Slope Detection and Toroidal Manifolds

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Background

Three properties

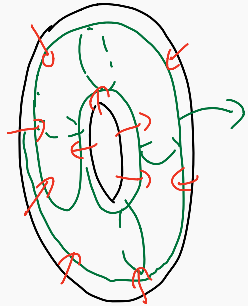
Y is a connected, orientable, irreducible 3-manifold.

(LO) $\pi_1(Y)$ is left-orderable.

$\exists$ a strict total order $<$ on $\pi_1(Y)$ satisfying $a < b \Leftrightarrow c \cdot a < c \cdot b$ for any $a, b, c \in \pi_1(Y)$.

(CTF) Y admits a co-orientable taut foliation.

- A (co-)orientable foliation $\mathcal{F}$ is a decomposition of Y into a disjoint union of orientable surfaces.
- $\mathcal{F}$ is **taut**, if for any $p \in Y$, $\exists$ a transverse loop passing p.



$\cong \mathbb{R}^2$

Reeb foliation.

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(NLS) Y is not an L-space.

Y is an L-space, if $|H_1(Y)| < \infty$ and $\widehat{HF}(Y)$ is "minimal".

The L-space conjecture

Conjecture (Boyer-Gordon-Watson, Juhász)

Y is a connected, compact, orientable, irreducible 3-manifold.

$$LO \iff CTF \iff NLS$$

Known:

- $CTF \implies NLS$ (Ozsváth-Szabó; Kazez-Roberts, Bowden)
- If $b_1(Y) > 0$, then Y is

  CTF (Gabai);
  LO (Boyer-Rolfsen-Wiest);
  NLS by definition.

Example: If $\partial Y \neq \emptyset$ and $Y \neq B^3$, then $b_1(Y) > 0$.

Slope detection

Rough ideas

Y is a toroidal $\mathbf{Q}$-homology sphere ($b_1(Y) = 0$).

essential $\sqcup T^2 \subset Y$

$\downarrow$ cut Y open along $\sqcup T^2$

$Y = M_1 \cup M_2 \cup \cdots \cup M_n$

$\partial M_i \neq \phi \Rightarrow M_i$ is CTF, LO, NLS

Question:

Under what boundary conditions is Y LO, CTF or NLS?

slope detection (Boyer-Clay, graph manifolds)

Slopes

M is a **knot manifold** (compact, connected, orientable, irreducible, $\partial M \cong T^2$ and $M \neq D^2 \times S^1$.)

A **rational slope** on ∂M is an isotopic class of simple closed curves on $\partial M \cong T^2$.

The **longitudinal slope** λ is unique.

- $[\lambda]$ has order $k > 0$ in $H_1(M)$;
- k copies of λ bounds an orientable surface F in M.

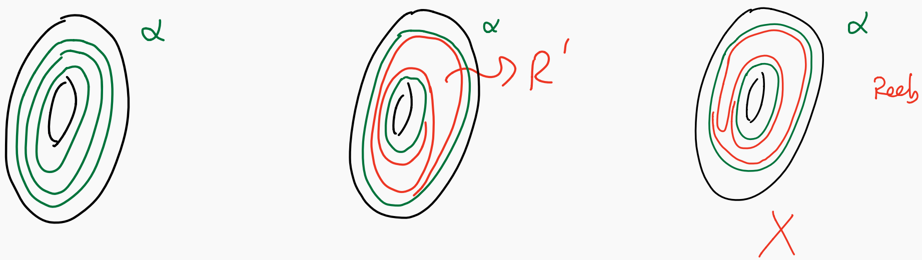
Any slope μ with $|\mu \cdot \lambda| = 1$ is called a **meridional slope**.

Slope detection

M is a knot manifold. A **rational slope** α on ∂M is $*$**-detected** if the following holds

$* =$ **CTF**:

$\exists$ a CTF $\mathcal{F}$ on M transverse to ∂M so that $\mathcal{F} \cap \partial M$ is **Reebless** and contains **a closed leaf of slope** α.



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M is a knot manifold. A **rational slope** α on ∂M is **$*$-detected** if the following holds

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$\exists$ a CTF $\mathcal{F}$ on M transverse to ∂M so that $\mathcal{F} \cap \partial M$ is **Reebless** and contains **a closed leaf of slope** α.

$* =$ LO:

$\exists$ a representation $\rho : \pi_1(M) \to \mathrm{Homeo}_+(\mathbb{R})$ satisfying that $\rho|_{\pi_1(\partial M)}$ has **no fixed point**, but $\rho(\alpha)$ **has a fixed point**. (NOT THE FORMAL DEFN)

$* =$ NLS:

$\alpha \notin$ the interior of **the set of the L-space slopes**.

Remark: $M(\alpha)$ is $* \Rightarrow \alpha$ is $*$-detected.

Basic gluing properties

Theorem

M_1 and M_2 are knot manifolds ($\neq D^2 \times S^1$). $Y = M_1 \cup_f M_2$ is irreducible. If $f(\alpha_1) = \alpha_2$ with both α_i are $*$-detected, then Y has property $*$.

- $* = NLS$;
  (Hanselman-Rasmussen-Watson, Rasmussen-Rasmussen)
- $* = LO$; (Boyer - Clay)
- $* = CTF$; (Boyer - Gordon - H.)

Conjecture

The converse of the gluing theorem holds.

Known if $*$ =NLS (HRW, RR)

Results

Detected slopes

Theorem (Boyer-Gordon-H.)

$M \neq D^2 \times S^1$ is an irreducible $\mathbb{Z}$-homology $D^2 \times S^1$. Then

- *all meridional slopes are $*$-detected if $* \in \{LO, NLS\}$.*
- *all meridional slopes are CTF-detected if M is fibered.*

Proof.

When $* =$ **NLS**, we use Rasmussen-Rasmussen.

When $* =$ **LO**,

$$\begin{tikzcd} \pi_1(M) \arrow[r, "\tilde{\rho}"] \arrow[dr, "\rho_{univ}"'] & \mathrm{Homeo}_+(\mathbb{R}) \arrow[d] \\ & \mathrm{Homeo}_+(S^1) \end{tikzcd}$$

$\tilde{\rho}|_{\partial M}$ has no fixed point but $\tilde{\rho}(\alpha)$ has one on $\mathbb{R}^1$

$\rho_{univ}|_{\partial M}$ has fixed points on S^1

Detected slopes

Theorem (Boyer-Gordon-H.)

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- *all meridional slopes are $*$-detected if $* \in \{LO, NLS\}$.*
- *all meridional slopes are CTF-detected if M is fibered.*

Proof.

When $* =$ **NLS**, we use Rasmussen-Rasmussen.

When $* =$ **LO**, we obtain a Universal circle action $\rho : \pi_1(M) \to \mathrm{Homeo}_+(S^1)$ from a finite depth foliation, satisfying that $\rho|_{\partial M}$ has a **fixed point**.

When $* =$ **CTF**, one needs a laminar branched surface B in M such that ∂B carries all **meridional slopes**. $\square$

Toroidal $\mathbb{Z}$-homology sphere

Conjecture (Ozsváth-Szabó)

Y is an irreducible $\mathbb{Z}$-homology sphere. If Y is an L-space, then Y is either S^3 or the Poincare sphere.

Theorem (Eftekhary, Hanselman-Rasmussen-Watson)

Y is an irreducible toroidal $\mathbb{Z}$-homology sphere. Then Y is not an L-space.

Theorem (Boyer-Gordon-H.)

$Y = M_1 \cup_f M_2$ is an irreducible toroidal $\mathbb{Z}$-homology sphere. Then

- *Y is LO.*
- *Y is CTF if one of M_i is fibered.*

<u>Remark:</u> Similar statements can be made if $|H_1(Y)| \leq 4$.

Cyclic branched covers of toroidal knots

Conjecture (Gordon-Lidman)

The n-fold cyclic branched cover of S^3 over a prime satellite knot is LO, CTF and NLS for $n \geq 2$.

Theorem (Boyer-Gordon-H.)

Suppose that $P(K)$ is a prime satellite knot with pattern P and companion K. Then the n-fold cyclic branched cover of S^3 branched cover $P(K)$ is

- *NLS and LO for $n \geq 2$;*
- *CTF if the companion knot K is fibered for $n \geq 2$.*

An application to knot theory

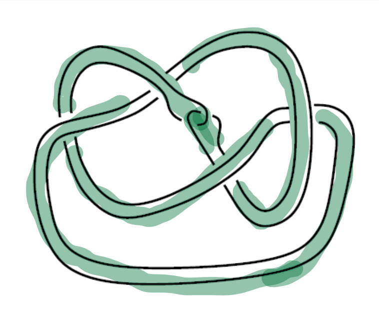

Figure 1: A satellite knot

Theorem (Menasco)

A prime satellite link is not alternating.

Theorem (Boyer-Gordon-H.)

A prime satellite link is not quasi-alternating.

Thank you