CE3-08 – Process Model Solution and Optimization

Problem Sheet for Topics #4 and #5

1. Consider the allocation LP model on slides #4 of Topic 4.
   (a) Implement and solve this model using GAMS. Analyze its solutions in terms of the optimal decision variables, optimal cost, and active constraints.
   (b) Modify your GAMS model in order for the decision variables to be integers (as opposed to continuous variables), and solve the resulting integer linear program (ILP). Compare the optimal solution with that of the LP model.

2. A cargo plane has three compartments for storing cargo: front, center, and rear. These compartments have the following limits on both weight and space:

Compartment	**Weight Capacity** [tonne]	**Volume Capacity** [cubic meter]
Front	10	6,800
Center	16	8,700
Rear	8	5,300

   In order to maintain the balance of the plane, the weight of the cargo in the respective compartments must be the same proportion of that compartment's weight capacity.

   The following four cargoes are available for shipment on the next flight:

Cargo	**Weight** [tonne]	**Density** [cubic meter/tonne]	**Profit** [$/tonne]
C1	18	480	310
C2	15	650	380
C3	23	580	350
C4	12	390	285

   It is assumed that any proportion of these cargoes can be accepted.

   The objective is to determine how much (if any) of each cargo C1, C2, C3 and C4 should be accepted and how to distribute each among the compartments so that the total profit for the flight is maximized.

   (a) Formulate the above problem as a *linear* program (LP); in particular,
      i. Motivate your choice of decision variables
      ii. Give the mathematical expressions of all the objective and constraint functions
   (b) Discuss the assumptions that are made in formulating this problem as a linear program.

(c) Implement and solve this problem using GAMS. Make sure that you use the full features of the GAMS modeling language, including sets, parameters, tables, etc.

(d) Describe and interpret the optimal solution.

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3. A company produces paper from new wood pulp, from recycled office paper, and from recycled newsprint. New pulp cost \$100 per ton; recycled office paper, \$50 per ton; and recycled newsprint, \$20 per ton. Four processes are available:

   ◦ **Process 1** uses 3 tons of pulp to make 1 ton of paper;

   ◦ **Process 2** uses 1 ton of pulp and 4 tons of recycled office paper to make 1 ton of paper;

   ◦ **Process 3** uses 1 ton of pulp and 12 tons of recycled newsprint to make 1 ton of paper;

   ◦ **Process 4** uses 8 tons of recycled office paper to make 1 ton of paper.

   At the moment, only 80 tons of pulp is available. The company wishes to produce 100 tons of new paper at minimum total cost.

   One of your colleagues provided the following GAMS™ implementation:

paper.gms

```
SETS
 I 'Paper Source' / PULP, RECOFF, RECNEW /
 J 'Process'      / P1*P4 /;

PARAMETER
 C(I) 'Raw Material Costs ($/ton)'
 / PULP    100.0
   RECOFF   50.0
   RECNEW   20.0 /;

TABLE
 Y(I,J) 'Raw Material Usage (ton/ton)'
            P1    P2    P3    P4
   PULP      3     1     1     0
   RECOFF    0     4     0     8
   RECNEW    0     0    12     0;

SCALARS
 PRODMIN 'Production Target (ton)' / 100.0 /
 PULPMAX 'Available Pulp (ton)'    /  80.0 /;

VARIABLES
 R(I) 'Amount of Raw Material Used (ton)'
 P(J) 'Amount of Paper Produced (ton)'
 Z    'Cost ($)';

POSITIVE VARIABLE R;
POSITIVE VARIABLE P;
R.UP('PULP') = PULPMAX;

EQUATIONS
 COST       'Objective function'
 BALANCE(I) 'Material Balance'
```

```
 TARGET     'Production Target';

COST..       Z =E= SUM(I, C(I)*R(I));
BALANCE(I).. R(I) =E= SUM(J, Y(I,J)*P(J));
TARGET..     SUM(J, P(J)) =G= PRODMIN;

MODEL PAPER / ALL /;
PAPER.OPTFILE=1;

SOLVE PAPER USING LP MINIMIZING Z;
```

He then solved the model using the LP solver CPLEX and obtained the following report:

paper.lst

```
G e n e r a l   A l g e b r a i c   M o d e l i n g   S y s t e m
Solution Report     SOLVE PAPER Using LP From line 43

---- VAR R  Amount of Raw Material Used (ton)

              LOWER          LEVEL          UPPER         MARGINAL

PULP            .            80.0000        80.0000      -100.0000
RECOFF          .           480.0000        +INF             .
RECNEW          .              .            +INF             .

---- VAR P  Amount of Paper Produced (ton)

          LOWER          LEVEL          UPPER         MARGINAL

P1          .              .            +INF         200.0000
P2          .            80.0000        +INF             .
P3          .              .            +INF          40.0000
P4          .            20.0000        +INF             .

                          LOWER          LEVEL          UPPER         MARGINAL

---- VAR Z                -INF        32000.0000        +INF             .

  Z  Cost ($)

**** REPORT SUMMARY :        0     NONOPT
                             0 INFEASIBLE
                             0  UNBOUNDED

EQUATION NAME                            LOWER          CURRENT            UPPER
-------------                            -----          -------            -----
COST                                      -INF                0             +INF
BALANCE(PULP)                              -20                0               80
BALANCE(RECOFF)                           -480                0             +INF
BALANCE(RECNEW)                              0                0             +INF
TARGET                                      80              100             +INF


VARIABLE NAME                            LOWER          CURRENT            UPPER
-------------                            -----          -------            -----
R(PULP)                                   -INF              100              200
R(RECOFF)                                   25               50               60
R(RECNEW)                              16.6667               20             +INF
```

```
P(P1)                                   -200           -0            +INF
P(P2)                                   -INF           -0              40
P(P3)                                    -40           -0            +INF
P(P4)                                   -100           -0            +INF
Z                                       -INF            1            +INF


                           LOWER          LEVEL          UPPER         MARGINAL

---- EQU COST                .              .              .             1.0000

  COST  Objective function

---- EQU BALANCE  Material Balance

              LOWER          LEVEL          UPPER         MARGINAL

PULP            .              .              .           200.0000
RECOFF          .              .              .            50.0000
RECNEW          .              .              .            20.0000

                           LOWER          LEVEL          UPPER         MARGINAL

---- EQU TARGET          100.0000       100.0000        +INF          400.0000

  TARGET  Production Target
```

Answer each of the following questions, as well as possible, from the results given in the GAMS report. When you report numerical values, make sure to also report the correct units.

(a) What are the optimal solution point and cost value? Which constraints are active at the optimum?

(b) What is the marginal cost of paper production at the optimum?

(c) Determine or bound as well as possible how much optimal cost would change if the number of tons of new paper needed (i) increased to 200; (ii) decreased to 60.

(d) Determine or bound as well as possible how much optimal cost would change if the price of pulp increased to \$150 per ton and that of recycled office paper decreased to \$40 per ton at the same time.

(e) Determine or bound as well as possible how much optimal cost would change if the price of recycled newsprint paper decreased to \$15 per ton.

(f) How much should we be willing to pay to obtain an extra ton of pulp?

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4. You are the Operations Manager of a biorefinery producing biofuel by blending two raw oils and a filler to provide bulk. One kg of produced biofuel must contain a minimum quantity of each of four chemicals (denoted generically as A, B, C and D for confidentiality issues) as below:

	A	B	C	D
Content	90 g	50 g	20 g	2 g

The raw oils have the following chemical contents and unit costs:

	A	B	C	D	Unit Cost
Raw Oil 1	100 g/kg	80 g/kg	40 g/kg	10 g/kg	£0.1 /kg
Raw Oil 2	200 g/kg	150 g/kg	20 g/kg	–	£0.15 /kg

Moreover, the filler does not contain any of these chemicals and its cost is negligible compared to that of the raw oils. The blend sells at £0.5/kg.

(a) You are to determine the amounts of raw oils and filler in one kg of biofuel in order to maximize the biorefinery profit. Formulate this problem as a linear program (LP). Your model will be based on the following main decisions variables:

$$x_1 := \text{amount (kg) of Raw Oil 1 in one kg of biofuel blend}$$
$$x_2 := \text{amount (kg) of Raw Oil 2 in one kg of biofuel blend}$$
$$x_F := \text{amount (kg) of Filler in one kg of biofuel blend}$$

Guidance:

- The use of alternative/additional decision variables is not permitted in this question, and all of the decision variables must appear in your model
- Make sure to carefully justify the resulting expressions for the cost and constraints in your model, and to include all relevant constraints

(b) One of your collaborators implemented an LP model for the above optimization problem in GAMS. The following lines can be found in the GAMS report:

report.lst

```
VARIABLE NAME                               LOWER        CURRENT        UPPER
-------------                               -----        -------        -----
X2                                           -0.2          -0.15        -0.05

                    LOWER         LEVEL         UPPER         MARGINAL
---- VAR X2           .           0.1000        +INF              .
```

Determine or bound as well as possible what the change in profit would be if the unit cost of Raw Oil 2 were to: (i) increase to £0.18 /kg? (ii) increase to £0.22 kg? Make sure to carefully justify your answers and explain any calculations.

(c) You have just obtained additional information from your subordinates. Due to processing and safety concerns, the use of any of Raw Oil 2 incurs a fixed cost of £0.03 (relative to one kg of biofuel). Moreover, the blend need not satisfy all four chemical constraints, but need only satisfy three of them (i.e., the blend could only meet any three of these constraints and violate the remaining one if it is worthwhile to do so). Revise your model to account for these additional specifications.

Guidance:

- Because the biorefinery does not have access to a nonlinear solver (MINLP), you are to formulate your model as a mixed-integer linear program (MILP). You can now choose any binary variable you want in your model, but since more binary variables can lead to longer computations, you will of course try and use as few as possible.

- Make sure to carefully justify the selected decision variables as well as the additional constraints in your model

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5. In the planning of the monthly production for the next 6 months, a chemical company must, in each month, operate either a normal shift or an extended shift – if it produces at all. The cost incurred by either type of shift is fixed by a union guarantee agreement and so is independent of the amount produced:

   - a normal shift costs £100,000 per month and can produce up to 5,000 units per month;
   - an extended shift costs £180,000 per month and can produce up to 7,500 units per month.

   It is estimated that changing from a normal shift in one month to an extended shift in the next month costs an extra £15,000; no other cost is incurred, on the other hand, in changing from an extended shift in one month to a normal shift in the next month. Moreover, production constraints are such that, if the company produces anything in a particular month, it must produce at least 2,000 units.

   The cost of holding stock is estimated to be £2 per unit and per month (based on the stock held at the end of each month), and the initial stock is 3,000 units (produced by a normal shift). The demand for the company's product in each of the next 6 months is estimated to be as shown below:

Month, t	1	2	3	4	5	6
Demand, D_t	6,000	6,500	7,500	7,000	6,000	6,000

   Stockouts are not tolerated at any time. Finally, the amount in stock at the end of month 6 should not be less than 2,000 units.

   You are to formulate a mixed-integer program that models the company's production plan for the next 6 months: minimize production costs, while satisfying production constraints.

   - Your model will be based on the following main decisions variables:

   $$x_t := \begin{cases} 1, & \text{if we operate a normal shift in month } t \ (t = 1, 2, \ldots, 6) \\ 0, & \text{otherwise} \end{cases}$$

   $$y_t := \begin{cases} 1, & \text{if we operate an extended shift in month } t \ (t = 1, 2, \ldots, 6) \\ 0, & \text{otherwise} \end{cases}$$

   $$P_t \ (\geq 0) := \text{the amount produced in month } t \ (t = 1, 2, \ldots, 6)$$

   Three additional variables are also defined to ease model formulation:

   $$z_t := \begin{cases} 1, & \text{if we switch from a normal shift in month } t-1 \text{ to an extended} \\ & \text{shift in month } t \ (t = 1, 2, \ldots, 6) \\ 0, & \text{otherwise} \end{cases}$$

   $$w_t := \begin{cases} 1, & \text{if we produce anything in month } t \ (t = 1, 2, \ldots, 6) \\ 0, & \text{otherwise} \end{cases}$$

   $$I_t \ (\text{URS}) := \text{the closing inventory (amount of stock left) at the end of month } t \ (t = 1, 2, \ldots, 6)$$

   The use of alternative/additional decision variables is not permitted.

- Make sure to *carefully* justify the cost and constraint expressions in your model. Also make sure that *all* production constraints are enforced in your model.
- In formulating a relation between decision variables x, y and z, you may end up with a *nonlinear* constraint.

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6. Prof. Sargent is trying to decide which of six needed tasks ($j = 1, \ldots, 6$) he will assign to his two GTAs ($i = 1, 2$) for his next Optimization course in the Spring. The estimated time needed to accomplish each task is t_j (considered the same for both GTAs) and the corresponding deadline is d_j. Each GTA undertaking a given task must complete it prior to considering another one, so no overlap is possible. Moreover, tasks 5 and 6 are related in that task 5 must be completed before starting task 6 and both should thus be assigned to the same assistant. Naturally, one GTA would probably be better at some tasks and the other GTA better at others; Prof. Sargent's scoring of their potentials is $p_{i,j}$ (a higher value of $p_{i,j}$ means a better potential).

   Help Prof Sargent and formulate an MIP model to decide an optimal schedule for the GTA work. The goal here is to maximize the potential of the assignment chosen, while satisfying the deadlines as well as the non-overlapping and precedence constraints. In particular, your model will use the decision variables:

$$x_j := \text{start time for task } j,$$

$$y_{i,j} := \begin{cases} 1, & \text{if task } j \text{ is carried out by GTA } i, \\ 0, & \text{otherwise} \end{cases}$$

$$z_{j,j'} := \begin{cases} 1, & \text{if tasks } j \text{ and } j' \text{ are carried out by the same GTA,} \\ 0, & \text{otherwise} \end{cases}$$

$$\omega_{j,j'} := \begin{cases} 1, & \text{if task } j \text{ starts before task } j', \\ 0. & \text{otherwise} \end{cases}$$

   Hint: You will need the big-M method as a means to reformulate disjuntive constraints. Moreover, you mayend up with nonlinear constraints in your MIP model.
