Discussion 2A

1. Use truth tables to show that $\neg(A \vee B) \equiv \neg A \wedge \neg B$ and $\neg(A \wedge B) \equiv \neg A \vee \neg B$. These two equivalences are known as DeMorgan's Law.

2. Which of the following statements are true? Let $Q(n)$ be the statement "n is divisible by 2." $\mathbb{N}$ denotes the set of natural numbers.

   (a) $\exists\, k \in \mathbb{N},\ Q(k) \wedge Q(k+1)$.

   (b) $\forall\, k \in \mathbb{N},\ Q(k) \implies Q(k^2)$.

   (c) $\exists\, x \in \mathbb{N}, \neg(\exists\, y \in \mathbb{N}, y < x)$.

3. Write the following statements using the notation covered in class. Use $\mathbb{N}$ to denote the set of natural numbers and $\mathbb{Z}$ to denote the set of integers. Also write $P(n)$ for the statement "n is odd".

   (a) For all natural numbers n, $2n$ is even.

   (b) For all natural numbers n, n is odd if n^2 is odd.

   (c) There are no integer solutions to the equation $x^2 - y^2 = 10$.

4. You are on an island inhabited by two types of people: the Liars and the Truthtellers. Liars always lie, and Truthtellers always tell the truth. In all other respects, the two types are indistinguishable. You meet a very attractive local and ask him/her on a date. The local responds, "I will go on a date with you if and only if I am a Truthteller." Is this good news?

5. Which of the following implications is/are true?

   (a) $\forall x \forall y\, P(x,y)$ implies $\forall y \forall x\, P(x,y)$.

   (b) $\exists x \exists y\, P(x,y)$ implies $\exists y \exists x\, P(x,y)$.

   (c) $\forall x \exists y\, P(x,y)$ implies $\exists y \forall x\, P(x,y)$.

   (d) $\exists x \forall y\, P(x,y)$ implies $\forall y \exists x\, P(x,y)$.

   Also, for the implication in part (c), what is its converse? And its contrapositive?

6. Prove or disprove each of the following:

   (a) $\forall x \big(P(x) \wedge Q(x)\big) \equiv \forall x P(x) \wedge \forall x Q(x)$

   (b) $\forall x \big(P(x) \vee Q(x)\big) \equiv \forall x P(x) \vee \forall x Q(x)$

   (c) $\forall x \big(P(x) \Rightarrow Q(x)\big) \equiv \big(\forall x P(x)\big) \Rightarrow \big(\forall x Q(x)\big)$

   (d) $\exists x \big(P(x) \wedge Q(x)\big) \equiv \exists x P(x) \wedge \exists x Q(x)$

   (e) $\exists x \big(P(x) \vee Q(x)\big) \equiv \exists x P(x) \vee \exists x Q(x)$

   (f) $\exists x \big(P(x) \Rightarrow Q(x)\big) \equiv \big(\exists x P(x)\big) \Rightarrow \big(\exists x Q(x)\big)$

7. Complete the following expression so that it states that: “There is one and only one natural number n for which the proposition formula $P(n)$ holds.”

$$(\exists n \in \mathbb{N}) \dots$$

8. A valid tiling for a chessboard is an arrangement of tiles such that no two tiles overlap and every square of the board is covered by a tile.

   (a) A domino is a tile consisting of two contiguous squares. Is there a valid domino tiling for the 8×8 chessboard where the squares in the bottom left and top right corners have been removed?

   (b) A straight tetromino is a tile consisting of four contiguous squares. Prove or disprove: A 10×10 chessboard can be tiled with straight tetrominoes.