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Identification and estimation of dynamic structural models with unobserved choices ☆

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ABSTRACT

This paper develops identification and estimation methods for dynamic discrete choice models when agents' actions are unobserved by econometricians. We provide conditions under which choice probabilities and latent state transition rules are nonparametrically identified with a continuous state variable in a single-agent dynamic discrete choice model. Our identification strategy from the baseline model can extend to models with serially correlated unobserved heterogeneity, cases in which choices are partially unavailable, and dynamic discrete games. We propose a sieve maximum likelihood estimator for primitives in agents' utility functions and state transition rules. Monte Carlo simulation results support the validity of the proposed approach.

1. Introduction

In a revealed preference framework, choices made by agents reflect their underlying preferences, thus are the key ingredients to further economic analysis. In reality, however, agents' decisions may not always be observed by researchers, and there are multiple reasons why this may occur. For example, in many panel survey datasets, some choice variables of interest (e.g., individuals' investment decisions on human capital, health, and child development) are not included as a result of the survey design. Data on consumer choices could be proprietary or highly regulated by the government due to privacy concerns. Moreover, there may be more than one dimension of choices (e.g., whether individuals search and the intensity of search) but not all are observed. In some other scenarios, individuals may have inherent incentives *not* to disclose (or truthfully report) their choices. For instance, an executive officer or a politician may not be willing to reveal the actual amount of time and effort they spend promoting the growth of the company or the economy. In such contexts where a potential moral hazard problem exists, it is even more difficult for researchers to observe or collect data on agents' choices. An important research question therefore arises: when choices are

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unobserved (or at least not fully observed), can we still uncover the decision-making process and infer agents' preferences from the data?

In this paper, we provide novel identification results for dynamic structural models with unobserved choice variables, which have received little attention in the literature.[1] We focus on identifying the conditional choice probabilities (CCPs) and state transition rules when agents' choices are not observed by econometricians. These are the "first-step" objects in the sequential approach for estimating dynamic discrete choice models pioneered in Hotz and Miller (1993), Hotz et al. (1994).[2] In the baseline analysis, we consider a single-agent finite-horizon dynamic discrete choice model (the choice variable could be either binary or take multiple discrete values) with a continuous state variable. The state transition process is specified through a nonparametric regression model with an additive error. We assume that the unobserved choices are independent with the error term conditional on the current state. The key intuition of our identification results is as follows. In a finite-horizon model, agents' choice probabilities are inherently *time-varying*. If the state transition process conditional on choice is stationary, which is typically assumed in the literature, then the differences in the observed state transition process across periods are driven purely by the differences in choice probabilities. Therefore, exploiting variations in moments of observed future state distributions across periods helps identify the unobserved choice probabilities and the latent state transition rules.

We consider several extensions to our baseline model. First, we incorporate individual serially correlated unobserved heterogeneity into the dynamic discrete choice model when choices are unobserved. We use the joint distribution of the observed state variable at four consecutive periods to identify the transition of the observed state conditional on the unobserved heterogeneity, to which we can further apply our method to deal with unobserved choices. Second, we discuss identification for infinite-horizon models. In finite-horizon models, time essentially serves as an exclusion restriction. We show that as long as there exists an excluded variable that only shifts choice probabilities but does not affect the latent state transition process, the identification strategy remains valid for infinite-horizon models. Thirdly, we show that similar identification arguments apply given partial observability of choices.

Our identification results are not limited to single-agent dynamic models. We show that the proposed approach can be extended to dynamic discrete games of incomplete information when choice data are not available. In a game setting, multiple players interact with each other and make decisions simultaneously. Their choices naturally depend on the actions and states of other players; however, the state transition process for a player may only depend on his own actions and state variables in the past. In that case, the state of other players can be treated as an excluded variable (i.e., it only affects the choice probabilities, but not the state transition process); hence, our identification arguments for single-agent models can be applied to recover unobserved choice probabilities in dynamic discrete games.

Following the identification results, we propose a sieve maximum likelihood estimation strategy to jointly estimate primitives in agent's utility functions and state transition rules. We conduct Monte Carlo simulations to examine the finite sample performances of our estimator under different data generating processes. Overall, our Monte Carlo simulations perform well. Compared to the estimation results assuming that the choices are observed by the econometricians, the results given unobserved choices exhibit slightly larger finite sample biases and mean squared errors. In addition to the simulations, for illustration purposes, we apply our method to two empirical settings. We first estimate the well-known engine replacement model studied in Rust (1987) assuming that agents' choices are not observed by econometricians. Our estimates are close to the ones given observable decisions and produce similar choice patterns. We also apply our method to a dataset containing all gubernatorial elections in the United States from 1950–2000. We estimate a model of governors' effort exerting decisions, which are not observed by econometricians. Our empirical analysis suggests that the probability of shirking increases as the governors approach the end of their terms.

Finally, we want to emphasize that our method fits better with the empirical applications where there is an intuitive or known set of discrete choices. For example, in Rust (1987), the manager's decision is known to be binary (i.e., replace the bus engine or not). Our method may also work well in settings where choices are partially observable. For instance in the context of school choice, we might observe that students go to college, but not observe whether the students choose STEM or non-STEM major. Partial observability of choices could provide useful information about the decision problem and hence help interpret discrete unobservables as actions. We discuss how our identification results apply to two types of partial observability of choices in Section 6.3.

When choices are completely unobserved or there is not an intuitive choice set *a priori*, more caution is needed when applying our method and interpreting the results. In Section 3.3, we provide theoretical results to identify the lower bound of the number of alternatives in the choice set from observable information under mild assumptions. When additional rank conditions are satisfied, we show that the number of alternatives is point identified. It is also possible that in some applications the unobservable actions are conflated with unobserved heterogeneity. One potential way to distinguish between the two is to exploit the restrictions implied by the dynamic structural model. The estimated probabilities of unobservable actions are endogenous objects derived from the agent's dynamic optimization problem. However, the probabilities of categories of different unobserved heterogeneity are exogenously determined. They do not necessarily satisfy the model's restrictions and can vary arbitrarily across time periods.

[1] Existing literature on dynamic discrete choice models mainly focuses on the cases in which choices are observable. Classic examples include engine replacement decisions in Rust (1987), parental contraceptive choices in Hotz and Miller (1993), occupational choices in Keane and Wolpin (1997), employee retirement decisions in Rust and Phelan (1997), retail firm inventory strategies in Aguirregabiria (1999), and water authority pricing behavior in Timmins (2002), etc. See Eckstein and Wolpin (1989), Rust (1994), Aguirregabiria and Mira (2010), and Arcidiacono and Ellickson (2011) for comprehensive surveys on dynamic discrete choice structural models.

[2] Once the CCPs and state transition rules are identified and estimated from the data in the first step, the structural parameters can be estimated in the second step. Magnac and Thesmar (2002) and Arcidiacono and Miller (2020) later provide the nonparametric identification of utility functions following the same CCP-based approach.

Literature review. Our paper is closely related to the literature on the identification of dynamic discrete choice models (Rust, 1994; Magnac and Thesmar, 2002; Abbring, 2010; Norets and Tang, 2014; Arcidiacono and Miller, 2020), which unexceptionally requires the observation of agents' choices. Arcidiacono and Miller (2020) summarize the necessary and sufficient conditions for identifying a certain class of models, where the utility function is time-separable, the unobserved states are conditionally independent and additively separable, and the agents' beliefs are rational.[3] Several papers have explored using additional assumptions (e.g., parametric assumptions on utility functions, exclusion restrictions on which state variables affect payoffs, availability of terminating actions) to identify the discount factor (Fang and Wang, 2015; Bajari et al., 2016; Komarova et al., 2018; Abbring and Daljord, 2020; Schneider, 2021), or counterfactual policies without normalizing per period payoffs (Aguirregabiria, 2010). Our paper in general fits into this literature and we focus on relaxing the assumption of full data coverage (i.e., observations of state and choice variables for a random set of agents for a sufficient period of time). Relatedly, Arcidiacono and Miller (2020) study identification of dynamic discrete choice models when the panels are *short*, so that the choices and state transitions after the sample period are not observed. Sasaki et al. (2023) develop the sharp identified sets of structural parameters when empirical data do not cover realizations of relevant future states. Aguirregabiria and Mira (2007), Kasahara and Shimotsu (2009), Arcidiacono and Miller (2011), Hu and Shum (2012), Hu and Sasaki (2018), Aguirregabiria et al. (2021), and Berry and Compiani (2023) focus on another type of incomplete data coverage: there exist latent types or a latent time-varying state variable so that even with the choices observed in the data, the CCPs cannot be directly recovered. Our paper adds to this literature on the identification of dynamic discrete choice models with incomplete data coverage. We propose assumptions on the state transition process to achieve identification of model primitives when the data do not cover agents' choices. In particular, we require the observation of a continuous state variable, the transition process of which is time-homogeneous and specified through a nonparametric regression model with an independent, additive error.

For the estimation of dynamic discrete choice models in the existing literature, agents' choices are usually needed to construct (pseudo) likelihood or to implement first-stage nonparametric estimation of the conditional choice probabilities and the state transition probabilities (see Rust, 1987; Hotz and Miller, 1993; Hotz et al., 1994; Aguirregabiria and Mira, 2002). Our paper complements this literature by developing estimation strategies that do not rely on the availability of agents' choices.

Our paper incorporates unobserved choice variables into a general framework of dynamic discrete choice models by imposing restrictions on the state transition process. Our identification assumption is different from other papers studying models with unobserved choices in various empirical settings. For example, Misra and Nair (2011) assume that the salesman allocates the same effort to different clients so that there are multiple proxies for the unobserved effort. Copeland and Monnet (2009) rely on special features of the firm's incentive pay scheme to estimate effort's effect on output. Gayle and Miller (2015) assume that an upper range of revenue may not be achieved if the agent shirks. Perrigne and Vuong (2011) focus on a model where there is a one-to-one mapping between the unobserved effort and observables (such as price).

The rest of the paper is organized as follows. We outline a standard dynamic discrete choice model in Section 2. Identification and estimation results for the baseline model are provided in Sections 3 and 4, respectively. Section 5 presents simulation results. We consider extensions to the baseline model in Section 6 and discuss two illustrating empirical examples in Section 7. Section 8 concludes.

2. A baseline model

Consider a finite-horizon model with $t = 1, 2, \dots, T < \infty$ being the time index. We use $s_t \in \mathcal{S}$ to represent the observed state variable and $y_t \in \mathcal{Y} = \{1, 2, \dots, J\}$ to denote the agent's choice. We first introduce the following assumption to restrict attention to certain classes of dynamic discrete choice models.

Assumption 1. The dynamic process of $\{s_t, y_t\}$ satisfies the following conditions.

(i) $f_{s_{t+1}, y_{t+1} | \Omega_{\le t}} = f_{y_{t+1}|s_{t+1}} f_{s_{t+1}|s_t, y_t}$, where $\Omega_{\le t} = \{s_\tau, y_\tau\}_{\tau=1}^{t}$.
(ii) $f_{s_{t_1+1}|s_{t_1}, y_{t_1}} = f_{s_{t_2+1}|s_{t_2}, y_{t_2}}$ for all (t_1, t_2).

Assumption 1(i) implies that the state transition is a first-order Markov process with limited feedback. $f_{y_{t+1}|s_{t+1}}$ represents the conditional choice probability, and $f_{s_{t+1}|s_t, y_t}$ represents the density of future state conditional on the current state and choice. This assumption is commonly adopted in the dynamic discrete choice literature and may be relaxed to allow for a higher-order Markov process or serially correlated unobserved heterogeneity. Assumption 1(ii) imposes stationarity on the state transition rule. Bajari et al. (2016) invoke the same assumption of time-homogeneous state transition for finite-horizon dynamic discrete choice models.

When the choice variable y_t is observed by the econometrician, the conditional choice probabilities $f_{y_{t+1}|s_{t+1}}$ and state transition rules $f_{s_{t+1}|s_t, y_t}$ are nonparametrically identified and can be directly estimated from the data. With these objects known, Magnac and Thesmar (2002) focus on identification of utility primitives for certain types of random utility models with geometric discounting. Specifically, they consider standard single-agent dynamic discrete choice models, where agents' per-period utility is additive separable: $u_t(s_t, \varepsilon_t, y_t) = u_t^*(s_t, y_t) + \varepsilon_t(y_t)$ and $\varepsilon_t = (\varepsilon_t(1), \varepsilon_t(2), \dots, \varepsilon_t(J)) \in \mathbb{R}^J$ represents the vector of i.i.d. preference shocks. The sum of the discounted utility stream of the agent is defined as $U(\boldsymbol{s}, \boldsymbol{\varepsilon}, \boldsymbol{y}) = \sum_{t=1}^{T} \beta^{t-1} u_t(s_t, \varepsilon_t, y_t)$, where $\boldsymbol{s} = (s_1, \dots, s_T)$, $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_T)$,

[3] An et al. (2021) studies identification of dynamic discrete choice models allowing subjective beliefs; Aguirregabiria and Magesan (2020) studies identification and estimation of dynamic discrete games allowing the players' beliefs are not in equilibrium.

$\boldsymbol{y} = (y_1, \ldots, y_T)$, and β is the discount factor. The agent's problem is to choose an optimal decision rule that maximizes the expected sum of the discounted utility. Using the Bellman's equation, the agent's optimization problem can be represented as

$$V_t(s_t, \varepsilon_t) = \max_y \left\{ u_t^*(s_t, y) + \varepsilon_t(y) + \beta \mathrm{E}[V_{t+1}(s_{t+1}, \varepsilon_{t+1})|s_t, y] \right\}. \tag{2.1}$$

Magnac and Thesmar (2002) show that utility functions in each alternative are nonparametrically identified if the distribution of the unobserved state, the discount factor, and the utility in one reference alternative are known. Arcidiacono and Miller (2020) characterize the identification of flow payoffs for a more general case nesting both the infinite and finite horizon models.

When the choice variable is *not* observed by econometricians, we cannot recover the decision rules nor the state transition probabilities directly from the data. These two objects are necessary for the identification of utility primitives in Magnac and Thesmar (2002) and Arcidiacono and Miller (2020); moreover, they are important second-step inputs for the CCP method developed by Hotz and Miller (1993) for estimating dynamic discrete choice models. To identify conditional choice probabilities and state transition rules when actions are not observed, one way is to impose stronger assumptions on the state transition process in addition to Assumption 1.

3. Identification

In this section, we provide identification results for the unobserved choice probabilities $f_{y_t|s_t}$ and latent state transition probabilities $f_{s_{t+1}|s_t,y_t}$ when only $\{s_t\}_{t=1}^{T+1}$ (with $T \geq 2$) are observed for a random sample of agents.[4] From now on, to follow notations in the literature, we use $p_t(y_t|s_t)$ to represent the unobserved conditional choice probabilities. We focus on the case in which s_t is an univariate continuous state variable. When only discrete state variables are available (which is also commonly seen in empirical settings), we show in Appendix B that the unobserved choice probabilities and state transition rules are identified when there are multiple states and their transition processes are independent conditional on the choice variable.

When agents' choices are unobserved, neither conditional choice probabilities nor state transition rules can be directly recovered from the data. However, these two sets of unknowns are connected through the observed state transition process as shown in the following equation under Assumption 1(i).

$$f_{s_{t+1}|s_t}(s'|s) = \sum_{y=1}^{J} f_{s_{t+1}|s_t,y_t}(s'|s, y) p_t(y|s), \tag{3.1}$$

where s' and s represent realized values of s_{t+1} and s_t, respectively. In Eq. (3.1), the probability density of the future state conditional on the current state is a mixture of the true latent state transition probabilities conditional on different alternatives, and the choice probabilities serve as the mixing weights. Under Assumption 1(ii), $f_{s_{t+1}|s_t,y_t}$ is time-invariant; while in finite-horizon models, $p_t(y|s)$ varies across different periods. The differences in $f_{s_{t+1}|s_t}$ across periods are therefore driven by the non-stationarity of the choice probabilities. Exploiting variations in moments of the observed state transition process enables us to identify choice probabilities and latent state transition rules, for which the following assumption is invoked.

Assumption 2 (*State Transition*)**.** Suppose that the state variable s_t is continuous.

$$s_{t+1} = m(y_t, s_t) + \eta_t,$$

where $m(\cdot, \cdot)$ is continuously differentiable in s_t, $\mathrm{E}(\eta_t|s_t) = 0$, $\eta_t \perp y_t|s_t$, and the conditional density function of the error term $f_{\eta_t|s_t}(\cdot|\cdot)$ is continuous in s_t.

Assumption 2 is the key identifying assumption we impose that is stronger than those imposed in the earlier literature on the state transition process. When choices are completely unobserved to the econometricians, we need to impose additional structures on the latent state transition process to connect choice probabilities with observed state transition to facilitate identification. In particular, Assumption 2 specifies the transition process of the continuous state variable s_t through a nonparametric regression model, where $m(\cdot, \cdot)$ is an unknown smooth function and η_t represents an additive random shock realized in the transition process with conditional mean equal to zero. It also requires that the regression error is independent of the unobserved choice conditional on the state variable. Similar assumptions have been used in previous empirical work. For example, Misra and Nair (2011) study the effect of unobserved effort (e_t) on sales (q_t), which is a continuous variable. They assume that $q_t = g(e_t, z_t) + \varepsilon_t$, where g is the sales production function, z_t is a vector of observed factors, and ε_t is a mean-zero random shock to demand realized at the end of each period.

The additivity and independence assumption of the error term does impose restrictions on the state transition. Specifically, the agent's choices affect the state transition process only through the deterministic part but not through the error term—conditional on the current state, the choice only shifts the mean of the future state distribution. In Appendix B, we present identification results for cases where such restrictions can be relaxed.

Combining Assumption 2 and Assumption 1(ii), we know that the conditional distribution of η_t is stationary. That is, for $t_1, t_2 \in \{1, \ldots, T\}$, $f_{\eta_{t_1}|s_{t_1}}(\eta|s) = f_{\eta_{t_2}|s_{t_2}}(\eta|s)$, $\forall \eta, s$. Furthermore, we assume that the conditional density function of η_t is continuous

[4] We assume that after the final period T, the realization of the state variable at $T+1$ is observed. Observing s_{T+1} is necessary for recovering the choice probabilities at T.

in s_t. By Assumption 2, the unknown function $m(\cdot,\cdot)$ and the conditional distribution of η_t jointly determine the state transition probabilities $f_{s_{t+1}|s_t,y_t}$, and thus are the key primitives to be identified in addition to the unobserved choice probabilities. In the rest of this section, we first provide identification results assuming the number of alternatives J is known by the econometricians. We then provide a theorem to identify J from data.

3.1. Binary choice

Suppose the choice variable takes binary values, i.e., $y_t \in \{0,1\}$. Identifying the function $m(y_t, s_t)$ is equivalent to identifying two functions of s_t, i.e., $m(y_t = 0, s_t)$ and $m(y_t = 1, s_t)$. For a fixed state s, let $m_1 = m(1,s)$ and $m_0 = m(0,s)$. Let $p_t = p_t(y_t = 1|s)$ and $1 - p_t = p_t(y_t = 0|s)$ denote the choice probabilities associated with choices 1 and 0, respectively, at period t. We define the first-, the second-, and the third-order conditional moments of the observed state variable at $t+1$ as follows.

$$\mu_{t+1} = \mathrm{E}\left[s_{t+1}|s_t = s\right], \quad \nu_{t+1} = \mathrm{E}\left[(s_{t+1} - \mu_{t+1})^2|s_t = s\right], \quad \xi_{t+1} = \mathrm{E}\left[(s_{t+1} - \mu_{t+1})^3|s_t = s\right].$$

All of these conditional moments can be directly estimated from the data, and are thus treated as known constants for identification purposes.

Note that μ_t, ν_t, and ξ_t are essentially weighted averages of moments of the future state conditional on the current state and the choice, where the choice probabilities $(p_t, 1-p_t)$ serve as the mixing weights. Given that $s_{t+1} = m(y_t, s_t) + \eta_t$ and η_t and y_t are independent conditional on s_t under Assumption 2, μ_t, ν_t, and ξ_t can be represented as functions of m_1, m_0, p_t, and moments of η_t conditional on $s_t = s$. For example,

$$\mu_{t+1} = p_t m_1 + (1-p_t)m_0 + \mathrm{E}(\eta_t|s).$$

By Assumption 2, the conditional mean of η_t equals 0, i.e., $\mathrm{E}(\eta_t|s) = 0$. Therefore, the choice probability

$$p_t = \frac{\mu_{t+1} - m_0}{m_1 - m_0}, \tag{3.2}$$

provided that $m_1 \neq m_0$.

Under Assumption 1(ii) and Assumption 2, the conditional distribution of η_t is stationary. This implies that the higher order moments of the error term η_t are time-invariant conditional on the same state s. Taking the difference of moments of the observed state variable (i.e., ν_{t+1} and ξ_{t+1}) across two periods t_1 and t_2 eliminates the unknown moments of η and leads to a system of equations for m_1 and m_0. We show that m_0 and m_1 are the two solutions to the following equation:

$$m^2 - \Delta_1 m + \Delta_2 = 0, \tag{3.3}$$

where

$$\Delta_1 = \frac{\nu_{t_1+1} - \nu_{t_2+1} + (\mu^2_{t_1+1} - \mu^2_{t_2+1})}{\mu_{t_1+1} - \mu_{t_2+1}},$$
$$\Delta_2 = \frac{\xi_{t_1+1} - \xi_{t_2+1} - (\mu_{t_1+1}(\Delta_1 - \mu_{t_1+1})(\Delta_1 - 2\mu_{t_1+1}) - \mu_{t_2+1}(\Delta_1 - \mu_{t_2+1})(\Delta_1 - 2\mu_{t_2+1}))}{2(\mu_{t_1+1} - \mu_{t_2+1})}.$$

Assumptions 1 and 2 guarantee that there exist two real roots of the quadratic equation (i.e., $\Delta_1^2 - 4\Delta_2 \geq 0$). This condition is empirically testable as Δ_1 and Δ_2 can be directly estimated using moments of the observed state variable provided that $\mu_{t_1+1} \neq \mu_{t_2+1}$. The condition that $\mu_{t_1+1} \neq \mu_{t_2+1}$ implies $m_1 \neq m_0$ and $p_{t_1} \neq p_{t_2}$, and is also empirically testable from the data. If $m_1 = m_0$, the identification of m functions at state s is trivial as $m_1 = m_0 = \mu_{t_1+1} = \mu_{t_2+1}$, but the choice probabilities are not identified as shown in Eq. (3.2). If $m_1 \neq m_0$ and $p_{t_1} = p_{t_2}$, we have no variations in observables from periods t_1 and t_2 to identify the m functions and choice probabilities. We invoke the following assumption to restrict our attention to cases where $\mu_{t_1+1} \neq \mu_{t_2+1}$ and to pin down the order of m_0 and m_1.

Assumption 3. The following conditions are satisfied:

(i) There exist two periods t_1 and t_2 such that $Pr(\tilde{S}) = 0$ with

$$\tilde{S} := \{s \in S : \mathrm{E}[s_{t_1+1}|s_{t_1} = s] = \mathrm{E}[s_{t_2+1}|s_{t_2} = s]\},$$

(ii) $\frac{\partial m(1,s)}{\partial s} \neq \frac{\partial m(0,s)}{\partial s}$ holds for any $s \in \bar{S} := \{s \in \tilde{S} : m(1,s) = m(0,s)\}$,

(iii) There exists an $s_0 \in S \backslash \bar{S}$ such that $m(1,s_0) > m(0,s_0)$.

Assumption 3(i), imposed on observables, ensures that the set of states where we cannot identify m functions from Eqs. (3.3) using data from periods t_1 and t_2 has a zero measure.[5] The values of m functions for $s \in \tilde{S}$ are, instead, identified using the continuity of $m(1,\cdot)$ and $m(0,\cdot)$ in s. Assumption 3(ii) rules out the possibility that the first-order derivatives of $m(1,\cdot)$ and $m(0,\cdot)$ are equal at

[5] For $s \in \tilde{S}$, the first-order conditional moments of the state variable are the same across two periods t_1 and t_2, i.e., $\mu_{t_1+1} = \mu_{t_2+1}$, so that Eq. (3.3) is not well defined.

the point they intersect (i.e., $m(1,\cdot)$ and $m(0,\cdot)$ are tangent to each other). Note that $\bar{S}$ also has zero probability as it is a subset of $\tilde{S}$. Under Assumption 3(i)–(ii), as long as there exists a state s at which we can order the two m functions, the smoothness condition (i.e., continuously differentiable) helps match $m(1,\cdot)$ and $m(0,\cdot)$ across all values of $s \in \mathcal{S}$. Assumption 3(iii) provides such an ordering condition as needed. Note that the assumption that $m(1,s_0) > m(0,s_0)$ is without loss of generality. Before linking the identified choice probabilities and conditional state transition probabilities to structural utility primitives, we can swap the labels of the two choices.

Once m_1 and m_0 are identified (and if they are not equal), the conditional choice probabilities are identified from Eq. (3.2). For $t \in \{t_1, t_2\}$, the observed state transition probability of $s_{t+1} = s'$ given $s_t = s$ can be written as a mixture of the conditional density of η_t evaluated at $s' - m_1$ and $s' - m_0$; again, the choice probabilities $(p_t, 1-p_t)$ serve as the mixing weights. With variations in choice probabilities across two periods (i.e., $p_{t_1} \neq p_{t_2}$) and the stationarity of η_t conditional on s_t, the conditional density function of η_t at $s' - m_1$ and $s' - m_0$ for all $t \in \{1, 2, \ldots, T\}$ and $s' \in \mathcal{S}$ is also identified, i.e.,

$$\begin{aligned} f_{\eta_t|s_t}(s'-m_1|s) &= \frac{f_{s_{t_1+1}|s_{t_1}}(s'|s)(1-p_{t_2}) - f_{s_{t_2+1}|s_{t_2}}(s'|s)(1-p_{t_1})}{p_{t_1}-p_{t_2}}, \\ f_{\eta_t|s_t}(s'-m_0|s) &= \frac{f_{s_{t_2+1}|s_{t_2}}(s'|s)p_{t_1} - f_{s_{t_1+1}|s_{t_1}}(s'|s)p_{t_2}}{p_{t_1}-p_{t_2}}. \end{aligned} \tag{3.4}$$

At the state s in which $p_{t_1}(\cdot|s) = p_{t_2}(\cdot|s)$, $f_{\eta_t|s_t}(\cdot|s)$ is instead identified by the continuity of the conditional density function of η_t in s_t as imposed in Assumption 2.[6]

We summarize the formal identification results in the following theorem.

Theorem 1 (*Identification – Binary Choice*). *If Assumptions 1, 2, and 3 hold for the dynamic process of* $\{s_t, y_t\}$ *with* $y_t \in \{0,1\}$*, then the observed conditional densities* $f_{s_{t_1+1}|s_{t_1}}(\cdot|\cdot)$ *and* $f_{s_{t_2+1}|s_{t_2}}(\cdot|\cdot)$ *for* t_1 *and* t_2 *defined in Assumption 3 uniquely determine:*

(i) the state transition $f_{s_{t+1}|s_t,y_t}$*, including* $m(1,s)$*,* $m(0,s)$*, and* $f_{\eta_t|s_t}(\cdot|s)$ *for all* $t \in \{1,2,\ldots,T\}$ *and* $s \in \mathcal{S}$*;*
(ii) choice probabilities $p_{t_1}(\cdot|s)$ *and* $p_{t_2}(\cdot|s)$ *for* $s \in \mathcal{S}\backslash\bar{\mathcal{S}}$ *with* $Pr(\bar{\mathcal{S}}) = 0$*.*

Proof. See Appendix A.1. □

Once the state transition rules are identified, choice probabilities $p_t(\cdot|s)$ are identified from $f_{s_{t+1}|s_t}(\cdot|s)$ for all $t \in \{1,2,\ldots,T\}$ and $s \in \mathcal{S}\backslash\bar{\mathcal{S}}$ with $Pr(\bar{\mathcal{S}}) = 0$. If additional smoothness assumptions are imposed on the utility functions, choice probabilities $p_t(\cdot|s)$ for $s \in \bar{\mathcal{S}}$ are also identified.

3.2. General multinomial choice

Next, we consider the case in which the choice variable takes multiple discrete values, i.e., $y_t \in \{1,2,\ldots,J\}$, with $J \geq 2$ known by the econometricians. Identifying the nonparametric function $m(y_t, s_t)$ for a fixed state s is equivalent to identifying the following J constants:

$$m_1 = m(y_t = 1, s), \quad m_2 = m(y_t = 2, s), \quad \cdots, \quad m_J = m(y_t = J, s).$$

In addition, we need to identify the conditional choice probabilities $p_t(y_t = j|s)$ for $j = 1,2,\ldots,J-1$ and $t = 1,2,\ldots,T$, as well as the distribution of η_t given $s_t = s$.

Under Assumptions 1 and 2, the observed state transition process can be represented as

$$f_{s_{t+1}|s_t}(s'|s) = \sum_{y=1}^{J} f_{s_{t+1}|s_t,y_t}(s'|s,y)p_t(y|s) = \sum_{y=1}^{J} f_{\eta_t|s_t}(s' - m(y,s)|s)p_t(y|s). \tag{3.5}$$

We show in Appendix A.2 that constructing the characteristic function of $s_{t+1}|s_t = s$ based on Eq. (3.5) yields

$$\phi_{s_{t+1}|s_t=s}(r) = \left|\phi_{\eta_t|s_t=s}(r)\right| \sum_{y=1}^{J} \exp(irm(y,s) + ib(r))p_t(y|s), \tag{3.6}$$

for all real-valued scalar $r \in \mathbb{R}$, where i represents the imaginary unit. $\phi_{s_{t+1}|s_t=s}(r)$ and $\phi_{\eta_t|s_t=s}(r)$ are the characteristic functions of $s_{t+1}|s_t = s$ and $\eta_t|s_t = s$, respectively, and

$$\left|\phi_{\eta_t|s_t=s}(r)\right| \equiv \sqrt{[\mathbf{Re}\{\phi_{\eta_t|s_t=s}(r)\}]^2 + [\mathbf{Im}\{\phi_{\eta_t|s_t=s}(r)\}]^2}, \quad b(r) \equiv \arccos \frac{\mathbf{Re}\{\phi_{\eta_t|s_t=s}(r)\}}{\left|\phi_{\eta_t|s_t=s}(r)\right|}.$$

[6] Note that $\{s \in \mathcal{S} : p_{t_1}(\cdot|s) = p_{t_2}(\cdot|s)\}$ is a subset of $\tilde{S}$, thus also has zero probability. Once $f_{\eta_t|s_t}(\cdot|s)$ is identified for all $s \in \mathcal{S}$ such that $p_{t_1}(\cdot|s) \neq p_{t_2}(\cdot|s)$, by the continuity of the density function, $f_{\eta_t|s_t}(\cdot|s)$ is identified for all $s \in \mathcal{S}$.

We now consider a set of time periods $\mathcal{T} = \{t_1, t_2, \ldots, t_J\}$. Define

$$P(\mathcal{T}) = \begin{pmatrix} p_{t_1}(1|s) & p_{t_1}(2|s) & \cdots & p_{t_1}(J|s) \\ p_{t_2}(1|s) & p_{t_2}(2|s) & \cdots & p_{t_2}(J|s) \\ \vdots & \vdots & \ddots & \vdots \\ p_{t_J}(1|s) & p_{t_J}(2|s) & \cdots & p_{t_J}(J|s) \end{pmatrix}_{J\times J}.$$

For a set of real-valued scalers $\mathcal{R} = \{0, r_2, \ldots, r_J\}$, let $D(\mathcal{R}) = Diag\{1, \left|\phi_{\eta_t|s_t=s}(r_2)\right|, \ldots, \left|\phi_{\eta_t|s_t=s}(r_J)\right|\}$,

$$\Phi(\mathcal{T},\mathcal{R}) = \begin{pmatrix} 1 & \phi_{s_{t_1+1}|s_{t_1}=s}(r_2) & \cdots & \phi_{s_{t_1+1}|s_{t_1}=s}(r_J) \\ 1 & \phi_{s_{t_2+1}|s_{t_2}=s}(r_2) & \cdots & \phi_{s_{t_2+1}|s_{t_2}=s}(r_J) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_{s_{t_J+1}|s_{t_J}=s}(r_2) & \cdots & \phi_{s_{t_J+1}|s_{t_J}=s}(r_J) \end{pmatrix}_{J\times J}, \quad \Gamma(\mathcal{R}) = \begin{pmatrix} 1 & \exp(ir_2m_1 + ib(r_2)) & \cdots & \exp(ir_Jm_1 + ib(r_J)) \\ 1 & \exp(ir_2m_2 + ib(r_2)) & \cdots & \exp(ir_Jm_2 + ib(r_J)) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \exp(ir_2m_J + ib(r_2)) & \cdots & \exp(ir_Jm_J + ib(r_J)) \end{pmatrix}_{J\times J}.$$

Eq. (3.6) implies that

$$\Phi(\mathcal{T},\mathcal{R}) = P(\mathcal{T})\Gamma(\mathcal{R})D(\mathcal{R}). \tag{3.7}$$

$\Phi(\mathcal{T},\mathcal{R})$ is a matrix that can be directly estimated from the data for any $\mathcal{T}$ and $\mathcal{R}$. On the right-hand side of Eq. (3.7), matrix $P(\mathcal{T})$ contains the unobserved conditional choice probabilities; $\Gamma(\mathcal{R})$ and $D(\mathcal{R})$ contain unknown structural parameters, including $m(\cdot,s)$ function, and the distribution of η_t (in the form of characteristic function). As we choose different sets of real-valued scalers in $\mathcal{R}$, Eq. (3.7) provides a sufficient number of equality constraints to identify the unknown structural parameters, provided that some rank conditions hold to rule out multiple solutions to these equations.

Let $A_{\mathcal{T},\mathcal{R}} := (\mathbf{Re}\{\Phi(\mathcal{T},\mathcal{R})\})^{-1}(\mathbf{Im}\{\Phi(\mathcal{T},\mathcal{R})\} + \Xi)$, where $\Xi := (\mathbf{1}_{J\times1}, \mathbf{0}_{J\times1}, \ldots, \mathbf{0}_{J\times1})$ is a $J\times J$ matrix with the elements in the first column equal to 1 and all other elements equal to 0. We impose the following rank condition.

Assumption 4 (*Rank*). There is a set of real numbers $\mathcal{R} = \{0, r_2, \ldots, r_J\}$ such that (i) $\mathbf{Re}\{\Phi(\mathcal{T},\mathcal{R})\}$ and $(\mathbf{Im}\{\Phi(\mathcal{T},\mathcal{R})\} + \Xi)$ are invertible, and (ii) for any real-valued $J\times J$ diagonal matrices $D_\iota = Diag\{0, d_{\iota,2}, \ldots, d_{\iota,J}\}$, if $D_1 + A_{\mathcal{T},\mathcal{R}}D_1A_{\mathcal{T},\mathcal{R}} + D_2A_{\mathcal{T},\mathcal{R}} - A_{\mathcal{T},\mathcal{R}}D_2 = 0$, then $D_\iota = 0$ for $\iota = 1, 2$.

Assumption 4 is similar to the rank condition proposed in Chen et al. (2009, Assumption 2.3).[7] It is analogous to the rank condition for identification in linear models and it is empirically testable as this assumption only involves the observable matrix $\Phi(\mathcal{T},\mathcal{R})$. As shown in the simulation exercises in Chen et al. (2009), Assumption 4 is generally satisfied with a continuous state and a finite number of alternatives. In Appendix A.2, we prove that $\Upsilon(\mathcal{R}) := (\mathbf{Re}\{\Gamma(\mathcal{R})D(\mathcal{R})\})^{-1}D_m\mathbf{Re}\{\Gamma(\mathcal{R})D(\mathcal{R})\}$ is identified under Assumption 4, where

$$D_m := Diag\{m_1, m_2, \ldots, m_J\}.$$

Moreover, we show that the following equality holds:

$$\mathbf{Re}\{\Phi(\mathcal{T},\mathcal{R})\}\Upsilon(\mathcal{R})(\mathbf{Re}\{\Phi(\mathcal{T},\mathcal{R})\})^{-1} = P(\mathcal{T})D_mP(\mathcal{T})^{-1}. \tag{3.8}$$

Eq. (3.8) leads to an eigenvalue-eigenvector decomposition of a known matrix on its left-hand side. $P(\mathcal{T})$ represents the matrix of eigenvectors and the diagonal elements in D_m are the corresponding eigenvalues. Additional assumptions are required to guarantee the uniqueness of the decomposition and to pin down the ordering of the eigenvectors. For example, we provide one such assumption below.

Assumption 5. $m(y_t, s) < \infty$ and $m(y_t, s) \neq 0$ for all $y_t \in \{1, 2, \ldots, J\}$. $m(y_t, s)$ strictly increases with $y_t \in \{1, 2, \ldots, J\}$.

Assumption 5 ensures that the eigenvalues $m_1, m_2, \ldots, m_J$ are bounded and distinct across all y_t. It provides an example of the shape restriction on the $m(\cdot,\cdot)$ function that allows us to link the identified eigenvalues in D_m and eigenvectors in $P(\mathcal{T})$ to its corresponding y_t. Other similar shape restrictions may also be imposed. We summarize the main identification theorem for the conditional choice probabilities and the structural parameters governing the state transition rules in the following theorem.

Theorem 2 (*Identification – General Multinomial Choice*). *If Assumptions 1, 2, 4, and 5 hold for the dynamic process of $\{s_t, y_t\}$ with $y_t \in \{1, 2, \ldots, J\}$ and $s \in \mathcal{S}$, then the observed conditional densities $f_{s_{t+1}|s_t}(\cdot|\cdot)$ for $t \in \{t_1, t_2, \ldots, t_J\}$ uniquely determine:*

(i) the state transition $f_{s_{t+1}|s_t,y_t}$, including $\{m(j,s)\}_{j=1}^J$ and $f_{\eta_t|s_t}(\cdot|s)$ for $t \in \{1, 2, \ldots, T\}$ and $s \in \mathcal{S}$;

(ii) the conditional choice probabilities $p_t(\cdot|s)$ for $t \in \{t_1, t_2, \ldots, t_J\}$ and $s \in \mathcal{S}$.

[7] Chen et al. (2009) studies identification of nonclassical errors-in-variables models where $Y = m(X^*) + \eta$ and X^* is the latent variable proxied by observable variable X. Our state transition process in Assumption 2 is similar to the nonparametric regression model in Chen et al. (2009). In our case, the unobserved choice y_t is the latent variable of interest. We exploit variations in observed state transition across time periods to identify the unobserved choice probabilities and primitives in the state transition rule. In contrast, Chen et al. (2009) exploit variations in the joint distribution of the outcome variable Y and the proxy X.

Proof. See Appendix A.2. □

Theorem 2 implies that identifying the primitives in the state transition rule specified in Assumption 2 requires the observation of the state variable for at least $J+1$ periods when $y_t \in \{1,2,\dots,J\}$. When more periods of data are available, we can relax the time-invariance assumption imposed on the state transition rules. For two sets of periods $\{t_1,t_2,\dots,t_J\}$ and $\{\tau_1,\tau_2,\dots,\tau_J\}$, Theorem 2 identifies the state transition rules for periods in these two sets separately. We can therefore empirically test whether the state transition rules are time invariant.

When $J+1$ periods of data are not available, having more than one state variables that are independent conditional on the choice variable helps to identify the latent state transition rules and unobserved conditional choice probabilities with two consecutive time periods. Intuitively, the future states can be viewed as proxies of the unobserved choice. The results from the measurement error literature (e.g., Hu (2008), Hu and Schennach (2008)) can then be applied. With additional state variables, the requirement of continuous state variable, the additivity and independence assumption imposed on the state transition process in Assumption 2 can be relaxed. We provide the identification results with two state variables in Appendix B.

Remark 1. In Sections 3.1 and 3.2, we provide two alternative sets of sufficient conditions for identification. Some high-level conditions in the general multinomial case have direct implications for the assumptions in the binary model. For example, Assumption 4(i) requires that the real part of matrix $\Phi(\mathcal{T},\mathcal{R})$ has full rank. In the special case where $J = 2$, this assumption essentially requires that $p_{t_1}(y|s) \neq p_{t_2}(y|s)$, which is also implied in Assumption 3(i) for the binary case. The shape restrictions on $m(y,s)$ are needed for both binary and multinomial cases.

3.3. Identifying the number of alternatives

So far, we have assumed that the econometrician knows the number of alternatives in the agent's choice set. We now provide theoretical results on identifying J. When J is unknown, we consider a generic set of time periods $\overline{\mathcal{T}} = \{t_1,t_2,\dots,t_{|\overline{\mathcal{T}}|}\}$ and a set of scaler values $\overline{\mathcal{R}} = \{0,r_2,\dots,r_{|\overline{\mathcal{R}}|}\}$, where $\left|\overline{\mathcal{T}}\right|$ and $\left|\overline{\mathcal{R}}\right|$ denote the cardinality of $\overline{\mathcal{T}}$ and $\overline{\mathcal{R}}$, respectively. $\overline{\mathcal{T}}$ and $\overline{\mathcal{R}}$ can be set as large as possible. For example, we can include all time periods available in the data in $\overline{\mathcal{T}}$. Note that Eq. (3.7) holds for any $\overline{\mathcal{T}}$ and $\overline{\mathcal{R}}$. The dimensions of matrices $\Phi(\overline{\mathcal{T}},\overline{\mathcal{R}})$, $P(\overline{\mathcal{T}})$, $\Gamma(\overline{\mathcal{R}})$, and $D(\overline{\mathcal{R}})$ are $\left|\overline{\mathcal{T}}\right| \times \left|\overline{\mathcal{R}}\right|$, $\left|\overline{\mathcal{T}}\right| \times J$, $J \times \left|\overline{\mathcal{R}}\right|$, and $\left|\overline{\mathcal{R}}\right| \times \left|\overline{\mathcal{R}}\right|$, respectively. The following corollary states that we can identify the lower bound of J from the rank of $\Phi(\overline{\mathcal{T}},\overline{\mathcal{R}})$, which is identified directly from the data. With additional rank conditions, J is point identified.

Corollary 1. *Suppose $\left|\overline{\mathcal{T}}\right| \geq J$, $\left|\overline{\mathcal{R}}\right| \geq J$, and Assumptions 1 and 2 hold for the dynamic process of $\{s_t,y_t\}$ with $y_t \in \{1,2,\dots,J\}$ and $s \in \mathcal{S}$. (i) $J \geq rank(\Phi(\overline{\mathcal{T}},\overline{\mathcal{R}}))$. (ii) If matrices $P(\overline{\mathcal{T}})$, $\Gamma(\overline{\mathcal{R}})$, and $D(\overline{\mathcal{R}})$ have full rank, $J = rank(\Phi(\overline{\mathcal{T}},\overline{\mathcal{R}}))$.*

Proof. See Appendix A.3. □

Our identifying assumption for J requires that we observe at least J periods of state transition ($\left|\overline{\mathcal{T}}\right| \geq J$), but the assumption that $\left|\overline{\mathcal{R}}\right| \geq J$ is innocuous because we can always choose a large set of real-valued scalers. Part (i) of Corollary 1 provides identification of the lower bound of J with mild assumptions. The full rank conditions on $P(\overline{\mathcal{T}})$, $\Gamma(\overline{\mathcal{R}})$ and $D(\overline{\mathcal{R}})$ are not directly testable from the data, but they essentially require that there are sufficient variations in choice probabilities and m functions with respect to different alternatives. When these rank conditions are satisfied, part (ii) of Corollary 1 shows the point identification of J.[8]

To choose J empirically, note that given certain sets $\overline{\mathcal{T}}$ and $\overline{\mathcal{R}}$, the matrix $\Phi(\overline{\mathcal{T}},\overline{\mathcal{R}})$ can be estimated from the data using the empirical characteristic function of $s_{t+1}|s_t$. Kleibergen and Paap (2006) develop a procedure to test the null hypothesis that the rank of $\Phi(\overline{\mathcal{T}},\overline{\mathcal{R}})$ is equal to J based on singular value decomposition. The choice of $\overline{\mathcal{T}}$ and $\overline{\mathcal{R}}$ could potentially affect the rank of $\Phi(\overline{\mathcal{T}},\overline{\mathcal{R}})$. In practice, including all available time periods in $\overline{\mathcal{T}}$ and setting $\left|\overline{\mathcal{R}}\right| = \left|\overline{\mathcal{T}}\right|$ might be preferable. The reason that we do not need to choose $\left|\overline{\mathcal{R}}\right| > \left|\overline{\mathcal{T}}\right|$ is because $rank(\Phi(\overline{\mathcal{T}},\overline{\mathcal{R}})) \leq \min\{\left|\overline{\mathcal{T}}\right|,\left|\overline{\mathcal{R}}\right|\}$. In cases where the rank of $\Phi(\overline{\mathcal{T}},\overline{\mathcal{R}})$ changes with the choice of real numbers in $\overline{\mathcal{R}}$, Corollary 1(i) implies that it is preferable to set J to be the largest rank of $\Phi(\overline{\mathcal{T}},\overline{\mathcal{R}})$.[9]

3.4. Discussion

We have provided identification results (Theorem 1 in Section 3.1 and Theorem 2 in Section 3.2) for the conditional choice probabilities and state transition rules when the agents' choice are not observed by econometricians up to *label swapping*.[10] To associate choice probabilities and state transition rules with specific alternatives faced by the agent, various assumptions arising

8 Corollary 1 is closely related to identifying the number of mixture components in a finite mixture. Kasahara and Shimotsu (2014) focus on k-variate, M-component finite mixture models and they show that when $k \geq 2$, a lower bound on the number of components is nonparametrically identified.

9 In our empirical example in Section 7.2, we include all available time periods in $\overline{\mathcal{T}}$ and set $\left|\overline{\mathcal{R}}\right| = \left|\overline{\mathcal{T}}\right|$. We do not find that the rank of $\Phi(\overline{\mathcal{T}},\overline{\mathcal{R}})$ is sensitive to the choices of real numbers in $\overline{\mathcal{R}}$.

10 Similar matching problem has been noted in the study of sequential identification of structural models with unobserved heterogeneity and/or multiple equilibria. See Aguirregabiria and Mira (2019) and Luo et al. (2022) for more discussions on this problem.

from the model or consistent with the economic intuition could be imposed. For example, consider a scenario where s_t represents the realized revenue of a loan and $y_t \in \{0,1\}$ represents the borrower's choice. Suppose we denote $y_t = 1$ if the borrower exerts effort to pay off the debt, and 0 otherwise; the agent's mean utility function has the following form $u^*(s_t, y_t) = \omega s_t - \rho \mathbf{1}\{y_t = 1\}$, where ρ represents the cost of exerting effort. It is reasonable to assume that when the borrower exerts effort, the revenue distribution first-order stochastically dominates the one when he exerts no effort, so that $m(1,s) \geq m(0,s)$ for all $s \in \mathcal{S}$. With this assumption, we can label the larger value of the m functions at state s as $m(1,s)$ and identify the probability that the borrower exerts effort accordingly.[11]

Our identification results for the state transition rules and choice probabilities can be applied to models that are more general than the single agent dynamic discrete choice model described in Magnac and Thesmar (2002). Our results do not rely on specific assumptions such as geometric discount, or even the existence of a random utility model. Instead, we only need some structures imposed on the underlying dynamic process as shown in Assumptions 1 and 2. Once the conditional choice probabilities and state transition rules are recovered for each alternative, the identification of per-period utility functions follows immediately from Magnac and Thesmar (2002) and Arcidiacono and Miller (2020). The details are hence omitted in this paper. Note that if additional assumptions are imposed on the per-period utility functions, the data requirement of $\{s_t\}_{t=1}^{T+1}$ (with $T \geq 2$) may be relaxed. For example, if we restrict the flow utility to not depend on t, then choice probabilities at a certain period t_1 are needed, which require the observations of state variables at (t_1, t_1+1) and another pair (t_2, t_2+1).

An important extension of the identification results in Sections 3.1 and 3.2 is to incorporate serially correlated unobserved heterogeneity into the model. Intuitively, when the unobserved heterogeneity is present, in order to apply our identification strategy, the key is to first recover the transition process of the observed state *conditional on* the unobserved heterogeneity. We show in Section 6.1 that the state transition process given the unobserved heterogeneity is identified from the joint distribution of state variables at four consecutive periods. We discuss several other extensions in Section 6, including infinite-horizon models (Section 6.2), cases in which choice data are partially unavailable (Section 6.3), and dynamic discrete games (Section 6.4).

4. Estimation

In this section, we propose to use a sieve maximum likelihood approach to jointly estimate the utility primitives, the nonparametric function $m(\cdot,\cdot)$ and the distribution of the error term $f_{\eta_t|s_t}$ in the state transition process. We follow the random utility framework in Magnac and Thesmar (2002) and consider a case where the per-period utility functions are parametrized by a finite-dimensional parameter α.[12]

For $\theta = \{\alpha, m, f_{\eta_t|s_t}\} \in \Theta$, the log-likelihood evaluated at a single observation $D_i = \{s_{it}\}_{t=1}^{T+1}$ is derived in the following equation.

$$\begin{aligned} l(D_i;\theta) &= \sum_{t=1}^{T} \log\left(f_{s_{t+1}|s_t}(s_{i,t+1}|s_{it};\theta)\right) \\ &= \sum_{t=1}^{T} \log\left(\sum_{y_{it}=1}^{J} f_{\eta_t|s_t}(s_{i,t+1} - m(y_{it}, s_{it})|s_{it}) p_t(y_{it}|s_{it};\theta)\right). \end{aligned} \tag{4.1}$$

In Eq. (4.1), $p_t(y_{it}|s_{it};\theta)$ represents the choice probability conditional on state s_{it} given the parameter value θ (including utility parameters, the nonparametric functions m, and $f_{\eta_t|s_t}$ in the state transition rules), which can be derived via the agent's optimization problem. The population criterion function $Q : \Theta \to \mathbb{R}$ is hence defined by

$$Q(\theta) = \mathrm{E}(l(D_i;\theta)). \tag{4.2}$$

A sample counterpart of the objective function in Eq. (4.2) is

$$\hat{Q}_n(\theta) = \frac{1}{n}\sum_{i=1}^{n} l(D_i;\theta). \tag{4.3}$$

In light of a finite sample, instead of searching parameters over an infinite-dimensional parameter space Θ, we maximize the empirical criterion function over a sequence of approximating sieve spaces Θ_k. The sieve maximum likelihood estimator $\hat{\theta}_k$ is defined as

$$\hat{\theta}_k = \arg\sup_{\theta\in\Theta_k} \hat{Q}_n(\theta). \tag{4.4}$$

We discuss the details of constructing the sieve spaces and the asymptotic properties of the proposed estimator in Appendix C.

[11] If we swap the label and denote $y_t = 0$ if the borrower exerts effort to pay off the debt, the utility function can be written as $u^*(s_t, y_t) = \omega s_t - \rho \mathbf{1}\{y_t = 0\}$, and the larger value of the m functions is labeled as $m(0,s)$.

[12] The estimation strategy in this section can be extended to allow the nonparametric estimation of per-period utility functions.

Table 1
Monte Carlo simulation results.

	DGP 1		DGP 2		DGP 3		DGP 4	
	Unobs	Obs	Unobs	Obs	Unobs	Obs	Unobs	Obs
Bias(ω)	0.0049	−0.0009	−0.0058	0.0019	−0.0032	0.0000	−0.0174	−0.0030
Std(ω)	0.0042	0.0031	0.0549	0.0062	0.0348	0.0055	0.0690	0.0060
RMSE(ω)	0.0065	0.0033	0.0551	0.0065	0.0349	0.0055	0.0711	0.0067
Bias(ρ)	−0.0005	0.0055	0.0646	0.0064	0.0033	0.0031	0.0288	−0.0022
Std(ρ)	0.0135	0.0158	0.2528	0.0086	0.0642	0.0098	0.1149	0.0092
RMSE(ρ)	0.0135	0.0167	0.2607	0.0107	0.0642	0.0102	0.1184	0.0094
IBias^2(m_0)	0.0006	0.0000	0.0003	0.0000	0.0004	0.0001	0.0068	0.0049
IMSE(m_0)	0.0008	0.0002	0.0012	0.0001	0.0102	0.0002	0.0236	0.0050
IBias^2(m_1)	0.0000	0.0000	0.0035	0.0000	0.0005	0.0002	0.0086	0.0032
IMSE(m_1)	0.0001	0.0001	0.0373	0.0001	0.0086	0.0003	0.0593	0.0034
IBias^2(f_η)	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
IMSE(f_η)	0.0000	0.0000	0.0004	0.0000	0.0002	0.0000	0.0004	0.0000

Note: In all simulation exercises, we use third-degree polynomials to approximate m_0 and m_1. For the square root of the density function f_η, we use fifth-degree polynomials. The IBias^2 of a function h is calculated as follows. Let $\hat{h}_r$ be the estimate of h from the rth simulated dataset, and $\bar{h}(x) = \frac{1}{R}\sum_{r=1}^{R}\hat{h}_r(x)$ be the point-wise average over R simulations. The integrated squared bias is calculated by numerically integrating the point-wise squared bias $(\bar{h}(x) - h(x))^2$ over the distribution of x. The integrated MSE is computed in a similar way.

Remark 2. It is also possible to estimate the model primitives using a two-step sequential approach. In the first-step, we can estimate the primitives in the state transition rules and the conditional choice probabilities nonparametrically using sieve MLE. The log-likelihood evaluated at D_i given $(m, f_{\eta_t|s_t})$ and the observed state transition rule $f_{s_{t+1}|s_t}$ is:

$$l(D_i; m, f_{\eta_t|s_t}, f_{s_{t+1}|s_t}) = \sum_{t=1}^{T}\log\left(\sum_{y_{it}=1}^{J} f_{\eta_t|s_t}(s_{i,t+1} - m(y_{it}, s_{it})|s_{it})p_t(y_{it}|s_{it}; m, f_{s_{t+1}|s_t})\right), \tag{4.5}$$

where the closed-form solution for the conditional choice probabilities $p_t(y_t|s_{it}; m, f_{s_{t+1}|s_t})$ are provided in Eq. (3.2) (binary case) and Eq. (3.8) (general multinomial case). Once the conditional choice probabilities and the state transition rules are obtained, in the second-step, we can estimate structural utility primitives using exiting sequential estimation approach in Hotz and Miller (1993), Hotz et al. (1994), and Aguirregabiria and Mira (2002).

5. Simulation

In this section, we present Monte Carlo simulation results for our baseline model with one continuous state variable s_t. Let $y_t = 1$ if the agent chooses to exert effort, and 0 otherwise. We assume that the mean utility function is time-invariant and takes the following form:

$$u^*(s_t, y_t) = 1 - \exp(-\omega s_t) - \rho y_t,$$

where $\omega = 0.8$, and $\rho = 0.3$ measures the marginal cost of exerting more effort. The utility shocks $\varepsilon_t(0)$ and $\varepsilon_t(1)$ independently follow the type I extreme value distribution and the discount factor is fixed at 0.95. We consider four data generating processes for the state transition.

– DGP 1: $s_{t+1} = 0.5s_t + 0.5y_t + 0.5s_t y_t + \eta_t$.
– DGP 2: $s_{t+1} = 0.2s_t + 0.05s_t^2 + 0.5y_t + \eta_t$.
– DGP 3: $s_{t+1} = \frac{1}{1+\exp(-(s_t+0.5y_t))} + \eta_t$.
– DGP 4: $s_{t+1} = \exp(-0.2(s_t - y_t)^2) + \eta_t$.

In the first specification, $m_0(s_t) = 0.5s_t$ and $m_1(s_t) = 0.5 + s_t$, both taking a linear form, but the marginal effects of the current state on the future state vary with alternatives. For DGP's 2–4, we assume that the transition rule is nonlinear in the current state s_t. DGP's 2 and 4 also allow that the m functions are non-monotonic in s_t. For all specifications, we assume $\eta_t \sim N(0,1)$ and $T = 10$.

Table 1 presents our simulation results for sample size $N = 5{,}000$ and the number of Monte Carlo replications $R = 500$. We report the Monte Carlo biases, standard deviations, and root mean squared errors for parameters in the flow utility (i.e., ω and ρ). For nonparametric functions m_0, m_1, and f_η, we use polynomial basis functions to approximate them and report their integrated squared biases (IBias^2) and integrated mean squared errors (IMSE). In Table 1, we also present estimation results when choices are observed by the econometricians.

For all data generating processes, our Monte Carlo simulations generally perform well. In specification 2, where we add a quadratic term in the m functions, the bias and variance of the cost parameter is larger compared to other cases. When there is some degree of sieve approximation error for m_0 and m_1 (as shown in specifications 3–4), the biases and root mean squared errors of the utility parameters, as well as the integrated biases and mean squared errors of the nonparametric functions are slightly larger. In Fig. 1, we plot our estimates of m_0 and m_1 (averaged over 500 simulations), and their true values. Our nonparametric estimates of

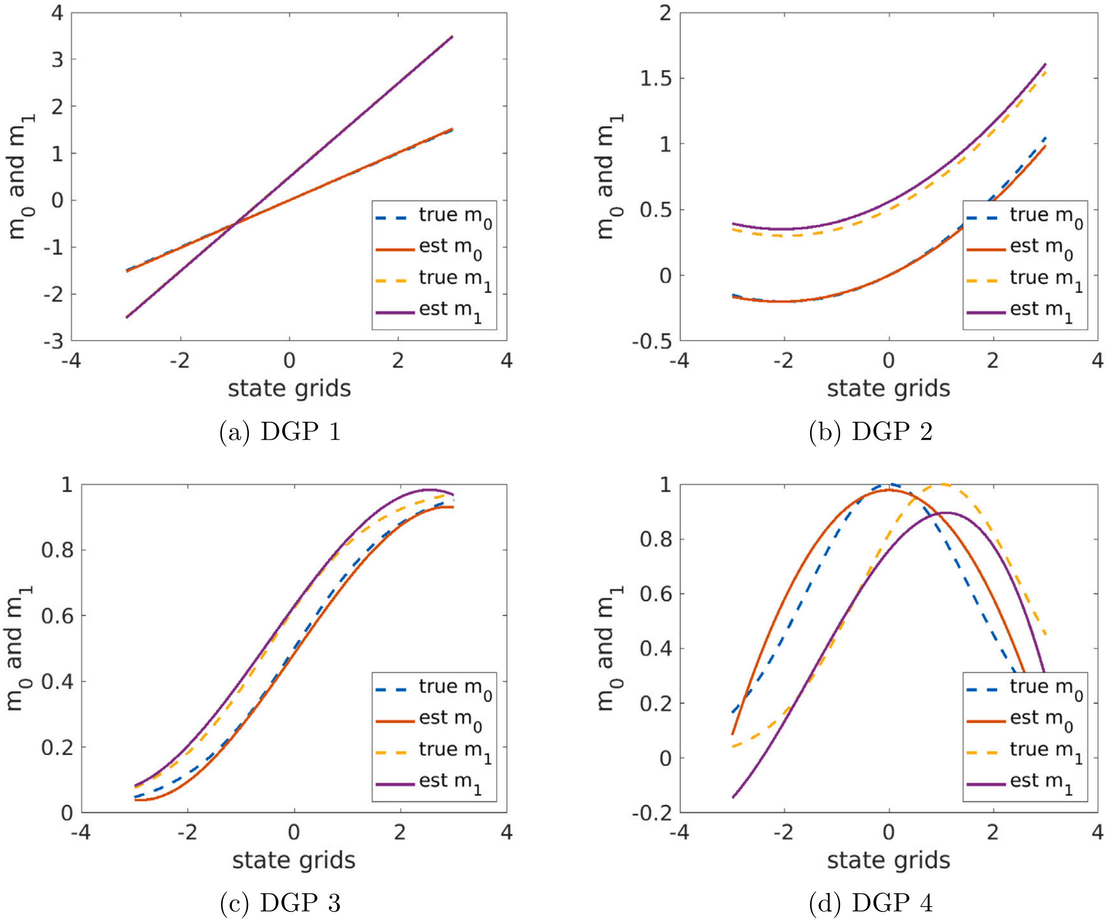

Fig. 1. Plot m_0 and m_1 using estimates and their true values.

m_0 and m_1 are close to their true values, especially when there is no sieve approximation error. In specifications 3–4, our estimates still capture the shape of the m functions reasonably well.

Comparing to the case where choices are observed, we find that estimation results with unobserved choices generally exhibit larger biases and mean squared errors. This result is expected because choices reveal important information about agents' preferences and the state transition rules can be more precisely estimated when choices are available in the data.

6. Extensions

We focus on a single-agent finite-horizon dynamic discrete choice model with one continuous state variable to illustrate our main identification and estimation methods. In this section, we discuss extensions to the baseline identification results. In particular, we consider scenarios in which: (1) serially correlated unobserved heterogeneity is allowed, (2) the model has infinite horizon, (3) choice data are partially observable, and (4) multiple players make simultaneous decisions in a game. We focus on binary choices throughout this section, which contains the types of assumptions that may be used to establish identification along the lines of Section 3.1, without providing formal identification results. Although we do not discuss extensions of the baseline model with multinomial choices, we expect that related conditions along the lines of Section 3.2 could be stated for the models considered below.

6.1. Serially correlated unobserved heterogeneity

Consider a model with a serially correlated unobserved state variable, denoted by $x_t^* \in \mathcal{X}$, entering the per-period utility. We impose the following assumptions on the dynamic process.[13]

[13] To address issues related to initial conditions when serially correlated unobserved heterogeneity is included, we assume that the structural dynamic discrete choice model does not apply to pre-sample periods.

Assumption 6. The dynamic process of $\{s_t, x_t^*, y_t\}$ satisfies the following conditions.

(i) $f_{s_{t+1}, y_{t+1}, x_{t+1}^* | \Omega_{\le t}} = f_{y_{t+1}|s_{t+1}, x_{t+1}^*} f_{s_{t+1}|s_t, x_t^*, y_t} f_{x_{t+1}^*|s_{t+1}, x_t^*}$, where $\Omega_{\le t} = \{s_\tau, x_\tau^*, y_\tau\}_{\tau=1}^t$.
(ii) $f_{s_{t_1+1}|s_{t_1}, x_{t_1}^*, y_{t_1}} = f_{s_{t_2+1}|s_{t_2}, x_{t_2}^*, y_{t_2}}$ for all (t_1, t_2).

In general, Assumption 6 is very similar to Assumption 1 invoked for the baseline model. The main difference is that Assumption 6 imposes additional restrictions on the dynamic process related to the unobserved heterogeneity x_t^*. Specifically, Assumption 6(i) allows that the transition of the observed state s_t depends on the unobserved heterogeneity in the previous period, and the unobserved heterogeneity is serially correlated (the distribution of x_{t+1}^* depends on (s_{t+1}, x_t^*)). Assumption 6 nests a special case where the unobserved heterogeneity is fixed over time, i.e., $x_{t+1}^* = x_t^*$.

Under Assumption 6(i), the transition rule of the observed state variable can be written as

$$f_{s_{t+1}|s_t, x_t^*}(s'|s, x^*) = \sum_y f_{s_{t+1}|s_t, x_t^*, y_t}(s'|s, x^*, y) p_t(y|s, x^*), \tag{6.1}$$

where $p_t(y|s, x^*)$ represents the choice probability of alternative y given $s_t = s$ and $x_t^* = x^*$. Unlike Eq. (3.1), both sides of Eq. (6.1) consist of unobserved terms. On the left-hand side of this equation, the transition probability of the future state given the current state is not directly estimable from the data due to the existence of the unobserved heterogeneity x_t^*. If we can first identify $f_{s_{t+1}|s_t, x_t^*}$, arguments similar to those used in Section 3 yield identification of the latent CCP's $p_t(y|s, x^*)$ and state transition rules $f_{s_{t+1}|s_t, x_t^*, y_t}$.[14]

In order to identify $f_{s_{t+1}|s_t, x_t^*}$, we consider the joint distribution of the observed state variable at four consecutive periods $(s_{t+2}, s_{t+1}, s_t, s_{t-1})$:

$$f_{s_{t+2}, s_{t+1}, s_t, s_{t-1}} = \int_{x_t^*} f_{s_{t+2}|s_{t+1}, x_t^*} f_{s_{t+1}|s_t, x_t^*} f_{x_t^*, s_t, s_{t-1}} \, dF_{x_t^*}. \tag{6.2}$$

The derivation of Eq. (6.2) is provided in Appendix D. In this equation, the unobserved heterogeneity x_t^* is assumed to be a continuous variable. In many empirical studies, it is also common to have discrete latent state variable (e.g., a latent class). An equation similar to (6.2) can be derived for discrete unobserved heterogeneity.

The key insight of Eq. (6.2) is that correlation among observed state variables across different periods is induced by the underlying individual heterogeneity. We can treat $(s_{t+2}, s_{t+1}, s_t, s_{t-1})$ as measurements of x_t^*. Using the spectral decomposition technique developed by Hu and Schennach (2008), $f_{s_{t+2}|s_{t+1}, x_t^*}$, $f_{s_{t+1}|s_t, x_t^*}$, and $f_{x_t^*, s_t, s_{t-1}}$ are nonparametrically identified from the joint distribution of the observed state variable at four periods: $t+2, t+1, t$, and $t-1$. We discuss the sufficient conditions to invoke Theorem 1 in Hu and Schennach (2008) in Appendix D. When x_t^* takes discrete values, the eigenvalue-eigenvector decomposition used in Hu (2008) can be similarly applied.

Given that $f_{s_{t+1}|s_t, x_t^*}$ is identified from the joint distribution of $(s_{t+2}, s_{t+1}, s_t, s_{t-1})$, the density function on the left-hand side of Eq. (6.1) is identified and can be treated as known. Now in order to apply the identification results in Section 3, we need to find another period τ. Suppose $\tau = t+1$. Then, with the state variable at $t+3, t+2, t+1$, and t, we are able to identify $f_{s_{\tau+1}|s_\tau, x_\tau^*}$. The main takeaway here is that the identification of latent choice and state transition probabilities when serially correlated unobserved heterogeneity is present requires the availability of at least five periods of data, i.e., $\{s_t\}_{t=1}^{T+1}$ (with $T \ge 4$).

Remark 3. Identification of models with time-invariant unobserved heterogeneity, such as individual fixed effects, is a special case of our results in Section 6.1. If the unobserved heterogeneity is constant over time (denoted by x^*), we can identify $f_{s_{t+2}|s_{t+1}, x^*}$ and $f_{s_{t+1}|s_t, x^*}$ from Eq. (6.2). This result indicates that state variables at period $t+2, t+1, t, t-1$ are needed to identify the latent choice and state transition probabilities conditional on individual fixed effects at period t and $t+1$.

6.2. Infinite horizon

In a finite-horizon model, the agent's choice probabilities vary over time. As a result, when the latent state transition rule is assumed to be time-homogeneous, variations in the moments of the future state distribution conditional on the same current state can be attributed to changes in choice probabilities across different periods. In other words, in a finite-horizon model, time serves as an exclusion restriction as it only affects the choice probabilities but not the latent state transition process. However, in an infinite-horizon model, agents' choice probabilities across different periods are the same conditional on the same state variable. Consequently, time cannot be used as an excluded variable any more.

In an infinite-horizon model, we need to have an additional variable z_t that satisfies the following exclusion restriction.

[14] When there exists an unobserved heterogeneity x_t^*, we need to modify the state transition process in Assumption 2 to $s_{t+1} = m(y_t, s_t, x_t^*) + \eta_t$. For any fixed x_t^*, similar conditions for m functions and η distribution can be stated.

Assumption 7 (*Exclusion Restriction*). z_t enters agents' flow utility, i.e., $u(s_t, z_t, y_t, \varepsilon_t)$, but the transition rule of s_t does not depend on z_t.

Assumption 7 ensures that the agent's choice probabilities vary with the values of z_t. The condition that the transition rule of s_t does not depend on z_t is an analogy to the time-invariance assumption in the baseline model. To see this, for two distinct values of z_t, $\bar{z}$ and $\hat{z}$, we obtain the following two equations under Assumption 7.

$$f_{s_{t+1}|s_t,z_t}(s'|s,\bar{z}) = \sum_y f_{s_{t+1}|s_t,y_t}(s'|s,y)p_t(y|s,\bar{z}), \quad f_{s_{t+1}|s_t,z_t}(s'|s,\hat{z}) = \sum_y f_{s_{t+1}|s_t,y_t}(s'|s,y)p_t(y|s,\hat{z}). \tag{6.3}$$

From Eq. (6.3), we can see that the variations in the moments of $f_{s_{t+1}|s_t,z_t}$ given different values of z_t are due to the differences in the choice probabilities. Identification arguments similar to those used in Section 3 can be stated; hence, the details are omitted.

Remark 4. In an infinite horizon setting, solving the value function for a continuous state variable is non-trivial. Arcidiacono et al. (2013) propose a sieve value function iteration approach to approximate the integrated value function of high-dimensional dynamic models with a nonparametric sieve function. For any sieve function, we can evaluate the Bellman equation and compute the distance between the approximation and its contraction. The parameters of the sieve approximation function can be estimated by minimizing the distance in the Bellman operator for only a subset of the states in the state space. The sieve value function iteration method can be embedded within our estimation routine in Section 4.

6.3. Partial observability of choices

Our framework is also applicable to cases where there are multiple dimensions of the choices made by the agents, but not all are observed in the data. For instance, researchers may know whether a consumer searches for a product or an apartment, but it is difficult to get information on the intensity of the search effort. It might be relatively easy to collect data on whether an employee goes to work on time, but it is rather difficult to observe the degree to which one works diligently.

To take partial observability of choices into account, we denote the vector of choices made by the agent at period t by $y_t = (y_{1t}, y_{2t})$, where y_{1t} represents the vector of choices observed by the econometrician, and y_{2t} represents the one not observed. Under Assumption 1(i), the observed state transition process can be factorized in the following equation:

$$f_{s_{t+1}|s_t,y_{1t}}(s'|s,y_{1t}) = \sum_{y_{2t}} f_{s_{t+1}|s_t,y_t}(s'|s,y_t)p_t(y_{2t}|s,y_{1t}). \tag{6.4}$$

In Eq. (6.4), $f_{s_{t+1}|s_t,y_t}$ denotes the conditional state transition rules and $p_t(y_{2t}|s,y_{1t})$ denotes the probabilities of the unobserved choice conditional on the state and other dimensions of the choices observable to researchers.

Eq. (6.4) is a direct extension of Eq. (3.1) in the baseline model. The only difference is that now the state transition probabilities are also conditional on the observable part of the choices. Imposing similar restrictions on the latent state transition rules helps to identify the unknown primitives on the right-hand side of Eq. (6.4). The choice probabilities for y_t is therefore identified:

$$p_t(y_t|s_t) = p_t(y_{2t}|s_t,y_{1t}) \cdot p_t(y_{1t}|s_t),$$

where $p_t(y_{1t}|s_t)$ can be directly estimated from the data since both y_{1t} and s_t are observables. Following our identification strategies given partial observability of choices, one may use the observed state transition rules (i.e., $f_{s_{t+1}|s_t,y_{1t}}$) to test if other dimensions of the choices (e.g., the "intensity" margin of effort choices) are relevant for the empirical application before going all the way to the full structural estimation. Intuitively, if the moments of s_{t+1} conditional on s_t and y_{1t} do not vary across periods (in a finite-horizon model), it suggests that unobserved dimensions of the choices might not play an important role in the agent's problem.

Our method can also be applied to another type of partially observed choices. Consider a choice variable $y_t \in \mathcal{Y}_1 \cup \mathcal{Y}_2$. Suppose that y_t is observed if $y_t \in \mathcal{Y}_1$, but it is not observed if $y_t \in \mathcal{Y}_2$. For example, $\mathcal{Y}_1 = \{\text{not purchase car}\}$ and $\mathcal{Y}_2 = \{\text{pay cash, finance}\}$. We can derive the following state transition probabilities conditioning on $y_t \in \mathcal{Y}_2$:

$$f_{s_{t+1}|s_t,y_t\in\mathcal{Y}_2}(s'|s) = \sum_{y\in\mathcal{Y}_2} f_{s_{t+1}|s_t,y_t}(s'|s,y)p_t(y|s). \tag{6.5}$$

Based on this equation, the state transition and choice probabilities are identified for $y_t \in \mathcal{Y}_2$ following similar arguments in Section 3.

6.4. Dynamic discrete games

In the baseline model and extensions discussed in Sections 6.1–6.3, we focus on single-agent dynamic discrete choice models. Our identification strategy can be extended to dynamic discrete games. We first describe a standard modeling framework of dynamic discrete games of incomplete information and then provide identification results for conditional choice probabilities and state transition rules when players' choices are unobserved by econometricians.

Consider a game with I players, where $i = 1, 2, \dots, I$ is the index of each individual. Players choose an action from the choice set $\mathcal{Y}$ simultaneously at each period $t = 1, 2, \dots, \infty$. We use y_{it} to represent player i's action at t, so the action profile is denoted by $\boldsymbol{y}_t = (y_{1t}, y_{2t}, \dots, y_{It}) \in \mathcal{Y}^I$. We use $s_{it} \in \mathcal{S}_i$ to denote the player's state variable that is publicly observed and $\varepsilon_{it} \in \mathcal{E}_i$ to denote the utility shock that is privately observed by player i (not by i's rivals or econometricians). Assume that ε_{it}'s are independently distributed over time and across players. Let $\boldsymbol{s}_t = (s_{1t}, s_{2t}, \dots, s_{It}) \in \times_{i=1}^{I} \mathcal{S}_i$ and $\boldsymbol{\varepsilon}_t = (\varepsilon_{1t}, \varepsilon_{2t}, \dots, \varepsilon_{It}) \in \times_{i=1}^{I} \mathcal{E}_i$ be the vector of observed states and private utility shocks at t, respectively.

Unlike the single-agent case, a player's utility now depends on the action profile and state variables of all players and her own private information ε_{it}. We use $u(\boldsymbol{s}_t, \varepsilon_{it}, \boldsymbol{y}_t)$ to represent the player's per period flow utility. At each period t, all players choose their actions simultaneously to maximize their own expected sum of the discounted utility, where the expectation is taken over other players' current and future actions, the future observed states, and i's private shocks in the future. We invoke the following assumption on the state transition process.

Assumption 8. The dynamic process of $\{\boldsymbol{s}_t, \boldsymbol{y}_t\}$ satisfies

(i) $f_{\boldsymbol{s}_{t+1}, \boldsymbol{y}_{t+1} | \boldsymbol{\Omega}_{\le t}} = f_{\boldsymbol{y}_{t+1}|\boldsymbol{s}_{t+1}} f_{\boldsymbol{s}_{t+1}|\boldsymbol{s}_t, \boldsymbol{y}_t}$, where $\boldsymbol{\Omega}_{\le t} = \{\boldsymbol{s}_\tau, \boldsymbol{y}_\tau\}_{\tau=1}^{t}$.
(ii) $f_{\boldsymbol{s}_{t+1}|\boldsymbol{s}_t, \boldsymbol{y}_t}(\boldsymbol{s}_{t+1}|\boldsymbol{s}_t, \boldsymbol{y}_t) = \prod_{i=1}^{I} f_{s_{i,t+1}|s_{it}, y_{it}}(s_{i,t+1}|s_{it}, y_{it})$.

Assumption 8(i) imposes a first-order Markov process for state transitions similarly as in Assumption 1(i). Assumption 8(ii) implies that the transition process of the observed state variable only depends on player i's own action and state in the last period, not on other players' actions or states. For example, in dynamic oligopoly competition, the transition of firm's capacity level only depends on its own investment decisions and capacities in the past (see Ryan (2012)).[15]

In the game described above, we consider pure strategy Markov Perfect Equilibrium (MPE) as our equilibrium concept, in which case players' actions only depend on the value of current states and utility shocks. In addition, we focus on stationary Markov strategies, so subscript t is dropped in the following definitions. We define a Markov strategy for player i as $a_i(\boldsymbol{s}_t, \varepsilon_{it})$ and i's belief that $\boldsymbol{y}_t$ is chosen at state $\boldsymbol{s}_t$ as $\sigma_i(\boldsymbol{y}_t|\boldsymbol{s}_t)$. Under Assumption 8, the value function for player i given belief σ_i is

$$V_i(\boldsymbol{s}_t, \varepsilon_{it}; \sigma_i) = \max_{y \in \mathcal{Y}} \sum_{\boldsymbol{y}_{-i} \in \mathcal{Y}^{I-1}} \sigma_i(\boldsymbol{y}_{-i}|\boldsymbol{s}_t) \Big[u(\boldsymbol{s}_t, \varepsilon_{it}, (y, \boldsymbol{y}_{-i})) + \beta \mathrm{E}[V_i(\boldsymbol{s}_{t+1}, \varepsilon_{i,t+1}; \sigma_i)|\boldsymbol{s}_t, (y, \boldsymbol{y}_{-i})] \Big], \tag{6.6}$$

where $\boldsymbol{y}_{-i}$ represents the profile of actions for all other players except i. The optimal strategy of player i given state variable $\boldsymbol{s}_t$ and private utility shock ε_{it} under belief σ_i is therefore

$$a_i(\boldsymbol{s}_t, \varepsilon_{it}; \sigma_i) = \arg\max_{y \in \mathcal{Y}} V_i(\boldsymbol{s}_t, \varepsilon_{it}; \sigma_i). \tag{6.7}$$

After integrating out the player's private information, we can define i's choice probabilities given state variable $\boldsymbol{s}_t$ and belief σ_i as

$$p_i(y_{it}|\boldsymbol{s}_t; \sigma_i) = \int \mathbf{1}\{y_{it} = a_i(\boldsymbol{s}_t, \varepsilon_{it}; \sigma_i)\} dG(\varepsilon_{it}), \tag{6.8}$$

where G is the cumulative distribution function of ε_{it}. In an MPE, players' beliefs are consistent with their strategies, leading to a fixed point of a mapping in the space of conditional choice probabilities. We define player i's equilibrium choice probabilities conditional on $\boldsymbol{s}_t$ as $p_i^*(y_{it}|\boldsymbol{s}_t)$, which cannot be directly estimated from data when agent's choices are unobserved.[16]

Under Assumption 8, we achieve the following equation for i's state transition process:

$$f_{s_{i,t+1}|\boldsymbol{s}_t}(s_i'|\boldsymbol{s}) = \sum_{y_{it} \in \mathcal{Y}} f_{s_{i,t+1}|s_{it}, y_{it}}(s_i'|s_i, y_{it}) p_i^*(y_{it}|\boldsymbol{s}). \tag{6.9}$$

In a game setting, all players interact with each other, so i's choices naturally depend on all other players' state variables as highlighted in Eq. (6.9). $\boldsymbol{s}_{-i}$ therefore can be used as an excluded variable. For two values of $\boldsymbol{s}_{-i}$, $\bar{\boldsymbol{s}}_{-i}$ and $\hat{\boldsymbol{s}}_{-i}$, we obtain the following two equations

$$\begin{aligned} f_{s_{i,t+1}|\boldsymbol{s}_t}(s_i'|s_i, \bar{\boldsymbol{s}}_{-i}) &= \sum_{y_{it} \in \mathcal{Y}} f_{s_{i,t+1}|s_{it}, y_{it}}(s_i'|s_i, y_{it}) p_i^*(y_{it}|s_i, \bar{\boldsymbol{s}}_{-i}), \\ f_{s_{i,t+1}|\boldsymbol{s}_t}(s_i'|s_i, \hat{\boldsymbol{s}}_{-i}) &= \sum_{y_{it} \in \mathcal{Y}} f_{s_{i,t+1}|s_{it}, y_{it}}(s_i'|s_i, y_{it}) p_i^*(y_{it}|s_i, \hat{\boldsymbol{s}}_{-i}). \end{aligned} \tag{6.10}$$

Identification arguments similar to those used in Section 3 yield the identification of the state transition probabilities and equilibrium choice probabilities for players $i = 1, 2, \dots, I$.

[15] Our framework is suitable for standard game setup in IO, where there are two types of state variables (see discussions in Kalouptsidi et al. (2021)). Let z_{it} denote the state variables that are specific to each player (e.g., past actions, capacity levels of the firm), and w_t denote the macro variables (e.g., aggregate demand) that are relevant for all players. $s_{it} = (z_{it}, w_t)$. The conditional independence restriction in Assumption 8(ii) can be modified to $f_{\boldsymbol{s}_{t+1}|\boldsymbol{s}_t, \boldsymbol{y}_t}(\boldsymbol{s}_{t+1}|\boldsymbol{s}_t, \boldsymbol{y}_t) = \prod_{i=1}^{I} f_{z_{i,t+1}|z_{it}, y_{it}, w_t}(z_{i,t+1}|z_{it}, y_{it}, w_t) f_{w_{t+1}|w_t}(w_{t+1}|w_t)$. $f_{w_{t+1}|w_t}$ captures the exogenous transition of the macro variables, which can be directly estimated from data.

[16] Under certain regularity conditions, at least one Markov perfect equilibrium exists for dynamic discrete games of incomplete information (Doraszelski and Satterthwaite, 2010), but multiplicity of equilibria may be possible. In this paper, our goal is to analyze situations when players' actions are unobserved by econometricians, so we focus on the case where the same equilibrium is played in the data.

Table 2
Estimation results for bus engine replacement model.

	Observed choices		Unobserved choices	
Parameters	Estimates	Std. Err.	Estimates	Std. Err.
ρ_0	9.2514	(0.8590)	7.9864	(0.5652)
ρ_1	−0.8874	(0.3386)	−0.7381	(0.3252)
ω	0.0022	(0.0004)	0.0017	(0.0002)
μ_η	1.1389	(0.0069)	1.1406	(0.0068)
σ_η	0.4825	(0.0025)	0.4789	(0.0025)

7. Empirical illustration

We apply our identification and estimation methods to two empirical settings. We first use the well-known dataset on Harold Zucher's engine replacement decisions studied in Rust (1987). We estimate Rust's model assuming that the agent's replacement decisions are not observed by econometricians. Our second example is related to politicians' hidden effort-exerting decisions. Using a dataset on gubernatorial elections, we estimate governors' preferences for policy outcome variables and their costs of exerting effort.

7.1. Rust's engine replacement model

In Rust (1987), the manager makes an infinite-horizon dynamic choice for each bus engine by trading off between an immediate lump sum cost of replacing it and higher maintenance costs for keeping it at each period. Let $s_t \in [0, \infty)$ denote the accumulated mileage of the bus observed at the beginning of period t and ϵ_t denote an unobserved idiosyncratic shock which follows a Type I extreme value distribution. $y_t = 1$ represents the case where the agent replaces the engine, and $y_t = 0$ otherwise. For a bus with observable characteristics z_t, the agent's per-period utility (net of the error) is defined as

$$u^*(s_t, y_t) = \begin{cases} -(\rho_0 + \rho_1 z_t) & \text{if } y_t = 1 \\ -\omega s_t & \text{if } y_t = 0, \end{cases} \tag{7.1}$$

where $\rho_0 + \rho_1 z_t$ represents the engine replacement cost and ω is the per-mileage maintenance cost. Given the current mileage and the agent's choice, the future mileage is updated based on the following equation:

$$s_{t+1} = \begin{cases} \eta_t & \text{if } y_t = 1 \\ s_t + \eta_t & \text{if } y_t = 0, \end{cases} \tag{7.2}$$

where $\eta_t \in [0, \infty)$ can be interpreted as the incremental mileage realized within period t. If the agent replaces the engine, the accumulated mileage is reset to 0, so the future mileage $s_{t+1} = \eta_t$. Instead of discretizing the mileage as in Rust (1987), we assume that s_t is a continuous variable and the incremental mileage η_t is randomly drawn from a distribution F_η. The state transition in Eq. (7.2) satisfies Assumption 2 with $m(1, s_t) = 0$ and $m(0, s_t) = s_t$. In this infinite-horizon model, z_t serves as an excluded variable that only affects the agent's utility, but not the state transition rule. We use bus model (i.e., newer buses vs. older 1975 GMC buses, see Rust (1987) Table III for more details), which is associated with high or low replacement cost, as our excluded variable. z_t in this case is a binary variable that does not vary with time. But more generally, z_t could be non-binary and time-varying in other applications.

We estimate this model's parameters $(\rho_0, \rho_1, \omega, F_\eta)$ under two scenarios: replacement decisions are observed or unobserved by the econometricians. Due to small sample size, instead of estimating F_η nonparametrically, we assume that η_t follows a log-normal distribution with parameters (μ_η, σ_η). From the estimation results in Table 2, we can see that our approach which does not require the observation of choices generates qualitatively similar estimates of structural parameters. We plot the probabilities of replacing the engine (conditional on mileage) computed using the two sets of estimates in Fig. 2. Since our estimates of replacement costs are slightly smaller when choices are not available, we observe a slightly higher replacement probability when accumulated mileage is low.

7.2. Governors' effort exerting decisions

Our second illustrating example is motivated by the political economy literature studying the impact of term limits on politician's behavior (e.g., Besley and Case (1995), Alt et al. (2011), Sieg and Yoon (2017)). As the politician approaches her term limit, the incentives of exerting effort may become lower. It is challenging to directly test this hypothesis because politician's effort levels are rarely available in a dataset.

Using identification and estimation strategies developed in this paper, we estimate a finite-horizon dynamic model of politician's effort exerting decisions using a dataset containing all gubernatorial elections between 1950 and 2000 in the United States.[17] The

[17] Data source: https://dataverse.harvard.edu/dataset.xhtml?persistentId=hdl:1902.1/14838. For more discussions on this dataset, see Alt et al. (2011).

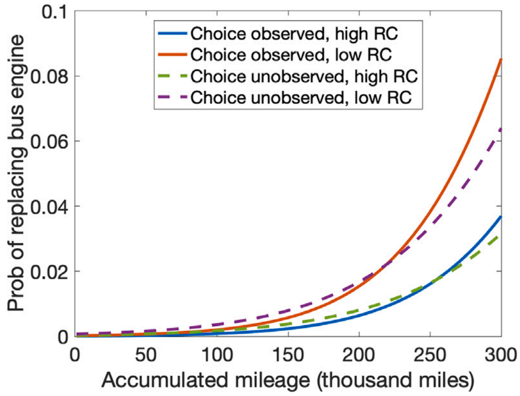

Fig. 2. Comparing CCP when choices are observed or unobserved.

Table 3
Estimation results for governors' effort-exerting decisions.

Panel (A): Estimates

Parameters	Estimates	Std. Err.
m_0: intercept	0.0014	(0.0193)
m_0: slope	1.0012	(0.0037)
m_1: intercept	0.4574	(0.0465)
m_1: slope	0.9438	(0.0102)
m_2: intercept	0.9762	(0.0959)
m_2: slope	0.8785	(0.0176)
ω	1.4901	(1.1234)
ρ_1	1.0005	(0.3674)
ρ_2	1.8598	(0.5516)
σ_η	0.0468	(0.0020)

Panel (B): Predicted choice probabilities

	Year 1	Year 2	Year 3	Year 4
$Pr(y_t = 0)$	0.5616	0.5941	0.6257	0.6564
$Pr(y_t = 1)$	0.2744	0.2649	0.2538	0.2414
$Pr(y_t = 2)$	0.1640	0.1410	0.1205	0.1022

model we estimate is mainly for illustration purposes and abstracts away from many important features of gubernatorial elections (such as political power, macro environment, bipartisan/unilateral approach, reelection incentives). In the dataset, we observe the characteristics of the elected governors and a few policy outcome variables (e.g., spending, and unemployment rate) at the state level; but the politicians' choices are not observed. There are a total of 142 governors in the sample. We focus on those who were in their last four-year term—these governors were essentially "lame ducks" and were not eligible for reelections.

We use log of per capita spending (reported in constant 1982 dollars) as the continuous state variable. Let $t \in \{0, 1, \dots, 4\}$ be the index of years within a term. $t = 1$ refers to the year when a governor was elected (or reelected); $t = 0$ refers to the year before the term began. We impose Assumption 2 on the transition process, that is $s_{t+1} = m(s_t, y_t) + \eta_t$, where η_t is independent with the choice $y_t \in \mathcal{Y}$. To determine the number of alternatives in $\mathcal{Y}$, we use the result in Corollary 1 and compute the rank of the $\Phi(\overline{\mathcal{T}}, \overline{\mathcal{R}})$ matrix. Given the state transition process observed in the data, $rank(\Phi(\overline{\mathcal{T}}, \overline{\mathcal{R}})) = 3$.[18] Therefore without loss of generality, we assume that the agents choose from $\mathcal{Y} = \{0, 1, 2\}$. The per-period mean utility of the governor is assumed to take the following linear form:

$$u_t^*(s_t, y_t) = \omega s_t - \rho_1 \mathbf{1}\{y_t = 1\} - \rho_2 \mathbf{1}\{y_t = 2\}, \tag{7.3}$$

where ω is the preference parameter for spending, ρ_1 and ρ_2 are the costs associated with choices 1 and 2 (the cost of choosing alternative 0 is normalized to 0). In addition to the deterministic part, the governor receives a random utility shock ϵ_t drawn from a Type I extreme value distribution.

We estimate this model's parameters $(\omega, \rho_1, \rho_2, m_0, m_1, m_2, F_\eta)$. Due to small sample size, we parameterize (m_0, m_1, m_2, F_η). Specifically, m functions are assumed to take a linear form, and we assume that η_t is drawn from a normal distribution $N(0, \sigma_\eta^2)$. The estimation results for this model are presented in Table 3 Panel (A). In the estimation, we impose the ordering assumption that $m_2(s) > m_1(s) > m_0(s)$ without loss of generality. Our estimates of utility parameters suggest that agents may prefer a higher level of s_t, and it is more costly to choose $y_t = 2$, which generates a higher average level of future states. Based on these estimates, we could potentially interpret $y_t = 0, 1$, and 2 as "exerting no effort", "exerting low effort", and "exerting high effort" to increase spending, respectively. Panel (B) of Table 3 shows the choice probabilities for each alternative in years 1–4. From this table we can see that as the governor approaches her term limit, the probability of exerting lower level of effort increases.

[18] In this application, we include all available time periods in $\overline{\mathcal{T}}$ and compute the rank of $\Phi(\overline{\mathcal{T}}, \overline{\mathcal{R}})$ when different sets of real numbers are included in $\overline{\mathcal{R}}$. We do not find that the rank of $\Phi(\overline{\mathcal{T}}, \overline{\mathcal{R}})$ is sensitive to the choices of $\overline{\mathcal{R}}$.

8. Conclusion

In this paper, we provide new identification and estimation methods for dynamic discrete choice models when agents' choices are unobserved by econometricians. We leverage variations in the observed state transition process across different periods. In finite-horizon models, time serves as an exclusion restriction because it only affects the choice probabilities but not the state transition rules. Our identification strategy from the baseline model can extend to infinite-horizon models, models with serially correlated unobserved heterogeneity, cases in which choices are partially unavailable, and dynamic discrete games. We propose a sieve maximum likelihood estimator for primitives in agents' utility functions and state transition rules. Monte Carlo simulations under various specifications demonstrate the good performance of the proposed approach. The identification and estimation results developed in this paper contribute to the body of our knowledge. Under mild assumptions on the state transition process, our methods can be applied to various empirical contexts in labor economics, industrial organization, political economy, and other related fields, when agents' choice data are not readily available.

Appendix A. Proofs

A.1. Proof of Theorem 1

We rewrite the first-order conditional mean of the state variable at period $t+1$ by replacing s_{t+1} with $m(y_t, s_t) + \eta_t$. Specifically,

$$\mu_{t+1} = \sum_{y\in\{0,1\}} p_t(y|s)\mathrm{E}[m(y,s)+\eta_t|s,y] = p_t m_1 + (1-p_t)m_0, \tag{A.1}$$

where the second equality holds because under Assumption 2, η_t and y_t are independent conditional on the state and $\mathrm{E}(\eta_t|s)=0$. In Eq. (A.1), μ_{t+1} is a weighted average of m_1 and m_0 with the choice probabilities $(p_t, 1-p_t)$ serving as the mixing weights. Following similar arguments, we rewrite the second- and the third-order conditional moments of the state variable as follows.

$$\begin{aligned} v_{t+1} &= \sum_{y\in\{0,1\}} p_t(y|s)\mathrm{E}\left[(m(y,s)+\eta_t-\mu_{t+1})^2|s,y\right] \\ &= \sum_{y\in\{0,1\}} p_t(y|s)\left[(m(y,s)-\mu_{t+1})^2 + 2(m(y,s)-\mu_{t+1})\mathrm{E}[\eta_t|s] + \mathrm{E}[\eta_t^2|s]\right] \\ &= p_t(m_1-\mu_{t+1})^2 + (1-p_t)(m_0-\mu_{t+1})^2 + \mathrm{E}\left[\eta_t^2|s\right], \end{aligned} \tag{A.2}$$

$$\begin{aligned} \xi_{t+1} &= \sum_{y\in\{0,1\}} p_t(y|s)\mathrm{E}\left[(m(y,s)+\eta_t-\mu_{t+1})^3|s,y\right] \\ &= \sum_{y\in\{0,1\}} p_t(y|s)\left[(m(y,s)-\mu_{t+1})^3 + \mathrm{E}[\eta_t^3|s] + 3(m(y,s)-\mu_{t+1})^2\mathrm{E}[\eta_t|s] + 3(m(y,s)-\mu_{t+1})\mathrm{E}[\eta_t^2|s]\right] \\ &= p_t(m_1-\mu_{t+1})^3 + (1-p_t)(m_0-\mu_{t+1})^3 + E\left[\eta_t^3|s\right]. \end{aligned} \tag{A.3}$$

In Eqs. (A.2) and (A.3), $\mathrm{E}\left[\eta_t^2|s\right]$ and $\mathrm{E}\left[\eta_t^3|s\right]$ are the second- and third-order conditional moments of the error η_t, respectively, but the values of these terms are not known.

To identify m_1, m_0, and the choice probabilities, we consider two periods t_1 and t_2 ($t_1 \neq t_2$) along the dynamic process. Eq. (A.1) identifies the choice probability for any given m_0 and m_1 as long as $m_0 \neq m_1$. Specifically, the choice probabilities at period t_1 and t_2 are

$$p_{t_1} = \frac{\mu_{t_1+1}-m_0}{m_1-m_0}, \quad p_{t_2} = \frac{\mu_{t_2+1}-m_0}{m_1-m_0}. \tag{A.4}$$

Under Assumption 1(ii) and Assumption 2, the conditional distribution of η_t is stationary. This implies that the higher order moments of the error term are time-invariant conditional on the same state s, i.e., $\mathrm{E}\left[\eta_{t_1}^2|s\right] = \mathrm{E}\left[\eta_{t_2}^2|s\right]$, and $\mathrm{E}\left[\eta_{t_1}^3|s\right] = \mathrm{E}\left[\eta_{t_2}^3|s\right]$. Taking the difference of Eqs. (A.2) and (A.3) across the two periods t_1 and t_2, we eliminate the unknown moments of η_t and achieve the following two equations.

$$\begin{aligned} v_{t_1+1} - v_{t_2+1} &= p_{t_1}(m_1-\mu_{t_1+1})^2 + (1-p_{t_1})(m_0-\mu_{t_1+1})^2 - p_{t_2}(m_1-\mu_{t_2+1})^2 - (1-p_{t_2})(m_0-\mu_{t_2+1})^2 \\ &= (p_{t_1}-p_{t_2})(m_1+m_0)(m_1-m_0) - (\mu_{t_1+1}^2 - \mu_{t_2+1}^2), \end{aligned} \tag{A.5}$$

$$\xi_{t_1+1} - \xi_{t_2+1} = p_{t_1}(m_1-\mu_{t_1+1})^3 + (1-p_{t_1})(m_0-\mu_{t_1+1})^3 - p_{t_2}(m_1-\mu_{t_2+1})^3 - (1-p_{t_2})(m_0-\mu_{t_2+1})^3. \tag{A.6}$$

We further plug the expressions of p_{t_1} and p_{t_2} in Eq. (A.4) into Eqs. (A.5) and (A.6), which leads to a system of equations for the unknown primitives m_1 and m_0:

$$v_{t_1+1} - v_{t_2+1} = (\mu_{t_1+1} - \mu_{t_2+1})\Delta_1 - (\mu_{t_1+1}^2 - \mu_{t_2+1}^2), \tag{A.7}$$

$$\xi_{t_1+1} - \xi_{t_2+1} = (\mu_{t_1+1}\Delta_1 - \Delta_2 - \mu_{t_1+1}^2)(\Delta_1 - 2\mu_{t_1+1}) - (\mu_{t_2+1}\Delta_1 - \Delta_2 - \mu_{t_2+1}^2)(\Delta_1 - 2\mu_{t_2+1}), \tag{A.8}$$

where $\Delta_1 = m_1 + m_0$ and $\Delta_2 = m_1 m_0$. We obtain analytical solutions for Δ_1 and Δ_2 from Eqs. (A.7)–(A.8), provided that $\mu_{t_1+1} \neq \mu_{t_2+1}$.

$$\Delta_1 = \frac{\nu_{t_1+1} - \nu_{t_2+1} + (\mu_{t_1+1}^2 - \mu_{t_2+1}^2)}{\mu_{t_1+1} - \mu_{t_2+1}}, \tag{A.9}$$

$$\Delta_2 = \frac{\xi_{t_1+1} - \xi_{t_2+1} - (\mu_{t_1+1}(\Delta_1 - \mu_{t_1+1})(\Delta_1 - 2\mu_{t_1+1}) - \mu_{t_2+1}(\Delta_1 - \mu_{t_2+1})(\Delta_1 - 2\mu_{t_2+1}))}{2(\mu_{t_1+1} - \mu_{t_2+1})}. \tag{A.10}$$

We now focus on the case where $\mu_{t_1+1} \neq \mu_{t_2+1}$. With Δ_1 and Δ_2 identified using the moments of the observed state transition process as shown in Eqs. (A.9)–(A.10), m_0 and m_1 are the two distinctive solutions to the following quadratic equation:

$$m^2 - \Delta_1 m + \Delta_2 = 0. \tag{A.11}$$

Without loss of generality, we label the larger root of Eq. (A.11) by m_1 and the smaller one by m_0, i.e.,

$$m_1 = \frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_2}}{2}, \quad m_0 = \frac{\Delta_1 - \sqrt{\Delta_1^2 - 4\Delta_2}}{2}.$$

Once m_1 and m_0 are identified (and they are not equal), the conditional choice probabilities p_{t_1} and p_{t_2} are identified from Eq. (A.4). Given the additive structure of the state transition process and the independence of η_t and y_t conditional s_t, the observed state transition probability of $s_{t+1} = s'$ given $s_t = s$ can be written as a mixture of the conditional density of η_t evaluated at $s' - m_1$ and $s' - m_0$ for $t \in \{t_1, t_2\}$:

$$\begin{aligned} f_{s_{t_1+1}|s_{t_1}}(s'|s) &= p_{t_1} f_{\eta_{t_1}|s_{t_1}}(s' - m_1|s) + (1 - p_{t_1}) f_{\eta_{t_1}|s_{t_1}}(s' - m_0|s), \\ f_{s_{t_2+1}|s_{t_2}}(s'|s) &= p_{t_2} f_{\eta_{t_2}|s_{t_2}}(s' - m_1|s) + (1 - p_{t_2}) f_{\eta_{t_2}|s_{t_2}}(s' - m_0|s). \end{aligned} \tag{A.12}$$

Given the stationarity of η_t conditional on s_t,

$$f_{\eta_{t_1}|s_{t_1}}(s' - m_1|s) = f_{\eta_{t_2}|s_{t_2}}(s' - m_1|s), \quad f_{\eta_{t_1}|s_{t_1}}(s' - m_0|s) = f_{\eta_{t_2}|s_{t_2}}(s' - m_0|s).$$

Eq. (A.12) identifies the conditional density function of η_t at $s' - m_1$ and $s' - m_0$ when $p_{t_1} \neq p_{t_2}$ for all $t \in \{1, 2, \dots, T\}$ and $s' \in \mathcal{S}$, i.e.,

$$\begin{aligned} f_{\eta_t|s_t}(s' - m_1|s) &= \frac{f_{s_{t_1+1}|s_{t_1}}(s'|s)(1 - p_{t_2}) - f_{s_{t_2+1}|s_{t_2}}(s'|s)(1 - p_{t_1})}{p_{t_1} - p_{t_2}}, \\ f_{\eta_t|s_t}(s' - m_0|s) &= \frac{f_{s_{t_2+1}|s_{t_2}}(s'|s)p_{t_1} - f_{s_{t_1+1}|s_{t_1}}(s'|s)p_{t_2}}{p_{t_1} - p_{t_2}}. \end{aligned} \tag{A.13}$$

So far, we have proved that for a state s at which $\mu_{t_1+1} \neq \mu_{t_2+1}$ and the order of the m functions is known, $m(0, s)$, $m(1, s)$, choice probabilities p_{t_1} and p_{t_2}, and the conditional density function of the error term η_t are identified. Assumption 3(i) ensures that the set of states at which $\mu_{t_1+1} = \mu_{t_2+1}$ has a zero measure. Therefore, by the continuity of m functions, we also identify $m(1, s)$ and $m(0)$ for $s \in \tilde{\mathcal{S}}$. Assumption 3(iii) guarantees that there exists a state s_0 such that we can distinguish between $m(0, s_0)$, $m(1, s_0)$. Starting from s_0, by the continuity of the first-order derivatives of the m functions imposed in Assumption 2 and the condition that $m(1, \cdot)$ and $m(0, \cdot)$ intersect with different derivatives imposed in Assumption 3(ii), we match $m(1, \cdot)$ and $m(0, \cdot)$ across all values of $s \in \mathcal{S}$.

Eq. (A.13) identifies $f_{\eta_t|s_t}(\cdot|s)$ for any $s \in \mathcal{S}$ such that $p_{t_1}(\cdot|s) \neq p_{t_2}(\cdot|s)$. Given that $\{s \in \mathcal{S} : p_{t_1}(\cdot|s) = p_{t_2}(\cdot|s)\}$ has zero probability and the continuity of the density function, $f_{\eta_t|s_t}(\cdot|s)$ is identified for all $s \in \mathcal{S}$. The identification of m functions and the conditional density function of the error term η_t implies the identification of the state transition rules, i.e., $f_{s_{t+1}|s_t, y_t}$. Choice probabilities for periods t_1 and t_2 are identified using Eq. (A.4) for all $s \in \mathcal{S}/\bar{\mathcal{S}}$ with $Pr(\bar{\mathcal{S}}) = 0$. This completes the proof of Theorem 1. □

A.2. Proof of Theorem 2

Derivation of Eq. (3.6). Constructing the characteristic function of $s_{t+1}|s_t = s$, we have

$$\begin{aligned} \phi_{s_{t+1}|s_t=s}(r) &= \int \exp(irs') f_{s_{t+1}|s_t}(s'|s) ds' \\ &= \int \exp(irs') \sum_{y=1}^{J} f_{\eta_t|s_t=s}(s' - m(y, s)|s) p_t(y|s) ds' \\ &= \sum_{y=1}^{J} \int \exp(ir(\eta' + m(y, s))) f_{\eta_t|s_t=s}(\eta'|s) d\eta' p_t(y|s) \\ &= \sum_{y=1}^{J} \phi_{\eta_t|s_t=s}(r) \exp(irm(y, s)) p_t(y|s) = \left|\phi_{\eta_t|s_t=s}(r)\right| \sum_{y=1}^{J} \exp(irm(y, s) + ib(r)) p_t(y|s). \end{aligned} \tag{A.14}$$

Derivation of Eq. (3.8). Consider the real part of $\Phi(\mathcal{T},\mathcal{R})$. Based on Eq. (3.7) we have

$$\mathbf{Re}\{\Phi(\mathcal{T},\mathcal{R})\} = P(\mathcal{T})\mathbf{Re}\{\Gamma(\mathcal{R})D(\mathcal{R})\}. \tag{A.15}$$

Assumption 4(i) implies that $P(\mathcal{T})$, $\mathbf{Re}\{\Gamma(\mathcal{R})\}$, and $D(\mathcal{R})$ are invertible. We define

$$\Upsilon(\mathcal{R}) := (\mathbf{Re}\{\Gamma(\mathcal{R})D(\mathcal{R})\})^{-1} D_m \mathbf{Re}\{\Gamma(\mathcal{R})D(\mathcal{R})\},$$

where $D_m := Diag\{m_1, m_2, \dots, m_J\}$. Based on Eq. (A.15), Eq. (3.8) is derived as follows:

$$\begin{aligned}
&\mathbf{Re}\{\Phi(\mathcal{T},\mathcal{R})\}\Upsilon(\mathcal{R})(\mathbf{Re}\{\Phi(\mathcal{T},\mathcal{R})\})^{-1} \\
=&P(\mathcal{T})\mathbf{Re}\{\Gamma(\mathcal{R})D(\mathcal{R})\}\Upsilon(\mathcal{R})(\mathbf{Re}\{\Gamma(\mathcal{R})D(\mathcal{R})\})^{-1}P(\mathcal{T})^{-1} \\
=&P(\mathcal{T})D_m P(\mathcal{T})^{-1}.
\end{aligned}$$

Eq. (3.8) leads to an eigenvalue-eigenvector decomposition of the matrix on its left-hand side. $P(\mathcal{T})$ represents the matrix of eigenvectors and the diagonal elements in D_m are the corresponding eigenvalues. Since $\mathbf{Re}\{\Phi(\mathcal{T},\mathcal{R})\}$ is identified from observable state transition process, we next prove that $\Upsilon(\mathcal{R})$ is also identified from data.

Identification of $\Upsilon(\mathcal{R})$. We follow Chen et al. (2009) to show the identification of $\Upsilon(\mathcal{R})$. Taking the derivative of the characteristic function $\phi_{s_{t+1}|s_t=s}(r)$ in Eq. (3.6) with respect to r yields

$$\begin{aligned}
\frac{\partial}{\partial r}\phi_{s_{t+1}|s_t=s}(r) =& \left(\frac{\partial}{\partial r}\left|\phi_{\eta_t|s_t=s}(r)\right|\right)\sum_{y=1}^{J}\exp(irm(y,s)+ib(r))p_t(y|s) \\
&+ i\left(\frac{\partial}{\partial r}b(r)\right)\left|\phi_{\eta_t|s_t=s}(r)\right|\sum_{y=1}^{J}\exp(irm(y,s)+ib(r))p_t(y|s) \\
&+ i\left|\phi_{\eta_t|s_t=s}(r)\right|\sum_{y=1}^{J}\exp(irm(y,s)+ib(r))m(y,s)p_t(y|s).
\end{aligned} \tag{A.16}$$

For a set of time periods $\mathcal{T} = \{t_1, t_2, \dots, t_J\}$ and real-valued scalers $\mathcal{R} = \{0, r_2, \dots, r_J\}$, define

$$\partial\Phi(\mathcal{T},\mathcal{R}) = \begin{pmatrix}
iE(s_{t_1+1}|s_{t_1}=s) & \frac{\partial}{\partial r}\phi_{s_{t_1+1}|s_{t_1}=s}(r_2) & \cdots & \frac{\partial}{\partial r}\phi_{s_{t_1+1}|s_{t_1}=s}(r_J) \\
iE(s_{t_2+1}|s_{t_2}=s) & \frac{\partial}{\partial r}\phi_{s_{t_2+1}|s_{t_2}=s}(r_2) & \cdots & \frac{\partial}{\partial r}\phi_{s_{t_2+1}|s_{t_2}=s}(r_J) \\
\vdots & \vdots & \ddots & \vdots \\
iE(s_{t_J+1}|s_{t_J}=s) & \frac{\partial}{\partial r}\phi_{s_{t_J+1}|s_{t_J}=s}(r_2) & \cdots & \frac{\partial}{\partial r}\phi_{s_{t_J+1}|s_{t_J}=s}(r_J)
\end{pmatrix}.$$

Eq. (A.16) is equivalent to

$$\partial\Phi(\mathcal{T},\mathcal{R}) = P(\mathcal{T})\Gamma(\mathcal{R})D_{\partial|\phi|}(\mathcal{R}) + iP(\mathcal{T})\Gamma(\mathcal{R})D(\mathcal{R})D_{\partial b}(\mathcal{R}) + iP(\mathcal{T})D_m\Gamma(\mathcal{R})D(\mathcal{R}), \tag{A.17}$$

where

$$\begin{aligned}
D_{\partial|\phi|}(\mathcal{R}) &= Diag\{0, \frac{\partial}{\partial r}\left|\phi_{\eta_t|s_t=s}(r_2)\right|, \dots, \frac{\partial}{\partial r}\left|\phi_{\eta_t|s_t=s}(r_J)\right|\}, \\
D_{\partial b}(\mathcal{R}) &= Diag\{0, \frac{\partial}{\partial r}b(r_2), \dots, \frac{\partial}{\partial r}b(r_J)\}.
\end{aligned}$$

Next, we consider the real and imaginary parts of $\Phi(\mathcal{T},\mathcal{R})$ and $\partial\Phi(\mathcal{T},\mathcal{R})$. Eq. (A.15) presents the real part of $\Phi(\mathcal{T},\mathcal{R})$. For the imaginary part of $\Phi(\mathcal{T},\mathcal{R})$, we have

$$\mathbf{Im}\{\Phi(\mathcal{T},\mathcal{R})\} + \Xi = P(\mathcal{T})(\mathbf{Im}\{\Gamma(\mathcal{R})\} + \Xi)D(\mathcal{R}), \tag{A.18}$$

where $\Xi := (\mathbf{1}_{J\times 1}, \mathbf{0}_{J\times 1}, \dots, \mathbf{0}_{J\times 1})$ is a $J \times J$ matrix with the elements in the first column equal to 1 and all other elements equal to 0. Adding Ξ to the imaginary part of $\Phi(\mathcal{T},\mathcal{R})$ is because $\mathbf{Im}\{\Phi(\mathcal{T},\mathcal{R})\}$ is not invertible (the elements in its first column are all zeros). The real part of $\partial\Phi(\mathcal{T},\mathcal{R})$ is

$$\mathbf{Re}\{\partial\Phi(\mathcal{T},\mathcal{R})\} = P(\mathcal{T})\mathbf{Re}\{\Gamma(\mathcal{R})\}D_{\partial|\phi|}(\mathcal{R}) - P(\mathcal{T})\mathbf{Im}\{\Gamma(\mathcal{R})\}D(\mathcal{R})D_{\partial b}(\mathcal{R}) - P(\mathcal{T})D_m\mathbf{Im}\{\Gamma(\mathcal{R})\}D(\mathcal{R}). \tag{A.19}$$

Subtracting $\Xi_m := P(\mathcal{T})D_m\Xi$ from both sides of Eq. (A.19) yields

$$\mathbf{Re}\{\partial\Phi(\mathcal{T},\mathcal{R})\} - \Xi_m = P(\mathcal{T})\mathbf{Re}\{\Gamma(\mathcal{R})\}D_{\partial|\phi|}(\mathcal{R}) - P(\mathcal{T})(\mathbf{Im}\{\Gamma(\mathcal{R})\}+\Xi)D(\mathcal{R})D_{\partial b}(\mathcal{R}) - P(\mathcal{T})D_m(\mathbf{Im}\{\Gamma(\mathcal{R})\}+\Xi)D(\mathcal{R}). \tag{A.20}$$

Note that this equality holds because $\Xi D(\mathcal{R})D_{\partial b}(\mathcal{R}) = 0$ and $\Xi D(\mathcal{R}) = \Xi$. Also note that Ξ_m is observed in the data because

$$\Xi_m = \begin{pmatrix}
E(s_{t_1+1}|s_{t_1}=s) & 0 & \cdots & 0 \\
E(s_{t_2+1}|s_{t_2}=s) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
E(s_{t_J+1}|s_{t_J}=s) & 0 & \cdots & 0
\end{pmatrix}.$$

For the imaginary part of $\partial\Phi(\mathcal{T},\mathcal{R})$, we have

$$\mathbf{Im}\{\partial\Phi(\mathcal{T},\mathcal{R})\} = P(\mathcal{T})(\mathbf{Im}\{\Gamma(\mathcal{R})\} + \Xi)D_{\partial|\phi|}(\mathcal{R}) + P(\mathcal{T})\mathbf{Re}\{\Gamma(\mathcal{R})\}D(\mathcal{R})D_{\partial b}(\mathcal{R}) + P(\mathcal{T})D_m\mathbf{Re}\{\Gamma(\mathcal{R})\}D(\mathcal{R}). \tag{A.21}$$

This equality holds because $\Xi D_{\partial|\phi|}(\mathcal{R}) = 0$.

Assumption 4(i) guarantees that $\mathbf{Re}\{\Phi(\mathcal{T},\mathcal{R})\}$ is invertible. Combining the real and imaginary parts of $\Phi(\mathcal{T},\mathcal{R})$ in Eqs. (A.15) and (A.18), we define

$$\begin{aligned} A_{\mathcal{T},\mathcal{R}} &:= \Big(\mathbf{Re}\{\Phi(\mathcal{T},\mathcal{R})\}\Big)^{-1}\Big(\mathbf{Im}\{\Phi(\mathcal{T},\mathcal{R})\} + \Xi\Big) \\ &= \Big(\mathbf{Re}\{\Gamma(\mathcal{R})\}D(\mathcal{R})\Big)^{-1}(\mathbf{Im}\{\Gamma(\mathcal{R})\} + \Xi)D(\mathcal{R}). \end{aligned} \tag{A.22}$$

Combining the real parts of $\Phi(\mathcal{T},\mathcal{R})$ and $\partial\Phi(\mathcal{T},\mathcal{R})$ in Eqs. (A.15) and (A.20), we define

$$\begin{aligned} A_{Re} &:= \Big(\mathbf{Re}\{\Phi(\mathcal{T},\mathcal{R})\}\Big)^{-1}\Big(\mathbf{Re}\{\partial\Phi(\mathcal{T},\mathcal{R})\} - \Xi_m\Big) \\ &= \Big(\mathbf{Re}\{\Gamma(\mathcal{R})\}D(\mathcal{R})\Big)^{-1}\mathbf{Re}\{\Gamma(\mathcal{R})\}D_{\partial|\phi|}(\mathcal{R}) \\ &- \Big(\mathbf{Re}\{\Gamma(\mathcal{R})\}D(\mathcal{R})\Big)^{-1}(\mathbf{Im}\{\Gamma(\mathcal{R})\} + \Xi)D(\mathcal{R})D_{\partial b}(\mathcal{R}) \\ &- \Big(\mathbf{Re}\{\Gamma(\mathcal{R})\}D(\mathcal{R})\Big)^{-1}D_m(\mathbf{Im}\{\Gamma(\mathcal{R})\} + \Xi)D(\mathcal{R}) \\ &= D(\mathcal{R})^{-1}D_{\partial|\phi|}(\mathcal{R}) - A_{\mathcal{T},\mathcal{R}}D_{\partial b}(\mathcal{R}) - Y(\mathcal{R})A_{\mathcal{T},\mathcal{R}}. \end{aligned} \tag{A.23}$$

Combining the imaginary parts of $\Phi(\mathcal{T},\mathcal{R})$ and $\partial\Phi(\mathcal{T},\mathcal{R})$ in Eqs. (A.18) and (A.21), we define

$$\begin{aligned} A_{Im} &:= \Big(\mathbf{Im}\{\Phi(\mathcal{T},\mathcal{R})\} + \Xi\Big)^{-1}\mathbf{Im}\{\partial\Phi(\mathcal{T},\mathcal{R})\} \\ &= \Big((\mathbf{Im}\{\Gamma(\mathcal{R})\} + \Xi)D(\mathcal{R})\Big)^{-1}(\mathbf{Im}\{\Gamma(\mathcal{R})\} + \Xi)D_{\partial|\phi|}(\mathcal{R}) \\ &+ \Big((\mathbf{Im}\{\Gamma(\mathcal{R})\} + \Xi)D(\mathcal{R})\Big)^{-1}\mathbf{Re}\{\Gamma(\mathcal{R})\}D(\mathcal{R})D_{\partial b}(\mathcal{R}) \\ &+ \Big((\mathbf{Im}\{\Gamma(\mathcal{R})\} + \Xi)D(\mathcal{R})\Big)^{-1}D_m\mathbf{Re}\{\Gamma(\mathcal{R})\}D(\mathcal{R}) \\ &= D(\mathcal{R})^{-1}D_{\partial|\phi|}(\mathcal{R}) + A^{-1}_{\mathcal{T},\mathcal{R}}D_{\partial b}(\mathcal{R}) + A^{-1}_{\mathcal{T},\mathcal{R}}Y(\mathcal{R}). \end{aligned} \tag{A.24}$$

Combining Eqs. (A.23) and (A.24), we can eliminate $Y(\mathcal{R})$.

$$A_{Re} + A_{\mathcal{T},\mathcal{R}}A_{Im}A_{\mathcal{T},\mathcal{R}} = D(\mathcal{R})^{-1}D_{\partial|\phi|}(\mathcal{R}) + A_{\mathcal{T},\mathcal{R}}D(\mathcal{R})^{-1}D_{\partial|\phi|}(\mathcal{R})A_{\mathcal{T},\mathcal{R}} + D_{\partial b}(\mathcal{R})A_{\mathcal{T},\mathcal{R}} - A_{\mathcal{T},\mathcal{R}}D_{\partial b}(\mathcal{R}). \tag{A.25}$$

In Eq. (A.25), matrices A_{Re}, $A_{\mathcal{T},\mathcal{R}}$, and A_{Im} are all observable. Our goal is to identify $D(\mathcal{R})^{-1}D_{\partial|\phi|}(\mathcal{R})$ and $D_{\partial b}(\mathcal{R})$. First, note that both of these D matrices are diagonal. Moreover, the diagonal values of $D_{\partial b}(\mathcal{R})A_{\mathcal{T},\mathcal{R}} - A_{\mathcal{T},\mathcal{R}}D_{\partial b}(\mathcal{R})$ are equal to 0. Let $diag(\cdot)$ be a function generating all the diagonal elements of its argument. Eq. (A.25) implies that

$$diag(A_{Re} + A_{\mathcal{T},\mathcal{R}}A_{Im}A_{\mathcal{T},\mathcal{R}}) = [A_{\mathcal{T},\mathcal{R}}\circ A'_{\mathcal{T},\mathcal{R}} + I]diag(D(\mathcal{R})^{-1}D_{\partial|\phi|}(\mathcal{R})), \tag{A.26}$$

where $\circ$ represents a point-wise product. As a result, the diagonal elements of $D(\mathcal{R})^{-1}D_{\partial|\phi|}(\mathcal{R})$ are identified. From Eq. (A.25), we then identify $D_{\partial b}(\mathcal{R})A_{\mathcal{T},\mathcal{R}} - A_{\mathcal{T},\mathcal{R}}D_{\partial b}(\mathcal{R})$. Let $e_1 := (1,0,\dots,0)'$ be a $J\times 1$ vector. We have

$$e_1'\Big(D_{\partial b}(\mathcal{R})A_{\mathcal{T},\mathcal{R}} - A_{\mathcal{T},\mathcal{R}}D_{\partial b}(\mathcal{R})\Big) = -e_1'A_{\mathcal{T},\mathcal{R}}D_{\partial b}(\mathcal{R}). \tag{A.27}$$

Eq. (A.27) holds because $e_1'D_{\partial b}(\mathcal{R}) = 0$ (the first element of $D_{\partial b}(\mathcal{R})$ is 0). Note that the left-hand side of Eq. (A.27) is identified. On the right-hand side, $A_{\mathcal{T},\mathcal{R}}$ is also observable, and Assumption 4(ii) implies that all elements in $e_1'A_{\mathcal{T},\mathcal{R}}$ are nonzero. Therefore the diagonal values of $D_{\partial b}(\mathcal{R})$ are also identified. Since we have shown that both $D(\mathcal{R})^{-1}D_{\partial|\phi|}(\mathcal{R})$ and $D_{\partial b}(\mathcal{R})$ are identified from Eq. (A.25), plugging these matrices to Eq. (A.24) identifies $Y(\mathcal{R})$ as

$$Y(\mathcal{R}) = A_{\mathcal{T},\mathcal{R}}\Big(A_{Im} - D(\mathcal{R})^{-1}D_{\partial|\phi|}(\mathcal{R})\Big) - D_{\partial b}(\mathcal{R}). \tag{A.28}$$

Eigenvalue-eigenvector decomposition. Once $Y(\mathcal{R})$ is identified, all terms on the left-hand side of Eq. (3.8) are identified. Eq. (3.8) then leads to an eigenvalue-eigenvector decomposition of the matrix $\mathbf{Re}\{\Phi(\mathcal{T},\mathcal{R})\}Y(\mathcal{R})(\mathbf{Re}\{\Phi(\mathcal{T},\mathcal{R})\})^{-1}$, where $P(\mathcal{T})$ represents the matrix of eigenvectors and the diagonal elements in D_m are the corresponding eigenvalues. Assumption 5 guarantees that all the eigenvalues are distinctive and non-zero in the diagonalization in Eq. (3.8). Specifically, m_j is identified as the jth smallest root of $det(Y(\mathcal{R}) - \lambda I) = 0$. Eq. (3.8) also implies that the jth column in matrix $P(\mathcal{T})$, i.e., $(p_{t_1}(y_t = j|s),\dots,p_{t_J}(y_t = j|s))^T$ is the eigenvector corresponding to the eigenvalue m_j. Therefore, the conditional choice probabilities, $p_t(y_t = j|s)$ for $j = 1, 2. \cdots, J$ and $t = t_1, t_2, \dots, t_J$ are also identified.

Once the $m(\cdot, \cdot)$ function and the conditional choice probabilities are identified, we can nonparametrically identify the distribution for $\eta_t|s_t = s$ from Eq. (3.5). Specifically, for any $s' \in \mathcal{S}$,

$$\begin{pmatrix} f_{\eta_t|s_t}(s' - m_1|s) \\ f_{\eta_t|s_t}(s' - m_2|s) \\ \vdots \\ f_{\eta_t|s_t}(s' - m_J|s) \end{pmatrix} = P(\mathcal{T})^{-1} \begin{pmatrix} f_{s_{t_1+1}|s_{t_1}}(s'|s) \\ f_{s_{t_2+1}|s_{t_2}}(s'|s) \\ \vdots \\ f_{s_{t_J+1}|s_{t_J}}(s'|s) \end{pmatrix}. \tag{A.29}$$

This completes the proof of Theorem 2. □

A.3. Proof of Corollary 1

We prove Corollary 1 using the rank inequalities: for any $m \times n$ matrix A and $n \times k$ matrix B,

$$rank(A) + rank(B) - n \leq rank(AB) \leq \min\{rank(A), rank(B)\}.$$

When $\left|\overline{\mathcal{T}}\right| \geq J$ and $\left|\overline{\mathcal{R}}\right| \geq J$,

$$rank(P(\overline{\mathcal{T}})\Gamma(\overline{\mathcal{R}})) \leq \min\{rank(P(\overline{\mathcal{T}})), rank(\Gamma(\overline{\mathcal{R}}))\} \leq \min\{J, J\} = J.$$

Similarly,

$$rank(\Phi(\overline{\mathcal{T}}, \overline{\mathcal{R}})) \leq \min\{rank(P(\overline{\mathcal{T}})\Gamma(\overline{\mathcal{R}})), rank(D(\overline{\mathcal{R}}))\} \leq J.$$

To prove the second part of Corollary 1, note that the full rank assumption of $P(\overline{\mathcal{T}})$, $\Gamma(\overline{\mathcal{R}})$ and $D(\overline{\mathcal{R}})$ implies that $rank(P(\overline{\mathcal{T}})) = rank(\Gamma(\overline{\mathcal{R}})) = J$ and $rank(D(\overline{\mathcal{R}})) = \left|\overline{\mathcal{R}}\right|$. Again using the rank inequalities, we first show that $rank(P(\overline{\mathcal{T}})\Gamma(\overline{\mathcal{R}})) = J$. Applying the rank inequalities on $P(\overline{\mathcal{T}})\Gamma(\overline{\mathcal{R}})$ and $D(\overline{\mathcal{R}})$ again, we prove that $rank(\Phi(\overline{\mathcal{T}}, \overline{\mathcal{R}})) = J$. □

Appendix B. Identification with two state variables

As we shown in Section 3.2, when $y_t \in \{1, 2, \dots, J\}$, identifying the state transition rules specified in Assumption 2 in general requires the observation of the state variable for at least $J + 1$ periods. When $J + 1$ periods of data are not available, having an additional state variable helps us identify the latent state transition rule with two consecutive time periods. When two (or more) state variables are available, the requirement of continuous state variable, the additivity and independence assumption imposed on the state transition process in Assumption 2 can be relaxed. In this section, we provide the identification results for the latent state transition rule when we have two discrete state variables $\{s_t, z_t\}$ for t and $t + 1$ that satisfy the following assumption.

Assumption 9 (*Conditional Independence*). $f_{s_{t+1},z_{t+1}|s_t,z_t,y_t} = f_{s_{t+1}|s_t,y_t} f_{z_{t+1}|z_t,y_t}$.

Assumption 9 implies that the transition process of the two state variables are independent conditional on the choice variable. Specifically, s_t is excluded from the transition of z_t and vice versa, but the choice probability depends on both state variables. Under Assumption 9, the observed joint distribution of $\{s_{t+1}, z_{t+1}, s_t, z_t\}$ can be factorized as follows.

$$f_{s_{t+1},z_{t+1},s_t,z_t}(s', z', s, z) = \sum_{y=1}^{J} f_{s_{t+1}|s_t,y_t}(s'|s, y) f_{z_{t+1}|z_t,y_t}(z'|z, y) f_{y_t,s_t,z_t}(y, s, z). \tag{B.1}$$

Intuitively, the future states can be viewed as proxies of the unobserved choice. Following the results in the measurement error literature (Hu, 2008; Hu and Shum, 2012), we now construct the matrix form of Eq. (B.1). Let $j_s = 1, \dots, J_s$, $j_z = 1, \dots, J_z$, and $j = 1, \dots, J$ index the categories of s_t, z_t and y_t, respectively. Assume that $J_s \geq J$ and $J_z \geq J$. We define the following matrices for fixed (s, z):

$$M_{s_{t+1},z_{t+1},s,z} = \left[f_{s_{t+1},z_{t+1},s_t,z_t}(s_{t+1}, z_{t+1}, s, z) \Big|_{s_{t+1}=j_s, z_{t+1}=j_z} \right]_{j_s, j_z},$$

$$M_{s_{t+1}|s,y_t} = \left[f_{s_{t+1}|s_t,y_t}(s_{t+1}|s, y_t) \Big|_{s_{t+1}=j_s, y_t=j} \right]_{j_s, j},$$

$$M_{y_t,s,z} = diag\left\{ \left[f_{y_t,s_t,z_t}(y_t, s, z) \Big|_{y_t=j} \right]_{j=1,2,\dots,J} \right\},$$

$$M_{z_{t+1}|z,y_t} = \left[f_{z_{t+1}|z_t,y_t}(z_{t+1}|z, y_t) \Big|_{y_t=j, z_{t+1}=j_z} \right]_{j, j_z}.$$

Eq. (B.1) in matrix form is therefore

$$M_{s_{t+1},z_{t+1},s,z} = M_{s_{t+1}|s,y_t} M_{y_t,s,z} M_{z_{t+1}|z,y_t}. \tag{B.2}$$

Similar to the results in Corollary 1, we can identify the number of alternatives $J = rank(M_{s_{t+1},z_{t+1},s,z})$ provided the following full rank condition.

Assumption 10 (*Rank*). $M_{s_{t+1}|s,y_t}$, $M_{y_t,s,z}$, and $M_{z_{t+1}|z,y_t}$ have full rank for all (s,z).

The rank condition on $M_{s_{t+1}|s,y_t}$ and $M_{z_{t+1}|z,y_t}$ requires that there are sufficient variations in the future state distributions of s_t and z_t with respect to the choice variable y_t. If the choice probability of each alternative is nonzero, the invertibility of $M_{y_t,s,z}$ is guaranteed. Once we identify the number of alternatives J from Eq. (B.2), it is always possible to regroup s_t and z_t so that $J_s = J_z = J$. We focus on this case for the rest of our analysis.

Consider four combinations of observed states at t: $(\bar{s},\bar{z})$, $(\hat{s},\bar{z})$, $(\bar{s},\hat{z})$, $(\hat{s},\hat{z})$, and construct the following equation:

$$\begin{aligned} &\left(M_{s_{t+1},z_{t+1},\bar{s},\bar{z}} M^{-1}_{s_{t+1},z_{t+1},\hat{s},\bar{z}}\right)\left(M_{s_{t+1},z_{t+1},\bar{s},\hat{z}} M^{-1}_{s_{t+1},z_{t+1},\hat{s},\hat{z}}\right)^{-1} \\ &= M_{s_{t+1}|\bar{s},y_t}\left(M_{y_t,\bar{s},\bar{z}} M^{-1}_{y_t,\hat{s},\bar{z}} M_{y_t,\hat{s},\hat{z}} M^{-1}_{y_t,\bar{s},\hat{z}}\right) M^{-1}_{s_{t+1}|\bar{s},y_t} \\ &\equiv MQM^{-1}. \end{aligned} \tag{B.3}$$

Under Assumption 10, Eq. (B.3) leads to an eigenvalue-eigenvector decomposition of the observed matrix on its left-hand side, where M represents the matrix of eigenvectors and the diagonal elements in Q are the corresponding eigenvalues. Additional assumptions are required to guarantee the uniqueness of the decomposition and to pin down the ordering of the eigenvectors. We provide one such assumption below.

Assumption 11. The following conditions are satisfied: (1) (uniqueness) let $Q(i)$ denote the ith diagonal element in Q. $Q(i) \neq Q(j)$ for any $i \neq j$, and (2) (ordering) $\mathrm{E}(s_{t+1}|\bar{s}, y_t)$ increases with $y_t \in \{1,2,\dots,J\}$.

Assumption 11(i) rules out the possibility of duplicated eigenvalues. Assumption 11(ii) imposes restrictions on the state transition process given different choices to determine which eigenvector corresponds to which y_t. We summarize the identification results for the unknown densities in M in the following theorem.

Theorem 3 (*Identification – Two State Variables*). *Suppose Assumptions 1, 9, 10, and 11 hold for the Markov process of* $\{s_t, z_t, \varepsilon_t, y_t\}$. *The joint distribution of* $\{s_{t+1}, z_{t+1}, s_t, z_t\}$ *uniquely determines the state transition rules* $f_{s_{t+1}|\bar{s},y_t}$.

The proof of Theorem 3 is a direct application of Hu (2008) on Eq. (B.3), hence is omitted in this paper. Similar to Eq. (B.3), we can derive the eigenvalue-eigenvector decompositions to identify matrices $M_{s_{t+1}|\hat{s},y_t}$, $M_{z_{t+1}|\bar{z},y_t}$, and $M_{z_{t+1}|\hat{z},y_t}$, which essentially lead to the identification of the state transition rules $f_{s_{t+1}|s_t,y_t}$ and $f_{z_{t+1}|z_t,y_t}$ given $s_t = \{\bar{s},\hat{s}\}$ and $z_t = \{\bar{z},\hat{z}\}$. The matrix $M_{y_t,s,z}$ is therefore identified from Eq. (B.2); the diagonal elements of this matrix correspond to the unobserved choice probabilities $f_{y_t|s_t,z_t}$.

When two continuous state variables are available, we can generalize our identification results to allow for continuous choice variables. Instead of using eigenvalue-eigenvector decompositions, spectral decompositions proposed by Hu and Schennach (2008) are applied.

Remark 5. The conditional independence assumption imposed on the two state variables are important for the identification results in this section. When Assumption 9 does not hold (for example, if there exists only one discrete state variable), we can connect the unobserved choice probabilities and the latent state transition rules through (1) the observed state transition process, and (2) the agent's optimization problem. This provides us with a system of nonlinear equations, through which we can locally identify the CCPs and the state transition rules conditional on the choice. Imposing parametric assumptions on the utility function and/or the state transition rules may further reduce the dimension of the parameter space.

Appendix C. Sieve maximum likelihood estimation

In this section, we provide details of constructing the sieve spaces and the asymptotic properties of the estimator proposed in Section 4. For the simplicity of the notations, we focus on the binary choice case, where $y_t \in \{0,1\}$. The extension of the results in this section to allow for general multinomial choice is straightforward.

Let $\theta^0 = (\alpha^0, m_0^0, m_1^0, f^0_{\eta_t|s_t})$ represent the vector of true parameter values of interest. We consider a case where the per-period utility functions are parametrized by a finite-dimensional parameter $\alpha^0 \in \mathcal{A}$. $m_0^0 : \mathcal{S} \to \mathcal{S}$ and $m_1^0 : \mathcal{S} \to \mathcal{S}$ are two nonparametric functions in the state transition rules when $y_t = 0$ or 1, respectively. $f^0_{\eta_t|s_t} : \mathbb{R} \times \mathcal{S} \to \mathbb{R}^+$ is the probability density function of the error term conditional on the state variable.

We impose the following smoothness restrictions on m_0^0, m_1^0, and the density function $f^0_{\eta_t|s_t}$. To strengthen the definition of continuity, we introduce the notation for the space of Hölder continuous functions. If Ψ is an open set in $\mathbb{R}^n$, $\kappa \in \mathbb{N}$, and $\zeta \in (0,1]$, then $\Gamma^{\kappa,\zeta}(\Psi)$ consists of all functions $m : \Psi \to \mathbb{R}$ with continuous partial derivatives in Ψ of order less than or equal to κ whose κ-th partial derivatives are locally uniformly Hölder continuous with exponent ζ in Ψ. Define a Hölder ball, which is a compact subset of $\Gamma^{\kappa,\zeta}(\Psi)$, as $\Gamma_c^{\kappa,\zeta}(\Psi) \equiv \left\{ m \in \Gamma^{\kappa,\zeta}(\Psi) \middle| \|m\|_{\Gamma^{\kappa,\zeta}(\Psi)} \le c < \infty \right\}$ with respect to the norm

$$\|m\|_{\Gamma^{\kappa,\zeta}(\Psi)} \equiv \max_{|r|\le\kappa} \sup_{\Psi} \|\partial^r m\|_e + \max_{|r|=\kappa} [\partial^r m]_{\zeta,\Psi}.$$

In the norm definition for the Hölder ball, $\|\cdot\|_e$ represents the Euclidean norm, and

$$[m]_{\zeta,\Psi} \equiv \sup_{x,x'\in\Psi, x\neq x'} \frac{\|m(x) - m(x')\|_e}{\|x - x'\|_e^{\zeta}}.$$

$\partial^r m$ represents the multi-index notation for partial derivatives with $r = (r_1, r_2, \dots, r_{\dim(\Psi)})$ and $|r| = r_1 + r_2 + \dots + r_{\dim(\Psi)}$. With notations for the space of Hölder continuous functions, we define the functional space of m_0 and m_1 by $\mathcal{H} = \Gamma_c^{\kappa_1,\zeta_1}(\mathcal{S})$ with supremum norm $\|m\|_{\mathcal{H}} = \sup_{x\in\mathcal{S}} |m(x)|$. The space of the density function is

$$\mathcal{F} = \left\{ f_{\eta_t|s_t}(\cdot|\cdot) \in \Gamma_c^{\kappa_2,\zeta_2}(\mathbb{R}\times\mathcal{S}) : f_{\eta_t|s_t}(\cdot|s) > 0, \int_{\mathbb{R}} f_{\eta_t|s_t}(\eta|s)d\eta = 1, \mathrm{E}(\eta_t|s) = 0, \forall s \in \mathcal{S} \right\},$$

with norm defined by $\|f\|_{\mathcal{F}} = \sup_{x\in\mathbb{R}\times\mathcal{S}} \left| f(x)(1 + \|x\|_e^2)^{-\psi/2} \right|$, $\psi > 0$. Notice that the conditional mean of η_t for all density functions in $\mathcal{F}$ are equal to 0, which is consistent with Assumption 2. $\Theta = \mathcal{A}\times\mathcal{H}\times\mathcal{H}\times\mathcal{F}$ denotes the space for all parameters of interest. Θ is an infinite-dimensional space as it contains nonparametric functions m_0, m_1, and $f_{\eta_t|s_t}$. The metric on Θ is defined by

$$d(\theta, \tilde{\theta}) = \|\alpha - \tilde{\alpha}\|_e + \|m_0 - \tilde{m}_0\|_{\mathcal{H}} + \|m_1 - \tilde{m}_1\|_{\mathcal{H}} + \left\| f_{\eta_t|s_t} - \tilde{f}_{\eta_t|s_t} \right\|_{\mathcal{F}}.$$

In light of a finite sample, instead of searching parameters over an infinite-dimensional parameter space Θ, we maximize the empirical criterion function in Eq. (4.3) over a sequence of approximating sieve spaces $\Theta_k = \mathcal{A}\times\mathcal{H}_{k_1}\times\mathcal{H}_{k_2}\times\mathcal{F}_{k_3}$, where

$$\mathcal{H}_{k_1} = \left\{ m \in \mathcal{H} \middle| m : \mathcal{S} \to \mathbb{R}, m(s) = \sum_{q=1}^{k_1} \gamma_q h_q(s), \gamma_q \in \mathbb{R}, \forall q \right\},$$

$$\mathcal{H}_{k_2} = \left\{ m \in \mathcal{H} \middle| m : \mathcal{S} \to \mathbb{R}, m(s) = \sum_{q=1}^{k_2} \gamma_q h_q(s), \gamma_q \in \mathbb{R}, \forall q \right\},$$

$$\mathcal{F}_{k_3} = \left\{ f \in \mathcal{F} \middle| f : \mathbb{R}\times\mathcal{S} \to \mathbb{R}^+, \sqrt{f(\eta|s)} = \mathbf{g}^{k_3}(\eta, s)^T \lambda, \lambda \in \mathbb{R}^{k_3} \right\}.$$

In the definition of sieve spaces, $(h_1(\cdot), h_2(\cdot), h_3(\cdot), \dots)$ represents a sequence of known basis functions, such as Hermite polynomials, power series, splines, etc. We use linear sieves to approximate the square root of densities and $\mathbf{g}^{k_3}(\cdot,\cdot)$ is a $k_3\times 1$ vector of the tensor product of spline basis functions on $\mathbb{R}\times\mathcal{S}$. Notice that it is standard to generate linear sieves of multivariate functions using a tensor product of linear sieves of univariate functions.

Chen (2007; Ch. 3) provides a general consistency theorem for sieve extremum estimators for various semi-/non-parametric models. Following Chen et al. (2008) and Carroll et al. (2010), we provide sufficient conditions tailored to our model for consistency of the sieve maximum likelihood estimator in Eq. (4.4).[19]

Assumption 12 (*Consistency*)**.** The following conditions are satisfied: (i) all assumptions in Theorem 1 hold; (ii) D_i is i.i.d. across i; (iii) m_0 and $m_1 \in \mathcal{H}$ with $\kappa_1 + \zeta_1 > 1/2$; $f_{\eta|S} \in \mathcal{F}$ with $\kappa_2 + \zeta_2 > 1$; (iv) $\left|Q(\theta^0)\right| < \infty$ and $Q(\theta)$ is upper semicontinuous on Θ under the metric $d(\cdot,\cdot)$; (v) the sieve spaces, Θ_k, are compact under $d(\cdot,\cdot)$; (vi) There is a finite $\sigma > 0$ and a random variable $c(D_i)$ with $\mathrm{E}(c(D_i)) < \infty$ such that $\sup_{\theta\in\Theta_k : d(\theta,\theta^0)\le\epsilon} \left| l(D_i;\theta) - l(D_i;\theta^0) \right| \le \epsilon^{\sigma} c(D_i)$; (vii) k_1, k_2, and $k_3 \to \infty$, $k_1/n, k_2/n$, and $k_3/n \to 0$.

Assumption 12(i) ensures the identification of the model primitives. Assumption 12 overall provides a set of assumptions that imply the conditions of Chen (2007; Ch. 3, Theorem 3.1). The following theorem for the consistency of our sieve maximum likelihood estimator is a direct application, therefore the proof is omitted.

Theorem 4 (*Consistency*)**.** *For $y_t \in \{0,1\}$, if Assumption 12 is satisfied, then the sieve maximum likelihood estimator in Eq. (4.4) is consistent with respect to the metric $d(\cdot,\cdot)$, i.e.,*

$$d(\hat{\theta}_k, \theta^0) = o_P(1).$$

For general results on convergence rates, root-n asymptotic normality, and semiparametric efficiency of sieve maximum likelihood estimators, see Shen and Wong (1994), Chen and Shen (1996), Shen (1997), Chen and Shen (1998), Ai and Chen (2001), and Chen (2007; Theorem 3.2 and Theorem 4.3).

[19] Chen et al. (2008) study identification and estimation of a nonparametric regression model with discrete covariates measured with error. Carroll et al. (2010) consider a general nonlinear errors-in-variables model using two samples.

Appendix D. Identifying state transition conditional on unobserved heterogeneity

In this section, we first derive Eq. (6.2). Assume that x_t^* is a continuous variable. The joint distribution of the observed state variable at four consecutive periods $(s_{t+2}, s_{t+1}, s_t, s_{t-1})$ can be decomposed as follows.

$$
\begin{aligned}
& f_{s_{t+2},s_{t+1},s_t,s_{t-1}} \\
&= \int_{x_{t+1}^*}\int_{x_t^*}\int_{x_{t-1}^*}\int_{y_{t+1}}\int_{y_t}\int_{y_{t-1}} f_{s_{t+2},y_{t+1},x_{t+1}^*,s_{t+1},y_t,x_t^*,s_t,y_{t-1},x_{t-1}^*,s_{t-1}}\, dF_{x_{t+1}^*}\cdots dF_{y_{t-1}} \\
&= \int_{x_{t+1}^*}\int_{x_t^*}\int_{x_{t-1}^*}\left(\int_{y_{t+1}} f_{s_{t+2}|s_{t+1},x_{t+1}^*,y_{t+1}} \times f_{y_{t+1}|s_{t+1},x_{t+1}^*}\, dF_{y_{t+1}^*}\right) \times f_{x_{t+1}^*|s_{t+1},x_t^*} \\
&\qquad\qquad \times \left(\int_{y_t} f_{s_{t+1}|s_t,x_t^*,y_t} \times f_{y_t|s_t,x_t^*}\, dF_{y_t}\right) \times f_{x_t^*|s_t,x_{t-1}^*} \\
&\qquad\qquad \times \left(\int_{y_{t-1}} f_{s_t|s_{t-1},x_{t-1}^*,y_{t-1}} \times f_{y_{t-1}|s_{t-1},x_{t-1}^*}\, dF_{y_{t-1}}\right) \times f_{x_{t-1}^*,s_{t-1}}\, dF_{x_{t+1}^*}\cdots dF_{x_{t-1}^*} \\
&= \int_{x_{t+1}^*}\int_{x_t^*}\int_{x_{t-1}^*} f_{s_{t+2}|s_{t+1},x_{t+1}^*} \times f_{x_{t+1}^*|s_{t+1},x_t^*} \times f_{s_{t+1}|s_t,x_t^*} \times f_{x_t^*|s_t,x_{t-1}^*} \times f_{s_t,x_{t-1}^*,s_{t-1}}\, dF_{x_{t+1}^*}\cdots dF_{x_{t-1}^*} \\
&= \int_{x_t^*}\left(\int_{x_{t+1}^*} f_{s_{t+2}|s_{t+1},x_{t+1}^*} \times f_{x_{t+1}^*|s_{t+1},x_t^*}\, dF_{x_{t+1}^*}\right) \times f_{s_{t+1}|s_t,x_t^*} \\
&\quad \times \left(\int_{x_{t-1}^*} f_{x_t^*|s_t,x_{t-1}^*} \times f_{s_t,x_{t-1}^*,s_{t-1}}\, dF_{x_{t-1}^*}\right) dF_{x_t^*} \\
&= \int_{x_t^*} f_{s_{t+2}|s_{t+1},x_t^*} \times f_{s_{t+1}|s_t,x_t^*} \times f_{x_t^*,s_t,s_{t-1}}\, dF_{x_t^*}.
\end{aligned}
\tag{D.1}
$$

The second equality in Eq. (D.1) holds under the first-order Markov property of the dynamic process and the conditional independence imposed in Assumption 6(i). By integrating out the unobserved choice variables (y_{t+1}, y_t, y_{t-1}), the third equality holds. We further integrate out the unobserved heterogeneity (x_{t+1}^*, x_{t-1}^*), which yields the last line of Eq. (D.1).

Based on this equation, we apply the spectral decomposition technique developed by Hu and Schennach (2008) to nonparametrically identify $f_{s_{t+2}|s_{t+1},x_t^*}$, $f_{s_{t+1}|s_t,x_t^*}$, and $f_{x_t^*,s_t,s_{t-1}}$. To invoke Theorem 1 in Hu and Schennach (2008), we need to assume that the following operators constructed from densities in Eq. (D.1) are invertible. Specifically, denote the support of s_t and x_t^* as $\mathcal{S}$ and $\mathcal{X}$, respectively. The linear operator $L_{s_{t-1},\bar{s}_t,\bar{s}_{t+1},s_{t+2}}$ is a mapping from the $\mathcal{L}^p$ space of functions of s_{t+2} to the $\mathcal{L}^p$ space of functions s_{t-1} defined as

$$
\left(L_{s_{t-1},\bar{s}_t,\bar{s}_{t+1},s_{t+2}} h\right)(s_{t-1}) = \int f_{s_{t-1},s_t,s_{t+1},s_{t+2}}(s_{t-1}, \bar{s}_t, \bar{s}_{t+1}, s_{t+2}) h(s_{t+2}) ds_{t+2},
$$

for all functions $h \in \mathcal{L}^p(\mathcal{S})$, $\bar{s}_t \in \mathcal{S}$, and $\bar{s}_{t+1} \in \mathcal{S}$. Similarly, the operators $L_{s_{t+2}|\bar{s}_{t+1},x_t^*}$, $L_{x_t^*,\bar{s}_t,s_{t-1}}$, and the diagonal operator $D_{\bar{s}_{t+1}|\bar{s}_t,x_t^*}$ are defined as

$$
\begin{aligned}
&\left(L_{s_{t+2}|\bar{s}_{t+1},x_t^*} h\right)(s_{t+2}) = \int f_{s_{t+2}|s_{t+1},x_t^*}(s_{t+2}|\bar{s}_{t+1}, x_t^*) h(x_t^*) dx_t^*, \\
&\left(L_{x_t^*,\bar{s}_t,s_{t-1}} h\right)(s_{t-1}) = \int f_{x_t^*,s_t,s_{t-1}}(x_t^*, \bar{s}_t, s_{t-1}) h(x_t^*) dx_t^* \\
&\left(D_{\bar{s}_{t+1}|\bar{s}_t,x_t^*} h\right)(x_t^*) = f_{s_{t+1}|s_t,x_t^*}(\bar{s}_{t+1}|\bar{s}_t, x_t^*) h(x_t^*)
\end{aligned}
$$

for all functions $h \in \mathcal{L}^p(\mathcal{X})$, $\bar{s}_t \in \mathcal{S}$, and $\bar{s}_{t+1} \in \mathcal{S}$. Assumptions similar to Hu and Schennach (2008, Assumptions 4 and 5) can also be invoked to guarantee the uniqueness of the spectral decomposition and the ordering of the eigenfunctions.

Appendix E. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.jeconom.2024.105806.

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