Motivation: prove the Continuity Theorem

Continuity Theorem: Let F, F_n be prob. distribution functions with characteristic function φ, φ_n

(a) If $F_n \xrightarrow{D} F$, then $\varphi_n(t) \longrightarrow \varphi(t)$

(b) If $\varphi_n(t) \to g(t)$ for each t and $\{F_n\}$ is tight, then there exists a prob. distr. function F s.t. $F_n \xrightarrow{D} F$ and $g(t)$ is the c.f. of F.

(c) If $\varphi_n(t) \to g(t)$ for each t, where g is continuous at 0, then there exists a prob. distr. function F s.t. $F_n \xrightarrow{D} F$ and g is the c.f. of F.

<proof of (a)>

If $F_n \xrightarrow{D} F$, i.e. $X_n \xrightarrow{D} X$ then let $h_t(X) = e^{itX}$, so $h_t(X)$ is a continuous function, then by Continuous Mapping Theorem,

$h_t(X_n) \xrightarrow{D} h_t(X)$ since $\varphi_n(t) = \mathbb{E}h_t(X_n)$ and $\varphi(t) = \mathbb{E}h_t(X)$.

since $|h_t(X)| \le 1$ is bounded, by Dominated Convergence Theorem,

$\mathbb{E}h_t(X_n) \longrightarrow \mathbb{E}h_t(X)$ i.e. $\varphi_n(t) \to \varphi(t)$. $\boxtimes$

<proof of (b)> 1. How to show the existence of such a F?

2. Why we need tightness of $\{F_n\}$?

existence of such a distr. function F is proved by Helly's Thm + Tightness

Helly's Thm: For every sequence $\{F_n\}$ of prob. distr. functions, there exists a subseq. $\{F_{n_k}\}$ and a non-decreasing right-continuous function F s.t. $F_{n_k}(x) \to F(x)$ at continuity point x of F.

<proof> Diagonal Method:

Let $\mathbb{Q}$ = set of rational #s in $\mathbb{R}$, so $\mathbb{Q}$ is countable: enumerate the rationals $r_1, r_2, r_3, \ldots$

[We can approximate any #s in $\mathbb{R}$ by rational #s!]

Now: Construct a subseq. $\{F_{n_k}\}$ s.t. $F_{n_k}(r_j) \to G(r_j)$ for all $j=1,2,\cdots$ for some function G.

$0 \le F_n(r_1) \le 1$ for $\forall n$. Since $\forall$ Bounded Sequence, there exists a subseq. converges, denote it as $\{F_{n_k^{(1)}}(r_1)\}$. so $F_{n_k^{(1)}}(r_1) \to G(r_1)$ $(k \to \infty)$. for some function $G(r_1)$

$$\begin{array}{lllll}
F_{n_1^{(1)}}(q_1), & F_{n_2^{(1)}}(q_1), & F_{n_3^{(1)}}(q_1), & \dots, & F_{n_k^{(1)}}(q_1), \dots \\
F_{n_1^{(2)}}(q_2), & F_{n_2^{(2)}}(q_2), & F_{n_3^{(2)}}(q_2), & \dots, & F_{n_k^{(2)}}(q_2), \dots \\
F_{n_1^{(3)}}(q_3), & F_{n_2^{(3)}}(q_3), & F_{n_3^{(3)}}(q_3), & \dots, & F_{n_k^{(3)}}(q_3), \dots \\
\vdots & \vdots & \vdots & & \\
F_{n_1^{(k)}}(q_k), & F_{n_2^{(k)}}(q_k), & F_{n_3^{(k)}}(q_k), & \dots, & F_{n_k^{(k)}}(q_k), \dots \\
\vdots & \vdots & \vdots & & \vdots \quad \ddots
\end{array}$$

$\{F_{n_k^{(1)}}(q_2)\}$ is a bounded seq. then there exists a subseq. $\{F_{n_k^{(2)}}(q_2)\}$ converges to $G(q_2)$

$\vdots$

Choose the diagonal indices.

$\{n_1^{(1)}, n_2^{(2)}, \dots, n_k^{(k)}, \dots\}$

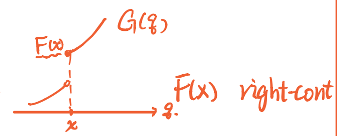
denote it as $\{n_1, n_2, \dots, n_k\}$.

so $\{F_{n_k}\}$ is a subseq. of $\{F_n\}$ and $F_{n_k}(q_j) \to G(q_j)$ $\forall j = 1, 2, 3, \dots$

so $F_{n_k}(q) \to G(q)$ for any $q \in \mathbb{Q}$. Define $F(x) = \inf_{q > x} \{G(q)\}$ for any $x \in \mathbb{R}$.

$F(x)$ is non-decreasing, $x \uparrow \{q : q > x\} \downarrow$ $\inf_{q>x}\{G(q)\} \uparrow \Rightarrow F(x) \uparrow$

$F(x)$ is right-continuous : $G(q)$ is also non-decreasing in $\mathbb{Q}$.



Now to show $F_{n_k}(x) \to F(x)$ for all x continuous pt of $F(x)$.

If x is a continuous pt of $F(x)$:

① $x \in \mathbb{Q}$: $F(x) = \inf_{q > x}\{G(q)\} = G(x)$

② $x \notin \mathbb{Q}$: By x is a cont-s pt:

for $\forall \varepsilon > 0$: $\exists\, q^- < x < q^+$ s.t. $F(x) - F(q^-) < \varepsilon$ and $F(q^+) - F(x) < \varepsilon$

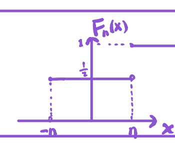
$$\begin{cases} \limsup_k F_{n_k}(x) \le \limsup_k F_{n_k}(q^+) = \lim_{k \to \infty} F_{n_k}(q^+) = F(q^+) < F(x) + \varepsilon. \\ \liminf_k F_{n_k}(x) \ge \liminf_k F_{n_k}(q^-) = \lim_{k\to\infty} F_{n_k}(q^-) = F(q^-) > F(x) - \varepsilon \end{cases}$$

$\Rightarrow$ limit exists and $\lim_{k \to \infty} F_{n_k}(x) = F(x)$

$\Rightarrow$ $F_{n_k}(x) \to F(x)$, $\forall x$. $F(x)$ non-decreasing & right continuous ☒

However, only by Helly's Thm, $F(x)$ might not be a valid CDF:

Counter-example : $P(X_n = -n) = P(X_n = n) = \frac{1}{2}$ then



So $F_n(x) \to F(x)$ and $F(x) = \frac{1}{2}$.

But $F(x) = \frac{1}{2}$ for $\forall x$, $F(x)$ is not a valid CDF!

We need more assumption on $\{F_n\}$: Tightness!

Tightness : A family Π of probability distribution function is tight if for every $\varepsilon > 0$, $\exists a, b$ real, s.t. $F(b) - F(a) > 1 - \varepsilon$ for all $F \in \Pi$.

[An Equivalent Definition: A family Π of prob. distr. func. is tight if $\forall \varepsilon>0$, $\exists K_\varepsilon>0$ s.t.

$$F(K_\varepsilon) - F(-K_\varepsilon) > 1-\varepsilon \quad \text{for all } F\in\Pi. \quad \text{i.e.}$$

$P(|X_n| < K_\varepsilon) > 1-\varepsilon$ i.e. $P(|X_n|\geq K_\varepsilon)<\varepsilon$. for all $X_n \sim F_n$ and $F_n\in\Pi$.]

Tightness and Relative Compactness.

Thm. Tightness of the seq. $\{F_n\}$ of prob. distr. func. is a necessary & sufficient condition that for $\forall$ subseq. $\{F_{n_k}\}$, there exists a further subseq. $\{F_{n_{k(j)}}\}$ and a prob. distr. func. F s.t. $F_{n_{k(j)}}(x) \xrightarrow{D} F(x)$ as $j\to+\infty$. [Why we need $\forall$subseq's $\exists$ further subseq? Necessity!]

<proof>

Sufficiency: $\forall$ subseq. of $\{F_n\}$: $\{F_{n_k}\}$, Apply Helly's Thm: $\exists$ a further subseq. $\{F_{n_{k(i)}}\}$ & F s.t. $F_{n_{k(i)}}(x) \to F(x)$ ($i\to+\infty$) for $\forall$ continuous pt x of F.

F is non-decreasing & right continuous.

By tightness, $\forall\varepsilon>0$, $\exists a_\varepsilon, b_\varepsilon$ s.t. $a_\varepsilon, b_\varepsilon$ cont-s pts of F and $1\geq F_n(b_\varepsilon)-F_n(a_\varepsilon)>1-\varepsilon$ for $\forall n$

$\Rightarrow 1\geq F(b_\varepsilon)-F(a_\varepsilon)\geq 1-\varepsilon$ by taking limit $n\to+\infty$. Also $\begin{cases} 0\leq F(b_\varepsilon)\leq 1 \\ 0\leq F(a_\varepsilon)\leq 1\end{cases}$

$$\Rightarrow \quad 1\geq \lim_{b\to+\infty}F(b) - \lim_{a\to-\infty}F(a) \geq 1-\varepsilon. \quad (\forall\varepsilon>0).$$

$$\Rightarrow \quad \lim_{b\to+\infty}F(b) - \lim_{a\to-\infty}F(a) = 1 \Rightarrow 1\geq \lim_{b\to+\infty}F(b) = 1+\lim_{a\to-\infty}F(a)\geq 1$$

$$\Rightarrow \lim_{b\to+\infty}F(b)=1 \text{ and } \lim_{a\to-\infty}F(a)=0.$$

$\Rightarrow F(x)$ is a distribution func.

Necessity: If a seq. $\{F_n\}$, $\forall$subseq. $\{F_{n_k}\}$, $\exists$ a further subseq. $\{F_{n_{k(j)}}\}$ s.t.

$F_{n_{k(j)}}(x) \xrightarrow{D} F(x)$. $F(x)$ a distr. function. then $\{F_n\}$ is tight.

proof by contradiction: If $\{F_n\}$ is not tight: $\exists\varepsilon>0$, $\forall(a,b]$: $F_n(b)-F_n(a)\leq 1-\varepsilon$ for some n. Choose n_k s.t. $F_n(K)-F_n(-K)\leq 1-\varepsilon$. so a further subseq. $F_{n_{k(j)}} \xrightarrow{D} F$, then

$$F(b)-F(a) = \lim_{k\to+\infty}[F_{n_k}(b)-F_{n_k}(a)]$$

$$\leq \lim_{k\to+\infty}[F_{n_k}(K)-F_{n_k}(-K)] \leq 1-\varepsilon.$$

$\Rightarrow F(b)-F(a) \le 1-\varepsilon$ for $\forall a,b$ But since $F(x)$ is a distr. func. so $b\to+\infty$ / $a\to-\infty$ then LHS $=1 > 1-\varepsilon$ Contradiction!

$\Rightarrow \{F_n\}$ is tight! $\boxtimes$

Corollary: If $\{F_n\}$ is a tight seq. of prob. distr. func., and if each subseq. $\{F_{n_k}\}$, $\exists$ further subseq $\{F_{n_{k(j)}}\}$ s.t. $\{F_{n_{k(j)}}\}$ converges weakly to the prob. distr. func. F (i.e. $F_{n_{k(j)}}(x) \xrightarrow{D} F(x)$), then $F_n \xrightarrow{D} F$

pick x cont-s pt of F, then suppose $F_n(x) \not\to F(x)$. then $\exists$ subseq $\{F_{n_k}\}$ s.t. $\forall k$: $|F_{n_k}(x)-F(x)|>\varepsilon$, $\nexists$ further subseq. $\to F(x)$. contradicts!

Back to the proof of Continuity Theorem part (b)!

<proof of (b)> If $\varphi_n(t)\to g(t)$ for each t and that $\{F_n\}$ is tight, for $\{F_n\}$, By Tightness & Relative Compactness Thm: $\forall$ a subseq. $\{F_{n_k}\}$, $\exists$ a further subseq. $\{F_{n_{k(j)}}\}$ s.t. $F_{n_{k(j)}} \xrightarrow{D} G$ and G is a distr. func.

then by (a), we'll have $\varphi_{n_{k(j)}}(t) \to \varphi_G(t)$ $\forall t$.

Also, since $\varphi_n(t)\to g(t)$ $\forall t$ $\Rightarrow$ $\varphi_{n_{k(j)}}(t)\to g(t)$ $\forall t$

$\Rightarrow \varphi_G(t)=g(t) \Rightarrow g(t)$ is the c.f. of G. [for any sub-subseq! all converges to $g(t)$]

so By Inversion Thm: $\varphi_X(t)$ and $F_X(x)$ is a one-to-one mapping [uniqueness].

so any sub-subseq. $F_{n_{k(j)}}(x) \xrightarrow{D} G(x)$ $\forall$ cont-s pt x of $G(x)$. Since all sub-subseq's c.f. $\to g(t)$ $(j\to+\infty)$

Therefore, each subseq. $\{F_{n_k}\}$, $\exists$ a further subseq. $\{F_{n_{k(j)}}\}$ s.t.

$F_{n_{k(j)}}(x) \xrightarrow{D} G(x)$ for all x cont-s pt of $G(x)$. and $\varphi_G(t)=g(t)$ $\Rightarrow$ By Corollary

$$F_n(x) \xrightarrow{D} F \quad \boxtimes$$

<proof of (c)> If $\varphi_n(t)\to g(t)$ for each t where g is cont-s at 0, then $\exists$ a prob. distr. func. F s.t. $F_n \xrightarrow{D} F$ and g is the c.f. of F.

We've already proved (b). so the difference betwn (b) & (c) is g is cont-s at 0 / $\{F_n(x)\}$ is tight. So we only need to prove g is cont-s at 0 can imply tightness of $\{F_n(x)\}$, then we can use (b)'s result.

Let X, X_n be r.v.'s w/. distr. F, F_n respectively.

$$\frac{1}{u}\int_{-u}^{u}(1-\varphi_n(t))\,dt = \mathbb{E}\left[\frac{1}{u}\int_{-u}^{u}(1-e^{itX_n})dt\right] = \mathbb{E}\left[\frac{1}{u}\int_{-u}^{u}(1-\cos tX_n - i\sin tX_n)\,dt\right]$$

$$= \mathbb{E}\left[\frac{1}{u}\int_{-u}^{u}(1-\cos tX_n)\,dt\right] = 2\,\mathbb{E}\left[\frac{1}{u}\int_{0}^{u}(1-\cos tX_n)\,dt\right] = 2\,\mathbb{E}\left[\frac{1}{u}\left(u-\frac{\sin uX_n}{X_n}\right)\right]$$

$$= 2\,\mathbb{E}\left[1-\frac{\sin uX_n}{uX_n}\right] \qquad \left|\frac{\sin x}{x}\right| = \left|\frac{e^{ix}-e^{-ix}}{2ix}\right| \le |e^{-ix}|\cdot\left|\frac{e^{2ix}-1}{2x}\right| = \left|\frac{e^{2ix}-1}{2x}\right| \le \frac{|2x|}{|2x|}=1$$

$$\ge 2\,\mathbb{E}\left[\left(1-\frac{1}{|uX_n|}\right)\right] \qquad \text{since } 1-\frac{\sin uX_n}{uX_n} \ge 1-\frac{1}{|uX_n|} \text{ for any } X_n \,\&\, u$$

$$\ge 2\,\mathbb{E}\left[\left(1-\frac{1}{|uX_n|}\right)\cdot\mathbb{1}\{|X_n|\ge 2/u\}\right] \qquad \left[\text{since } \frac{\sin uX_n}{uX_n} \le \frac{|\sin uX_n|}{|uX_n|} \le \frac{1}{|uX_n|}\right]$$

$$\ge 2\,\mathbb{E}\left[\frac{1}{2}\cdot\mathbb{1}\{|X_n|\ge 2/u\}\right] \qquad \begin{cases} \text{when } |uX_n|\ge 2: & 1-\frac{1}{|uX_n|}\ge\frac{1}{2}. \\ \text{when } |uX_n|<2: & 0. \end{cases}$$

$$= P\left(|X_n|\ge\frac{2}{u}\right)$$

and $\frac{1}{u}\int_{-u}^{u}(1-\varphi_n(t))\,dt \to \frac{1}{u}\int_{-u}^{u}(1-g(t))\,dt$ $(n\to+\infty)$ and

$g(t)\to g(0)=1$ when $t\to 0$.

So $\frac{1}{u}\int_{-u}^{u}(1-\varphi_n(t))\,dt < \frac{1}{u}\int_{-u}^{u}(1-g(t))\,dt + \varepsilon \to 0+\varepsilon=\varepsilon$.
for large $n\ge N$.

$\Rightarrow P\left(|X_n|\ge\frac{2}{u}\right) < \frac{1}{u}\int_{-u}^{u}(1-\varphi_n(t))\,dt \le \varepsilon$ for large $n.\ge N$

for $1\le i<N$:

Choose u_i: $P\left(|X_i|\ge\frac{2}{u_i}\right) < \varepsilon$. for $1\le i<N$.

Take $\max\left\{\frac{2}{u}, \frac{2}{u_i}, 1\le i<N\right\} = K_\varepsilon$.

$\Rightarrow P(|X_n|\ge K_\varepsilon) < \varepsilon \Rightarrow \{F_n\}$ is tight by second definition.

$\Rightarrow$ by (b). $F_n(x)\xrightarrow{\mathcal{D}}F(x)$ for cont-s pt x of F, a distr. func. with c.f. $g(t)$