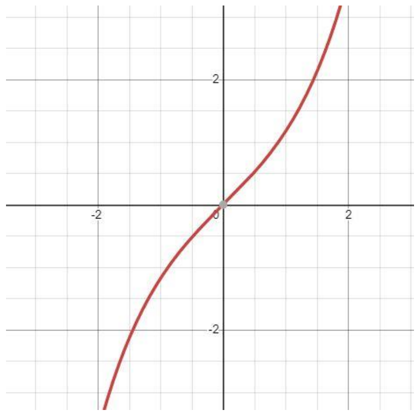
3.4 쌍곡선 함수

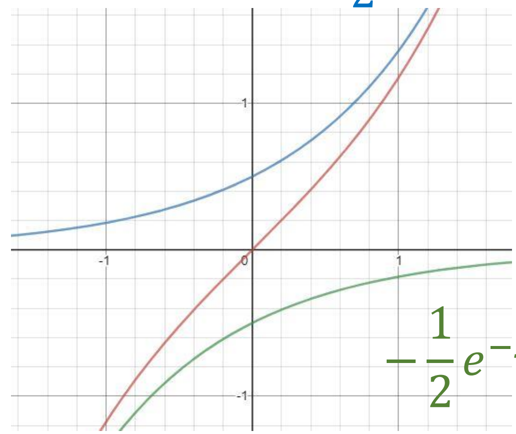
지수함수의 특별한 형태

e^{x} $\quad$ e^{-x} $\quad$ $f(x) = \dfrac{e^{x} - e^{-x}}{2}$ $\quad$ 특별한 명칭을 부여

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

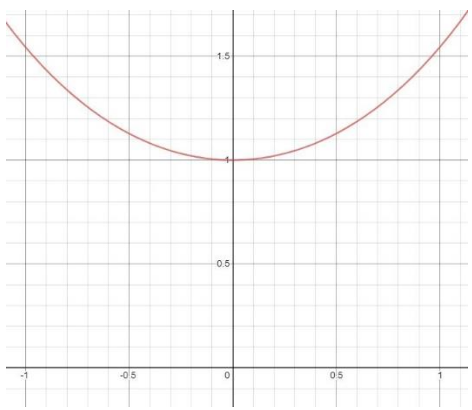


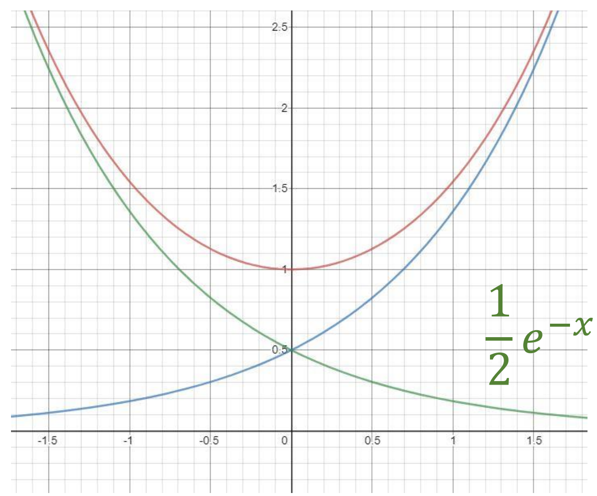
$\frac{1}{2}e^{x}$



$-\frac{1}{2}e^{-x}$

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$





$\frac{1}{2}e^{-x}$

쌍곡선 함수(Hyperbolic Functions)

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh x}{\cosh x}$$

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x}$$

$$\operatorname{sech} x = \frac{1}{e^x + e^{-x}} = \frac{1}{\cosh x}$$

$$\operatorname{csch} x = \frac{1}{e^x - e^{-x}} = \frac{1}{\sinh x}$$

(예제) 다음 값은?

(1) $\sinh 0 = \quad 0$

(2) $\cosh 0 = \quad 1$

(3) $\tanh 1 = \dfrac{e - e^{-1}}{e + e^{-1}}$

(4) $\operatorname{sech} 0 = \quad 1$

(5) $\cosh(\ln 5) = \dfrac{e^{\ln 5} + e^{-(\ln 5)}}{2} = \dfrac{5 + \frac{1}{5}}{2} = 2.6$

(6) $tanh\,(\ln 6) =$

$$= \frac{e^{\ln 6} - e^{-(\ln 6)}}{e^{\ln 6} + e^{-ln6}} = \frac{6 - \frac{1}{6}}{6 + \frac{1}{6}} = \frac{35}{37}$$

$e^{\ln x} = x$

$$\cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = e^x$$

$$\cosh x - \sinh x = \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} = e^{-x}$$

항등식

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = 1$$

양변을 $\cosh^2 x$ 로 나누면 $1 - \tanh^2 x = \mathrm{sech}^2 x$

양변을 $\sinh^2 x$ 로 나누면 $\coth^2 x - 1 = \mathrm{csch}^2 x$

$\cos^2 x + \sin^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$\cot^2 x + 1 = csc^2 x$

$$\sinh(x+y) = \sinh x \ \cosh y + \cosh x \sinh y \qquad \sin(x+y) = \sin x \ \cos y + \cos x \sin y$$

$$= \frac{e^{(x+y)} - e^{-(x+y)}}{2} \qquad = \frac{1}{2}(e^{x}e^{y} - e^{-x}\, e^{-y})$$

$$= \frac{1}{2}((\cosh x + \sinh x)\,(\cosh y + \sinh y)\ - (\cosh x\ - \sinh x)(\cosh y - \sinh y))$$

$$= \frac{1}{2}((\cancel{\cosh x \cosh y} + \sinh x \cosh y + \cosh x\ \sinh y + \cancel{\sinh x \sinh y})$$

$$-(\cancel{\cosh x \cosh y}\ - \sinh x \cosh y\ - \cosh x \sinh y + \cancel{\sinh x \sinh y})\ )$$

$$\cosh(x+y) = \cosh x\ \cosh y \boxed{+} \sinh x \sinh y \qquad \cos(x+y) = \cos x\ \cos y - \sin x \sin y$$

쌍곡선 함수의 도함수

$$(\sinh x)' = \left(\frac{e^x - e^{-x}}{2}\right)' = \left(\frac{e^x + e^{-x}}{2}\right) = \cosh x$$

$$(\cosh x)' = \left(\frac{e^x + e^{-x}}{2}\right)' = \left(\frac{e^x - e^{-x}}{2}\right) = \sinh x$$

$$(\tanh x\,)' = \left(\frac{\sinh x}{\cosh x}\right)' = \frac{\cosh^2 x\, - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

+

$$(\coth x\,)' = \left(\frac{\cosh x}{\sinh x}\right)' = \frac{\sinh^2 x\, - \cosh^2 x}{\sinh^2 x} = \frac{-1}{\sinh^2 x} = -csch^{\,2}\, x$$

$$(\operatorname{sech} x)' = -\operatorname{sech} x \tanh x$$

$$(\operatorname{csch} x)' = -\operatorname{csch} x \coth x$$

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p. 165 역쌍곡선 함수

1. $y = \sinh^{-1} x$

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$2x = e^y - e^{-y}$ 의 양변에 e^y 을 곱하면 $e^{2y} - 2x\, e^y - 1 = 0$ 을 얻는다.

$X = e^y$ 라 두면 주어진 식은 X 에 관한 2차 방정식 $X^2 - 2xX - 1 = 0$ 이 된다.

$X = \frac{2x \pm \sqrt{4x^2+4}}{2}$ $\quad X = x \pm \sqrt{x^2+1}$ 이때 $X = e^y > 0$ 이므로

$X = e^y = x + \sqrt{x^2+1}$ 이 되어 $y = \ln(x + \sqrt{x^2+1})$ 이다.

$$y = \sinh^{-1} x = \ln(x + \sqrt{x^2+1})$$

2. $y = \cosh x$ 의 역함수 $y = \cosh^{-1} x$

$$x = \cosh y = \frac{e^{y} + e^{-y}}{2}$$

$2x = e^{y} + e^{-y}$ 의 양변에 e^{y} 을 곱하면 $e^{2y} - 2x\, e^{y} + 1 = 0$ 을 얻는다.

$X = e^{y}$ 라 두면 주어진 식은 X 에 관한 2차 방정식 $X^2 - 2xX + 1 = 0$ 이 된다.

$$X = \frac{2x \pm \sqrt{4x^2 - 4}}{2} \qquad X = x \pm \sqrt{x^2 - 1} \Longrightarrow X = x + \sqrt{x^2 - 1}$$

$X = e^{y} = x + \sqrt{x^2 - 1}$ 이 되어 $y = \ln(x + \sqrt{x^2 - 1})$ 이다.

$$y = \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$$

3. $y = \tanh x$ 의 역함수

$$x = \tanh y = \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}}$$

윗 식의 분모 분자에 e^{y} 을 곱하면 $x = \dfrac{e^{2y} - 1}{e^{2y} + 1}$ $\quad xe^{2y} + x = e^{2y} - 1$

$$(x-1)e^{2y} = -x - 1 \quad \Rightarrow \quad e^{2y} = \frac{-x-1}{x-1} = \frac{1+x}{1-x}$$

$2y = \ln\left(\frac{1+x}{1-x}\right)$ 이다.

$$y = \ tanh^{-1}\, x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$\tanh^{-1} x = \frac{1}{2}(\ln(1+x) - \ln(1-x))$$

4. $y = \coth x$ 의 역함수

$$x = \coth y = \frac{e^y + e^{-y}}{e^y - e^{-y}}$$

윗 식의 분모 분자에 e^y 을 곱하면 $x = \dfrac{e^{2y}+1}{e^{2y}-1}$ $xe^{2y} - x = e^{2y} + 1$

$(x-1)e^{2y} = x + 1$ $\Rightarrow$ $e^{2y} = \dfrac{x+1}{x-1}$

$$y = \ coth^{-1}x \ = \underline{\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)}$$

$$\underline{\coth^{-1} x = \frac{1}{2}(\ln(x+1) - \ln(x-1))}$$

5. $y = \operatorname{sech} x$ 의 역함수

$$y = \cosh^{-1}\frac{1}{x} \iff \frac{1}{x} = \cosh y \iff x = \operatorname{sech} y \iff y = \operatorname{sech}^{-1} x$$

$$y = \cosh^{-1}\frac{1}{x} = \ln\left(\frac{1}{x} + \sqrt{\left(\frac{1}{x}\right)^2 - 1}\right) = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$$

6. $y = \operatorname{csch} x$ 의 역함수

$$y = sinh^{-1}\frac{1}{x} \iff \frac{1}{x} = \sinh y \iff x = \operatorname{csch} y \iff y = \operatorname{csch}^{-1} x$$

$$y = \sinh^{-1}\frac{1}{x} = \ln\left(\frac{1}{x} + \sqrt{\left(\frac{1}{x}\right)^2 + 1}\right) = \ln\left(\frac{1+\sqrt{1+x^2}}{x}\right)$$

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$$y' = (\sinh^{-1} x)' = (\ln(x + \sqrt{x^2+1}))' = \frac{1}{x+\sqrt{x^2+1}} \cdot \left(1 + \frac{2x}{2\sqrt{x^2+1}}\right)$$

$$= \frac{1}{\cancel{x + \sqrt{x^2+1}}} \cdot \frac{\cancel{\sqrt{x^2+1} + x}}{\sqrt{x^2+1}}$$

$$= \frac{1}{\sqrt{x^2+1}}$$

$$y = \cosh^{-1} x = \ln(x + \sqrt{x^2-1}) \qquad = \frac{1}{x+\sqrt{x^2-1}} \cdot \left(1 + \frac{2x}{2\sqrt{x^2-1}}\right)$$

$$= \frac{1}{\cancel{x + \sqrt{x^2-1}}} \cdot \frac{\cancel{\sqrt{x^2-1} + x}}{\sqrt{x^2-1}}$$

$$= \frac{1}{\sqrt{x^2-1}}$$

$$y' = (\tanh^{-1} x)' = \left(\frac{1}{2}(\ln(1+x) - \ln(1-x))\right)'$$

$$= \frac{1}{2}\left(\frac{1}{1+x} - \frac{-1}{1-x}\right) = \frac{1}{2}\left(\frac{2}{1-x^2}\right) = \frac{1}{1-x^2}$$

(예제) $\frac{d}{dx}(\tanh^{-1}(\sin x)) = \dfrac{1}{1-(\sin x)^2} \times \cos x = \dfrac{\cos x}{\cos^2 x} = \sec x$

$$(\cosh^{-1}(e^{-x}))' = \frac{1}{\sqrt{(e^{-x})^2 - 1}} \times (-e^{-x})$$

$(\coth^{-1} x)' = \dfrac{1}{1-x^2}$ $\quad$ $(\mathrm{sech}^{-1} x)' = -\dfrac{1}{|x|\sqrt{x^2+1}}$ $\quad$ $(\mathrm{csch}^{-1} x)' = -\dfrac{1}{|x|\sqrt{1-x^2}}$

- $(x^n)' = nx^{n-1}$
- $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$
- $(\log_a x)' = \frac{1}{x \ln a}$

$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$

$(\ln x)' = \frac{1}{x}$

$(e^x)' = e^x \quad (a^x)' = a^x \ln a$

- $(\sin x)' = \cos x$
- $(\cos x)' = -\sin x$
- $(\tan x)' = \sec^2 x$
- $(\cot x)' = -\csc^2 x$
- $(\sec x)' = \sec x \tan x$
- $(\csc x)' = -\csc x \cot x$

$(\sinh x)' = \cosh x$

$(\cosh x)' = \sinh x$

$(\tanh x)' = \operatorname{sech}^2 x$

$(\coth x)' = -\operatorname{csch}^2 x$

$(\operatorname{sech} x)' = -\operatorname{sech} x \tanh x$

$(\operatorname{csch} x)' = -\operatorname{csch} x \coth x$

$$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\tan^{-1} x)' = \frac{1}{1+x^2}$$

$$(\cot^{-1} x)' = -\frac{1}{1+x^2}$$

$$(\sec^{-1} x)' = \frac{1}{x\sqrt{x^2-1}}$$

$$(\csc^{-1} x)' = -\frac{1}{x\sqrt{x^2-1}}$$

$$(\sinh^{-1} x)' = \frac{1}{\sqrt{1+x^2}}$$

$$(\cosh^{-1} x)' = \frac{1}{\sqrt{x^2-1}}$$

$$(\tanh^{-1} x)' = \frac{1}{1-x^2}$$

$$(\coth^{-1} x)' = -\frac{1}{x^2-1}$$

$$(\operatorname{sech}^{-1} x)' = -\frac{1}{x\sqrt{1-x^2}}$$

$$(\operatorname{csch}^{-1} x)' = -\frac{1}{|x|\sqrt{1+x^2}}$$