = \sum_{(m_1,\cdots,m_n)} \text{sign}(m_1,\cdots,m_n)\, a_{m_1 1} a_{m_2 2} \cdots a_{m_n n}$$

① 1 entry from each col.

② no two factors come from the same row.

Ex. • $A = (10)_{1\times 1} \Rightarrow \det(A) = 10.$

• $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $\qquad \det\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = 1\cdot 2 - 0\cdot 1 = 2.$

$$\det A = \sum_{\{(1,2),\,(2,1)\}} \text{sign}(\quad)\, a_{m_1 1} a_{m_2 2}$$

$$= +\underbrace{a_{11}}_{\substack{\uparrow \\ m_1}}\underbrace{a_{22}}_{\substack{\uparrow \\ m_2}} + -\underbrace{a_{21}}_{\substack{\uparrow \\ m_1}}\underbrace{a_{12}}_{\substack{\uparrow \\ m_2}} = a_{11}a_{21} - a_{12}a_{21}$$

$\{(1 \quad 2)\} \qquad\qquad (2, \; 1)$

(10.35). $A = \begin{pmatrix} a_{11} & * & * \cdots * \\ 0 & & \\ 0 & & \\ \vdots & & A' \\ 0 & & \end{pmatrix}$ or $A = \begin{pmatrix} a_{11} & 0 & 0 \cdots 0 \\ * & & \\ \vdots & & A' \\ * & & \end{pmatrix}$

$$\det(A) = a_{11} \det A'$$

Row / col operations:

① (10.36) If A is obtained from B by switching 2 rows (or col) of B. then

$$\det A = -\det(B).$$

(*) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 1 \\ 2 & 5 & 0 \\ 3 & 7 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 5 & 0 \\ 1 & 4 & 1 \\ 3 & 7 & 0 \end{pmatrix}$

② If A is obtained from B by adding a multiple of a row in B to another row of B.

$$\det(A) = \det(B).$$

$\begin{pmatrix} 1 & 4 & 1 \\ 2 & 5 & 0 \\ 3 & 7 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 9 & 4 & 1 \\ 12 & 5 & 0 \\ 17 & 7 & 0 \end{pmatrix}$

$\times 2$

Ex.

$$A = \left|\begin{pmatrix} 1 & 4 & 1 \\ 2 & 5 & -1 \\ 3 & 7 & 0 \end{pmatrix}\right| \qquad \det(A) = -6.$$

$$\downarrow$$

$$= -\left|\begin{pmatrix} 1 & 4 & 1 \\ -1 & 5 & 2 \\ 0 & 7 & 3 \end{pmatrix}\right| \longrightarrow -\left|\begin{pmatrix} 1 & 4 & 1 \\ 0 & 9 & 3 \\ 0 & 7 & 3 \end{pmatrix}\right|$$

$$= -1 \cdot \begin{vmatrix} 9 & 3 \\ 7 & 3 \end{vmatrix}$$

$$= -(27 - 21) = -6$$

04/21/2020 [10B]

<u>Recall:</u> V $T \in \mathcal{L}(V)$

- $\underset{\overset{\shortparallel}{A}}{\mathcal{M}(T, v_1, \dots, v_n)} \underset{\text{similar}}{\sim} \mathcal{M}(T, u_1, \dots, u_n)$

$$A = S^{-1} B S$$

- <u>Trace of a matrix</u>

$$tr(A) = \sum a_{ii}$$

If $A \sim B$, then $tr(A) = tr(B)$

$$tr\big(\mathcal{M}(T, v_1, \dots, v_n)\big) = tr\big(\mathcal{M}(T, u_1, \dots, u_n)\big)$$

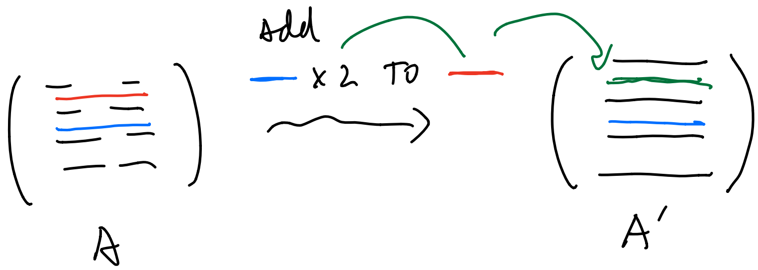
$$:= tr(T) = \sum_{i=1}^{m} d_i \lambda_i$$

$d_i = \dim G(T, \lambda_i)$ multiplicity

- [10B] Determinant of a matrix

① <u>Def'n:</u> using permutation

② $\left|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| = ad - cb$



$$\det(A) = \det(A')$$

$$(10.40) \qquad \boxed{\det(AB) = \det(A)\det(B)}$$

Pf A special case,

$$A = \begin{pmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & & * \\ & \ddots & \\ 0 & & b_{nn} \end{pmatrix}$$

$$AB = \begin{pmatrix} a_{11}b_{11} & & * \\ & \ddots & \\ 0 & & a_{nn}b_{nn} \end{pmatrix}$$

$$\det(AB) = \prod_{i=1}^{n} a_{ii}b_{ii} = \prod_{i=1}^{n} a_{ii} \prod_{i=1}^{n} b_{ii} = \det(A)\det(B)$$

<u>Recall</u>

$$\left| \begin{pmatrix} a_{11} & * & * \\ 0 & & \\ 0 & & A' \\ \vdots & & \end{pmatrix} \right| = a_{11} \det(A')$$

$$A = \begin{pmatrix} 1 & & * \\ & 2 & \\ 0 & & 3 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & & * \\ & 10 & \\ 0 & & 3 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & * & * \\ 0 & 2 & * \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & * & * \\ 0 & 10 & * \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1\cdot(-1) & * & * \\ 0 & 2\cdot 10 & * \\ 0 & 0 & 3\cdot 3 \end{pmatrix}$$

$$\det(A) = \left| \begin{pmatrix} 1 & * & * \\ 0 & 2 & * \\ 0 & 0 & 3 \end{pmatrix} \right| = 1 \cdot \begin{vmatrix} 2 & * \\ 0 & 3 \end{vmatrix}$$

$$= 1 \cdot 2 \cdot |(3)| = 1 \cdot 2 \cdot 3 = 6$$

$\boxed{\text{In general, (sketch)}}$

$$A \xrightarrow[\text{ROW OPERATIONS}]{\text{SEQUENCE OF}} \begin{pmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} = P \cdot A.$$

$$B \xrightarrow[\text{column operations}]{\text{SEQUENCE OF}} \begin{pmatrix} b_{11} & & * \\ & \ddots & \\ 0 & & b_{nn} \end{pmatrix} = B \cdot Q$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 10 & -1 \\ 0 & 5 & 4 \end{pmatrix}^{\times -2} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 6 & -7 \\ 0 & 5 & 4 \end{pmatrix}_{\times(-1)} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -11 \\ 0 & 5 & 4 \end{pmatrix}^{\times(-5)}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -11 \\ 0 & 0 & 59 \end{pmatrix}$$

$$\left| (PA)(BQ) \right| = \left| \begin{pmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & & * \\ & \ddots & \\ 0 & & b_{nn} \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} \right| \cdot \left| \begin{pmatrix} b_{11} & & * \\ & \ddots & \\ 0 & & b_{nn} \end{pmatrix} \right|$$

$$= \det(PA) \quad \det(BQ)$$

$$\det(PABQ) = \det(PA)\ \det(BQ)$$

$$\Downarrow \ \cdots\cdots$$

$$\det(AB) = \det(A)\ \det(B)$$

$\square$

(10.41) $T \in \mathcal{L}(V)$ $\{u_i\}, \{v_i\}$ two bases of V

$$\det\big(\underset{\substack{\shortparallel \\ A}}{\mathcal{M}(T, v_i)}\big) = \det\big(\underset{\substack{\shortparallel \\ B}}{\mathcal{M}(T, u_i)}\big)$$

<u>pf</u>

$$A = S^{-1} B S$$

$$\det A = \det(S^{-1} B S)$$

$$= \underline{\det(S^{-1})} \det(B) \underline{\det(S)} = \det(B).$$

<u>Remark</u> $\det(S^{-1}) = \dfrac{1}{\det(S)}$

$$\det(S^{-1}) \cdot \det(S) = \det(S^{-1} \cdot S) = \det(I) = 1$$

2.

<u>Def'n</u>

$$\det(T) := \det(\mathcal{M}(T))$$

$$= \det\begin{pmatrix} \text{Jordan} \\ \text{normal} \\ \text{form} \end{pmatrix}$$

$$= \prod_{i=1}^{m} \lambda_i^{d_i}$$

$(\ast)$

If $T \in \mathcal{L}(V)$ and V is real vector space. We apply $\ast$ to $T_{\mathbb{C}} \in \mathcal{L}(V_{\mathbb{C}})$

Ex. $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

WRT $\{e_1, e_2, e_3\}$

$$\mathcal{M}(T) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & -1 & 1 \end{pmatrix} = A.$$

HAS 1 Real eigenvalue

$\lambda_1 = 1$

$v_1 = e_1$

$$\det(A) = 1 \cdot \det\begin{pmatrix} 1 & 4 \\ -1 & 1 \end{pmatrix} = 1 \cdot (1 \cdot 1 + 4) = 5.$$

$T_{\mathbb{C}} : \mathbb{C}^3 \longrightarrow \mathbb{C}^3.$

WRT $\{e_1, e_2, e_3\}$

$T_{\mathbb{C}}(e_1) = e_1 \qquad T_{\mathbb{C}}(e_2) = e_2 - e_3 \qquad T_{\mathbb{C}}(e_3) = e_1 + 4e_2 + e_3$

$$\mathcal{M}(T_{\mathbb{C}}) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & -1 & 1 \end{pmatrix} : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$$

$\lambda_1 = 1 \qquad \lambda_2 = 1 + 2i \qquad \lambda_3 = \overline{\lambda_2} = \overline{(1+2i)} = 1 - 2i$

$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -i \\ -4i \\ 2 \end{pmatrix} \qquad v_3 = \overline{v_2} = \begin{pmatrix} i \\ 4i \\ 2 \end{pmatrix}$

W.R.T

$$\mathcal{M}(T_{\mathbb{C}}, v_1, v_2, v_3) = \begin{pmatrix} 1 & & \\ & 1+2i & \\ & & 1-2i \end{pmatrix}$$

$$\det\left(\mathcal{M}(T_{\mathbb{C}}, v_1, v_2, v_3) \right) = \det\left(\mathcal{M}(T_{\mathbb{C}}, e_1, e_2, e_3) \right)$$

$$\underbrace{\begin{vmatrix} 1 & & \\ & 1+2i & \\ & & 1-2i \end{vmatrix}}_{} = \underbrace{\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & -1 & 1 \end{vmatrix}}_{\Uparrow \atop A}$$

$$\det(A) = \lambda_1 \lambda_2 \lambda_3 = 1\,(1+2i)\,(1-2i) = 5$$

12

(10.25) $T \in \mathcal{L}(V)$. then

$$P_{char}(T) = \det\left(zI - \mathcal{M}(T) \right)$$

Ex. $A = \begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix}$ find the eigen value of A

Before: $\cancel{\begin{pmatrix} x \\ y \end{pmatrix} \neq 0 \quad A\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}}$

$\cancel{\begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}}$ ......

$$P_{char}(A) = \det\left(z \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix} \right)$$

$$= \det \begin{pmatrix} z-1 & -2 \\ -4 & z-1 \end{pmatrix}$$

$$= (z-1)^2 - (-2)\cdot(-4)$$

$$= (z-1)^2 - 8$$

$$(z-1)^2 - 8 = 0 \Rightarrow \lambda = \pm 2\sqrt{2} + 1$$ 12

$$\begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1-\lambda & 2 \\ 4 & 1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

solve $\emptyset$

(10.25) $T \in \mathcal{L}(V)$. then

$$P_{char}(T) = \det\left(zI - \mathcal{M}(T) \right)$$

Pf Recall

$$P_{char}(T) = \prod_{i=1}^{m} (z-\lambda_i)^{d_i}$$

$\exists$ $v_1, \dots, v_n$ such that

$$M(T, v_1, \dots, v_n) = \begin{pmatrix} \lambda_1 & & * \\ & \ddots \lambda_1 \ddots & \\ 0 & & \lambda_m \end{pmatrix} \leftarrow \text{Jordan form}$$

$\nwarrow J$

$\exists$ S such that

$$M(T) = S^{-1} J S$$

$$\begin{aligned} zI - M(T) &= zI - S^{-1} J S \\ &= z S^{-1} \cdot S - S^{-1} J S \\ &= S^{-1} \begin{pmatrix} z & & \\ & \ddots & \\ & & z \end{pmatrix} S - S^{-1} J S \\ &= S^{-1} \left(\begin{pmatrix} z & & \\ & \ddots & \\ & & z \end{pmatrix} - J \right) S \end{aligned}$$

$$
\begin{aligned}
\det\left(z I - M(T)\right) &= \det\left(\begin{pmatrix} z & & \\ & \ddots & \\ & & z \end{pmatrix} - J\right) \\
&= \det\begin{pmatrix} z-\lambda_1 & & * \\ & \ddots & \\ 0 & & z-\lambda_m \end{pmatrix} \\
&= \prod_{i=1}^{m} (z-\lambda_i)^{d_i} \\
&= P_{char}(T).
\end{aligned}
$$

Q.