Weak Convergence: $X_n \xrightarrow{D} X$ **iff** $\mathbb{E}f(X_n) \to f(X)$ for $\forall$ bounded & cont-s functions f.

"$\Rightarrow$" Skorohod's Theorem & DCT.

"$\Leftarrow$" construct 2 cont-s & bounded func. g_1 & g_2 s.t. $g_1(x) \le \mathbb{1}\{x\le t\} \le g_2(x)$

<proof> (Skorohod's Theorem.) "$\Rightarrow$" Get Y_n and Y on same space s.t. $Y_n(w) \to Y(w)$ for all w. $X_n(w)$'s and $Y_n(w)$'s have the same distribution $F_n(x)$. $X(w)$ and $Y(w)$ have the same distribution $F(x)$. f is continuous, so $f(Y_n(w)) \to f(Y(w))$ everywhere.

and $\mathbb{E}f(X_n) = \mathbb{E}f(Y_n)$ as $X_n \overset{d}{=} Y_n$. for $\forall f$ bounded & continuous. so does $\mathbb{E}f(X) = \mathbb{E}f(Y)$

To prove $\lim_{n\to+\infty} \mathbb{E}f(Y_n) = \mathbb{E}\lim_{n\to\infty} f(Y_n) = \mathbb{E}f(Y)$ use Dominated Convergence Theorem.

$\underbrace{\lim_{n\to\infty} f(Y_n) = f(Y)}$ everywhere.

since f is bounded: $f(Y_n) \to f(Y)$ and $|f(Y_n)| < M$. for some $M>0$. and $EM = M < \infty$. $\Rightarrow$

$$\lim_{n\to+\infty} \mathbb{E}f(Y_n) = \mathbb{E}\lim_{n\to\infty} f(Y_n) = \mathbb{E}f(Y)$$

$$\Rightarrow \lim_{n\to\infty} \mathbb{E}f(X_n) = \mathbb{E}f(X) \Rightarrow \mathbb{E}f(X_n) \to \mathbb{E}f(X).$$

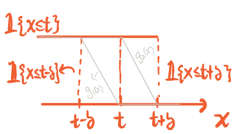
"$\Leftarrow$" Need to show that $F_n(t) \to F(t)$ for $\forall t$ s.t. $F(t)$ is cont-s at t.

Let $F(t)$ is continuous at t, then we'll have: $\forall \varepsilon>0$, get $\delta>0$, s.t. $F(t+\delta) - F(t-\delta) < \varepsilon$ ♦.

Now we need to show $\lim_{n\to\infty} F_n(t) = F(t)$.

$F(t) = \mathbb{P}(X\le t) = \mathbb{E}\,\mathbb{1}\{X\le t\}$ $\quad F_n(t) = \mathbb{P}(X_n\le t) = \mathbb{E}\,\mathbb{1}\{X_n\le t\}$ But $\mathbb{1}(\cdot)$ is not continuous!

Hence we'll use the bounded and continuous function $g_1(x)$ and $g_2(x)$ to approach $\mathbb{1}\{x\le t\}$



construct $g_1(x)$ and $g_2(x)$ as left-side figure. Then g_1 & g_2 are cont-s and bounded and

$$\mathbb{1}\{x\le t-\delta\} \le g_1(x) \le \mathbb{1}\{x\le t\} \le g_2(x) \le \mathbb{1}\{x\le t+\delta\} \quad \forall x.$$

take expectation for both sides: $F(t-\delta) \le \mathbb{E}g_1(X) \le F(t) \le \mathbb{E}g_2(X) \le F(t+\delta)$ $\quad$ *.

$\Rightarrow \mathbb{1}\{X_n\le t-\delta\} \le g_1(X_n) \le \mathbb{1}\{X_n\le t\} \le g_2(X_n) \le \mathbb{1}\{X_n\le t+\delta\}$ (take expectation for all r.v.s)

$\Rightarrow F_n(t-\delta) \le \mathbb{E}g_1(X_n) \le F_n(t) \le \mathbb{E}g_2(X_n) \le F_n(t+\delta)$

since g_1 & g_2 are bounded and continuous. $\Rightarrow$ $\mathbb{E}g_1(X_n) \to \mathbb{E}g_1(X)$ and $\mathbb{E}g_2(X_n) \to \mathbb{E}g_2(X)$

$\forall \varepsilon>0$, $\exists N>0$ s.t. $\forall n\ge N$: $|\mathbb{E}g_1(X_n) - \mathbb{E}g_1(X)| < \varepsilon$ & $|\mathbb{E}g_2(X_n) - \mathbb{E}g_2(X)| < \varepsilon$.

$\exists N>0$ s.t. $\forall n\ge N$: ♥. $\mathbb{E}g_1(X) - \mathbb{E}g_2(X) - \varepsilon \le \mathbb{E}g_1(X_n) - \mathbb{E}g_2(X) \le F_n(t) - F(t) \le \mathbb{E}g_2(X_n) - \mathbb{E}g_1(X) \le \mathbb{E}g_2(X) - \mathbb{E}g_1(X) + \varepsilon$

Also by *. & ♥. we have

$$\Rightarrow |F_n(t) - F(t)| \overset{\text{triangle ineq.}}{\le} |\mathbb{E}g_1(X) - \mathbb{E}g_2(X)| + \varepsilon = \mathbb{E}g_2(X) - \mathbb{E}g_1(X) + \varepsilon < F(t+\delta) - F(t-\delta) + \varepsilon \underset{\text{by } ♦.}{<} \varepsilon + \varepsilon = 2\varepsilon.$$

$\forall \varepsilon>0$, $\exists N>0$ s.t. $\forall n\ge N$: $|F_n(t) - F(t)| < 2\varepsilon$. $\Rightarrow F_n(t) \to F(t)$. for $\forall t$ cont-s pt. of $F(\cdot)$ ⊠.

Multivariate Extension

- Converge to r.v.

$$
\begin{array}{ccc}
 & \underline{X}_n \xrightarrow{s} \underline{X} & (\mathbb{E}\|\underline{X}_n-\underline{X}\|^s \to 0) \\
(\mathbb{P}(\limsup_n \{\|\underline{X}_n-\underline{X}\|>\varepsilon\})=0) & \Downarrow & \\
\underline{X}_n \xrightarrow{a.e.} \underline{X} \;\Rightarrow & \underline{X}_n \xrightarrow{P} \underline{X} & (\mathbb{P}(\|\underline{X}_n-\underline{X}\|>\varepsilon)\to 0) \\
 & \Downarrow & \\
 & \underline{X}_n \xrightarrow{D} \underline{X} & (F_n(\underline{x}) \to F(\underline{x}) \text{ for } \underline{x} \text{ cont-s pt. of } F(\cdot)\,)
\end{array}
$$

- Converge to constant.

$$
\begin{array}{cc}
 & \underline{X}_n \xrightarrow{s} \underline{c} \\
 & \Downarrow \\
\underline{X}_n \xrightarrow{a.e.} \underline{c} \;\Rightarrow & \underline{X}_n \xrightarrow{P} \underline{c} \\
 & \Updownarrow \\
 & \underline{X}_n \xrightarrow{D} \underline{c}
\end{array}
$$

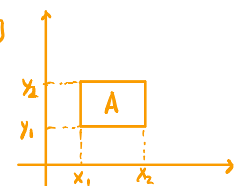
Note that in Multivariate case, CDF of $\underline{X}$ should satisfy 4 properties: (e.g. 2-dim)

1. $\lim_{x_1, x_2 \to \infty} F(x_1, x_2) = 1$. $\lim_{x_1 \text{ or } x_2 \to -\infty} F(x_1, x_2) = 0$. and $F(\cdot,\cdot) \in [0,1]$

2. $F(x_1, x_2)$ right-continuous on each coordinate.

3. $F(x_1, x_2)$ non-decreasing on each coordinate

4*. $P((X_1, X_2) \in A) = F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1) \geq 0$



[Construct an example of $F(\cdot,\cdot)$ satisfying 1-3 but 4 fails!].

Weak Law of Large Numbers. [sample mean converge to mean in probability]

*. Under which conditions is WLLN true ?

1. $\{X_n\}$ i.i.d. r.v. with finite mean,

a. If $\sigma^2 = Var(X_1) < \infty$ finite variance, then $\bar{X}_n \xrightarrow{P} \mu$

<proof> $P(|\bar{X}_n - \mu| > \varepsilon) \overset{\text{Chebyshev's}}{\leq} \frac{Var(\bar{X}_n)}{\varepsilon^2} = \frac{Var(\frac{\sum_{i=1}^n X_i}{n})}{\varepsilon^2} = \frac{n \cdot \sigma^2}{n^2 \varepsilon^2} = \frac{\sigma^2}{n \varepsilon^2} \to 0 \quad (n \to \infty)$

$\Rightarrow \lim_{n\to+\infty} P(|\bar{X}_n - \mu| > \varepsilon) = 0$. ($\bar{X}_n \xrightarrow{P} \mu$). ∎

b. If $\mathbb{E}X_1^4 < \infty$, then the sample variance $S_n^2 \xrightarrow{P} \sigma^2$.

<proof>
$S_n^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n)^2 = \frac{1}{n}\left[\sum_{i=1}^n (X_i-\mu)^2 - n(\mu - \bar{X}_n)^2\right] = \frac{1}{n}\sum_{i=1}^n (X_i-\mu)^2 - (\mu-\bar{X}_n)^2$ since $\mathbb{E}X_1^4 < \infty$ Let $Y_i = (X_i-\mu)^2$.

[If $\mathbb{E}|X|^r < \infty$ then $\forall\, 0<s<r$: $\mathbb{E}|X|^s \leq \max\{1, \mathbb{E}|X|^r\} \leq 1 + \mathbb{E}|X|^r < \infty$, $-\infty < -\mathbb{E}|X|^s \leq \mathbb{E}X^s \leq \mathbb{E}|X|^s < \infty$]

then $Var(Y_i) = \mathbb{E}(X_i-\mu)^4 + [\mathbb{E}(X_i-\mu)^2]^2 < \infty$. and $\mathbb{E}Y_i = \sigma^2 < \infty$. By a. $(\mu - \bar{X}_n) \xrightarrow{P} 0$ and also by a. we have

$\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i \xrightarrow{P} \sigma^2$ and $S_n^2 = \underset{\downarrow P \atop \sigma^2}{\bar{Y}} - \underset{\downarrow P \atop 0}{(\mu - \bar{X}_n)^2} \xrightarrow{P} \sigma^2$ ⊠

[If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$ then $X_n + Y_n \xrightarrow{P} X+Y$]
$P(|X_n + Y_n - X - Y| > \varepsilon)$
$\le P(|X_n - X| > \varepsilon$ or $|Y_n - Y| > \varepsilon)$
$\le P(|X_n - X| > \varepsilon) + P(|Y_n - Y| > \varepsilon) \to 0$.]

2. $\{X_n\}$ independent but not indentically distributed; $EX_i = \mu$, $Var(\bar{X}) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) \to 0$

<proof> By Chebyshev's $P(|\bar{X}_n - \mu| > \varepsilon) \le \frac{Var(\bar{X})}{\varepsilon^2} \to 0$. $\Rightarrow \bar{X}_n \xrightarrow{P} \mu$.

3. Non-independent but identically distributed case. Assume $EX_i = \mu$. $Var(X_i) = \sigma^2$.
[Eg. Time Series Setting] $Var(\bar{X}_n) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} cov(X_i, X_j) = \frac{2}{n^2}\sum_{1 \le i < j \le n} cov(X_i, X_j) + \frac{1}{n}\sigma^2$.
If $Var(\bar{X}_n) \to 0$, then $\bar{X}_n \xrightarrow{P} \mu$.

Eg.1: Stationary process ($cov(X_i, X_{i+k}) = C_k \ \forall i$) : $Var(\bar{X}_n) = \frac{\sigma^2}{n} + \frac{2}{n^2}\sum_{k=1}^{n-1} cov(X_1, X_{1+k}) \cdot (n-k) \le \frac{\sigma^2}{n} + \frac{2}{n}\sum_{k=1}^{n-1} cov(X_1, X_{1+k})$
provided $cov(X_1, X_n) \to 0$ as $n \to +\infty$. $\Rightarrow \frac{2}{n}\sum_{k=1}^{n-1} cov(X_1, X_{1+k}) \to 0$ (If $Y_k \to Y$ then $S_k = \frac{1}{n}\sum_{k=1}^{n} Y_k \to Y$)
$\Rightarrow Var(\bar{X}_n) \le \frac{\sigma^2}{n} + \frac{2}{n} \cdot \sum_{k=1}^{n-1} cov(X_1, X_{1+k}) \to 0$. then $\bar{X}_n \xrightarrow{P} \mu$

Eg.2: m-dependent case: $(X_1, \dots, X_i)$ and $(X_j, X_{j+1}, \dots)$ are independent if $j - i > m$. then
$Cov(X_k, X_n) = 0$ If $(n-k) > m$ since $X_k \perp\!\!\!\perp X_n$ If $n-k > m$. then $\frac{1}{n^2}\sum_{|i-j| \le m} Cov(X_i, X_j) \le \frac{1}{n^2} \cdot n \cdot 2mC \to 0$

Strong Law of Large Numbers [Sample mean converge to mean a.e.]

(SLLNs holds for pairwise iid. r.v.'s with finite mean ($E|X_1| < \infty$), then $\bar{X}_n \to EX_1$ a.e.

1. Let X_i be i.i.d. r.v's. If $E|X_1| < \infty$, $EX_1^4 < \infty$, then $\bar{X}_n \to EX_1$ a.e.

<proof> Chebyshev's $P(|\bar{X}_n - EX_1| > \varepsilon) \le \frac{E(\bar{X}_n - EX_1)^4}{\varepsilon^4} = \frac{E(\sum_{i=1}^{n}(X_i - EX_1))^4}{n^4 \cdot \varepsilon^4} = \frac{E(\sum_{i=1}^{n} Y_i)^4}{n^4 \cdot \varepsilon^4}$. (Let $Y_i = X_i - EX_1$, Y_i have zero mean)

$E(\sum_{i=1}^{n} Y_i)^4 = nEY_i^4 + \underbrace{\binom{n}{2}}_{n \text{ choose } i \ne j} \cdot \underbrace{\binom{4}{2}}_{\sum\sum\sum\sum_{i\,j\,k\,h} \text{ 4 choose 2/2.}} EY_i^2 EY_j^2 = n\tau + \frac{n(n-1)}{2} \cdot 6 \cdot \sigma^2$ [Let $EY_i^4 = \tau$ and $EY_i^2 = \sigma$]

$\Rightarrow P(|\bar{X}_n - EX_1| > \varepsilon) \le \frac{n\tau + 3n(n-1)\sigma^2}{n^4 \varepsilon^4} \le \frac{3n(n-1)(\sigma^2 + \tau)}{n^4 \varepsilon^4} \le \frac{3C}{n^2 \varepsilon^4}$ for some constant $C > 0$.
$\sum_{n=1}^{\infty} P(|\bar{X}_n - EX_1| > \varepsilon) \le \sum_{n=1}^{\infty} \frac{3C}{n^2 \varepsilon^4} < \infty$ By Borel-Cantelli Lemma: $P(\limsup_n |\bar{X}_n - EX_1| > \varepsilon) = 0$

2. [Kolmogorov's SLLN]: Let X_n's be independent mean zero r.v.s If

$$\sum_{i=1}^{n} \frac{1}{i^2} Var(X_i) < \infty \quad \text{then} \quad \bar{X}_n \to 0 \text{ a.e.}$$

<proof> Let $Y_k = \max_{2^{k-1} \le n < 2^k} |\bar{X}_n|$ [Recall Kolmogorov's Ineqn.]
[X_i's indep with zero mean, $S_n = \sum_{i=1}^{n} X_i$ then: $P(\max_{1 \le k \le n} |S_k| > \varepsilon) \le \frac{Var(S_n)}{\varepsilon^2}$]

$P(Y_k > \varepsilon) = P(\max_{2^{k-1} \le n < 2^k} \frac{1}{n}|S_n| > \varepsilon) \le P(\max_{2^{k-1} \le n < 2^k} \frac{1}{2^{-k} \cdot 2}|S_n| > \varepsilon) = P(\max_{2^{k-1} \le n < 2^k} |S_n| > 2^{(k-1)} \cdot \varepsilon)$
since $2^{-k} \le \frac{1}{n} \le 2^{-(k-1)}$, $\varepsilon < \frac{1}{n}|S_n| \le 2^{-(k-1)}|S_n|$

$\le P\left(\max_{1\le n<2^k} |S_n| > 2^{(k-1)}\cdot\varepsilon\right) \overset{\text{Kolmogorov's Ineq.}}{\le} \frac{Var(S_{2^k})}{2^{2(k-1)}\cdot\varepsilon^2} = \frac{4}{4^k\cdot\varepsilon^2}\sum_{i=1}^{2^k} Var(X_i)$

$k=1$: $\frac{1}{4}Var(X_1)$ $\frac{1}{4}Var(X_2)$

$k=2$: $\frac{1}{4^2}Var(X_1)$ $\frac{1}{4^2}Var(X_2)$ $\frac{1}{4^2}Var(X_3)$ $\frac{1}{4^2}Var(X_4)$

$\Rightarrow \sum_{k=1}^{\infty} P(Y_k>\varepsilon) \le \sum_{k=1}^{\infty}\frac{4}{4^k\varepsilon^2}\sum_{i=1}^{2^k}Var(X_i) = \frac{4}{\varepsilon^2}\sum_{k=1}^{\infty}\sum_{i=1}^{2^k}Var(X_i)\cdot\frac{1}{4^k} = \frac{4}{\varepsilon^2}\sum_{i=1}^{\infty}\sum_{\substack{k\ge\log_2 i\\(i\le 2^k)}}\frac{1}{4^k}Var(X_i) = \frac{4}{\varepsilon^2}\sum_{i=1}^{\infty}\left[\sum_{k=m_i}^{\infty}\frac{1}{4^k}\right]Var(X_i)$ $*.$

since non-neg. value's infinite sums

Where m_i is the smallest integer number such that. $m_i\ge\log_2 i$

$2^{m_i}\ge i \Rightarrow 4^{m_i}\ge i^2 \Rightarrow \frac{1}{4^{m_i}}\le\frac{1}{i^2} \Rightarrow *. = \frac{4}{\varepsilon^2}\sum_{i=1}^{\infty}\left(\frac{\frac{1}{4^{m_i}}}{1-\frac{1}{4}}\right)Var(X_i) \le \frac{4}{\varepsilon^2}\sum_{i=1}^{\infty}\frac{4}{3i^2}Var(X_i) = \frac{16}{3\varepsilon^2}\sum_{i=1}^{\infty}\frac{Var(X_i)}{i^2}$

given $\sum_{i=1}^{\infty}\frac{Var(X_i)}{i^2}<\infty \Rightarrow \sum_{k=1}^{\infty}P(Y_k>\varepsilon)\le\frac{16}{3\varepsilon^2}\sum_{i=1}^{\infty}\frac{Var(X_i)}{i^2}<\infty \Rightarrow$ Borel-Cantelli Lemma:

$Y_k \xrightarrow{a.e.} 0 : 1 = P(\liminf_k |Y_k|<\varepsilon) = P\left(\liminf_k \max_{2^{k-1}\le n\le 2^k}|\bar{X}_n|<\varepsilon\right) \le P(\liminf_n |\bar{X}_n|<\varepsilon) \le 1$

$\max_{2^{k-1}\le n<2^k}|\bar{X}_n|<\varepsilon \Rightarrow |\bar{X}_n|<\varepsilon$ for $\forall n\in 2^{k-1}\le n<2^k$.

$\Rightarrow \bar{X}_n \xrightarrow{a.e.} 0$

3. $\{Y_i\}$ are indep. and $\sum_{i=1}^{\infty}\frac{1}{i^2}Var(Y_i)<\infty$ then $\frac{1}{n}\sum_{i=1}^{n}(Y_i-\mathbb{E}Y_i)\to 0$ a.e..

<proof> Let $X_i = Y_i - \mathbb{E}Y_i$ then X_i's indep. & zero-means. $\Rightarrow$ KLLNs: $\bar{X}_n\to 0$ a.e.

i.e. $\frac{1}{n}\sum_{i=1}^{n}(Y_i-\mathbb{E}Y_i)\to 0$ a.e.

4. X_n's are i.i.d. r.v.'s and $\mathbb{E}|X_1|<\infty$. then $\bar{X}_n\to\mathbb{E}X_1$ a.s.

① truncated $x_i\to Z_i$ s.t. $\sum_{i=1}^{n}\frac{1}{i^2}Var(Z_i)<\infty$.
Z_i's are indep. KLLNs: $\frac{1}{n}\left(\sum_{i=1}^{n}(Z_i-\mathbb{E}Z_i)\right)\to 0$ a.e.
② $\frac{1}{n}\sum_{i=1}^{n}Z_i-\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}Z_i\to 0$ a.e.
since $X_n-Z_n\to 0$ a.e. $\Rightarrow \frac{1}{n}\left(\sum_{i=1}^{n}X_i-\sum_{i=1}^{n}Z_i\right)\to 0$ a.e.
③ $\mathbb{E}Z_i\to\mathbb{E}X_1$ By DCT Theorem. ④ $\frac{1}{n}\sum_{i=1}^{n}Z_i-\frac{1}{n}\sum_{i=1}^{n}X_i\to 0$ a.e.
$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}X_i\to\mathbb{E}X_1$ a.e.

<proof> ① Truncation $Z_i = X_i\mathbb{1}\{|X_i|\le i\}$. therefore

$Var(Z_i) = \mathbb{E}Z_i^2 - [\mathbb{E}Z_i]^2 \le \mathbb{E}Z_i^2 = \mathbb{E}X_i^2\cdot\mathbb{1}\{|X_i|\le i\} \overset{\text{since } X_i\overset{d}{=}X_1}{=} \mathbb{E}X_1^2\mathbb{1}\{|X_1|\le i\}$

② We want to firstly show $\frac{1}{n}\sum_{i=1}^{n}(Z_i-\mathbb{E}Z_i)\to 0$ a.e., If using KLLNs, we need to bound $\sum_{i=1}^{\infty}\frac{1}{i^2}Var(Z_i)$.

$\sum_{i=1}^{\infty}\frac{1}{i^2}Var(Z_i) \le \sum_{i=1}^{\infty}\frac{1}{i^2}\cdot\mathbb{E}X_1^2\mathbb{1}\{|X_1|\le i\} = \mathbb{E}\sum_{i=1}^{\infty}\frac{1}{i^2}X_1^2\mathbb{1}\{|X_1|\le i\} = \mathbb{E}X_1^2\cdot\left(\sum_{i\ge|X_1|}^{\infty}\frac{1}{i^2}\right)$

To show: $\sum_{i=1}^{\infty}\frac{1}{i^2}\mathbb{E}X_1^2\mathbb{1}\{|X_1|\le i\} = \mathbb{E}\sum_{i=1}^{\infty}\frac{1}{i^2}X_1^2\mathbb{1}\{|X_1|\le i\}$ use Monotone Convergence Theorem:

Since $\mathbb{E}\sum_{i=1}^{n}\underbrace{\frac{1}{i^2}X_1^2\mathbb{1}\{|X_1|\le i\}}_{Y_i} = \mathbb{E}\sum_{i=1}^{n}Y_i > 0$ and $0<\sum_{i=1}^{n}Y_n \nearrow \sum_{i=1}^{\infty}Y_i = \sum_{i=1}^{\infty}\frac{1}{i^2}X_1^2\mathbb{1}\{|X_1|\le i\}$ then by MCT:

$\lim_{n\to\infty}\mathbb{E}\sum_{i=1}^{n}Y_i = \mathbb{E}\sum_{i=1}^{\infty}Y_n$ i.e. $\lim_{n\to\infty}\sum_{i=1}^{n}\mathbb{E}Y_i = \sum_{i=1}^{\infty}\mathbb{E}Y_i = \mathbb{E}\sum_{i=1}^{\infty}Y_n$

① $|X_1|\le 1$: $X_1^2\le 1$ and $\mathbb{E}X_1^2\sum_{i\ge|X_1|}^{\infty}\frac{1}{i^2}\cdot I(|X_1|\le 1) \le \mathbb{E}\,1\cdot\underbrace{\sum_{i=1}^{\infty}\frac{1}{i^2}}_{\frac{\pi^2}{6}} I(|X_1|\le 1) = c_1\cdot P(|X_1|\le 1) \le c_1$

② $|X_1|>1$: $\frac{1}{i^2} = \int_{i-1}^{i}\frac{1}{i^2}dx \le \int_{i-1}^{i}\frac{1}{x^2}dx$ $\Rightarrow \sum_{i\ge|X_1|}^{\infty}\frac{1}{i^2} \le \int_{|X_1|-1}^{\infty}\frac{1}{x^2}dx = \frac{1}{|X_1|-1}$

since $\frac{1}{i^2}\le\frac{1}{x^2}\le\frac{1}{(i-1)^2}$

since $|X_1|\to\infty$ then $\frac{1}{|X_1|-1}\sim\frac{1}{|X_1|}$

$\Rightarrow \mathbb{E}X_1^2\left(\sum_{i\ge|X_1|}^{\infty}\frac{1}{i^2}\right)\cdot I(|X_1|>1) \le \mathbb{E}X_1^2\cdot\frac{1}{|X_1|-1}\cdot I(|X_1|>1) \le \mathbb{E}X_1^2\frac{1}{|X_1|-1} \overset{?}{\le} \mathbb{E}X_1^2\cdot\frac{c}{|X_1|} = \mathbb{E}|X_1|\cdot c_2$

$\Rightarrow \mathbb{E}X_1^2\left(\sum_{i\ge|X_1|}^{\infty}\frac{1}{i^2}\right)\cdot I(|X_1|>1) \le c_2\,\mathbb{E}|X_1|$

$\Rightarrow \mathbb{E}X_1^2 \left(\sum_{i\ge|X_1|}^{\infty} \frac{1}{i^2}\right) \le C(\mathbb{E}|X_1|+1) < \infty$. for $C=\max\{C_2, C_1\} > 0$.

$\Rightarrow \sum_{i=1}^{\infty} \frac{1}{i^2}\mathrm{Var}(Z_i) < \infty$ To show $\frac{1}{n}\sum_{i=1}^{n} Z_i \xrightarrow{a.e.} \mathbb{E}X_1$ we have to show $\mathbb{E}Z_i \to \mathbb{E}X_1$.

③ since $|Z_i| = |X_i \mathbb{1}\{|X_i|\le i\}| \le |X_i|$ and $\mathbb{E}|X_i| = \mathbb{E}|X_1| < \infty$. By DCT:

$\lim_{i\to+\infty} \mathbb{E}Z_i = \mathbb{E}\lim_{i\to+\infty} Z_i = \mathbb{E} \lim_{i\to+\infty} X_i \mathbb{1}\{|X_i|\le i\} = \mathbb{E}X_i = \mathbb{E}X_1$. $\Rightarrow \mathbb{E}Z_i \to \mathbb{E}X_1$

By $\sum_{i=1}^{\infty}\frac{1}{i^2}\mathrm{Var}(Z_i)<\infty$ and Z_i indep. and $\mathbb{E}Z_i\to \mathbb{E}X_1$ $\Rightarrow \frac{1}{n}\sum_{i=1}^{n} Z_i \to \mathbb{E}X_1$ a.e.

$\frac{1}{n}\sum_{i=1}^{n} (Z_i - X_i) = \frac{1}{n}\sum_{i=1}^{n} Z_i - \frac{1}{n}\sum_{i=1}^{n} X_i$ $\quad P(X_i\ne Z_i) = P(|X_i|>i) = P(|X_1|>i)$

④ and $\sum_{i=1}^{\infty} P(X_i\ne Z_i) = \sum_{i=1}^{\infty} P(|X_1|>i) \le \sum_{i=1}^{\infty} P(|X_1|\ge i) \le \mathbb{E}|X_1| < \mathbb{E}|X_1|+1<\infty$.

Any non-neg. r.v. Y $\quad \sum_{i=1}^{\infty} P(Y\ge i) \le \mathbb{E}Y \le \sum_{i=1}^{\infty} P(Y\ge i)+1$

By Borel-Cantelli: $P(X_i\ne Z_i \text{ i.o.}) = 0$.

⑤ $\Rightarrow (X_n - Z_n)\to 0$ a.e. $\Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} Z_i = \frac{1}{n}\left(\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} Z_i\right) \to 0$ $(n\to+\infty)$ a.e.

[$\exists N>0$ s.t. $\forall n\ge N$: $X_n = Z_n$]

$\Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} Z_i \to 0$ a.e. $\Rightarrow$ as $\frac{1}{n}\sum_{i=1}^{n} Z_i \xrightarrow{a.e.} \mathbb{E}X_1$ $\Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i \to \mathbb{E}X_1$ a.e.

$\Rightarrow$ If $X_1,\dots,X_n,\dots$ iid and $\mathbb{E}|X_1|<\infty$ then $\frac{1}{n}\sum_{i=1}^{n} X_i \to \mathbb{E}X_1$ a.e.

Converse of SLLN [Stronger version]

$X_1,\dots,X_n$ i.i.d. and If $\frac{S_n}{n}\to c$ a.e. for some c real, then $\mathbb{E}|X_1|<\infty$ and $c=\mathbb{E}X_1$

<proof> $X_n = S_n - S_{n-1}$ and $\bar{X}_n = \frac{S_n}{n}$.

$$\Rightarrow \frac{X_n}{n} = \frac{S_n - S_{n-1}}{n} = \frac{S_n}{n} - \frac{S_{n-1}}{n-1}\cdot\frac{n-1}{n} = \bar{X}_n - \bar{X}_{n-1}\cdot\frac{n-1}{n} \to c - c\cdot 1 = 0 \quad \text{a.s.}$$

i.e. $\forall \varepsilon>0$, $\exists N>0$ s.t. $\forall n\ge N$: $\left|\frac{X_n}{n}\right| < \varepsilon$.

proof by contradiction : $\sum_{n=1}^{\infty} P(|X_n|>n) < \infty$.

suppose $\sum_{n=1}^{\infty} P(|X_n|>n) = \infty$, then since X_i's iid, By Borel-Cantelli theorem, $\Rightarrow P(\limsup_n \{|X_n|>n\}) = 1$ i.e. $P(|X_n|>n \text{ i.o.}) = 1$

i.e. $P\left(\left|\frac{X_n}{n}\right|>1 \text{ i.o.}\right)=1$ contradicts to $\frac{X_n}{n}\to 0$ a.s.

therefore $\sum_{n=1}^{\infty} P(|X_n|>n) < \infty$ $\Rightarrow$ Also by the property of non-negative r.v.'s,

$$\sum_{n=1}^{\infty} P(|X_n|>n) < \mathbb{E}|X_n| \overset{iid}{=} \mathbb{E}|X_1| < 1+\sum_{n=1}^{\infty} P(|X_n|>n) < \infty$$

$\Rightarrow \mathbb{E}|X_1|<\infty$ so By SLLN: $\frac{S_n}{n}\to \mathbb{E}X_1$ a.e. $\Rightarrow c=\mathbb{E}X_1$.

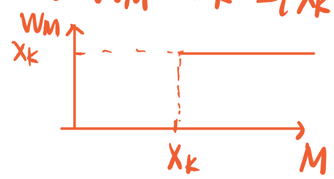
Infinite mean case Theorem:

$X_i > 0$ and X_i's iid with $\mathbb{E}X_1 = \infty$, then $\frac{S_n}{n} \to \infty$ a.s.

proof: Let $Y_k = X_k \cdot \mathbb{1}\{X_k < M\}$, $0 < \mathbb{E}Y_k = \mathbb{E}Y_1 < M$

By SLLN: $\frac{\sum_{i=1}^{n} Y_k}{n} \to \mathbb{E}Y_1$ a.s. Also Let $W_M = X_k \cdot \mathbb{1}\{X_k < M\}$

$W_M > 0$ and $W_M \nearrow X_k$ as $M \to +\infty$.



so By MCT: $\mathbb{E}W_M \to \mathbb{E}X_k = \mathbb{E}X_1 = \infty$.

$$\frac{\sum_{i=1}^{n} X_i}{n} \geq \frac{\sum_{i=1}^{n} Y_i}{n} \xrightarrow[a.s.]{SLLN} \mathbb{E}Y_1 = \mathbb{E}W_M \longrightarrow +\infty \quad (M \to +\infty)$$

$\Rightarrow \frac{S_n}{n} \to +\infty$ a.s.

Therefore, X_i iid Cauchy r.v.s $\mathbb{E}X_i = \int_{-\infty}^{\infty} x \cdot \frac{1}{1+x^2}\,dx$ is not defined!

$\mathbb{E}|X| = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|x|}{1+x^2}\,dx = \frac{2}{\pi}\int_0^{\infty} \frac{x}{1+x^2}\,dx = \infty$ By Infinite mean case Thm:

$$\frac{\sum_{i=1}^{n} |X_i|}{n} \to \infty \quad \text{a.s.}$$

Charateristic Functions (Fourier Transform) i.e. $\int e^{itx}\,dF(x)$ $[e^{ix} = \cos x + i \sin x]$

[Motivation: MGF is finite then it has all the moments! But what if MGF is not finite? Charateristic Functions always exists.]

$\varphi_F(t) = \mathbb{E}e^{itX} = \varphi_X(t)$ where $X \sim F$.

$\mathbb{E}[e^{it(aX+b)}] = e^{itb}\varphi_X(at)$

1. $|\varphi(t)| \leq 1$ $\quad |\mathbb{E}e^{itx}| \overset{\text{Jencen's}}{\leq} \mathbb{E}|e^{itx}| = \mathbb{E}|\cos tx + i \sin tx| = \mathbb{E}1 = 1$
2. $\varphi(0) = 1$ $\quad \varphi(0) = \mathbb{E}e^0 = 1.$ $\quad \varphi(-t) = \mathbb{E}e^{-itx} = \cos tx - i\sin tx = \overline{\mathbb{E}e^{itx}} = \overline{\varphi(t)}$
3. $\varphi(-t) = \overline{\varphi(t)}$ $\quad$ [If X has a symmetric distribution, then φ_X is a real fun]
4. $\sum_{r,s=1}^{k} a_r \bar{a}_s \varphi(t_r - t_s) \geq 0$ for $t_1, \dots, t_k$ real, complex $a_1, \dots, a_k$.

(1–4) $\Leftrightarrow$ Characteristic Function of a r.v.

Basic Complex Operations: $\overline{e^{ix}} = \overline{\cos x + i\sin x} = \cos x - i\sin x = e^{-ix}$

$$\overline{xy} = \bar{x}\cdot\bar{y}$$
$$\overline{x+y} = \bar{x}+\bar{y}$$

$$\sum_{r=1}^{k}\sum_{s=1}^{k} a_r \bar{a}_s \varphi(t_r - t_s) = \sum_{r=1}^{k}\sum_{s=1}^{k} a_r \bar{a}_s \mathbb{E} e^{i(t_r-t_s)X} = \mathbb{E}\sum_{r=1}^{k} e^{it_rX}\cdot a_r \sum_{s=1}^{k} \bar{a}_s e^{-it_sX}$$

$$= \mathbb{E}\Big[\underbrace{\sum_{r=1}^{k} e^{it_rX}\cdot a_r}_{z} \underbrace{\sum_{s=1}^{k} \overline{a_s e^{it_sX}}}_{\bar{z}}\Big] = \mathbb{E}\, z\cdot\bar{z} = \mathbb{E}\underbrace{|z|^2}_{\text{real}} \geq 0$$

Properties of Characteristic Functions.

(a) φ is uniformly continuous:

$$|\varphi(t+h) - \varphi(t)| = |\mathbb{E}e^{itX}(e^{ihX}-1)| \overset{\text{Jensen's}}{\leq} \mathbb{E}|e^{itX}(e^{ihX}-1)| \overset{\text{C-S Ineq.}}{\leq} \mathbb{E}|e^{itX}|\cdot|e^{ihX}-1|$$

$$= \mathbb{E}\, 1\cdot|e^{ihX}-1| = \mathbb{E}|e^{ihX}-1| \quad \text{since } |e^{ihX}-1| \leq |e^{ihX}|+1 \leq 2$$

and $\lim_{h\to 0}|e^{ihX}-1| = |\lim_{h\to 0} e^{ihX}-1| = 0$ By DCT: $\mathbb{E}|e^{ihX}-1| \to 0$ as $h\to 0$

so $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall |x-y|<\delta$, $x\in\mathbb{R}$, $y\in\mathbb{R}$:

$$\sup_{x,y} |\varphi(x) - \varphi(y)| < \varepsilon$$

($\forall \varepsilon>0, \exists\delta>0$ s.t. $|h|<\delta$: $\mathbb{E}|e^{ihX}-1|<\varepsilon$)

[No t here!]

(b) $\varphi_{X+Y} = \varphi_X\varphi_Y$ for $X \perp\!\!\!\perp Y$. [prove $X_i \overset{iid}{\sim}$ Cauchy then $\bar{X}_n \overset{D}{=} X_1$]

(c) ① $\forall x > 0$: $|e^{ix}-1| = \left|\int_0^x ie^{iy}\,dy\right| \overset{\text{Jensen's}}{\leq} \int_0^x |ie^{iy}|dy = \int_0^x 1\,dy = x = |x|$

Also $|e^{ix}-1| \leq |e^{ix}| + |1| = 1+1 = 2$

$\Rightarrow |e^{ix}-1| \leq \min\{2, |x|\}$

$\forall x<0$: $|e^{ix}-1| = |1-e^{ix}| = \left|\int_x^0 ie^{iy}\,dy\right| \overset{\text{Jensen's}}{\leq} \int_x^0 1\,dy = -x = |x|$.

$\Rightarrow |e^{ix}-1| \leq \min\{2, |x|\}$

② $|e^{ix}-1-ix| = \left|\int_0^x i(e^{iy}-1)\,dy\right| \overset{\text{Jensen's}}{\leq} \int_0^{|x|} |ie^{iy}-i|\,dy = \int_0^{|x|} |e^{iy}-1|\,dy$

$\overset{\text{by ①}}{\leq} \int_0^{|x|} \min(2,|y|)\,dy = \min\{2|x|, \frac{1}{2}x^2\}$

③ $\left|e^{ix} - \sum_{k=0}^{n}\frac{(ix)^k}{k!}\right| \leq \min\left\{\frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!}\right\}$

<pf by induction> $\left|e^{ix}-\sum_{k=0}^{n+1}\frac{(ix)^k}{k!}\right| = \left|\int_0^x i\left(e^{iy}-\sum_{k=0}^{n}\frac{(iy)^k}{k!}\right)dy\right| \overset{\text{Jensen's}}{\le} \int_0^{|x|}\left|e^{iy}-\sum_{k=0}^{n}\frac{(iy)^k}{k!}\right|dy$

$\overset{\text{by③}}{\le} \int_0^{|x|} \min\left\{\frac{|y|^{n+1}}{(n+1)!}, \frac{2|y|^n}{n!}\right\}dy = \min\left\{\frac{|x|^{n+2}}{(n+2)!}, \frac{2|x|^{n+1}}{(n+1)!}\right\}$

Moments and Derivatives of a Characteristic Function.

Theorem. If $E(|X|^n)<\infty$, then n^{th} derivative $\varphi^{(n)}$ of φ exists,

① $\varphi^{(n)}(t)=E((iX)^n e^{itX})$ and $\varphi(t)=\sum_{k=0}^{n}\frac{(it)^k}{k!}EX^k+o(t^n)$ as $t\to 0$ ②

$\varphi^{(k)}(0)=i^k EX^k$ and $\varphi^{(n)}(t)$ is uniformly continuous. ③

In particular, if $EX^2<\infty$, then $\varphi(t)=1+itEX-\frac{t^2}{2}EX^2+h(t)\cdot t^2$, where

② $|h(t)|\le E\min(|t|\cdot|X|^3, X^2)\to 0$ as $t\to 0$. By DCT.

Also note: ④ $\frac{1}{h}(\varphi(t+h)-\varphi(t))-EiXe^{itX} = E\left(e^{itX}\left[\frac{1}{h}(e^{ihX}-1-ihX)\right]\right)$

<proof> prove ①:

$$\frac{\varphi(t+h)-\varphi(t)}{h} = E\left[\frac{e^{i(t+h)X}-e^{itX}}{h}\right] = E\left[\frac{e^{itX}(e^{ihX}-1)}{h}\right]$$

$$\left|\frac{e^{itX}(e^{ihX}-1)}{h}\right| \overset{C\text{-}S}{\le} |e^{itX}|\cdot\left|\frac{e^{ihX}-1}{h}\right| \le 1\cdot\left|\frac{e^{ihX}-1}{h}\right| \le \frac{\min\{2,|hX|\}}{h} \le \frac{|Xh|}{h}\le|X|$$

and $E|X|<\infty$.

$$\lim_{h\to 0}\frac{\varphi(t+h)-\varphi(t)}{h} = \varphi'(t) = \lim_{h\to 0}E\left[\frac{e^{i(t+h)X}-e^{itX}}{h}\right]$$

use DCT: $= E\lim_{h\to 0}\left[\frac{e^{i(t+h)X}-e^{itX}}{h}\right] = E\left[iX\cdot e^{itX}\right]$

$\Rightarrow \varphi'(t)=EiXe^{itX}$ exists if $E|X|<\infty$.

similarly. $\frac{\varphi'(t+h)-\varphi'(t)}{h} = E\left[\frac{iXe^{itX}(e^{ihX}-1)}{h}\right]$

$\left|\frac{iXe^{itX}(e^{ihX}-1)}{h}\right| \overset{C\text{-}S}{\le} \frac{1\cdot|X|\cdot\min\{2,|hX|\}}{h} \le |X|^2$ and $E|X|^2=EX^2<\infty$

By DCT: $\varphi''(t)=E(iX)^2e^{itX}$ ... ① $\varphi^{(k)}(t)=E(iX)^k e^{itX}$. If $E|X|^k<\infty$ ∎

prove ③:

$$|\varphi^{(k)}(t+h) - \varphi^{(k)}(h)| = |\mathbb{E}(iX)^k \cdot e^{itX}[e^{ihX}-1]| \overset{\text{Jensen's}}{\le} \mathbb{E}|(iX)^k \cdot e^{itX}[e^{ihX}-1]|$$

$$\overset{C-S}{\le} \mathbb{E}|X|^k \cdot 1 \cdot |e^{ihX}-1|$$

since $|X|^k \cdot |e^{ihX}-1| \le |X|^k \cdot \min\{2, |hX|\} \le |X|^k \cdot 2$ and $\mathbb{E}|X|^k < \infty$

By DCT: $\lim_{h\to 0} \mathbb{E}|X|^k \cdot |e^{ihX}-1| = \mathbb{E} \lim_{h\to 0} |X|^k \cdot |e^{ihX}-1| = 0$.

(not depends on t!) $\forall \varepsilon>0, \exists \delta>0$, s.t. $|h|<\delta$: $\mathbb{E}|X|^k|e^{ihX}-1|<\varepsilon$ $\to$ so as long as $|x-y|<\delta$, it holds!

$\Rightarrow \forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall |y_1 - y_2| < \delta$: $\sup_{y_1, y_2} |\varphi^{(k)}(y_1) - \varphi^{(k)}(y_2)|$

$\Rightarrow \varphi^{(k)}(t)$ uniformly continuous.

③ ☒

prove ②

$$\varphi(t) - \sum_{k=0}^{2} \mathbb{E}X^k \cdot \frac{(it)^k}{k!} = \varphi(t) - 1 - \mathbb{E}X \cdot it - \mathbb{E}X^2 \cdot \frac{(it)^2}{2}$$

$$= \mathbb{E}\left(e^{itX} - 1 - itX - \frac{(it)^2}{2}X^2\right)$$

and $\left|e^{itX} - 1 - itX - \frac{(it)^2}{2}X^2\right| \le \min\left\{\frac{|tX|^3}{3!}, \frac{2|tX|^2}{2}\right\} = t^2 \min\left\{\frac{|t||X|^3}{6}, |X|^2\right\}$

1st term is for $\lim_{t\to 0} \frac{|t|\cdot|X|^3}{6} = 0$ (for $\forall$ realization X) $\to$ 0 as $t \to 0$.

2nd term is for DCT: $|\cdots| \le t^2|X|^2$ and $\mathbb{E}|X|^2 < \infty$.

By DCT:

$\Rightarrow \varphi(t) - 1 - \mathbb{E}X \cdot it - \mathbb{E}X^2 \cdot \frac{(it)^2}{2} = o(t^2) \to 0$ as $t \to 0$

i.e. $\varphi(t) - 1 - \mathbb{E}X \cdot it - \mathbb{E}X^2 \cdot \frac{(it)^2}{2} = o(t^2)$ and $\lim_{t\to 0} \frac{o(t^2)}{t^2} = 0$ ☒

④ $\frac{1}{h}[\varphi(t+h) - \varphi(t)] - \mathbb{E}[iXe^{itX}] = \mathbb{E}e^{itX}\left(\frac{1}{h}(e^{ihX} - 1 - ihX\right)$

LHS $= \mathbb{E}e^{itX}\left[\frac{e^{ihX}}{h} - \frac{1}{h} - iX\right] = \mathbb{E}e^{itX} \cdot \left(\frac{1}{h}(e^{ihX} - 1 - ihX)\right) =$ RHS ☒

Inversion Theorem:

Let F be the distr. of a r.v. of X. Then

$$\lim_{T\to\infty} I_T = P(a<X<b) + \frac{1}{2}P(X=a) + \frac{1}{2}P(X=b) \quad \text{where}$$

$$I_T = \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it}\varphi_X(t)\,dt.$$

$a<b$.

$\boxed{\text{Inversion Thm implies uniqueness.}}$

<proof>

$$I_T = \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita}-e^{-itb}}{it}\cdot E e^{itX}\,dt = \frac{1}{2\pi}\int_{-T}^{T} E\,\frac{e^{it(X-a)}-e^{it(X-b)}}{it}\,dt$$

By Fubini's

$$= \frac{1}{2\pi} E\int_{-T}^{T} \frac{e^{it(X-a)}-e^{it(X-b)}}{it}\,dt$$

Since $\left|\frac{e^{it(X-a)}-e^{it(X-b)}}{it}\right| \le b-a$

$\left|e^{itX}\cdot\frac{e^{-ita}-e^{-itb}}{it}\right| \le 1\cdot(b-a)$

Also $$\int_{-T}^{T}\frac{e^{itY}}{it}\,dt = \int_{-T}^{T}\underbrace{\frac{\cos tY}{it}}_{\text{odd}}\,dt + \int_{-T}^{T}\underbrace{\frac{\sin tY}{t}}_{\text{even}}\,dt$$

$b>a$ since $\left|\frac{e^{-ita}-e^{-itb}}{it}\right| \le b-a.$

<proof> $\left|\frac{e^{-it(a-b)}-1}{it}\right|\cdot\left|e^{-itb}\right| \le b-a$

$\Leftrightarrow \left|\frac{e^{-it(a-b)}-1}{it}\right| \le b-a.$

$$= 2\int_0^T \frac{\sin tY}{t}\,dt = \begin{cases} -2\int_0^T \frac{\sin t|Y|}{t}\,dt & Y<0 \\ 2\int_0^T \frac{\sin t|Y|}{t}\,dt & Y>0 \end{cases}$$

Let $t|Y| = s$, $\begin{cases} t = \frac{s}{|Y|} \\ s\in[0, T|Y|] \end{cases}$

$$= \begin{cases} -2\int_0^{T|Y|}\frac{\sin s}{s}\,ds \\ 2\int_0^{T|Y|}\frac{\sin s}{s}\,ds \end{cases}$$

Bounded. $\Rightarrow$ interchange of integrals.

$\Rightarrow$ Fubini's Thm.

$\Rightarrow \left|\frac{e^{-itx}-1}{it}\right| \le x$ for $x>0$.

LHS $= \left|\int_0^x e^{-ity}\,dy\right|$

LHS $= \left|\frac{e^{ity}}{-it}\Big|_0^x\right| = \left|\frac{e^{-itx}}{-it}+\frac{1}{it}\right| \overset{\text{Jensen's}}{\le} \int_0^x \left|e^{-ity}\right|dy = \int_0^x 1\,dy = x$

$\Rightarrow \left|\frac{e^{-it(a-b)}-1}{it}\right| \le b-a.$

Let $S(x) = \int_0^x \frac{\sin s}{s}\,ds.$ $\Rightarrow \int_{-T}^{T}\frac{e^{itY}}{it}\,dt = S(T|Y|)\cdot 2\cdot \operatorname{sgn}(Y).$

$$\Rightarrow I_T = \frac{1}{2\pi}E\int_{-T}^{T}\frac{e^{it(X-a)}-e^{it(X-b)}}{it}\,dt$$

$$= \frac{1}{\pi}E\left[\operatorname{sgn}(X-a)\cdot S(T|X-a|) - \operatorname{sgn}(X-b)\cdot S(T|X-b|)\right] \triangleq E\psi_T(X)$$

$$\lim_{x\to+\infty} S(x) = \lim_{x\to+\infty}\int_0^x \frac{\sin s}{s}\,ds = \int_0^{+\infty}\frac{\sin s}{s}\,ds = \underbrace{\int_0^{+\infty}\frac{1}{1+x^2}\,dx}_{\text{Cauchy}} = \arctan(x)\Big|_0^{\infty} = \frac{\pi}{2} \quad \bigstar$$

How to interchange limit & expectation? Dominated Conv. Thm!

$$\left|\operatorname{sgn}(X-a)\cdot S(T|X-a|) - \operatorname{sgn}(X-b) S(T|X-b|)\right| \le |S(T|X-a|)| + |S(T|X-b|)|$$

[How to show that it is bounded?] since $\lim_{x\to+\infty} S(x) = \frac{\pi}{2}$ so $|S(x)|$ is bounded

① $a \quad b \quad X$ then $\psi_T(X) = \frac{1}{\pi}\left[S(T|X-a|) - S(T|X-b|)\right] \to 0$ as $T\to+\infty$

② $a \quad X \quad b$ then $\psi_T(X) = \frac{1}{\pi}\left[S(T|X-a|) + S(T|X-b|)\right] \to 1$ as $T\to+\infty$

③ $\overset{X}{a}$ or $\overset{X}{b}$ then $\psi_T(X) = \begin{cases} \frac{1}{\pi}S(T|X-b|) & X=a & \to \frac{1}{2} \text{ as } T\to\infty \\ \frac{1}{\pi}S(T|X-a|) & X=b & \to \frac{1}{2} \text{ as } T\to\infty \end{cases}$

$$\Rightarrow \psi_T(X) \longrightarrow \mathbb{1}\{a<X<b\} + \frac{1}{2}\mathbb{1}\{X=a\} + \frac{1}{2}\mathbb{1}\{X=b\} \quad \text{as } T\to+\infty.$$

By DCT $\lim_{T\to+\infty} I_T = \lim_{T\to+\infty} \mathbb{E}\,\psi_T(X) \overset{DCT}{=} \mathbb{E} \lim_{T\to+\infty} \psi_T(X) = \mathbb{E}\, \mathbb{1}\{a<X<b\} + \frac{1}{2}\mathbb{1}\{X=a\} + \frac{1}{2}\mathbb{1}\{X=b\}$

$= P(a<X<b) + \dfrac{P(X=a)+P(X=b)}{2}$ ☒

Why Inversion Theorem implies uniqueness?

If $\varphi_X(t)=\varphi_Y(t)$, then $P_X=P_Y$:

Let a/b be continuous point of F_X & F_Y

$\Rightarrow P_X(X\in(a,b]) = P_X(X\in(a,b)) = \lim_{T\to\infty} I_{T,X} = \lim_{T\to\infty} I_{T,Y} = P_Y(Y\in(a,b])$

Let $a\to-\infty \Rightarrow F_X(b)=F_Y(b)$ for continuous pt b.

If $P_X=P_Y$ then by c.f.'s definition: $\varphi_X(t)=\varphi_Y(t)$

Therefore $\varphi_X(t)=\varphi_Y(t) \Leftrightarrow P_X=P_Y$

Relation between characteristic func. and density:

Corollary: If $\int_{-\infty}^{\infty} |\varphi_X(t)|\,dt < \infty$, then F is differentiable everywhere and its density f is given by $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\varphi_F(t)\,dt$.

[If $X_n \xrightarrow{D} X \Rightarrow h_t(x)=e^{itx}$ is cont-s & bounded $\forall t \Rightarrow \varphi_{X_n}(t)\to\varphi_X(t)$]

⟨proof⟩ 1. $F(x)$ is continuous everywhere:

$$|I_T| \le \frac{1}{2\pi}\int_{-T}^{T} \left|\frac{e^{ita}-e^{-itb}}{it}\right| \cdot |\varphi(t)|\,dt \le \frac{b-a}{2\pi}\int_{-T}^{T}|\varphi(t)|\,dt$$

$$\lim_{T\to+\infty}|I_T| \le \frac{b-a}{2\pi}\int_{-\infty}^{\infty}|\varphi(t)|\,dt < \infty$$

$$\frac{1}{2}P(X=a) \le \lim_{T\to+\infty} I_T \le C\cdot(b-a) \longrightarrow 0 \quad \text{as } b\to a.$$

$\Rightarrow P(X=a)=0. \quad \Rightarrow F(X)$ is continuous at a. ($\forall a$.)

$\Rightarrow F(X)$ is continuous.

$P(x \le X \le x+h) = \lim_{T\to\infty} I_T(x, x+h)$.

then
$$\frac{F(x+h)-F(x)}{h} = \lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-itx}-e^{-it(x+h)}}{ith}\phi_X(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-itX}(1-e^{-ith})}{ith}\phi_X(t)\,dt.$$

since
$$\left|\frac{F(x+h)-F(x)}{h}\right| \overset{\text{Jensen's}}{\le} \frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\left|e^{-itX}\right|}_{=1}\cdot\underbrace{\left|\frac{1-e^{-ith}}{ith}\right|}_{\le 1}\cdot|\phi_X(t)|\,dt$$

≤ 1. since $\frac{|1-e^{-ith}|}{|th|} \le \frac{\min\{2, |th|\}}{|th|} \le 1$.

$$\le \frac{1}{2\pi}\int_{-\infty}^{\infty}|\phi_X(t)|\,dt < \infty.$$

so $\left|\frac{e^{-itx}(1-e^{-ith})}{ith}\phi_X(t)\right| \le |\phi_X(t)|$ and $\int_{-\infty}^{\infty}|\phi_X(t)|\,dt < \infty$ is integrable.

Therefore, by Dominated Convergence Theorem:

$$f(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-itX}(1-e^{-ith})}{ith}\phi_X(t)\,dt.$$

$$\overset{\text{D.C.T.}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itX}\lim_{h\to 0}\frac{(1-e^{-ith})}{ith}\phi_X(t)\,dt$$

$$\overset{\text{L'Hospital}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itX}\cdot\phi_X(t)\cdot\lim_{h\to 0}\frac{e^{-ith}(it)}{it}\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itX}\cdot\phi_X(t)\,dt$$

$$\Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itX}\cdot\phi_X(t)\,dt.$$