이상적분

Improper Integral

$$\int_{-1}^{1} \frac{1}{x^2}\,dx \neq \left[-\frac{1}{x}\right]_{-1}^{1} = -2$$

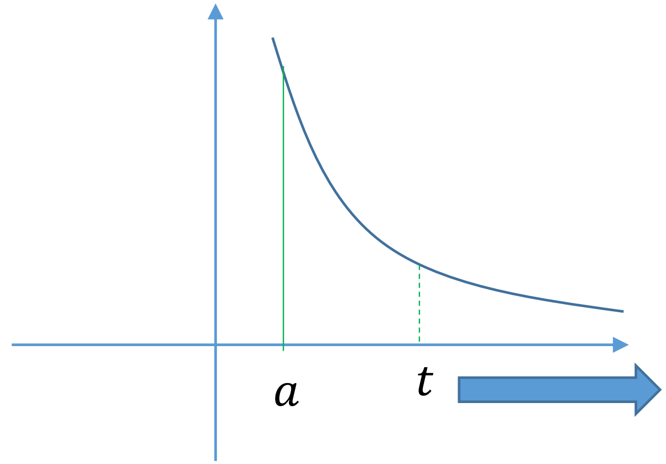
$\int_a^b f(x)\ dx$ 닫힌 구간 $[a, b]$ 에서, 연속인 함수 $f(x)$: 정적분

이상적분 (특이적분, Improper Integrals)

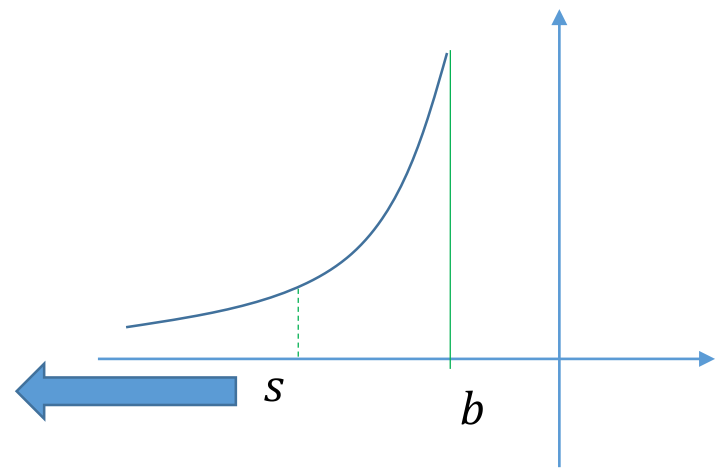
- 유형1. 닫힌 구간이 아닌 경우
- 유형2. 연속이 아닌 경우

정적분

(1) $$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$



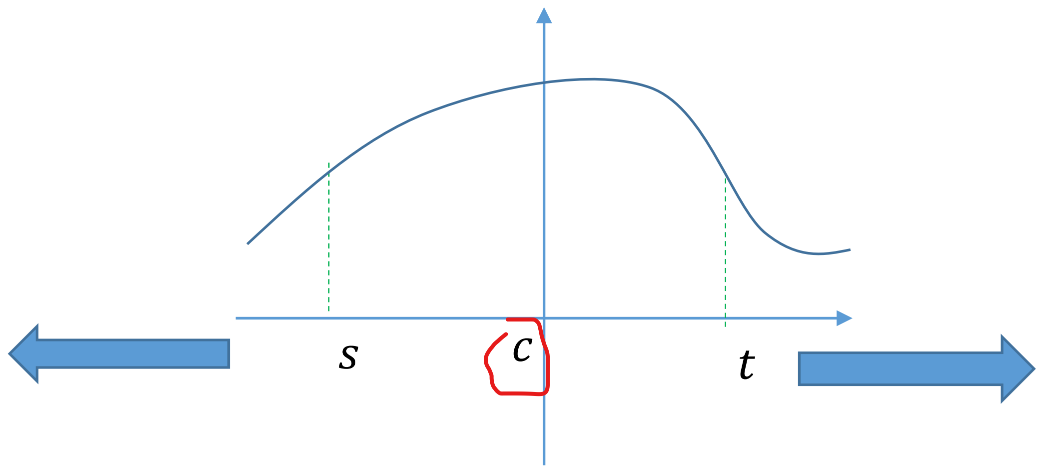
(2) $$\int_{-\infty}^{b} f(x)\,dx = \lim_{s\to-\infty}\int_s^b f(x)\,dx$$



$$(3) \quad \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

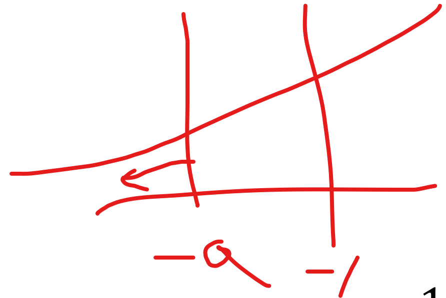
$$= \lim_{s\to-\infty}\int_{s}^{c} f(x)\,dx + \lim_{t\to\infty}\int_{c}^{t} f(x)\,dx$$

$$= \cancel{\lim_{\substack{s\to-\infty \\ t\to\infty}}\int_{s}^{t} f(x)\,dx}$$



(예제5.2, p420)

(1) $$\int_{-\infty}^{-1} xe^{-x^2}\,dx$$



(풀이) $$= \lim_{a\to-\infty}\int_a^1 x\,e^{-x^2}\,dx = \lim_{a\to-\infty}\left(-\frac{1}{2}e^{-1}+\frac{1}{2}e^{-a^2}\right) = -\frac{1}{2e}$$

$$-x^2 = u \qquad -2xdx = du$$

$$\int xe^{-x^2}\,dx \Rightarrow -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u$$

$$= \left[-\frac{1}{2}e^{-x^2}\right]_a^{-1} = -\frac{1}{2}e^{-1}+\frac{1}{2}e^{-a^2}$$

(3) $$\int_1^{\infty}\frac{1}{x}\,dx$$

(풀이) $$= \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty}[\ln x]_1^t = \lim_{t\to\infty}(\ln t - \ln 1) = \infty$$

예제 5.4 , p. 422 실수 p 에 대하여 $\int_1^\infty \frac{1}{x^p}dx$

풀이: $p=1$ 일 때 $\int_1^\infty \frac{1}{x}dx$ 은 발산

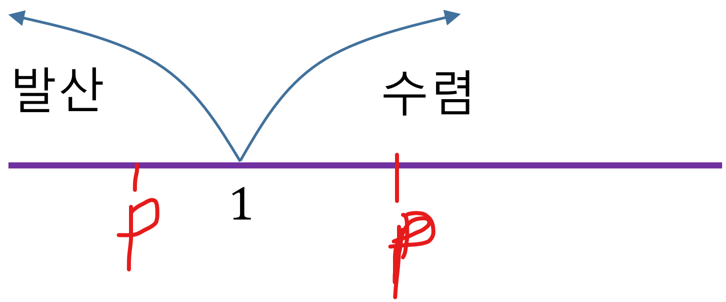
$p \neq 1$ 이라 하자.

$$\int_1^\infty \frac{1}{x^p}dx = \lim_{b\to\infty}\int_1^b x^{-p}\,dx = \lim_{b\to\infty}\left[\frac{1}{-p+1}x^{-p+1}\right]_1^b$$

$$= \frac{1}{1-p}\lim_{b\to\infty}\left(\frac{1}{b^{p-1}} - 1\right)$$

$$= \begin{cases} \infty, & p-1<0, \quad p<1 \quad \text{: 발산} \\ \dfrac{1}{p-1}, & p-1>0, \quad p>1, \quad \text{: 수렴} \end{cases}$$

2학기 p – 급수에서 활용

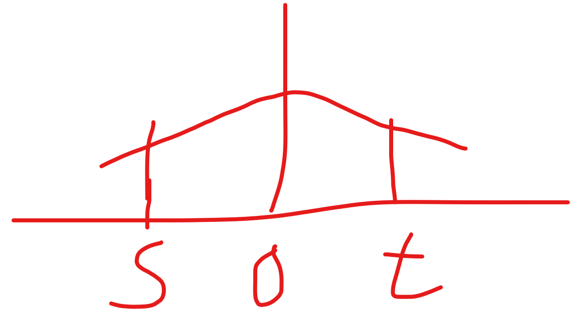


$$\int_1^{\infty} x^{-p}\, dx$$

$$\int_1^{\infty} \frac{1}{x^{1.4}}\, dx \qquad \int_1^{\infty} \frac{1}{x^{2}}\, dx \qquad \int_1^{\infty} \frac{1}{x^{3}}\, dx$$

$$\int_1^{\infty} \frac{1}{\sqrt{x}}\, dx \qquad \int_1^{\infty} x^{-0.7}\, dx \qquad \int_1^{\infty} x^{0.5}\, dx$$

(예제5.5, p422) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$



(풀이) $$= \int_{-\infty}^{0} \frac{1}{1+x^2} dx + \int_{0}^{\infty} \frac{1}{1+x^2} dx$$

$$= \lim_{s \to -\infty} \int_{s}^{0} \frac{1}{1+x^2} dx + \lim_{t \to \infty} \int_{0}^{t} \frac{1}{1+x^2} dx$$

$$= \lim_{s \to -\infty} [\tan^{-1} x]_s^0 + \lim_{t \to \infty} [\tan^{-1} x]_0^t$$

$$= \lim_{s \to -\infty} (\tan^{-1} 0 - \tan^{-1} s) + \lim_{t \to \infty} (\tan^{-1} t - \tan^{-1} 0)$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Cauchy 의 확률밀도함수 : $f(x) = \dfrac{k}{1+x^2}$ 여기서 $k = \dfrac{1}{\pi}$

$\displaystyle\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = 1$ 이중 적분, 극형식 : 2학기 $\displaystyle\int_{0}^{\infty} f(x)\, e^{-sx} dx$ Laplace 변환

(예제) $\int_{-\infty}^{\infty} f(x)\,dx \qquad \lim_{t\to\infty}\int_{-t}^{t} f(x)\,dx$

$$\int_0^{\infty} \frac{2x}{x^2+1}\,dx = \lim_{t\to\infty}[\ln(x^2+1)]_0^t = \lim_{t\to\infty}\ln(t^2+1) = \infty$$

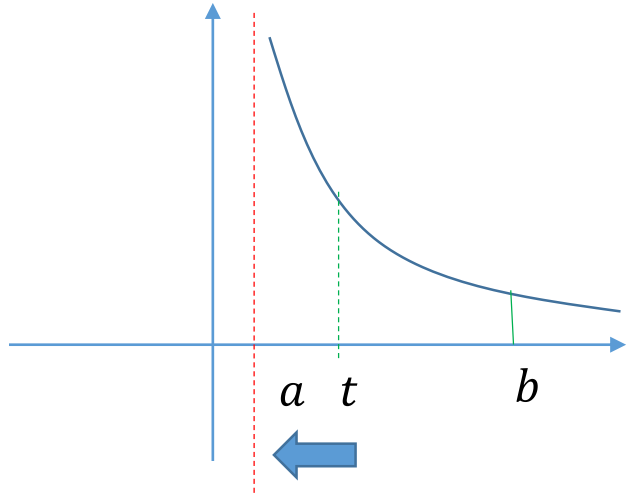
$$\lim_{t\to\infty}\int_{-t}^{t} \frac{2x}{x^2+1}\,dx = \lim_{t\to\infty}[\ln(x^2+1)]_{-t}^{t}$$

$$= \lim_{t\to\infty}(\ln(t^2+1) - \ln((-t)^2+1) \qquad = 0$$

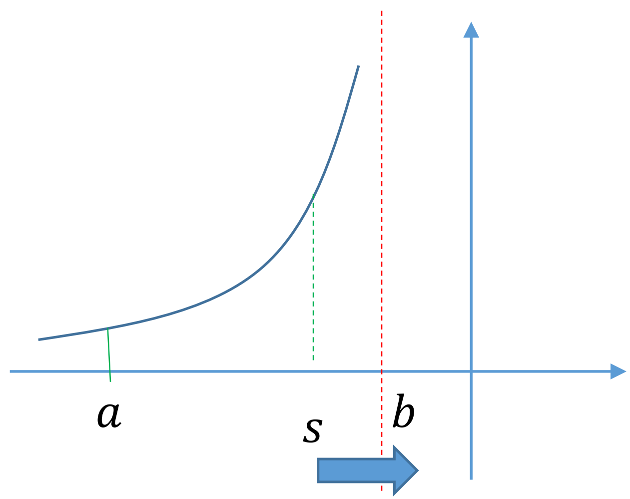
어떤 계산이 맞습니까? 1

정적분

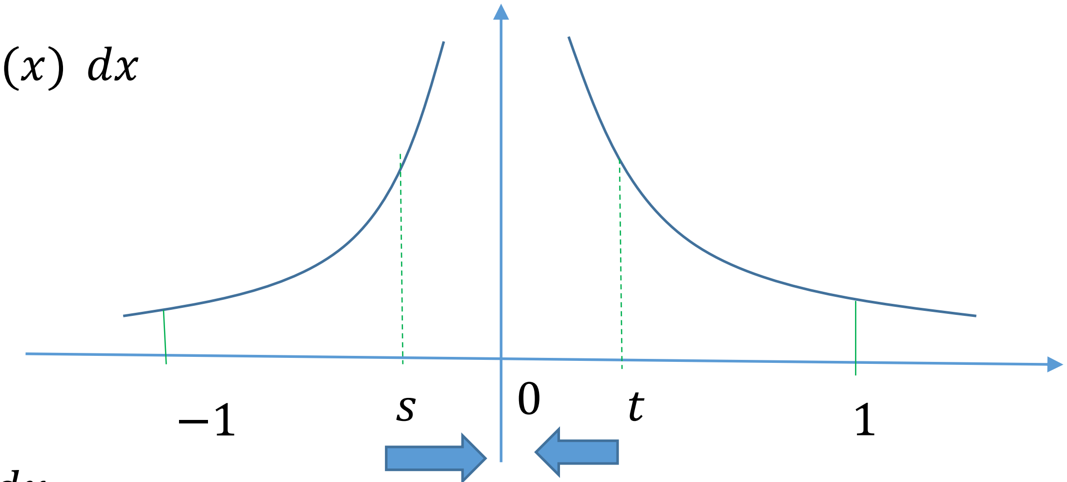
(4) $$\int_a^b f(x)\,dx = \lim_{t \to a} \int_t^b f(x)\,dx$$



(5) $$\int_a^b f(x)\,dx = \lim_{s \to b} \int_a^s f(x)\,dx$$



(6) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx$$

$$= \lim_{s\to 0} \int_{-1}^{s} \frac{1}{x^2}\,dx + \lim_{t\to 0} \int_{t}^{1} \frac{1}{x^2}\,dx$$

$$= \lim_{s\to 0} \left[-\frac{1}{x}\right]_{-1}^{s} + \lim_{t\to 0} \left[-\frac{1}{x}\right]_{t}^{1}$$

$$= \lim_{s\to 0} \left(-\frac{1}{s} - 1\right) + \lim_{t\to 0} \left(-1 + \frac{1}{t}\right) = \infty$$

발산 (diverge)

수렴 (converge)

예제 5.8, p 426

$$(1) \int_0^1 \frac{1}{\sqrt{x}}\,dx \neq [2\sqrt{x}]_0^1 = 2$$

$$= \lim_{a\to 0+}\left[\int_a^1 \frac{1}{\sqrt{x}}\,dx\right] = \lim_{a\to 0+}[2-2\sqrt{a}] = 2$$

$$(2) \int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx \quad = \lim_{b\to 2^-}\int_0^b \frac{1}{\sqrt{4-x^2}}\,dx = \lim_{b\to 2^-}\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^b = \frac{\pi}{2}$$

$$(4) \int_0^3 \frac{1}{(x-1)^{2/3}}\,dx \;\neq \left[3(x-1)^{\frac{1}{3}}\right]_0^3 = 3\sqrt[3]{2} - 3\sqrt[3]{-1}$$

$$= \int_0^1 \frac{1}{(x-1)^{2/3}}\,dx + \int_1^3 \frac{1}{(x-1)^{2/3}}\,dx$$

$$= \lim_{t\to 1^-}\int_0^t \frac{1}{(x-1)^{2/3}}\,dx + \lim_{s\to 1^+}\int_s^3 \frac{1}{(x-1)^{2/3}}\,dx$$

$$= \lim_{t\to 1^-}\left[3(x-1)^{\frac{1}{3}}\right]_0^t + \lim_{s\to 1^+}\left[3(x-1)^{1/3}\right]_s^3 = 3 + 3\sqrt[3]{2}$$

(예제) $$\int_0^3 \frac{1}{x-1}\,dx$$

(풀이) $$\cancel{= \left[\ln|x-1|\ \right]_0^3 = \ln 2 - \ln 1 = \ln 2}$$

$$= \int_0^1 \frac{1}{x-1}\,dx + \int_1^3 \frac{1}{x-1}\,dx$$

$$= \lim_{t\to 1} \int_0^t \frac{1}{x-1}\,dx + \lim_{t\to 1} \int_t^3 \frac{1}{x-1}\,dx$$

$$= \lim_{t\to 1}\left[\ln|x-1|\right]_0^t = \lim_{t\to 1} \ln|t-1| = -\infty$$

$\int_0^1 \frac{1}{x-1}\,dx$ 가 발산 하므로 $\int_0^3 \frac{1}{x-1}\,dx$ 도 발산한다.

Comparison theorem $\quad$ f 와 g가 $x = a$ 에서 연속이고 $\quad f(x) \geq g(x) \geq 0$

(1) $\displaystyle\int_a^\infty f(x)\ dx$ 가 수렴하면 $\displaystyle\int_a^\infty g(x)\ dx$ 도 수렴한다.

(2) $\displaystyle\int_a^\infty g(x)\ dx$ 가 발산하면 $\displaystyle\int_a^\infty f(x)\ dx$ 도 발산한다.

(예제) $\displaystyle\frac{1+e^x}{x} > \frac{1}{x} \longrightarrow \int_1^\infty \frac{1}{x} dx = \infty$ 따라서 $\displaystyle\int_1^\infty \frac{1+e^x}{x} dx = \infty$

(예제) $\displaystyle\frac{1+\sin^2 x}{\sqrt{x}} > \frac{1}{\sqrt{x}} \longrightarrow \int_1^\infty \frac{1}{\sqrt{x}} dx = \infty$ 따라서 $\displaystyle\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}} dx = \infty$

혼자 해보기

1. $\displaystyle\int_e^{\infty} \frac{1}{x(\ln x)^3} dx$ $\qquad \ln x = t \qquad \frac{1}{2}$

2. $\displaystyle\int_0^{1} \frac{3}{x^5} dx$ $\qquad \infty \qquad \int_0^1 \frac{1}{x^p} dx$

3. $\displaystyle\int_{-\infty}^{2} \frac{2}{x^2+4} dx$ $\qquad \frac{3\pi}{4}$

4. $\displaystyle\int_0^{\pi} \frac{\sin^2 x}{\sqrt{x}} dx$ 수렴, 발산 판정 수렴