Nonparametric Identification and Estimation of Level-k Auctions

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February 14, 2011

Bidding Strategies in First-price Sealed-bid Auctions

1. Bidder i forms her own valuation of the object: v_i
   - Bidders' values are private and independent
   - Common knowledge: value distribution, number of bidders
2. Bidder i maximizes her expected utility function

$$U_i = (v_i - b_i) \Pr(\max_{j \neq i} b_j < b_i)$$

   - Winning probability $\Pr(\max_{j \neq i} b_j < b_i)$ depends on bidder i's belief about her opponents' bidding behavior

- Bidders' beliefs about their opponents' bidding behavior?
  - Approach 1: perfectly correct beliefs $\rightarrow$ Nash equilibrium
    - Nash equilibrium is too demanding: seldom observed e.g. experimental auctions (Cox, Smith, and Walker, 1983, 1988); p-beauty contest (Nagel, 1995; Ho, Camerer, and Weigelt, 1998)

The Level-k Auction Model: Heterogeneous Beliefs

- Bidders have different levels of sophistication $\Rightarrow$ Heterogenous (possibly incorrect) beliefs about others' behavior
- Beliefs (types) have a hierarchical structure (Stahl and Wilson, 1994, 1995; Nagel 1995)

Type	Belief about other bidders' behavior
1	all other bidders are type-$L0$ (bid naïvely)
2	all other bidders are type-1
$\vdots$	$\vdots$
k	all other bidders are type-$(k-1)$

  - Specification of type-$L0$ is crucial, assumed by the researchers

- Explains overbidding in experimental auctions (Crawford and Iriberri, 2007)
- Level-k belief explains non-equilibrium behavior in other games (Costa-Gomes and Crawford, '06; Crawford and Iriberri, '07a)

Motivation and Contributions

- Existing evidence on the level-k auction model are mixed (Crawford and Iriberri, 2007, Georganas, 2009, Gillen 2009; Ivanov, Levin, and Niederle, 2010) because the evidence
  - relies on experimental data and parametric assumptions
  - depends on (arbitrarily) specification of type-$L0$
- Motivation: two questions
  - Can we identify the model nonparametrically for field data?
  - Do field data support the level-k auction model?
- Contributions of this paper
  - A methodology for nonparametric analysis of level-k auctions
    - Applicable to other settings, e.g., Goldfarb and Xiao (2010)
  - Empirical evidence from field data that support the model
    - bidders' behavior in field auctions, policy implications

Preview of the Results

- A *nonparametric* methodology that identifies and estimates
  - The number of bidders' types
  - Probability of each type
  - Distribution of bidders' values
  - Specification of type-$L0$
- Empirical evidence from US Forest Service Timber Auctions
  - Bidders are of three types
  - Level-k auction model is supported by field data

Road Map

1. Motivation and contributions, preview of results, literature
2. Theoretical model of level-k auctions
3. Nonparametric identification
4. Nonparametric estimation
5. Monte Carlo evidence
6. Empirical application: US Forest Service timber auctions

- Basic setup
  - Risk-neutral bidders, indexed by $i \in \{1, 2, ..., I\}$, $I \geq 2$
  - Independent private value (IPV), $v_i \sim F(\cdot), v_i \perp v_j, v_i \in [\underline{v}, \overline{v}]$
  - A bidder's type $\tau \in \{(L1, \omega), (L2, \omega), ..., (Lm, \omega)\}$
    - Specification of type-$L0$: ω
    - Simplification of notation: $\tau \in \{L1, L2, ..., Lm\}$
- Bidding strategy of type-Lk bidders
  - Utility maximization problem $\max_{b_i}(v_i - b_i) \Pr(\max_{j \neq i} b_j \leq b_i)$
  - $b_i = s_{Lk}(v_i, I, F)$

The Theoretical Model

A regularity condition and properties on support of bid distribution

- Bidding strategy of bidder i (of type Lk): $b_i = s_{Lk}(v_i, I, F)$
- A regularity condition: the bidding strategy is monotonically increasing in value v_i, i.e., $s'_{Lk}(\cdot, I, F) > 0$
- Properties on supports of bid distribution functions
  - Lower bounds of bids' supports for all types are equal to $\underline{v}$
  - Upper bounds are monotonically decreasing in type
  - Properties on supports can be used to identify the model

The Data Structure

- Independent, homogeneous auctions with number of bidders I
- All the bids in each auction
- Each bidders' identity
- Three bids (in three auctions) of each bidder

Data structure

Identity	Bid 1	Bid 2	Bid 3
Bean Lumber Co.	88.51	81.32	86.69
Simmons Lumber Co.	74.53	61.35	60.47
$\vdots$	$\vdots$	$\vdots$	$\vdots$

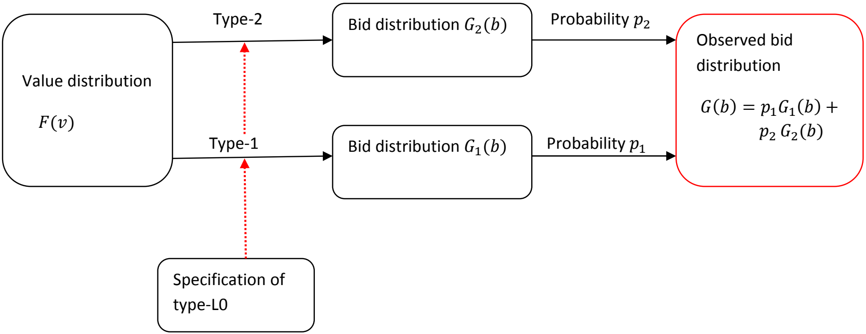
- Bids for two bidders are not necessarily from the same auction
- Number of bidders, N, is the sample size

Assumption 1. The econometrician observes data described above.

- Independent, homogeneous auctions with number of bidders I
- Bidders' are of type $L1, L2, ..., Lm$ $(m \geq 1)$:
  $p(\tau = Lk) > 0, \sum_{\tau} p(\tau) = 1$, $1 \leq k \leq m$
- Type does not change across auctions
- Value distribution is independent of type distribution
- Bidders' values: i.i.d. $v_i \sim F(v)$, $v \in [\underline{v}, \overline{v}]$
- Reserve price is non-binding

The Econometric Model

An example (two types) of the model structure



The Identification Problem

1. Observed distribution of bids $G(b)$ (b_1, b_2, or b_3): mixed behavior of different-type bidders

$$G(b) = \sum_{\tau \in \mathcal{K}} G(b|\tau)p(\tau) \tag{1}$$

   - $G(b|\tau)$: bid distribution for type τ; $p(\tau)$: probability of type τ

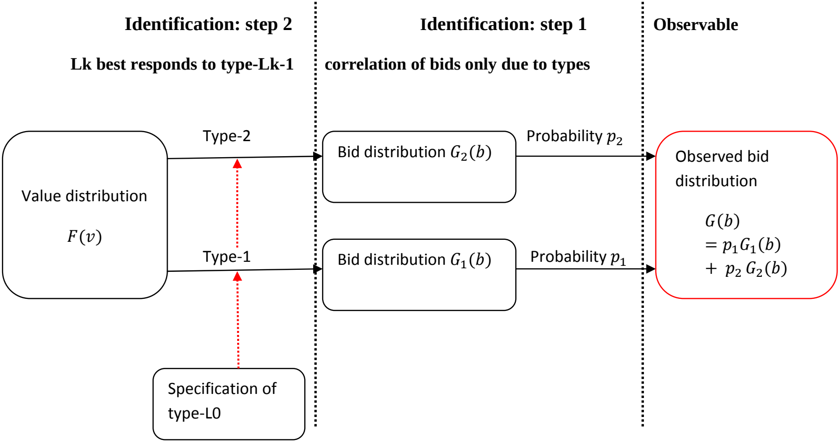
2. Bidding strategy of type τ, $s_\tau(\cdot)$, maps value distribution $F(\cdot)$ to bid distribution $G(b|\tau)$

$$G(b|\tau) = F\big(s_\tau^{-1}(b)\big) \tag{2}$$

Definition: The level-k auction model is identified if for a given observed bid distribution $G(b)$, there is a unique pair for distributions of types and values $\big(p(\cdot), F(\cdot)\big)$, and a unique specification of type-$L0$ such that Eq.(1), Eq(2) hold.

Preview of Identification

Two-step identification procedure



Preview of Identification

Intuition and methodology of identification

1. Identify number of type m, bid distribution conditional on type, $G(b|\tau)$, from observed bid distribution $G(b)$
   - Intuition: correlation of bids is only due to correlation of types
   - Methodology: Hu (2008) on misclassification

	Misclassification model	level-k auction model
true value	X^* (discrete)	τ
dependent variable	Y	3rd bid
measurement of X^*	$X : X^* + \epsilon$	discretized 1st bid
Instrument	Z	discretized 2nd bid

2. Identify value distribution and specification of type-$L0$
   - Intuition: type-Lk bidders' bidding strategy is a best response to value distribution and type-$L(k-1)$ bidders' behavior
   - Methodology: the model implied relationship between value and bid

The law of total probability

$$
\begin{aligned}
g\left(b_1, b_2, b_3\right) &= \sum_{\tau \in \mathcal{K}} g\left(b_1, b_2, b_3, \tau\right) \\
&= \sum_{\tau \in \mathcal{K}} g\left(b_2 | \tau, b_1, b_3\right) g\left(b_1 | \tau, b_3\right) g\left(\tau, b_3\right).
\end{aligned}
$$

Assumption 2. The joint density of (b_1, b_2, b_3, τ) exists and is bounded away from zero and infinity.

Assumption 3. $g(b_1, b_2, b_3|\tau) = g(b_1|\tau)g(b_2|\tau)g(b_3|\tau)$.

- Bids of different bidders with the same type are independent
- Bids for each bidder are independent across auctions

$$
g\left(b_1, b_2, b_3\right) = \sum_{\tau \in \mathcal{K}} g\left(b_1 | \tau\right) g\left(b_2 | \tau\right) g\left(\tau, b_3\right)
$$

I need two "measurements" for unobserved type τ. Choose bids b_1 and b_3, and discrete them to be the measurements.

- Discretize b_1 and b_3 as d_1 and d_3, respectively:

$$d = \begin{cases} 1 & \text{if } b \in [\underline{b}, b^1], \\ 2 & \text{if } b \in (b^1, b^2], \\ \vdots & \\ M & \text{if } b \in (b^{M-1}, \bar{b}], \end{cases}$$

$\underline{b} < b^1 < b^2 < \ldots < b^{M-1} < \bar{b}$

- Both bids are disretized to M integers
- The methods of discretization for two bids can be different

The equation associating observable and unknowns

$$g\left(b_1, b_2, b_3\right) = \sum_{\tau \in \mathcal{K}} g\left(b_1|\tau\right) g\left(b_2|\tau\right) g\left(\tau, b_3\right)$$

$\Downarrow$ discretized version

$$g\left(d_1, b_2, d_3\right) = \sum_{\tau \in \mathcal{K}} g\left(d_1|\tau\right) g\left(b_2|\tau\right) g\left(\tau, d_3\right)$$

$\Downarrow$ in matrix form

$$B_{b_2,d_1,d_3} = B_{d_1|\tau} D_{b_2|\tau} B_{\tau,d_3}$$

The diagonal matrix $D_{b_2|\tau}$ consists m bid distribution conditional on type: $g(b_2|L1), ..., g(b_2|Lm)$

Nonparametric Identification

Definition of matrices

$$
\begin{aligned}
B_{b_2,d_1,d_3} &\equiv [g(b_2, d_1 = i, d_3 = j)]_{i,j}\,, \\
B_{d_1|\tau} &\equiv [g\,(d_1 = i|\tau = Lk)]_{i,k}\,, \\
B_{\tau,d_3} &\equiv [g\,(\tau = Lk, d_3 = j)]_{k,j}\,, \\
B_{d_1,d_3} &\equiv [g\,(d_1 = i, d_3 = j)]_{i,j}\,, \\
D_{b_2|\tau} &\equiv \mathsf{diag}\Big[g(b_2|L1)\;\; g(b_2|L2)\;\; \ldots\;\; g(b_2|Lm)\Big].
\end{aligned}
$$

$$
\begin{aligned}
B_{d_1,d_3} &= B_{d_1|\tau} B_{\tau,d_3} \\
g(d_1, d_3) &= \sum_{\tau\in\mathcal{K}} g(d_1, d_3, \tau) = \sum_{\tau\in\mathcal{K}} g(d_1|\tau, d_3) g(\tau, d_3) \\
&= \sum_{\tau\in\mathcal{K}} g(d_1|\tau) g(\tau, d_3).
\end{aligned}
$$

Lemma (number of types)

The number of types, m, is equal to rank of the matrix $B_{d_1,d_3} = B_{d_1|\tau} B_{\tau,d_3}$ under the following two conditions (1) $M \geq m$; (2) bid distribution of any type is not a linear combination of those for other types.

Exemplify the lemma above: three types: $L1$, $L2$, and $L3$

$$B_{d_1|\tau} = \begin{pmatrix} \Pr(d_1 = 1|L1) & \Pr(d_1 = 1|L2) & \Pr(d_1 = 1|L3) \\ \Pr(d_1 = 2|L1) & \Pr(d_1 = 2|L2) & \Pr(d_1 = 2|L3) \\ \vdots & \vdots & \vdots \\ \Pr(d_1 = M|L1) & \Pr(d_1 = M|L2) & \Pr(d_1 = M|L3) \end{pmatrix}$$

The rank is equal to three whenever $M \geq m = 3$

Assumption 4. There exists a method of discretization such that $Rank(B_{d_1,d_3}) = m$.

- The number of types, m, is equal to the largest value of M such that the matrix B_{d_1,d_3} is full rank
- Choose $M = m$, then all the matrices are $m \times m$
- $B^{-1}_{d_1,d_3} = B^{-1}_{\tau,d_3} B^{-1}_{d_1|\tau}$ and $B_{b_2,d_1,d_3} = B_{d_1|\tau} D_{b_2|\tau} B_{\tau,d_3}$ lead to

Main Equation (Eigenvalue-eigenvector decomposition)

$$B_{b_2,d_1,d_3} B^{-1}_{d_1,d_3} = B_{d_1|\tau} D_{b_2|\tau} B^{-1}_{d_1|\tau}$$

Is the decomposition unique?

Nonparametric Identification

Uniqueness of eigenvalue-eigenvector decomposition

Assumption 5. For any two different types $Lk, Lj \in \mathcal{K}$, the set $\{b : g(b|\tau = Lk) \neq g(b|\tau = Lj)\}$ has nonzero Lebesgue measure, where $k, j \in \{1, 2, ...m\}$.

- This assumption rules out identical eigenvalues
- Normalization of eigenvectors: dividing each column by the column sum since sum of each column for $B_{d_1|\tau}$ is one
- Ordering of m eigenvalues: monotonic supports of bids for different types
- Probability of each type is identified from the mixture model

Conclusion (the first step of identification):

Bid distribution conditional on type, $g(b|\tau)$, is identified from observed bid distribution for each of the m identifed types.

Nonparametric Identification

A value-bid relationship

Proposition (A crucial value-bid relationship)

$$v \equiv \xi_{Lk}(b, G(\cdot|Lk-1), I) = b + \frac{1}{I-1}\frac{G(b|Lk-1)}{g(b|Lk-1)}.$$

Proof

- Type Lk's bidding strategy is associated with type $L(k-1)$'s bid distribution
- Intuition: type-Lk bidders' bidding strategy depends on type-$L(k-1)$'s bidding strategy and value distribution; bid distribution of type-$L(k-1)$ contains the same information

Proposition (bidding strategies and value distribution)

The inverse of bidding strategy for type-Lk bidders, $s_{Lk}^{-1}(b, F, I)$, can be identified as $s_{Lk}^{-1}(b, F, I) = \xi_{Lk}(b, G(b|Lk-1), I)$, where $k = 2, 3, ..., m$. Moreover, the value distribution $F(\cdot)$ can be uniquely determined as $F(\cdot) = G(\xi_{Lk}(\cdot, G(\cdot|Lk-1), I)|Lk)$ by bidding strategy and the corresponding bid distribution of any type-Lk, where $k = 2, ..., m$.

- Intuition: bidding strategy is a (one-to-one, monotonic) mapping from value distribution to bid distribution

$$\left.\begin{array}{c} G(b|Lk-1) \Rightarrow \xi_{Lk}(\cdot) \\ G(b|Lk) \end{array}\right\} \Rightarrow F(\cdot)$$

- Bidding strategy for bidders of type-$L1$, $s_{L1}(\cdot)$, is identified from $F(\cdot)$ and $G(\cdot|L1)$

Type-$L1$ bidders' belief: bid distribution $Z(\cdot)$, with density $z(\cdot)$

- Type-$L1$ bidder's maximization problem
  $\max_{b_i}(v_i - b_i)\Pr(\max_{j\neq i} b_j \leq b_i) = \max_{b_i}(v_i - b_i)Z^{I-1}(b_i)$
- Optimal bidding strategy $s_{L1}(\cdot)$: $s_{L1}^{-1}(b) = b + \frac{1}{I-1}\frac{Z(b)}{z(b)}$
- $s_{L1}^{-1}(b)$ is identified $\Longrightarrow Z(\cdot)$ is identified

Nonparametric Identification

Identification of type-$L0$: Cont'

$$
\begin{tikzcd}
G(b|L1) \arrow[d] \arrow[rr] & & \xi_{L1}(.)=s_{L1}^{-1}(.) \arrow[r] & z(.) \\
\xi_{L2}(.) \arrow[r] & F(.) \arrow[ur] & & \\
G(b|L2) \arrow[ur] & & &
\end{tikzcd}
$$

$$
Z(b) = \begin{cases} 0, \ b = \underline{b} \\ \exp\left\{-\frac{1}{I-1}\int_b^{\bar{b}} \frac{du}{F^{-1}(G(u|L1))-u}\right\}, otherwise \end{cases}
$$

- A testable implication (when bidders are of three or more types): CDFs of values for different types are the same

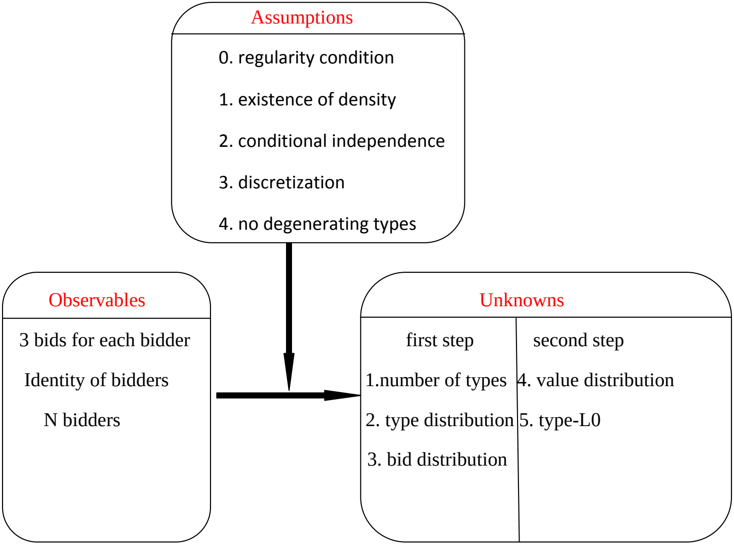
Conclusion (the second step of identification):

Value distribution and specification of type-$L0$ are both identified.

The results provide a testable implication of the model.

Nonparametric Identification

Main result on identification



Nonparametric Estimation

Overview

The identification procedure is constructive, and it implies a two-step procedure of estimation

① Estimating bid distribution and it density $G(b|\tau)$, $g(b|\tau)$, boundaries of bid distribution

$$B_{Eb_2,d_1,d_3}B^{-1}_{d_1,d_3} = B_{d_1|\tau}D_{Eb_2|\tau}B^{-1}_{d_1|\tau}$$

② Pseudo-value $\widehat{\xi}(\cdot)$, value and type distributions $F(v)$ and $p(\tau)$

$$v \equiv \xi_{k_\theta}(b, G(\cdot|j_\theta), I) = b + \frac{1}{I-1}\frac{G(b|j_\theta)}{g(b|j_\theta)}$$

Nonparametric Estimation

The first step

1. Estimate eigenvector matrix $B_{d_1|\tau}$ by eigen-decomposition

$$\widehat{B}_{d_1|\tau} := \phi\left(\widehat{B}_{Eb_2,d_1,d_3}\widehat{B}^{-1}_{d_1,d_3}\right),$$

   where $\widehat{B}_{Eb_2,d_1,d_3} = \left(\frac{1}{N}\sum_{i=1}^{N} b_{i2}\mathbf{1}(d_{i1}=j, d_{i3}=k)\right)_{j,k}$

2. Estimate joint distribution of bid and type $G(b_2,\tau)$

$$G(b_2,d_1) = B_{d_1|\tau}G(b_2,\tau) \to \widehat{G}(b_2,\tau) = \widehat{B}^{-1}_{d_1|\tau}\widehat{G}(b_2,d_1),$$

   where $\widehat{G}\left(b_2, d_1 = j\right) = \frac{1}{N}\sum_{i=1}^{N}\mathbf{1}(b_{i2} < b_2, d_{i1} = k)$

3. Estimate conditional bid distribution $G(b_2|\tau) = G(b_2,\tau)/p(\tau)$

Nonparametric Estimation

The second step

- Estimates of boundaries
  - All types share the same lower bound $\widehat{\underline{b}} = \inf \{b_{it}\}$
  - Upper bound
    $$\widehat{\overline{b}}_{Lk} = \begin{cases} \sup \{b_{it}, i = 1, 2, ..., N, t = 1, 2, 3\} & \text{if } k = 1 \\ \inf \{b : \widehat{G}(b|Lk) = 1\} & \text{if } k > 1 \end{cases}$$
- Boundary effects
  - $\widehat{G}(\cdot)/\widehat{g}(\cdot)$ is asymptotically biased
  - $\widehat{\xi}_{Lk}(b) = \begin{cases} b + \frac{1}{I-1} \frac{\widehat{G}(b|Lj)}{\widehat{g}(b|Lj)} & \text{if } \hat{\underline{b}} + \frac{\rho h}{2} \leq b_{it} \leq \widehat{\overline{b}} - \frac{\rho h}{2}, \\ +\infty & \text{otherwise,} \end{cases}$
- Estimate of $p(\tau)$: $\widehat{p}(\tau = Lk) = \widehat{B}_{d_1|\tau}^{-1} \widehat{p}(d_1)$.
- Estimate of $F(v)$
  - $\widehat{F}\big(\widehat{\xi}_{Lk}(b)\big) = \widehat{G}(b|Lk)$

Nonparametric Test of Level-k Auctions

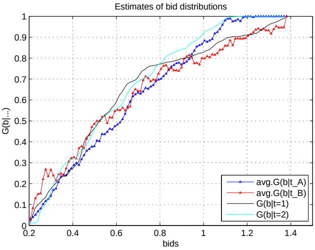
- A testable implication of level-k auctions: value distribution is identical for each type
  - Condition: three (or more) consecutive types
- $H_0 : F_{Lk}(v) = F_{Lj}(v)$
- Test statistic $\widehat{S}_n = \frac{1}{n}\sum_{i=1}^{n}\left[\widehat{F}_{Lk}(v_i) - \widehat{F}_{Lj}(v_i)\right]^2$
  - The asymptotic distribution of $\widehat{S}_n$ is not clear
- Generate T samples by bootstrap, compute test statistic $\{\widehat{S}_{n,(i)}\}_{i=1}^{T}$, $\{\widehat{S}_{n,i}\}_{i=1}^{T}$ arranged in increasing order of magnitude
- A bootstrap critical region of significance level α: $\widehat{S}_n > \widehat{S}_{n,([(1-\alpha)T])}$

Monte Carlo

1. Two types: random type-1 (55%), random type-2 (45%)
    - $\log v \sim \mathcal{N}(0,1), v \in [0.2, 2]$
    - Sample size $N = 200, 500, 5000$
    - Three bidders in each auction
    - Replications $S = 400$
2. Three types: random type-1 (50%), random type-2 (30%), truthful type-1 (20%)
    - $\log v \sim \mathcal{N}(5,3), v \in [0.2, 5]$
    - Sample size $N = 500, 5000$
    - Three bidders in each auction
    - Replications $S = 400$

- First identify number of types: test whether $det(B_{d_1,d_3}) = 0$

Figure: Results of estimation: two-types, $N = 200$



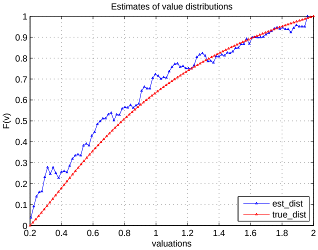
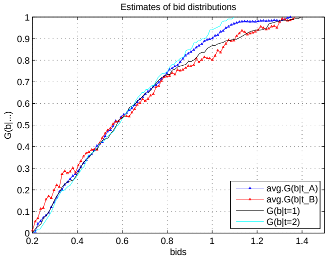


Figure: Results of estimation: two-types, $N = 500$



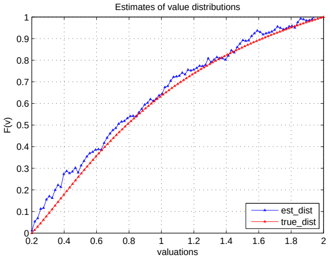
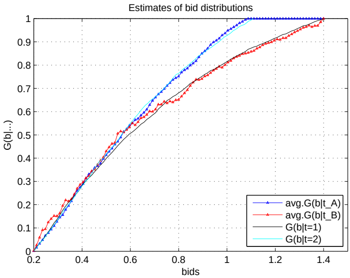


Figure: Results of estimation: two-types, $N = 5000$



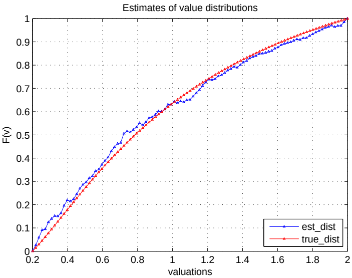
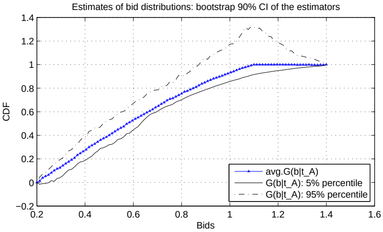


Figure: Estimated bid distribution: bootstrap 90% CI, $N = 5000$



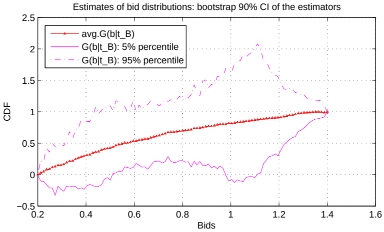


Table: Estimates of type distributions: two-type setting

		True Value	Estimate
$N = 200$	$p(L1, R)$	0.55	0.68
		-	(0.42)
	$p(L2, R)$	0.45	0.32
		-	(0.42)
$N = 5000$	$p(L1, R)$	0.55	0.56
		-	(0.35)
	$p(L2, R)$	0.45	0.44
		-	(0.35)

Figure: Results of estimation: three-types, $N = 500$

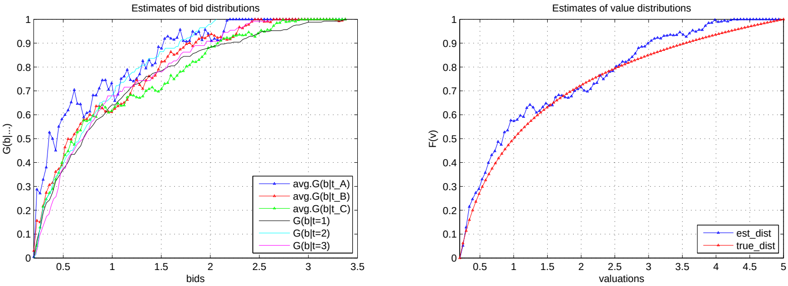
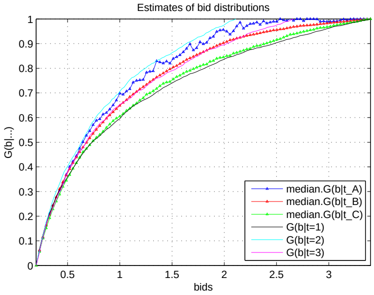


Figure: Results of estimation: three-types, $N = 5000$



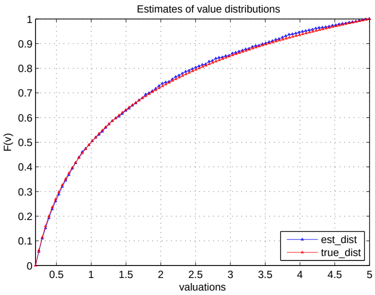


Table: Estimates of type distributions: three-type setting

		True Value	Estimate
$N = 500$	$p(L1, R)$	0.50	0.44
		-	(0.43)
	$p(L2, R)$	0.30	0.40
		-	(0.40)
	$p(L1, T)$	0.20	0.16
		-	(0.28)
$N = 5000$	$p(L1, R)$	0.50	0.46
		-	(0.24)
	$p(L2, R)$	0.30	0.27
		-	(0.39)
	$p(L1, T)$	0.20	0.27
		-	(0.39)

Application: USFS Timber Auctions

Application: USFS Timber Auctions

Data: USFS timber auction data

- Studies used this data set
  - Nonequilibrium behavior for risk neutral bidders
    - Risk-aversion: Baldwin (1995), Athey and Levin (2001), Campo, Guerre, Perrigne, and Vuong (2003)
    - Risk-aversion $\Longleftrightarrow$ level-k: Dohmen, Falk, Huffman, Sunde (2010)
  - Independent private value paradigm
    - Haile, Hong, and Shum (2003) finds evidence for "scaled sale"
  - Reserve price is nonbinding: Haile (2001) among others
- Sample description
  - Sealed bid, scaled sale: 1982-1993 in all regions
  - Eliminate salvage and small-business set-aside auctions
  - Choose those bidders who participate in at least three auctions
  - Choose those auctions with three bidders

Application: USFS Timber Auctions

Estimation: a preliminary regression

- Control for auction-specific heterogeneity
  - The approach: run a preliminary regression of bids on auction characteristics, and use the residuals for analysis
    Haile, Hong, and Shum (2003) and Bajari, Houghton, Tadelis, and Berkeley (2007)
  - Appraisal value captures the heterogeneity: Haile (2001) and Campo, Guerre, Perrigne, and Vuong (2003)
- Summary statistics

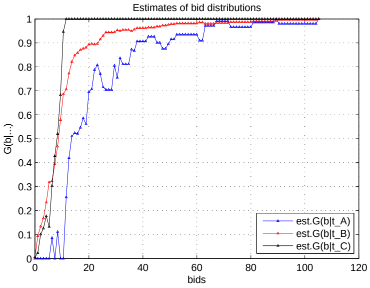
Variable	Mean	Std	Minimum	Maximum
Auctions participated	4.99	4.56	3	65
Bids ($/MBF)	11.14	11.98	0.02	105.16

Sample size 462: number of identified bidders

- Hypothesis testing $\det(B_{d_1,d_3}) = 0$, 5% significance level
  - Reject $\det(B_{d_1,d_3}) = 0$ for M=2, 3
  - Fail to reject $\det(B_{d_1,d_3}) = 0$ for M=4, M=5
- Conclusion: data support three types

Application: USFS Timber Auctions

Estimation of bid and value distributions, sample size 462



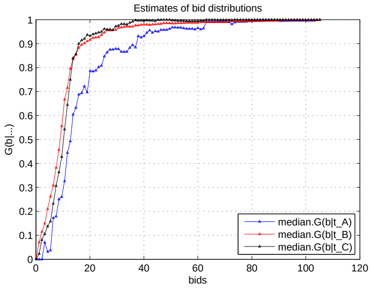


Table: Estimated type distribution

	Estimate	Median
p(the lowest type)	0.07	0.03
	(0.22)	–
p(the higher type)	0.80	0.71
	(0.33)	–
p(the highest type)	0.13	0.16
	(0.28)	–

Application: USFS Timber Auctions

Value distributions and hypothesis testing

- No specification of lowest type
- Estimate two value distribution functions $F_2(\cdot)$, $F_3(\cdot)$
- Hypothesis testing, $H_0 : F_2(\cdot) = F_3(\cdot)$
  - Test statistic S_{23}: average "distance" between two distribution functions
  - Result: $\widehat{S}_{23} = 0.0072 \pm 0.0583$
  - Fail to reject the hypothesis under 5% significance level
- Implications:
  - Can not reject the level-k auction model
  - Bidders are more sophisticated than those in experiments

Application: USFS Timber Auctions

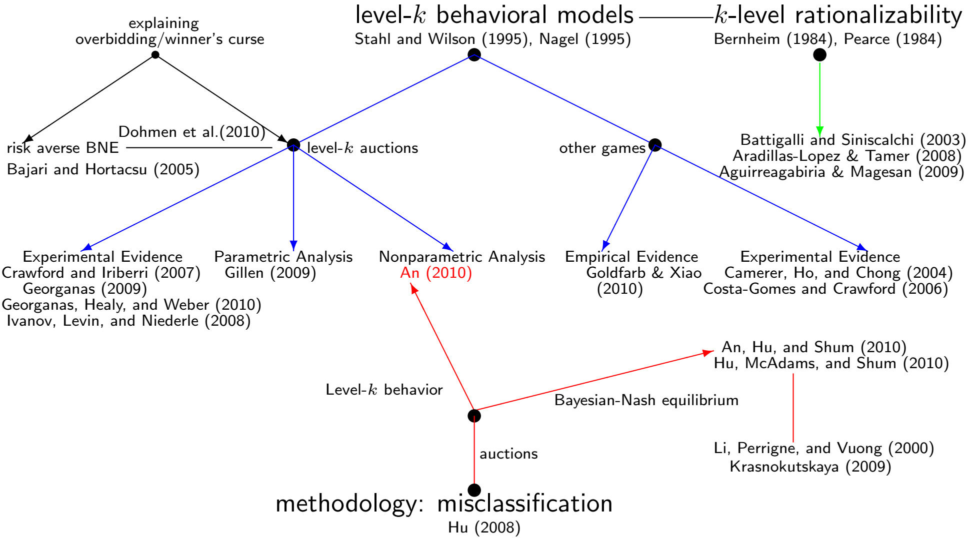
Hypothesis testing with specification of the lowest type

- Specification the lowest type: $\tau = (L1, R)$
  - three types: $(L1, R), (L2, R), (L3, R)$
- Hypothesis testing: $F_1(\cdot) = F_2(\cdot)$, $F_1(\cdot) = F_3(\cdot)$
  - Test statistic $\widehat{S}_{12} = 0.0459 \pm 0.1105$, $\widehat{S}_{13} = 0.0431 \pm 0.1539$
  - Fail to reject both hypotheses under 5% significance level
  - $\widehat{S}_{23} = 0.0072 \pm 0.0583$ is much smaller than both $\widehat{S}_{12}$ and $\widehat{S}_{13}$
- Implications:
  - Specification of the lowest type might not correct
  - Bidders are more sophisticated than subjects in experiments

Summary

- Results
  1. Nonparametric methodology that identifies and estimates
     - Number of types
     - Probability of each type
     - Underlying value distribution
     - Specification of type-$L0$
  2. Empirical evidence
     - Bidders are of three types: understand bidders' behavior in field auctions
     - Data support level-k auction model: policy implications
- Possible extensions
  - More general beliefs
  - More general settings in which heterogenous behavior is due to (discrete) unobserved heterogeneity, e.g., entry games in I.O.

Related Literature



Both $s_k(\cdot)$ and $s_j(\cdot)$ are strictly increasing, for each $b \in [\underline{v}, s_k(\bar{v})]$, where $s_k(v_1) = b$, $v_1 \in [\underline{v}, \bar{v}]$, there exists a unique valuation $v_2 \in [\underline{v}, \bar{v}]$ such that $s_k(v_1) = s_j(v_2) = b$.

$$\begin{aligned} v_1 &= b + \frac{1}{I-1} \frac{F(s_j^{-1}(s_k(v_1)))}{f(s_j^{-1}(s_k(v_1)))} \frac{1}{\frac{ds_j^{-1}(t)}{dt}|_{t=s_k(v_1)}} \\ &= b + \frac{1}{I-1} \frac{F(s_j^{-1}(s_j(v_2)))}{f(s_j^{-1}(s_j(v_2)))} \frac{1}{\frac{ds_j^{-1}(t)}{dt}|_{t=s_j(v_2)}} \\ &= b + \frac{1}{I-1} \frac{F(v_2)}{f(v_2)} \frac{1}{\frac{ds_j^{-1}(t)}{dt}|_{t=s_j(v_2)}}. \end{aligned} \quad (3)$$

The change of variables for type j implies

$$G(b|j) = F(v_2)g(b|j) = f(v_2)\frac{ds_j^{-1}(t)}{dt}|_{t=s_j(v_2)}.$$

Combine these two equations,

$$\frac{G(b|j)}{g(b|j)} = \frac{F(v_2)}{f(v_2)}\left(\frac{ds_j^{-1}(t)}{dt}|_{t=s_j(v_2)}\right)^{-1}. \quad (4)$$

Return

Nonparametric Identification

Extension: two dimensional unobserved heterogeneity

- Multiple specifications of type-$L0$
  - Eigenvalue-eigenvector decomposition still applies
  - Supports of bids are not monotonic
- Exemplifying identification
  - $\tau \in \{(L1, R), (L2, R), (L1, T)\}$, $(L1, \cdot)$ is identified
  - $G_A(b)$, $G_B(b)$, $G_C(b)$: three anonymous bid distribution functions, $G_A(b)$ has the largest upper bound
  - $G_A(b)$ is bid distribution of type-$(L1, R)$ or type-$(L1, T)$
  - Two alternative hypotheses: (1) $G^R_{L1}(b) = G_A(b)$; (2) $G^T_{L1}(b) = G_A(b)$
  - Under hypothesis (1)
    - $G_A(b) \Rightarrow F_A(\cdot)$
    - If $G_B(\cdot) = G^R_{L2}(\cdot)$, $\Rightarrow F_B(\cdot)$

Figure: Estimated specification of type-$L0$: $N = 5000$

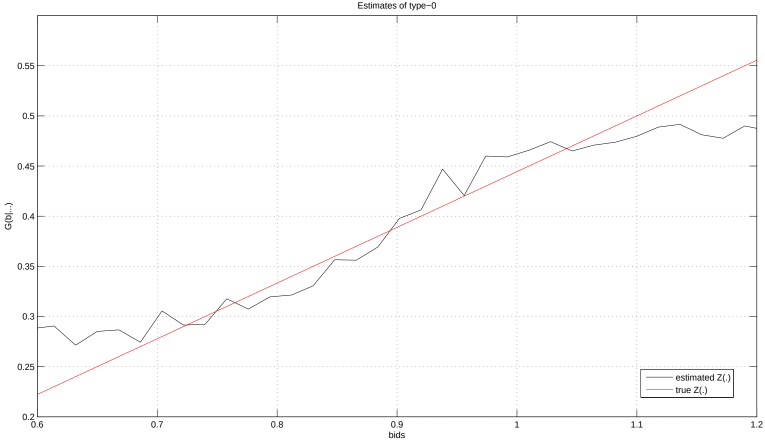


Figure: Estimated bid distribution: bootstrap 90% CI

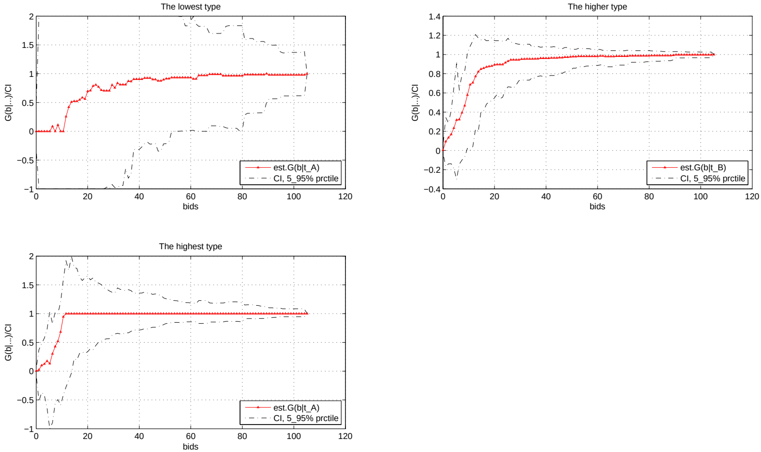


Figure: Estimated value distribution

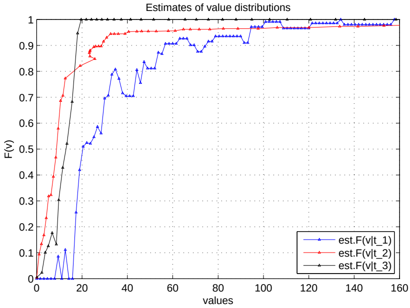


Figure: Distribution of statistic