1. **Balls and bins**

You have n bins and you throw balls into them one by one randomly. A collision is when a ball is thrown into a bin which already has another ball.

(a) What is the probability that the first ball thrown will cause the first collision?

(b) What is the probability that the second ball thrown will cause the first collision?

(c) What is the probability that, given the first two balls are not in collision, the third ball thrown will cause the first collision?

(d) What is the probability that the third ball thrown will cause the first collision?

(e) What is the probability that, given the first $m-1$ balls are not in collision, the m^{th} ball thrown will cause the first collision?

(f) What is the probability that the m^{th} ball thrown will cause the first collision?

Solutions:

(a) 0

(b) $\frac{1}{n}$

(c) $\frac{2}{n}$

(d) Basically: $P(\text{Ball 3 collides} \mid \text{Ball 1, 2 do not collide}) \cdot P(\text{Ball 1, 2 do not collide})$
Which is $\frac{2}{n} \cdot \frac{n-1}{n}$

(e) $\frac{m-1}{n}$

(f) Similar to (d), $\frac{m-1}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \ldots \cdot \frac{n-m+2}{n} = \frac{m-1}{n} \cdot \prod_{i=0}^{m-2} \frac{n-i}{n}$

2. **Another Stirling Approximation**

(a) Consider a deck with M many distinct cards. Suppose one removes a card from this deck at random x many times, recording the ordered sequence of drawn cards. How many possible sequences could one end up with?

(b) Suppose instead that, after each draw, the drawn card is reinserted into the deck. Now how many possible sequences could one get (when drawing randomly with replacement x many times from a deck of M cards)?

(c) What is the probability that no card gets drawn more than once, if one draws randomly with replacement x many times from a deck of M cards?

(d) What happens to this probability as x is held constant but M grows very large?

(e) Show that, for any natural numbers x and N, we have that $x! = \dfrac{(N+x)!}{N!} \displaystyle\prod_{i=1}^{N} \frac{i}{x+i}$.

(f) Combining the previous parts, show that $x! = \lim_{N\to\infty} (N+x)^x \prod_{i=1}^{N} \frac{i}{x+i}$. (*Hint: Think of a deck with $N+x$ many cards.*)

Note that this last formula can be made sense of even for non-integer x; this is often used to generalize the factorial function in mathematics.

Solutions:

(a) $M!/(M-x)!$.

(b) M^x.

(c) $\frac{M!/(M-x)!}{M^x} = \prod_{i=0}^{x-1} \frac{M-i}{M} = \prod_{i=0}^{x-1}(1-i/M)$.

(d) As seen in at the end of the last answer sketch, this is the product of a constant number of factors, each of which approaches 1 as M grows very large. Thus, this probability itself approaches 1 as M grows very large.

(e) Simply note that the product on the right is $\frac{N!}{(N+x)!/x!}$, and then cancel out the $N!$ and $(N+x)!$.

(f) Take the result from the previous section, and subsitute $(N+x)^x$ for $\frac{(N+x)!}{N!}$, using the fact that their limiting ratio is 1 for large N (as established two sections ago).