Identifying the Connectivity of Feasible Regions for Optimal Decentralized Control Problems

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LQR Problem

- Consider a continuous-time linear system:

$$\dot{x}(t) = Ax(t) + Bu(t),$$

  with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$.
- Assume the initial state $x(0)$ is a random variable subject to some distribution $\mathcal{D}$.
- The goal is to find the optimal control input $u(t)$ minimizing the quadratic cost

$$\mathbb{E}_{x(0) \sim \mathcal{D}} \int_0^{+\infty} (x(t)^T Q x(t) + u(t)^T R u(t)) \mathrm{d}t,$$

  where Q is positive semidefinite and R are positive definite.

LQR Problem

$$
\begin{aligned}
\min \quad & \mathbb{E}_{x(0)\sim\mathcal{D}}\, J(x(0), K) \\
\text{s.t.} \quad & A - BK \text{ is a stable matrix.}
\end{aligned}
$$

- $J(x(0), K)$ is the cost with initial state $x(0)$ and control input $u(t) = -Kx(t)$.
- The constraint guarantees that the closed-loop system

$$\dot{x}(t) = (A - BK)x(t)$$

is stable and thus the objective is finite.

LQR Problem

$$\begin{aligned} \min \quad & \mathbb{E}_{x(0)\sim\mathcal{D}}\, J(x(0), K) \\ \mathrm{s.\,t.} \quad & A - BK \text{ is a stable matrix.} \end{aligned}$$

Solving by local search methods such as the gradient method:

- Low computational and memory complexities.
- Can be implemented without explicitly establishing the underlying model.

Optimality of Local Search Methods[1]

$$
\begin{aligned}
\min \quad & \mathbb{E}_{x(0)\sim\mathcal{D}}\, J(x(0), K) \\
\text{s.t.} \quad & A - BK \text{ is a stable matrix.}
\end{aligned}
$$

- This is a nonconvex optimization problem due to the constraint.
- Despite the nonconvexity, the problem has no spurious local minimums.
- Gradient descent will converge to the global optimal solution.

[1] M. Fazel, R. Ge, S. M. Kakade, and M. Mesbahi, "Global convergence of policy gradient methods for the linear quadratic regulator," in *Proc. Mach. Learn. Res.*, vol. 80, Jul. 2018, pp. 1467–1476.

Optimal Distributed Control

The optimal distributed control (ODC) problem with communication constraints:

$$\begin{aligned} \min \quad & \mathbb{E}_{x(0)\sim\mathcal{D}}\, J(x(0), K) \\ \mathrm{s.\,t.} \quad & A - BK \text{ is a stable matrix}, \\ & K_{ij} = 0, \quad \forall (i,j) \notin \mathscr{P}. \end{aligned}$$

- $\mathscr{P} \subseteq \{(i,j) | 1 \le i \le m, 1 \le j \le n\}$ is the set of free variables in the feedback gain K.
- For ODC problems, will local search methods converge to near-optimal solutions?
- In the rest of the talk, focus on the case when $m = n$, A is arbitrary and B is the identity matrix.

Notations

When B is identity, a pattern $\mathscr{P}$ can be represented by both a square matrix and a directed graph with possible self-loops.

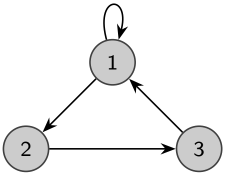
Example

$$\mathscr{P} = \{(1,1),(1,2),(2,3),(3,1)\}$$

can be described in the matrix form

$$\begin{bmatrix} * & * & 0 \\ 0 & 0 & * \\ * & 0 & 0 \end{bmatrix}$$

or equivalently in the graph form on the right.



Connectivity Properties of the Feasible Region

- The effectiveness of the local search methods depends on the connectivity properties of the feasible region.
- If the feasible region is connected, the trajectory of local search methods can always take feasible directions.
- Even with the presence of spurious local minimums, approaches such as stochastic gradient method can jump out these local minimums.
- If the feasible region is disconnected, to jump between different connected components the trajectory has to take infeasible directions.

An ODC Problem with Disconnected Feasible Region[2]

- The feasible region of the problem with $A = 0_n, B = I_n$ and an $n \times n$ pattern

$$\mathscr{P} = \begin{bmatrix} * & * & * & \dots & \dots & * \\ * & 0 & * & \dots & \dots & * \\ 0 & * & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & * \\ 0 & \dots & \dots & 0 & * & 0 \end{bmatrix}$$

  has 2^{n-1} connected components.
- The local search method cannot be effective for this problem since there could be an exponential number of local minimums that are far away from each other.

[2] H. Feng and J. Lavaei, "Connectivity properties of the set of stabilizing static decentralized controllers," *SIAM J. Control Optim.*, vol. 58, no. 5, pp. 2790–2820, 2020.

Checking Connectivity

Given an ODC problem with some pattern $\mathscr{P}$, how to check the connectivity of its feasible region?

- The feasible region is connected if the complement graph of $\mathscr{P}$ does not have self-loops.
- For more general patterns, can check the connectivity by reducing the pattern to its subpatterns.

The Connectivity Criterion

An ODC problem has connected feasible region if there exists a partition $\{S_1, \ldots, S_m\}$ of the vertices in its pattern $\mathscr{P}$ satisfying:

1. For any $1 \leq k < l \leq m$, there is no edge from S_l to S_k in the complement graph $\mathscr{P}^c$.
2. The complement graph of the subpattern S_1 does not have self-loops.
3. For each $k > 1$, if d_k is the number of vertices i in the original pattern $\mathscr{P}$ with the property

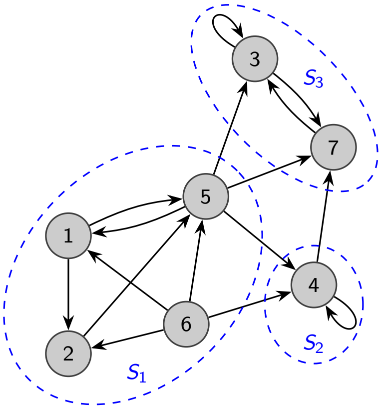
$$i \notin S_k \text{ and } \exists j \in S_k \text{ s.t. } (i, j) \in \mathscr{P}^c,$$

and r_k is the number of vertices in S_k, then

$$\sum_{l=1}^{k-1} r_l > d_k + r_k.$$

Example

A 7×7 pattern $\mathscr{P}$ whose complement graph $\mathscr{P}^c$ is given by:



Here

$$r_1 = 4, \quad r_2 = 1, \quad r_3 = 2, \quad d_2 = 2, \quad d_3 = 2.$$

Collararies of the Connectivity Criterion

- The connectivity criterion can be verified by an efficient algorithm based on topological sorting over the strongly connected components of the complement graph $\mathscr{P}^c$.
- Assume $\mathscr{P}$ is a pattern of size n. If the largest strongly connected component of its complement graph $\mathscr{P}^c$ has size r and the number of edges

$$|\mathscr{P}^c| \leq n - \max\{r, 2\},$$

  then the corresponding feasible region is connected.
- There are exponential number of connected patterns in which approximately half of the entries are "0".