Euler class of taut foliations on QHS & Dehn Filling

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Euler class of taut foliations and Dehn filling, https://arxiv.org/abs/1912.01645. Dec 03, 2019 🎂

M is closed oriented irreducible 3-manifold.

M is a rational homology sphere (QHS)
$\Longleftrightarrow H_*(M, \mathbb{Q}) = H_*(S^3, \mathbb{Q})$.

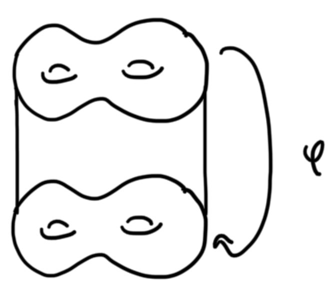
Codimension 1 foliation (co-orientable)

A decomposition of M into a disjoint union of of orientable surfaces. (leaves of the foliation)

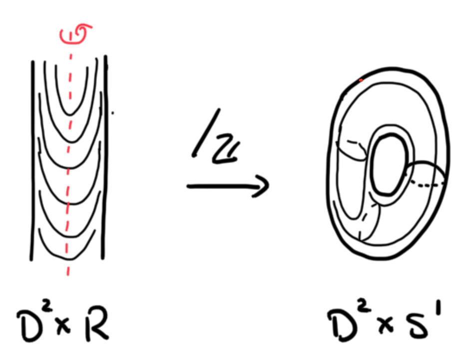
Assume leaves are C^∞ [Calegari].

Examples:

$F \to M \to S^1$



Reeb solid torus



* $\mathcal{F}$ is taut, if given any point p in M there exists a closed transverse loop passing p.

$T\mathcal{F}$: the tangent plane field of the foliation $\mathcal{F}$.

$T\mathcal{F} \hookrightarrow TM$, $\quad T\mathcal{F}|_p$ is tangent to the leaf at p

By assumption, $T\mathcal{F}$ is orientable.

* Foliation = integrable tangent plane field over M

Euler class e of an oriented plane field $E \to M$

A cohomology class in $H^2(M)$

= The obstruction to $\exists$ nowhere vanishing $\sigma : M \to E$

* $e(E \to M) = 0 \iff \exists \sigma \iff E \to M \underset{\text{iso}}{\cong} M \times \mathbb{R}^2 \to M.$

Question : When is $e(T\mathcal{F}) = 0$?

Question: When does $\dot{M}$ have a taut foliation $\mathcal{F}$ with $e(T\mathcal{F}) = 0$?

L-space conjecture

(Boyer-Gordon-Watson, Juhász, Ozsváth-Szabó)

M is an irreducible $\mathbb{Q}HS$. Then M admits a taut foliation $\Longleftrightarrow$ $\pi_1(M)$ has a left order. (LO)

an total order $<$ on $\pi_1(M)$ satisfying $a<b \Leftrightarrow ca<cb$

universal circle action $\pi_1(M) \curvearrowright S^1$

$\pi_1(M) \curvearrowright L$ 1-manifold simply connected non-Hausdorff.

$\Longleftrightarrow$ $\pi_1(M) \curvearrowright \mathbb{R}^1$ nontrivially (Boyer-Rolfsen-Wiest)

Thm: M is a $\mathbb{Q}HS$.
$\exists$ taut foliation $\mathcal{F}$ with $e(T\mathcal{F})=0$ $\Rightarrow$ $\pi_1(M)$ has a left-order.

(Thurston; Calegari-Dunfield, Candel, Plante. Boyer-H.)

Prop. (H.) $E \xrightarrow{\xi} M$ is a tangent plane field of M. If $H^2(M) = \bigoplus \mathbb{Z}_2$, then $e(\xi)=0$.

Cor. (H.) If $H^2(M) = \bigoplus \mathbb{Z}_2$ (including $H^2(M)=0$), then M admits a taut foliation $\Rightarrow$ $\pi_1(M)$ is LO.

Pf TM is trivial

$\Downarrow$

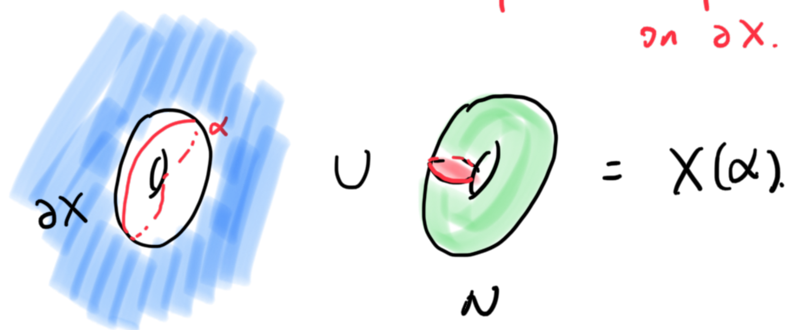
$e(\xi)$ is an "even" class $\in \mathrm{Im}\left(H^2(M,\mathbb{Z}) \xrightarrow{\cdot 2} H^2(M,\mathbb{Z})\right)$

$\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$.

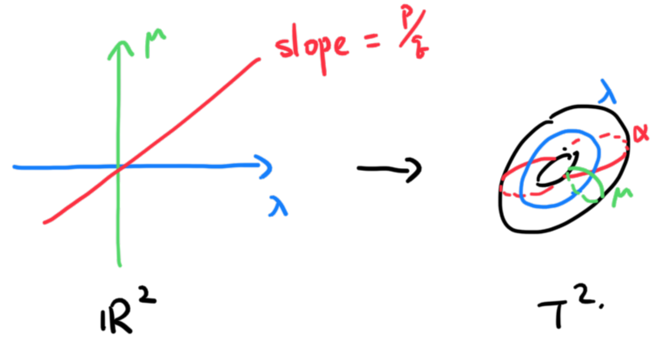
Dehn filling of X, $\partial X = T^2$

α - Dehn filling $X(\alpha)$

slope = simple closed curve on ∂X.



* $\alpha \longleftrightarrow p/q \in \mathbb{Q} \cup \{\frac{1}{0}\}$



* X is an $\mathbb{Z}H$ solid torus (i.e. $H_*(X) \cong H_*(S^1 \times D^2)$)

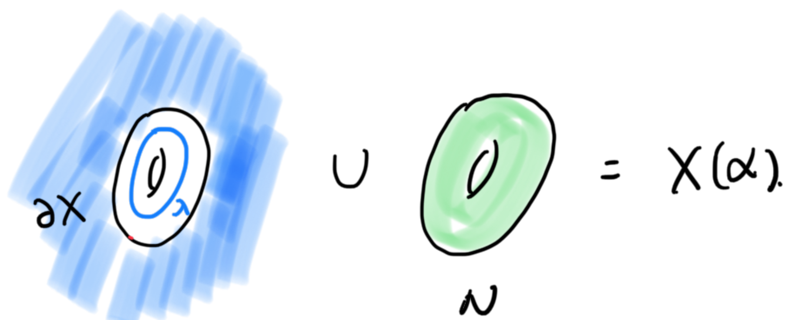
$H_2(X, \partial X) = \mathbb{Z} \cong [F]$. $\quad F \cap \partial X = \lambda$.

* $H^2(X(p/q)) \cong \mathbb{Z}_p$. $\quad (p \geq 0)$

Thm (H.) X is an $\mathbb{Z}H$ solid torus. $\mathcal{F}$ is a foliation on $X(\frac{p}{q}) = X \cup$  that is transverse to the core . Then

$e(T\mathcal{F}) = 0 \iff aq = 1 \pmod p$

Intuition:



$H^2(X) = 0$ $\qquad$ $H^2(N) = 0$

" $aq = 1 \pmod p \iff \exists\ \sigma_1 = X \to T\mathcal{F}|_X$

$\sigma_2 = N \to T\mathcal{F}|_N$

so that $\sigma_1 \cup \sigma_2 : X(\alpha) \to T\mathcal{F}$. "

$\uparrow$ measures the rotation of σ_1 along ∂F.

* If $\mathcal{F}$ is taut, then

$|a| \leq |\chi(F)|$ and a is odd. (Thurston)

Applications.

I. Compute the Euler class of taut foliations on Dehn fillings of the exterior .

① Fibered knots (Roberts , Krisha)

② Alternating knots (Roberts)

③ Any nontrivial knots (Li - Roberts)

$a = \pm \chi(F)$.

$\Rightarrow$ many new LO slopes for these knots

④ Persistently foliar knots (Delman - Roberts)

$a = ?$

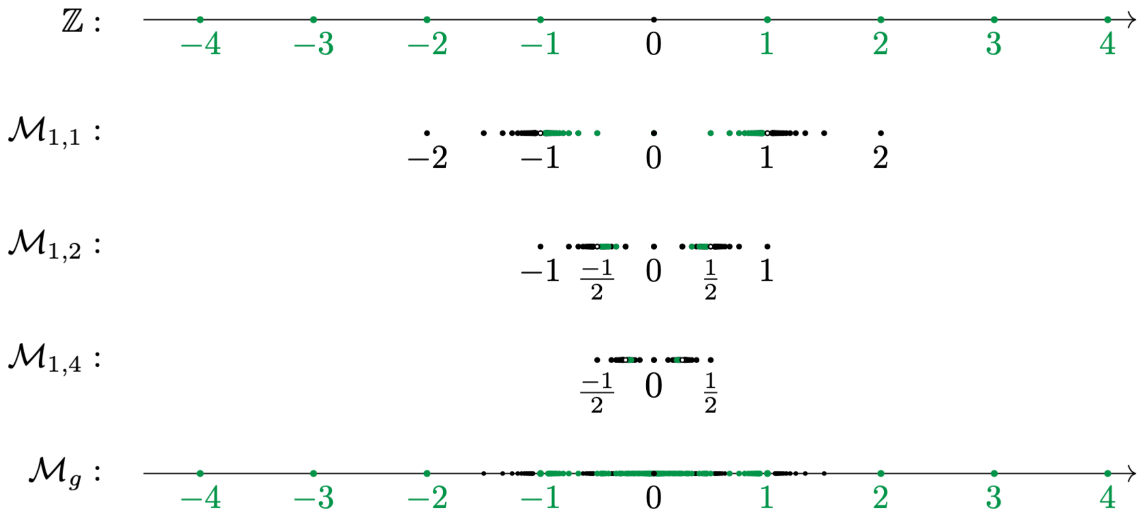
II Restriction on slopes.

S = The set of slopes α so that

$\exists \mathcal{F}$ on $X(\alpha)$, $e(T\mathcal{F}) = 0$ $\mathcal{F} \pitchfork$ core of N,

Thm (H.)

- S is nowhere dense
- $S \setminus (-2g, 2g) \subseteq \mathbb{Z}$, $g = g(F)$



$\mathbb{Z}$:
-4 -3 -2 -1 0 1 2 3 4
$\mathcal{M}_{1,1}$:
-2 -1 0 1 2
$\mathcal{M}_{1,2}$:
-1 $\frac{-1}{2}$ 0 $\frac{1}{2}$ 1
$\mathcal{M}_{1,4}$:
$\frac{-1}{2}$ 0 $\frac{1}{2}$
$\mathcal{M}_g$:
-4 -3 -2 -1 0 1 2 3 4