Identification and Estimation of Incomplete Information Games with Multiple Equilibria

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Motivation

- In theory, games generally admit multiple equilibria
- In empirical, the same equilibrium assumption is imposed
  - Data example data
  - Problems in the existing literature:
    - Obtain inconsistent estimates
    - Counterfactual analysis is impossible

Question

- Question: can we still recover players' payoff without imposing assumptions on the equilibrium?
- Main results:
  - Nonparametrically identify finite games with incomplete information in static and dynamic settings
  - Four elements are identified:
    - The number of equilibria
    - The equilibrium selection mechanism
    - Players equilibrium strategies
    - Payoff primitives

Contributions

- A novel methodology to tackle multiple equilibria
  - The number of equilibria (Bajari et al.(2010))
  - Nonparametric eq selection (Bajari et al.(2010) and jia (2008))
  - Point identification (Ciliberto and Tamer (2009))
  - Consistent estimation of payoff
  - Enables counterfactual analysis
  - New direction for identifying games with multiple equilibria
- Empirical application
  - Provides evidence on existence of multiple equilibria in reality
  - Provides evidence on how equilibria evolve over time

Related Literature

- Multiple Equilibria
  - Degenerated equilibrium selection: Seim (2006)
  - Functional form eq selection: Bajari, Hong and Ryan (2010)
  - Partial identification: Ciliberto and Tamer (2009)
  - Multiple equilibria versus unobserved heterogeneity: Aguirregabiria and Mira (2013)
- Measurement error
  - Misclassification: Hu (2008)
  - Dynamic models with unobserved heterogeneity: Hu and Shum (2012)

Outline of The Talk

- Static game
  - Setup
  - Identification with cross-sectional data
  - Identification with panel data
- Dynamic game: identification
- Monte Carlo evidence in the static setting
- Empirical application

Static Game with Incomplete Information

- Players: $i = 1, ..., n$
- Actions: $a_i \in \mathscr{A} \equiv \{0, 1, ..., K\}$
- Action vector: $a = (a_1, ..., a_n) \in \mathscr{A}^n$
- Action-related profit shocks: $\epsilon_i = (\epsilon_i(0), ..., \epsilon_i(K))$
- ϵ_i: only observable to player i
- Density $f(\epsilon_i)$: common knowledge
- State variables $s \in \mathscr{S}$, observables

Bayesian Nash Equilibrium (BNE)

- The payoff function for player i with action profile a: (example)

$$u_i(a, s, \epsilon_i) = \pi_i(a_i, a_{-i}, s) + \epsilon_i(a_i)$$

- Player i's decision rule: δ_i: $(s, \epsilon_i) \rightarrow a_i$
- Conditional choice probability (CCP):

$$\sigma_i(a_i = k|s) = \int I\{\delta_i(s, \epsilon_i) = k\} f(\epsilon_i) d\epsilon_i$$

- Choice specific utility:

$$\Pi_i(a_i, s; \sigma_{-i}(a_{-i}|s)) = \sum_{a_{-i}} \pi_i(a_i, a_{-i}, s)\sigma_{-i}(a_{-i}|s)$$

- The BNE is defined using CCPs instead of decision rules

$$\sigma_i(a_i = k|s) = Pr\{\Pi_i(a_i = k, s) + \epsilon_i(k) > \Pi_i(a_i = j, s) + \epsilon_i(j) \quad \forall j \in \mathscr{A}\}$$

Identification without Multiple Equilibria

- General mapping (following Hotz-Miller (1993)):

$$(\sigma_i(0|s), ..., \sigma_i(K|s)) = \Gamma_i(\Pi_i(1,s) - \Pi_i(0,s), ..., \Pi_i(K,s) - \Pi_i(0,s))$$

- $\Pi_i(k,s) - \Pi_i(0,s)$ is identified if $\sigma_i(k|s)$ can be observed
- With multiple equilibria, equilibrium CCPs are unobserved

Identification of Static Games

Cross-sectional Data

- Data Structure
  - Suppose we observe n players play in M games: $\{a_1^m, ..., a_n^m, s^m\}_{m=1}^M$
  - Games might employ different equilibria
- Identification procedures
  1. Equilibrium relevant components
     - The number of equilibria
     - CCPs of players in each equilibrium
     - Equilibrium selection mechanism
  2. Payoff primitives

Identification of Static Games (continued)

Intuition

Data generating process (if $n = 3$)

$$\begin{array}{ccc} & e^* & \\ \swarrow & \downarrow & \searrow \\ a_1 & a_2 & a_3 \\ \uparrow & \uparrow & \uparrow \\ \epsilon_1 & \epsilon_2 & \epsilon_3 \end{array}$$

Key assumption: shocks are independent across actions and players

a_1, a_2 and a_3 are independent conditional on the equilibrium e^*

$$\begin{aligned} \Pr(a_1, ..., a_n) &= \sum_{e^*} \Pr(a_1 ... a_n | e^*) \Pr(e^*) && \text{by law of total probability} \\ &= \sum_{e^*} \Pr(a_1|e^*) ... \Pr(a_n|e^*) \Pr(e^*) && \text{by independence} \end{aligned}$$

Identification of Static Games (continued)

Matrix Representation

- The joint distribution between actions of player 1 and player 2:

$$\Pr(a_1, a_2) = \sum_{e^*} \Pr(a_1|e^*) \times \Pr(e^*) \times \Pr(a_2|e^*)$$

matrix representation

$$F_{a_1,a_2} = A_{a_1|e^*} \times D_{e^*} \times A^T_{a_2|e^*}$$

an example for binary choice games ($a_i \in \{0,1\}$) and $e^* \in \{1,2\}$

$$\underbrace{\Pr\begin{pmatrix} (0,0) & (0,1) \\ (1,0) & (1,1) \end{pmatrix}}_{\text{Observable joint distribution}} = \underbrace{\Pr\begin{pmatrix} (0|1) & (0|2) \\ (1|1) & (1|2) \end{pmatrix}}_{\text{CCPs of player 1}} \times \underbrace{\Pr\begin{pmatrix} (1) & 0 \\ 0 & (2) \end{pmatrix}}_{\text{Equilibrium selection}} \times \underbrace{\Pr\begin{pmatrix} (0|1) & (1|1) \\ (0|2) & (1|2) \end{pmatrix}}_{\text{CCPs of player 2}}$$

Identification of Static Games (continued)

Matrix Representation-an example

Data example:

observed		latent
a_1	a_2	e^*
1	1	1
1	0	2
0	1	1
0	0	2

$$\underbrace{\begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}}_{\text{Observable joint distribution}} = \underbrace{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}}_{\text{CCPs of player 1}} \times \underbrace{\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}}_{\text{Equilibrium selection}} \times \underbrace{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}}_{\text{CCPs of player 2}}$$

Identification of Static Games (continued)

The Number of Equilibria

Matrix equation links unknowns with knowns

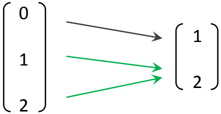
$$\underbrace{F_{a_1,a_2}}_{(K+1)\times(K+1)} = \underbrace{A_{a_1|e^*}}_{(K+1)\times Q} \times \underbrace{D_{e^*}}_{Q\times Q} \times \underbrace{A^T_{a_2|e^*}}_{Q\times(K+1)}$$

- The number of equilibria Q is identified as $Rank(F_{a_1,a_2})$, if
  1. The equilibrium being selected with positive probability
  2. Enough variation of actions: $K+1 > Q$
  3. No equilibrium is redundant: CCPs are linearly independent
- Lower bound: $Rank(F_{a_1,a_2}) \leq Q$

Identification of Static Games (continued)

Reducing Dimension

- If $K+1>Q$, information from two players are sufficient
- Partition the action space to be a dimension of Q
  - For example, $K=2$, $\mathscr{A}=\{0,1,2\}$, and $Q=2$, then



  - Matrix F_{a_1,a_2} has a dimension of $Q\times Q$ and rank of Q
  - Matrix F_{a_1,a_2} is invertible

Identification of Static Games (continued)

CCPs of Different Equilibria

- The identification requires at least three players:
- By the conditional independence:

$$\Pr(a_1, a_2, a_3) \;=\; \sum_{e^*} \Pr(a_1|e^*)\Pr(a_2|e^*)\Pr(a_3|e^*)\Pr(e^*)$$

- Fixing $a_3 = k$, matrix representation:

$$F_{a_1,a_2} \;=\; A_{a_1|e^*} \times D_{e^*} \times A_{a_2|e^*}^T$$

$$\underbrace{F_{a_1,a_2,a_3=k}}_{\text{Observed joint dis}} \;=\; \underbrace{A_{a_1|e^*}}_{\text{CCPs of player 1}} \times \underbrace{D_{a_3=k|e^*}}_{\text{CCPs of } a_3 = k} \times \underbrace{D_{e^*}}_{\text{Eq selection}} \times \underbrace{A_{a_2|e^*}^T}_{\text{CCPs of player 2}}$$

$$\Rightarrow F_{a_1,a_2,a_3=k} F_{a_1,a_2}^{-1} = A_{a_1|e^*} D_{a_3=k|e^*} A_{a_1|e^*}^{-1}$$

Identification of Static Games (continued)

Eigenvalue-eigenvector Decomposition

- Main equation

$$\underbrace{F_{a_1,a_2,a_3=k}F_{a_1,a_2}^{-1}}_{\text{Observed joint distribution}} = \underbrace{A_{a_1|e^*}}_{\text{CCPs for player 1}} \underbrace{D_{a_3=k|e^*}}_{\text{CCPs for } a_3=k} \underbrace{A_{a_1|e^*}^{-1}}_{\text{CCPs for player 1}}$$

- A unique decomposition requires distinct eigenvalues
- Matrix of CCPs for player 1: eigenvectors up to some scales
- Normalization: column sum equals 1
- The equilibrium selection: $\Pr(a_1) = \sum_{e^*} \Pr(a_1|e^*) \Pr(e^*)$

The First Step of Identification

Equilibrium-specific Components

- Conditions
  - The payoff shocks are i.i.d across actions and players
  - The number of actions or players is relatively large
  - Not a single equilibrium is redundant
  - The probability of choosing certain action varies with equilibria
  - At least three players in the game
- Identification of equilibrium-specific components
  - The number of equilibria
  - The CCPs of players in different equilibria
  - The equilibrium selection mechanism

Static Games with Panel Data

Question:

- Will players select the same equilibrium over time?

More general question:

- How do equilibria evolve over time?

Assumption:

- The equilibrium evolution follows a 1st-order Markov Process
- The Markovian assumption nests the assumption that the same equilibrium is employed over time

Static Games with Panel Data (continued)

- a_t: action vector of all players in period t
- e_t^* : the equilibrium index in period t
- $\epsilon_t = \{\epsilon_{1t}, ..., \epsilon_{nt}\}$
- Data Generating Process

Cross-sectional data | | Panel data

$$
\begin{array}{ccc}
 & e^* & \\
\swarrow & \downarrow & \searrow \\
a_1 & a_2 & a_3 \\
\uparrow & \uparrow & \uparrow \\
\epsilon_1 & \epsilon_2 & \epsilon_3
\end{array}
\quad \text{versus} \quad
\begin{array}{ccccc}
e_t^* & \longrightarrow & e_{t+1}^* & \longrightarrow & e_{t+2}^* \\
\downarrow & & \downarrow & & \downarrow \\
a_t & & a_{t+1} & & a_{t+2} \\
\uparrow & & \uparrow & & \uparrow \\
\epsilon_t & & \epsilon_{t+1} & & \epsilon_{t+2}
\end{array}
$$

Note: shocks are allowed to be correlated across players

The First Step of Identification (panel data)

Conditions

- The first-order Markov equilibrium evolution
- The number of actions or players are relatively large
- Not a single equilibrium is redundant
- The probability of a certain action vector varies with equilibria
- Observe three periods of data

Identification

- The number of equilibria
- The equilibrium transition
- The CCPs of players in different equilibria

Test

- Equilibrium evolution process: $Pr(e_{t+1}^* = e | e_t^* = e) = 1$

Multiple Equilibria in Dynamic Games

Multiple equilibria occur in dynamic games

- A single equilibrium is selected in market-level time series (Pesendorfer and Schmidt-Dengler (2008))
- Estimation can be conducted in individual markets
  - Data demanding
  - Loss of efficiency by using only part of the data
  - Impossible counterfactual analysis without the equilibrium selection

The Identification in Dynamic Games

Data Generating Process

- Consider only Markov Perfect Equilibria
- Denote $w_t = \{a_t, s_t\}$ all observables in period t
- In a given market (e^*), w_t follows a first-order Markov process
- Pooling cross-sectional games together

$$
\begin{array}{ccccccccc}
 & \text{Static} & & & & & \text{Dynamic} & & \\
 & e^* & & & & & e^* & & \\
\swarrow & \downarrow & \searrow & & \text{versus} & \swarrow & \downarrow & & \searrow \\
a_1 & a_2 & a_3 & & & w_t & \rightarrow \; w_{t+1} & \rightarrow & w_{t+2}
\end{array}
$$

The Identification in Dynamic Games (continued)

The Number of Equilibria

- Three periods of data are needed:

$$\Pr(w_{t+2}, w_{t+1}, w_t) = \sum_{e^*} \Pr(w_{t+2}|w_{t+1}, e^*) \Pr(w_{t+1}, w_t|e^*) \Pr(e^*)$$

- Fixing w_{t+1}, the correlation between w_t and w_{t+2} is from e^*

$$F_{w_{t+2},\bar{w}_{t+1},w_t} = A_{w_{t+2}|\bar{w}_{t+1},e^*} D_{e^*} A_{\bar{w}_{t+1},w_t|e^*}$$

- $Q = rank(F_{w_t,\bar{w}_{t+1},w_{t+2}})$ if
  - Enough variation: dimension of $w_t > Q$
  - Non equilibrium is redundant
  - All $\Pr(e^*) > 0$

more

The First Step Identification (dynamic games)

Assumptions

- Only consider Markov Perfect Equilibria
- Sufficient variation in actions and states
- No equilibrium is redundant (full rank conditions)
- Observe four periods of data

Identification

- The number of equilibria
- The equilibrium selection
- The CCPs of players of different equilibria
- State transition that is allowed to be equilibrium specific

Monte Carlo Evidence—Static Setting

- n homogeneous retailers make location decision in markets $s \in \mathscr{S}$
- Assume that the market attribute is discrete (market types)
- Payoff functions of location (1) or location (0) :

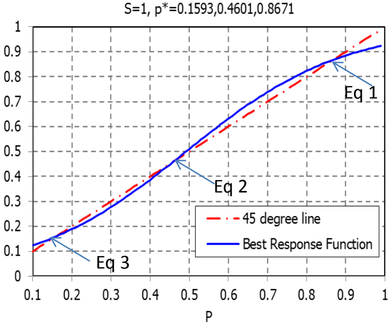
$$\begin{aligned}\pi(a_i = 1, a_{-i}; s) &= \alpha s + \beta \frac{\sum_{-i} I(a_{-i} = 1)}{n-1} + \epsilon_{i1} \\ \pi(a_i = 0, a_{-i}; s) &= \beta \frac{\sum_{-i} I(a_{-i} = 0)}{n-1} + \epsilon_{i0}\end{aligned}$$

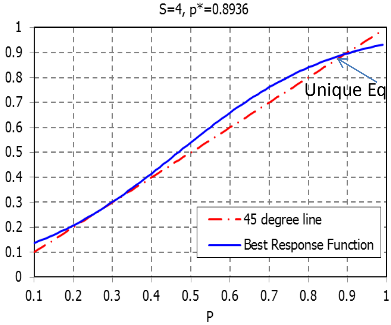
- ϵ_{i1} and ϵ_{i0} are private shocks
- ϵ is independent and follows extreme value distribution
- $\alpha = 0.04$, $\beta = 2.5$

Monte Carlo Evidence (continued)

The Best Response Function for Different Values of s

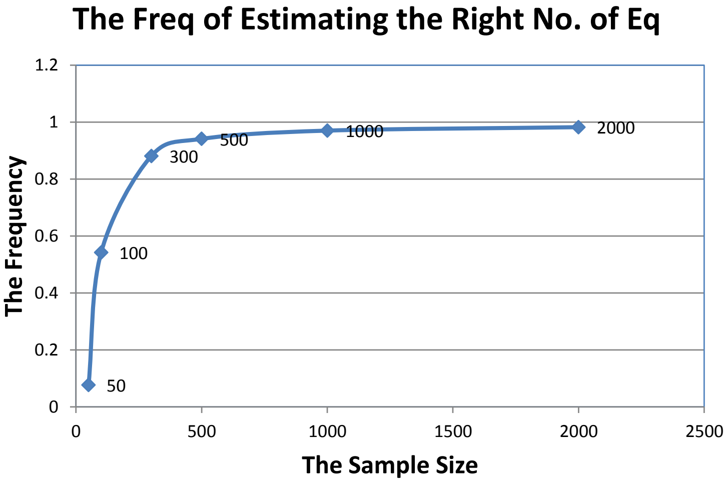
- Let $p_s \equiv Pr(a_i = 1|s)$, consider symmetric equilibria
- Equilibrium condition: $p_s = \frac{exp(\alpha s + \beta p_s)}{exp(\alpha s + \beta p_s) + exp(\beta(1 - p_s))}$





Monte Carlo Evidence (continued)

Estimation of The Number of Equilibria



Monte Carlo Evidence (continued)

CCPs of Each Equilibrium: Cross-sectional Data

	s=1		s=4	
	DGP	Estimates	DGP	Estimates
The Number of Equilibria	2	2	1	1
$\Pr(a=0\|s, e^*=1)$	0.159	0.159	n/a	
		(0.012)		
$\Pr(a=0\|s, e^*=2)$	0.867	0.867	0.894	0.893
		(0.013)		(0.004)
The Equilibrium Selection	0.5	0.5004	0	0
		(0.017)		(0)

Monte Carlo Evidence (continued)

Game Primitives: Multiple versus Unique

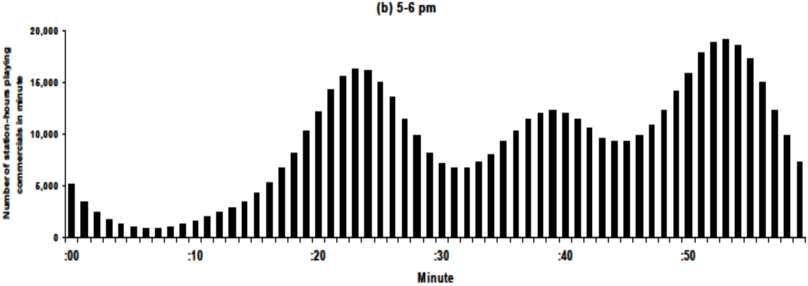
	DGP	Cross-section		Panel data	
		Unique Eq	Multiple Eq	Unique Eq	Multiple Eq
Strategic Interaction β	2.5	2.825	2.505	2.276	2.509
		(0.041)	(0.019)	(0.043)	(0.044)
Market Effect α	0.04	-0.024	0.040	0.009	0.040
		(0.006)	(0.005)	(0.007)	(0.009)

Conclusion:

- Ignoring multiple equilibria does result in estimation inconsistency

The Industry Background

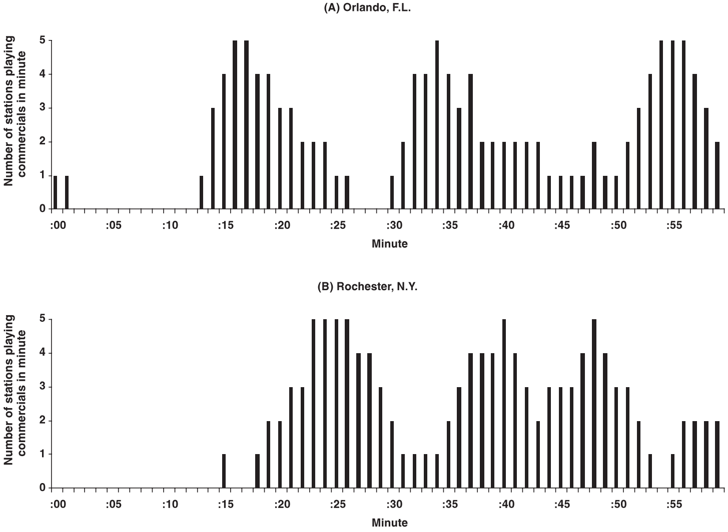
Figure: The number of radio stations playing commercials each minute



The Industry Background (continued)

Timing Patterns for Commercials in Different Markets

TIMING OF COMMERCIALS IN ORLANDO, F.L., AND ROCHESTER, N.Y., ON OCTOBER 30, 2001 5-6 P.M.



Rationalizations

The observed pattern can be explained by:

- Common unobserved factors
- Multiple equilibria

Modeling

- Homogeneous stations decide 1(:48-:52) and 0(:53-:57)
- Payoff of choosing time block $t \in \{0, 1\}$:

$$\pi_{it} = \alpha_t + \beta P_{-it} + \epsilon_{it}$$

- P_{-it} : proportion of stations choosing the same timing.
- α_t: different average profit for stations with different timing t
- β : coordination incentives
- Private shocks are independent and follow extreme value distributions
- Normalization: $\alpha_0 = 0$

Summary Statistics

Table: Summary Statistics

Variable	Obs	Mean	std. Dev	Min	Max
No. Players	108,554	12.9	3.2	2	20
Timing	108,554	.499	.501	0	1
Day	108,554	31.42	17.6	1	59
Hour	108,554	16.47	3.15	12	21
Market(large=1)	108,554	.517	.4997	0	1

Estimation Using Cross-sectional Data

	Market Size			Time	
	All market	Large	Small	Drive-time	Non drive-time
				5-6 PM	9-10PM
The Number of Eq	2	1	2	2	1
The Eq Selection: $\Pr(e^* = 1)$	0.285	1	0.273	0.419	1
	(0.129)	-	(0.123)	(0.128)	-
CCPs: $\Pr(a = 0 \mid e^* = 1)$	0.657	0.506	0.658	0.669	0.492
	(0.188)	(0.010)	(0.117)	(0.214)	(0.004)
$\Pr(a = 0 \mid e^* = 2)$	0.429	-	0.429	0.370	-
	(0.063)	-	(0.050)	(0.111)	-
α	-0.006	-0.024	-0.006	-0.005	0.032
	(0.326)	(0.039)	(0.101)	(0.220)	(0.017)
β	2.052	0	2.053	2.067	0
	(0.315)	-	(0.109)	(0.200)	-

Estimation Using Panel Data

		Market Size		Time	
	All Market	Large	Small	Drive-time	Non drive-time
				5-6 PM	9-10PM
The Number of Eq	2	1	2	2	1
Eq Evolution: $\Pr(e^*_{t+1} = 1 \vert e^*_t = 1)$	1.000	1	1.000	1.000	1
	(0.019)	-	(0.431)	(0.049)	-
$\Pr(e^*_{t+1} = 2 \vert e^*_t = 2)$	0.871	-	0.978	0.924	-
	(0.019)	-	(0.408)	(0.091)	-
CCPs: $\Pr(a = 0 \vert e^* = 1)$	0.561	0.512	0.658	0.643	0.498
	(0.023)	(0.009)	(0.152)	(0.041)	(0.008)
$\Pr(a = 0 \vert e^* = 2)$	0.362	-	0.438	0.379	-
	(0.037)	-	(0.061)	(0.030)	-
α	0.004	-0.047	-0.002	0.010	
	(0.015)	(0.034)	(0.062)	(0.030)	(0.034)
β	2.040	0	2.006	2.050	0
	(0.034)	-	(0.072)	(0.034)	-

Empirical Findings

- Multiple equilibria exist
  - Two equilibria exist in smaller markets or drive-time
  - A unique equilibrium exists in large markets or non drive-time
- The same equilibrium is employed over time

Conclusions

- Conclusions
  - Identify and estimate finite games with incomplete information allowing for possibly multiple equilibria
  - The methodology applies in both static and dynamic settings
  - Empirical application provides evidence of multiple equilibria
  - The same equilibrium is employed over time
- Further research
  - Extensions
    - Consider both common unobserved heterogeneity and multiple equilibria together
    - Games with continuous actions
  - New project
    - Chinese Auto insurance market

Data Example

- Two radio stations choose to place commercials in two time slots
- Suppose two equilibria being employed in the data

Market	Player 1	Player 2	True Eq	Ass	Ass
Market 1	option 1	option 1	1	1	2
Market 2	option 1	option 2	2	1	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Market $m-1$	option 2	option 2	1	1	2
Market m	option 2	option 1	2	1	2

Back to main.

Game Matrix

- Payoff structure:

		Player 2	
		$a_2 = 1$	$a_2 = 0$
Player 1	$a_1 = 1$	$(\alpha_1 + \epsilon_{11}, \beta_1 + \epsilon_{21})$	$(\alpha_2 + \epsilon_{11}, \beta_2 + \epsilon_{20})$
	$a_1 = 0$	$(\alpha_3 + \epsilon_{10}, \beta_3 + \epsilon_{21})$	$(\alpha_4 + \epsilon_{10}, \beta_4 + \epsilon_{20})$

- Player 1 only observes own shock (ϵ_{11}, ϵ_{10}), but not her rival's (ϵ_{21},ϵ_{20})

Back to main.

Static Games

- No sufficient variation $K + 1 \leq Q$, (e.g., binary choice games)
- If n is greater than 3, aggregation of information helps
  - For example, $n = 4$ and $\mathscr{A} = \{0, 1\}$
  - Divide players into two groups and expand the action space

$$\mathscr{A} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \mathscr{A}^2 = \begin{pmatrix} (0\ 0) \\ (0\ 1) \\ (1\ 0) \\ (1\ 1) \end{pmatrix}$$

Dynamic Game

State law of motion

- Four periods of data $(w_t, w_{t+1}, w_{t+2}, w_{t+3})$ are needed.
- w_{t+3} and w_t

$$\Pr(w_{t+3}, w_{t+2}, w_{t+1}, w_t) =$$

$$\sum_{e^*} \Pr(w_{t+3}|w_{t+2}, e^*) \Pr(w_{t+2}|w_{t+1}, e^*) \Pr(w_{t+1}, w_t|e^*) \Pr(e^*)$$

- Fix w_{t+2} and w_{t+1}, matrix representation:

$$F_{w_{t+3}, w_{t+2}, w_{t+1}, w_t} = A_{w_{t+3}|w_{t+2}, e^*} D_{w_{t+2}|w_{t+1}, e^*} D_{e^*} A_{w_{t+1}, w_t|e^*}$$

The Identification in Dynamic Games (continued)

Matrix Representations

$$\begin{aligned}
for\ \ (w_{t+2}, w_{t+1}) &: \quad F_{w_{t+3}, w_{t+2}, w_{t+1}, w_t} &=& \quad A_{w_{t+3}|w_{t+2}, e^*} D_{w_{t+2}|w_{t+1}, e^*} D_{e^*} A_{w_{t+1}, w_t|e^*} \\
for\ \ (\bar{w}_{t+2}, w_{t+1}) &: \quad F_{w_{t+3}, \bar{w}_{t+2}, w_{t+1}, w_t} &=& \quad A_{w_{t+3}|\bar{w}_{t+2}, e^*} D_{\bar{w}_{t+2}|w_{t+1}, e^*} D_{e^*} A_{w_{t+1}, w_t|e^*} \\
for\ \ (w_{t+2}, \bar{w}_{t+1}) &: \quad F_{w_{t+3}, w_{t+2}, \bar{w}_{t+1}, w_t} &=& \quad A_{w_{t+3}|w_{t+2}, e^*} D_{w_{t+2}|\bar{w}_{t+1}, e^*} D_{e^*} A_{\bar{w}_{t+1}, w_t|e^*} \\
for\ \ (\bar{w}_{t+2}, \bar{w}_{t+1}) &: \quad F_{w_{t+3}, \bar{w}_{t+2}, \bar{w}_{t+1}, w_t} &=& \quad A_{w_{t+3}|\bar{w}_{t+2}, e^*} D_{\bar{w}_{t+2}|\bar{w}_{t+1}, e^*} D_{e^*} A_{\bar{w}_{t+1}, w_t|e^*}
\end{aligned}$$

post-multiply inverse of equation 1 to equation 1, to obtain:

$$\begin{aligned}
Y &\equiv F_{w_{t+3}, w_{t+2}, w_{t+1}, w_t} F^{-1}_{w_{t+3}, \bar{w}_{t+2}, w_{t+1}, w_t} \\
&= A_{w_{t+3}|w_{t+2}, e^*} D_{w_{t+2}|w_{t+1}, e^*} D^{-1}_{\bar{w}_{t+2}|w_{t+1}, e^*} A^{-1}_{w_{t+3}|\bar{w}_{t+2}, e^*}
\end{aligned}$$

Similarly,

$$\begin{aligned}
Z &\equiv F_{w_{t+3}, \bar{w}_{t+2}, \bar{w}_{t+1}, w_t} F^{-1}_{w_{t+3}, w_{t+2}, \bar{w}_{t+1}, w_t} \\
&= A_{w_{t+3}|\bar{w}_{t+2}, e^*} D_{\bar{w}_{t+2}|\bar{w}_{t+1}, e^*} D^{-1}_{w_{t+2}|\bar{w}_{t+1}, e^*} A^{-1}_{w_{t+3}|w_{t+2}, e^*}
\end{aligned}$$

The Identification in Dynamic Games (continued)

The Main Equation

Main Equation (Eigenvalue-eigenvector decomposition)

$$
\begin{aligned}
YZ &= A_{w_{t+3}|w_{t+2},e^*} D_{w_{t+2}|w_{t+1},e^*} D^{-1}_{\bar{w}_{t+2}|w_{t+1},e^*} \\
&\quad D_{\bar{w}_{t+2}|\bar{w}_{t+1},e^*} D^{-1}_{w_{t+2}|\bar{w}_{t+1},e^*} A^{-1}_{w_{t+3}|w_{t+2},e^*} \\
&\equiv A_{w_{t+3}|w_{t+2},e^*} D_{w_{t+2},\bar{w}_{t+2},w_{t+1},\bar{w}_{t+1}|e^*} A^{-1}_{w_{t+3}|w_{t+2},e^*}
\end{aligned}
$$

- Distinctive eigenvalues are needed, which is empirically testable.
- For each w_{t+2}, $A_{w_{t+3}|w_{t+2},e^*}$ are identified as eigenvectors
- Normalization uses the fact that column sum equals one

Back to main.

Complete Versus Incomplete Information Games

Incomplete Information

$$
\begin{array}{ccc}
 & e^* & \\
\swarrow & & \searrow \\
a_1 & & a_2 \\
\uparrow & & \uparrow \\
\epsilon_1 & & \epsilon_2
\end{array}
$$

versus

Complete Information

$$
\begin{array}{ccc}
 & e^* & \\
\swarrow & & \searrow \\
a_1 & & a_2 \\
\nwarrow & & \nearrow \\
 & (\epsilon_1, \epsilon_2) &
\end{array}
$$