Recall

V is a finite dim vector space. $T \in \mathcal{L}(V)$

$\mathcal{M}(T, v_1, \cdots, v_n) = B$, $\mathcal{M}(T, u_1, \cdots, u_n) = A$

A and B are similar.

$$A = S^{-1} B S$$

$$S = \mathcal{M}(Id_V, (u_1, \cdots, u_n), v_1, \cdots, v_n)$$

$u_1 = S_{11} v_1 + \cdots + S_{n1} v_1$

$- - - -$

$u_n = S_{1n} v_1 + \cdots + S_{nn} v_n$

$$\begin{array}{ccc} u_1 & & u_n \\ \downarrow & & \downarrow \\ \left(\begin{matrix} S_{11} \\ \vdots \\ S_{n1} \end{matrix} \right. & \cdots & \left. \begin{matrix} S_{1n} \\ \vdots \\ S_{nn} \end{matrix} \right) \end{array}$$

Trace of a matrix.

$$A_{n \times n} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{22} \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}_{n \times n}$$

$$tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}.$$

(10.14). If A, B are both $n \times n$ matrices

$$\boxed{tr(AB) = tr(BA)} \qquad (AB \neq BA)$$

Pf. $tr(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} \left(a_{i1} b_{1i} + a_{i2} b_{2i} + \cdots + a_{in} b_{ni} \right)$

$$
\begin{array}{c} \\ i\text{th} \end{array}
\begin{pmatrix} & & \\ a_{i1} & \cdots & a_{in} \\ & & \end{pmatrix}
\begin{array}{c} i\text{th} \\ \begin{pmatrix} & b_{1i} & \\ & \vdots & \\ & b_{ni} & \end{pmatrix} \end{array}
$$

$A \qquad\qquad B$

$$= \sum_{i=1}^{n} \left(\sum_{k=1}^{n} a_{ik} b_{ki} \right)$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ki} a_{ik}$$

$$= \sum_{k=1}^{n} \sum_{i=1}^{n} b_{ki} a_{ik}$$

$$= \sum_{k=1}^{n} (BA)_{kk} = \underline{\underline{tr(BA)}}$$

(10.15) $T \in \mathcal{L}(V)$ $\{v_1, \cdots, v_n\}$ & $\{u_1, \cdots, u_n\}$ are two basis of V.

$$tr(\underbrace{\mathcal{M}(T, v_1, \cdots, v_n)}_{= A}) = tr(\underbrace{\mathcal{M}(T, u_1, \cdots, u_n)}_{= B})$$

Define $tr(\mathcal{M}(T))$ is the trace of T $\underline{\underline{tr(T)}}$

Pf

$A = S^{-1}BS$

$\underline{\underline{tr(A)}} = tr(S^{-1}BS) = tr(BS \cdot S^{-1}) = \underline{\underline{tr(B)}}$

$\square$

Remark $T \in \mathcal{L}(V)$. If $T: V \to V$ where V is a real vector space. Then we will consider $T_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$. in the following definition

"Def'n" $\quad tr(T) = \sum_{i=1}^{m} d_i \cdot \lambda_i$

where $\lambda_1, \dots, \lambda_m$ are distinction eigenvalues

& $d_i = \dim G(\lambda_i, T)$ multiplicity of λ_i.

pf. $\quad T \in \mathcal{L}(V) \qquad V$ is complex vector space.

$\exists$ a basis $\{v_1, \dots, v_n\}$ such that.

$$\mathcal{M}(T, v_1, \dots, v_n) = \begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_1 \end{matrix}} & & & \\ & \boxed{\begin{matrix} \lambda_2 & & \\ & \ddots & \\ & & \lambda_2 \end{matrix}} & & \\ & & \ddots & \\ & & & \lambda_m \end{pmatrix} \begin{matrix} d_1 \\ \\ d_2. \\ \\ \end{matrix}$$

$$tr(\mathcal{M}(T, v_1, \dots, v_n)) = \sum_{i=1}^{m} d_i \lambda_i$$

Q.

Ex. $\quad T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3.$

- $\mathcal{M}(T, e_1, e_2, e_3) = \underset{A}{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & -1 & 1 \end{pmatrix}} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3.$

$tr(A) = 1+1+1 = 3. = tr(T)$

- $T_{\mathbb{C}} : \mathbb{C}^3 \longrightarrow \mathbb{C}^3. \qquad tr(T) = \lambda_1 + \lambda_2 + \lambda_3 = 3$

$$\begin{cases} \lambda_1 = 1 \longrightarrow v_1 = e_1 \\ \lambda_2 = 1+2i \longrightarrow v_2 = \begin{pmatrix} -i \\ -4i \\ 2 \end{pmatrix} \\ \lambda_3 = 1-2i \end{cases}$$

$$v_3 = \begin{pmatrix} i \\ 4i \\ 2 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & & \\ & 1+2i & \\ & & 1-2i \end{pmatrix}}_{\mathcal{M}(T_{\mathbb{C}},\, e_1, v_2, v_3)} = S^{-1} \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & -1 & 1 \end{pmatrix}}_{\mathcal{M}(T_{\mathbb{C}},\, e_1\, e_2\, e_3)} S$$

$$S = \begin{array}{c} \begin{pmatrix} 1 & -i & i \\ 0 & -4i & 4i \\ 0 & 2 & 2 \end{pmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ e_1 & v_2 & v_3 \end{array} \end{array}$$

$e_1 = e_1$

$$v_2 = \begin{pmatrix} -i \\ -4i \\ 2 \end{pmatrix} = -i\, e_1 + (-4i) e_2 + 2 e_3$$

$$v_3 = \begin{pmatrix} i \\ 4i \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -i & i \\ 0 & -4i & 4i \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}} & \begin{matrix} \\ 1+2i \\ \\ \end{matrix} & \begin{matrix} \\ \\ 1-2i \end{matrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}} & \begin{matrix} -i \\ -4i \\ 2 \end{matrix} & \begin{matrix} i \\ 4i \\ 2 \end{matrix} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -i & i \\ 0 & -4i & 4i \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \lambda_1 e_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\boxed{10\text{B}}$ Determinant.

- <u>Determinant of a matrix.</u>

(10.27). • A permutation of $(1, \dots, n)$ is a list $(m_1, \dots, m_n)$ that contains each of the number $1, \dots, n$ once.

- Perm n = the set of all permutation of $(1, \dots, n)$.

  $|\text{Perm } n| = n!$

- sign of a permutation is

  $+$ if the natural order has been changed even # of time.

  $-$ otherwise.

e.g. $n=2$: $\{(1,2), (2,1)\}$ = perm 2.
$+$ $-$

$n=3$:

$(1,2,3)$ $+$ $\quad$ $(1,3,2)$ $-$

$(2,1,3)$ $-$ $\rightarrow$ $(2,3,1)$ $+$

$(3,2,1)$ $-$ $\rightarrow$ $(3,1,2)$ $+$

(10.33). Det (A)

Suppose that A is an $n \times n$ matrix.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

Then

$$\det A = \sum_{(m_1, \cdots, m_n)} \operatorname{sign}(m_1, \cdots, m_n)\, a_{m_1 1} a_{m_2 2} \cdots a_{m_n n}$$

① 1 entry from each col.

② no two factors come from the same row.

Ex. • $A = (10)_{1\times 1} \Rightarrow \det(A) = 10.$

• $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $\qquad \det\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = 1\cdot 2 - 0\cdot 1 = 2.$

$$\det A = \sum_{\{(1,2),\,(2,1)\}} \operatorname{sign}(\quad)\, a_{m_1 1} a_{m_2 2}$$

$$= + \underset{\substack{\uparrow \\ m_1 \\ \{(1}}}{a_{11}} \underset{\substack{\uparrow \\ m_2 \\ 2)\}}}{a_{22}} + - \underset{\substack{\uparrow \\ m_1 \\ (2,}}{a_{21}} \underset{\substack{\uparrow \\ m_2 \\ 1)}}{a_{12}} = a_{11} a_{21} - a_{12} a_{21}$$

(10.35). $A = \left(\begin{array}{c|ccc} a_{11} & * & * \cdots & * \\ \hline 0 & & & \\ 0 & & A' & \\ \vdots & & & \\ 0 & & & \end{array}\right)$ or $A = \left(\begin{array}{c|cccc} a_{11} & 0 & 0 & \cdots & 0 \\ \hline * & & & & \\ \vdots & & A' & & \\ * & & & & \end{array}\right)$

$$\det(A) = a_{11} \det A'$$

Row / col operations:

① (10.36) If A is obtained from B by switching 2 rows (or col) of B. then

$$\det A = -\det(B).$$

(*) $$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 1 \\ 2 & 5 & 0 \\ 3 & 7 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 5 & 0 \\ 1 & 4 & 1 \\ 3 & 7 & 0 \end{pmatrix}$$

② If A is obtained from B by adding a multiple of a row in B to another row of B.

$$\det(A) = \det(B).$$

$$\begin{pmatrix} 1 & 4 & 1 \\ 2 & 5 & 0 \\ 3 & 7 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 9 & 4 & 1 \\ 12 & 5 & 0 \\ 17 & 7 & 0 \end{pmatrix}$$

$\times 2$

Ex. $A = \left|\begin{pmatrix} 1 & 4 & 1 \\ 2 & 5 & -1 \\ 3 & 7 & 0 \end{pmatrix}\right| \qquad \det(A) = -6.$

$$\downarrow$$

$$= -\left|\begin{pmatrix} 1 & 4 & 1 \\ -1 & 5 & 2 \\ 0 & 7 & 3 \end{pmatrix}\right| \longrightarrow -\left|\begin{pmatrix} 1 & 4 & 1 \\ 0 & 9 & 3 \\ 0 & 7 & 3 \end{pmatrix}\right|$$

$$= -1 \cdot \begin{vmatrix} 9 & 3 \\ 7 & 3 \end{vmatrix}$$

$$= -(27 - 21) = -6$$