CE3-08 – Process Model Solution and Optimization

Problem Sheet for Topics #4 and #5

1. Consider the allocation LP model on slides #4 of Topic 4.

   (a) Implement and solve this model using GAMS. Analyze its solutions in terms of the optimal decision variables, optimal cost, and active constraints.

   Answers. A possible GAMS model implementation making use of SET, PARAMETER, TABLE, etc is as follows:

allocation1.gms

```
SETS
 P 'Products' / P1*P5 /
 M 'Machines' / M1*M4 /;

PARAMETER
 NM(M) 'Number of machines'
 / M1   4
   M2   5
   M3   3
   M4   7 /
 UP(P) 'Unit profit'
 / P1   18
   P2   25
   P3   10
   P4   12
   P5   15 /;

TABLE
 PT(M,P) 'Process times (hour)'
          P1    P2    P3    P4    P5
   M1     1.2   1.3   0.7   0.0   0.5
   M2     0.7   2.2   1.6   0.5   1.0
   M3     0.9   0.7   1.3   1.0   0.8
   M4     1.4   2.8   0.5   1.2   0.6;

SCALARS
 TTOTAL 'Weekly total time (hour)' / 40.0 /;

VARIABLES
 X(P)  'Weekly production quantity'
 Z     'Profit';

POSITIVE VARIABLE X;

EQUATIONS
 PROFIT     'Objective function'
 AVAIL(M)   'Maximal available time constraints';

PROFIT..    Z =E= SUM(P, UP(P)*X(P));
AVAIL(M)..  SUM(P, PT(M,P)*X(P)) =l= NM(M)*TTOTAL;

MODEL ALLOCATION / ALL /;
```

```
SOLVE ALLOCATION USING LP MAXIMIZING Z;
```

The following results can be found in the GAMS report:

allocation1.lst

```
LP status(1): optimal

                           LOWER          LEVEL          UPPER         MARGINAL

---- EQU PROFIT              .              .              .             1.0000

  PROFIT  Objective function

---- EQU AVAIL  Maximal available time constraints

          LOWER          LEVEL          UPPER         MARGINAL

M1        -INF          160.0000       160.0000         4.8195
M2        -INF          200.0000       200.0000         5.2016
M3        -INF          120.0000       120.0000         8.9635
M4        -INF          280.0000       280.0000         0.3631

---- VAR X  Weekly production quantity

          LOWER          LEVEL          UPPER         MARGINAL

P1          .            58.9614        +INF             .
P2          .            62.6346        +INF             .
P3          .              .            +INF          -13.5303
P4          .            10.5763        +INF             .
P5          .            15.6428        +INF             .

          LOWER          LEVEL          UPPER         MARGINAL

---- VAR Z                 -INF        2988.7270        +INF             .
  Z  Profit

**** REPORT SUMMARY :        0     NONOPT
                             0 INFEASIBLE
                             0  UNBOUNDED

EXECUTION TIME      =       0.001 SECONDS      2 MB  24.4.6 r52609 LEX-LEG
```

- The optimal solution value (profit) is $z^* \approx £2,989$.
- The corresponding optimal decisions are about 59.0 units for product 1, 62.6 units for product 2, 10.6 units for product 4, 15.6 units for product 5, and no production for product 3. This is not surprising as product 3 is indeed the one with the lowest unit price.
- All 4 maximal available time constraints are active, together with the minimal production (0) for product 3. Since the problem has 5 decision variables, the optimal solution is indeed at a extreme point of the feasible region (5 active constraints). The marginal prices for the time constraints are all positive, since increasing the

production timespan (or the number of machine) would result in a larger profit.

(b) Modify your GAMS model in order for the decision variables to be integers (as opposed to continuous variables), and solve the resulting integer linear program (ILP). Compare the optimal solution with that of the LP model.

Answers. A possible GAMS model implementation with minimal changes is as follows—lines containing modifications are highlighted in red:

allocation2.gms

```
SETS
 P 'Products' / P1*P5 /
 M 'Machines' / M1*M4 /;

PARAMETER
 NM(M) 'Number of machines'
 / M1   4
   M2   5
   M3   3
   M4   7 /
 UP(P) 'Unit profit'
 / P1   18
   P2   25
   P3   10
   P4   12
   P5   15 /;

TABLE
 PT(M,P) 'Process times (hour)'
          P1    P2    P3    P4    P5
   M1     1.2   1.3   0.7   0.0   0.5
   M2     0.7   2.2   1.6   0.5   1.0
   M3     0.9   0.7   1.3   1.0   0.8
   M4     1.4   2.8   0.5   1.2   0.6;

SCALARS
 TTOTAL 'Weekly total time (hour)' / 40.0 /;

VARIABLES
 X(P)  'Weekly production quantity'
 Z     'Profit';

INTEGER VARIABLE X;

EQUATIONS
 PROFIT     'Objective function'
 AVAIL(M)   'Maximal available time constraints';

PROFIT..    Z =E= SUM(P, UP(P)*X(P));
AVAIL(M)..  SUM(P, PT(M,P)*X(P)) =l= NM(M)*TTOTAL;

MODEL ALLOCATION / ALL /;
ALLOCATION.OPTCR = 0.0;

SOLVE ALLOCATION USING MIP MAXIMIZING Z;
```

The following results can be found in the GAMS report:

allocation2.lst

```
MIP status(101): integer optimal solution

Best possible:        2984.000000
Absolute gap:            0.000000
Relative gap:            0.000000

                           LOWER          LEVEL          UPPER         MARGINAL

---- EQU PROFIT              .              .              .             1.0000

  PROFIT  Objective function

---- EQU AVAIL  Maximal available time constraints

          LOWER          LEVEL          UPPER         MARGINAL

M1        -INF          159.2000       160.0000          .
M2        -INF          200.0000       200.0000          .
M3        -INF          120.0000       120.0000          .
M4        -INF          277.6000       280.0000          .

---- VAR X  Weekly production quantity

          LOWER          LEVEL          UPPER         MARGINAL

P1          .            58.0000        +INF           18.0000
P2          .            62.0000        +INF           25.0000
P3          .              .            +INF           10.0000
P4          .            10.0000        +INF           12.0000
P5          .            18.0000        +INF           15.0000

                           LOWER          LEVEL          UPPER         MARGINAL

---- VAR Z                 -INF         2984.0000        +INF             .
  Z  Profit

**** REPORT SUMMARY :        0     NONOPT
                             0 INFEASIBLE
                             0  UNBOUNDED
```

The integer optimal solution is close the fractional solution found previously. Moreover, the profit decrease slightly (from £2,989 down to £2,984), which is expected since the integrality constraint is indeed a restriction of the feasibility domain. The optimal quantities of products 1, 2 and 4 are all rounded down, whereas that of product 5 increases by a few units. This result illustrate the fact that rounding a fractional solution to the nearest integer may not provide an optimal solution to a mixed-integer program!

---

2. A cargo plane has three compartments for storing cargo: front, center, and rear. These compartments have the following limits on both weight and space:

Compartment	Weight Capacity [tonne]	Volume Capacity [cubic meter]
Front	10	6,800
Center	16	8,700
Rear	8	5,300

In order to maintain the balance of the plane, the weight of the cargo in the respective compartments must be the same proportion of that compartment's weight capacity.

The following four cargoes are available for shipment on the next flight:

Cargo	Weight [tonne]	Density [cubic meter/tonne]	Profit [£/tonne]
C1	18	480	310
C2	15	650	380
C3	23	580	350
C4	12	390	285

It is assumed that any proportion of these cargoes can be accepted.

The objective is to determine how much (if any) of each cargo C1, C2, C3 and C4 should be accepted and how to distribute each among the compartments so that the total profit for the flight is maximized.

(a) Formulate the above problem as a *linear* program (LP); in particular,

i. Motivate your choice of decision variables

ii. Give the mathematical expressions of all the objective and constraint functions

Answers First, we define two sets $I = \{\mathrm{f}, \mathrm{c}, \mathrm{r}\}$ and $J = \{1, 2, 3, 4\}$, which indexes the compartment location and cargo number, respectively. The decision variables for this optimization problem will be the fraction of cargo $j \in J$ in compartment $i \in I$ or x_{ij}. By considering the fraction of cargo in each compartment, in addition to the total profit, we can compute the amount of weight and space (if any) taken up by each piece of cargo in a specific compartment using linear calculations.

An possible formulation of the optimization problem is shown below.

$$
\begin{aligned}
\max_{x_{ij}} \quad & z = \sum_{j \in J} \sum_{i \in I} x_{ij} W_j P_j \\
\text{s.t.} \quad & \sum_{j \in J} x_{ij} W_j \le WC_i, \quad \forall i \in I, \\
& \sum_{j \in J} x_{ij} W_j \rho_j \le VC_i, \quad \forall i \in I, \\
& \frac{\sum_{j \in J} x_{ij} W_j}{\sum_{j \in J} \sum_{k \in I} x_{kj} W_j} = \frac{WC_i}{\sum_{k \in I} WC_k}, \quad \forall i \in I, \\
& \sum_{i \in I} x_{ij} \le 1, \quad \forall j \in J, \\
& x_{ij} \ge 0, \quad \forall i \in I, j \in J.
\end{aligned}
$$

Please note the definitions of W_j, ρ_j, P_j, WC_i, and VC_i from the table headings in the problem definition. Also worth noting is that the balance constraints can be re-written as follows to conform to an LP.

$$\sum_{j\in J} x_{ij}W_j = \frac{WC_i}{\sum_{k\in I} WC_k} \sum_{j\in J}\sum_{k\in I} x_{kj}W_j, \quad \forall i \in I.$$

An alternative LP formulation is shown below, whereby the decision variables are taken as the weight of cargo j in compartment i (as opposed to the fraction). Denoting these decision variables as y_{ij}, we have:[1]

$$\begin{aligned}
\max_{y_{ij}} \quad & z = \sum_{j\in J}\sum_{i\in I} y_{ij}P_j \\
\text{s.t.} & \sum_{j\in J} y_{ij} \le WC_i, \quad \forall i \in I, \\
& \sum_{j\in J} y_{ij}\rho_j \le VC_i, \quad \forall i \in I, \\
& \sum_{j\in J} y_{ij} = \frac{WC_i}{\sum_{k\in I} WC_k} \sum_{j\in J}\sum_{k\in I} y_{kj}, \quad \forall i \in I, \\
& \sum_{i\in I} y_{ij} \le W_j, \quad \forall j \in J, \\
& y_{ij} \ge 0, \quad \forall i \in I, j \in J.
\end{aligned}$$

(b) *Discuss the assumptions that are made in formulating this problem as a linear program.*

Answers One assumption made is that the weight and volume of a specific piece of cargo that is accepted can be represented as a fraction of the total weight and volume of that piece of cargo. For example, consider cargo 1, in some instances, it may not be possible to exactly accept the weight of cargo 1 represented by $\sum_i x_{i1}W_1$ or $\sum_i y_{i1}$. The same argument could be made for the volume. It is also assumed that the entire space in the compartments can be packed with cargo. In reality, there is likely to be dead space in each compartment which cannot be filled.

(c) Implement and solve this problem using GAMS. Make sure that you use the full features of the GAMS modeling language, including sets, parameters, tables, etc.

Answers A possible GAMS implementation is as follows:

Tutorial1-Q1c.gms

```
SETS
  i  compartment location / f,c,r /
  j  carge number         / C1*C4 /;
ALIAS (i,k);

```

[1] Note that $y_{ij} = x_{ij}W_j$

```
PARAMETERS
  WC(i)  weight capacity [tonne]
   / f  10
     c  16
     r   8 /
  VC(i)  volume capacity [cubic meter]
   / f  6800
     c  8700
     r  5300 /
  W(j)   weight [tonne]
   / C1  18
     C2  15
     C3  23
     C4  12 /
  D(j)   density [cubic meter per tonne]
   / C1  480
     C2  650
     C3  580
     C4  390 /
  P(j)   profit [pounds per tonne]
   / C1  310
     C2  380
     C3  350
     C4  285 /;

VARIABLES
  x(i,j)  fraction of cargo j in compartment i
  f       profit;

POSITIVE VARIABLES x;

EQUATIONS
  profit    objective function
  WCmax(i)  maximum weight capacity
  VCmax(i)  maximum volume capacity
  frac(j)   maximum fractions
  prop(i)   compartment proportions;

profit..    f =E= SUM((i,j),x(i,j)*W(j)*P(j));
WCmax(i)..  SUM(j,x(i,j)*W(j)) =L= WC(i);
VCmax(i)..  SUM(j,x(i,j)*W(j)*D(j)) =L= VC(i);
frac(j)..   SUM(i,x(i,j)) =L= 1;
prop(i)..   SUM(j,x(i,j)*W(j)) =E= SUM((k,j),x(k,j)*W(j))*WC(i)/SUM(k,WC(k));

MODEL cargo /ALL/;
SOLVE cargo USING LP MAXIMIZING f;
```

The following results can be found in the GAMS report:

Tutorial1-Q1c.lst

```
                           LOWER          LEVEL          UPPER         MARGINAL

---- EQU profit              .              .              .             1.0000
  profit  objective function

---- EQU WCmax  maximum weight capacity

         LOWER          LEVEL          UPPER         MARGINAL
```

```
f        -INF           10.0000        10.0000       350.0000
c        -INF           16.0000        16.0000       151.5789
r        -INF            8.0000         8.0000       350.0000

---- EQU VCmax  maximum volume capacity

         LOWER          LEVEL          UPPER         MARGINAL

f        -INF         6290.0000      6800.0000          .
c        -INF         8700.0000      8700.0000         0.3421
r        -INF         5200.0000      5300.0000          .

---- EQU frac  maximum fractions

          LOWER          LEVEL          UPPER         MARGINAL

C1        -INF             .             1.0000          .
C2        -INF            1.0000         1.0000       450.0000
C3        -INF            0.6934         1.0000          .
C4        -INF            0.2544         1.0000          .

---- EQU prop  compartment proportions

         LOWER          LEVEL          UPPER         MARGINAL

f          .              .              .              .
c          .              .              .              .
r          .              .              .              .

---- VAR x  fraction of cargo j in compartment i

            LOWER          LEVEL          UPPER         MARGINAL

f.C1          .              .            +INF        -720.0000
f.C2          .             0.4667        +INF            .
f.C3          .             0.1304        +INF            .
f.C4          .              .            +INF        -780.0000
c.C1          .              .            +INF        -104.2105
c.C2          .              .            +INF        -359.2105
c.C3          .             0.5629        +INF            .
c.C4          .             0.2544        +INF            .
r.C1          .              .            +INF        -720.0000
r.C2          .             0.5333        +INF            .
r.C3          .              .            +INF           EPS
r.C4          .              .            +INF        -780.0000

                          LOWER          LEVEL          UPPER         MARGINAL

---- VAR f                -INF        12151.5789        +INF             .
  f  profit

**** REPORT SUMMARY :        0     NONOPT
                             0 INFEASIBLE
                             0  UNBOUNDED

EXECUTION TIME       =        0.001 SECONDS      2 Mb  LEX237-237 Aug 23, 2011
```

(d) Describe and interpret the optimal solution.

Answers

- The optimal solution value (profit) is $z^* \approx £12,152$;
- The corresponding optimal decisions are as follows (only nonzero cargo fractions are reported):
  - front: $x_{\mathrm{f}2} \approx 0.4667$, $x_{\mathrm{f}3} \approx 0.1304$;
  - center: $x_{\mathrm{c}3} \approx 0.5629$, $x_{\mathrm{c}4} \approx 0.2544$;
  - rear: $x_{\mathrm{r}2} \approx 0.5333$, $x_{\mathrm{f}3} \approx 0.1304$;
- The optimal active set is such that:
  - all three compartments are at their maximum weight capacities;
  - the center compartment is at its maximum volume capacity;
  - all of Cargo 2 is shipped;
  - the front compartment contains no Cargo 1 or 4;
  - the center compartment contains no Cargo 1 or 2;
  - the rear compartment contains no Cargo 1, 3 or 4.

  All three equality constraints on the proportions of each compartment are naturally active too.

It is not surprising that all of Cargo 2 is shipped, given that this cargo generates the highest profit, although less dense than the other three cargoes. The fact that none of Cargo 1 is shipped, on the other hand, seems to result from the fact that its density is quite lower than that of Cargo 4, yet the corresponding difference in profit is rather small.

---

3. A company produces paper from new wood pulp, from recycled office paper, and from recycled newsprint. New pulp cost \$100 per ton; recycled office paper, \$50 per ton; and recycled newsprint, \$20 per ton. Four processes are available:

   - **Process 1** uses 3 tons of pulp to make 1 ton of paper;
   - **Process 2** uses 1 ton of pulp and 4 tons of recycled office paper to make 1 ton of paper;
   - **Process 3** uses 1 ton of pulp and 12 tons of recycled newsprint to make 1 ton of paper;
   - **Process 4** uses 8 tons of recycled office paper to make 1 ton of paper.

   At the moment, only 80 tons of pulp is available. The company wishes to produce 100 tons of new paper at minimum total cost.

   One of your colleagues provided the following GAMS™ implementation:

paper.gms

```
SETS
 I 'Paper Source' / PULP, RECOFF, RECNEW /
 J 'Process'      / P1*P4 /;

PARAMETER
 C(I) 'Raw Material Costs ($/ton)'
 / PULP    100.0
```

```
   RECOFF   50.0
   RECNEW   20.0 /;

TABLE
 Y(I,J) 'Raw Material Usage (ton/ton)'
            P1    P2    P3    P4
   PULP      3     1     1     0
   RECOFF    0     4     0     8
   RECNEW    0     0    12     0;

SCALARS
 PRODMIN 'Production Target (ton)' / 100.0 /
 PULPMAX 'Available Pulp (ton)'    /  80.0 /;

VARIABLES
 R(I) 'Amount of Raw Material Used (ton)'
 P(J) 'Amount of Paper Produced (ton)'
 Z    'Cost ($)';

POSITIVE VARIABLE R;
POSITIVE VARIABLE P;
R.UP('PULP') = PULPMAX;

EQUATIONS
 COST       'Objective function'
 BALANCE(I) 'Material Balance'
 TARGET     'Production Target';

COST..       Z =E= SUM(I, C(I)*R(I));
BALANCE(I).. R(I) =E= SUM(J, Y(I,J)*P(J));
TARGET..     SUM(J, P(J)) =G= PRODMIN;

MODEL PAPER / ALL /;
PAPER.OPTFILE=1;

SOLVE PAPER USING LP MINIMIZING Z;
```

He then solved the model using the LP solver CPLEX and obtained the following report:

paper.lst

```
G e n e r a l   A l g e b r a i c   M o d e l i n g   S y s t e m
Solution Report     SOLVE PAPER Using LP From line 43

---- VAR R  Amount of Raw Material Used (ton)

              LOWER          LEVEL          UPPER         MARGINAL

PULP            .            80.0000        80.0000      -100.0000
RECOFF          .           480.0000        +INF             .
RECNEW          .              .            +INF             .

---- VAR P  Amount of Paper Produced (ton)

          LOWER          LEVEL          UPPER         MARGINAL

P1          .              .            +INF         200.0000
P2          .            80.0000        +INF             .
P3          .              .            +INF          40.0000
```

```
P4          .           20.0000        +INF             .

                           LOWER          LEVEL          UPPER         MARGINAL

---- VAR Z                 -INF        32000.0000        +INF             .

  Z  Cost ($)

**** REPORT SUMMARY :        0     NONOPT
                             0 INFEASIBLE
                             0  UNBOUNDED

EQUATION NAME                       LOWER          CURRENT            UPPER
-------------                       -----          -------            -----
COST                                 -INF                0             +INF
BALANCE(PULP)                         -20                0               80
BALANCE(RECOFF)                      -480                0             +INF
BALANCE(RECNEW)                         0                0             +INF
TARGET                                 80              100             +INF


VARIABLE NAME                       LOWER          CURRENT            UPPER
-------------                       -----          -------            -----
R(PULP)                              -INF              100              200
R(RECOFF)                              25               50               60
R(RECNEW)                         16.6667               20             +INF
P(P1)                                -200               -0             +INF
P(P2)                                -INF               -0               40
P(P3)                                 -40               -0             +INF
P(P4)                                -100               -0             +INF
Z                                    -INF                1             +INF


                           LOWER          LEVEL          UPPER         MARGINAL

---- EQU COST                .              .              .             1.0000

  COST  Objective function

---- EQU BALANCE  Material Balance

              LOWER          LEVEL          UPPER         MARGINAL

PULP            .              .              .           200.0000
RECOFF          .              .              .            50.0000
RECNEW          .              .              .            20.0000

                           LOWER          LEVEL          UPPER         MARGINAL

---- EQU TARGET          100.0000       100.0000         +INF          400.0000

  TARGET  Production Target$
```

Answer each of the following questions, as well as possible, from the results given in the GAMS report. When you report numerical values, make sure to also report the correct units.

(a) What are the optimal solution point and cost value? Which constraints are active at the optimum?

Answers. The optimal solution point is as follows.

- **Raw Materials:** Use 80 tons of pulp (`PULP` $=$ `80`) and 480 tons of recycled office paper (`RECOFF` $=$ `480`), but no recycle newsprint (`RECNEW` $=$ `0`)
- **Paper Produced:** 80 tons by process 2 (`P2` $=$ `80`) and 20 tons by process 4 (`P4` $=$ `20`), but none by process 1 (`P1` $=$ `0`) and process 3 (`P3` $=$ `0`)

The corresponding minimum total cost is $\$32,000$. Seven constraints are active at the optimum solution point:

- The production target constraint of 100 tons;
- All three material balance constraints;
- The lower bound on the amount of recycle newsprint used;
- The upper bound on the amount of pulp used;
- The lower bounds on the amount of paper produced by process 1 and process 3;

(b) What is the marginal cost of paper production at the optimum?

Answers. The marginal cost of paper production is \$400/ton. (This value can be found under the section `---- EQU TARGET` of the GAMS report.)

(c) Determine or bound as well as possible how much optimal cost would change if the number of tons of new paper needed (i) increased to 200; (ii) decreased to 60.

Answers. In the case of an increase to 200 tons, one can apply quantitative sensitivity analysis since the allowable upper range is unlimited (`+INF`); the *exact* change in cost is: $100 \times 400 = \$40,000$.
In the case of a decrease to 60 tons, quantitative sensitivity analysis may no longer provides exact results since the allowable lower range in RHS is 80 tons. However, one can conclude that the change in cost is *at least* $(100 - 80) \times 400 = -\$8,000$, but *at most* $(100 - 60) \times 400 = -\$16,000$.

(d) Determine or bound as well as possible how much optimal cost would change if the price of pulp increased to \$150 per ton and that of recycled office paper decreased to \$40 per ton at the same time.

Answers. Since several cost coefficients change simultaneously, one can check the '100% rule' to decide whether or not quantitative information can be inferred from sensitivity analysis. The percentages of change with respect to maximal allowable changes are:

- $\frac{150-100}{200-100} = 50\%$ for the price of pulp;
- $\frac{40-50}{25-50} = 40\%$ for the price of recycled office paper.

Adding both changes give 90%, therefore quantitative sensitivity analysis *does* apply. The exact change in optimal cost is: $50 \times 80 - 10 \times 480 = -\$800$.

(e) Determine or bound as well as possible how much optimal cost would change if the price of recycled newsprint paper decreased to \$15 per ton.

Answers. In the case of a reduction to \$15/ton, one cannot infer exact quantitative changes from sensitivity analysis. Though it can be concluded, with absolute certainty, that such a reduction can only be beneficial in terms of the total cost—no reduction in total cost will be obtained in the worst case.

(f) How much should we be willing to pay to obtain an extra ton of pulp?

Answers. The marginal cost of pulp maximum availability is −\$100/ton. Paying \$200 for an extra ton of pulp would thus leave the total cost unchanged. This is the maximum we should be willing to pay for it.

---

4. You are the Operations Manager of a biorefinery producing biofuel by blending two raw oils and a filler to provide bulk. One kg of produced biofuel must contain a minimum quantity of each of four chemicals (denoted generically as A, B, C and D for confidentiality issues) as below:

	A	B	C	D
Content	90 g	50 g	20 g	2 g

The raw oils have the following chemical contents and unit costs:

	A	B	C	D	Unit Cost
Raw Oil 1	100 g/kg	80 g/kg	40 g/kg	10 g/kg	£0.1 /kg
Raw Oil 2	200 g/kg	150 g/kg	20 g/kg	–	£0.15 /kg

Moreover, the filler does not contain any of these chemicals and its cost is negligible compared to that of the raw oils. The blend sells at £0.5/kg.

(a) You are to determine the amounts of raw oils and filler in one kg of biofuel in order to maximize the biorefinery profit. Formulate this problem as a linear program (LP). Your model will be based on the following main decisions variables:

$$
\begin{aligned}
x_1 &:= \text{amount (kg) of Raw Oil 1 in one kg of biofuel blend}\\
x_2 &:= \text{amount (kg) of Raw Oil 2 in one kg of biofuel blend}\\
x_F &:= \text{amount (kg) of Filler in one kg of biofuel blend}
\end{aligned}
$$

Guidance:

- The use of alternative/additional decision variables is <u>not</u> permitted in this question, and <u>all</u> of the decision variables must appear in your model.
- Make sure to <u>carefully</u> justify the resulting expressions for the cost and constraints in your model, and to include all relevant constraints.

Answers. We first express each constraint in words, and then in terms of the imposed decision variables.

- One kg of produced biofuel must contain a minimum quantity of the chemicals A, B, C and D:

$$
\begin{aligned}
100x_1 + 200x_2 &\geq 90 \\
80x_1 + 150x_2 &\geq 50 \\
40x_1 + 20x_2 &\geq 30 \\
10x_1 &\geq 2.
\end{aligned}
$$

  Note the use of an inequality rather than an equality in these constraints since what we want are lower limits on the chemical contents in the blend. Note also that x_{F} does not participate in these constraints since the filler does not contain any of the chemicals.

- Implicit to the definition of the decision variables is the following balancing constraint:

$$x_1 + x_2 + x_{\mathrm{F}} = 1,$$

  so that the fractions of raw oils and filler sum up to exactly one kg.

- Nonnegativity constraints on the decision variables:

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_{\mathrm{F}} \geq 0.$$

- Profit function:

$$\max\ 0.5 - 0.1x_1 - 0.15x_2.$$

  The total profit (for one kg) is expressed as the benefit from product sales (first term), minus the cost of buying the raw oils (second term). This expression assumes that the cost of filler is negligible in comparison to that of the raw oils.

The complete linear program is as follows:

$$
\begin{aligned}
\max_{x_1, x_2, x_{\mathrm{F}}} \quad & 0.5 - 0.1x_1 - 0.15x_2 \\
\text{s.t.} \quad & x_1 + x_2 + x_{\mathrm{F}} = 1 \\
& 100x_1 + 200x_2 \geq 90 \\
& 80x_1 + 150x_2 \geq 50 \\
& 40x_1 + 20x_2 \geq 30 \\
& 10x_1 \geq 2 \\
& x_1 \geq 0, \quad x_2 \geq 0, \quad x_{\mathrm{F}} \geq 0.
\end{aligned}
$$

(b) One of your collaborators implemented an LP model for the above optimization problem in GAMS. The following lines can be found in the GAMS report:

report.lst

```
VARIABLE NAME                          LOWER         CURRENT           UPPER
-------------                          -----         -------           -----
X2                                      -0.2           -0.15           -0.05

              LOWER          LEVEL          UPPER          MARGINAL
---- VAR X2     .            0.1000         +INF              .
```

Determine or bound as well as possible what the change in profit would be if the unit cost of Raw Oil 2 were to: (i) increase to $£0.18$ /kg? (ii) increase to $£0.22$ kg? Make sure to carefully justify your answers and explain any calculations.

Answers. From inspection of the GAMS report, the cost coefficient for variable x_2 is -0.15. The negative sign here results from the fact that we are subtracting the price of raw material in the objective function.

- In the case of an increase of the unit price to $£0.18$/kg, the new cost coefficient would be $-0.18 \in [-0.20, -0.05]$, therefore one can apply quantitative sensitivity analysis. The exact change in cost is obtained by multiplying the nominal optimal value for x_2 (i.e., 0.1) with the variation in cost coefficient (i.e., $-£0.03/kg$), giving:

$$\Delta P = 0.1 \times (-0.03) = -0.003 \text{ (in } £/\text{kg)}.$$

  Note that an increase in raw oil price indeed incurs a decrease in profit.
- In the case of a further increase of the unit price to $£0.22$/kg, the new cost coefficient is $-0.22 \notin [-0.20, -0.05]$. Therefore, the exact change in cost can no longer be inferred from the sensitivity information. However, one can still conclude with certainty that the change in profit is:
  - **at least** $\Delta P = 0.1 \times (-0.05) = -0.005$ (in $£$/kg);
  - **at most** $\Delta P = 0.1 \times (-0.07) = -0.007$ (in $£$/kg).

(c) You have just obtained additional information from your subordinates. Due to processing and safety concerns, the use of any of Raw Oil 2 incurs a fixed cost of $£0.03$ (relative to one kg of biofuel). Moreover, the blend need not satisfy all four chemical constraints, but need only satisfy three of them (i.e., the blend could only meet any three of these constraints and violate the remaining one if it is worthwhile to do so). Revise your model to account for these additional specifications.

Guidance:

- Because the biorefinery does not have access to a nonlinear solver (MINLP), you are to formulate your model as a mixed-integer linear program (MILP). You can now choose any binary variable you want in your model, but since more binary variables can lead to longer computations, you will of course try and use as few as possible.
- Make sure to carefully justify the selected decision variables as well as the additional constraints in your model.

Answers. The additional information is incorporated by making the following additions/changes in the linear program of Question 1.

- To cope with the condition that if $x_2 \geq 0$ (i.e., when we use any of raw oil 2) we have a fixed cost of $£0.03$ incurred, we can use the standard trick of introducing a binary variable $y \in \{0, 1\}$ defined by:

$$y = \begin{cases} 1, & \text{if } x_2 \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

  Then, we subtract the term $0.03y$ to the objective function and add the constraint

$$x_2 \leq [\text{largest value } x_2 \text{ can take}]\, y.$$

In this case, it is easy to see that x_2 can never be greater than one, therefore our additional contraint reads:

$$x_2 \leq y.$$

- To cope with the condition that we need only satisfy three of the four chemical content constraints, we can introduce four binary variables $z_\mathsf{A}, z_\mathsf{B}, z_\mathsf{C}, z_\mathsf{D} \in \{0,1\}$ defined by:

$$z_i = \begin{cases} 1, & \text{if constraint on chemical content } i \in \{\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}\} \text{ is satisfed,} \\ 0, & \text{otherwise.} \end{cases}$$

  Then, we add the constraint:

$$z_\mathsf{A} + z_\mathsf{B} + z_\mathsf{C} + z_\mathsf{D} \geq 3,$$

  and alter the constraints on chemical contents to be

$$\begin{aligned} 100x_1 + 200x_2 &\geq 90\, z_\mathsf{A} \\ 80x_1 + 150x_2 &\geq 50\, z_\mathsf{B} \\ 40x_1 + 20x_2 &\geq 30\, z_\mathsf{C} \\ 10x_1 &\geq 2\, z_\mathsf{D}. \end{aligned}$$

The complete mixed-integer linear program is as follows:

$$\begin{aligned} \max_{\substack{x_1,x_2,x_\mathrm{F},y,\\ z_\mathsf{A},z_\mathsf{B},z_\mathsf{C},z_\mathsf{D}}} \quad & 0.5 - 0.1x_1 - 0.15x_2 - 0.03y \\ \text{s.t.} \quad & x_1 + x_2 + x_\mathrm{F} = 1 \\ & 100x_1 + 200x_2 \geq 90\, z_\mathsf{A} \\ & 80x_1 + 150x_2 \geq 50\, z_\mathsf{B} \\ & 40x_1 + 20x_2 \geq 30\, z_\mathsf{C} \\ & 10x_1 \geq 2\, z_\mathsf{D} \\ & x_2 \leq y \\ & z_\mathsf{A} + z_\mathsf{B} + z_\mathsf{C} + z_\mathsf{D} \geq 3 \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_\mathrm{F} \geq 0 \\ & y, z_\mathsf{A}, z_\mathsf{B}, z_\mathsf{C}, z_\mathsf{D} \in \{0,1\}. \end{aligned}$$

---

5. In the planning of the monthly production for the next 6 months, a chemical company must, in each month, operate either a normal shift or an extended shift – if it produces at all. The cost incurred by either type of shift is fixed by a union guarantee agreement and so is independent of the amount produced:

   - a normal shift costs £100,000 per month and can produce up to 5,000 units per month;
   - an extended shift costs £180,000 per month and can produce up to 7,500 units per month.

It is estimated that changing from a normal shift in one month to an extended shift in the next month costs an extra £15,000; no other cost is incurred, on the other hand, in changing from an extended shift in one month to a normal shift in the next month. Moreover, production constraints are such that, if the company produces anything in a particular month, it must produce at least 2,000 units.

The cost of holding stock is estimated to be £2 per unit and per month (based on the stock held at the end of each month), and the initial stock is 3,000 units (produced by a normal shift). The demand for the company's product in each of the next 6 months is estimated to be as shown below:

Month, t	1	2	3	4	5	6
Demand, D_t	6,000	6,500	7,500	7,000	6,000	6,000

Stockouts are not tolerated at any time. Finally, the amount in stock at the end of month 6 should not be less than 2,000 units.

You are to formulate a mixed-integer program that models the company's production plan for the next 6 months: minimize production costs, while satisfying production constraints.

- Your model will be based on the following main decisions variables:

$$x_t := \begin{cases} 1, & \text{if we operate a normal shift in month } t \ (t = 1, 2, \dots, 6) \\ 0, & \text{otherwise} \end{cases}$$

$$y_t := \begin{cases} 1, & \text{if we operate an extended shift in month } t \ (t = 1, 2, \dots, 6) \\ 0, & \text{otherwise} \end{cases}$$

$$P_t \ (\geq 0) := \text{the amount produced in month } t \ (t = 1, 2, \dots, 6)$$

  Three additional variables are also defined to ease model formulation:

$$z_t := \begin{cases} 1, & \text{if we switch from a normal shift in month } t-1 \text{ to an extended} \\ & \text{shift in month } t \ (t = 1, 2, \dots, 6) \\ 0, & \text{otherwise} \end{cases}$$

$$w_t := \begin{cases} 1, & \text{if we produce anything in month } t \ (t = 1, 2, \dots, 6) \\ 0, & \text{otherwise} \end{cases}$$

$$I_t \ (\text{URS}) := \text{the closing inventory (amount of stock left) at the end of month } t \ (t = 1, 2, \dots, 6)$$

  The use of alternative/additional decision variables is not permitted.

- Make sure to *carefully* justify the cost and constraint expressions in your model. Also make sure that *all* production constraints are enforced in your model.
- In formulating a relation between decision variables x, y and z, you may end up with a *nonlinear* constraint.

Answers. We first express each constraint in words, and then in terms of the variables defined above.

- At most one shift a month:

$$x_t + y_t \leq 1, \quad t = 1, 2, \dots, 6$$

  This constraint expresses a condition of mutual exclusiveness (see: knapsack models, in MILP formulation handouts)

- Production limits not exceeded:

$$P_t \le 5000x_t + 7500y_t, \quad t = 1, 2, \ldots, 6$$

Note the use of addition in the RHS of this constraint where we are making use of the fact that at most one of x_t and y_t can be one and other must be zero.

- Production either 0 or at least 2,000 units:

$$P_t \le 7500w_t, \quad t = 1, 2, \ldots, 6$$
$$P_t \ge 2000w_t, \quad t = 1, 2, \ldots, 6$$

It is easily verified that in the case that $w_t = 1$ then $2000 \le P_t \le 7500$, otherwise $P_t = 0$ (no production); note that the value of 7500 is used since it represents the most one can produce in a given period.

- No stockouts allowed:

$$I_t \ge 0, \quad t = 1, 2, \ldots, 6$$

- Amount of at least 2,000 units in stock at the end of month $t = 6$:

$$I_6 \ge 2000$$

- Balance constraints on inventory:

$$I_1 = 3000 + P_1 - D_1$$
$$I_t = I_{t-1} + P_t - D_t, \quad t = 2, 3, \ldots, 6$$

These balance constraints express the requirements that, during each period,

$$\text{closing stock} = \text{opening stock} + \text{production} - \text{demand}$$

In particular, it assumes that (i) opening stock in period t is equal to close stock in period $t-1$, and that (ii) production in period t is available to meet demand in that same period — i.e., no lag time between goods being produced and becoming available to meet demand. The balance constraint in period $t = 1$ uses the information that the initial stock is 3,000 units.

- Switching from normal to extended shift:

$$z_1 = y_1$$
$$z_t = x_{t-1}y_t, \quad 2, 3, \ldots 6$$

Note that z_t can only take the value of 1 when both x_{t-1} and y_t are themselves equal to 1; any other scenario results in z_t being equal to 0. The equality between z_1 and y_1 during the first time period follows from the knowledge that the initial stock has been produced by a normal shift.

- Cost objective function:

$$\min \sum_{t=1}^{6} 100000x_t + 180000y_t + 15000z_t + 2I_t$$

The total cost takes into account the normal/extended shift costs, the cost for change from normal to extended shift, and the cost of holding stock.

6. Prof. Sargent is trying to decide which of six needed tasks ($j = 1, \ldots, 6$) he will assign to his two GTAs ($i = 1, 2$) for his next Optimization course in the Spring. The estimated time needed to accomplish each task is t_j (considered the same for both GTAs) and the corresponding deadline is d_j. Each GTA undertaking a given task must complete it prior to considering another one, so no overlap is possible. Moreover, tasks 5 and 6 are related in that task 5 must be completed before starting task 6 and both should thus be assigned to the same assistant. Naturally, one GTA would probably be better at some tasks and the other GTA better at others; Prof. Sargent's scoring of their potentials is $p_{i,j}$ (a higher value of $p_{i,j}$ means a better potential).

   Help Prof Sargent and formulate an MIP model to decide an optimal schedule for the GTA work. The goal here is to maximize the potential of the assignment chosen, while satisfying the deadlines as well as the non-overlapping and precedence constraints. In particular, your model will use the decision variables:

$$x_j := \text{start time for task } j,$$
$$y_{i,j} := \begin{cases} 1, & \text{if task } j \text{ is carried out by GTA } i, \\ 0, & \text{otherwise} \end{cases}$$
$$z_{j,j'} := \begin{cases} 1, & \text{if tasks } j \text{ and } j' \text{ are carried out by the same GTA}, \\ 0, & \text{otherwise} \end{cases}$$
$$\omega_{j,j'} := \begin{cases} 1, & \text{if task } j \text{ starts before task } j', \\ 0. & \text{otherwise} \end{cases}$$

   Hint: You will need the big-M method as a means to reformulate disjunctive constraints. Also, some of the constraints in your MIP model may be nonlinear.

   Answers. We express each function in words and in terms of the variables defined above.

   - **Maximizing assignment potential:**
     The Objective function can be formulated as
     $$\max \sum_{i=1}^{2} \sum_{j=1}^{6} p_{i,j} y_{i,j}.$$
   - **Meeting deadlines:**
     $$x_j + t_j \leq d_j, \quad j = 1, 2, \ldots, 6$$
     This constraint expresses the condition that the completion time for each task, $x_i + t_i$, must be less than the corresponding deadline, d_i, for that task.
   - **Assigning tasks to GTAs:**
     $$\sum_{i=1}^{2} y_{i,j} = y_{1,j} + y_{2,j} = 1, \quad j = 1, 2, \ldots, 6$$
     This constraint enforces that each task must be assigned to exactly one GTA.
   - **Satisfying non-overlapping requirement:**
     Let j and j' be 2 tasks carried by the same GTA, i. The non-overlapping requirement can be expressed in disjunctive form as:
     $$x_j + t_j \geq x_{j'} \quad \vee \quad x_{j'} + t_{j'} \leq x_j.$$

Using the binary $\omega_{j,j'}$, a big-M reformulation of the previous disjunctive constraint yields the following pair of inequality constraints:

$$\begin{aligned} x_j + t_j &\leq x_{j'} + M\left(1 - \omega_{j,j'}\right) \\ x_{j'} + t_{j'} &\leq x_j + M\omega_{j,j'}, \end{aligned}$$

with M a large positive constant; e.g., $M = \max_j\{d_j\}$ (assuming that the tasks start at time $t = 0$).
Now, enforcing non-overlapping for those tasks $j < j'$ carried out the same GTA only can be done by premultiplying the left-hand side of the previous constraints by $z_{j,j'}$,

$$\begin{aligned} z_{j,j'}\left(x_j + t_j\right) &\leq x_{j'} + M\left(1 - \omega_{j,j'}\right) \\ z_{j,j'}\left(x_{j'} + t_{j'}\right) &\leq x_j + M\omega_{j,j'}. \end{aligned}$$

This way, the constraints remain unchanged in the case that $z_{j,j'} = 1$, whereas they are trivially satisfied if $z_{j,j'} = 0$ (since $x_j \geq 0$). Note that the foregoing constraints are no longer linear due to the presence of bilinear product terms $z_{j,j'}x_j$ and $z_{j,j'}x_{j'}$.

- **Addressing task 5 before task 6:**

$$x_5 \leq x_6$$

- **Assigning tasks 5 and 6 to the same GTA:**

$$z_{5,6} = 1$$

- **Relating $y_{i,j}$ and $z_{j,j'}$:**
One is not free to set the variable $y_{i,j}$ and $z_{j,j'}$ independently from each other. Instead, the following constraint must be satisfied:

$$z_{j,j'} = y_{1,j}y_{1,j'} + y_{2,j}y_{2,j'} = \sum_{i=1}^{2} y_{i,j}y_{i,j'},$$

for each $j, j' = 1, 2, \ldots, 6$. Observe that this constraint is again nonlinear in that it contains bilinear product terms.

On the whole, the (nonlinear) MIP model reads:

$$\boxed{\begin{aligned} \max_{\mathbf{x},\mathbf{y},\mathbf{z},\boldsymbol{\omega}} \quad & \sum_{i=1}^{2}\sum_{j=1}^{6} p_{i,j}y_{i,j} \\ \text{s.t.} \quad & \sum_{i=1}^{2} y_{i,j} = 1, \quad \forall j = 1, 2, \ldots, 6 \\ & z_{j,j'} = \sum_{i=1}^{2} y_{i,j}y_{i,j'}, \quad \forall j, j' = 1, 2, \ldots 6 \\ & z_{j,j'}\left(x_j + t_j\right) \leq x_{j'} + M\left(1 - \omega_{j,j'}\right), \quad \forall j, j' = 1, 2, \ldots 6 \\ & z_{j,j'}\left(x_{j'} + t_{j'}\right) \leq x_j + M\omega_{j,j'}, \quad \forall j, j' = 1, 2, \ldots 6 \\ & x_5 \leq x_6 \\ & z_{5,6} = 1 \\ & x_i \geq 0; \qquad y_{i,j}, z_{j,j'}, \omega_{j,j'} \in \{0, 1\}. \end{aligned}}$$

Notice that the number of binary decision variables could be easily reduced by considering $z_{j,j'}$ and $\omega_{j,j'}$ for $j < j'$ only.