[8A] Generalized eigenvector & Generalized eigenspace.

$G(\lambda, T) = \{ v \in V \mid (T-\lambda I)^j (v) = 0 \text{ FOR some } j \}$

Eigenvalue $= \ker((T-\lambda I)^n)$ $\quad n = \dim(V)$

#1 $\boxed{8A}$ $T \in \mathcal{L}(\mathbb{C}^2)$ $\quad T(w, z) = (z, 0)$

Find all generalized eigenvectors of T.

Pf WRT the standard basis $e_1 = (1, 0)$ $e_2 = (0, 1)$

$\mathcal{M}(T) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Eigenvalue $\lambda = 0$ $\quad n = \dim(\mathbb{C}^2) = 2$

$G(0, T) = \ker\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2\right) = \ker\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) = \mathbb{C}^2.$

nilponent $\rightarrow$ $\boxed{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow$ All non-zero vectors are generalized eigenvector.

(8.16) An operator N in $\mathcal{L}(V)$ is called nilpotent if $N^k = 0$ for some $k \in \mathbb{Z}^+$. (Def'n)

(8.18) If $N \in \mathcal{L}(V)$ is a nilponent operator, then $N^{\dim(V)} = 0$.

PR $G(0, N) = \{ v \in V \mid \underline{N}^j (v) = 0 \text{ for some } j \}$

$\because$ N is nilpotent

$\therefore$ $\exists\, k$ s.t. $N^k = 0$ by definition.

$$\left.\begin{array}{l} \Rightarrow \; G(0, N) = V \\ \Rightarrow \; G(0, N) = \ker(N^{\dim V}) \end{array}\right\} \Rightarrow$$

$\ker(N^{\dim V}) = V \Rightarrow N^{\dim V} = 0$

$\square$

(8.19). N is a nilpotent operator. then $\exists$ a basis of V such that w.r.t. that basis

$$M(N) = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \qquad n = \dim(V)$$

Pf.

$$\begin{array}{ccccccc} & & & & & & V \\ & & & & & & \shortparallel \\ Null(N) & \subseteq & Null(N^2) & \subseteq & Null(N^3) \subseteq \cdots & \subseteq & Null(N^n) = \cdots \\ \shortparallel & & \shortparallel & & & & \shortparallel \\ span\{\underline{v_1}, v_2\} & & span\{v_1, v_2, v_3, v_4\} & & \cdots\cdots & & span\{v_1, v_2, v_3, v_4, v_5\} \end{array}$$

Claim. w.r.t. $\{v_1, v_2, \ldots, v_5\}$ a basis of V.

$M(N)$ in the desired form.

$N(v_1) = 0 \qquad N(v_2) = 0$

$N(v_3) \in Null(N) \Leftarrow N(N(v_3)) = N^2(v_3) = 0$

$\Leftarrow v_3 \in Null(N^2) \qquad N(v_4) \in Null(N)$

$N(v_3) = a_1 v_1 + a_2 v_2 \qquad N(v_4) = * v_1 + * v_2$

$v_5 \in Null(N^3)$ $\qquad$ $N(v_5) \in Null(N^2) = span\{v_1, v_2, v_3, v_4\}$

$N^2(N(v_5)) = N^3(v_5) = 0$

$N(v_5) = * v_1 + * v_2 + * v_3 + * v_4$

$$\mathcal{M}(N) = \begin{pmatrix} 0 & 0 & a_1 & * & * \\ 0 & 0 & a_2 & * & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{5\times 5}$$

Q.

8B1.

(8.21) Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \dots, \lambda_m$ be distinct eigenvalues of T. Then

a) $V = G(\lambda_1, T) \oplus G(\lambda_2, T) \oplus \dots \oplus G(\lambda_m, T)$

b) Each $G(\lambda_i, T)$ is invariant under T.

c) $(T - \lambda_i I)\big|_{G(\lambda_i, T)}$ is nilpotent, over $G(\lambda_i, T)$

(8.20) $T \in \mathcal{L}(V)$ $p(x)$ is a polynomial. Then $\text{Null}(p(T))$ & $\text{Range}(p(T))$ ARE BOTH INVARIANT under T.

Remark: FOR us $p(x) = (x - \lambda_i)^n$ $p(T) = (T - \lambda I)^n$

Pf: $v \in \text{Null}(p(T))$ WTS $T(v) \in \text{Null}(p(T))$

$$P(T) \cdot (T(v)) = T \cdot P(T)(v) = T(0) = 0$$

$v \in \text{Range}(P(T))$ WTS $T(v) \in \text{Range}(P(T))$

$\exists\, w \in V$ s.t.

$$v = P(T)(w) \Rightarrow T(v) = T \cdot P(T)(w)$$
$$= P(T)(T(w))$$
$$\in \text{Range}(P(T))$$

$\square$.

(8.21) Suppose V is a complex vector space and $T \in \mathcal{L}(V)$

Let $\lambda_1, \cdots, \lambda_m$ be distinct eigenvalues of T. Then

a) $V = G(\lambda_1, T) \oplus G(\lambda_2, T) \oplus \cdots \oplus G(\lambda_m, T)$?

b) Each $G(\lambda_i, T)$ is invariant under T. ✓

c) $(T - \lambda_i I)|_{G(\lambda_i, T)}$ is nilpotent. over $G(\lambda_i, T)$ ✓

pf b). $G(\lambda_i, T) = Null\left(\left(T - \lambda_i I\right)^n\right)$

$= Null\left(p(T)\right)$ w/ $p(x) = (x - \lambda_i)^n$

is invariant under T by (8.20).

c) Claim that

$$(T - \lambda_i I)^n \Big|_{G(\lambda_i, T)} = 0$$

$\Leftarrow \quad G(\lambda_i, T) = Null\left(\left(T - \lambda_i I\right)^n\right)$

$\Rightarrow \quad T - \lambda_i I \Big|_{G(\lambda_i, T)}$ is nilpotent over $G(\lambda_i, T)$

$\square$

a). Remark We know in general

$$G(\lambda_1, T) \oplus G(\lambda_2, T) \oplus \cdots \oplus G(\lambda_m, T) \neq V$$

① BASE CASE. (HW)

$n=1$. $T \in \mathcal{L}(V) \Rightarrow \underline{\underline{T(v) = \lambda v}}$ FOR some λ.

$V = G(\lambda, T)$ (DONE).

② Assume a) holds for V w/ $\dim(V) = n$

WTS a) holds for V of $\dim(V) = n+1$.

$\because$ V is a vector space over complex number,

$\exists$ an eigenvector v corresponding to an eigenvalue λ_1 of T.

$G(\lambda_1, T) \neq \{0\} \Rightarrow \dim(G(\lambda_1, T)) > 0$.

Note that

$$G(\lambda_1, T) = \text{null}((T - \lambda_1 I)^n)$$

$$U = \text{Range}((T - \lambda_1 I)^n)$$

Claim: $V = \underset{\text{inv.}}{G(\lambda_1, T)} \oplus \underset{\text{inv.}}{U}$.

<u>LEM</u> (8.5) $S \in \mathcal{L}(V)$ $n = \dim V$

$V = \text{null}(S^n) \oplus \text{Range}(S^n)$.

IN our case $S = (T - \lambda_1 I)$