6A. Inner product space. (vector space w/ $\langle v, w \rangle$)
$\downarrow$
$\mathbb{F}$.

- ($\mathbb{R}^n$, dot product)
$\uparrow$

- Inner product

$\langle u, v \rangle \longrightarrow \lambda \in \mathbb{R}$.

$\rightsquigarrow$ size of a vector $\|\cdot\|$ norm.

"angles" between v & u.

v, u "independent" $v \perp u$

chap 3F
$\downarrow$

$\rightsquigarrow$ concrete map $V \longrightarrow V^* \cong \mathcal{L}(V, \mathbb{F})$

$v \longmapsto \langle v, \cdot \rangle$

(6.3) $\langle \cdot, \cdot \rangle$

An inner product $\langle \cdot, \cdot \rangle$ on V is a function.

$\langle \ , \ \rangle : V \times V \longrightarrow \mathbb{F}$. ($\langle u, v \rangle \in \mathbb{F}$), satisfying

① Positivity : $\langle v, v \rangle \geq 0 \quad \forall v \in V$ $\in \mathbb{R}$

② $\langle v, v \rangle = 0 \iff v = 0$

③ $\langle u, v \rangle = \overline{\langle v, u \rangle}$ ($\overline{x+iy} = x - iy$)

④ $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$.

$\langle \lambda u, w \rangle = \lambda \langle u, w \rangle$

$f : V \to \mathbb{F}$.

$\begin{cases} f(u+v) = f(u) + f(v) \\ f(\lambda u) = \lambda f(u) \end{cases}$

(6.7). Basic properties.

① $< w, u+v > = < w, u > + < w, v >$

② $< u, \lambda v > = \bar{\lambda} < u, v >$

Pf. $< u, \lambda v > = \overline{< \lambda v, u >} = \overline{\lambda < v, u >}$

$= \bar{\lambda} \overline{< v, u >} = \bar{\lambda} < u, v >.$

③ $< 0, u > = < u, 0 > = 0$

Pf $< 0, u > = < 0+0, u >$ . . . . ☐.

(6.5). An inner product space is a vector space w/ $< \cdot, \cdot >$.

(6.8) Norm $\|v\|$.

$\forall v \in V$. $\|v\| = \sqrt{< v, v >}$

$\dim(\overset{\mathbb{R}}{\underset{sy}{U}}) = 2$

$|\alpha|$ = absolute #.

(6.10). a) $\|v\| = 0 \iff v = 0$

b) $\|\lambda v\| = |\lambda| \|v\|$

$\lambda = x+iy$ $|\lambda| = \sqrt{x^2+y^2}$

(6.18) Triangle inequality (Cauchy - Schwartz)

$\|u+v\| \leq \|u\| + \|v\|$.

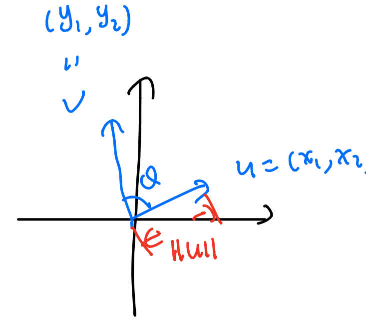
(Inner product. norm.).

① Dot product $\bullet$ on $\mathbb{R}^n$.

$v = (x_1, \dots, x_n)$, $\quad u = (y_1, \dots, y_n)$

$$\langle v, u \rangle = v \bullet u = \sum_{k=1}^{n} x_k y_k .$$

$$= (x_1, \dots, x_n) \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$



$$\|v\| = \sqrt{x_1^2 + \dots + x_n^2}$$

$$|\langle u, v \rangle| = \|u\| \cdot \|v\| \cdot \cos\theta$$

$\|u\|$

② Other inner product in $\mathbb{R}^n$.

$$\langle u, v \rangle = (x_1, \dots, x_n) \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

A

$$\langle u, v \rangle = (x_1, x_2) \begin{pmatrix} 2 & 0 \\ 0 & 10^{-5} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\langle u, u \rangle = (x_1, x_2) \begin{pmatrix} 2 & 0 \\ 0 & 10^{-5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2x_1^2 + 10^{-5} x_2^2 \geq 0$$

$\Updownarrow$

* A symmetric $\quad A^T = A \quad \begin{pmatrix} a & c \\ c & b \end{pmatrix}$

* A positively definite $\iff$ all the eigenvalues of A real > 0.

② (Hermitian) inner product. $\mathbb{C}^n$

$v = (x_1, \dots, x_n)$ $\qquad I_n$

$u = (y_1, \dots, y_n)$

$$\langle v, u \rangle = \sum_{k=1}^{n} x_k \overline{y_k} = (x_1, \dots, x_n) \begin{pmatrix} \overline{y_1} \\ \vdots \\ \overline{y_n} \end{pmatrix}$$

$$\langle v, v \rangle = \sum_{k=1}^{n} x_k \overline{x_k} = \sum_{k=1}^{n} |x_k|^2 \geq 0$$

⑥ $C^0[0,1]$ = space of continuous functions over $[0,1]$

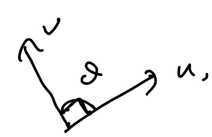
over $\mathbb{R}$

$$\langle f, g \rangle = \int_0^1 f(x)\, g(x)\, dx$$

$$\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_0^1 f^2(x)\, dx}$$

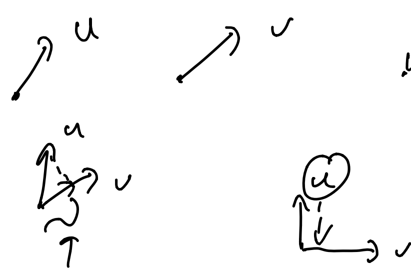
(6.11) $v \perp u$ if $\langle v, u \rangle = 0$

e.g. $\langle u, v \rangle = 0$



$|\langle u, v \rangle| = \|u\| \|v\| \cdot \cos\theta$

$\theta = 90$



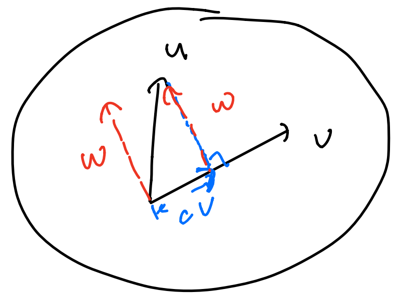
$u = \lambda v$

(6.14) Orthogonal decomposition

$u, v \in V \quad (v \neq 0)$

$c = \dfrac{\langle u, v\rangle}{\|v\|^2}$ and $w = u - cv$. Then

$\underline{\langle w, v\rangle = 0}$ $\quad (w \perp v)$ and $\boxed{u = w + cv}$



Pf $\langle w, v\rangle = 0$

$\langle u - cv, v\rangle$

$= \langle u, v\rangle - \langle cv, v\rangle$

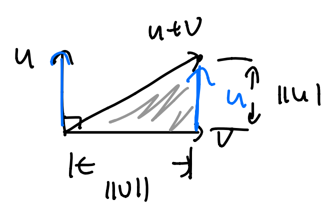
$= \langle u, v\rangle - c\,\langle v, v\rangle$

$= \langle u, v\rangle - \dfrac{\langle u, v\rangle}{\|v\|^2} \cdot \|v\|^2 = 0$

Q.

(6.13) $\underset{\Updownarrow}{\overset{\langle u, v\rangle = 0}{}} u \perp v$ Then $\|u+v\|^2 = \|u\|^2 + \|v\|^2$

(Pythagorean thm)

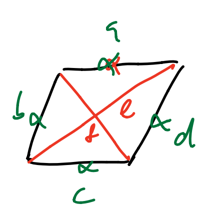


Pf $\langle u+v, u+v\rangle = \|u+v\|^2$

$= \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle$

$= \|u\|^2 + 0 + 0 + \|v\|^2$

(6.22) $a^2 + b^2 + c^2 + d^2 = e^2 + f^2$



(6.15) Cauchy - Schwartz inequality

$$|\langle u, v\rangle| \leq \|u\| \|v\|$$

The inequality is achieved $\Leftrightarrow$ $u = \lambda v$ for some $\lambda \in \mathbb{R}$.

Ex. ① $(\mathbb{R}^n, \bullet)$ $(x_1, \dots, x_n)$ $(y_1, \dots, y_n)$

$$\left|\sum_{k=1}^{n} x_k y_k\right| \leq \sqrt{x_1^2 + \cdots + x_n^2} \cdot \sqrt{y_1^2 + \cdots + y_n^2}$$

② $C[0,1]$ = cont. function over $[0,1]$

$$\langle f, g\rangle = \int_0^1 fg\, dx$$

$$\left|\int_0^1 fg\, dx\right| \leq \sqrt{\int_0^1 f^2 dx} \cdot \sqrt{\int_0^1 g^2 dx}$$

Pf $|\langle u, v\rangle| \leq \|u\| \|v\|$

① $v = 0$ trivially holds ✓

② $v \neq 0$

$u = w + cv$ where $\langle w, v\rangle = 0$, $c = \dfrac{\langle u, v\rangle}{\|v\|^2}$

$$\|u\|^2 = \|w + cv\|^2$$

$$= \|w\|^2 + \|cv\|^2$$

$$\geq \|cv\|^2$$

$\langle w, v\rangle = 0 \Rightarrow \langle w, cv\rangle$
$= \bar{c}\langle w, v\rangle = 0$

$$= |c|^2 \|v\|^2 = \frac{|\langle u,v\rangle|^2}{\|v\|^4} \cdot \|v\|^2$$

$$= \frac{|\langle u,v\rangle|^2}{\|v\|^2}$$

$$\|u\|^2 \geq \frac{|\langle u,v\rangle|^2}{\|v\|^2} \Rightarrow \|u\|^2\|v\|^2 \geq |\langle u,v\rangle|^2.$$

It is equal! $\Leftrightarrow$ $w=0$ $\Leftrightarrow$ $u=cv$.

$\square$