This homework is due April 21 2014, at 12:00 noon.

1. Independent Random Variables

Find four random variables taking values in $\{-1,1\}$ so that any three are independent but all four are not.

Solution: Let X_1,X_2,X_3,X_4 be i.i.d random variables with $P(X_i=1)=P(X_i=-1)=1/2$. Let $X_4=X_1X_2X_3$. Check that X_1,X_2,X_3,X_4 are four random variables such that any three are independent but all four are not. For example, they are not all independent because

$$P(X_1=1,X_2=1,X_3=1,X_4=1)=1/8\neq P(X_1=1)P(X_2=1)P(X_3=1)P(X_4=1)$$

2. Expectation Basics

For discrete random variables X and Y:

a) Show that for constants c and d, $\mathbb{E}[cX+dY]=c\mathbb{E}[X]+d\mathbb{E}[Y]$.
Solution: Beginning with the left-hand-side:

$$\begin{aligned}
\mathbb{E}[cX+dY] &= \sum_x\sum_y(cx+dy)\mathbb{P}(x,y) && \text{definition of expectation}\\
&= c\sum_x\sum_y x\mathbb{P}(x,y)+d\sum_x\sum_y y\mathbb{P}(x,y) && \text{expand and break up the sum into two parts}\\
&= c\sum_x x\sum_y\mathbb{P}(x,y)+d\sum_y y\sum_x\mathbb{P}(x,y) && \text{reorder terms that depend on } x \text{ or } y\\
&= c\sum_x x\mathbb{P}(x)+d\sum_y y\mathbb{P}(y) && \text{marginalize over } y \text{ in the first term and } x \text{ in the second term}\\
&= c\mathbb{E}[X]+d\mathbb{E}[Y]. && \text{arrive at the definition of expectation for } X \text{ and } Y
\end{aligned}$$

One of the key steps was using the fact that

$$\sum_y\mathbb{P}(x,y)=\mathbb{P}(x).$$

This is often phrased as "marginalizing out" a variable, to retain only the variables of interest. In this case we are marginalizing out y and retaining only the probability of x.

b) Show that $\mathbb{E}[\min(X,Y)]+\mathbb{E}[\max(X,Y)]=\mathbb{E}[X]+\mathbb{E}[Y]$.
Solution:

$$\begin{aligned}
\mathbb{E}[\min(X,Y)]+\mathbb{E}[\max(X,Y)] &= \mathbb{E}[\min(X,Y)+\max(X,Y)] && \text{By linearity of expectation}\\
&= \mathbb{E}[X+Y]\\
&= \mathbb{E}[X]+\mathbb{E}[Y] && \text{By linearity of expectation}
\end{aligned}$$

3. Variance of Random Variables

The variance of a random variable tells us information about how far the random variable is spread outside its expectation. For this problem, we will use the definition $Var[X]=\mathbb{E}[X^2]-\mathbb{E}[X]^2$. We already know how to compute the expectation of a sum of random variables because expectation is linear ($\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$). In this problem, we will compute the variance of Z, where Z is a sum of random variables:

$$Z = \sum_{i=1}^{n} X_i$$

and $X_1, X_2, \cdots, X_n$ are pair-wise independent (this means $\forall i \neq j$ $\Pr[X_i = a \cap X_j = b] = Pr[X_i = a]Pr[X_j = b]$).

a) In order to compute the variance of the sum of these random variables, first prove the following lemma:
Lemma: If X and Y are independent random variables, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.
(Hint: Let $\mathscr{X}$ be the set of all possible values of X and let $\mathscr{Y}$ be the set of all possible values of Y. Now use the definition of expectation, $\mathbb{E}[XY] = \sum\limits_{x \in \mathscr{X}} \sum\limits_{y \in \mathscr{Y}} xy \cdot Pr[X = x \cap Y = y]$). **Solution:** We start with the hint.
$\mathbb{E}[XY] = \sum\limits_{x \in \mathscr{X}} \sum\limits_{y \in \mathscr{Y}} xy \cdot Pr[X = x \cap Y = y]$
$= \sum\limits_{x \in \mathscr{X}} \sum\limits_{y \in \mathscr{Y}} xy \cdot Pr[X = x]Pr[Y = y]$ Because X and Y are independent
$= \sum\limits_{x \in \mathscr{X}} Pr[X = x] \cdot x \sum\limits_{y \in \mathscr{Y}} y \cdot Pr[Y = y]$ Because x does not change in the inner summation.
$= \sum\limits_{x \in \mathscr{X}} Pr[X = x] \cdot x\mathbb{E}[Y]$ By definition of expectation
$= \mathbb{E}[Y] \cdot \sum\limits_{x \in \mathscr{X}} Pr[X = x] \cdot x = \mathbb{E}[Y]\mathbb{E}[X]$

b) Prove that $\mathbb{E}[Z^2] = \sum_{i=1}^{n} \mathbb{E}[X_i^2] + \sum_{i=1}^{n} \sum_{j \neq i} \mathbb{E}[X_i X_j]$.
(Hint: What happens when you expand out $(X_1 + X_2 + \cdots X_n)^2$?) **Solution:** We start with the hint, we see that $Z = (X_1 + X_2 + \cdots X_n)$, therefore $Z^2 = (X_1 + X_2 + \cdots X_n)^2$. When we expand this out we get
$Z^2 = (X_1 + X_2 + \cdots X_n) \cdot (X_1 + X_2 + \cdots X_n)$ To see what happens here we try a smaller example,
$(X_1 + X_2 + X_3)(X_1 + X_2 + X_3) = (X_1^2 + X_2^2 + X_3^2 + X_1X_2 + X_1X_3 + X_2X_1 + X_2X_3 + X_3X_1 + X_3X_2)$ In general, we have the squared term of each X_i we are adding, and we have all the possible products of X_i and X_j where order matters and $i \neq j$. Therefore, we can generalize as
$(X_1 + X_2 + \cdots X_n) \cdot (X_1 + X_2 + \cdots X_n) = (X_1^2 + X_2^2 + \cdots X_n^2 + X_1X_2 + X_1X_3 + \cdots X_1X_n + X_2X_1 + \cdots X_2X_n + \cdots X_nX_{n-1})$
$= \sum_{i=1}^{n} X_i^2 + \sum_{i=1}^{n} \sum_{j \neq i} X_iX_j$. Therefore, by linearity of expectation.
$\mathbb{E}[Z^2] = \sum_{i=1}^{n} \mathbb{E}[X_i^2] + \sum_{i=1}^{n} \sum_{j \neq i} \mathbb{E}[X_iX_j]$

c) Prove that $\mathbb{E}[Z]^2 = \sum_{i=1}^{n} \mathbb{E}[X_i]^2 + \sum_{i=1}^{n} \sum\limits_{j \neq i} \mathbb{E}[X_i]\mathbb{E}[X_j]$. **Solution:** We calculate $\mathbb{E}[Z]$ and then square it. By linearity of expectation, $\mathbb{E}[Z] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n]$. So, now we wish to calculate $(\mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n])^2$. This calculation is similar to what we did before. We will have a term for each i of $\mathbb{E}[X_i]\mathbb{E}[X_i]$ and we will have on term for every possible product of $\mathbb{E}[X_i]$ and $\mathbb{E}[X_j]$ where order matters and $i \neq j$. Ultimately,
$\mathbb{E}[Z]^2 = \sum_{i=1}^{n} \mathbb{E}[X_i]\mathbb{E}[X_i] + \sum_{i=1}^{n} \sum\limits_{j \neq i} \mathbb{E}[X_i]\mathbb{E}[X_j]$
$\mathbb{E}[Z]^2 = \sum_{i=1}^{n} \mathbb{E}[X_i]^2 + \sum_{i=1}^{n} \sum\limits_{j \neq i} \mathbb{E}[X_i]\mathbb{E}[X_j]$

d) Combine b) and c) to show $Var[Z] = \sum_{i=1}^{n} \mathbb{E}[X_i^2] + \sum_{i=1}^{n} \sum\limits_{j \neq i} \mathbb{E}[X_iX_j] - \sum_{i=1}^{n} \mathbb{E}[X_i]^2 - \sum_{i=1}^{n} \sum\limits_{j \neq i} \mathbb{E}[X_i]\mathbb{E}[X_j]$. Now use the lemma from part a) and rearrange to show that $Var[Z] = \sum_{i=1}^{n} \mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2$.
Solution: Since $Var[Z] = \mathbb{E}[Z^2] - \mathbb{E}[Z]^2$ We subtract our answer from c from our answer from b.
$Var[Z] = \sum_{i=1}^{n} \mathbb{E}[X_i^2] + \sum_{i=1}^{n} \sum_{j \neq i} \mathbb{E}[X_iX_j] - \sum_{i=1}^{n} \mathbb{E}[X_i]^2 + \sum_{i=1}^{n} \sum\limits_{j \neq i} \mathbb{E}[X_i]\mathbb{E}[X_j]$ We group our summations together
$Var[Z] = \sum_{i=1}^{n} (\mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2) + \sum\limits_{j \neq i} (\mathbb{E}[X_iX_j] - \mathbb{E}[X_i]\mathbb{E}[X_j])$ Since X_i and X_j are pair-wise independent, we use our lemma that $\mathbb{E}[X_iX_j] = \mathbb{E}[X_i]\mathbb{E}[X_j]$
$Var[X] = \sum_{i=1}^{n} (\mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2)$

e) Finally reason that $Var[Z] = Var[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} Var[X_i]$. This is true if for all $i \neq j$, X_i and X_j are pair-wise independent. **Solution:** We look at the previous expression and we realize that $Var[X_i] = \mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2$. So, $Var[X] = \sum_{i=1}^{n} (\mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2) = \sum_{i=1}^{n} Var[X_i]$

f) The proof above does not hold if X_i and X_j are not pair-wise independent, because then it is possible that $\mathbb{E}[X_iX_j] \neq \mathbb{E}[X_i]\mathbb{E}[X_j]$ Come up with two random variables, X and Y such that $Var[X+Y] \neq Var[X] + Var[Y]$.

Similarly, come up with two random variables that are *not independent* but for which the variance of the sum is nonetheless the sum of the variances.

Solution: First we come up with two random variables X and Y such that $Var[X+Y] \neq Var[X] + Var[Y]$. Let us let X count the number of heads after 1 flip of a fair coin and let Y count the number of tails after the same 1 flip of that coin. Then $X+Y=1$ always, and therefore the Variance is 0. The Variance of a constant number is 0. However,
$Var[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = Pr[X=1] \cdot 1^2 + Pr[X=0] \cdot 0^2 - (Pr[X=1] \cdot 1 + Pr[X=0] \cdot 0)^2$
$= 1/2 \cdot 1 - (1/2 \cdot 1)^2 = 1/2 - 1/4 = 1/4$
The calculation for $Var[Y]$ is the exact same.
Therefore, $Var[X] + Var[Y] = 1/4 + 1/4 = 1/2 \neq 0 = Var[X+Y]$
Now we wish to come up with two variables X and Y that are dependent but for which $Var[X+Y] = Var[X] + Var[Y]$. From before, we realize this is true if $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$. Therefore we wish to find two variables for which this is true. We propose the following random variables:
$X : i \in \mathbb{Z} \; -2 \leq i \leq -2 \qquad Pr[X=i] = 1/5.$
Therefore $\mathbb{E}[X] = -2 \cdot 1/5 + -1 \cdot 1/5 + 0 \cdot 1/5 + 1 \cdot 1/5 + 2 \cdot 1/5 = 0.$
$Y: \; Y = -1$ iff $X = 0 \vee X = 2$. Then, $Pr[Y=-1] = 2/5$
$Y = 0$ iff $X = -2 \vee X = -1$. Then, $Pr[Y=0] = 2/5$
$Y = 2$ iff $X = 1$. Then, $Pr[Y=2] = 1/5$
Therefore $\mathbb{E}[Y] = -1 \cdot 2/5 + 0 \cdot 2/5 + 2 \cdot 1/5 = 0$. So, $\mathbb{E}[X]\mathbb{E}[Y] = 0$.
Now let us calculate $\mathbb{E}[XY]$. Since Y is completely determined by X, we have the following five possibles values of $(X,Y) = (-2,0),(-1,0),(0,-1),(1,2),(2,-1)$ Each of these occur with probability $1/5$. Therefore, we have $Pr[XY=0] = 1/5 + 1/5 + 1/5$ and $Pr[XY=2] = 1/5$ and $Pr[XY=-2] = 1/5$
$\mathbb{E}[XY] = 0 \cdot 3/5 + 2 \cdot 1/5 + -2 \cdot 1/5 = 0.$
In this case, $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ and therefore $Var[X+Y] = Var[X] + Var[Y]$ since the above arguments in parts b-e used the fact that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$

4. Binomial Distribution In this question we will derive the Binomial Distribution. First it might help if we define the word "distribution." The distribution of a random variable specifies the probability of every possible value of that random variable. For instance, for a random variable that counts the value of a standard 6-sided die after 1 roll, the distribution is $i \in \{1,2,3,4,5,6\}$ $Pr[X=i] = 1/6$. The Binomial Distribution is used to describe the number of heads that come up when we toss a potentially biased coin (with prob p of coming up heads) n times. If this sounds familiar, it's been the focus of almost all of our labs! Remember that $0 < p < 1$.

a) What is the probability that after n trials, we see exactly n heads? Recall that each coin flip has probability p of being a head and different tosses of the coin are independent of each other.

Solution: Each trial is independent, and the probability of a head for each trial is p so we have probability p^n

b) What is the probability that after n trials, we see exactly 0 heads?

Solution: Each trial is independent, and the probability of a head for each trial is $1-p$ so we have probability $(1-p)^n$

c) How many sequences of n coin flips have exactly 1 head?

Solution: n. We simply choose 1 of the n flips to be a head so we get $\binom{n}{1}$

d) How many sequences of n coin flips have exactly k heads? Let $0 \leq k \leq n$

Solution: This time, we count the ways to choose k of the n flips to be a head, so we get $\binom{n}{k}$

e) Let $0 \leq k \leq n$. What is the probability of a particular outcome of k heads occurring? For example, what is the probability the first k trials are heads, and the rest of the trials are tails? Does it matter which order the heads and tails are as long as there are exactly k heads out of exactly n tosses?

Solution: Again, because the coin tosses are independent, we multiply the probability of the result of each coin toss. Since we have k heads, the probability of those k heads is p^k, but we still have count the probability the rest of the $n-k$ coins are tails which is $(1-p)^{n-k}$. Therefore we get $p^k(1-p)^{n-k}$. It does not matter what order the heads and tails are because multiplication is commutative (that is $a * b = b * a$).

f) Combine part d and e to compute the probability that we get k heads in n coin tosses. Call this $\Pr[X=k]$ since our random variable X counts the number of heads after n flips.

Solution: Part e computes the probability of a particular sequence of k heads and $n-k$ tails. Part d counts *how many* of these sequences exist. Therefore, to count the probability of getting k heads in n coin tosses we multiply our answer to part d and part e. $Pr[X=k] = \binom{n}{k}p^k(1-p)^{n-k}$

g) We have now specified the distribution of X where X counts the number of heads after n flips of a biased coin with a probability p of flipping a head. In order to prove that this is indeed a valid probability distribution for a random variable, we must verify that $\sum_{k=0}^{n} Pr[X=k] = 1$. Verify that your distribution is valid (all the probabilities sum to 1) when $n=2$ and $p=1/3$.

Solution: We wish to show that $Pr[X=0]+Pr[X=1]+Pr[X=2]=1$
$\sum_{k=0}^{2} Pr[X=k] = \binom{2}{0}(1/3)^0(1-(1/3))^{2-0} + \binom{2}{1}(1/3)^1(1-(1/3))^{2-1} + \binom{2}{2}(1/3)^2(1-(1/3))^{2-2}$
$= \binom{2}{0}(1/3)^0(2/3)^2 + \binom{2}{1}(1/3)^1(2/3)^1 + \binom{2}{2}(1/3)^2(2/3)^0$
$= \binom{2}{0}(4/9) + \binom{2}{1}(2/9) + \binom{2}{2}(1/9) = 1(4/9)+2(2/9)+1(1/9) = 4/9+4/9+1/9$
$= 1$

h) Now we want to prove mathematically that this holds for all values of n and p. The Binomial Theorem states that $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$. We will use the Binomial Theorem in our proof, so give a brief combinatorial argument to support the Binomial Theorem. Why is the coefficient of each term $\binom{n}{k}$? It may help to try some examples for $n=3$ and $n=4$.

Solution: Let's examine the case $n=4$. In this case $(a+b)^4 = (a+b)(a+b)(a+b)(a+b)$ When we expand out these terms, we get $a^4 + a^3b + a^3b + a^3b + a^3b + \ldots$ and some other terms. We have 4 terms of a^3b. Where do these four terms come from? Well, one comes from multiplying the b in the first parentheses and the a's in the other parentheses. Another one comes from multiplying the b in the second parentheses and the a's in the other parentheses. And so on. So, we have 4 choices which b we pick. Therefore there are $\binom{4}{1}$ terms of a^3b, just like the binomial theorem suggests. In general, if we have a term $a^{n-k}b^k$ we have to choose k of the parentheses to multiply the b term in. This can be done $\binom{n}{k}$ so we expect that many terms of $a^{n-k}b^k$. This completes our brief combinatorial argument.

i) Use your result from part (h) to show that $\sum_{k=0}^{n} Pr[X=k] = 1$ in general, where $Pr[X=k]$ is the probability you calculated in (f).

Solution: We plug in $a=(1-p)$ and $b=p$. Then we get
$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$ By the binomial theorem
$= \sum_{k=0}^{n}\binom{n}{k}(1-p)^{n-k}p^k$ plugging in the values of a and b
$= \sum_{k=0}^{n} Pr[X=k]$ using our answer to part f
Therefore $\sum_{k=0}^{n} Pr[X=k] = (a+b)^n = (1-p+p)^n = 1^n = 1$
Thus this is a valid probability distribution.

j) We can say $X \sim B(n,p)$ Which is read as "X is a binomial random variable with parameters n and p. We lastly want to calculate $\mathbb{E}[X]$. An easy way to do this is to see X as as sum of independent random variables. Each X_i is an indicator variable that is 1 if the i^{th} flip is heads and 0 if the i^{th} flip is tails. Argue briefly that $X = \sum_{i=1}^{n} X_i$.

Solution: If X is a count of the total number of heads, we get this count by counting each trial being a head or not. Since each X_i is 1 if and only if the i^{th} flip is a head, the X_i count each trial being a head or not. This technique of breaking a random variable into a sum of simpler random variables is a widely-used technique and you should remember to this trick, especially when computing expectations.

k) Use the linearity of expectation to calculate $\mathbb{E}[X]$.

Solution: By the linearity of expectation, if $X = \sum_{i=1}^{n} X_i$, then $\mathbb{E}[X] = \sum_{i=1}^{n}\mathbb{E}[X_i]$. So now we must compute the expectation of each X_i. We notice that each X_i is distributed identically, therefore it is sufficient to compute the expectation of only one X_i and the rest will be identical.
Each X_i is 1 (flip heads on i^{th} flip) with probability p and 0 (flip tails on i^{th} flip) with probability $1-p$.
Therefore, $\mathbb{E}[X_i] = 1\cdot p + 0\cdot(1-p) = p$ by definition of expectation.
$\mathbb{E}[X] = \sum_{i=1}^{n}\mathbb{E}[X_i] = \sum_{i=1}^{n} p = np$
If $X \sim B(n,p)$ then, $\mathbb{E}[X] = np$

l) Looking at what this expectation is, remark on one empirical finding from a virtual lab or the lab we did during discussion that is explained by the expectation you just calculated. Congratulate yourself for finishing a problem by GSI lots-of-parts.

Solution: In Discussion 8A, we saw that after n flips d-sided dice, the most common value (counting the number of times we roll a 1) was close to n/d. We see that this is similar to flipping a coin with $p = 1/d$ since we were only counting the times we roll a 1. The most common value is not the same thing as expectation. (Some random variables never take on their expectation). However, we do know that expectation is the weighted-average. So, this finding explains why the histograms we found in Discussion 8A were symmetrical around the expectation. If it weren't symmetrical around the expectation, then the expectation wouldn't actually be the weighted average!

Dear GSI lots-of-parts: please don't write the final...

5. **Winning the Lottery**

Suppose that every day, Lily buys a lottery ticket, and will only stop buying lottery tickets when she wins. Let p be the probability that on any given day, Lily wins the lottery. Let X represent the totaly number of lottery tickets Lily buys.

a) What is the probability that Lily only buys 1 lottery ticket (i.e. $X = 1$)? What is the probability $X = 2$? What is the probability $X = x$? We can denote this $f(x)$. Show that $f(x)$ is a proper probability mass function.

Solution: The probability of getting $(x-1)$ non-winning lottery tickets is $(1-p)^{x-1}$. Therefore the probability of getting $(x-1)$ non-winning tickets, then a winning ticket is

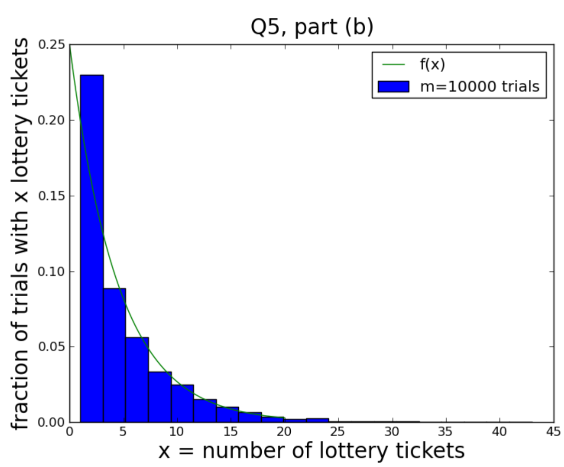
$$f(x) = \mathbb{P}(X = x) = (1-p)^{x-1}p.$$

To show this is a valid probability distribution, we need to make sure the sum of $f(x)$ over all possible x is 1. Using the formula for an infinite geometric series,

$$\sum_{x=1}^{\infty} f(x) = p\sum_{x=1}^{\infty}(1-p)^{x-1} = p \cdot \frac{1}{1-(1-p)} = 1.$$

b) (Lab) Using a computer, simulate $m = 10,000$ trials of Lily buying lottery tickets until she gets a winner. Use $p = 0.2$, and plot a histogram with the number of lottery tickets on the x-axis and the fraction of trials with each of these outcomes on the y-axis. Overlay $f(x)$ from part (a). What is the average number of lottery tickets Lily has to buy?

Solution: Example average: 5.057.



c) Compute $\mathbb{E}(X)$, the expected value of X. How does this relate to the average number of lottery tickets from part (b)? (Hint: use the relation $\mathbb{E}(X) = (1-p)\mathbb{E}(X) + 1$, but first explain where this relation comes from.)

Solution: This relation comes from the fact that with probability p, Lily will only need to buy 1 lottery ticket. But if she is not so lucky the first time, with probability $(1-p)$ she will have to try again. When she tries again, her expectation from that point hasn't changed, so in a sense she is just starting over. Thus we get 1 plus the original expectation multiplied by the probability of failure on the first day:

$$\mathbb{E}(X) = 1 + (1-p)\mathbb{E}(X).$$

Solving for $\mathbb{E}(X)$, we get

$$\mathbb{E}(X) - (1-p)\mathbb{E}(X) = 1 \quad \Rightarrow \quad \boxed{\mathbb{E}(X) = \frac{1}{p}}$$

This fits with our answer from part (b), since $\frac{1}{0.2} = 5$, and we got an average about about 5 days before Lily won the lottery.

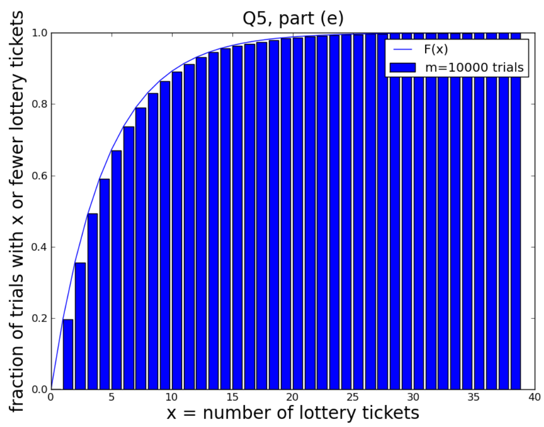
d) Compute the cumulative mass function (cmf) for X, $F(x) = \mathbb{P}(X \leq x)$.

Solution: To compute $F(x)$ we need to add up $f(y)$, for all the y less than or equal to x. We can use the formula for a finite geometric series:

$$F(x) = \sum_{y=1}^{x} f(x) = p\sum_{y=1}^{x}(1-p)^{y-1} = p \cdot \frac{1-(1-p)^x}{1-(1-p)} = 1-(1-p)^x.$$

e) (Optional Lab) Use your histogram from part (b) to plot an "empirical cmf" (i.e. for each number of tickets x, plot the fraction of trials where Lily bought x or fewer tickets). Then overlay $F(x)$.

Solution:



6. 007

Mr. Bond is imprisoned in a cell from which there are three possible ways to escape: an air-conditioning duct, a sewer pipe and the door (which is unlocked). The air-conditioning duct leads him on a two-hour trip whereupon he falls through a trap door onto his head, much to the amusement of his captors. The sewer pipe is similar but takes five hours to traverse. Each fall produces amnesia and he is returned to the cell immediately after each fall. Assume that he always immediately chooses one of the three exits from the cell with probability 1/3. On the average, how long does it take before he opens the unlocked door and escapes?

Equivalent story: A certain chainsmoker is at a party. There are three things that she can do. She can go to the bathroom, in which case she will take two selfies (one with a cigarette, and one without). After this, she again has to decide what to do. Or she can fiddle with Instagram, in which case she will take five selfies. After which, she again has to decide what to do. Or, she can visit her friend Jason, at which point the party ends. When she has to decide, she chooses equally likely between her three choices: bathroom, Instagram, or Jason. How many selfies will she take on average at a party?

[HINT: Think about how to adapt the recursive trick used to calculate the expectation of a Geometric Random Variable to this situation.]

Solution: Using the notation in the extremely helpful hint, $E[T] = E[T\,|\,A]\Pr[A] + E[T\,|\,S]\Pr[S] + E[T\,|\,D]\Pr[D] = \frac{1}{3}(E[T\,|\,A] + E[T\,|\,S] + E[T\,|\,D])$. Note that $E[T\,|\,A] = E[T]+2$ by the memorylessness of the situation. Similarly, $E[T\,|\,S] = E[T]+5$. Lastly, $E[T\,|\,D] = 0$ because Mr. Bond escapes immediately. So we have that $E[T] = \frac{1}{3}(E[T]+2+E[T]+5+0)$. Solving this for $E[T]$, we get that $E[T] = \boxed{\mathbf{007}}$.

7. **Write your own problem**
   Write your own problem related to this week's material and solve it. You may still work in groups to brainstorm problems, but each student should submit a unique problem. What is the problem? How to formulate it? How to solve it? What is the solution?

8. **Midterm question 3**
   Re-do midterm question 3.

9. **Midterm question 4**
   Re-do midterm question 4.

10. **Midterm question 5**
    Re-do midterm question 5.

11. **Midterm question 6**
    Re-do midterm question 6.

12. **Midterm question 7**
    Re-do midterm question 7.

13. **Midterm question 8**
    Re-do midterm question 8.

14. **Midterm question 9**
    Re-do midterm question 9.

15. **Midterm question 10**
    Re-do midterm question 10.

16. **Midterm question 11**
    Re-do midterm question 11.

17. **Midterm question 12**
    Re-do midterm question 12.

18. **Midterm question 13**
    Re-do midterm question 13.

19. **Midterm question 14**
    Re-do midterm question 14.