



Contents lists available at ScienceDirect

Journal of Management Science and Engineering

journal homepage: www.keaipublishing.com/en/journals/journal-of-management-science-and-engineering/

Pricing vulnerable European options with dynamic correlation between market risk and credit risk

Huawei Niu ^{a, b, *}, Yu Xing ^b, Yonggan Zhao ^c^a School of Economics and Management, China University of Mining and Technology, Xuzhou, 221116, China^b School of Finance, Nanjing Audit University, Nanjing, 211815, China^c Rowe School of Business, Dalhousie University, Halifax, B3H 4R2, Canada

ARTICLE INFO

Article history:

Available online 8 April 2020

Keywords:

Vulnerable options
Reduced-form model
Credit risk
Fourier transform
Affine jump-diffusion

ABSTRACT

In this paper, we study the valuation of vulnerable European options incorporating the reduced-form approach, which models the credit default of the counterparty. We provide an analytical pricing model in which the components of the state processes, including the dynamics of the underlying asset value and the intensity process corresponding to the default event, are cross-exciting and they could facilitate the description of complex structure of events dependence. To illustrate how our model works, we present an application when the state variables follow specific affine jump-diffusion processes. Semi-analytical pricing formulae are obtained through a system of matrix Riccati equations. The derived formula can be implemented numerically, and we give numerical analysis to investigate the impact of the dynamic correlation between jump risk of the underlying asset value and default risk of the counterparty.

© 2020 China Science Publishing & Media Ltd. Publishing Services by Elsevier B.V. on behalf of KeAi Communications Co. Ltd. This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

1. Introduction

The main purpose of this paper is to study the valuation of vulnerable European options if the risky factors appearing in the pricing models are dynamically correlated. Following Johnson and Stulz (1987), an option subject to credit risk is referred to as a vulnerable option. Recently, lots of the derivative trading activities have been moving from standardized products quoted on exchange-traded markets towards customized products traded on the over-the-counter (OTC) markets. Unlike the options traded in regulated exchanges, OTC markets require that the option positions should be marked to market on a daily basis and sufficient collateral be posted. When the OTC option writer (the counterparty) defaults, the option holder would receive only a fraction of the contract promised payoff. Thus, the potential credit risk induced by the counterparty may cause a large loss to the option holder.

The derivatives with credit risk are mainly related to the credit default events. Hence, the key point to price credit derivatives is how to get the probability of the default. Currently, there are two basic approaches to model credit risk, the structural approach and the reduced-form approach. The former aims at providing an intuitive understanding of credit risk by specifying a firm's asset value process. An important class of structural models is the class of first-passage time models,

* Corresponding author. School of Economics and Management, China University of Mining and Technology, Xuzhou, 221116, China.

E-mail addresses: niuhuawei@cumt.edu.cn, niuhuawei@nau.edu.cn (H. Niu).

specifying default as the first time the firm's asset value falls below a default boundary.¹ While the reduced-form models provide a simple framework to postulate default as a single jump time. See, e.g., Jarrow and Turnbull (1995), Madan (1998), Duffie and Singleton (1999) and the references therein.

Based on these two approaches on modeling credit risk, there are lots of literatures studying the valuation of vulnerable options.² Under the assumption of independence between the underlying asset of the option and the credit risk of the counterparty, Hull and White (1995) and Jarrow and Turnbull (1995) work in the reduced-form approach to investigate the vulnerable options. Kang and Kim (2005) also derive a simple closed-form pricing model for vulnerable option under the reduced-form framework proposed by Jarrow and Yu (2001). Under the structural models, Klein (1996) extends the results of Johnson and Stulz (1987) and Hull and White (1995) to propose a pricing model with more realistic assumptions. Based on Klein (1996), Klein and Inglis (1999) consider the stochastic default boundary instead of constant barrier. Hui, Lo, and Lee (2003) propose a valuation model with a dynamic default barrier, and obtain closed-form solutions for vulnerable option values through partial differential equation techniques.³ However, the evolutions of the two risky assets, i.e., the underlying asset value and the firm value of the counterparty, are modeled by the diffusion processes in all above papers. But this treatment is contrasted with business situations, such as sudden drops in the firm value.

To overcome this shortcoming, there have been some extensions on this subject. For instance, based on the structural approach, Xu, Xu, Li, and Xiao (2012) present an improved method of pricing vulnerable option by allowing both of the underlying stock price and the firm's asset value to follow jump-diffusion processes. Subject to the early default condition presented in Hull and White (1995), Niu and Wang (2015) derive a simple analytical pricing formula via two-dimensional Laplace transforms when the firm value of the counterparty follows a jump-diffusion process and correlates with the underlying asset value which just follows a diffusion process. Because both risky assets are governed by jump-diffusion processes and the default only happens at the option maturity, Tian, Wang, Wang, and Wang (2014) model that the jumps in these two processes correlate with each other besides diffusion parts, and they obtain a closed-form pricing formula when the jump size follows normal distribution. Moreover, Niu and Wang (2016) consider a more complex case that the two risky assets follow correlated jump-diffusion processes under Markov-modulated regime switching, and they provide an analytical solution which can be numerically implemented. Han, Song, Niu, and Zhou (2018) also investigate the valuation of vulnerable European option considering the market prices of common systematic jump risks under regime-switching jump-diffusion models. Meanwhile, under the reduced-form approach, Fard (2015) formulates the dynamics of the spot price in a completely random generalized jump-diffusion model, and derives the closed-form pricing formula.

From the definition of vulnerable option, we know that pricing vulnerable options include two issues, the market risk (i.e., the value of the underlying asset of the option) and the credit risk (i.e., the default event of the counterparty of the option). The issue of dependence between the market risk and the credit risk has already attracted a lot of attention, however, most studies just propose simple and static structures of the correlation. For example, Cherubini and Luciano (2003) use the copula functions to describe the dependence relationship between these two risks statically, and other pricing models, including all the above mentioned studies, just only use the correlation coefficient ρ of two diffusion parts in risky asset value processes to describe the dependence under the structural approach. Incorporating the long-term and short-term stochastic volatilities in the dynamics of both the underlying asset and the asset of the counterparty, Wang, Wang, and Zhou (2017) derive a vulnerable European option pricing formula, but they also use the constant correlation coefficient to capture the static correlation between the two risky asset value processes.

Recently, to model the dynamic correlation between the underlying asset and the counterparty's asset, Wang (2018) has proposed a pricing model under the structural approach with time-varying variances of the diffusion parts, so that the model could capture the stochastic correlation. Moreover, in the real economy, the value of the underlying asset and the default event of the counterparty would be dynamically affected by each other. Specifically, besides the diffusion parts, the jump risk in the value of the underlying asset as a risk factor would also have impacts on the counterparty's default policy (or the firm value of the counterparty), and vice versa. In this paper, we would like to adopt the reduced-form framework to model the dynamic correlation between the underlying asset value and the intensity of the point process in the reduced-form approach, which could describe the default probability of the counterparty.

Generally, if the underlying asset and the firm of the counterparty in the vulnerable option belong to two correlated industries, when there are jumps in the underlying asset value, it will feedback to the counterparty's industry and the firm value of the counterparty would also be affected suddenly. Since the jumps of the firm value of the counterparty directly affect the default timing, and the default timing is associated with the intensity of the point process in the reduced-form approach, the jumps in the underlying asset value would have an obvious impact on the intensity process via the above affecting channel. Meanwhile, the change of the intensity would also affect the underlying asset value. Thus, as the state processes

¹ For example, see Merton (1974), Black and Cox (1976), Longstaff and Schwartz (1995), Leland and Toft (1996), Kou and Wang (2003).

² There are also several studies on pricing the specific options that could be affected by credit risk besides vulnerable options. For example, Wang (2016) and Bi, Wang, and Wang (2019) study the valuation of catastrophe equity put options with default risk.

³ Besides these studies, there are other extensions for the vulnerable option pricing models. See, for example, Huang and Liu (2005) study the vulnerable option pricing by assuming non-tradability of the assets at the counterparty incomplete markets. Cao and Wei (2001) consider the default boundary is the function of the bonds and the option, and examined the price of vulnerable option by the Monte-Carlo simulation. Liang and Ren (2007) develop a pricing model for the OTC options incorporating a practical default condition, and obtain the pricing formulae by using Green's function. Klein and Yang (2010) extend the models of Johnson and Stulz (1987) and Klein (1996) to price vulnerable American options.

including these correlated risk factors, the intensity process and the underlying asset value process are cross-exciting,⁴ and we could provide a model of dynamic correlation between the market risk and credit risk incorporating the feedback phenomena, which is one of the contributions of this paper.

Note that in valuing financial securities including options, one inevitably faces a trade-off between the analytical and computational tractability of pricing and estimation, and the complexity of the probability model for the state vector X . Following the existing studies on the reduced-form approach, we would let the state vector X follow affine jump-diffusion (AJD) processes,⁵ which have proved to be particularly fruitful in developing tractable and are crucial to obtaining the explicit vulnerable option pricing formulae. Moreover, the estimation of the parameters would also be tractable in practice. And that is why we would like to price vulnerable options incorporating the reduced-form framework.

This paper will investigate the valuation of vulnerable European options when the dynamic of the value of the underlying asset, which is driven by a double exponential jump-diffusion processes (for example, see [Chen and Kou \(2009\)](#)), and the intensity process with jumps corresponding to the default timing of the counterparty are cross-exciting. Thus, another contribution of this paper is that, subject to the early default condition, we develop a general analytical pricing formula of the vulnerable European option under the reduced-form framework when both the market risk and the credit risk contain unpredictable jump risks. Specifically, as an application, we let the state vector X follow the AJD processes and obtain a closed-form pricing formula of the vulnerable European option, which can be numerically implemented.

The rest of the paper is organized as follows. Section 2 presents the model setting and basic assumptions. In Section 3, we derive the analytical pricing formula of vulnerable European options by using the martingale technique and two-dimensional Laplace transform. Section 4 presents numerical results and analysis about the prices of vulnerable European options. Section 5 concludes.

2. Basic setting of the model

As we know, a vulnerable European option is an European option whose counterparty may default at any time prior to the option maturity. Thus the value of the vulnerable European option is determined by both the underlying asset value of the option at expiration and the default probability of the counterparty, which is modeled by the reduced-form approach in this paper. To be convenient, we first briefly introduce the setting of the reduced-form model, i.e., the intensity-based model.

2.1. Intensity-based model of credit risk

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with right-continuous and complete information filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$, on which default stopping times $0 < \tau_1 < \tau_2 < \dots$ are defined. The nature of the probability \mathbb{P} depends on the application. In risk management applications, \mathbb{P} is the actual or statistical measure. In valuation applications, \mathbb{P} is a risk-neutral pricing measure, relative to which the discounted price of a traded security is a martingale. Since we mainly study the pricing problem, unless otherwise stated, the probability \mathbb{P} used in the rest of this paper is risk-neutral.

The reduced-form approach assumes that the defaults occur unexpectedly. For the credit risk induced by the default, the default stopping time τ is equivalent to counting processes N_t , i.e., $N_t = 1_{\{\tau \leq t\}}$, and we specify the counting processes N_t through a conditional intensity λ_t and unit jump size. The intensity λ_t follows a strictly positive stochastic process that describes the conditional mean arrival rate in the sense that $E[N_{t+\Delta t} - N_t | \mathcal{F}_t] \approx \lambda_t \Delta t$, for small $\Delta t > 0$, which means $\{N_t - \int_0^t \lambda_s ds : t \geq 0\}$ is a local martingale relative to \mathbb{P} . Under the intensity-based model, the process followed by the intensity determines the distribution of stopping time completely.

Moreover, in the reduced-form approach, we denote full filtration $\mathcal{F}_t = \mathcal{H}_t \vee \mathcal{G}_t$, where \mathcal{H}_t is the natural filtration of the default indicator process $N_t = 1_{\{\tau \leq t\}}$ and \mathcal{G}_t denotes the “market filtration”. For the counting process N_t , we make doubly stochastic assumption on its intensity λ_t . Thus N_t is a Cox process, i.e. conditional on \mathcal{G}_t available at time t , for $0 \leq s \leq t$, $P(N_t - N_s = k | \mathcal{F}_s \vee \mathcal{G}_\infty) = \frac{(\Lambda_t - \Lambda_s)^k}{k!} e^{-(\Lambda_t - \Lambda_s)}$, where $\Lambda_t = \int_0^t \lambda_u du$. Specifically, under this setting, we get the probability of the default time: for $0 \leq t \leq T$, we have $P_t(\tau > T) = \hat{E}_t \left[e^{-\int_t^T \lambda_u du} \right]$, where $\hat{E}_t[\cdot] = \hat{E}[\cdot | \mathcal{G}_t]$ is the expectation conditional on the current market information \mathcal{G}_t .

2.2. Underlying asset models

To generalize the model of the risky asset value with jumps, we use a double exponential jump-diffusion process ([Kou, 2002](#); [Kou & Wang, 2003](#)) for the underlying asset of the option. Essentially, it replaces the jump size distribution in Merton's (1976) normal jump-diffusion model by a double exponential distribution. Besides offering an explanation of two empirical phenomena on the risky asset return—the asymmetric leptokurtic feature, and the implied volatility smile, a main

⁴ Actually, [Errasi, Giesecke and Goldberg \(2010\)](#) also propose the similar idea. They treat point processes as risk factors so that the processes describing the state variables are self-exciting and cross-exciting.

⁵ For example, see [Duffie, Pan, and Singleton \(2000\)](#) etc.

advantage of the double exponential distribution is that it leads to analytical solutions for many option pricing problems, while they are difficult under the normal jump-diffusion model. Another main advantage is that it can describe unexpected up and down changes in the underlying asset value, which provides sufficient economic insights about the mechanism of the vulnerable option pricing.

Under the double exponential jump-diffusion model, Kou (2002) has shown that when using a HARA type utility function for a representative agent, the rational-expectations equilibrium price of an asset is given by the expectation of the discounted asset payoff under a risk-neutral measure \mathbb{P} . Under such a risk-neutral measure \mathbb{P} , the value of the underlying asset S_t follows a double exponential jump-diffusion process

$$\frac{dS_t}{S_t} = (r - \varphi)dt + \tilde{\sigma}_1 dW_t^1 + d\left(\sum_{i=1}^{\tilde{N}_t} (e^{Y_i} - 1)\right) - \tilde{\lambda}\xi dt, \quad (2.1)$$

whose solution is obtained by $S_t = e^{Z_t} := \exp\left\{\ln S_0 + \left(r - \varphi - \frac{1}{2}\tilde{\sigma}_1^2 - \tilde{\lambda}\xi\right)t + \tilde{\sigma}_1 W_t^1 + \sum_{i=1}^{\tilde{N}_t} Y_i\right\}$, where

$$Z_t = \ln S_0 + \left(r - \varphi - \frac{1}{2}\tilde{\sigma}_1^2 - \tilde{\lambda}\xi\right)t + \tilde{\sigma}_1 W_t^1 + \sum_{i=1}^{\tilde{N}_t} Y_i \quad (2.2)$$

is the return process of the value of the underlying asset S_t and $Z_0 = \ln S_0$, r the constant risk-free interest rate, φ the total payment rate to the firm's investor, $\tilde{\sigma}_1$ the volatility of the firm's asset value, W_t^1 standard Brownian motion, $\{\tilde{N}_t, t \geq 0\}$ a Poisson process with constant intensity rate $\tilde{\lambda}$, Y_i i.i.d. jumps with double exponential density

$$f_Y(y) = p_1 \eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + p_2 \eta_2 e^{\eta_2 y} 1_{\{y < 0\}}, \quad \eta_1 > 1, \eta_2 > 0,$$

ξ the mean percentage jump size given by

$$\xi = E[e^Y - 1] = \frac{p_1 \eta_1}{\eta_1 - 1} + \frac{p_2 \eta_2}{\eta_2 + 1} - 1,$$

and $p_1, p_2 \geq 0$, $p_1 + p_2 = 1$, represent the upward and downward jump probabilities. We assume that all sources of the randomness $\{W_t^1; t \geq 0\}$, $\{\tilde{N}_t; t \geq 0\}$ and $\{Y_1, Y_2, \dots\}$ are independent. Note that, under the risk-neutral measure \mathbb{P} , S_t is a martingale after proper discounting: $S_t = E[e^{-(r-\varphi)(T-t)S_t} | \mathcal{F}_t]$, where \mathcal{F}_t is the information up to time t .

As we have analyzed before, the jump $J_t := \sum_{i=1}^{\tilde{N}_t} Y_i$ in the underlying asset value can also affect the intensity process λ_t of the counting process N_t . Thus, we treat the jump J_t as a risk factor. And to describe the cross-exciting between the intensity process λ_t and the underlying asset value, we let the dynamic of λ_t satisfy the following stochastic differential equation (SDE)

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \tilde{\sigma}_2 dW_t^2 + \delta dJ_t, \quad (2.3)$$

where κ and θ are parameters characterizing the mean-reverting of the intensity process, volatility $\tilde{\sigma}_2$ is a function of λ_t with $\tilde{\sigma}_2 = \sigma_2 \sqrt{\lambda_t}$ and $\sigma_2 > 0$ is constant, W_t^2 is a standard Brownian motion which is correlated with W_t^1 by

$$dW_t^1 dW_t^2 = \rho dt.$$

Besides, parameter δ reflects the sensitivity of the intensity to the jump in the underlying asset value. Thus, the market risk and credit risk are correlated just not only by the diffusion parts with the static correlation coefficient ρ , but also dynamically correlated by the cross-exciting jumps. Moreover, we assume $\{(W_t^1, W_t^2); t \geq 0\}$ and $\{\tilde{N}_t; t \geq 0\}$ are mutually independent.

2.3. Vulnerable European option pricing model

In this paper, we extend the vulnerable European option pricing model proposed by Hull and White (1995) and Klein (1996), especially our model follows the early default and payoff conditions which are also proposed in Hull and White (1995). Recall that τ is the time when the counterparty defaults, and we denote the option maturity by T , then the payoff of a vulnerable European call option is given as follows

$$C_\tau = (S_T - K)^+ 1_{\{\tau \geq T\}} + \alpha P_\tau(S_T, K) 1_{\{\tau < T\}},$$

where K is the strike price of the option, $P_\tau(S_T, K)$ represents the current value of the option assuming no default happens at time τ , α is the proportional recovery made in the event of default, i.e., only the proportion α of the current value of the option at default time τ is paid out by the counterparty.

By the standard risk-neutral arguments, the price of a vulnerable European option at time 0 is given as

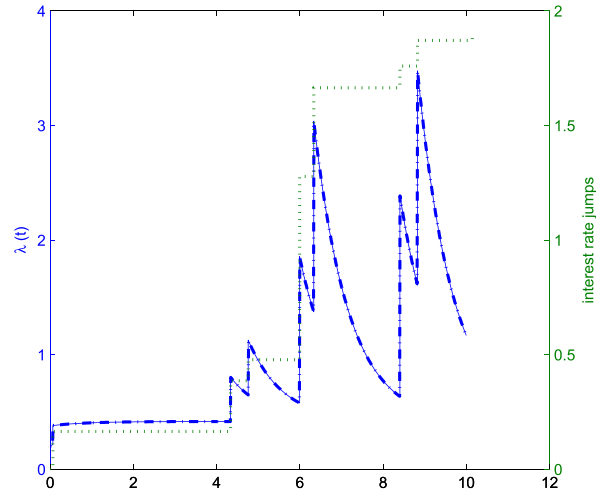


Fig. 1. Sample paths of the specified intensity and the pure jump process. Sample paths of the cumulative jump of jump process J_t and the intensity (2.7) with no other pure jump process, $i = 1$ and $R_t = \kappa(\theta - \lambda_t)$ without the diffusion fluctuation. The reversion rate $\kappa = 1.2$, the reversion level $\theta = 0.5$ and the sensitive parameter $\delta = 1$. J_t is a pure jump process with independently distributed jumps at Poisson times with intensity $\tilde{\lambda} = 2$ and independent jump sizes drawn from an exponential distribution with mean $J = 2$.

$$C_0 = E[e^{-rT}(S_T - K)^+ 1_{\{\tau > T\}}] + E[e^{-r\tau} \alpha P_\tau(S_T, K) 1_{\{\tau \leq T\}}]. \quad (2.4)$$

Since the option price $P_\tau(S_T, K)$ at time τ satisfies $P_\tau(S_T, K) = E_\tau[e^{-r(T-\tau)}(S_T - K)^+]$, substituting this formula into the formula (2.4), we can easily get

$$C_0 = E[e^{-rT}(S_T - K)^+ 1_{\{\tau > T\}}] + \alpha E[e^{-rT}(S_T - K)^+ 1_{\{\tau \leq T\}}]. \quad (2.5)$$

Formula (2.5) implies that the value of the vulnerable option is determined by the underlying asset value at the option maturity T , the default probability at time τ and the correlation between the default event and the underlying asset value. Without jumps, i.e. $\tilde{\lambda} = 0$, equation (2.5) can be reduced to vulnerable option pricing model in Hull and White (1995), and they give the explicit pricing formula with the independent assumption under the reduced-form approach. However, when considering the jumps and the dependent assumption under the reduced-form approach, all the existing methods cannot be directly applied to derive the vulnerable option pricing formula. The main contribution of this paper is that, subject to the early default condition shown in (2.4) and considering dynamic dependence between the market risk and the credit risk, we will obtain the analytical vulnerable option pricing formula under the reduced-form approach with affine jump-diffusion state processes.

2.4. Pricing model with cross-exciting jumps under the reduced-form approach

As mentioned above, our proposed pricing model is based on the reduced-form approach with affine jump-diffusion (AJD) state-process. To introduce our model, we thus first present the AJD state-process framework briefly.⁶

2.4.1. Review of AJD state-vectors

As before, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an information filtration, $\{\mathcal{F}_t : t \geq 0\}$ ⁷ and suppose X is a n -dimensional Markov process of *cadlag* state variables drawn from some state space $D \subset \mathbb{R}^n$. The dynamics of the state vector X follow the stochastic differential equation (SDE)

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t + dM_t. \quad (2.6)$$

⁶ Besides the AJD state-process, the state vector can also be supposed to follow linear-quadratic jump-diffusion (LQJD) processes, which have also been proved to be fruitful in developing tractable option pricing formulae under the reduced-form framework. For example, see Chen, Filipovic, and Poor (2004), Jiang and Yan (2009). Since this paper mainly focuses on how to apply the dynamic correlation structure to price vulnerable options, and the result obtained under the LQJD framework would not have intrinsic difference with which got under the ADJ framework, without generally, we omit the details about LQJD for the space consideration and it is available upon request.

⁷ The filtration \mathcal{F}_t is assumed to be satisfy the usual conditions, and X is assumed to be Markov relative to \mathcal{F}_t . For technical details, see for example, Ethier and Kurtz (1986).

Here, for AJD state-process model, M_t is a pure process whose jumps have a fixed probability distribution $\nu(dy, t)$ on \mathbb{R}^n and arrive with intensity $\hat{\lambda}(t, X_t)$, W_t is a n -dimensional \mathcal{F}_t -standard Brownian motion vector, the drift $\mu : D \rightarrow \mathbb{R}^n$, $\sigma : D \rightarrow \mathbb{R}^{n \times n}$, and the entries of μ , $\hat{\lambda}$ and covariance matrix $\Omega = \sigma(t, X_t)\sigma(t, X_t)^\top$ are all affine in X . To get more knowledge about the affine jump-diffusion, one can check [Cheng and Scaillet \(2007\)](#) and the references therein.

2.4.2. Standard transforms for the state vectors

Recall that $J_t = \sum_{i=1}^{N_t} Y_i$ is the jump component of the underlying asset value S_t . According to the above analysis, J_t itself can be treated as a risk factor. To model the dynamic dependent relationship of the jumps in the underlying asset value and the probabilities of default time of the counterparty, we let the intensity process of the corresponding counting process of the default satisfy the following SDE

$$d\lambda_t = dR_t + \delta dJ_t + \sum_{j=1}^{m-1} \delta_j dM_j(t), \quad (2.7)$$

where R_t is the continuous and diffusion part, and δ is a function of jump cumulative sizes expressing the sensitivity of stopping times to jumps in the underlying asset value. $M_j(t), j = 1, \dots, m-1$, are the other pure jump processes which make the SDE (2.7) be formally similar to equation (2.6) and have unpredicted impacts on the intensity process, here $m > 1$ is a integer number. $\{\delta_j\}_{j=1}^{m-1}$ are sensitive parameters to jumps $\{M_j(t)\}_{j=1}^{m-1}$, respectively. An illustration, a sample path of the specified intensity is provided in [Fig. 1](#), which shows the dynamic dependent relationship between the intensity process and the jumps in the underlying asset value.

Generally, we consider an extension of SDE (2.7) that the state vector X_t is driven by multivariate pure jump processes, whose jump times are either temporally or non-temporally consistent with each other, and X_t solves the following SDEs

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t + \sum_{j=1}^m \delta_j dM_j(t), \quad (2.8)$$

where $M_j(t)$ is a pure jump process whose jumps have a fixed probability distribution ν_j on \mathbb{R}^n and arrive with intensity $\lambda_j(X_t) : t \geq 0$, for some function $\lambda_j : D \rightarrow [0, \infty)$, and $\{\delta_j\}_{j=1}^m$ are n -dimensional diagonal matrixes, their each component is a sensitive parameter to the component of $\{M_j\}_{j=1}^m$.

To price the derivatives with credit risk including vulnerable option, we need the standard transform of the state vector X , which is defined by

$$\psi(g; X_t, t, T) = E_t \left[\exp \left(- \int_t^T R(s, X_s) ds \right) e^{g(T, X_T)} \right], \quad (2.9)$$

where the expectation $E_t[\cdot] = E[\cdot | \mathcal{F}_t]$ is taken conditional on information $\{\mathcal{F}_t, t \leq T < \infty\}$ up to time t , $R(t, X_t)$ and $g(t, X_t)$ are all continuous differentiable functions of t and X_t . If the process $\Psi(t, X_t) = \exp \left(- \int_0^t R(s, X_s) ds \right) e^{g(t, X_t)}$ is a martingale, then $\psi(g; X_t, t, T) = e^{g(t, X_t)}$, and this implies that the function $g(t, X_t)$ satisfies a partial integral differential equation (PIDE) in the following lemma.

Lemma 1. *If the technical integrability conditions hold and the function $g(t, X_t)$ satisfies the following PIDE*

$$\frac{\partial g}{\partial t} + \mu^\top \frac{\partial g}{\partial x} + \frac{1}{2} \text{tr} \left[\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial g}{\partial x} \left(\frac{\partial g}{\partial x} \right)^\top \right) \sigma \sigma^\top \right] + \sum_{j=1}^m \lambda_j (\theta_j - 1) = R, \quad (2.10)$$

where $\theta_j = \int_{\mathbb{R}^n} [e^{g(t, x + \delta_j z) - g(t, x)}] d\nu_j(z)$ whenever the integral is well defined, $j = 1, 2, \dots, m$, then Ψ_t is a martingale.

The Proof method is similar to the methods used in [Errais, Giesecke, and Goldberg \(2010\)](#), and the proof details of this lemma is given in [Appendix A](#).

Remark 2.1. To get the solution to PIDE (2.10) under the AJD framework, interest rate r_t and intensity processes $\{\lambda_j(X_t)\}_{j=1}^m$ should be affine functions of X_t . Hence the term R and g in formula (2.9) are also affine functions of X_t . Denote $\pi = g, R, r$ and $\{\lambda_j(X_t)\}_{j=1}^m$, then $\pi(t, x) = A_\pi(t)^\top x + B_\pi(t)$.

2.4.3. Analytical solution of the standard transform for AJD state-vectors

For a specific case, we will derive an analytical solution of the standard transform when the state vectors are constituted of AJD processes. Based on the equation (2.10) in Lemma 1 proposed above, we can show that the standard transform (2.9) is known as an exponential affine function in closed-form up to the solutions to ordinary differential equations (ODEs).

To achieve this, we first reset the drift μ , intensities $\{\lambda_j\}_{j=1}^m$ and covariance matrix $\Omega = \sigma\sigma^\top$ and fix an affine discount-rate function $R: D \rightarrow \mathbb{R}$. The affine dependence of μ , Ω , $\{\lambda_j\}_{j=1}^m$ and R determined by coefficients $(K, \Omega, \{l_j\}_{j=1}^m, \rho)$ are defined by⁸

$$\begin{aligned} \mu &= K_0 + K_1 X, & K &= (K_0, K_1) \in \mathbb{R}^n \times \mathbb{R}^{n \times n}, \\ (\Omega)_{ij} &= (\Omega_0)_{ij} + (\Omega_1)_{ij} \cdot X, & \Omega &= ((\Omega_0)_{ij}, (\Omega_1)_{ij}) \in \mathbb{R} \times \mathbb{R}^n, i, j = 1, \dots, n, \\ R &= \rho_1(t) \cdot X + \rho_0(t), & \rho &= (\rho_0, \rho_1) \in \mathbb{R} \times \mathbb{R}^n, \\ \lambda_j &= l_{1j}(t) \cdot X + l_{0j}(t), & l_j &= (l_{0j}, l_{1j}) \in \mathbb{R} \times \mathbb{R}^n, j = 1, \dots, m. \end{aligned} \quad (2.11)$$

Substituting the formula (2.11) into the PIDE (2.10) and denote $g_{xx^\top} = \frac{\partial g}{\partial x} \left(\frac{\partial g}{\partial x} \right)^\top$, then the function $g(t, X_t)$ satisfies the following PIDE

$$\frac{\partial g}{\partial t} + (K_0 + K_1 X) \cdot \frac{\partial g}{\partial x} + \frac{1}{2} \text{tr}[\Omega g_{xx^\top}] + \sum_{j=1}^m [l_{1j}(t) \cdot X + l_{0j}(t)] (\theta_j - 1) = R. \quad (2.12)$$

We may solve the above PIDE up to a system of ODEs using the method of undetermined coefficients. To achieve this, we assume that the affine function $g(t, X_t) = \alpha(t) + \beta(t) \cdot X_t$ is the solution to the PIDE (2.12), with terminal condition $g(T, x) = g_0(x)$, here $\beta = (\beta_1, \dots, \beta_n)^\top$. By substituting the partial derivatives of g into (2.12) and collecting the terms with the same powers of the state vector X , note that the PIDE holds for any values of X , then the sum of coefficients of any power of X in (2.12) must be zero. Thus this leads to the following system of complex-valued ODEs

$$\begin{aligned} -\frac{d\alpha(t)}{dt} &= -\rho_0 + K_0 \cdot \beta(t) + \frac{1}{2} \beta(t)^\top \Omega_0 \beta(t) + \sum_{j=1}^m l_{0j} (\theta_j - 1), \\ -\frac{d\beta(t)}{dt} &= -\rho_1 + K_1^\top \beta(t) + \frac{1}{2} \sum_{i,j} \beta_i(t) \beta_j(t) (\Omega_1)_{ij} + \sum_{j=1}^m l_{1j} (\theta_j - 1) \end{aligned} \quad (2.13)$$

Therefore, we can now provide the following Lemma 2 based on the above analysis.

Lemma 2. Suppose that the technical integrability conditions hold, then the standard transform can be given by

$$\begin{aligned} \psi(g; X_t, t, T) &= E_t \left[\exp \left(- \int_t^T R(s, X_s) ds \right) e^{g(T, X_T)} \right] \\ &= e^{g(t, X_t)} = e^{\alpha(t) + \beta(t) \cdot X_t}, \end{aligned} \quad (2.14)$$

where the coefficient functions $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^n$ admit a unique solution through the system of Riccati equation (2.13) with the terminal condition. $g(T, x) = g_0(x)$.

Specifically, to price options, we usually should set $g_0(X_T) = u \cdot X_T$, $u \in \mathbb{C}^n$. Then, for $u \in \mathbb{C}^n$, the set of n – tuples of complex numbers, θ_j is rewritten as

$$\theta_j(u) = \int_{\mathbb{R}^n} \exp(\delta_j u \cdot z) d\nu_j(z) \quad (2.15)$$

whenever the integral is well-defined, these “jump transforms” θ_j determine the jump-size distributions, $j = 1, 2, \dots, m$. Thus $(K, \Omega, \{l_j\}_{j=1}^m, \{\theta_j\}_{j=1}^m, \rho)$ captures both the distribution of X as well as the effects of any discounting, and determines a standard transform $\psi: \mathbb{C}^n \times D \times \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{C}$ conditional on \mathcal{F}_t , when well defined at $t \leq T$, by

$$\psi(u; X_t, t, T) = E_t \left[\exp \left(- \int_t^T R(s, X_s) ds \right) e^{u \cdot X_T} \right]. \quad (2.16)$$

Then based on Lemma 2 we have

⁸ The following expressions can also be expressed by Kronecker product operator of matrices, see Cheng and Scaillet (2007) etc.

$$\psi(u; X_t, t, T) = e^{\alpha(t) + \beta(t) \cdot X_t}, \quad (2.17)$$

and the coefficient functions α and β still satisfies ODEs (2.13), but with boundary conditions $\alpha(T) = 0$ and $\beta(T) = u$.

3. Vulnerable European option pricing

In this section, we start to derive the analytical solution of vulnerable European option pricing formula, based on the above analysis and results incorporating with the reduced-form approach. As an application, we present a closed-form of pricing formula under the specific SDEs which are followed by the state variables.

3.1. Pricing formula

Recall that the price of the vulnerable European call option at time 0 is

$$C_0 = E \left[e^{-rT} (e^{Z_T} - K)^+ 1_{\{\tau > T\}} \right] + \alpha E \left[e^{-rT} (e^{Z_T} - K)^+ 1_{\{\tau \leq T\}} \right], \quad (3.1)$$

where the dynamic of Z_t follows (8). Specifically, in the previous model setting, Z_t is given by (2.2), i.e.,

$$dZ_t = \left(r - \varphi - \frac{1}{2} \tilde{\sigma}_1^2 - \tilde{\lambda} \xi \right) dt + \tilde{\sigma}_1 dW_t^1 + dJ_t, \quad (3.2)$$

Then, we have

$$\begin{aligned} C_0 &= E \left[e^{-rT} (e^{Z_T} - K)^+ 1_{\{\tau \geq T\}} \right] + \alpha E \left[e^{-rT} (e^{Z_T} - K)^+ (1 - 1_{\{\tau > T\}}) \right] = (1 - \alpha) E \left[e^{-rT} (e^{Z_T} - K)^+ 1_{\{\tau > T\}} \right] \\ &+ \alpha E \left[e^{-rT} (e^{Z_T} - K)^+ \right] \quad := C_1 + C_2, \end{aligned} \quad (3.3)$$

where

$$\begin{aligned} C_1 &= (1 - \alpha) E \left[E \left[e^{-rT} (S_T - K)^+ 1_{\{\tau > T\}} \middle| \mathcal{H}_0 \vee \mathcal{G}_\infty \right] \right] \\ &= (1 - \alpha) E \left[e^{-rT} e^{-\int_0^T \lambda_t dt} (e^{Z_T} - K)^+ \right], \end{aligned} \quad (3.4)$$

and C_2 can be directly given by the Black-Scholes formula. Hence it is just only necessary to derive the formula of C_1 to get the pricing formula of vulnerable European call option analytically.

We let the AJD vector $X_t = (Z_t, \lambda_t)^\top$ with initial value $X_0 = (Z_0, \lambda_0)$, where λ_t is also governed by (2.8). Thus, we have

$$\begin{aligned} C_1 &= (1 - \alpha) E \left[e^{-\int_0^T (r + \lambda_t) dt} (e^{Z_T} - K)^+ \right] \\ &= (1 - \alpha) e^{-rT} \Phi(u, v, X_0, 0, T) \\ &= (1 - \alpha) E \left[\exp \left(-\int_0^T (r + \lambda_t) dt \right) (e^{Z_T} - K) 1_{\{Z_T \geq \ln K\}} \right] \\ &= (1 - \alpha) E \left[\exp \left(-\int_0^T R(t, X_t) dt \right) (e^{u \cdot X_T} - v) 1_{\{u \cdot X_T \geq \ln(v)\}} \right], \end{aligned} \quad (3.5)$$

where we denote u and v by $u = (1, 0)^\top$ and $v = K$ respectively, and $R(X_t) = r + \lambda_t = \rho_0 + \rho_1 \cdot X_t$ with $\rho_0 = r$ and $\rho_1 = (0, 1)^\top$.

To pricing the formula (3.5), it calls for the calculation of expected presented value of the option at maturity $\Phi : \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S} \times \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ defined as follows

$$\Phi(u, v, X_t, t, T) = E_t \left[\exp \left(- \int_t^T R(s, X_s) ds \right) (e^{u \cdot X_T} - v)^+ \right]. \quad (3.6)$$

For the above equality, we have

$$\begin{aligned} \Phi(u, v, X_t, t, T) &= E_t \left[\exp \left(- \int_t^T R(s, X_s) ds \right) (e^{u \cdot X_T} - v) 1_{\{u \cdot X_T \geq \ln(v)\}} \right] \\ &= G_{u, -u}(-\ln(v); X_t, t, T) - v G_{0, -u}(-\ln(v); X_t, t, T), \end{aligned} \quad (3.7)$$

where given some $(x, T, a, b) \in \mathcal{S} \times [0, \infty)$, $G_{a,b}(\cdot; x, t, T) : \mathbb{R} \rightarrow \mathbb{R}^+$ is given by

$$G_{a,b}(y; X_t, t, T) = E_t \left[\exp \left(- \int_t^T R(s, X_s) ds \right) e^{a \cdot X_T} 1_{\{b \cdot X_T \leq y\}} \right]. \quad (3.8)$$

Thus we can rewrite (3.5) given the formulae (3.7) and (3.8) as follows

$$C_1 = (1 - \alpha) [G_{u, -u}(-\ln(v); X_0, 0, T) - v G_{0, -u}(-\ln(v); X_0, 0, T)]. \quad (3.9)$$

Note that based on the results on the Lévy inversion formula⁹ (from the typical case of a proper cumulative distribution function), we could extend the Lévy inversion formula to obtain the Proposition 1 proposed in Appendix B.¹⁰ Then the functions $G_{u, -u}$ and $G_{0, -u}$ in (26) can be analytically provided by taking advantage of the formula (5.2) in Proposition 1, i.e.,

$$G_{u, -u}(-\ln(v); X_0, 0, T) = \frac{\psi(u, X_0, 0, T)}{2} - \frac{1}{\pi} \int_0^\infty \frac{\text{Im}[\psi(u - iku, X_0, 0, T) e^{ik \ln(v)}]}{k} dk \quad (3.10)$$

and

$$G_{0, -u}(-\ln(v); X_0, 0, T) = \frac{\psi(0, X_0, 0, T)}{2} - \frac{1}{\pi} \int_0^\infty \frac{\text{Im}[\psi(-iku, X_0, 0, T) e^{ik \ln(v)}]}{k} dk, \quad (3.11)$$

where $\psi(a + ikb, X_t, t, T)$ is the Fourier-Stieltjes transform of $G_{a,b}(\cdot; x, t, T)$ given by

$$\psi(a + ikb, X_t, t, T) = \int_{\mathbb{R}} e^{iky} dG_{a,b}(y; x, t, T) = E_t \left[\exp \left(- \int_t^T R(s, X_s) ds \right) \exp((a + ikb) \cdot X_T) \right].$$

According to Lemma 2, we can obtain the functions $G_{u, -u}(-\ln(v); X_0, 0, T)$ and $G_{0, -u}(-\ln(v); X_0, 0, T)$ by solving ODEs (2.13) and (3.9), which determine the functions ψ . Therefore, we would obtain the pricing formula of vulnerable European option.

Based on the above calculations and analysis, The following theorem provides an analytical pricing formula of vulnerable European call option when its risk factors are cross-exciting, which is the main result of this paper.

Theorem 1. Suppose that the dynamics of the value of the underlying asset and the intensity of the counting process corresponding to the default time are both affine jump-diffusion processes, which are driven by (2.8). Then the pricing formula of the vulnerable European call option at time 0, C_0 satisfies

$$\begin{aligned} C_0 &= C_1 + C_2 \\ &= (1 - \alpha) S_0 E \left[e^{-rT} \left(e^{Z_T} - \frac{K}{S_0} \right)^+ 1_{\{\tau > T\}} \right] + \alpha E \left[e^{-rT} (S_T - K)^+ \right], \end{aligned} \quad (3.12)$$

⁹ See, for example, Gil-Pelaez (1951) and Williams (1991) for a treatment of the Lévy inversion formula.

¹⁰ Duffie et al. (2000) also provide a similar proposition.

where C_1 is given by the formulae (3.9)–(3.11) with the function ψ solved through Lemma 2, and $C_2 = \alpha E[e^{-rT}(S_T - K)^+]$ can be directly given by Black-Scholes formula in the jump model¹¹

$$C_2 = \alpha \sum_{m=0}^{\infty} e^{-\tilde{\lambda}T} \frac{\tilde{\lambda}^m T^m}{m!} E \left[f \left(0, S_0 e^{-\tilde{\lambda}\xi T + \sum_{i=1}^m Y_i} \right) \right] \quad (3.13)$$

where $f(t, x) = xN(d_+(t, x)) - Ke^{-r(T-t)}N(d_-(t, x))$, with $d_{\pm}(t, x) = \frac{\ln\left(\frac{x}{K}\right) + \left(r - \varphi \pm \frac{1}{2}\sigma_1^2\right)(T-t)}{\sigma_1\sqrt{T-t}}$ and $N(y) = \int_{-\infty}^y \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$ is the cumulative standard normal distribution function.

Remark 3.1. For the vulnerable European put option, we can also get an analytical pricing formula with the similar analysis and calculation.

Note that the above pricing formula C_1 is derived formally, and $X_t = (Z_t, \lambda_t)$ is the general affine jump-diffusion state vector. To illustrate how to apply our model to price vulnerable European options, we make a specification on the state vector X as a concrete example, and thus propose a closed-form pricing formula in the next subsection.

3.2. Pricing formula with a specific cross-exciting intensity process

We consider a case that the intensity of \tilde{N}_t in J_t is a constant $\tilde{\lambda} > 0$. We simply set the risk-free interest rate r to be constant.¹² Except the pure jump process $J_t = \sum_{i=1}^{N_t} Y_i$ given in (2.2), there is no any other pure jump driving the intensity process λ_t , i.e., $j = 1, M_t = (J_t, J_t)^T$ and the intensity of M_t is $\lambda_1 = \tilde{\lambda}$. Thus, we let the state vector $X = (Z_t, \lambda_t)$. The dynamics of the state vectors X are governed by the following processes:

$$\begin{aligned} dZ_t &= \hat{\mu}dt + \sigma_1 \sqrt{\lambda_t} dW_t^1 + dJ_t, \\ d\lambda_t &= \kappa(c - \lambda_t)dt + \sigma_2 \sqrt{\lambda_t} dW_t^2 + \delta dJ_t, \end{aligned}$$

where $\hat{\mu} = \bar{\mu} - \frac{1}{2}\sigma_1^2\lambda_t$ and $\bar{\mu} := r - \varphi - \tilde{\lambda}\xi$, W^1 and W^2 are standard Brownian motions with correlation coefficient ρ , i.e., $dW_t^1 dW_t^2 = \rho dt$, δ is a sensitive parameter and it could be a function of time. While in this example, we simply assume that δ is a constant parameter, for instance $\delta = 1$. The above processes indicate that the intensity λ_t is strictly positive, and the market risk and credit risk in vulnerable options are cross-excited, since both of the two processes are affected by each other.

To take advantage of Theorem 1, we rewrite the above SDEs as follows:

$$dX_t = \mu_X dt + \sigma_X d\overrightarrow{W}_t + \zeta dM_t, \quad (3.14)$$

where $\mu_X = K_0 + K_1 X = \begin{pmatrix} \bar{\mu} \\ \kappa c \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{2}\sigma_1^2 \\ 0 & -\kappa \end{pmatrix} \begin{pmatrix} Z_t \\ \lambda_t \end{pmatrix}$, $\sigma_X = \sqrt{\lambda_t} \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix}$, $\overrightarrow{W}_t = (W^1, W^2)^T$, $\zeta = \text{diag}(1, \delta)$ is a

sensitive parameter matrix and the pure jump process $M_t = (J_t, J_t)^T$. The covariance matrix is $\sigma\sigma^T = \lambda_t \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$. As

mentioned above, we let $R(t, X_t)$ in formula (2.9) be given by $R(X_t) = r + \lambda_t = \rho_0 + \rho_1 \cdot X_t$ with $\rho_0 = r$ and $\rho_1 = (0, 1)^T$, and the intensity of M_t , i.e., λ_1 is given by $\lambda_1 = \tilde{\lambda} = l_0 + l_1 \cdot X_t$ with $l_0 = \tilde{\lambda}$ and $l_1 = (0, 0)^T$.

To make the formula (26) be tractable, we have to calculate $\psi(\tilde{u}, X_t, t, T) = E_t \left[e^{-\int_t^T (r+\lambda_s) ds} e^{g(T, X_T)} \right]$, where $g(t, X_t) = \alpha(t) + \beta(t) \cdot X_t$ with the terminal condition $g(T, X_T) = \tilde{u} \cdot X_T$ and $\tilde{u} = (u, 0)^T \in \mathbb{C}^2$, i.e., $\alpha(T) = 0$ and $\beta(T) = (u, 0)^T$.

Thus, by the result of Lemma 2 we have

$$\psi(u, X_t, t, T) = \psi(\tilde{u}, X_t, t, T) = e^{\alpha(t) + \beta(t) \cdot X_t}, \quad (3.15)$$

where $\alpha(t), \beta(t)$ are the solutions to the following system of Riccati equations

¹¹ We omit the Proof here, one can refer to Shreve (2004, Chapter 11, Theorem 11.7.5) for the details of deriving the formula (3.13).

¹² If the risk-free interest rate is driven by a affine jump-diffusion process, in which the jump part affects other processes, then we could also get the vulnerable option pricing formula analytically by using our general pricing model. Moreover, this model could reflect much more realistic business situation and is more complex in calculation and analysis. For the space consideration, we omit the detail of deriving the pricing formula under this case and it is available upon request.

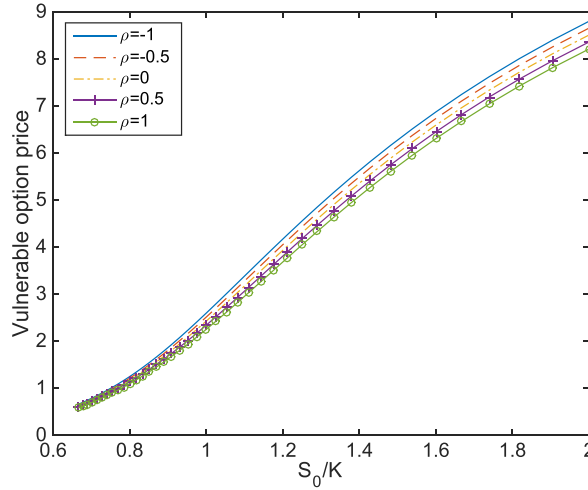


Fig. 2. Vulnerable European call option price against spot-to-strike ratio for different correlations. This figure uses the basic parameters listed in Table 1. The solid, dashed, dash-dotted, solid with plus, and solid with circle lines correspond to $\rho = -1$, $\rho = -0.5$, $\rho = 0$, $\rho = 0.5$ and $\rho = 1$, respectively.

$$\begin{aligned}
 -\frac{d\alpha(t)}{dt} &= -r + \hat{\mu}\beta_1(t) + \kappa\beta_2(t) + \tilde{\lambda} \left(\int_{\mathbb{R}^2} e^{(\zeta\beta(t))^\top z} dv(z) - 1 \right), -\frac{d\beta(t)}{dt} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} + (0 - \kappa\beta_2(t)) \\
 &+ \frac{1}{2} \left[\beta_1^2 \begin{pmatrix} 0 \\ \sigma_1^2 \end{pmatrix} + \beta_2^2 \begin{pmatrix} 0 \\ \sigma_2^2 \end{pmatrix} + 2\beta_1\beta_2 \begin{pmatrix} 0 \\ \rho\sigma_1\sigma_2 \end{pmatrix} \right],
 \end{aligned} \tag{3.16}$$

with boundary conditions $\beta(T) = (u, 0)^\top$ and $\alpha(T) = 0$.

It is obvious to find out that $\beta_1(t) = u$, for any $t \geq 0$. Thus, letting $\tau = T - t$, both the functions α and β_2 are dependent on u and τ , i.e., $\alpha(t) = \alpha(\tau, u)$ and $\beta(t) = \beta(\tau, u)$. Then the above ODEs can be rewritten as follows:

$$\begin{aligned}
 \frac{d\alpha(\tau, u)}{d\tau} &= (r - \hat{\mu}u) - \kappa\beta_2(\tau) - \tilde{\lambda} \left(\int_{\mathbb{R}^2} \exp\{(u, \delta\beta_2(\tau)) \cdot z\} dv(z) - 1 \right) \frac{d\beta_2(\tau, u)}{d\tau} = \left(1 - \frac{1}{2}u^2\sigma_1^2 \right) \\
 &- (\rho u\sigma_1\sigma_2 - \kappa)\beta_2(\tau) - \frac{1}{2}\sigma_2^2\beta_2^2,
 \end{aligned} \tag{3.17}$$

subject to the boundary conditions $\alpha(T) = 0$ and $\beta_2(T) = 0$. The transform of ψ of the log-price state variable Z_t should be rewritten as

$$\psi(u, X_t, t, T) = e^{\alpha(\tau, u) + uZ_t + \beta_2(\tau, u)\lambda_t}. \tag{3.18}$$

It is easy to solve the above Riccati equations with closed-form solutions subject to the boundary conditions $\alpha(T) = 0$ and $\beta_2(T) = 0$. By the Lemma 3 in Appendix C, the solution of $\beta_2(t)$ is given as follows,

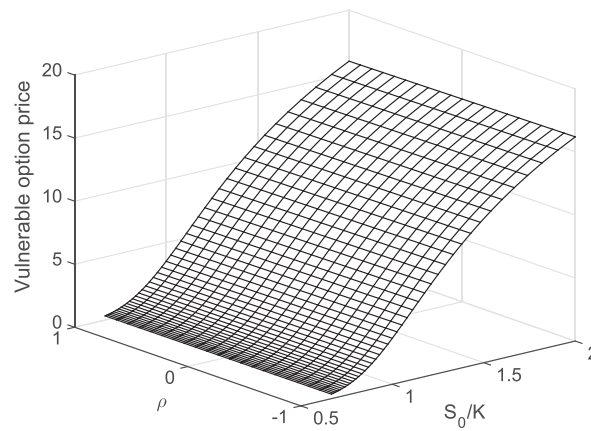


Fig. 3. Vulnerable European call option price against spot-to-strike ratio and correlation. This figure uses the basic parameters listed in Table 1. Here, spot-to-strike ratio S_0/K changes from 0.6 to 2.0, and correlation coefficient ρ changes from -1 to 1 .

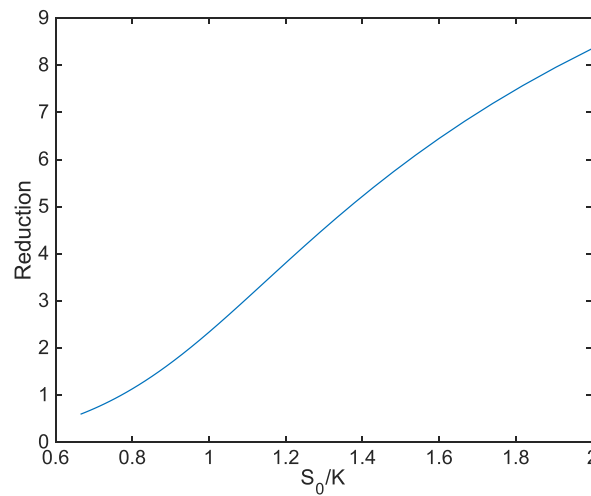


Fig. 4. Influence of spot-to-strike ratio to the reduction in Vulnerable European call option price. This figure uses the basic parameters listed in Table 1.

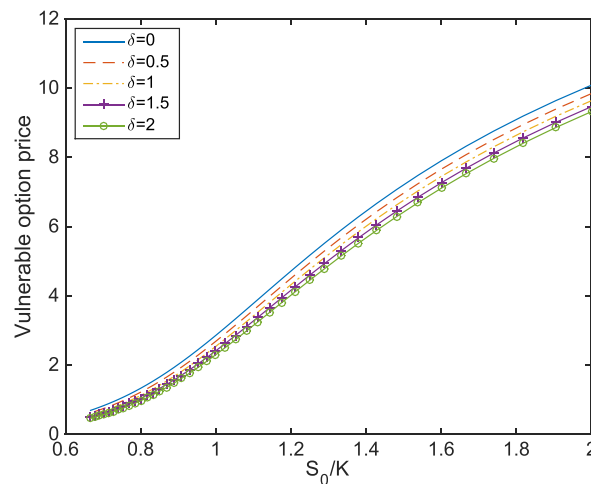


Fig. 5. Vulnerable European call option price against spot-to-strike ratio for different sensitivity of intensity. This figure uses the basic parameters listed in Table 1. The solid, dashed, dash-dotted, solid with plus, and solid with circle lines correspond to $\delta = 0$, $\delta = 0.5$, $\delta = 1$, $\delta = 1.5$ and $\delta = 2$, respectively.

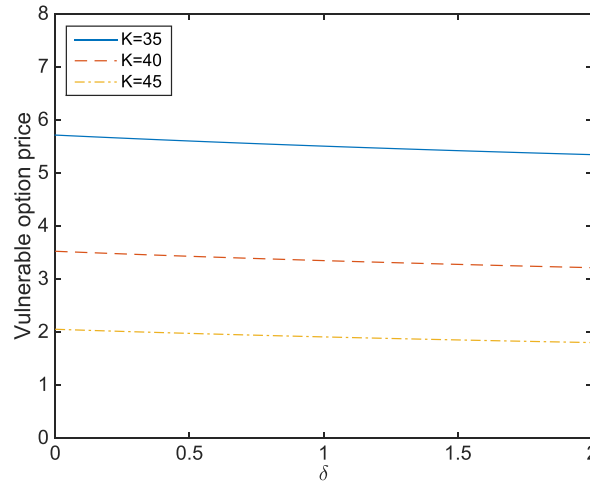


Fig. 6. Vulnerable European call option price against sensitivity of intensity δ for different strike prices. This figure uses the basic parameters listed in Table 1. The solid, dashed, and dash-dotted lines correspond to $K = 35$, $K = 40$, and $K = 45$, respectively.

$$\beta_2(\tau, u) = \frac{g}{\sigma_2^2} + \frac{\gamma}{\sigma_2^2} \tanh\left(\frac{-\gamma\tau}{2} + q\right), \quad (3.19)$$

where $\gamma = \sqrt{\kappa^2 + (\rho^2 - 1)u^2\sigma_1^2\sigma_2^2 + 2\sigma_2^2 - 2\rho u\kappa\sigma_1\sigma_2}$,¹³ $g = \kappa - \rho u\sigma_1\sigma_2$, $q = \frac{1}{2} \ln\left(\frac{\gamma - g}{\gamma + g}\right)$, and the function $\tanh(\cdot)$ is defined by $\tanh(x) \equiv \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$. Moreover, letting $a = 2 - u^2\sigma_1^2$, $b = \rho\sigma_1\sigma_2u - \kappa$, then $\gamma = \sqrt{a\sigma_2^2 + b^2}$, and the formula of $\beta_2(\tau, u)$ (3.19) can be rewritten by

$$\beta_2(\tau, u) = \frac{a(e^{-\gamma\tau} - 1)}{2\gamma - (b + \gamma)(1 - e^{-\gamma\tau})}. \quad (3.20)$$

Thus, we can get the analytical formula of $\alpha(t)$ by substituting the above closed-form solution (3.19) or (3.20) into the first equation of (3.17), and we have¹⁴

$$\begin{aligned} \alpha(\tau, u) &= \alpha(T) + (r - \hat{\mu})(t - T) + \kappa c \int_0^\tau \beta_2(s, u) ds + \int_0^\tau \tilde{\lambda} \cdot (\theta(u, \beta_2(s, u)) - 1) ds = (\hat{\mu} - r)\tau + \kappa c \int_0^\tau \beta_2(s, u) ds \\ &+ \int_0^\tau \tilde{\lambda} \cdot (\theta(u, \beta_2(s, u)) - 1) ds = (\hat{\mu} - r)\tau + \kappa c \frac{g}{\sigma_2^2} \tau + \kappa c \frac{2}{\sigma_2^2} \ln\left(\cosh\left(\frac{-\gamma s}{2} + q\right)\right) \Big|_{s=0}^{\tau} \\ &+ \int_0^\tau \tilde{\lambda} \cdot (\theta(u, \beta_2(s, u)) - 1) ds = (\hat{\mu} - r)\tau + \kappa c \frac{g}{\sigma_2^2} \tau + \frac{2\kappa c}{\sigma_2^2} \ln \frac{\cosh(q)}{\cosh\left(\frac{-\gamma\tau}{2} + q\right)} + \int_0^\tau \tilde{\lambda} \cdot (\theta(u, \beta_2(s, u)) - 1) ds. \end{aligned}$$

Since the function $\cosh(x) = \frac{e^x + e^{-x}}{2}$, the above function $\alpha(\tau, u)$ can be rewritten by

$$\alpha(\tau, u) = \left(\hat{\mu} - r - \frac{\kappa c a}{\gamma - b}\right)\tau - \frac{2\kappa c}{\sigma_2^2} \ln\left(1 - \frac{\gamma + b}{2\gamma}(1 - e^{-\gamma\tau})\right) + \int_0^\tau \tilde{\lambda} \cdot (\theta(1, \beta_2(s, u)) - 1) ds. \quad (3.21)$$

Recall that $\theta(u, \beta_2(\tau, u)) = \int \exp\{u, \delta\beta_2(\tau, u) \cdot y\} d\nu(y)$, and $\nu(dy)$ is the fixed probability distribution of a two-dimensional vector of jump processes $M_t = (J_t, J_t)$ with arrival intensity $\tilde{\lambda}$. As the specific jump process used in Kou and

¹³ To be more precise, $\gamma^2 = |\gamma^2|^{1/2} \exp(i \arg(\gamma^2)/2)$, where $\gamma^2 = \kappa^2 + (\rho^2 - 1)u^2\sigma_1^2\sigma_2^2 + 2\sigma_2^2 - 2\rho u\kappa\sigma_1\sigma_2$. Note that for any $z \in \mathbb{C}$, $\arg(z)$ is defined such that $z = |z| \exp(i \arg(z))$, with $-\pi \leq \arg(z) \leq \pi$.

¹⁴ For any $z \in \mathbb{C}$, $\ln(z) = \ln|z| + i \arg(z)$, as define on the "principal branch".

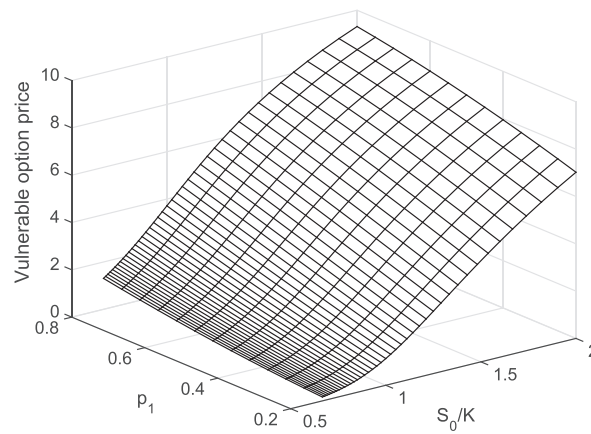


Fig. 7. Vulnerable European call option price against upward jump probability and spot-to-strike ratio. This figure uses the basic parameters listed in Table 1. Here, upward jump probability p_1 changes from 20% to 80%, and spot-to-strike ratio S_0/K changes from 0.6 to 2.0.

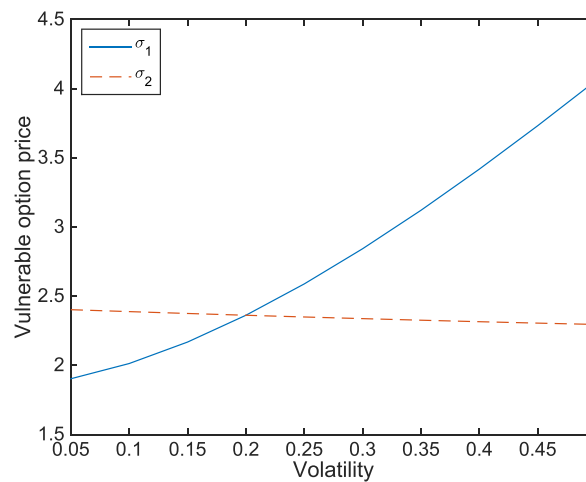


Fig. 8. Vulnerable European call option price against volatility. This figure uses the basic parameters listed in Table 1. The solid, and dashed lines correspond to σ_1 and σ_2 , respectively.

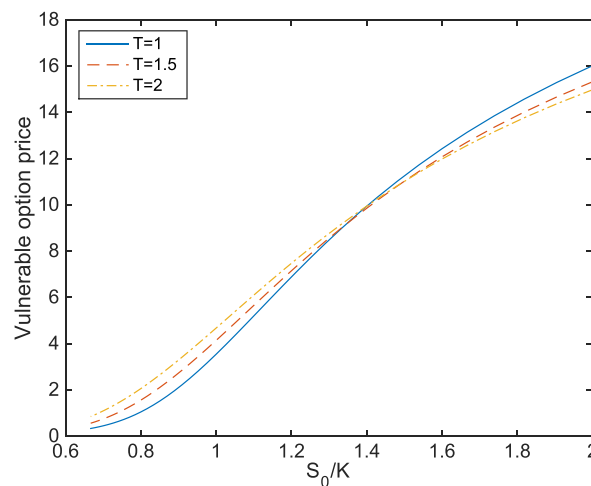


Fig. 9. Vulnerable European call option price against spot-to-strike ratio for different maturities. This figure uses the basic parameters listed in Table 1. The solid, dashed, and dash-dotted lines correspond to $T = 1$, $T = 1.5$, and $T = 2$, respectively.

Table 1
Baseline parameters for numerical illustration.

Parameter	Value	Parameter	Value
Volatility of S	$\sigma_1 = 0.2$	Volatility of intensity process	$\sigma_2 = 0.3$
Risk-free interest rate	$r = 0.058$	Payment rate	$\varphi = 0.02$
Initial price	$S_0 = 40$	Strike price	$K = 35$
Maturity	$T = 1$	Deadweight cost of bankruptcy	$\alpha = 0.6$
Correlation coefficient	$\rho = 0.6$	Mean jump size of S	$\xi = -0.0388$
Annual jump intensity of \tilde{N}_t in J_t	$\lambda = 2.0$	Sensitivity of intensity	$\delta = 1.0$
Reversal rate	$\kappa = 1.1$	Reversal level of intensity	$c = 0.5$
Upward jump probability	$p_1 = 0.5$	Downward jump probability	$p_2 = 0.5$
Mean of upward jump size	$\frac{p_1}{\eta_1} = \frac{1}{6.8}$	Mean of downward jump size	$\frac{p_2}{\eta_2} = \frac{1}{3}$

Wang (2003), and Chen and Kou (2009), we could assume that the jumps size of Y in J_t has a double exponential density¹⁵ $f_Y(y) = p_1 \eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + p_2 \eta_2 e^{\eta_2 y} 1_{\{y < 0\}}$, where $p_1, p_2 \geq 0$ are constants, $p_1 + p_2 = 1$, and $\eta_1 > 0, \eta_2 > 0$. Note that the means of the two exponential distributions are $\mu_1 = 1/\eta_1$ and $\mu_2 = 1/\eta_2$ respectively, then for any $c \in \mathbb{C}$, the moment generating function of jump size Y is given by. $E[e^{cY}] = \frac{p_1 \eta_1}{\eta_1 - c} + \frac{p_2 \eta_2}{\eta_2 - c} = \frac{p_1}{1 - \mu_1 c} + \frac{p_2}{1 + \mu_2 c}$.

Based the above results, we can get $\theta(u, \beta_2(\tau, u))$ through the direct calculation and have

$$\begin{aligned} \theta(u, \beta_2(\tau, u)) &= \frac{p_1}{1 - \mu_1(u + \delta \beta_2(\tau, u))} + \frac{p_2}{1 + \mu_2(u + \delta \beta_2(\tau, u))} \\ &= \frac{p_1}{1 - \mu_1 \left(u + \frac{\delta a(e^{-\gamma\tau} - 1)}{2\gamma - (b + \gamma)(1 - e^{-\gamma\tau})} \right)} + \frac{p_2}{1 + \mu_2 \left(u + \frac{\delta a(e^{-\gamma\tau} - 1)}{2\gamma - (b + \gamma)(1 - e^{-\gamma\tau})} \right)}. \end{aligned} \quad (3.22)$$

Therefore, we could get that the following formulae

$$\int_0^\tau \theta(u, \beta_2(s, u)) ds = p_1 F_1(u, \tau) + p_2 F_2(u, \tau), \quad (3.23)$$

with $F_1(u, \tau) = \frac{(\gamma-b)\tau}{(\gamma-b+\delta\mu_1 a)-\mu_1 u(\gamma-b)} - \frac{2\delta\mu_1 a}{f_{11}(u)} \ln \left(1 - \frac{f_{11}(u)}{2\gamma f_{12}(u)} (1 - e^{-\gamma\tau}) \right)$, and

$F_2(u, \tau) = \frac{(\gamma-b)\tau}{(\gamma-b-\delta\mu_2 a)+\mu_2 u(\gamma-b)} + \frac{2\delta\mu_2 a}{f_{21}(u)} \ln \left(1 - \frac{f_{21}(u)}{2\gamma f_{22}(u)} (1 - e^{-\gamma\tau}) \right)$, where the functions $f_{11}(u), f_{12}(u), f_{21}(u)$, and $f_{22}(u)$ are given as follows $f_{11}(u) = \gamma^2 - (b - \delta\mu_1 a)^2 + \mu_1^2 u^2 (\gamma^2 - b^2) - 2(\gamma^2 - b^2)\mu_1 u - 2\mu_1^2 abu\delta$, $f_{12}(u) = \gamma - b + \delta\mu_1 a + \mu_1^2 u^2 (\gamma - b) - 2\mu_1 (\gamma - b)u - \mu_1^2 au\delta$, $f_{21}(u) = \gamma^2 - (b + \delta\mu_2 a)^2 + \mu_2^2 u^2 (\gamma^2 - b^2) + 2(\gamma^2 - b^2)\mu_2 u - 2\mu_2^2 abu\delta$, $f_{22}(u) = \gamma - b - \delta\mu_2 a + \mu_2^2 u^2 (\gamma - b) + 2\mu_2 (\gamma - b)u - \mu_2^2 au\delta$.

Substituting the above formulae (3.23) into (3.21), then $\alpha(\tau, u)$ can be solved analytically through the direct calculation. Therefore, we get the closed-form of $\psi(u, X_t, t, T)$, i.e., the formula (3.15). Applying the above results in the formulae given in Theorem 1 for the special functions $\psi(1, X_0, 0, T)$, $\psi(0, X_0, 0, T)$, $\psi(1 - ik, X_0, 0, T)$ and $\psi(-ik, X_0, 0, T)$, we can obtain the analytical pricing formula of the vulnerable European option with dynamic correlated risk factors.

Remark 3.1. As a simple example, if the parameter $u = 0$ and $t = 0$, which corresponds to the function $\psi(0, X_0, 0, T)$, we can derive the more simple expressions about the formulae (3.22) and (3.23). In fact, in this case and suppose $\delta = 1$, (3.22) is given as follows $\theta(u, \beta_2(\tau, u)) = \frac{p_1}{1 + \frac{\mu_1 a(1 - e^{-\gamma\tau})}{2\gamma - (b + \gamma)(1 - e^{-\gamma\tau})}} + \frac{p_2}{1 - \frac{\mu_2 a(1 - e^{-\gamma\tau})}{2\gamma - (b + \gamma)(1 - e^{-\gamma\tau})}}$, and the formula (3.23) is $\int_0^\tau \theta(u, \beta_2(s, u)) ds = p_1 f_1(u, \tau) + p_2 f_2(u, \tau)$, where

$$f_1(u, \tau) = \frac{\gamma - b}{\gamma - b + a\mu_1} - \frac{2a\mu_1}{\gamma^2 - (b - a\mu_1)^2} \ln \left(1 - \frac{(\gamma + b) - a\mu_1}{2\gamma} (1 - e^{-\gamma\tau}) \right),$$

and

¹⁵ Besides the double exponentially distributed jump size, we could also assume the jump size has normal density with mean μ_Y and variance σ_Y^2 , or an exponential density with mean μ_e . Here we adopt the double exponential distribution of jump size for analytical tractable in calculation, and it can also reflect more realistic in the up and down of the asset values.

Table 2
Sensitivity analysis for some key parameters.

Parameter	Effect on option prices	Parameter	Effect on option prices
S_0	Inverted S-shaped	T	?
K	–	ξ	+
δ	–	σ_2	–
σ_1	+		
ρ	–		

$$f_2(u, \tau) = \frac{\gamma - b}{\gamma - b - a\mu_2} + \frac{2a\mu_2}{\gamma^2 - (b + a\mu_2)^2} \ln \left(1 - \frac{(\gamma + b) + a\mu_2}{2\gamma} (1 - e^{-\gamma\tau}) \right).$$

4. Numerical results

In this section, for the theoretical results derived in Subsection 4.2, we present numerical results of the vulnerable European call option price under reduced-form credit risk model with correlated risk factors, including jump risk in the underlying asset value and the intensity of the counting process describing the occurrence of credit risk of the counterparty. Since the functions $\psi(1, X_0, 0, T)$, $\psi(0, X_0, 0, T)$, $\psi(1 - ik, X_0, 0, T)$ and $\psi(-ik, X_0, 0, T)$ in pricing formula (3.12) and (3.13) are analytically given, we could provide the numerical pricing via Laplace inversion based on the results in Proposition 1. Since almost formulae in our pricing model are analytical with simple functions, the speed and efficiency of the calculation algorithm with Laplace inversion would be fast and high.

We mainly discuss the effects of some basic parameters on vulnerable European options (see Figs. 2–9), including time to maturity, correlation, spot-to-strike ratio, sensitivity of intensity, volatility, upward(downward) jump probability. To facilitate our analysis, we use the set of baseline parameters given in Table 1.

We just choose these parameters to be broadly consistent with those used in the literatures to calibrate standard credit risk models and option models based on real financial data, such as Chen and Kou (2009), He and Xiong (2012), Tian et al. (2014). Preference parameters listed in Table 1 represent a typical business situation. In the basic case, the vulnerable option is at the money, and is written by a highly leveraged firm. Time to maturity T is assumed to be one year. The risk-free interest rate r is assumed to be constant and we let $r = 0.058$, and payment rate to the investors is also constant with $\varphi = 0.02$. The market value of the asset underlying the option and the intensity process corresponding to default are correlated with instantaneous correlation coefficient $\rho = 0.6$, and the deadweight cost of bankruptcy $\alpha = 0.6$. We also assume that the constant intensity of \tilde{N}_t in J_t is given by $\tilde{\lambda} = 2$, which means that the shocks to the underlying asset value happen two times one year. Following the volatility set in the above papers, we assume that the volatility of risky asset values $\sigma_1 = 0.2$, and the volatility of the intensity process $\sigma_2 = 0.3$. Besides, both the parameters describing the jumps in the underlying asset values including mean jump size, upward(downward) jump probability, and the parameters describing the intensity process are also given in Table 1.

By taking the parameters given in Table 1, Figs. 2 and 3 show how the vulnerable option prices behave against correlation coefficient ρ and spot-to-strike ratio S_0/K . In Fig. 2, the vulnerable option prices change as spot-to-strike ratio increases from 0.6 to 1.6 for five specific correlation coefficients $\rho = -1, \rho = -0.5, \rho = 0, \rho = 0.5$ and $\rho = 1$. From Figs. 2 and 3 we could find that, the European vulnerable call option price increases as S_0/K increases, and decrease as ρ increases. These means that the vulnerable option price is positively related to spot-to-strike ratio, and negatively related to correlation coefficient, which is consistent with the intuition. In fact, ρ describes the degree of consistency between option's value and the default of counterparty. Thus, for the bigger correlation coefficient, when the amplitude of the deep in-the-money gets more large, the default probability would also get larger correspondingly, which reduces the option's value.

On the other hand, both of Figs. 2 and 3 imply that the vulnerable option price would decrease as spot-to-strike ratio S_0/K decreases. This numerical result based on our pricing model is consistent with the theory of options, which shows that reducing the underlying asset value or increasing the strike price would reduce the option price. In Fig. 4, reduction in European vulnerable option price, i.e., the difference between the vanilla option price and the vulnerable option price, becomes larger as the spot-to-strike ratio increases. It means that the reduction amplitude of the value of European vulnerable call option, i.e., the reduction of the value of European call option price caused by the credit risk of the counterparty, would increase as spot-to-strike ratio increases, since the occurrence of default is more possible when the option is more deep in-the-money so that the reduction in the option price would be more large. Hence, Fig. 4 further supports the analysis result derived from Figs. 2 and 3.

Since our pricing model considers the cross-exciting between the underlying asset value process and the intensity process corresponding to the credit risk λ_t , and sensitivity of intensity δ reflects how the jumps in the underlying asset value affect the intensity process, we should assess the impact of δ on the vulnerable option prices. Fig. 5 shows vulnerable European call option price against spot-to-strike ratio for different sensitivity of intensity, which are given by $\delta = 0, \delta = 0.5, \delta = 1, \delta = 1.5$ and $\delta = 2$, respectively. The figure demonstrates that the vulnerable call option price decreases as the δ increases. Intuitively, when $\delta > 0$ is larger, the jumps in intensity process would become larger so that the intensity gets larger, thus the default probability would be higher which makes the option price be smaller to compensate the credit risk of the counterparty.

Moreover, Fig. 6 shows vulnerable call option price against sensitivity of intensity δ with three different strike prices $K = 35$, $K = 40$, and $K = 45$, respectively. From Fig. 6, we could also find that the vulnerable call option prices are negatively related sensitivity of intensity and strike price. In Figs. 5 and 6, the amplitude of change of the vulnerable option prices as δ increases from 0 to 2 is very small, and this is because mean jump size $\xi = -0.0388$ in both the underlying asset value process and the intensity process is close to 0, which depends on choice of the parameters of the jump. If the mean jump size ξ is large, the impacts of the cross-exciting between two risky factors in our proposed model will be more obvious. Hence, the vulnerable option prices are negatively correlated with δ and ξ . Note that our model can be reduced to the existing models if we set $\delta = 0$, thus Figs. 4–5 indicate that our proposed model considering the correlated risk factors is more realistic, flexible and tractable than the model presented under reduced-form credit risk approach. Hence our model improves the existing studies on vulnerable option pricing.

Recall that the upward jump probability p_1 and the downward jump probability p_2 appear in both the underlying asset value process and the intensity process, we could find that these two parameters are also important to reflect the dynamic correlation and to price the vulnerable options. Since $p_1 + p_2 = 1$, it is just necessary to discuss how the upward jump probability p_1 affects the vulnerable option prices. Fig. 7 shows that the vulnerable option prices increase as both the upward jump probability p_1 and the spot-to-strike ratio S_0/K increase. This numerical result implies that the vulnerable option price is positively affected by the upward jump probability. Moreover, when the upward jump probability p_1 increases, even though both the underlying asset value and the intensity are enhanced correspondingly, the impact of the increase of p_1 on the underlying asset value that raises the option value is larger than the impact of the increase of p_1 on the intensity process which decreases the option value.

Besides showing the effects of the correlation coefficient, spot-to-strike ratio, sensitivity of intensity, and upward jump probability to the vulnerable option prices, we also discuss the relationship between vulnerable option prices and other two key parameters of the options, i.e., maturity and volatility. Fig. 8 presents vulnerable call option prices against the volatilities of underlying asset value and the intensity process. When σ_1 and σ_2 change from 0.05 to 0.5 respectively, the range of the vulnerable option prices are (1.825, 4.078) and (2.307, 2.483). From this figure, we could find that the volatility of the underlying asset value σ_1 is positively correlated with the option prices, the volatility of the intensity process σ_2 is negatively correlated with the option price, and the impact of the volatility associated with market risk on the option price is more sensitive than the volatility associated credit risk. This result is consistent with the other model based on structural approach, such as Tian et al. (2014) etc.

In Fig. 9, vulnerable call option prices trace out a inverted S-shaped curve against spot-to-strike ratio for three maturities $T = 1$, $T = 1.5$, and $T = 2$. This figure indicates that when spot-to-strike ratio is greater than some fixed number, the vulnerable call option price becomes larger as time-to-maturity gets bigger. In contrast, vulnerable option price would get smaller as time-to-maturity increases when spot-to-strike ratio is greater than some fixed number. Actually, when spot-to-strike ratio is less than a fixed number, the option is out-the-money or at least not deep in-the-money, the default risk is relatively small for different time to maturity T , and the return would be larger with higher probability when T is bigger. Hence, in this case the option prices would be larger for the bigger maturity, and the margin value of the option increases as spot-to-strike ratio increases. Meanwhile, when spot-to-strike ratio increases over the fixed number, the option would be deep in-the-money at maturity with high probability. And this probability, which is positively related to default probability, would be higher as time-to-maturity increases. Thus, in this case the vulnerable option price is negatively correlated with time-to-maturity, and the margin value of the option decreases as spot-to-strike ratio increases.

Note that the effects of other parameters in vulnerable options have been studies in many papers, see e.g., Hull and White (1995), Tian et al. (2014). In conclusion, based on the above numerical analysis, the effects of some key parameters on the vulnerable European call option prices are summarized in Table 2.

Remark 4.1. For the vulnerable European put option, we can also use the analytical pricing formula and the same algorithm to present the similar numerical analysis.

5. Conclusion

Owing to the facts that risk factors associated with market risk and credit risk in vulnerable options could be dynamically correlated and the probability of default of the counterparty would be hence affected, this paper discusses a model of pricing vulnerable European options under the reduced-form framework. In the proposed vulnerable options pricing model, the dynamic of the value of the underlying asset is driven by a double exponential jump-diffusion processes, and the intensity process with jumps corresponding to the default timing of the counterparty are cross-exciting subject to the early default condition. As an application, we let the state vector X follow the AJD processes and obtain a closed-form pricing formula of the vulnerable European option through a system of matrix Riccati equations, and the formula can be implemented numerically. The numerical results indicate that our pricing formulae are both accurate and efficient.

Declaration of competing interest

The authors declare no conflict of interest.

Acknowledgements

The work of Huawei Niu in this paper was supported by National Natural Science Foundation of China (71871120, 71501099), Key Project of Philosophy and Social Science Research in Universities in Jiangsu Province (2018SJZD1101), Six Talent Peaks Project in Jiangsu Province (SZCY-012), and Qing Lan Project in Jiangsu Province. The work of Yu Xing was supported by Natural Science Foundation for Youths of Jiangsu of China (BK20171072).

Appendix

Appendix A. Proof of Lemma 1

Proof. By Itô's Formula, we have $\Psi_t = \Psi_0 + \int_0^t \mathcal{D}\Psi_s ds + \int_0^t \eta_s dW_s + \mathcal{J}_t$, where \mathcal{D} denotes the infinitesimal generator of Levy type on X_t , defined at a bounded C^2 function f with bounded first and second derivatives, by

$$\mathcal{D}f(t, x) = \frac{\partial f}{\partial t} + \mu(x)^\top \frac{\partial f}{\partial x} + \frac{1}{2} \text{tr} \left[\sigma(x) \sigma(x)^\top \frac{\partial^2 f}{\partial x^2} \right] + \sum_{j=1}^m \lambda_j(x) \int_{\mathbb{R}^n} [f(t, x + \delta_j z) - f(t, x)] d\nu_j(z),$$

and

$$\eta_t = \left(\frac{\partial \Psi_t}{\partial x} \right)^\top,$$

$$\mathcal{J}_t = \sum_{j=1}^m \left(\sum_{0 < \tau_j(i) < t} (\Psi_{\tau_j(i)} - \Psi_{\tau_j(i)^-}) - \int_0^t \gamma_s^j ds \right),$$

with $\tau_j(i) = \inf\{t : N_t^j = i\}$ denoting the i th jump time of $Z_j, j = 1, 2, \dots, m$, and $\gamma_t^j = \lambda_j(x) \int_{\mathbb{R}^n} [\Psi(t, x + \delta_j z) - \Psi(t, x)] d\nu_j(z)$. Assume the following technical integrability conditions satisfied:

1. $E \left[\int_0^T |\gamma_s| ds \right] < \infty$;
2. $E[|\Psi_t|] < \infty$;
3. $E \left[\left(\int_0^T \eta_s \eta_s^\top ds \right)^{\frac{1}{2}} \right] < \infty$.

Under integrability condition 3, $\int_0^t \eta_s dW_s$ is a martingale. Next we Proof \mathcal{J}_t is a martingale as well. Let $t < s$ and fix j , we have

$$\begin{aligned} E_t \left[\sum_{t < \tau_j(i) < s} (\Psi_{\tau_j(i)} - \Psi_{\tau_j(i)^-}) \right] &= E_t \left[\sum_{t < \tau_j(i) < s} E(\Psi_{\tau_j(i)} - \Psi_{\tau_j(i)^-} | X_{\tau_j(i)^-}, \tau_j(i)) \right] = E_t \left[\sum_{t < \tau_j(i) < s} \Psi_{\tau_j(i)^-} (\theta_{j, \tau_j(i)} - 1) \right] \\ &= E_t \left[\sum_{t < \tau_j(i) < s} \int_{\tau_j(i-1)^+}^{\tau_j(i)} \Psi_{u-} (\theta_{j, u} - 1) dN_u \right] = E_t \left[\int_t^s \Psi_{u-} (\theta_{j, u} - 1) dN_u \right], \end{aligned}$$

Because the jump-counting process $N_{j,t}$ has intensity $\lambda_j(t, X_t)$, integrability condition 1 implies that $E_t \left[\int_t^s \Psi_{u-} (\theta_{j, u} - 1) dN_u \right] = E_t \left[\int_t^s \Psi_{u-} (\theta_{j, u} - 1) \lambda_j(u, X_u) du \right]$. Thus $E_t[\mathcal{J}_s] = \mathcal{J}_t$, i.e. \mathcal{J}_t is a martingale.

Hence, the PIDE is obtained by computing $\mathcal{D}\Psi_t$, setting it to zero, and dividing through the resulting equation by Ψ_t . *Q.E.D.*

Appendix B. Proof of Proposition 1

Proposition 1. (Transform Inversion) For fixed $T \in (0, \infty)$, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$, we assume that $((K, \Omega, \{l_j\}_{j=1}^m, \{\theta_j\}_{j=1}^m, \rho))$ is well-behaved at $(a + ikb, T)$ for any $k \in \mathbb{R}$, and that

$$\int_{\mathbb{R}} |\psi(a + ikb, X_t, t, T)| dk < \infty. \quad (5.1)$$

Then $G_{a,b}(\cdot; x, t, T)$ is well defined by (3.8) and given by

$$G_{a,b}(y; X_t, t, T) = \frac{\psi(a, X_t, t, T)}{2} - \frac{1}{\pi} \int_0^\infty \frac{\text{Im}[\psi(a + ikb, X_t, t, T)e^{-iky}]}{k} dk, \quad (5.2)$$

where $\text{Im}(c)$ denotes the imaginary part of $c \in \mathbb{C}$.

Proof. For a fixed $y \in \mathbb{R}$ and $0 < h < \infty$,

$$\begin{aligned} \frac{1}{2\pi} \int_{-h}^h \frac{e^{iky}\psi(a - ikb, x, t, T) - e^{-iky}\psi(a + ikb, x, t, T)}{ik} dk &= \frac{1}{2\pi} \int_{-h}^h \int_{\mathbb{R}} \frac{e^{-ik(z-y)} - e^{ik(z-y)}}{ik} dG_{a,b}(z; x, t, T) dk \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{-h}^h \frac{e^{-ik(z-y)} - e^{ik(z-y)}}{ik} dk dG_{a,b}(z; x, t, T), \end{aligned}$$

here Fubini theorem is applicable because of the fact that $|e^{iu} - e^{iv}| \leq |u - v|$, for any $u, v \in \mathbb{R}$, and $\lim_{y \rightarrow +\infty} G_{a,b}(y; x, t, T) = \psi(a, x, t, T) < \infty$.

New we note that, for $h > 0$, $\frac{1}{2\pi} \int_{-h}^h \frac{e^{-ik(z-y)} - e^{ik(z-y)}}{ik} dk = -\frac{\text{sgn}(z-y)}{\pi} \int_{-h}^h \frac{\sin(k|z-y|)}{k} dk$ is bounded simultaneously in z and h , for each fixed y .¹⁶ By the bounded convergence theorem,

$$\begin{aligned} \lim_{h \rightarrow \infty} \frac{1}{2\pi} \int_{-h}^h \frac{e^{iky}\psi(a - ikb, x, t, T) - e^{-iky}\psi(a + ikb, x, t, T)}{ik} dk &= - \int_{\mathbb{R}} \text{sgn}(z - y) dG_{a,b}(z; x, t, T) \\ &= -\psi(a, x, t, T) + (G_{a,b}(y; x, t, T) + G_{a,b}(y-; x, t, T)), \end{aligned}$$

where $G_{a,b}(y-; x, t, T) = \lim_{z \rightarrow y-} G_{a,b}(z; x, t, T)$. Using the integrability condition (5.1), by the dominated convergence theorem we have

$$G_{a,b}(y; x, t, T) = \frac{\psi(a, x, t, T)}{2} + \frac{1}{4\pi} \int_{-\infty}^\infty \frac{e^{iky}\psi(a - ikb, x, t, T) - e^{-iky}\psi(a + ikb, x, t, T)}{ik} dk.$$

Since $\psi(\cdot, x, t, T)$ is an exponential function, thus $\psi(a - ikb, x, t, T)$ is the complex conjugate of $\psi(a + ikb, x, t, T)$, and we obtain (5.2).

Remark B.1. For the discounted rate $R = 0$, the formula (5.2) gives us the probability distribution function of $b \cdot X_T$. Thus the associated transition density of X is derived by differentiation of $G_{a,b}$.

Appendix C. Lemma 3 and its Proof

Lemma 3. The solution to the Riccati equation

$$y'(t) = fy^2(t) + gy(t) + h \quad (5.3)$$

for $t \leq T$ and $fh < 0$ with boundary condition $y(T) = k$ is given by

¹⁶ We define $\text{sgn}(x)$ to be 1 if $x > 0$, -1 if $x < 0$, and 0 if $x = 0$.

$$y(t) = l + d \tanh(-\gamma(T-t)/2 + e), \quad (5.4)$$

where $\gamma = \sqrt{g^2 - 4fh}$, $l = -g/(2f)$, $d = -\gamma/(2f)$, $e = \operatorname{atanh}((k-l)/d)$.

Proof. First we can check that $a = -\frac{g+\gamma}{2f}$ is a solution to the Riccati equation, where $\gamma^2 = g^2 - 4fh$. Then we transform the y to $z = \frac{1}{y-a}$, thus $y = a + \frac{1}{z}$, and $y' = -\frac{z'}{z^2}$. Substitute the above two formulae into the Riccati equation, we get the following equation on z :

$$z' = \frac{1}{2}(-2f) + \gamma z, \quad (5.5)$$

which is a first order ordinary differential equation. By the condition $y(T) = k$, we obtain that z satisfies $z(T) = \frac{1}{k-a}$. Then it can easily solve the equation (5.5) with the integral path from T to t , $z = \frac{-f \int_T^t e^{\gamma(T-s)} ds + C}{e^{\gamma(T-t)}} = \frac{-f}{\gamma e^{\gamma(T-t)}} [1 - e^{\gamma(T-t)}] + C e^{-\gamma(T-t)}$, by the condition $z(T) = \frac{1}{k-a}$, we get the constant $C = \frac{1}{k-a}$. Hence, $z = \frac{-f}{\gamma} [e^{-\gamma(T-t)} - 1] + \frac{1}{k-a} e^{-\gamma(T-t)}$. By the above solution to the equation (5.5) we can solve the Riccati equation as follows: $y = a + \frac{1}{z} = -\frac{g+\gamma}{2f} + \frac{2\gamma(a-k)}{(a-k)(-2f)(e^{-\gamma(T-t)}-1)-2\gamma e^{-\gamma(T-t)}}$, by the following calculation, we get that

$$\begin{aligned} y &= l + \frac{\gamma}{-2f} + \frac{2\gamma(a-k)e^{\gamma(T-t)}}{((g+\gamma) - (-2f)k)(1 - e^{\gamma(T-t)}) - 2\gamma} \\ &= l + \frac{(2a\gamma - 2k\gamma)(-2f)e^{\gamma(T-t)} + \gamma((g+\gamma) - (-2f)k)(1 - e^{\gamma(T-t)}) - 2\gamma^2}{(-2f)((g+\gamma) - (-2f)k)(1 - e^{\gamma(T-t)}) - 2(-2f)\gamma} \\ &= l + \frac{(2\gamma(g+\gamma) - 2(-2f)k\gamma)e^{\gamma(T-t)} + ((g\gamma + \gamma^2) - (-2f)k\gamma)(1 - e^{\gamma(T-t)}) - 2\gamma^2}{2f[((g+\gamma) - (-2f))e^{\gamma(T-t)} - 1] + 2\gamma} \\ &= l + \frac{\gamma}{-2f} \frac{e^{-\gamma(T-t)} \left(\frac{\gamma-g}{-2f} + k \right) - \left(\frac{\gamma+g}{-2f} - k \right)}{e^{-\gamma(T-t)} \left(\frac{\gamma-g}{-2f} + k \right) + \left(\frac{\gamma+g}{-2f} - k \right)} \\ &= l + d \tanh \left(\frac{-\gamma(T-t)}{2} + \frac{1}{2} \log((d+(k-l))/(d-(k-l))) \right) \\ &= l + d \tanh \left(-\frac{\gamma(T-t)}{2} + e \right) \end{aligned}$$

Finally we get the solution to the Riccati equation given in Lemma 3 with the analytical formulation.

References

- Bi, H., Wang, G., & Wang, X. (2019). Valuation of catastrophe equity put options with correlated default risk and jump risk. *Finance Research Letters*, 29, 323–329.
- Black, F., & Cox, J. (1976). Valuing corporate securities: Some effects of bond indenture provisions. *The Journal of Finance*, 31, 351–367.
- Cao, M., & Wei, J. (2001). Vulnerable options, risky corporate bond, and credit spread. *Journal of Futures Markets*, 21, 301–327.
- Chen, L., Filipovic, D., & Poor, H. V. (2004). Quadratic term structure models for risk-free and defaultable rates. *Mathematical Finance*, 14, 515–536.
- Cheng, P., & Scaillet, O. (2007). Linear-quadratic jump-diffusion modeling. *Mathematical Finance*, 17, 575–598.
- Chen, N., & Kou, S. G. (2009). Credit spreads, optimal capital structure, and implied volatility with endogenous default and jump risk. *Mathematical Finance*, 19, 343–378.
- Cherubini, U., & Luciano, E. (2003). Pricing vulnerable options with copulas. *The Journal of Risk Finance*, 5, 27–39.
- Duffie, D., Pan, J., & Singleton, K. (2000). Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, 68(6), 1343–1376.
- Duffie, D., & Singleton, K. (1999). Modelling term structures of default bonds. *Review of Financial Studies*, 12, 687–720.
- Errais, E., Giesecke, K., & Goldberg, L. R. (2010). Affine point processes and portfolio credit risk. *Journal of Financial Mathematics*, 1, 642–665.
- Ethier, S., & Kurtz, T. (1986). *Markov processes, characterization and convergence*. New York: John Wiley & Sons.
- Fard, F. A. (2015). Analytical pricing of vulnerable options under a generalized jump-diffusion model. *Insurance: Mathematics and Economics*, 60, 19–28.
- Gil-Pelaez, J. (1951). Note on the inversion theorem. *Biometrika*, 38, 481–482.
- Han, M., Song, X., Niu, H., & Zhou, S. (2018). Pricing vulnerable options with market prices of common jump risks under regime-switching models. *Discrete Dynamics in Nature and Society*, 2018, 1–15.
- He, Z., & Xiong, W. (2012). Rollover risk and credit risk. *The Journal of Finance*, 67, 391–429.
- Huang, M. W., & Liu, Y. H. (2005). Pricing vulnerable options in incomplete markets. *Journal of Future Markets*, 25, 135–170.
- Hui, C. H., Lo, C. F., & Lee, H. C. (2003). Pricing vulnerable Black-Scholes options with dynamic default barriers. *Journal of Derivatives*, 10(4), 62–69.

- Hull, J., & White, A. (1995). The impacts of default risk on the prices of options and other derivative securities. *Journal of Banking & Finance*, 19, 299–322.
- Jarrow, R. A., & Turnbull, S. (1995). Pricing derivatives on financial securities subject to credit risk. *The Journal of Finance*, 50, 53–85.
- Jarrow, R. A., & Yu, F. (2001). Counterparty risk and the pricing of defaultable securities. *The Journal of Finance*, 56, 175–1799.
- Jiang, G., & Yan, S. (2009). Linear-quadratic term structure models: Toward the understanding of jumps in interest rates. *Journal of Banking & Finance*, 33, 473–485.
- Johnson, H., & Stulz, R. (1987). The pricing of options with default risk. *The Journal of Finance*, 42, 267–280.
- Kang, J., & Kim, H. S. (2005). Pricing counterparty default risks: Application to FRNs and vulnerable options. *International Review of Financial Analysis*, 14, 376–392.
- Klein, P. (1996). Pricing Black-Scholes options with correlated credit risk. *Journal of Banking & Finance*, 20, 1211–1229.
- Klein, P., & Inglis, M. (1999). Pricing vulnerable European options when the option's payoff can increase the risk of financial distress. *Journal of Banking & Finance*, 25, 993–1012.
- Klein, P., & Yang, J. (2010). Vulnerable American options. *Managerial Finance*, 36, 414–430.
- Kou, S. G. (2002). A jump diffusion model for option pricing. *Management Science*, 48, 1086–1101.
- Kou, S. G., & Wang, H. (2003). First passage time of a jump diffusion process. *Advances in Applied Probability*, 35, 504–531.
- Leland, H., & Toft, K. B. (1996). Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads. *The Journal of Finance*, 51, 987–1019.
- Liang, G. C., & Ren, X. M. (2007). The credit risk and pricing OTC options. *Asia-Pacific Financial Markets*, 14, 45–68.
- Longstaff, F., & Schwartz, E. (1995). A simple approach to valuing risky fixed and floating rate debt. *The Journal of Finance*, 50, 789–819.
- Madan, D. B., & Unal, H. (1998). Pricing the risks of default. *Review of Derivatives Research*, 2, 121–160.
- Merton, R. C. (1974). On the pricing of corporate debt: The risk structure of interest rates. *The Journal of Finance*, 29, 449–470.
- Niu, H., & Wang, D. (2015). Pricing vulnerable European options under a two-sided jump model via Laplace transforms. *Scientia Sinica Mathematica*, 45, 199–212.
- Niu, H., & Wang, D. (2016). Pricing vulnerable options with correlated jump-diffusion processes depending on various states of the economy. *Quantitative Finance*, 16(7), 1129–1145.
- Shreve, S. (2004). *Stochastic calculus for finance II: Continuous-time models*. New York: Springer-Verlag.
- Tian, L., Wang, G., Wang, W., & Wang, Y. (2014). Pricing vulnerable options with correlated credit risk under jump-diffusion processes. *Journal of Future Markets*, 33, 957–979.
- Wang, X. (2016). The pricing of catastrophe equity put options with default risk. *International Review of Finance*, 16, 181–201.
- Wang, X. (2018). Pricing vulnerable European options with stochastic correlation. *Probability in the Engineering and Informational Sciences*, 32(1), 67–95.
- Wang, G., Wang, X., & Zhou, K. (2017). Pricing vulnerable options with stochastic volatility. *Physica A: Statistical Mechanics and its Applications*, 485, 91–103.
- Williams, D. (1991). *Probability with Martingales*. Cambridge: Cambridge University Press.
- Xu, W. D., Xu, W., Li, H., & Xiao, W. A. (2012). Jump-diffusion approach to modelling vulnerable option pricing. *Finance Research Letters*, 9, 48–56.