6. 무리함수의 적분 (삼각치환법)

$\sqrt{a^2 - x^2}$ $\qquad x = a \sin t \qquad$ $\sqrt{a^2 - (a \sin t)^2} = \sqrt{a^2(\,1 - \sin^2 t)} = a\sqrt{\cos^2 t} = a \cos t$

$\sqrt{a^2 + x^2}$ $\qquad x = a\, tan\, t \qquad$ $\sqrt{a^2 + (a \tan t)^2} = \sqrt{a^2(\,1 + \tan^2 t)} = a\sqrt{\sec^2 t} = a \sec t$

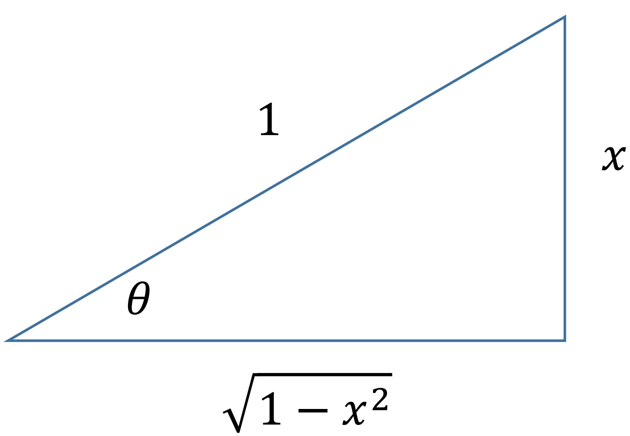
$\sqrt{x^2 - a^2}$ $\qquad x = a \sec t \qquad$ $\sqrt{(a \sec t)^2 - a^2} = \sqrt{a^2(\sec^2 t - 1)} = a\sqrt{\tan^2 t} = a \tan t$

(예제6.2, p 359) $\int \sqrt{1-x^2}\ dx$

$x = \sin\theta$ 로 치환 $\quad dx = \cos\theta\ d\theta$

$$= \int \sqrt{1-\sin^2\theta}\cos\theta\, d\theta \qquad = \int \cos^2\theta\ d\theta$$

$$= \int \cos\theta \cdot \cos\theta\, d\theta \qquad = \int \frac{1+\cos 2\theta}{2}\ dt = \frac{1}{2}\int (1+\cos 2\theta) d\theta$$



$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C$$

$$= \frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{1-x^2} + C$$

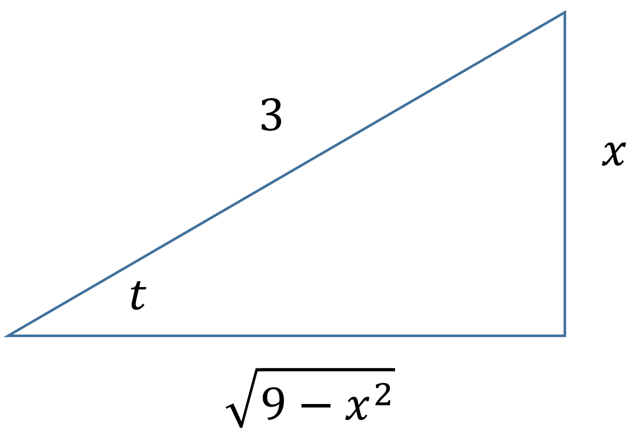
(예제) $$\int \frac{\sqrt{9-x^2}}{x}dx$$

$x = 3\sin t$ 로 치환 $\qquad dx = 3\cos t \ dt$

$$= \int \frac{3\cos t}{3\sin t} 3\cos t \ dt \qquad = \int \frac{3\cos^2 t}{\sin t} \ dt$$

$$= 3\int \frac{(1-\sin^2 t)}{\sin t} \ dt = 3\int (\csc t \ -\sin t)dt$$

$$= 3\ (-\ln(\csc t + \cot t\ ) + \cos t)$$

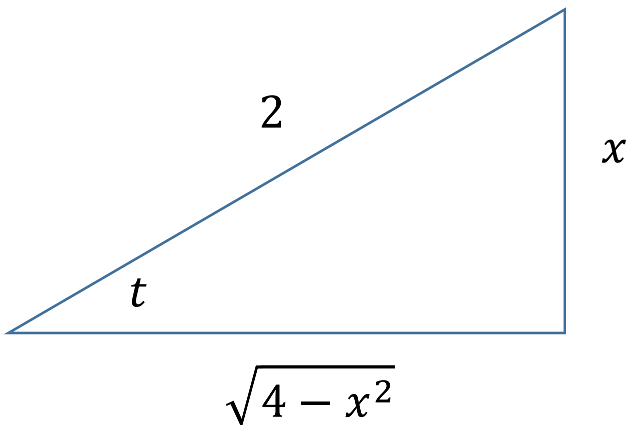


$$= -3 \ \cdot \ln\left(\frac{3}{x} + \frac{\sqrt{9-x^2}}{x}\right) + 3 \cdot \frac{\sqrt{9-x^2}}{3} + C$$

(예제) $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx$

$x = 2\sin t$ 로 치환 $\qquad dx = 2\cos t \; dt$

$$= \int \frac{1}{2\cos t}\,2\cos t \; dt \quad = \int 1 \; dt = t \quad = \sin^{-1}\frac{x}{2}$$



$$\int \frac{1}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$\int \frac{x}{\sqrt{4-x^2}}\,dx = -\sqrt{4-x^2}$$

(예제) $$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$$

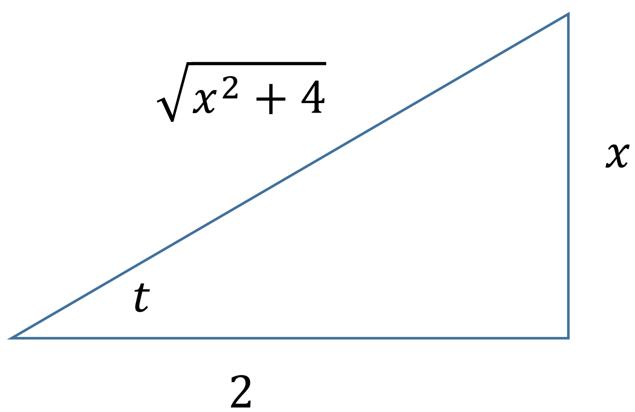
$x = 2\tan t$ 로 치환 $\qquad dx = 2\sec^2 t\ dt$

$$= \int \frac{2\sec^2 t}{4\tan^2 t \cdot \sqrt{4\tan^2 t + 4}}\,dt = \int \frac{2\sec^2 t}{4\tan^2 t \cdot 2\sec t}\,dt = \frac{1}{4}\int \frac{\sec t}{\tan^2 t}\,dt$$

$$= \frac{1}{4}\int \frac{\cos t}{\sin^2 t}\,dt$$

$u = \sin t$ 로 치환

$du = cos\, t\, dt$



$$= \frac{1}{4}\int \frac{1}{u^2}\,du = \frac{1}{4}\left(-\frac{1}{u}\right) + C$$

$$= -\frac{1}{4\sin t} + C$$

$$= -\frac{1}{4\dfrac{x}{\sqrt{x^2+4}}} + C \qquad = -\frac{\sqrt{x^2+4}}{4x} + C$$

(예제) $$\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$$

(풀이) $\int \frac{x}{\sqrt{4-(x+1)^2}}\,dx$ $x+1=u$ 로 치환하면 $x=u-1$ 이고 $dx=du$ 이므로

$= \int \frac{u-1}{\sqrt{4-u^2}}du$ 또, $u=2\sin t$ 로 치환하면 $du = 2\cos t\ dt$ 이고 $\sqrt{4-u^2}=2\cos t$ 이므로

$$= \int \frac{2\sin t - 1}{2\cos t} 2\cos t\,dt \quad = \int (2\sin t - 1)dt = -2\cos t - t$$

$$= -\sqrt{4-u^2} - \sin^{-1}\frac{u}{2}$$

$$= -\sqrt{3-2x-x^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C$$

(참고)

$$\int \frac{u-1}{\sqrt{4-u^2}}du = \boxed{\int \frac{u}{\sqrt{4-u^2}}du} - \int \frac{1}{\sqrt{4-u^2}}du$$

$4-u^2=t$ 로 치환

$$\sin^{-1}\frac{u}{2}$$

$$\int \frac{1}{\sqrt{a^2+x^2}}dx = sinh^{-1}\frac{x}{a} \qquad \int \frac{1}{\sqrt{a^2-x^2}}dx = sin^{-1}\frac{x}{a}$$

(예제) (1) $\displaystyle\int \frac{1}{\sqrt{x^2+6x+15}}dx \quad = \int \frac{1}{\sqrt{(x^2+6x+9)+6}}dx = \int \frac{1}{\sqrt{6+(x+3)^2}}dx$

$x+3=u$로 치환, $dx=du$

$$= \sinh^{-1}\frac{x+3}{\sqrt{6}} + C$$

$$= \int \frac{1}{\sqrt{6+u^2}}du \quad = \sinh^{-1}\frac{u}{\sqrt{6}}$$

(2) $\displaystyle\int \frac{1}{\sqrt{15+6x-x^2}}dx \quad = \int \frac{1}{\sqrt{24-(9-6x+x^2)}}dx = \int \frac{1}{\sqrt{24-(x-3)^2}}dx$

$x-3=u$로 치환, $dx=du$

$$= sin^{-1}\frac{x-3}{\sqrt{24}} + C$$

$$= \int \frac{1}{\sqrt{24-u^2}}du \quad = sin^{-1}\frac{u}{\sqrt{24}}$$