Recall:

$T \in \mathcal{L}(V)$ V is a complex vector space.

$n = \dim(V)$

$\exists$ a basis of V s.t.

Jordan normal form.

$G(T, \lambda_1)$

$$\mu(T) = \begin{pmatrix} \boxed{J_1} & & & \\ & \boxed{J_2} & & \\ & & \ddots & \\ & & & \boxed{J_m} \end{pmatrix}$$

$G(T, \lambda_i)$

$$\boxed{J_i = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}}$$

λ is an eigenvalue

Jordan Block.

① Characteristic Poly & the Cayley-Hamilton Thm.

(8.34) Def'n. V is a complex vector space.

$\lambda_1, \cdots, \lambda_m$ are distinct eigenvalues of $T \in \mathcal{L}(V)$

$d_i = \dim(G(T, \lambda_i))$ (muliplicity of λ_i)

Then the characteristic polynomial

$$P_{char}(x) = (z-\lambda_1)^{d_1}(z-\lambda_2)^{d_2} \cdots (z-\lambda_m)^{d_m}$$
$$= \prod_{i=1}^{m} (z-\lambda_i)^{d_i}$$

(8.36) • $\deg(P_{char}(x)) = d_1 + d_2 + \cdots + d_m$
$= \dim V.$ ←

Pf) $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$

$$\dim V = \sum_{i=1}^{m} \dim G(\lambda_i, T) = \sum_{i=1}^{m} d_i$$

• $\left\{ \begin{array}{c} \text{zeros of} \\ P_{char}(x) \end{array} \right\} = \left\{ \begin{array}{c} \text{Eigenvalues} \\ \text{of } T. \end{array} \right\}$

Remark $P_{char}(z) = \det(\mathcal{M}(T) - z I)$

e.g. $P_{char}(z) = (z-2)^3 (z-1)(z+1)$

is the characteristic poly of $T \in \mathcal{L}(V)$

① $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = -1$

② $\dim G(2, T) = 3$

$\dim G(1, T) = 1$

$\dim G(-1, T) = 1$

the Jordan normal form of T

$$\begin{pmatrix} \boxed{\begin{matrix} 2 & * & 0 \\ & 2 & * \\ 0 & & 2 \end{matrix}} & & & 0 \\ & \boxed{1} & & \\ 0 & & & \boxed{-1} \end{pmatrix}_{5\times 5}$$

② $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

all 3 matrices

$$P_{char}(z) = (z-2)^3$$

(8.37) Cayley-Hamilton Thm.

$P_{char}(T) = 0$ where $P_{char}(z)$ is the char poly of $T \in \mathcal{L}(V)$

pf (A different proof in the textbook)

$(T-\lambda_i I)|_{G(T,\lambda_i)}$ nil

$$\begin{pmatrix} \boxed{A} & 0 & 0 \\ 0 & \boxed{B} & 0 \\ 0 & 0 & \boxed{C} \end{pmatrix}^n = \begin{pmatrix} \boxed{A}^n & 0 & 0 \\ 0 & \boxed{B}^n & 0 \\ 0 & 0 & \boxed{C}^n \end{pmatrix}$$

e.g.
$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & \boxed{4} \end{pmatrix}^2 = \begin{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^2 & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \boxed{4}^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$\exists$ a basis $\{v_1, \ldots, v_n\}$ s.t. with respect to

$$M(T) = \begin{pmatrix} D_1 & & \\ & \ddots & \\ & & D_m \end{pmatrix} \qquad D_i = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$

λ is an eigenvalue.

$$M(P_{char}(T)) = 0 \iff P_{char}(M(T)) = 0$$

e.g.

$$M(T) = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_1 & 1 & & \\ & & \lambda_1 & & \\ & & & \lambda_2 & 1 \\ & & & & \lambda_2 \end{pmatrix} = A$$

$$P_{char}(z) = (z-\lambda_1)^3 (z-\lambda_2)^2 \qquad \text{WTS.}$$

$$P_{char}(A) = 0$$

$$P_{char}(A) = \underbrace{(A - \lambda_1 I)^3}\, \underbrace{(A - \lambda_2 I)^2} = 0 \text{ matrix.}$$

$$(A - \lambda_2 I)^2 = \left[\begin{pmatrix} \lambda_1 & & & & \\ & \lambda_1 & 1 & & \\ & & \lambda_1 & & \\ & & & \lambda_2 & 1 \\ & & & & \lambda_2 \end{pmatrix} - \begin{pmatrix} \lambda_2 & & \\ & \ddots & \\ & & \lambda_2 \end{pmatrix}_{5\times 5} \right]^2$$

$$= \left[\begin{pmatrix} \lambda_1-\lambda_2 & 1 & & & \\ & \ddots & \ddots & & \\ & & \lambda_1-\lambda_2 & & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} \right]^2$$

$$= \begin{pmatrix} \begin{pmatrix} \lambda_1-\lambda_2 & 1 & \\ & \ddots & \\ & & \lambda_1-\lambda_2 \end{pmatrix}^2 & \\ & \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}^2 \end{pmatrix} = \begin{pmatrix} \ast & \\ & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{pmatrix}$$

$$(A-\lambda_1 I)^3 = \begin{pmatrix} \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \\ & \ast \end{pmatrix}$$

$$(A-\lambda_1 I)^3 (A-\lambda_2 I)^2 = \begin{pmatrix} \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \\ & \ast \end{pmatrix} \begin{pmatrix} \ast & \\ & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{pmatrix} = 0$$

$\blacksquare$

Minimal Poly. V is a vector space. finite dim.

(8.40) $T \in \mathcal{L}(V)$ $\exists$ a unique monic poly

$P_{min}(x)$ of smallest degree s.t.

$P_{min}(T) = 0.$

the coefficient of the highest degree term = 1

e.g. $2x^2+1$ ✗

x^2+1 ✓

pf Existence:

$n = \dim(V)$

$\dim(\mathcal{L}(V)) = n^2$

$\{I, T, T^2, T^3, \ldots, T^{n^2}\}$ is linearly dependent.

$\exists$ $a_0, \ldots, a_m$ s.t.

$$a_0 + a_1 T + \cdots + \overset{1}{\underset{}{a_m}} T^m = 0$$

where m is the smallest integer s.t.

$$\underline{P_{min}(z) = a_0 + a_1 z + \cdots + \overset{1}{a_m} z^m.}$$

monic

Unique: $\deg q(z) = \deg P_{min}(z)$

$q(T)=0 \Rightarrow q(z) - P_{min}(z)$ has lower degree

&

$q(T) - P_{min}(T) = 0$

$f(x) = 2+3x$

$\frac{2}{3}+x$

$f(T) = 0 \Rightarrow \frac{1}{3} f(T) = 0$

(8.46) Given any polynomial $q(x)$.

$$\boxed{q(T) = 0 \iff P_{min}(x) \mid q(x)}$$

(8.48). V is a complex vector space

$T \in \mathcal{L}(V)$. $\quad P_{min}(x) \mid P_{char}(x)$.

Pf "$\Leftarrow$" $q(x) = f(x) P_{min}(x)$

zero map

$q(T) = f(T) P_{min}(T) = f(T) \cdot 0 = 0$

" $\Rightarrow$ " $g(T)=0$ WTS $P_{min}(x) \mid g(x)$.

By the division Algorithm. (Long division)

$$\boxed{\begin{gathered} g(x) = P_{min}(x)\cdot f(x) + r(x) \\ \deg r(x) < \deg g(x) \end{gathered}}$$

$g(x) = x^5 + x^4 - 2x + 1$

$P_{min}(x) = x^2+1$

$$\begin{array}{r} x^3 + x^2 - x - 1 \\ x^2+1 \overline{\big)\, x^5 + x^4 - 2x + 1} \\ -\ \underline{x^5 + x^3 \qquad\qquad} \\ x^4 - x^3 - 2x + 1 \\ \underline{x^4 + x^2 \qquad\quad} \\ -x^3 - x^2 - 2x + 1 \\ \underline{-x^3 \qquad - x \quad} \\ -x^2 - x + 1 \\ \underline{-x^2 \qquad - 1} \\ \boxed{-x+2} \end{array}$$

$g(x) =$

$f(x)$

$P_{min}(x^3+x^2-x-1)$

$+(-x+2)$

$r(x)$

WTS $r(x) = 0$

suppose that $r(x) \neq 0$.

$$q(x) = P_{min}(x) f(x) + r(x)$$

$$\underbrace{\cancel{q(T)}}_{0} = \underbrace{\cancel{P_{min}(T)}}_{0} f(T) + r(T)$$

$r(T) = 0$ contradiction $\because \deg(r(x)) < \deg P_{min}$

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$\boxed{\text{(8.49). } T \in \mathcal{L}(V) \quad V \text{ is a vector space. } \{\text{zeros of } P_{min}(z)\} = \{\text{eigenvalues of } T\}}$

Examples

① $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ $C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$P_{min} = (z-2)^{2}$ $P_{min}(z) = (z-2)^3$ $P_{min}(z) = z-2$

$$P_{char}(z) = (z-2)^3$$

P_{min} :

For C: $\quad q(z) = z-2$

$$q(C) = \begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 \end{pmatrix} - \begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 \end{pmatrix} = 0$$

$$P_{min}(z) \mid q(z)$$

$$\Rightarrow P_{min}(z) = z-2$$

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad P_{min}(z) \mid P_{char} = (z-2)^3$$

minimal Poly:

$\cancel{z-2}$

$(z-2)^2$ ✓ $\qquad (A-2I)^2 = 0 \qquad P_{min}(z) = (z-2)^2$

$$B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \qquad P_{min}(z) = (z-2)^3$$

(2) $T \in \mathcal{L}(V)$ $\quad$ $\dim V = 6$

$$P_{\underline{char}}(z) = (x-3)(x-2)^2(x+1)^3$$

<u>a)</u> $$P_{min}(z) = (x-3)(x-2)(x+1)$$

$\exists$ a basis of V. s.t.

$$M(T) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ & & & 0 & 2 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & 3 \end{pmatrix}$$

b) $P_{min}(z) = (x-3)(x-2)(x+1)^2$

$(x+1)^2$

$$M(T) = \begin{pmatrix} \boxed{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{3} \end{pmatrix} \quad \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{matrix}$$

$\begin{matrix} v_2 \\ v_3 \end{matrix} \leftrightarrow v_1$

$$\begin{cases} P_{min}(z) = (x-3)(x-2)^2(x+1)^3 \\ P_{min}(z) = (x-3)(x-2)^2(x+1)^2 \end{cases}$$

$$\begin{bmatrix} -1 & 1 & \\ & -1 & \\ & & -1 \end{bmatrix}$$

---

e.g. $P_{char}(z) = (x-2)^4$

$P_{min}(z) = (x-2)^2$

$$\begin{pmatrix} 2 & 1 & & \\ & 2 & & \\ & & 2 & 1 \\ & & & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 & & \\ & 2 & & \\ & & 2 & 1 \\ & & & 2 \end{pmatrix}$$