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Time Dependent Heston Model*

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Abstract. The use of the Heston model is still challenging because it has a closed formula only when the parameters are constant [S. Heston, *Rev. Financ. Stud.*, 6 (1993), pp. 327–343] or piecewise constant [S. Mikhailov and U. Nogel, *Wilmott Magazine*, July (2003), pp. 74–79]. Hence, using a small volatility of volatility expansion and Malliavin calculus techniques, we derive an accurate analytical formula for the price of vanilla options for any time dependent Heston model (the accuracy is less than a few bps for various strikes and maturities). In addition, we establish tight error estimates. The advantage of this approach over Fourier-based methods is its rapidity (gain by a factor 100 or more) while maintaining a competitive accuracy. From the approximative formula, we also derive some corollaries related first to equivalent Heston models (extending some work of Piterbarg on stochastic volatility models [V. Piterbarg, *Risk Magazine*, 18 (2005), pp. 71–75]) and second, to the calibration procedure in terms of ill-posed problems.

Key words. asymptotic expansion, Malliavin calculus, small volatility of volatility, time dependent Heston model

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1. Introduction.

Stochastic volatility modeling emerged in the late nineties as a way to manage the smile. In this work, we focus on the Heston model, which is a lognormal model where the square of volatility follows a Cox–Ingersoll–Ross (CIR)[1] process. The call (and put) price has a closed formula in this model thanks to a Fourier inversion of the characteristic function (see Heston [22], Lewis [27], and Lipton [29]). When the parameters are piecewise constant, one can still derive a recursive closed formula using a PDE method (see Mikhailov and Nogel [31]) or a Markov argument in combination with affine models (see Elices [16]), but formula evaluation becomes increasingly time consuming. However, for general time dependent parameters there is no analytical formula and one usually has to perform Monte Carlo simulations. This explains the interest of recent works for designing more efficient Monte Carlo simulations: see Broadie and Kaya [13] for an exact simulation and bias-free scheme based on Fourier integral inversion; see Andersen [4] based on a Gaussian moment matching method and a user friendly algorithm; see Smith [40] relying on an almost exact scheme;

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[1] Nice properties for the CIR process are derived by Dufresne [15], Göing-Jaeschke and Yor [20], Diop [14], Alfonsi [1], and Miri [32].

see Alfonsi [2] using higher order schemes and a recursive method for the CIR process. For numerical PDEs, we refer the reader to Kluge's doctoral dissertation [25].

Comparison with the literature. A more recent trend in the quantitative literature has been the use of the so-called approximation method to derive analytical formulas. This has led to an impressive number of papers, with many original ideas. For instance, Alòs, León, and Vives [3] have been studying the short time behavior of implied volatility for stochastic volatility using an extension of Itô's formula. Another trend has focused on analytical techniques to derive the asymptotic expansion of the implied volatility near expiry (see, for instance, Berestycki, Busca, and Florent [10], Labordère [26], Hagan et al. [21], Lewis [28], Osajima [34], or Forde [17]). But in these works the implied volatility near expiry does not have a closed formula because the related geodesic distance is not explicit. It can, however, be approximated by a series expansion [28]. The drawback to these methods is their inability to handle nonhomogeneous (that is, time dependent) parameters. For long maturities, another approach has been the asymptotic expansion w.r.t. the mean reversion parameter of the volatility as shown in [18]. In the case of zero correlation, averaging techniques as exposed in [36] and [35] can be used. Antonelli and Scarlatti take another view in [5] and have suggested price expansion w.r.t. correlation. For all of these techniques, the domain of availability of the expansion is restricted to short or long maturities, to zero correlation, or to homogeneous parameters. However, in [19] regular and singular expansions are generalized to the case of time dependent parameters (with volatilities of Ornstein–Uhlenbeck type). The PDE approach used in [19] could also be used in the present situation and would lead formally to the same expansion we derive. In our work, we aim to give an analytical formula which covers both short and long maturities that also handles time inhomogeneous parameters as well as nonnull correlations. As a difference with several previously quoted papers, our purpose consists also of justifying our approximation mathematically.

The results closest to ours are probably those based on an expansion w.r.t. the volatility of volatility by Lewis [27]: it is based on formal analytical arguments and is restricted to constant parameters. Our formula can be viewed as an extension of Lewis' formula in order to address a time dependent Heston model, using a direct probabilistic approach. In addition, we prove an error estimate which shows that our approximation formula for call/put is of order 2 w.r.t. the volatility of volatility. The advantage of this current approximation is that the evaluation is about 100 to 1000 times quicker than a Fourier-based method (see our numerical tests).

Comparison with our previous works [8] and [7]. Our approach here consists of expanding the price w.r.t. the volatility of volatility and of computing the correction terms using Malliavin calculus. In these respects, the current approach is similar to our previous works [8] and [7]; however, the techniques for estimating error are different. Indeed, we use the fact that the price of vanilla options can be expressed as an expectation of a smooth price function for stochastic volatility models. This is based on a conditioning argument as in [38]. Consequently, the smoothness hypotheses (H_1, H_2, H_3) of our previous papers are no longer required. Note also that the square root function arising in the martingale part of the CIR process is not Lipschitz continuous. Hence, the Heston model does not fit the smoothness framework used previously. Therefore, to overcome this difficulty, we derive new technical results in order to prove the accuracy of the formula.

Contribution of the paper. We give an explicit analytical formula for the price of vanilla options in a time dependent Heston model. Our approach is based on an expansion w.r.t. a small volatility of volatility. This is practically justified by the fact that this parameter is usually quite small (of order 1 or less; see [6], [27], or [13], for instance). The resulting formula is the sum of two terms: the leading term is the Black–Scholes price for the model without volatility of volatility, while the correction term is a combination of Greeks of the leading term with explicit weights depending only on the model parameters. Proving the accuracy of the expansion is far from straightforward, but with some technicalities and a relevant analysis of error we succeed in giving tight error estimates. Our expansion enables us to obtain averaged parameters for the dynamic Heston model.

Formulation of the problem. We consider the solution of the stochastic differential equation (SDE)

$$\mathrm{d}X_t = \sqrt{v_t}dW_t - \frac{v_t}{2}\mathrm{d}t, \quad X_0 = x_0, \tag{1.1}$$

$$\mathrm{d}v_t = \kappa(\theta_t - v_t)\mathrm{d}t + \xi_t\sqrt{v_t}\mathrm{d}B_t, \quad v_0, \tag{1.2}$$
$$\mathrm{d}\langle W, B\rangle_t = \rho_t \mathrm{d}t,$$

where $(B_t, W_t)_{0\leq t\leq T}$ is a two-dimensional correlated Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ with the usual assumptions on filtration $(\mathcal{F}_t)_{0\leq t\leq T}$. In our setting, $(X_t)_t$ is the log of the forward price and $(v_t)_t$ is the square of the volatility which follows a CIR process with an initial value $v_0 > 0$, a positive mean reversion κ, a positive long-term level $(\theta_t)_t$, a positive volatility of volatility $(\xi_t)_t$, and a correlation $(\rho_t)_t$. These time dependent parameters are assumed to be measurable and bounded on $[0, T]$.

To develop our approximation method, we will examine the following perturbed process w.r.t. $\epsilon \in [0, 1]$:

$$\begin{aligned} \mathrm{d}X_t^\epsilon &= \sqrt{v_t^\epsilon}dW_t - \frac{v_t^\epsilon}{2}\mathrm{d}t, \quad X_0^\epsilon = x_0, \\ \mathrm{d}v_t^\epsilon &= \kappa(\theta_t - v_t^\epsilon)\mathrm{d}t + \epsilon\xi_t\sqrt{v_t^\epsilon}\mathrm{d}B_t, \quad v_0^\epsilon = v_0, \end{aligned} \tag{1.3}$$

so that our perturbed process coincides with the initial one for $\epsilon = 1$: $X_t^1 = X_t, v_t^1 = v_t$. For the existence of the solution v^ϵ, we refer the reader to Chapter IX in [39] (moreover, the process is nonnegative for $k\theta_t \geq 0$; see also the proof of Lemma 4.2). Our main purpose is to give an accurate analytic approximation, in a certain sense, of the expected payoff of a put option:

$$g(\epsilon) = e^{-\int_0^T r_t \mathrm{d}t}\mathbb{E}[(K - e^{\int_0^T (r_t - q_t)\mathrm{d}t + X_T^\epsilon})_+], \tag{1.4}$$

where r (resp., q) is the risk-free rate (resp., the dividend yield), T is the maturity, and $\epsilon = 1$. Extensions to call options and other payoffs are discussed later.

Outline of the paper. In section 2, we explain the methodology of the small volatility of volatility expansion. An approximation formula is then derived in Theorem 2.2 and its accuracy stated in Theorem 2.3. This section ends by explicitly expressing the formula's coefficients for general time dependent parameters (constant, smooth, and piecewise constant).

Our expansion allows us to give equivalent constant parameters for the time dependent Heston model (see subsection 2.5). As a second corollary, the options calibration for Heston's model using only one maturity becomes an ill-posed problem; we give numerical results to confirm this situation. In section 3, we provide numerical tests to benchmark our formula with the closed formula in the case of constant and piecewise constant parameters. In section 4, we prove the accuracy of the approximation stated in Theorem 2.3: this section is the technical core of the paper. In section 5, we establish lemmas used to make the calculation of the correction terms explicit (those derived in Theorem 2.2). In section 6, we conclude this work and give a few extensions. In the appendix, we recall details about the closed formula (of Heston [22] and Lewis [27]) in the case of constant (and piecewise constant) parameters.

2. Smart Taylor expansion.

2.1. Notations.

Notation 2.1 (extremes of deterministic functions). *For a càdlàg function* $l : [0, T] \to \mathbb{R}$*, we denote* $l_{Inf} = \inf_{t\in[0,T]} l_t$ *and* $l_{Sup} = \sup_{t\in[0,T]} l_t$.

Notation 2.2 (differentiation).

(i) *For a smooth function* $x \mapsto l(x)$*, we denote by* $l^{(i)}(x)$ *its ith derivative.*

(ii) *Given a fixed time t and for a function* $\epsilon \to f_t^\epsilon$*, we denote (if it has a meaning) the ith derivative at* $\epsilon = 0$ *by* $f_{i,t} = \frac{\partial^i f_t^\epsilon}{\partial \epsilon^i}|_{\epsilon=0}$.

2.2. About the CIR process.

Assumptions. In order to bound the approximation errors, we need a positivity assumption for the CIR process.

Assumption P. The parameters of the CIR process (1.2) verify the following conditions:

$$\xi_{Inf} > 0, \quad \left(\frac{2\kappa\theta}{\xi^2}\right)_{Inf} \geq 1.$$

This assumption is crucial to ensure the positivity of the process on $[0, T]$, which is stated in detail in Lemma 4.2 (remember that $v_0 > 0$). We have

$$\mathbb{P}(\forall t \in [0, T] : \ v_t > 0) = 1.$$

When the functions θ and ξ are constant, Assumption (P) coincides with the usual Feller test condition $\frac{2\kappa\theta}{\xi^2} \geq 1$ (see [24]).

Note that the above assumption ensures that the positivity property also holds for the perturbed CIR process (1.3): for any $\epsilon \in [0, 1]$, we have

$$\mathbb{P}(\forall t \in [0, T] : \ v_t^\epsilon > 0) = 1$$

(see Lemma 4.2). We also need a uniform bound of the correlation in order to preserve the nondegeneracy of the SDE (1.1) conditionally on $(B_t)_{0\leq t\leq T}$.

Assumption R. The correlation is bounded away from -1 and $+1$:

$$|\rho|_{Sup} < 1.$$

2.3. Taylor development. In this section, we present the main steps leading to our results. Complete proofs are given later.

If $(\mathcal{F}_t^B)_t$ denotes the filtration generated by the Brownian motion B, the distribution of X_T^ϵ conditionally to $\mathcal{F}_T^B$ is a Gaussian distribution with mean $x_0 + \int_0^T \rho_t \sqrt{v_t^\epsilon} \mathrm{d}B_t - \frac{1}{2}\int_0^T v_t^\epsilon \mathrm{d}t$ and variance $\int_0^T (1-\rho_t^2) v_t^\epsilon \mathrm{d}t$ ($\epsilon \in [0,1]$). Therefore, the function (1.4) can be expressed as follows:

$$g(\epsilon) = \mathbb{E}\left[P_{BS}\left(x_0 + \int_0^T \rho_t \sqrt{v_t^\epsilon}\mathrm{d}B_t - \int_0^T \frac{\rho_t^2}{2} v_t^\epsilon \mathrm{d}t, \int_0^T (1-\rho_t^2) v_t^\epsilon \mathrm{d}t\right)\right], \tag{2.1}$$

where the function $(x,y) \to P_{BS}(x,y)$ is the put function price in a Black–Scholes model with spot e^x, strike K, total variance y, risk-free rate $r_{eq} = \frac{\int_0^T r(t)\mathrm{d}t}{T}$, dividend yield $q_{eq} = \frac{\int_0^T q(t)\mathrm{d}t}{T}$, and maturity T. For the sake of completeness, we recall that $P_{BS}(x,y)$ has the following explicit expression:

$$Ke^{-r_{eq}T}\mathcal{N}\left(\frac{1}{\sqrt{y}}\log\left(\frac{Ke^{-r_{eq}T}}{e^x e^{-q_{eq}T}}\right) + \frac{1}{2}\sqrt{y}\right) - e^x e^{-q_{eq}T}\mathcal{N}\left(\frac{1}{\sqrt{y}}\log\left(\frac{Ke^{-r_{eq}T}}{e^x e^{-q_{eq}T}}\right) - \frac{1}{2}\sqrt{y}\right).$$

In the following, we expand $P_{BS}(.,.)$ w.r.t. its two arguments. For this, we note that P_{BS} is a smooth function (for $y > 0$). In addition, there is a simple relation between its partial derivatives:

$$\frac{\partial P_{BS}}{\partial y}(x,y) = \frac{1}{2}\left(\frac{\partial^2 P_{BS}}{\partial x^2}(x,y) - \frac{\partial P_{BS}}{\partial x}(x,y)\right) \quad \forall x \in \mathbb{R}, \forall y > 0, \tag{2.2}$$

which can be proved easily by a standard calculation left to the reader.

Under Assumption (P), for any t, v_t^ϵ is C^2 w.r.t ϵ at $\epsilon = 0$ (differentiation in the L_p sense). This result will be shown later. In addition, v^ϵ does not vanish (for any $\epsilon \in [0,1]$). Hence, by putting $v_{i,t}^\epsilon = \frac{\partial^i v_t^\epsilon}{\partial \epsilon^i}$, we get

$$\mathrm{d}v_{1,t}^\epsilon = -\kappa v_{1,t}^\epsilon \mathrm{d}t + \xi_t \sqrt{v_t^\epsilon}\mathrm{d}B_t + \epsilon\xi_t \frac{v_{1,t}^\epsilon}{2\sqrt{v_t^\epsilon}}\mathrm{d}B_t, \qquad v_{1,0}^\epsilon = 0,$$

$$\mathrm{d}v_{2,t}^\epsilon = -\kappa v_{2,t}^\epsilon \mathrm{d}t + \xi_t \frac{v_{1,t}^\epsilon}{\sqrt{v_t^\epsilon}}\mathrm{d}B_t + \epsilon\xi_t \frac{v_{2,t}^\epsilon}{2\sqrt{v_t^\epsilon}}\mathrm{d}B_t - \epsilon\xi_t \frac{[v_{1,t}^\epsilon]^2}{4[v_t^\epsilon]^{3/2}}\mathrm{d}B_t, \qquad v_{2,0}^\epsilon = 0.$$

From the definitions $v_{i,t} \equiv \frac{\partial^i v_t^\epsilon}{\partial \epsilon^i}|_{\epsilon=0}$, we easily deduce

$$v_{0,t} = e^{-\kappa t}\left(v_0 + \int_0^t \kappa e^{\kappa s}\theta_s \mathrm{d}s\right),$$

$$v_{1,t} = e^{-\kappa t}\int_0^t e^{\kappa s}\xi_s \sqrt{v_{0,s}}\mathrm{d}B_s, \tag{2.3}$$

$$v_{2,t} = e^{-\kappa t}\int_0^t e^{\kappa s}\xi_s \frac{v_{1,s}}{(v_{0,s})^{\frac{1}{2}}}\mathrm{d}B_s. \tag{2.4}$$

Note that $v_{0,t}$ coincides also with the expected variance $\mathbb{E}(v_t)$ because of the linearity of the drift coefficient of $(v_t)_t$. Now, to expand $g(\epsilon)$, we use the Taylor formula twice, first applied to $\epsilon \to v_t^\epsilon$ and $\sqrt{v_t^\epsilon}$ at $\epsilon = 1$ using derivatives computed at $\epsilon = 0$:

$$v_t^1 = v_{0,t} + v_{1,t} + \frac{v_{2,t}}{2} + \cdots ,$$
$$\sqrt{v_t^1} = \sqrt{v_{0,t}} + \frac{v_{1,t}}{2(v_{0,t})^{\frac{1}{2}}} + \frac{v_{2,t}}{4(v_{0,t})^{\frac{1}{2}}} - \frac{v_{1,t}^2}{8(v_{0,t})^{\frac{3}{2}}} + \cdots ;$$

second, it is applied to the smooth function P_{BS} at the second order w.r.t. the first and second variables around $(x_0 + \int_0^T \rho_t \sqrt{v_{0,t}} \mathrm{d}B_t - \int_0^T \frac{\rho_t^2}{2} v_{0,t} \mathrm{d}t, \int_0^T (1-\rho_t^2) v_{0,t} \mathrm{d}t)$. For convenience, we simply write

$$\tilde{P}_{BS} = P_{BS}\left(x_0 + \int_0^T \rho_t \sqrt{v_{0,t}} \mathrm{d}B_t - \int_0^T \frac{\rho_t^2}{2} v_{0,t} \mathrm{d}t, \int_0^T (1-\rho_t^2) v_{0,t} \mathrm{d}t\right), \tag{2.5}$$
$$\frac{\partial^{i+j} \tilde{P}_{BS}}{\partial x^i y^j} = \frac{\partial^{i+j} P_{BS}}{\partial x^i y^j}\left(x_0 + \int_0^T \rho_t \sqrt{v_{0,t}} \mathrm{d}B_t - \int_0^T \frac{\rho_t^2}{2} v_{0,t} \mathrm{d}t, \int_0^T (1-\rho_t^2) v_{0,t} \mathrm{d}t\right).$$

Then, one gets

$$g(1) = \mathbb{E}[\tilde{P}_{BS}] \tag{2.6}$$
$$+ \mathbb{E}\Bigg[\frac{\partial \tilde{P}_{BS}}{\partial x}\Bigg(\int_0^T \rho_t \left(\frac{v_{1,t}}{2(v_{0,t})^{\frac{1}{2}}} + \frac{v_{2,t}}{4(v_{0,t})^{\frac{1}{2}}} - \frac{v_{1,t}^2}{8(v_{0,t})^{\frac{3}{2}}}\right) \mathrm{d}B_t \tag{2.7}$$
$$- \int_0^T \frac{\rho_t^2}{2}\left(v_{1,t} + \frac{v_{2,t}}{2}\right) \mathrm{d}t\Bigg)\Bigg]$$
$$+ \mathbb{E}\left[\frac{\partial \tilde{P}_{BS}}{\partial y} \int_0^T (1-\rho_t^2)\left(v_{1,t} + \frac{v_{2,t}}{2}\right) \mathrm{d}t\right] \tag{2.8}$$
$$+ \frac{1}{2}\mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial x^2}\left(\int_0^T \rho_t \frac{v_{1,t}}{2(v_{0,t})^{\frac{1}{2}}} \mathrm{d}B_t - \int_0^T \frac{\rho_t^2}{2} v_{1,t} \mathrm{d}t\right)^2\right] \tag{2.9}$$
$$+ \frac{1}{2}\mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial y^2}\left(\int_0^T (1-\rho_t^2) v_{1,t} \mathrm{d}t\right)^2\right] \tag{2.10}$$
$$+ \mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial xy}\left(\int_0^T (1-\rho_t^2) v_{1,t} \mathrm{d}t\right)\left(\int_0^T \rho_t \frac{v_{1,t}}{2(v_{0,t})^{\frac{1}{2}}} \mathrm{d}B_t - \int_0^T \frac{\rho_t^2}{2} v_{1,t} \mathrm{d}t\right)\right] \tag{2.11}$$
$$+ \mathcal{E}, \tag{2.12}$$

where $\mathcal{E}$ is the error in our Taylor expansion. In fact, we notice that

$$\mathbb{E}[\tilde{P}_{BS}] = \mathbb{E}[\mathbb{E}[e^{-\int_0^T r_t \mathrm{d}t}(K - e^{x_0 + \int_0^T (r_t - q_t - \frac{v_{0,t}}{2})\mathrm{d}t + \int_0^T \sqrt{v_{0,t}}(\rho_t \mathrm{d}B_t + \sqrt{1-\rho_t^2}\mathrm{d}B_t^\perp)})_+ | \mathcal{F}_T^B]]$$
$$= P_{BS}\left(x_0, \int_0^T v_{0,t} \mathrm{d}t\right),$$

where $B^{\perp}$ is a Brownian motion independent on $\mathcal{F}_T^B$. Furthermore, the relation (2.2) remains the same for $\tilde{P}_{BS}$, and this enables us to simplify the expansion above. This gives the following proposition.

Proposition 2.1. *The approximation* (2.12) *is equivalent to*

$$g(1) = P_{BS}\left(x_0, \int_0^T v_{0,t}\mathrm{d}t\right) + \mathbb{E}\left[\frac{\partial \tilde{P}_{BS}}{\partial y}\int_0^T (v_{1,t}+v_{2,t})\mathrm{d}t\right] + \frac{1}{2}\mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial y^2}\left(\int_0^T v_{1,t}\mathrm{d}t\right)^2\right] + \mathcal{E}.$$

The details of the proof are given in subsection 5.3. At first sight, the above formula looks like a Taylor formula of P_{BS} w.r.t. the cumulated variance. In fact, it is different; note that the coefficient of $v_{2,t}$ is not $1/2$ but 1. We do not have any direct interpretation of this formula.

The next step consists of making explicit the correction terms as a combination of Greeks of the Black–Scholes price.

Theorem 2.2. *Under Assumptions* (P) *and* (R)*, the put*[2] *price is approximated by*

$$\begin{aligned} e^{-\int_0^T r_t \mathrm{d}t}\mathbb{E}[(K - e^{\int_0^T (r_t - q_t)\mathrm{d}t + X_T^1})_+] &= P_{BS}(x_0, var_T) + \sum_{i=1}^{2} a_{i,T}\frac{\partial^{i+1} P_{BS}}{\partial x^i y}(x_0, var_T) \\ &\quad + \sum_{i=0}^{1} b_{2i,T}\frac{\partial^{2i+2} P_{BS}}{\partial x^{2i} y^2}(x_0, var_T) + \mathcal{E}, \end{aligned} \tag{2.13}$$

where

$$var_T = \int_0^T v_{0,t}\mathrm{d}t, \qquad a_{1,T} = \int_0^T e^{\kappa s}\rho_s\xi_s v_{0,s} ds \int_s^T e^{-\kappa u} du,$$

$$a_{2,T} = \int_0^T e^{\kappa s}\rho_s\xi_s v_{0,s} ds \int_s^T \rho_t\xi_t dt \int_t^T e^{-\kappa u} du,$$

$$b_{0,T} = \int_0^T e^{2\kappa s}\xi_s^2 v_{0,s} ds \int_s^T e^{-\kappa t} dt \int_t^T e^{-\kappa u} du, \qquad b_{2,T} = \frac{a_{1,T}^2}{2}.$$

The proof is postponed to subsection 5.4. Finally, we give an estimate regarding the error $\mathcal{E}$ arising in the above theorem.

Theorem 2.3. *Under Assumptions* (P) *and* (R)*, the error in the approximation* (2.13) *is estimated as follows:*

$$\mathcal{E} = O\left(\xi_{Sup}^3 T^2\right).$$

In view of Theorem 2.3, we may refer to the formula (2.13) as a second order approximation formula w.r.t. the volatility of volatility.

[2]The approximation formula for the call price is obtained using the call/put parity relation: in (2.13), it consists of replacing on the left-hand side the put payoff by the call one, and on the right-hand side the put price function P_{BS} by the similar call price function, while the coefficients remain the same.

2.4. Computation of coefficients.

Constant parameters. The case of constant parameters (θ, ξ, ρ) gives us the coefficients a and b explicitly. Using Mathematica, we derive the following explicit expressions.

Proposition 2.4 (explicit computations). *For constant parameters, one has*

$$var_T = m_0 v_0 + m_1\theta, \qquad a_{1,T} = \rho\xi(p_0 v_0 + p_1\theta),$$
$$a_{2,T} = (\rho\xi)^2(q_0 v_0 + q_1\theta), \qquad b_{0,T} = \xi^2(r_0 v_0 + r_1\theta),$$

where

$$m_0 = \frac{e^{-\kappa T}\left(-1+e^{\kappa T}\right)}{\kappa}, \qquad m_1 = T - \frac{e^{-\kappa T}\left(-1+e^{\kappa T}\right)}{\kappa},$$
$$p_0 = \frac{e^{-\kappa T}\left(-\kappa T+e^{\kappa T}-1\right)}{\kappa^2}, \qquad p_1 = \frac{e^{-\kappa T}\left(\kappa T+e^{\kappa T}(\kappa T-2)+2\right)}{\kappa^2},$$
$$q_0 = \frac{e^{-\kappa T}\left(-\kappa T(\kappa T+2)+2e^{\kappa T}-2\right)}{2\kappa^3}, \qquad q_1 = \frac{e^{-\kappa T}\left(2e^{\kappa T}(\kappa T-3)+\kappa T(\kappa T+4)+6\right)}{2\kappa^3},$$
$$r_0 = \frac{e^{-2\kappa T}\left(-4e^{\kappa T}\kappa T+2e^{2\kappa T}-2\right)}{4\kappa^3}, \qquad r_1 = \frac{e^{-2\kappa T}\left(4e^{\kappa T}(\kappa T+1)+e^{2\kappa T}(2\kappa T-5)+1\right)}{4\kappa^3}.$$

Remark 2.1. In the case of constant parameters (θ, ξ, ρ), we retrieve the usual Heston model. In this particular case, our expansion coincides exactly with Lewis's volatility of volatility series expansion (see equation (3.4), p. 84 in [27] for Lewis's expansion formula and p. 93 in [27] for the explicit calculation of the coefficients $J^{(i)}$ with $\varphi = \frac{1}{2}$). Using his notation, we have $a_{1,T} = J^{(1)}$, $a_{2,T} = J^{(4)}$, and $b_{0,T} = J^{(3)}$.

Smooth parameters. In this case, we may use a Gauss–Legendre quadrature formula for the computation of the terms a and b.

Piecewise constant parameters. The computation of the variance var_T is straightforward. Thus, it remains to provide explicit expressions of a and b as a function of the piecewise constant data. Let $T_0 = 0 \le T_1 \le \cdots \le T_n = T$ such that θ, ρ, ξ are constant on each interval $]T_i, T_{i+1}[$ and are equal, respectively, to $\theta_{T_{i+1}}, \rho_{T_{i+1}}, \xi_{T_{i+1}}$. Before giving the recursive relation, we need to introduce the following functions: $\tilde{\omega}_{1,t} = \int_0^t e^{\kappa s}\rho_s\xi_s v_{0,s}ds$, $\tilde{\omega}_{2,t} = \int_0^t e^{2\kappa s}\xi_s^2 v_{0,s}ds$, $\alpha_t = \int_0^t e^{\kappa s}\rho_s\xi_s v_{0,s}ds \int_s^t \rho_u\xi_u du$, $\beta_t = \int_0^t e^{2\kappa s}\xi_s^2 v_{0,s}ds \int_s^t e^{-\kappa u}du$.

Proposition 2.5 (recursive calculations). *For piecewise constant coefficients, one has*

$$\begin{aligned}
a_{1,T_{i+1}} &= a_{1,T_i} + \tilde{\omega}^{-\kappa}_{T_i,T_{i+1}}\tilde{\omega}_{1,T_i} + \rho_{T_{i+1}}\xi_{T_{i+1}} f^1_{\kappa,v_{0,T_i}}(\theta_{T_{i+1}}, T_i, T_{i+1}),\\
a_{2,T_{i+1}} &= a_{2,T_i} + \tilde{\omega}^{-\kappa}_{T_i,T_{i+1}}\alpha_{T_i} + \rho_{T_{i+1}}\xi_{T_{i+1}}\tilde{\omega}^{0,-\kappa}_{T_i,T_{i+1}}\tilde{\omega}_{1,T_i} + (\rho_{T_{i+1}}\xi_{T_{i+1}})^2 f^2_{\kappa,v_{0,T_i}}(\theta_{T_{i+1}}, T_i, T_{i+1}),\\
b_{0,T_{i+1}} &= b_{0,T_i} + \tilde{\omega}^{-\kappa}_{T_i,T_{i+1}}\beta_{T_i} + \tilde{\omega}^{-\kappa,-\kappa}_{T_i,T_{i+1}}\tilde{\omega}_{2,T_i} + \xi^2_{T_{i+1}} f^0_{\kappa,v_{0,T_i}}(\theta_{T_{i+1}}, T_i, T_{i+1}),\\
\alpha_{T_{i+1}} &= \alpha_{T_i} + \rho_{T_{i+1}}\xi_{T_{i+1}}(T_{i+1}-T_i)\tilde{\omega}_{1,T_i} + \rho^2_{T_{i+1}}\xi^2_{T_{i+1}} g^1_{\kappa,v_{0,T_i}}(\theta_{T_{i+1}}, T_i, T_{i+1}),\\
\beta_{T_{i+1}} &= \beta_{T_i} + \tilde{\omega}^{-\kappa}_{T_i,T_{i+1}}\tilde{\omega}_{2,T_i} + \xi^2_{T_{i+1}} g^2_{\kappa,v_{0,T_i}}(\theta_{T_{i+1}}, T_i, T_{i+1}),\\
\tilde{\omega}_{1,T_{i+1}} &= \tilde{\omega}_{1,T_i} + \rho_{T_{i+1}}\xi_{T_{i+1}} h^1_{\kappa,v_{0,T_i}}(\theta_{T_{i+1}}, T_i, T_{i+1}),\\
\tilde{\omega}_{2,T_{i+1}} &= \tilde{\omega}_{2,T_i} + \xi^2_{T_{i+1}} h^2_{\kappa,v_{0,T_i}}(\theta_{T_{i+1}}, T_i, T_{i+1}),\\
v_{0,T_{i+1}} &= e^{-\kappa(T_{i+1}-T_i)}(v_{0,T_i} - \theta_{T_{i+1}}) + \theta_{T_{i+1}},
\end{aligned}$$

where

$$f^0_{\kappa,v_0}(\theta,t,T) = \frac{e^{-2\kappa T}\left(e^{2\kappa t}(\theta-2v_0)+e^{2\kappa T}((-2\kappa t+2\kappa T-5)\theta+2v_0)+4e^{\kappa(t+T)}((-\kappa t+\kappa T+1)\theta+\kappa(t-T)v_0)\right)}{4\kappa^3},$$

$$f^1_{\kappa,v_0}(\theta,t,T) = \frac{e^{-\kappa T}\left(e^{\kappa T}((-\kappa t+\kappa T-2)\theta+v_0)-e^{\kappa t}((\kappa t-\kappa T-2)\theta-\kappa t v_0+\kappa T v_0+v_0)\right)}{\kappa^2},$$

$$f^2_{\kappa,v_0}(\theta,t,T) = \frac{e^{-\kappa(t+3T)}\left(2e^{\kappa(t+3T)}((\kappa(T-t)-3)\theta+v_0)+e^{2\kappa(t+T)}((\kappa(\kappa(t-T)-4)(t-T)+6)\theta-(\kappa(\kappa(t-T)-2)(t-T)+2)v_0)\right)}{2\kappa^3},$$

$$g^1_{\kappa,v_0}(\theta,t,T) = \frac{2e^{\kappa T}\theta+e^{\kappa t}\left(\kappa^2(t-T)^2v_0-(\kappa(\kappa(t-T)-2)(t-T)+2)\theta\right)}{2\kappa^2},$$

$$g^2_{\kappa,v_0}(\theta,t,T) = \frac{e^{-\kappa T}\left(e^{2\kappa T}\theta-e^{2\kappa t}(\theta-2v_0)+2e^{\kappa(t+T)}(\kappa(t-T)(\theta-v_0)-v_0)\right)}{2\kappa^2},$$

$$h^1_{\kappa,v_0}(\theta,t,T) = \frac{e^{\kappa T}\theta+e^{\kappa t}((\kappa t-\kappa T-1)\theta+\kappa(T-t)v_0)}{\kappa},$$

$$h^2_{\kappa,v_0}(\theta,t,T) = \frac{\left(e^{\kappa t}-e^{\kappa T}\right)\left(e^{\kappa t}(\theta-2v_0)-e^{\kappa T}\theta\right)}{2\kappa},$$

and $\tilde{\omega}^{u}_{t,T} = \frac{-e^{tu}+e^{Tu}}{u}$, $\tilde{\omega}^{0,u}_{t,T} = \frac{e^{Tu}(-tu+Tu-1)+e^{tu}}{u^2}$, $\tilde{\omega}^{u,u}_{t,T} = \frac{\left(e^{tu}-e^{Tu}\right)^2}{2u^2}$.

Proof. According to Theorem 2.2, one has

$$\begin{aligned}
a_{1,T_{i+1}} &= \int_0^{T_i} e^{\kappa t}\rho_t\xi_t v_{0,t}\left(\int_t^{T_{i+1}} e^{-\kappa s}ds\right)\mathrm{d}t + \int_{T_i}^{T_{i+1}} e^{\kappa t}\rho_t\xi_t v_{0,t}\left(\int_t^{T_{i+1}} e^{-\kappa s}ds\right)\mathrm{d}t\\
&= a_{1,T_i} + \int_0^{T_i} e^{\kappa t}\rho_t\xi_t v_{0,t}\left(\int_{T_i}^{T_{i+1}} e^{-\kappa s}ds\right)\mathrm{d}t + \int_{T_i}^{T_{i+1}} e^{\kappa t}\rho_t\xi_t v_{0,t}\left(\int_t^{T_{i+1}} e^{-\kappa s}ds\right)\mathrm{d}t\\
&= a_{1,T_i} + \left(\int_{T_i}^{T_{i+1}} e^{-\kappa s}ds\right)\int_0^{T_i} e^{\kappa t}\rho_t\xi_t v_{0,t}\mathrm{d}t + \int_{T_i}^{T_{i+1}} e^{\kappa t}\rho_t\xi_t v_{0,t}\left(\int_t^{T_{i+1}} e^{-\kappa s}ds\right)\mathrm{d}t\\
&= a_{1,T_i} + \tilde{\omega}^{-\kappa}_{T_i,T_{i+1}}\tilde{\omega}_{1,T_i} + \rho_{T_{i+1}}\xi_{T_{i+1}} f^1_{\kappa,v_{0,T_i}}(\theta_{i+1},T_i,T_{i+1}),
\end{aligned}$$

where the functions f^1_{κ,v_0} and $\tilde{\omega}^{-\kappa}$ are calculated analytically using Mathematica. The other terms are calculated analogously. ∎

2.5. Corollaries of the approximation formula (2.13).

Averaging Heston's model parameters. We derive a first corollary of the approximation formula in terms of equivalent Heston models. As explained in [36], this averaging principle may facilitate efficient calibration. Namely, we search for equivalent constant parameters $\bar{\kappa}, \bar{\theta}, \bar{\xi}, \bar{\rho}$ for the Heston model[3]

$$\begin{aligned}
d\bar{X}_t &= \sqrt{\bar{v}_t}dW_t - \frac{\bar{v}_t}{2}\mathrm{d}t, \quad \bar{X}_0 = x_0,\\
d\bar{v}_t &= \bar{\kappa}(\bar{\theta}_t - \bar{v}_t)\mathrm{d}t + \bar{\xi}\sqrt{\bar{v}_t}\mathrm{d}B_t, \quad \bar{v}_0 = v_0,\\
\mathrm{d}\langle W, B\rangle_t &= \bar{\rho}\mathrm{d}t
\end{aligned}$$

that equalize the price of call/put options maturing at T in the time dependent model (equality up to the approximation error $\mathcal{E}$). The following rules give the equivalent parameters as a function of the variance var_T and the coefficients $a_{1,T}$, $a_{2,T}$, $b_{0,T}$ that are computed in the time dependent model.

[3] In this approach, we leave the initial value $\bar{v}_0$ equal to v_0. Indeed, it is not natural to modify its value since it is not a parameter but rather an unobserved factor.

Averaging rule in the case of zero correlation. If $\rho_t \equiv 0$, the equivalent constant parameters (for maturity T) are

$$\bar{\kappa} = \kappa, \quad \bar{\theta} = \frac{var_T - m_0 v_0}{m_1}, \quad \bar{\xi} = \sqrt{\frac{b_{0,T}}{r_0 v_0 + r_1 \bar{\theta}}}, \quad \bar{\rho} = 0.$$

Proof. Two sets of prices coincide at maturity T if they have the same approximation formula (2.13). In this case $a_{1,T} = a_{2,T} = b_{2,T} = 0$; thus the approximation formula depends only on two quantities var_T and $b_{0,T}$. It is quite clear that there is not a single choice of parameters to fit these two quantities. A simple solution results from the choices of $\bar{\kappa} = \kappa$ and $\bar{\rho} = 0$: then, using Proposition 2.4, we obtain the announced parameters $\bar{\theta}$ and $\bar{\xi}$. ■

Remark 2.2. In this case of zero correlation and $\theta = v_0 = \bar{\theta}$, we exactly retrieve Piterbarg's results for the averaged volatility of volatility $\bar{\xi}$ (see [36]).

Averaging rule in the case of nonzero correlation. We follow the same arguments as before. Now the approximation formula also depends on the four quantities var_T, $a_{1,T}$, $a_{2,T}$, and $b_{2,T}$. Thus, equalizing call/put prices at maturity T is equivalent to equalizing these four quantities in both models, by adjusting $\bar{\kappa}, \bar{\theta}, \bar{\xi}$, and $\bar{\rho}$. Unfortunately, we have not found a closed expression for these equivalent parameters. An alternative and simpler way of proceeding consists of modifying the unobserved initial value $\bar{v}_0$ of the variance process while keeping $\bar{\kappa} = \kappa$. For nonvanishing correlation $(\rho_t)_t$, it leads to two possibilities

$$\begin{aligned}
\bar{v}_0 &= b\frac{(b \pm \sqrt{b^2 - 4ac})}{2a} - \frac{p_1 var_T}{m_1 p_0 - m_0 p_1}, & \bar{\theta} &= \frac{var_T - m_0 \bar{v}_0}{m_1},\\
\bar{\xi} &= \sqrt{\frac{b_{0,T}}{r_0 \bar{v}_0 + r_1 \bar{\theta}}}, & \bar{\rho} &= -\frac{2a}{\bar{\xi}(b \pm \sqrt{b^2 - 4ac})},
\end{aligned}$$

such that

$$a = \frac{a_{2,T} m_1}{m_1 q_0 - m_0 q_1}, \quad b = -\frac{a_{1,T} m_1}{m_1 p_0 - m_0 p_1}, \quad c = var_T \left(\frac{p_1}{m_1 p_0 - m_0 p_1} - \frac{q_1}{m_1 q_0 - m_0 q_1} \right),$$

where $m_0, m_1, p_0, p_1, q_0, q_1, r_0$, and r_1 are given in Proposition 2.4.

In practice, only one solution gives realistic parameters. However, this rule is heuristic since there is a priori no guarantee that these averaged parameters satisfy Assumption (P), which is the basis for the argument's correctness.

Proof. Using Proposition 2.4, one has to solve the following system of equations:

$$\begin{aligned}
var_T &= m_0 \bar{v}_0 + m_1 \bar{\theta}, & a_{1,T} &= \bar{\rho}\bar{\xi}(p_0 \bar{v}_0 + p_1 \bar{\theta}),\\
a_{2,T} &= (\bar{\rho}\bar{\xi})^2 (q_0 \bar{v}_0 + q_1 \bar{\theta}), & b_{0,T} &= \bar{\xi}^2 (r_0 \bar{v}_0 + r_1 \bar{\theta}).
\end{aligned}$$

The first equation gives $\bar{\theta} = \frac{var_T - m_0 \bar{v}_0}{m_1}$. Replacing this identity in $a_{1,T}$ and $a_{2,T}$ gives

$$\bar{v}_0 = \left(\frac{a_{1,T}}{(\bar{\rho}\bar{\xi})} - \frac{p_1 var_T}{m_1} \right) \frac{m_1}{p_0 m_1 - p_1 m_0}, \qquad \bar{v}_0 = \left(\frac{a_{2,T}}{(\bar{\rho}\bar{\xi})^2} - \frac{q_1 var_T}{m_1} \right) \frac{m_1}{q_0 m_1 - q_1 m_0}.$$

It readily leads to a quadratic equation $ax^2 + bx + c = 0$ with $x = \frac{1}{\bar{\rho}\bar{\xi}}$. By solving this equation, we easily complete the proof of the result. ■

Collinearity effect in the Heston model. Another corollary of the approximation formula (2.13) is that we can obtain the same vanilla prices at time T with different sets of parameters. For instance, take on the one hand $v_0 = \theta = 4\%$, $\kappa_1 = 2$, and $\xi_1 = 30\%$ (model M_1) and on the other hand $v_0 = \theta = 4\%$, $\kappa_2 = 3$, and $\xi_2 = 38.042\%$ (model M_2), both models having zero correlation. The resulting errors between implied volatilities within the two models are presented in Table 1: they are so small that prices can be considered as equal. Actually, this kind of example is easy to create even with nonnull correlation: as before, in view of the approximation formula (2.13), it is sufficient to equalize the four quantities var_T, $a_{1,T}$, $a_{2,T}$, and $b_{2,T}$.

Table 1

Errors in implied Black–Scholes volatilities (in bps) between the closed formulas (see the appendix) of the two models M_1 and M_2 expressed as relative strikes. Maturity is equal to one year.

Strikes K	80%	90%	100%	110%	120%
Model M_1	20.12%	19.64%	19.50%	19.62%	19.92%
Model M_2	20.11%	19.65%	19.51%	19.62%	19.92%
Errors (bps)	0.69	−0.35	−0.81	−0.42	0.34

As a consequence, calibrating a Heston model using options with a single maturity is an ill-posed problem, which is not a surprising fact. From the work [16] by Elices, we know that it may be possible to compute in a unique manner the Heston parameters by adding other options in the set of calibrated instruments.

3. Numerical accuracy of the approximation.

We give numerical results of the performance of our method. In what follows, the spot S_0, the risk-free rate r, and the dividend yield q are set, respectively, to 100, 0%, and 0%. The initial value of the variance process is set to $v_0 = 4\%$ (initial volatility equal to 20%). Then we study the numerical accuracy w.r.t. K, T, κ, θ, ξ, and ρ by testing different values for these parameters.

In order to present more interesting results for various relevant maturities and strikes, we allow the range of strikes to vary over the maturities. The strike values evolve approximately as $S_0 \exp(c\sqrt{\theta T})$ for some real numbers c and $\theta = 6\%$. The extreme values of c are chosen to be equal to ± 2.57, which represents the 1%–99% quantile of the standard normal distribution. This corresponds to very out-of-the-money options or very deep-in-the-money options. The set of pairs (maturity, strike) chosen for the tests is given in Table 2.

Constant parameters. In Table 3, we report the numerical results when $\theta = 6\%$, $\kappa = 3$, $\xi = 30\%$, and $\rho = 0\%$, giving the errors of implied Black–Scholes volatilities between our approximation formula (see (2.13)) and the price calculated using the closed formula (see the appendix) for the maturities and strikes of Table 2. The table should be read as follows: for example, for one year maturity and strike equal to 170, the implied volatility is equal to 24.14% using the closed formula and 24.20% with the approximation formula, giving an error of −6.33 bps. In Table 3, we observe that the errors do not exceed 7 bps for a large range of strikes and maturities. We notice that the errors are surprisingly higher for short maturities. At first sight, it is counterintuitive, as one would expect our perturbation method to work better for short maturities and worse for long maturities, since the difference between our

Table 2

Set of maturities and relative strikes (in %) used for the numerical tests.

T/K								
3M	70	80	90	100	110	120	125	130
6M	60	70	80	100	110	130	140	150
1Y	50	60	80	100	120	150	170	180
2Y	40	50	70	100	130	180	210	240
3Y	30	40	60	100	140	200	250	290
5Y	20	30	60	100	150	250	320	400
7Y	10	30	50	100	170	300	410	520
10Y	10	20	50	100	190	370	550	730

Table 3

Implied Black–Scholes volatilities of the closed formula and of the approximation formula and related errors (in bps), expressed as a function of maturities in fractions of years and relative strikes. Parameters: $\theta = 6\%$, $\kappa = 3$, $\xi = 30\%$, and $\rho = 0\%$.

3M	23.24%	22.14%	21.43%	21.19%	21.39%	21.86%	22.14%	22.44%
	23.06%	22.19%	21.42%	21.19%	21.38%	21.88%	22.19%	22.49%
	18.01	**− 4.86**	**0.53**	**0.38**	**0.65**	**−2.68**	**−4.86**	**−4.71**
6M	24.32%	23.29%	22.55%	21.99%	22.10%	22.75%	23.17%	23.60%
	24.12%	23.36%	22.57%	21.98%	22.09%	22.79%	23.24%	23.65%
	19.69	**−7.17**	**−1.89**	**0.93**	**1.05**	**−3.97**	**−7.12**	**−4.57**
1Y	24.85%	24.06%	23.14%	22.90%	23.06%	23.66%	24.14%	24.38%
	24.78%	24.12%	23.14%	22.89%	23.06%	23.71%	24.20%	24.42%
	7.72	**−6.49**	**0.26**	**1.12**	**0.72**	**−4.54**	**−6.33**	**−4.27**
2Y	24.86%	24.36%	23.82%	23.61%	23.73%	24.16%	24.46%	24.76%
	24.86%	24.40%	23.82%	23.61%	23.72%	24.19%	24.50%	24.78%
	−0.21	**−3.51**	**−0.12**	**0.68**	**0.37**	**−2.54**	**−3.62**	**−1.71**
3Y	24.95%	24.53%	24.10%	23.89%	23.98%	24.27%	24.53%	24.74%
	24.94%	24.55%	24.10%	23.89%	23.98%	24.28%	24.55%	24.75%
	1.80	**−2.12**	**−0.33**	**0.39**	**0.19**	**−1.27**	**−2.12**	**−1.26**
5Y	24.88%	24.56%	24.20%	24.12%	24.17%	24.38%	24.53%	24.69%
	24.86%	24.57%	24.20%	24.12%	24.17%	24.39%	24.54%	24.70%
	1.38	**−0.96**	**0.03**	**0.17**	**0.10**	**−0.58**	**−0.95**	**−0.59**
7Y	25.03%	24.46%	24.30%	24.23%	24.27%	24.42%	24.54%	24.65%
	24.97%	24.46%	24.30%	24.22%	24.27%	24.42%	24.55%	24.66%
	5.72	**−0.43**	**−0.02**	**0.09**	**0.04**	**−0.33**	**−0.54**	**−0.35**
10Y	24.72%	24.51%	24.34%	24.30%	24.34%	24.44%	24.54%	24.62%
	24.71%	24.51%	24.34%	24.30%	24.34%	24.44%	24.54%	24.62%
	0.42	**−0.28**	**0.02**	**0.05**	**0.02**	**−0.17**	**−0.29**	**−0.19**

proxy model (BS with volatility $(v_{0,t})_t$) and the original one is increasing w.r.t. time. In fact, this intuition is true for prices but not for implied volatilities. When we compare the price errors (in price bp[4]) for the same data, we observe in Table 4 that the error terms are not any bigger for short maturities but vary slightly over time with two observed effects. The error term first increases over time as the error between the proxy and the original model increases

[4]Error price bp= $\frac{\text{Price Approximation} - \text{True Price}}{\text{Spot}} \times 10000$.

Table 4

Put prices of the closed formulas and of the approximation formula and related errors (in bps), expressed as a function of maturities in fractions of years and relative strikes. Parameters: $\theta = 6\%$, $\kappa = 3$, $\xi = 30\%$, and $\rho = 0\%$.

3M	30.00	20.08	10.87	4.22	1.14	0.24	0.10	0.04
	30.00	20.08	10.87	4.22	1.14	0.24	0.10	0.04
	0.03	**−0.11**	**0.06**	**0.08**	**0.09**	**−0.15**	**−0.14**	**−0.07**
6M	40.01	30.07	20.52	6.20	2.72	0.40	0.14	0.05
	40.01	30.08	20.52	6.19	2.71	0.40	0.14	0.05
	0.05	**−0.16**	**−0.18**	**0.26**	**0.26**	**−0.34**	**−0.29**	**−0.08**
1Y	50.01	40.11	21.84	9.12	3.08	0.51	0.15	0.09
	50.01	40.11	21.84	9.11	3.07	0.52	0.16	0.09
	0.04	**−0.21**	**0.06**	**0.44**	**0.23**	**−0.51**	**−0.29**	**−0.12**
2Y	60.03	50.20	32.08	13.26	4.71	0.79	0.28	0.11
	60.03	50.20	32.08	13.26	4.71	0.79	0.29	0.11
	0.00	**−0.18**	**−0.03**	**0.38**	**0.17**	**−0.43**	**−0.29**	**−0.06**
3Y	70.02	60.15	41.70	16.39	5.73	1.21	0.36	0.15
	70.02	60.15	41.70	16.39	5.73	1.21	0.37	0.15
	0.01	**−0.09**	**−0.08**	**0.27**	**0.11**	**−0.31**	**−0.22**	**−0.07**
5Y	80.01	70.15	43.80	21.26	8.50	1.61	0.58	0.21
	80.01	70.15	43.80	21.26	8.50	1.61	0.58	0.21
	0.01	**−0.04**	**0.01**	**0.15**	**0.08**	**−0.19**	**−0.15**	**−0.04**
7Y	90.00	70.42	53.15	25.14	9.32	1.97	0.66	0.26
	90.00	70.42	53.15	25.14	9.32	1.97	0.67	0.26
	0.00	**−0.04**	**−0.01**	**0.09**	**0.04**	**−0.14**	**−0.10**	**−0.03**
10Y	90.01	80.23	55.22	29.92	11.49	2.62	0.84	0.33
	90.01	80.23	55.22	29.92	11.49	2.62	0.84	0.33
	0.00	**−0.02**	**0.01**	**0.06**	**0.03**	**−0.09**	**−0.07**	**−0.02**

over time, as forecasted. But for long maturities, presumably because the volatility converges to its stationary regime, errors decrease. The mean absolute error over the 64 prices is 0.13 bps. When we convert these prices to implied Black–Scholes volatilities, these error terms are dramatically amplified for short maturities due to very small vega. Finally, note that for fixed maturity, price errors are quite uniform w.r.t. strike K.

Impact of the correlation. Results are analogous for various values of correlation. For instance for $\rho = 20\%$ (resp., -20% and -50%), the mean absolute error for the price is 0.28 bps (resp., 0.33 bps and 0.84 bps). We refer to Table 5 for the prices when $\rho = -50\%$ (for other values of ρ, results can be found in the first version of this work [9]). We notice that the errors are smaller for a correlation close to zero and become larger when the absolute value of the correlation increases. However, for realistic correlation values (-50%, for instance), the accuracy for the usual maturities and strikes remains excellent, except for very extreme strikes.

Impact of the volatility of volatility. In view of Theorem 2.3, the smaller the volatility of volatility, the more accurate the approximation. In the following numerical tests, we increase ξ. We consider as Heston parameters the calibrated parameters obtained in [6, Table III]: $\kappa = 1.15$, $\theta = 3.48\%$, $\xi = 39\%$, and $\rho = -64\%$. Moreover, we set $v_0 = 4\%$. We vary the value of ξ in the numerical tests from 0% to 100%. There are two important values for ξ:

- The positivity value $\sqrt{2\kappa\theta}$. For this value, the so-called positivity ratio is $\frac{2\kappa\theta}{\xi^2} = 1$.

Table 5

Put prices of the closed formula and of the approximation formula and related errors (in bp), expressed as a function of maturities in fractions of years and relative strikes. Parameters: $\theta = 6\%$, $\kappa = 3$, $\xi = 30\%$, and $\rho = -50\%$.

3M	30.01	20.14	11.01	4.21	0.95	0.12	0.03	0.01
	30.01	20.15	11.02	4.21	0.94	0.11	0.03	0.01
	0.21	**−0.47**	**−0.31**	**0.04**	**0.57**	**0.82**	**0.16**	**−0.36**
6M	40.02	30.15	20.70	6.16	2.43	0.19	0.04	0.01
	40.02	30.15	20.71	6.15	2.42	0.17	0.04	0.02
	0.37	**−0.59**	**−1.33**	**0.23**	**0.81**	**1.59**	**−0.09**	**−1.05**
1Y	50.04	40.21	22.11	9.03	2.59	0.22	0.03	0.01
	50.04	40.22	22.12	9.02	2.57	0.21	0.05	0.03
	0.36	**−0.88**	**−1.17**	**0.61**	**2.27**	**1.67**	**−1.05**	**−1.69**
2Y	60.08	50.33	32.38	13.11	4.06	0.39	0.08	0.02
	60.08	50.34	32.39	13.10	4.03	0.37	0.09	0.04
	0.09	**−1.00**	**−1.32**	**0.80**	**2.47**	**1.59**	**−0.84**	**−2.00**
3Y	70.05	60.25	41.99	16.20	4.98	0.69	0.13	0.03
	70.05	60.25	42.00	16.19	4.96	0.67	0.13	0.05
	0.17	**−0.54**	**−1.21**	**0.72**	**2.20**	**1.73**	**−0.74**	**−1.80**
5Y	80.03	70.23	44.06	21.01	7.65	0.99	0.25	0.06
	80.03	70.23	44.07	21.00	7.64	0.98	0.26	0.07
	0.11	**−0.30**	**−0.53**	**0.54**	**1.54**	**1.29**	**−0.38**	**−1.50**
7Y	90.00	70.54	53.40	24.84	8.36	1.28	0.31	0.09
	90.00	70.55	53.40	24.84	8.35	1.27	0.32	0.10
	0.06	**−0.41**	**−0.44**	**0.43**	**1.32**	**1.04**	**−0.45**	**−1.32**
10Y	90.02	80.30	55.42	29.57	10.43	1.82	0.44	0.13
	90.02	80.30	55.42	29.57	10.42	1.81	0.44	0.14
	0.03	**−0.18**	**−0.20**	**0.34**	**1.04**	**0.89**	**−0.42**	**−1.17**

Hence, the positivity assumption (Assumption (P)) is maintained while ξ is smaller than $\sqrt{2\kappa\theta}$. For the current example, the positivity value is 28.28%.

- The calibrated value. For this example, it is 39%. Remark that the related positivity ratio is $0.53 < 1$. Hence, the positivity assumption (Assumption (P)) is not satisfied for the calibrated parameters.

We plot in Figure 1 the mean absolute error for the prices[5] at each maturity (in bps) according to the volatility of volatility. As expected from Theorem 2.3, we observe that the mean absolute error is increasing w.r.t. the volatility of volatility and the maturity. For $\xi \leq \sqrt{2\kappa\theta}$ (when the positivity assumption (Assumption (P)) is satisfied), the accuracy is excellent. For the calibrated value, this is very satisfactory as well. Even for larger values of ξ, it is very accurate for nonlarge maturities.

Piecewise constant parameters. Heston's constant parameters have been set to $v_0 = 4\%$, $\kappa = 3$. In addition, the piecewise constant functions θ, ξ, and ρ are equal, respectively, at each interval of the form $]\frac{i}{4}, \frac{i+1}{4}[$ to $4\% + i \times 0.05\%$, $30\% + i \times 0.5\%$, and $-20\% + i \times 0.35\%$. In Tables 7 and 8, we report values using three different formulas. For a given maturity, the first row is obtained using the closed formula with piecewise constant parameters (see the appendix), the second row uses our approximation formula (2.13), and the third row uses the

[5]For higher values of ξ, we compute the price taking in consideration the no-arbitrage interval.

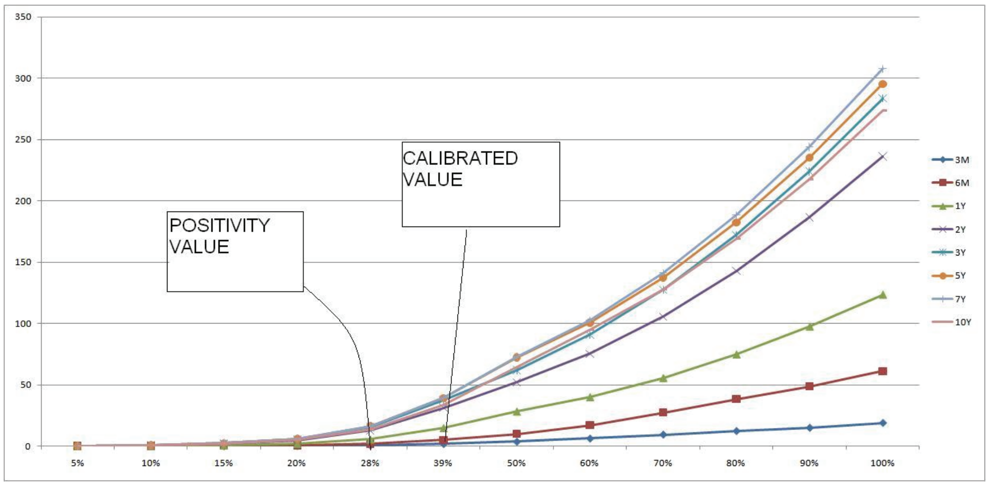

Figure 1. *The mean absolute error for the prices (in bps), expressed as a function of the volatility of volatility and computed for each maturity.*

Table 6
Equivalent averaged parameters.

T	$\bar{v}_0$	$\bar{\theta}$	$\bar{\xi}$	$\bar{\rho}$
3M	4 %	4 %	30 %	−20 %
6M	3.97 %	4.04 %	30.12 %	−19.93 %
1Y	3.28 %	4.38 %	30.89 %	−19.72 %
2Y	4.64 %	4.02 %	31.12 %	−18.95 %
3Y	56.24 %	4.04 %	32.10 %	−18.20 %
5Y	28.58 %	2.68 %	33.63 %	−16.52 %
7Y	84.92 %	0.59 %	35.41 %	−14.80 %
10Y	14.54 %	4.57 %	39.98 %	−12.32 %

closed formula with constant parameters computed by averaging (see subsection 2.5). In order to give complete information on our tests, we also report in Table 6 the values used for the averaging parameters (following subsection 2.5).

Of course, the quickest approach is the use of the approximation formula (2.13). As before, its accuracy is very good, except for very extreme strikes. It is quite interesting to observe that the averaging rules that we propose are extremely accurate.

Calibration to real market data. In this section, we will show how the calibration using time dependent parameters strongly reduces the calibration error.

As a benchmark for the pricing method (i.e., the Lewis formula for constant coefficients or our formula for time dependent coefficients), we choose a Monte Carlo approach using the very accurate QE scheme of Andersen [4] (with 100000 simulations and a monthly time step). Then, we consider the implied Black–Scholes volatility surface of the S&P 500 Index (see Table 9). The spot value is 1360.14. The equivalent risk-free rate is computed from the bond price curve on June 6, 2008. First, we calibrate the constant Heston model using the Lewis formula. The calibrated parameters are $\kappa = 41.95\%$, $\rho = -38.27\%$, $\xi = 59.34\%$, $\theta = 12.40\%$,

Table 7

Implied Black–Scholes volatilities of the closed formula, of the approximation formula, and of the averaging formula, expressed as a function of maturities in fractions of years and relative strikes. Piecewise constant parameters.

3M	23.45%	21.88%	20.58%	19.70%	19.39%	19.55%	19.74%	19.97%
	22.73%	21.96%	20.60%	19.69%	19.35%	19.53%	19.84%	20.28%
	23.45%	21.88%	20.58%	19.70%	19.39%	19.55%	19.74%	19.97%
6M	24.09%	22.59%	21.30%	19.63%	19.33%	19.58%	19.92%	20.31%
	23.09%	22.60%	21.43%	19.61%	19.30%	19.58%	20.19%	20.93%
	24.09%	22.59%	21.30%	19.63%	19.33%	19.58%	19.92%	20.31%
1Y	23.95%	22.66%	20.76%	19.70%	19.37%	19.69%	20.12%	20.36%
	23.12%	22.66%	20.81%	19.68%	19.32%	19.78%	20.62%	21.05%
	23.95%	22.66%	20.76%	19.70%	19.37%	19.69%	20.12%	20.35%
2Y	23.26%	22.30%	21.01%	19.99%	19.66%	19.83%	20.09%	20.37%
	22.84%	22.33%	21.04%	19.96%	19.62%	19.90%	20.43%	21.02%
	23.26%	22.30%	21.01%	19.98%	19.66%	19.83%	20.09%	20.37%
3Y	23.28%	22.40%	21.27%	20.26%	19.96%	20.02%	20.23%	20.43%
	22.81%	22.38%	21.33%	20.24%	19.93%	20.04%	20.47%	20.90%
	23.28%	22.40%	21.27%	20.26%	19.96%	20.02%	20.23%	20.42%
5Y	23.22%	22.46%	21.34%	20.77%	20.54%	20.54%	20.65%	20.80%
	22.88%	22.44%	21.35%	20.77%	20.52%	20.55%	20.76%	21.09%
	23.22%	22.46%	21.34%	20.77%	20.54%	20.54%	20.64%	20.79%
7Y	23.86%	22.36%	21.81%	21.26%	21.06%	21.06%	21.16%	21.27%
	23.25%	22.39%	21.82%	21.26%	21.05%	21.07%	21.23%	21.45%
	23.86%	22.37%	21.81%	21.26%	21.06%	21.06%	21.15%	21.26%
10Y	23.59%	22.96%	22.30%	21.97%	21.82%	21.83%	21.92%	22.02%
	23.46%	22.98%	22.30%	21.97%	21.81%	21.84%	21.96%	22.12%
	23.59%	22.96%	22.30%	21.97%	21.82%	21.83%	21.92%	22.01%

and $v_0 = 4.40\%$. We give in Table 10 the error of implied Black–Scholes volatilities between the calibrated price computed using the Monte Carlo scheme and the market price. The mean absolute error is 0.72%. Now, we calibrate the implied Black–Scholes volatility surface using our approximation formula (2.13) for piecewise constant parameters. The time break points are the surface time break points. The calibrated parameters are given in Table 11. For the reader interested in practical considerations in the calibration procedure, we refer him/her to [16] for interesting suggestions defining constraints on the parameters that ensure a more robust calibration. We give in Table 12 the error of implied Black–Scholes volatilities between the calibrated price computed using the Monte Carlo scheme and the market price. The mean absolute error is 0.19%. Hence, the accuracy is improved by a factor 3.8, which demonstrates the potential interest of using the time dependent Heston model. However, in this example, we observe that the time parameters vary greatly and the use of the Heston model (on these data) is questionable.

Computational time. Regarding the computational time, the approximation formula (2.13) yields essentially the same computational cost as the Black–Scholes formula, while the closed formula requires an additional space integration involving many exponential and trigonometric functions for which evaluation costs are higher. For instance, using a 2.6 GHz Pentium PC, the computations of the 64 numerical values in Table 3 (or 5) take 4.71 ms using the approximation formula and 301 ms using the closed formula. For the example with time

Table 8

Put prices of the closed formula, of the approximation formula, and of the averaging formula, expressed as a function of maturities in fractions of years and relative strikes. Piecewise constant parameters.

3M	30.00	20.07	10.78	3.93	0.87	0.13	0.05	0.02
	30.00	20.08	10.78	3.93	0.87	0.13	0.05	0.02
	30.00	20.07	10.78	3.93	0.87	0.13	0.05	0.02
6M	40.01	30.06	20.41	5.53	2.06	0.18	0.05	0.01
	40.00	30.06	20.42	5.53	2.05	0.18	0.05	0.02
	40.01	30.06	20.41	5.53	2.06	0.18	0.05	0.01
1Y	50.01	40.07	21.33	7.85	1.97	0.17	0.03	0.02
	50.01	40.07	21.35	7.84	1.95	0.18	0.04	0.02
	50.01	40.07	21.33	7.85	1.97	0.17	0.03	0.02
2Y	60.02	50.11	31.38	11.23	2.92	0.24	0.06	0.01
	60.01	50.11	31.39	11.23	2.90	0.25	0.07	0.02
	60.02	50.11	31.38	11.23	2.92	0.24	0.06	0.01
3Y	70.01	60.07	41.07	13.92	3.55	0.41	0.08	0.02
	70.01	60.07	41.08	13.92	3.54	0.42	0.09	0.03
	70.01	60.07	41.07	13.92	3.55	0.41	0.08	0.02
5Y	80.01	70.07	42.64	18.37	5.74	0.61	0.15	0.04
	80.01	70.07	42.64	18.36	5.72	0.61	0.16	0.04
	80.01	70.07	42.64	18.37	5.74	0.61	0.15	0.04
7Y	90.00	70.24	52.22	22.15	6.46	0.86	0.21	0.06
	90.00	70.24	52.22	22.15	6.45	0.86	0.21	0.07
	90.00	70.24	52.22	22.15	6.46	0.86	0.21	0.06
10Y	90.01	80.14	54.13	27.17	8.71	1.42	0.35	0.11
	90.01	80.14	54.13	27.16	8.70	1.42	0.36	0.12
	90.01	80.14	54.13	27.17	8.71	1.42	0.35	0.11

Table 9

Implied volatility surface for the S&P 500 *Index (on June* 16, 2008*), expressed as a function of maturities in fractions of years and relative strikes.*

T/K	90.00%	95.00%	97.50%	100.00%	102.50%	105.00%	110.00%
3M	22.48%	21.61%	21.11%	20.60%	20.18%	19.76%	18.88%
1Y	24.62%	23.25%	22.57%	21.87%	21.22%	20.60%	19.30%
1Y+6M	24.66%	23.51%	22.95%	22.38%	21.80%	21.29%	20.22%
2Y	24.83%	23.81%	23.33%	22.84%	22.34%	21.86%	20.93%

dependent coefficients (reported in Table 7), the computational time for the 64 prices is about 40.2 ms using the approximation formula and 2574 ms using the closed formula. Roughly speaking, the use of the approximation formula enables us to speed up the valuation (and thus the calibration) by a factor 100 to 600.

4. Proof of Theorem 2.3.

The proof is divided into several steps. In subsection 4.1 we give the upper bounds for derivatives of the put function P_{BS}, in subsection 4.2 the conditions for positivity of the squared volatility process v, in subsection 4.3 the upper bounds for the negative moments of the integrated squared volatility $\int_0^T v_t dt$, and in subsection 4.4 the upper bounds for derivatives of functionals of the squared volatility process v. Finally, in subsection 4.5, we complete the proof of Theorem 2.3 using the previous subsections.

Table 10

Error of implied volatility surface (in %) when calibrating the constant Heston model.

T/K	90.00%	95.00%	97.50%	100.00%	102.50%	105.00%	1.1
3M	−1.71%	−0.63%	−0.26%	−0.01%	0.17%	0.18%	−0.36%
1Y	1.27%	1.05%	0.88%	0.64%	0.40%	0.12%	−0.71%
1Y+6M	1.57%	1.12%	0.89%	0.62%	0.31%	0.03%	−0.68%
2Y	2.12%	1.42%	1.10%	0.76%	0.39%	0.04%	−0.69%

Table 11

The piecewise constant calibrated parameters.

T/Calibrated parameters	κ	ρ	ξ	v_0	θ
3M	8.63%	−99.30%	15.63%	3.49%	83.35%
1Y	8.63%	−99.84%	37.16%	3.49%	19.92%
1Y+6M	8.63%	−10.65%	16.85%	3.49%	15.83%
2Y	8.63%	40.91%	0.13%	3.49%	53.15%

Notation. In order to alleviate the proofs, we introduce some notation specific to this section.

Differentiation. For every process Z^ϵ, we write (if these derivatives have a meaning)

(i) $Z_{i,t} = \frac{\partial^i Z_t^\epsilon}{\partial \epsilon^i}|_{\epsilon=0}$,

(ii) the ith Taylor residual by $R_{i,t}^{Z^\epsilon} = Z_t^\epsilon - \sum_{j=0}^{i} \frac{\epsilon^j}{j!} Z_{j,t}$.

Generic constants. We keep the same notation C for all nonnegative constants

(i) depending on universal constants, on a number $p \geq 1$ arising in L_p estimates, on θ_{Inf}, v_0, and K;

(ii) depending in a nondecreasing way on κ, $\frac{1}{\sqrt{1-|\rho|_{Sup}^2}}$, θ_{Sup}, ξ_{Sup}, $\frac{\xi_{Sup}}{\xi_{Inf}}$, and T.

We write $A = O(B)$ when $|A| \leq CB$ for a generic constant.

Miscellaneous.

(i) We write $\sigma_t^\epsilon = \sqrt{v_t^\epsilon}$ for the volatility for the perturbed process.

(ii) If $(Z)_{t\in[0,T]}$ is a càdlàg process, we denote by Z^* its running extremum: $Z_t^* = \sup_{s\leq t} |Z_s|$ for all $t \in [0,T]$.

(iii) The L_p norm of a random variable is denoted, as usual, by $\|Z\|_p = \mathbb{E}[|Z|^p]^{1/p}$.

4.1. Upper bounds for put derivatives.

Lemma 4.1. *For every $(i,j) \in \mathbb{N}^2$, there exists a polynomial P with positive coefficients such that*

$$\sup_{x\in\mathbb{R}} \left| \frac{\partial^{i+j} P_{BS}}{\partial x^i y^j}(x,y) \right| \leq \frac{P(\sqrt{y})}{y^{\frac{(2j+i-1)_+}{2}}}.$$

Proof. Note that it is enough to prove the estimates for $j = 0$, owing to the relation (2.2). We now take $j = 0$. For $i = 0$, the inequality holds because P_{BS} is bounded. Thus consider

Table 12
Error of implied volatility surface (in %) when calibrating the time dependent Heston model.

T/K	90.00%	95.00%	97.50%	100.00%	102.50%	105.00%	110.00%
3M	0.41%	0.24%	0.13%	0.03%	0.04%	0.05%	0.07%
1Y	−0.20%	−0.25%	−0.25%	−0.25%	−0.18%	−0.07%	0.16%
1Y+6M	−0.27%	−0.18%	−0.11%	−0.05%	0.02%	0.17%	0.46%
2Y	−0.35%	−0.32%	−0.28%	−0.25%	−0.24%	−0.20%	−0.13%

$i \geq 1$. Then, by differentiating the payoff, one gets

$$\begin{aligned}\frac{\partial^i P_{BS}}{\partial x^i}(x,y) &= \partial_x^i \mathbb{E}[e^{-\int_0^T r_t \mathrm{d}t}(K - e^{x+\int_0^T (r_t-q_t)\mathrm{d}t - \frac{y}{2} + \sqrt{\frac{y}{T}}W_T})_+] \\ &= -\partial_x^{i-1}\mathbb{E}\left[\mathbb{1}_{(e^{x+\int_0^T (r_t-q_t)\mathrm{d}t-\frac{y}{2}+\sqrt{\frac{y}{T}}W_T} \leq K)} e^{x-\int_0^T q_t \mathrm{d}t - \frac{y}{2}+\sqrt{\frac{y}{T}}W_T}\right] \\ &= -\partial_x^{i-1}\mathbb{E}[\Psi(x+G)],\end{aligned}$$

where Ψ is a bounded function (by K) and G is a Gaussian variable with zero mean and variance equal to y. For such a function, we write $\mathbb{E}[\Psi(x+G)] = \int_{\mathbb{R}} \Psi(z)\frac{e^{-(z-x)^2/(2y)}}{\sqrt{2\pi y}}\mathrm{d}z$, and from this it follows by a direct computation that

$$\left|\partial_x^{i-1}\mathbb{E}[\Psi(x+G)]\right| \leq \frac{C}{y^{\frac{i-1}{2}}}$$

for any x and y. We have proved the estimate for $j = 0$ and $i \geq 1$. ■

4.2. Positivity of the squared volatility process v. For a complete review related to time homogeneous CIR processes, we refer the reader to [20]. For time dependent CIR processes, see [30], where the existence and representation using squared Bessel processes are provided.

To prove the positivity of the process v, we show that it can be bounded from below by a suitable time homogeneous CIR process, time scale being the only difference (see Definition 5.1.2 in [39]). The arguments are quite standard, but since we need a specific statement that is not available in the literature, we detail the result and its proof. The time change $t \mapsto A_t$ is defined by

$$t = \int_0^{A_t} \xi_s^2 \mathrm{d}s.$$

Because $\xi_{Inf} > 0$, A is a continuous, strictly increasing time change and its inverse A^{-1} enjoys the same properties.

Lemma 4.2. *Assume Assumption* (P) *and $v_0 > 0$. Denote by $(y_s)_{0\leq s\leq A_T^{-1}}$ the CIR process defined by*

$$\mathrm{d}y_t = \left(\frac{1}{2} - \frac{\kappa}{\xi_{Inf}^2}y_t\right)\mathrm{d}t + \sqrt{y_t}d\tilde{B}_t, \quad y_0 = v_0,$$

where $\tilde{B}$ is the Brownian motion given by

$$\tilde{B}_t = \int_0^{A_T} \xi_s \mathrm{d}B_s. \tag{4.1}$$

Then, a.s. one has $v_t \geq y_{A_t^{-1}}$ *for any* $t \in [0,T]$. *In particular,* $(v_t)_{0\leq t\leq T}$ *is a.s. positive.*

Proof. Note that $(\tilde{B}_t)_{0\leq t\leq A_T^{-1}}$ is really a Brownian motion because by Lévy's characterization theorem it is a continuous local martingale with $\langle \tilde{B}, \tilde{B}\rangle_t = t$ (see Proposition 5.1.5 in [39] for the computation of the bracket). Now that we have set $\tilde{v}_t = v_{A_t}$, our aim is to prove that $\tilde{v}_t \geq y_t$ for $t \in [0, A_T^{-1}]$. Using Propositions 5.1.4 and 5.1.5 in [39], we write

$$\tilde{v}_t = v_0 + \int_0^{A_t} (\kappa(\theta_s - v_s)\mathrm{d}s + \xi_s\sqrt{v_s}\mathrm{d}B_s) = v_0 + \int_0^t \left(\frac{\kappa}{\xi_{A_s}^2}(\theta_{A_s} - \tilde{v}_s)\mathrm{d}s + \sqrt{\tilde{v}_s}\mathrm{d}\tilde{B}_s\right).$$

Now we apply a comparison result for SDEs twice (see Proposition 5.2.18 in [24]).

1. First, one gets $\tilde{v}_t \geq n_t$, where $(n_s)_s$ is the (unique) solution of

$$n_t = 0 + \int_0^t -\frac{\kappa}{\xi_{A_s}^2} n_s \mathrm{d}s + \sqrt{n_s}\mathrm{d}\tilde{B}_s,$$

because $v_0 \geq 0$ and $\frac{\kappa}{\xi_{A_s}^2}(\theta_{A_s} - x) \geq -\frac{\kappa}{\xi_{A_s}^2}x$ for all $x \in \mathbb{R}$ and $s \in [0, A_T^{-1}]$. Of course $n_t = 0$; thus $\tilde{v}_t$ is nonnegative.

2. Second, using the nonnegativity of $\tilde{v}$, we need only compare drift coefficients for the nonnegative variable x. Under Assumption (P), since

$$\frac{\kappa}{\xi_{A_s}^2}(\theta_{A_s} - x) \geq \frac{1}{2} - \frac{\kappa}{\xi_{Inf}^2}x \quad \forall x \geq 0, \forall s \in [0, A_T^{-1}],$$

we obtain $\tilde{v}_t \geq y_t$ for $t \in [0, A_T^{-1}]$ a.s.

Moreover, the positivity of y (and consequently that of v) is standard: indeed, y is a two-dimensional squared Bessel process with a time/space scale change (see [20] or the proof of Lemma 4.3). ■

4.3. Upper bound for negative moments of the integrated squared volatility process $\int_0^T v_t dt$.

Lemma 4.3. *Assume Assumption* (P). *Then, for every* $p > 0$, *one has*

$$\sup_{0\leq\epsilon\leq 1} \mathbb{E}\left[\left(\int_0^T v_t^\epsilon \mathrm{d}t\right)^{-p}\right] \leq \frac{C}{T^p}.$$

Before proving the result, we mention that analogous estimates appear in [12] (Lemmas A.1 and A.2): some exponential moments are stated under stronger conditions than those in Assumption (P). In addition, the uniformity of the estimates w.r.t. ξ (or equivalently w.r.t. ϵ) is not emphasized. In our study, it is crucial to get uniform estimates w.r.t. ϵ.

Proof. Fix $p \geq \frac{1}{2}$ (for $0 < p < \frac{1}{2}$, we derive the result from the case $p = \frac{1}{2}$ using the Hölder inequality). The proof is divided into two steps. We first prove the estimates in the case of constant coefficients κ, θ, and ξ with $\kappa\theta = \frac{1}{2}$, $\epsilon = 1$, and $\xi = 1$. Then, using the time change of Lemma 4.2, we derive the result for $(v_t^\epsilon)_t$. The critical point is to get estimates that are uniform w.r.t. ϵ.

Step 1. Take $\theta_t \equiv \theta$, $\xi_t \equiv 1$, $\kappa\theta = \frac{1}{2}$, $\epsilon = 1$ and consider

$$\mathrm{d}y_t = \left(\frac{1}{2} - \kappa y_t\right)\mathrm{d}t + \sqrt{y_t}\mathrm{d}B_t, \quad y_0 = v_0$$

for a standard Brownian motion B. We represent y as a time/space transformed squared Bessel process (see [20])

$$y_t = e^{-\kappa t} z_{\frac{(e^{\kappa t}-1)}{4\kappa}},$$

where z is a two-dimensional squared Bessel process. Therefore, using a change of variable and the explicit expression of the Laplace transform for the integral of z (see [11, p. 377]), one obtains for any $u \geq 0$

$$\begin{aligned}\mathbb{E}\left[\exp\left(-u\int_0^T y_t \mathrm{d}t\right)\right] &\leq \mathbb{E}\left[\exp\left(-4ue^{-2\kappa T}\int_0^{\frac{(e^{\kappa T}-1)}{4\kappa}} z_s \mathrm{d}s\right)\right]\\ &\leq \cosh\left(\frac{\sqrt{2u}(1-e^{-\kappa T})}{2\kappa}\right)^{-1}\exp\left(-\sqrt{2u}e^{-\kappa T}v_0\tanh\left(\frac{\sqrt{2u}(1-e^{-\kappa T})}{2\kappa}\right)\right).\end{aligned}$$

Combining this with the identity $x^{-p} = \frac{1}{\Gamma(p)}\int_0^\infty u^{p-1}e^{-ux}\mathrm{d}u$ for $x = \int_0^T y_t\mathrm{d}t$, one gets

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^T y_t\mathrm{d}t\right)^{-p}\right] \leq \frac{1}{\Gamma(p)}\int_0^\infty u^{p-1}\cosh\left(\frac{\sqrt{2u}(1-e^{-\kappa T})}{2\kappa}\right)^{-1}\\ \times\exp\left(-\sqrt{2u}e^{-\kappa T}v_0\tanh\left(\frac{\sqrt{2u}(1-e^{-\kappa T})}{2\kappa}\right)\right)\mathrm{d}u.\end{aligned}$$

Define the parameter $\lambda^2 = \frac{(e^{\kappa T}-1)}{2\kappa v_0}$ and the new variable $n = \frac{\sqrt{2u}(1-e^{-\kappa T})}{2\kappa} = v_0 e^{-\kappa T}\lambda^2\sqrt{2u}$. It readily follows that

$$\mathbb{E}\left[\left(\int_0^T y_t\mathrm{d}t\right)^{-p}\right] \leq C\left(\frac{e^{\kappa T}}{\lambda^2}\right)^{2p}\int_0^\infty n^{2p-1}\cosh(n)^{-1}\exp\left(-\frac{\tanh(n)n}{\lambda^2}\right)\mathrm{d}n,$$

where C is a constant depending only on v_0 and p. We upper bound the above integral differently according to the value of λ.

(i) If $\lambda \geq 1$, then

$$\mathbb{E}\left[\left(\int_0^T y_t\mathrm{d}t\right)^{-p}\right] \leq C\left(\frac{e^{\kappa T}}{\lambda^2}\right)^{2p}\int_0^\infty n^{2p-1}\cosh(n)^{-1}\mathrm{d}n \leq Ce^{2p\kappa T}. \tag{4.2}$$

(ii) If $\lambda \leq 1$, split the integral into two parts, $n \leq \operatorname{arctanh}(\lambda)$ and $n \geq \operatorname{arctanh}(\lambda)$. For the first part, simply use $n \geq \tanh(n)$ for any n. For the second part, use $\tanh(n) \geq \lambda$ and

$\cosh(n)^{-1} \leq 1$. This gives

$$\mathbb{E}\left[\left(\int_0^T y_t \mathrm{d}t\right)^{-p}\right] \leq C\Bigg[\left(\frac{e^{\kappa T}}{\lambda^2}\right)^{2p} \int_0^{\operatorname{arctanh}(\lambda)} n^{2p-1}\cosh(n)^{-1}\exp\left(-\frac{\tanh^2(n)}{\lambda^2}\right)\mathrm{d}n$$
$$+\left(\frac{e^{\kappa T}}{\lambda^2}\right)^{2p}\int_{\operatorname{arctanh}(\lambda)}^{\infty} n^{2p-1}\exp\left(-\frac{n}{\lambda}\right)\mathrm{d}n\Bigg] := C[\mathcal{T}_1+\mathcal{T}_2]. \tag{4.3}$$

We upper bound the two terms separately.

1. Term $\mathcal{T}_1$. Using the change of variable $m = \frac{\tanh(n)}{\lambda}$, one has

$$\mathcal{T}_1 \leq e^{2p\kappa T}\lambda^{-4p+1}\int_0^1 \operatorname{arctanh}(\lambda m)^{2p-1}\cosh(\operatorname{arctanh}(\lambda m))\exp(-m^2)\mathrm{d}m.$$

Because of $\lambda \leq 1$, we have the following inequalities for $m \in [0,1[$:

$$\operatorname{arctanh}(\lambda m) \leq \lambda \operatorname{arctanh}(m), \quad \cosh(\operatorname{arctanh}(\lambda m)) \leq \cosh(\operatorname{arctanh}(m)).$$

Using $2p-1 \geq 0$, it readily follows that

$$\mathcal{T}_1 \leq \left(\frac{e^{2\kappa T}}{\lambda^2}\right)^{p}\int_0^1 \operatorname{arctanh}(m)^{2p-1}\cosh(\operatorname{arctanh}(m))\exp(-m^2)\mathrm{d}m. \tag{4.4}$$

2. Term $\mathcal{T}_2$. Clearly, we have

$$\mathcal{T}_2 \leq \left(\frac{e^{\kappa T}}{\lambda^2}\right)^{2p}\int_0^{\infty} n^{2p-1}\exp\left(-\frac{n}{\lambda}\right)\mathrm{d}n = \left(\frac{e^{2\kappa T}}{\lambda^2}\right)^{p}\int_0^{\infty} v^{2p-1}e^{-v}\mathrm{d}v. \tag{4.5}$$

Combining (4.3), (4.4), and (4.5), we obtain $\mathbb{E}[(\int_0^T y_t\mathrm{d}t)^{-p}] \leq C(\frac{e^{2\kappa T}}{\lambda^2})^p$. In view of the inequality $(e^x - 1 \geq x, x \geq 0)$, we have $\lambda^2 = \frac{(e^{\kappa T}-1)}{2\kappa v_0} \geq \frac{T}{2v_0}$, which gives

$$\mathbb{E}\left[\left(\int_0^T y_t \mathrm{d}t\right)^{-p}\right] \leq C\frac{e^{2p\kappa T}}{T^p}, \tag{4.6}$$

available when $\lambda \leq 1$.

To sum up (4.2) and (4.6), we have proved that

$$\mathbb{E}\left[\left(\int_0^T y_t \mathrm{d}t\right)^{-p}\right] \leq Ce^{2p\kappa T}\left(1+\frac{1}{T^p}\right) \tag{4.7}$$

for a constant C depending only on p and v_0.

Step 2. Take $\epsilon \in]0,1]$. We apply Lemma 4.2 to v^ϵ, in order to write $v_t^\epsilon \geq y^\epsilon_{A^{-1}_{\epsilon,t}}$, where $t = \int_0^{A_{\epsilon,t}}(\epsilon\xi_s)^2\mathrm{d}s$ and $\mathrm{d}y_t^\epsilon = (\frac{1}{2} - \frac{\kappa}{(\epsilon\xi_{Inf})^2}y_t^\epsilon)\mathrm{d}t + \sqrt{y_t^\epsilon}d\tilde{B}_t^\epsilon, y_0^\epsilon = y_0$. Thus, we get $\int_0^T v_t^\epsilon \mathrm{d}t \geq$

$(\int_0^{A_{\epsilon,t}^{-1}} y_s^\epsilon \mathrm{d}s)/(\epsilon\xi_{Sup})^2$ and, in view of (4.7), it follows that

$$\mathbb{E}\left[\left(\int_0^T v_t^\epsilon \mathrm{d}t\right)^{-p}\right] \le (\epsilon\xi_{Sup})^{2p}\mathbb{E}\left(\int_0^{A_{\epsilon,T}^{-1}} y_s^\epsilon \mathrm{d}s\right)^{-p}$$
$$\le C(\epsilon\xi_{Sup})^{2p} e^{2p\frac{\kappa}{(\epsilon\xi_{Inf})^2}A_{\epsilon,T}^{-1}}\left(1+\frac{1}{[A_{\epsilon,T}^{-1}]^p}\right)$$
$$\le Ce^{2p\kappa\frac{\xi_{Sup}^2}{\xi_{Inf}^2}T}\left(\xi_{Sup}^{2p}+\frac{\xi_{Sup}^{2p}}{\xi_{Inf}^{2p}}\frac{1}{T^p}\right),$$

where we have used $\epsilon^2\xi_{Inf}^2 T \le A_{\epsilon,T}^{-1} \le \epsilon^2\xi_{Sup}^2 T$.

Note that the upper bound does not depend on $\epsilon \in]0,1]$. For $\epsilon = 0$, the upper bound in Lemma 4.3 is also true because $(v_t^0)_t$ is deterministic and

$$\max(v_0, \theta_{Sup}) \ge v_t^0 \ge \min(v_0, \theta_{Inf}) > 0. \qquad \blacksquare \tag{4.8}$$

4.4. Upper bound for residuals of the Taylor development of $g(\epsilon)$ defined in (1.4). Throughout the following section, we assume that Assumption (P) is in force. We define the variables:

$$P_T^\epsilon = \int_0^T \rho_t(\sigma_t^\epsilon - \sigma_{0,t})\mathrm{d}B_t - \int_0^T \frac{\rho_t^2}{2}(v_t^\epsilon - v_{0,t})\mathrm{d}t, \qquad Q_T^\epsilon = \int_0^T (1-\rho_t^2)(v_t^\epsilon - v_{0,t})\mathrm{d}t.$$

Notice that $(x_0+\int_0^T \rho_t\sqrt{v_{0,t}}\mathrm{d}B_t - \int_0^T \frac{\rho_t^2}{2}v_{0,t}\mathrm{d}t, \int_0^T(1-\rho_t^2)v_{0,t}\mathrm{d}t) + (P_T^1, Q_T^1) = (x_0+\int_0^T \rho_t\sqrt{v_t^1}\mathrm{d}B_t - \int_0^T \frac{\rho_t^2}{2}v_t^1\mathrm{d}t, \int_0^T(1-\rho_t^2)v_t^1\mathrm{d}t)$.

The main result of this subsection is the following proposition, the statement of which uses the notation introduced at the beginning of section 4.

Proposition 4.4. *One has the following estimates for every $p \ge 1$:*

$$\|P_T^1\|_p \le C(\xi_{Sup}\sqrt{T})\sqrt{T},$$
$$\|R_{2,T}^{P^1}\|_p \le C(\xi_{Sup}\sqrt{T})^3\sqrt{T},$$
$$\|R_{2,T}^{(P^1)^2}\|_p \le C(\xi_{Sup}\sqrt{T})^3 T,$$
$$\|Q_T^1\|_p \le C(\xi_{Sup}\sqrt{T})T,$$
$$\|R_{2,T}^{Q^1}\|_p \le C(\xi_{Sup}\sqrt{T})^3 T,$$
$$\|R_{2,T}^{(Q^1)^2}\|_p \le C(\xi_{Sup}\sqrt{T})^3 T^2,$$
$$\|R_{2,T}^{P^1Q^1}\|_p \le C(\xi_{Sup}\sqrt{T})^3 T^{\frac{3}{2}}.$$

To estimate the derivatives and the residuals for the variables P_T^ϵ and Q_T^ϵ, we first need to prove the existence of the derivatives and the residuals of the volatility process $\sigma_t^\epsilon = \sqrt{v_t^\epsilon}$ and its square v^ϵ. Finally we prove Proposition 4.4.

4.4.1. Upper bounds for derivatives of σ^ϵ and v^ϵ. Under Assumption (P), the volatility process σ_t^ϵ is governed by the SDE

$$d\sigma_t^\epsilon = \left(\left(\frac{\kappa\theta_t}{2} - \frac{\epsilon^2\xi_t^2}{8}\right)\frac{1}{\sigma_t^\epsilon} - \frac{\kappa}{2}\sigma_t^\epsilon\right)\mathrm{d}t + \frac{\epsilon\xi_t}{2}\mathrm{d}B_t, \quad \sigma_0^\epsilon = \sqrt{v_0}, \tag{4.9}$$

where we have used Itô's lemma and the positivity of v_t^ϵ (see Lemma 4.2).

In order to estimate $R_{0,t}^{\sigma^\epsilon}$, we are going to prove that it verifies a linear equation (Lemma 4.5) from which we deduce an a priori upper bound (Proposition 4.6). We iterate the same analysis for the residuals $R_{1,t}^{\sigma^\epsilon}$ (Proposition 4.7) and $R_{2,t}^{\sigma^\epsilon}$ (Proposition 4.8). Analogously, we give upper bounds for the residuals of v_t^ϵ (Corollary 4.9).

Lemma 4.5. *Under Assumption (P), the process $(R_{0,t}^{\sigma^\epsilon} = \sigma_t^\epsilon - \sigma_t^0)_{0\le t\le T}$ is given by*

$$R_{0,t}^{\sigma^\epsilon} = U_t^\epsilon \int_0^t (U_s^\epsilon)^{-1}\left(-\frac{\epsilon^2\xi_s^2}{8\sigma_{0,s}}\mathrm{d}s + \frac{\epsilon\xi_s}{2}\mathrm{d}B_s\right),$$

where

$$\mathrm{d}U_t^\epsilon = -\alpha_t^\epsilon U_t^\epsilon \mathrm{d}t, \quad U_t^\epsilon = 1,$$
$$\alpha_t^\epsilon = \left(\frac{\kappa\theta_t}{2} - \frac{\epsilon^2\xi_t^2}{8}\right)\frac{1}{\sigma_t^\epsilon\sigma_{0,t}} + \frac{\kappa}{2}.$$

Proof. From the definition $(\sigma_{0,t})_t = (\sigma_t^0)_t$ and (4.9), one obtains the SDE

$$d\sigma_{0,t} = \left(\frac{\kappa\theta_t}{2\sigma_{0,t}} - \frac{\kappa}{2}\sigma_{0,t}\right)\mathrm{d}t, \quad \sigma_{0,0} = \sqrt{v_0}.$$

Substitute this equation in (4.9) to obtain

$$\mathrm{d}R_{0,t}^{\sigma^\epsilon} = -\alpha_t^\epsilon R_{0,t}^{\sigma^\epsilon}\mathrm{d}t - \frac{\epsilon^2\xi_t^2}{8\sigma_{0,t}}\mathrm{d}t + \frac{\epsilon\xi_t}{2}\mathrm{d}B_t, \quad R_{0,0}^{\sigma^\epsilon} = 0. \tag{4.10}$$

Note that $R_{0,.}^{\sigma^\epsilon}$ is the solution of a linear SDE. Hence, it can be explicitly represented using the process U^ϵ (see Theorem 52 in [37]):

$$R_{0,t}^{\sigma^\epsilon} = U_t^\epsilon \int_0^t (U_s^\epsilon)^{-1}\left(-\frac{\epsilon^2\xi_s^2}{8\sigma_{0,s}}\mathrm{d}s + \frac{\epsilon\xi_s}{2}\mathrm{d}B_s\right). \qquad \blacksquare$$

Proposition 4.6. *Under Assumption (P), for every $p \ge 1$ one has*

$$\|(R_{0,.}^{\sigma^\epsilon})_t^*\|_p \le C\epsilon\xi_{Sup}\sqrt{t}.$$

In particular, the application $\epsilon \to \sigma_t^\epsilon$ is continuous[6] *at $\epsilon = 0$ in L_p.*

[6] Note that from the upper bound (4.11) in the proof we easily obtain that the continuity also holds a.s. and not only in L_p. Since only the latter is needed in what follows, we do not go into detail.

Proof. At first sight, the proof seems to be straightforward from Lemma 4.5. But actually, the difficulty lies in the fact that one cannot uniformly in ϵ upper bound U_t^ϵ in L_p (because of the term with $1/\sigma_t^\epsilon$ in α_t^ϵ).

Using Lemma 4.5 and Itô's formula for the product $(U_t^\epsilon)^{-1}(\int_0^t \frac{\epsilon\xi_s}{2}\mathrm{d}B_s)$, one has

$$R_{0,t}^{\sigma^\epsilon} = U_t^\epsilon \int_0^t (U_s^\epsilon)^{-1}\left(-\frac{\epsilon^2\xi_s^2}{8\sigma_{0,s}}\right)\mathrm{d}s + \int_0^t \frac{\epsilon\xi_s}{2}\mathrm{d}B_s - U_t^\epsilon \int_0^t \left(\int_0^s \frac{\epsilon\xi_u}{2}\mathrm{d}B_u\right) d(U_s^\epsilon)^{-1}.$$

Under Assumption (P), one has $\alpha_t^\epsilon \geq \kappa/2 > 0$, which implies that $t \mapsto U_t^\epsilon$ is decreasing and $t \mapsto (U_t^\epsilon)^{-1}$ is increasing. Thus, $0 \leq U_t^\epsilon (U_s^\epsilon)^{-1} \leq 1$ for $s \in [0,t]$. Consequently, we deduce

$$\begin{aligned} |R_{0,t}^{\sigma^\epsilon}| &\leq \int_0^t \frac{\epsilon^2\xi_s^2}{8\sigma_{0,s}}\mathrm{d}s + \left(\int_0^\cdot \frac{\epsilon\xi_s}{2}\mathrm{d}B_s\right)_t^* + \left(\int_0^\cdot \frac{\epsilon\xi_s}{2}\mathrm{d}B_s\right)_t^*(1-U_t^\epsilon) \\ &\leq \int_0^t \frac{\epsilon^2\xi_s^2}{8\sigma_{0,s}}\mathrm{d}s + \left(\int_0^\cdot \epsilon\xi_s \mathrm{d}B_s\right)_t^*. \end{aligned} \tag{4.11}$$

Now we easily complete the proof by observing that $\sigma_{0,s} \geq \min(\sqrt{\theta_{Inf}}, \sqrt{v_0})$ and $\|(\int_0^\cdot \xi_s \mathrm{d}B_s)_t^*\|_p \leq C\xi_{Sup}\sqrt{t}$. ∎

We define $(\sigma_{1,t})_{0\leq t\leq T}$ as the solution of the linear equation (4.12), which is obtained by taking derivatives w.r.t. ϵ of (4.9) and setting ϵ equal to zero thereafter:

$$d\sigma_{1,t} = -\left(\frac{\kappa\theta_t}{2(\sigma_{0,t})^2} + \frac{\kappa}{2}\right)\sigma_{1,t}\mathrm{d}t + \frac{\xi_t}{2}\mathrm{d}B_t, \quad \sigma_{1,0} = 0. \tag{4.12}$$

The solution of this SDE is obtained similarly as was done with (4.10) and is given by the equation

$$\sigma_{1,t} = U_t^0 \int_0^t (U_s^0)^{-1}\frac{\xi_s}{2}\mathrm{d}B_s.$$

For every $p \geq 1$, taking into account that $0 \leq U_t^0 (U_s^0)^{-1} \leq 1$ for $s \in [0,t]$, the upper bound is given by (4.13) (see now that there is a uniform bound w.r.t. ϵ):

$$\|(\sigma_{1,.})_t^*\|_p \leq C\xi_{Sup}\sqrt{t}. \tag{4.13}$$

Proposition 4.7. *Under Assumption (P), the process $(R_{1,t}^{\sigma^\epsilon} = \sigma_t^\epsilon - \sigma_t^0 - \epsilon\sigma_{1,t})_{0\leq t\leq T}$ fulfills the equality*

$$R_{1,t}^{\sigma^\epsilon} = U_t^\epsilon \int_0^t (U_s^\epsilon)^{-1}\left(-\frac{\epsilon^2\xi_s^2}{8\sigma_{0,s}} + \epsilon\sigma_{1,s}\left(\left(\frac{\alpha_s^\epsilon}{\sigma_{0,s}} - \frac{\kappa}{2\sigma_{0,s}}\right)R_{0,s}^{\sigma^\epsilon} + \frac{\epsilon^2\xi_s^2}{8\sigma_{0,s}}\right)\right)\mathrm{d}s.$$

Moreover, for every $p \geq 1$, one has

$$\|(R_{1,.}^{\sigma^\epsilon})_t^*\|_p \leq C(\epsilon\xi_{Sup}\sqrt{t})^2.$$

In particular, the application $\epsilon \to \sigma_t^\epsilon$ is $\mathcal{C}^1$ at $\epsilon = 0$ in the L_p sense with the first derivative at $\epsilon = 0$ equal to $\sigma_{1,t}$ (justifying a posteriori the definition $R_{1,.}^{\sigma^\epsilon}$).

Proof. From (4.10) and (4.12), it readily follows that

$$\mathrm{d}R^{\sigma^\epsilon}_{1,t} = -\alpha^\epsilon_t R^{\sigma^\epsilon}_{1,t}\mathrm{d}t - \epsilon\sigma_{1,t}\left(\alpha^\epsilon_t - \frac{\kappa\theta_t}{2(\sigma_{0,t})^2} - \frac{\kappa}{2}\right)\mathrm{d}t - \frac{\epsilon^2\xi_t^2}{8\sigma_{0,t}}\mathrm{d}t, \quad R^{\sigma^\epsilon}_{1,0} = 0.$$

Because of the identity

$$-\left(\alpha^\epsilon_t - \frac{\kappa\theta_t}{2(\sigma_{0,t})^2} - \frac{\kappa}{2}\right) = \left(\left(\frac{\alpha^\epsilon_t}{\sigma_{0,t}} - \frac{\kappa}{2\sigma_{0,t}}\right)R^{\sigma^\epsilon}_{0,t} + \frac{\epsilon^2\xi_t^2}{8(\sigma_{0,t})^2}\right),$$

one deduces the equality

$$R^{\sigma^\epsilon}_{1,t} = U^\epsilon_t\int_0^t (U^\epsilon_s)^{-1}\left(-\frac{\epsilon^2\xi_s^2}{8\sigma_{0,s}} + \epsilon\sigma_{1,s}\left(\left(\frac{\alpha^\epsilon_s}{\sigma_{0,s}} - \frac{\kappa}{2\sigma_{0,s}}\right)R^{\sigma^\epsilon}_{0,s} + \frac{\epsilon^2\xi_s^2}{8(\sigma_{0,s})^2}\right)\right)\mathrm{d}s.$$

Then

$$\begin{aligned}
|R^{\sigma^\epsilon}_{1,t}| &\le \int_0^t U^\epsilon_t(U^\epsilon_s)^{-1}\left(\frac{\epsilon^2\xi_s^2}{8\sigma_{0,s}} + \epsilon|\sigma_{1,s}|\left(\left(\frac{\alpha^\epsilon_s}{\sigma_{0,s}} + \frac{\kappa}{2\sigma_{0,s}}\right)|R^{\sigma^\epsilon}_{0,s}| + \frac{\epsilon^2\xi_s^2}{8(\sigma_{0,s})^2}\right)\right)\mathrm{d}s\\
&\le \int_0^t U^\epsilon_t(U^\epsilon_s)^{-1}\left(\frac{\epsilon^2\xi_s^2}{8\sigma_{0,s}} + \epsilon|\sigma_{1,s}|\left(\frac{\kappa}{2\sigma_{0,s}}|R^{\sigma^\epsilon}_{0,s}| + \frac{\epsilon^2\xi_s^2}{8(\sigma_{0,s})^2}\right)\right)\mathrm{d}s\\
&\quad + \epsilon\int_0^t U^\epsilon_t(U^\epsilon_s)^{-1}\frac{\alpha^\epsilon_s}{\sigma_{0,s}}|\sigma_{1,s}||R^{\sigma^\epsilon}_{0,s}|\mathrm{d}s\\
&\le \int_0^t \left(\frac{\epsilon^2\xi_s^2}{8\sigma_{0,s}} + \epsilon|\sigma_{1,s}|\left(\frac{\kappa}{2\sigma_{0,s}}|R^{\sigma^\epsilon}_{0,s}| + \frac{\epsilon^2\xi_s^2}{8(\sigma_{0,s})^2}\right)\right)\mathrm{d}s + \epsilon\left(\frac{\sigma_{1,.}R^{\sigma^\epsilon}_{0,.}}{\sigma_{0,.}}\right)^*_t,
\end{aligned}$$

where we have used $U^\epsilon_t(U^\epsilon_s)^{-1} \le 1$ for every $s \in [0,t]$ and $U^\epsilon_t\int_0^t \alpha^\epsilon_s(U^\epsilon_s)^{-1}\mathrm{d}s = 1 - U^\epsilon_t \le 1$ for the third inequality. Apply Proposition 4.6 and inequality (4.13) to complete the proof of the estimate of $\|(R^{\sigma^\epsilon}_{1,.})^*_t\|_p$. ∎

We define $(\sigma_{2,t})_{0\le t\le T}$ as the solution of the linear equation (4.14), which is obtained by differentiating twice (4.9) w.r.t. ϵ and setting ϵ equal to zero:

$$d\sigma_{2,t} = \left(-\left(\frac{\kappa\theta_t}{2(\sigma_{0,t})^2} + \frac{\kappa}{2}\right)\sigma_{2,t} + \kappa\theta_t\frac{(\sigma_{1,t})^2}{(\sigma_{0,t})^3} - \frac{\xi_t^2}{4\sigma_{0,t}}\right)\mathrm{d}t, \quad \sigma_{2,0} = 0. \tag{4.14}$$

Clearly, for $p \ge 1$, we have

$$\|(\sigma_{2,.})^*_t\|_p \le C(\xi_{Sup}\sqrt{t})^2. \tag{4.15}$$

Proposition 4.8. *Under Assumption (P), the process $(R^{\sigma^\epsilon}_{2,t} = \sigma^\epsilon_t - \sigma^0_t - \epsilon\sigma_{1,t} - \frac{\epsilon^2}{2}\sigma_{2,t})_{0\le t\le T}$ fulfills the equality*

$$\begin{aligned}
R^{\sigma^\epsilon}_{2,t} = U^\epsilon_t\int_0^t (U^\epsilon_s)^{-1}\Bigg[&\epsilon^2\left(\left(\frac{\alpha^\epsilon_s}{\sigma_{0,s}} - \frac{\kappa}{2\sigma_{0,s}}\right)R^{\sigma^\epsilon}_{0,s} + \frac{\epsilon^2\xi_s^2}{8(\sigma_{0,s})^2}\right)\left(\frac{\sigma_{2,s}}{2} - \frac{(\sigma_{1,s})^2}{\sigma_{0,s}}\right)\\
&+ \epsilon\left(\left(\frac{\alpha^\epsilon_s}{\sigma_{0,s}} - \frac{\kappa}{2\sigma_{0,s}}\right)R^{\sigma^\epsilon}_{1,s} + \frac{\epsilon^2\xi_s^2}{8(\sigma_{0,s})^2}\right)\sigma_{1,s}\Bigg]\mathrm{d}s.
\end{aligned}$$

Moreover, for every $p \geq 1$, one has

$$\|(R^{\sigma^\epsilon}_{2,.})^*_t\|_p \leq C(\epsilon\xi_{Sup}\sqrt{t})^3.$$

In particular, the application $\epsilon \to \sigma^\epsilon_t$ is $\mathcal{C}^2$ at $\epsilon = 0$ in the L_p sense with the second derivative at $\epsilon = 0$ equal to $\sigma_{2,t}$.

Proof. The equality is easy to check. The estimate is proved in the same way as in the proof of Proposition 4.7; we therefore skip the details. ■

Corollary 4.9. *The application $\epsilon \to v^\epsilon_t$ is $\mathcal{C}^2$ at $\epsilon = 0$ in the L_p sense. The residuals for the squared volatility satisfy the following inequalities: for every $p \geq 1$, one has*

$$\|(R^{v^\epsilon}_{0,.})^*_t\|_p \leq C\epsilon\xi_{Sup}\sqrt{t},$$
$$\|(R^{v^\epsilon}_{1,.})^*_t\|_p \leq C(\epsilon\xi_{Sup}\sqrt{t})^2,$$
$$\|(R^{v^\epsilon}_{2,.})^*_t\|_p \leq C(\epsilon\xi_{Sup}\sqrt{t})^3.$$

Proof. Note that $v^\epsilon_t = (\sigma^\epsilon_t)^2 = (\sigma_{0,t} + R^{\sigma^\epsilon}_{0,t})^2 = v_{0,t} + 2\sigma_{0,t}R^{\sigma^\epsilon}_{0,t} + (R^{\sigma^\epsilon}_{0,t})^2$. Thus, we have $R^{v^\epsilon}_{0,t} = 2\sigma_{0,t}R^{\sigma^\epsilon}_{0,t} + (R^{\sigma^\epsilon}_{0,t})^2$, which leads to the required estimate using $\sigma_{0,t} \leq \max(\sqrt{v_0}, \sqrt{\theta_{Sup}})$ and Proposition 4.6. The other estimates are proved analogously using Propositions 4.7 and 4.8 and inequalities (4.13) and (4.15). ■

4.4.2. Proof of Proposition 4.4.

We can write

$$P^1_T = \int_0^T \rho_t R^{\sigma^1}_{0,t}\mathrm{d}B_t - \int_0^T \frac{\rho_t^2}{2}R^{v^1}_{0,t}\mathrm{d}t, \quad R^{P^1}_{2,T} = \int_0^T \rho_t R^{\sigma^1}_{2,t}\mathrm{d}B_t - \int_0^T \frac{\rho_t^2}{2}R^{v^1}_{2,t}\mathrm{d}t.$$

Then, using Propositions 4.6 and 4.8 and Corollary 4.9, we prove the two first estimates of Proposition 4.4. The others inequalities are proved in the same way.

4.5. Proof of Theorem 2.3.

For convenience, we introduce the following notation for $\lambda \in [0, 1]$:

$$\bar{P}_{BS}(\lambda) = P_{BS}\Bigg(x_0 + \int_0^T \rho_t\left((1-\lambda)\sqrt{v_{0,t}} + \lambda\sqrt{v^1_t}\right)\mathrm{d}B_t - \int_0^T \frac{\rho_t^2}{2}((1-\lambda)v_{0,t} + \lambda v^1_t)\mathrm{d}t,$$
$$\int_0^T (1-\rho_t^2)((1-\lambda)v_{0,t} + \lambda v^1_t)\mathrm{d}t\Bigg),$$
$$\frac{\partial^{i+j}\bar{P}_{BS}}{\partial x^i y^j}(\lambda) = \frac{\partial^{i+j}P_{BS}}{\partial x^i y^j}\Bigg(x_0 + \int_0^T \rho_t\left((1-\lambda)\sqrt{v_{0,t}} + \lambda\sqrt{v^1_t}\right)\mathrm{d}B_t - \int_0^T \frac{\rho_t^2}{2}((1-\lambda)v_{0,t} + \lambda v^1_t)\mathrm{d}t,$$
$$\int_0^T (1-\rho_t^2)((1-\lambda)v_{0,t} + \lambda v^1_t)\mathrm{d}t\Bigg).$$

Notice that $\tilde{P}_{BS}$ (see (2.5)) is a particular case of $\bar{P}_{BS}$ for $\lambda = 0$:

$$\tilde{P}_{BS} = \bar{P}_{BS}(0), \quad \frac{\partial^{i+j}\tilde{P}_{BS}}{\partial x^i y^j} = \frac{\partial^{i+j}\bar{P}_{BS}}{\partial x^i y^j}(0).$$

Now, we represent the error $\mathcal{E}$ in (2.12) using the previous notation. A second order Taylor expansion leads to

$$g(1) = \mathbb{E}(\bar{P}_{BS}(1)) = \mathbb{E}\left(\bar{P}_{BS}(0) + \partial_\lambda \bar{P}_{BS}(0) + \frac{1}{2}\partial^2_\lambda \bar{P}_{BS}(0) + \int_0^1 \mathrm{d}\lambda \frac{(1-\lambda)^2}{2}\partial^3_\lambda \bar{P}_{BS}(\lambda)\right).$$

The first term $\mathbb{E}(\bar{P}_{BS}(0))$ is equal to (2.6). Approximations of the three above derivatives contribute to the error $\mathcal{E}$.

1. We have $\mathbb{E}(\partial_\lambda \bar{P}_{BS}(0)) = \mathbb{E}(\frac{\partial \tilde{P}_{BS}}{\partial x} P^1_T + \frac{\partial \tilde{P}_{BS}}{\partial y} Q^1_T)$. These two terms are equal to (2.7) and (2.8) plus an error equal to

$$\mathbb{E}\left(\frac{\partial \tilde{P}_{BS}}{\partial x} R^{P^1}_{2,T} + \frac{\partial \tilde{P}_{BS}}{\partial y} R^{Q^1}_{2,T}\right).$$

2. W.r.t. the second derivatives, we have $\mathbb{E}(\frac{1}{2}\partial^2_\lambda \bar{P}_{BS}(0)) = \mathbb{E}(\frac{1}{2}\frac{\partial^2 \tilde{P}_{BS}}{\partial x^2}(P^1_T)^2 + \frac{1}{2}\frac{\partial^2 \tilde{P}_{BS}}{\partial y^2}(Q^1_T)^2 + \frac{\partial^2 \tilde{P}_{BS}}{\partial xy} P^1_T Q^1_T)$. These terms are equal to (2.9), (2.10), and (2.11) plus an error equal to

$$\mathbb{E}\left(\frac{1}{2}\frac{\partial^2 \tilde{P}_{BS}}{\partial x^2} R^{(P^1)^2}_{2,T} + \frac{1}{2}\frac{\partial^2 \tilde{P}_{BS}}{\partial y^2} R^{(Q^1)^2}_{2,T} + \frac{\partial^2 \tilde{P}_{BS}}{\partial xy} R^{P^1 Q^1}_{2,T}\right).$$

3. The last term with $\partial^3_\lambda \bar{P}_{BS}$ is neglected and thus is considered as an error.

To sum up, we have shown that

$$\begin{aligned}\mathcal{E} = &\sum_{i=0}^{1} \mathbb{E}\left[\frac{\partial^1 \bar{P}_{BS}}{\partial x^i y^{1-i}}(0) R^{(P^1)^i (Q^1)^{1-i}}_{2,T}\right] + \sum_{i=0}^{2} \frac{C^i_2}{2}\mathbb{E}\left[\frac{\partial^2 \bar{P}_{BS}}{\partial x^i y^{2-i}}(0) R^{(P^1)^i (Q^1)^{2-i}}_{2,T}\right] \\ &+ \int_0^1 \frac{(1-\lambda)^2}{2} \sum_{i=0}^{3} C^i_3 \mathbb{E}\left[\frac{\partial^3 \bar{P}_{BS}}{\partial x^i y^{3-i}}(\lambda)(P^1_T)^i (Q^1_T)^{3-i}\right] \mathrm{d}\lambda.\end{aligned}$$

Using Lemma 4.1 and Assumption (R), one obtains for all $\lambda \in [0,1]$

$$\begin{aligned}\left\|\frac{\partial^{i+j} \bar{P}_{BS}}{\partial x^i y^j}(\lambda)\right\|_2 &\le C\left\|\left(\int_0^T ((1-\lambda)v_{0,t} + \lambda v^1_t)\mathrm{d}t\right)^{\frac{-(2j+i-1)_+}{2}}\right\|_4 \\ &\le C\left((1-\lambda)\left\|\left(\int_0^T v_{0,t}\mathrm{d}t\right)^{\frac{-(2j+i-1)_+}{2}}\right\|_4 + \lambda\left\|\left(\int_0^T v^1_t \mathrm{d}t\right)^{\frac{-(2j+i-1)_+}{2}}\right\|_4\right),\end{aligned}$$

where we have applied a convexity argument. Finally, apply Lemma 4.3 with $\epsilon = 0$ and $\epsilon = 1$ to conclude that

$$\left\|\frac{\partial^{i+j} \bar{P}_{BS}}{\partial x^i y^j}(\lambda)\right\|_2 \le \frac{C}{(\sqrt{T})^{(2j+i-1)_+}}$$

uniformly w.r.t. $\lambda \in [0,1]$. Combining this with Proposition 4.4 yields that

$$\begin{aligned}|\mathcal{E}| &\le C\left(\sum_{i=0}^{1}(\xi_{Sup}\sqrt{T})^3\frac{T^{1-i/2}}{(\sqrt{T})^{1-i}}+\sum_{i=0}^{2}(\xi_{Sup}\sqrt{T})^3\frac{T^{2-i/2}}{(\sqrt{T})^{3-i}}+\sum_{i=0}^{3}(\xi_{Sup}\sqrt{T})^3\frac{T^{3-i/2}}{(\sqrt{T})^{5-i}}\right)\\ &\le C\xi_{Sup}^3T^2.\end{aligned}$$

Theorem 2.3 is proved.

5. Proofs of Proposition 2.1 and Theorem 2.2.

5.1. Definitions. In order to make the approximation explicit, we introduce the following family of operators indexed by maturity T.

Definition 5.1 (integral operator). *We define the integral operator $\omega_{.,T}^{(.,.)}$ as follows:*

(i) *For any real number k and any integrable function l, we set*

$$\omega_{t,T}^{(k,l)} = \int_t^T e^{ku}l_u\mathrm{d}u \quad \forall t \in [0,T].$$

(ii) *For any real numbers $(k_1,\dots,k_n)$ and for any integrable functions $(l_1,\dots,l_n)$, the n-times iteration is given by*

$$\omega_{t,T}^{(k_1,l_1),\dots,(k_n,l_n)} = \omega_{t,T}^{(k_1,l_1\omega_{.,T}^{(k_2,l_2),\dots,(k_n,l_n)})} \quad \forall t \in [0,T].$$

(iii) *When the functions $(l_1,\dots,l_n)$ are equal to the unity constant function 1, we simply write*

$$\tilde{\omega}_{t,T}^{k_1,\dots,k_n} = \omega_{t,T}^{(k_1,1),\dots,(k_n,1)} \quad \forall t \in [0,T].$$

Note that with this short notation used in Theorem 2.2, we have $a_{1,T} = \omega_{0,T}^{(\kappa,\rho\xi v_{0,.}),(-\kappa,1)}$, $a_{2,T} = \omega_{0,T}^{(\kappa,\rho\xi v_{0,.}),(0,\rho\xi),(-\kappa,1)}$, and $b_{0,T} = \omega_{0,T}^{(2\kappa,\xi^2 v_{0,.}),(-\kappa,1),(-\kappa,1)}$.

5.2. Preliminary results. In this section, we bring together the results (and their proofs) which allow us to derive the explicit terms in the formula (2.13). Before stating the lemmas, we give some guidance about the way to express the correction terms as derivatives of the leading price (2.13):

- Lemma 5.2 is the equivalent for random variables of the integration by parts formula. It allows expressing the expectation of an Itô integral multiplied by a random variable. This will lead to taking derivatives of $\tilde{P}_{BS}$ (the objective of the reduction).
- Lemma 5.3 is a simpler version of Lemma 5.2 in order to simplify some calculus.
- Lemma 5.4 is a simple application of the integration by parts formula w.r.t. time.
- Lemma 5.5 computes the expectation of integrals of derivatives w.r.t. ϵ of the stochastic process v_t in terms of the expectation of the derivatives of $\tilde{P}_{BS}$.
- Lemma 5.6 computes the expectation of the derivatives of $\tilde{P}_{BS}$.

In the following, α_t (resp., β_t) is a square integrable and predictable process (resp., deterministic) and l is a smooth function with derivatives having, at most, exponential growth.

For the next Malliavin calculus computations, we freely use standard notation from [33].

Lemma 5.2 (see [33, Lemma 1.2.1]). *Let $G \in \mathbb{D}^{1,\infty}(\Omega)$. One has*

$$\mathbb{E}\left[G\int_0^t \alpha_s \mathrm{d}B_s\right] = \mathbb{E}\left[\int_0^t \alpha_s D_s^B(G)\mathrm{d}s\right],$$

where $D^B(G) = (D_s^B(G))_{s\geq 0}$ is the first Malliavin derivative of G w.r.t. B.

Taking $G = l(\int_0^T \rho_t\sqrt{v_{0,t}}\mathrm{d}B_t)$ gives $D_s^B(G) = l^{(1)}(\int_0^T \rho_t\sqrt{v_{0,t}}\mathrm{d}B_t)\rho_s\sqrt{v_{0,s}}\mathbf{1}_{s\leq T}$. Hence, we obtain immediately the following result.

Lemma 5.3. *One has*

$$\mathbb{E}\left[\left(\int_0^T \alpha_t \mathrm{d}B_t\right) l\left(\int_0^T \rho_t\sqrt{v_{0,t}}\mathrm{d}B_t\right)\right] = \mathbb{E}\left[\left(\int_0^T \rho_t\sqrt{v_{0,t}}\alpha_t \mathrm{d}t\right) l^{(1)}\left(\int_0^T \rho_t\sqrt{v_{0,t}}\mathrm{d}B_t\right)\right].$$

Lemma 5.4. *For any deterministic integrable function f and any continuous semimartingale Z vanishing at $t = 0$, one has*

$$\int_0^T f(t)Z_t\mathrm{d}t = \int_0^T \omega_{t,T}^{(0,f)}\mathrm{d}Z_t.$$

Proof. This is an application of the Itô formula to the product $\omega_{t,T}^{(0,f)}Z_t$. ∎

Lemma 5.5. *One has*

$$\mathbb{E}\left[l\left(\int_0^T \rho_t\sqrt{v_{0,t}}\mathrm{d}B_t\right)\int_0^T \beta_t v_{1,t}\mathrm{d}t\right] = \omega_{0,T}^{(\kappa,\rho\xi v_{0,.}),(-\kappa,\beta)}\mathbb{E}\left[l^{(1)}\left(\int_0^T \rho_t\sqrt{v_{0,t}}\mathrm{d}B_t\right)\right],$$

$$\begin{aligned}\mathbb{E}\left[l\left(\int_0^T \rho_t\sqrt{v_{0,t}}\mathrm{d}B_t\right)\int_0^T \beta_t v_{1,t}^2\mathrm{d}t\right] &= \omega_{0,T}^{(2\kappa,\xi^2 v_{0,.}),(-2\kappa,\beta)}\mathbb{E}\left[l\left(\int_0^T \rho_t\sqrt{v_{0,t}}\mathrm{d}B_t\right)\right]\\ &\quad + 2\omega_{0,T}^{(\kappa,\rho\xi v_{0,.}),(\kappa,\rho\xi v_{0,.}),(-2\kappa,\beta)}\mathbb{E}\left[l^{(2)}\left(\int_0^T \rho_t\sqrt{v_{0,t}}\mathrm{d}B_t\right)\right],\end{aligned}$$

$$\mathbb{E}\left[l\left(\int_0^T \rho_t\sqrt{v_{0,t}}\mathrm{d}B_t\right)\int_0^T \beta_t v_{2,t}\mathrm{d}t\right] = \omega_{0,T}^{(\kappa,\rho\xi v_{0,.}),(0,\rho\xi),(-\kappa,\beta)}\mathbb{E}\left[l^{(2)}\left(\int_0^T \rho_t\sqrt{v_{0,t}}\mathrm{d}B_t\right)\right].$$

Proof. Using Lemmas 5.3 ($f(t) = e^{-\kappa t}\beta_t$, $Z_t = \int_0^t e^{\kappa s}\xi_s\sqrt{v_{0,s}}\mathrm{d}B_s$) and 5.4, one has

$$\begin{aligned}\mathbb{E}\left[l\left(\int_0^T \rho_t\sqrt{v_{0,t}}\mathrm{d}B_t\right)\int_0^T \beta_t v_{1,t}\mathrm{d}t\right] &= \mathbb{E}\left[l\left(\int_0^T \rho_t\sqrt{v_{0,t}}\mathrm{d}B_t\right)\int_0^T e^{-\kappa t}\beta_t\int_0^t e^{\kappa s}\xi_s\sqrt{v_{0,s}}\mathrm{d}B_s\mathrm{d}t\right]\\ &= \mathbb{E}\left[l\left(\int_0^T \rho_t\sqrt{v_{0,t}}\mathrm{d}B_t\right)\int_0^T \omega_{t,T}^{(-\kappa,\beta)}e^{\kappa t}\xi_t\sqrt{v_{0,t}}\mathrm{d}B_t\right]\\ &= \mathbb{E}\left[l^{(1)}\left(\int_0^T \rho_t\sqrt{v_{0,t}}\mathrm{d}B_t\right)\right]\int_0^T \omega_{t,T}^{(-\kappa,\beta)}e^{\kappa t}\rho_t\xi_t v_{0,t}\mathrm{d}t,\end{aligned}$$

which gives the first equality. The second and the third equalities are proved in the same way. ∎

Lemma 5.6. *One has*

$$\mathbb{E}\left[\frac{\partial^{i+j}\tilde{P}_{BS}}{\partial x^i y^j}\right] = \frac{\partial^{i+j}P_{BS}}{\partial x^i y^j}\left(x_0, \int_0^T v_{0,t}\mathrm{d}t\right).$$

Proof. One has

$$\mathbb{E}\left[\frac{\partial^i \tilde{P}_{BS}}{\partial x^i}\right] = \partial^i_{x=x_0}\mathbb{E}\left[P_{BS}\left(x_0 + \int_0^T \rho_t\sqrt{v_{0,t}}\mathrm{d}B_t - \int_0^T \frac{\rho_t^2}{2}v_{0,t}\mathrm{d}t, \int_0^T (1-\rho_t^2)v_{0,t}\mathrm{d}t\right)\right]$$
$$= \frac{\partial^i P_{BS}}{\partial x^i}\left(x_0, \int_0^T v_{0,t}\mathrm{d}t\right).$$

Since $\tilde{P}_{BS}$ verifies the relation

$$\frac{\partial \tilde{P}_{BS}}{\partial y} = \frac{1}{2}\left(\frac{\partial^2 \tilde{P}_{BS}}{\partial x^2} - \frac{\partial \tilde{P}_{BS}}{\partial x}\right), \tag{5.1}$$

we immediately obtain the result. ∎

5.3. Proof of Proposition 2.1. One has

$$\mathbb{E}\left[\frac{\partial \tilde{P}_{BS}}{\partial x}\left(\int_0^T \rho_t\left(\frac{v_{1,t}}{2(v_{0,t})^{\frac{1}{2}}} + \frac{v_{2,t}}{4(v_{0,t})^{\frac{1}{2}}} - \frac{v_{1,t}^2}{8(v_{0,t})^{\frac{3}{2}}}\right)\mathrm{d}B_t - \int_0^T \frac{\rho_t^2}{2}\left(v_{1,t} + \frac{v_{2,t}}{2}\right)\mathrm{d}t\right)\right]$$
$$= \mathbb{E}\left[\frac{1}{2}\left(\frac{\partial^2 \tilde{P}_{BS}}{\partial x^2} - \frac{\partial \tilde{P}_{BS}}{\partial x}\right)\int_0^T \rho_t^2\left(v_{1,t} + \frac{v_{2,t}}{2}\right)\mathrm{d}t\right] - \mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial x^2}\int_0^T \frac{\rho_t^2 v_{1,t}^2}{8v_{0,t}}\mathrm{d}t\right]$$
$$= \mathbb{E}\left[\frac{\partial \tilde{P}_{BS}}{\partial y}\int_0^T \rho_t^2\left(v_{1,t} + \frac{v_{2,t}}{2}\right)\mathrm{d}t\right] - \mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial x^2}\int_0^T \frac{\rho_t^2 v_{1,t}^2}{8v_{0,t}}\mathrm{d}t\right],$$

where we have used Lemma 5.3 at the first equality and identity (5.1) at the second one. Plugging this relation into the approximation (2.12) and summing the second and third lines, one has

$$g(1) = \mathbb{E}[\tilde{P}_{BS}] + \mathbb{E}\left[\frac{\partial \tilde{P}_{BS}}{\partial y}\int_0^T \left(v_{1,t} + \frac{v_{2,t}}{2}\right)\mathrm{d}t\right]$$
$$- \mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial x^2}\int_0^T \frac{\rho_t^2 v_{1,t}^2}{8v_{0,t}}\mathrm{d}t\right] + \frac{1}{2}\mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial x^2}\left(\int_0^T \rho_t\frac{v_{1,t}}{2(v_{0,t})^{\frac{1}{2}}}\mathrm{d}B_t - \int_0^T \frac{\rho_t^2}{2}v_{1,t}\mathrm{d}t\right)^2\right]$$
$$+ \frac{1}{2}\mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial y^2}\left(\int_0^T (1-\rho_t^2)v_{1,t}\mathrm{d}t\right)^2\right]$$
$$+ \mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial xy}\left(\int_0^T (1-\rho_t^2)v_{1,t}\mathrm{d}t\right)\left(\int_0^T \rho_t\frac{v_{1,t}}{2(v_{0,t})^{\frac{1}{2}}}\mathrm{d}B_t - \int_0^T \frac{\rho_t^2}{2}v_{1,t}\mathrm{d}t\right)\right] + \mathcal{E}. \tag{5.2}$$

In addition, one has

$$
\begin{aligned}
&-\mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial x^2}\int_0^T \frac{\rho_t^2 v_{1,t}^2}{8v_{0,t}}\mathrm{d}t\right] + \frac{1}{2}\mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial x^2}\left(\int_0^T \rho_t \frac{v_{1,t}}{2(v_{0,t})^{\frac{1}{2}}}\mathrm{d}B_t - \int_0^T \frac{\rho_t^2}{2}v_{1,t}\mathrm{d}t\right)^2\right]\\
&= \mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial x^2}\int_0^T \left(\int_0^t \rho_s \frac{v_{1,s}}{2(v_{0,s})^{\frac{1}{2}}}\mathrm{d}B_s - \int_0^t \frac{\rho_s^2}{2}v_{1,s}\mathrm{d}s\right)\left(\rho_t \frac{v_{1,t}}{2(v_{0,t})^{\frac{1}{2}}}\mathrm{d}B_t - \frac{\rho_t^2}{2}v_{1,t}\mathrm{d}t\right)\right]\\
&= \mathbb{E}\left[\frac{1}{2}\left(\frac{\partial^3 \tilde{P}_{BS}}{\partial x^3} - \frac{\partial^2 \tilde{P}_{BS}}{\partial x^2}\right)\int_0^T \left(\int_0^t \rho_s \frac{v_{1,s}}{2(v_{0,s})^{\frac{1}{2}}}\mathrm{d}B_s - \int_0^t \frac{\rho_s^2}{2}v_{1,s}\mathrm{d}s\right)\rho_t^2 v_{1,t}\mathrm{d}t\right]\\
&= \mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial xy}\int_0^T \left(\int_0^t \rho_s \frac{v_{1,s}}{2(v_{0,s})^{\frac{1}{2}}}\mathrm{d}B_s - \int_0^t \frac{\rho_s^2}{2}v_{1,s}\mathrm{d}s\right)\rho_t^2 v_{1,t}\mathrm{d}t\right],
\end{aligned}
$$

where we have used Itô's lemma for the square at the first equality, Lemma 5.3 at the second one, and identity (5.1) at the third one. Substituting this relation in the approximation (5.2) and summing the second and fourth lines, one gets

$$
\begin{aligned}
g(1) = \mathbb{E}[\tilde{P}_{BS}] &+ \mathbb{E}\left[\frac{\partial \tilde{P}_{BS}}{\partial y}\int_0^T \left(v_{1,t} + \frac{v_{2,t}}{2}\right)\mathrm{d}t\right]\\
&+ \mathbb{E}\Bigg[\frac{\partial^2 \tilde{P}_{BS}}{\partial xy}\Bigg(\int_0^T \left(\int_0^t \rho_s \frac{v_{1,s}}{2(v_{0,s})^{\frac{1}{2}}}\mathrm{d}B_s - \int_0^t \frac{\rho_s^2}{2}v_{1,s}\mathrm{d}s\right)\rho_t^2 v_{1,t}\mathrm{d}t\\
&\qquad + \left(\int_0^T (1-\rho_t^2)v_{1,t}\mathrm{d}t\right)\left(\int_0^T \rho_t \frac{v_{1,t}}{2(v_{0,t})^{\frac{1}{2}}}\mathrm{d}B_t - \int_0^T \frac{\rho_t^2}{2}v_{1,t}\mathrm{d}t\right)\Bigg)\Bigg]\\
&+ \frac{1}{2}\mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial y^2}\left(\int_0^T (1-\rho_t^2)v_{1,t}\mathrm{d}t\right)^2\right] + \mathcal{E}.
\end{aligned}
\tag{5.3}
$$

First, we define

$$
H_t = \int_0^t \rho_s \frac{v_{1,s}}{2(v_{0,s})^{\frac{1}{2}}}\mathrm{d}B_s - \int_0^t \frac{\rho_s^2}{2}v_{1,s}\mathrm{d}s, \qquad dH_t = \rho_t \frac{v_{1,t}}{2(v_{0,t})^{\frac{1}{2}}}\mathrm{d}B_t - \frac{\rho_t^2}{2}v_{1,t}\mathrm{d}t.
$$

We now study the second term of (5.3). In the computations below, we use Itô's lemma for the second equality, Lemma 5.3 and identity (5.1) for the third equality, and Lemma 5.2

$(G = \frac{\partial^2 \tilde{P}_{BS}}{\partial xy} v_{1,t})$ for the fourth equality; it gives

$$
\begin{aligned}
A &= \mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial xy}\left(\int_0^T H_t \rho_t^2 v_{1,t}\mathrm{d}t + \left(\int_0^T (1-\rho_t^2) v_{1,t}\mathrm{d}t\right) H_T\right)\right] \\
&= \mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial xy}\left(\int_0^T H_t(\rho_t^2 + 1 - \rho_t^2) v_{1,t}\mathrm{d}t + \int_0^T \left(\int_0^t (1-\rho_s^2) v_{1,s}\mathrm{d}s\right) dH_t\right)\right] \\
&= \int_0^T \mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial xy} v_{1,t} H_t\right]\mathrm{d}t + \mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial y^2}\int_0^T \left(\int_0^t (1-\rho_s^2) v_{1,s}\mathrm{d}s\right)\rho_t^2 v_{1,t}\mathrm{d}t\right] \\
&= \int_0^T \mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial xy}\left(v_{1,t}\left(-\int_0^t \frac{\rho_s^2}{2} v_{1,s}\mathrm{d}s\right) + \int_0^t \rho_s \frac{v_{1,s}}{2\sqrt{v_{0,s}}} D_s^B v_{1,t}\mathrm{d}s\right)\right. \\
&\qquad\left. + \frac{\partial^3 \tilde{P}_{BS}}{\partial x^2 y} v_{1,t}\int_0^t \frac{\rho_s^2}{2} v_{1,s}\mathrm{d}s\right]\mathrm{d}t + \mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial y^2}\int_0^T \left(\int_0^t (1-\rho_s^2) v_{1,s}\mathrm{d}s\right)\rho_t^2 v_{1,t}\mathrm{d}t\right].
\end{aligned}
$$

From (2.3), one has $D_s^B v_{1,t} = e^{-kt}e^{ks}\xi_s\sqrt{v_{0,s}}$. Hence it is deterministic. Thus, using identity (5.1) and Lemma 5.3 for the first equality and (2.4) for the second equality, one has

$$
\begin{aligned}
A &= \mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial y^2}\left(\int_0^T \left(\int_0^t \rho_s^2 v_{1,s}\mathrm{d}s\right) v_{1,t}\mathrm{d}t + \left(\int_0^t (1-\rho_s^2) v_{1,s}\mathrm{d}s\right)\rho_t^2 v_{1,t}\mathrm{d}t\right)\right] \\
&\quad + \mathbb{E}\left[\frac{\partial \tilde{P}_{BS}}{\partial y}\int_0^T \left(\int_0^t \frac{v_{1,s}}{2v_{0,s}} e^{-kt}e^{ks}\xi_s\sqrt{v_{0,s}}\mathrm{d}B_s\right)\mathrm{d}t\right] \\
&= \mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial y^2}\left(\int_0^T \left(\int_0^t \rho_s^2 v_{1,s}\mathrm{d}s\right) v_{1,t}\mathrm{d}t + \left(\int_0^t (1-\rho_s^2) v_{1,s}\mathrm{d}s\right)\rho_t^2 v_{1,t}\mathrm{d}t\right)\right] \\
&\quad + \mathbb{E}\left[\frac{\partial \tilde{P}_{BS}}{\partial y}\int_0^T \frac{v_{2,t}}{2}\mathrm{d}t\right].
\end{aligned}
$$

Now, plug this last equality into (5.3) and use the easy identity

$$
\begin{aligned}
&\int_0^T \left(\left(\int_0^t \rho_s^2 v_{1,s}\mathrm{d}s\right) v_{1,t}\mathrm{d}t + \left(\int_0^t (1-\rho_s^2) v_{1,s}\mathrm{d}s\right)\rho_t^2 v_{1,t}\mathrm{d}t\right) + \frac{1}{2}\left(\int_0^T (1-\rho_t^2) v_{1,t}\mathrm{d}t\right)^2 \\
&= \int_0^T \left(\left(\int_0^t \rho_s^2 v_{1,s}\mathrm{d}s\right) v_{1,t}\mathrm{d}t + \left(\int_0^t (1-\rho_s^2) v_{1,s}\mathrm{d}s\right)(\rho_t^2 + 1 - \rho_t^2) v_{1,t}\mathrm{d}t\right) \\
&= \int_0^T \left(\left(\int_0^t (\rho_s^2 + 1 - \rho_s^2)\right) v_{1,s}\mathrm{d}s\right) v_{1,t}\mathrm{d}t = \frac{1}{2}\left(\int_0^T v_{1,t}\mathrm{d}t\right)^2;
\end{aligned}
$$

it immediately gives the result.

5.4. Proof of Theorem 2.2.

Proof. *Step* 1. We show the equality

$$
\mathbb{E}\left[\frac{\partial \tilde{P}_{BS}}{\partial y}\int_0^T (v_{1,t} + v_{2,t})\mathrm{d}t\right] = \sum_{i=1}^{2} a_{i,T}\frac{\partial^{i+1} P_{BS}(x_0, \int_0^T v_{0,t}\mathrm{d}t)}{\partial x^i y},
$$

where

$$a_{1,T} = \omega_{0,T}^{(\kappa,\rho\xi v_{0,.}),(-\kappa,1)}, \quad a_{2,T} = \omega_{0,T}^{(\kappa,\rho\xi v_{0,.}),(0,\rho\xi),(-\kappa,1)}.$$

Actually, the result is an immediate application of Lemmas 5.5 and 5.6.

Step 2. We show the equality

$$\frac{1}{2}\mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial y^2}\left(\int_0^T v_{1,t}\mathrm{d}t\right)^2\right] = \sum_{i=0}^{1} b_{2i,T}\frac{\partial^{2i+2}P_{BS}(x_0, \int_0^T v_{0,t}\mathrm{d}t)}{\partial x^{2i}y^2},$$

where

$$b_{0,T} = \omega_{0,T}^{(2\kappa,\xi^2 v_{0,.}),(-\kappa,1),(-\kappa,1)},$$
$$b_{2,T} = \omega_{0,T}^{(\kappa,\rho\xi v_{0,.}),(-\kappa,1),(\kappa,\rho\xi v_{0,.}),(-\kappa,1)} + 2\omega_{0,T}^{(\kappa,\rho\xi v_{0,.}),(\kappa,\rho\xi v_{0,.}),(-\kappa,1),(-\kappa,1)} = \frac{a_{1,T}^2}{2}.$$

Indeed, one has

$$\begin{aligned}
\frac{1}{2}\mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial y^2}\left(\int_0^T v_{1,t}\mathrm{d}t\right)^2\right] &= \mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial y^2}\int_0^T\left(\int_0^t v_{1,s}\mathrm{d}s\right)v_{1,t}\mathrm{d}t\right]\\
&= \mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial y^2}\int_0^T\left(\int_t^T e^{-\kappa s}\mathrm{d}s\right)\left(e^{\kappa t}v_{1,t}^2\mathrm{d}t + \xi_t\sqrt{v_{0,t}}e^{\kappa t}\left(\int_0^t v_{1,s}\mathrm{d}s\right)\mathrm{d}B_t\right)\right]\\
&= \mathbb{E}\left[\frac{\partial^2 \tilde{P}_{BS}}{\partial y^2}\int_0^T\left(\int_t^T e^{-\kappa s}\mathrm{d}s\right)e^{\kappa t}v_{1,t}^2\mathrm{d}t\right] + \mathbb{E}\left[\frac{\partial^3 \tilde{P}_{BS}}{\partial x y^2}\int_0^T \omega_{t,T}^{(\kappa,\rho\xi v_{0,.}),(-\kappa,1)}v_{1,t}\mathrm{d}t\right],
\end{aligned}$$

where we have used Lemma 5.4 ($f(t) = e^{-\kappa t}$, $Z_t = (\int_0^t v_{1,s}\mathrm{d}s)(e^{\kappa t}v_{1,t})$) for the second equality and Lemmas 5.3 and 5.4 ($f(t) = (\int_t^T e^{-\kappa s}\mathrm{d}s)\rho_t\xi_t v_{0,t}e^{\kappa t}$, $Z_t = \int_0^t v_{1,s}\mathrm{d}s$) for the third equality.

An application of the first and second equalities in Lemma 5.5 gives the announced result. Actually, it remains to show that $b_{2,T} = a_{1,T}^2/2$. Indeed, consider two càdlàg functions f and $g : [0,T] \to \mathbb{R}$. Then

$$\begin{aligned}
\frac{(\int_0^T f_t(\int_t^T g_s\mathrm{d}s)\mathrm{d}t)^2}{2} &= \frac{\int_0^T\int_0^T f_{t_1}(\int_{t_1}^T g_{t_3}\mathrm{d}t_3)f_{t_2}(\int_{t_2}^T g_{t_4}\mathrm{d}t_4)\mathrm{d}t_2\mathrm{d}t_1}{2}\\
&= \int_0^T f_{t_1}\left(\int_{t_1}^T\int_{t_1}^T g_{t_3}f_{t_2}\left(\int_{t_2}^T g_{t_4}\mathrm{d}t_4\right)\mathrm{d}t_3\mathrm{d}t_2\right)\mathrm{d}t_1\\
&= \int_0^T f_{t_1}\Bigg(\int_{t_1}^T f_{t_2}\int_{t_2}^T\int_{t_2}^T g_{t_3}g_{t_4}\mathrm{d}t_3\mathrm{d}t_4\mathrm{d}t_2\\
&\qquad + \int_{t_1}^T g_{t_3}\int_{t_3}^T f_{t_2}\int_{t_2}^T g_{t_4}\mathrm{d}t_4\mathrm{d}t_2\mathrm{d}t_3\Bigg)\mathrm{d}t_1\\
&= 2\int_0^T f_{t_1}\int_{t_1}^T f_{t_2}\int_{t_2}^T g_{t_3}\int_{t_3}^T g_{t_4}\mathrm{d}t_3\mathrm{d}t_4\mathrm{d}t_2\mathrm{d}t_1\\
&\qquad + \int_0^T f_{t_1}\int_{t_1}^T g_{t_3}\int_{t_3}^T f_{t_2}\int_{t_2}^T g_{t_4}\mathrm{d}t_4\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_1.
\end{aligned}$$

Putting $f(t) = \rho_t \xi_t v_{0,t} e^{kt}$ and $g(t) = e^{-kt}$ in the previous equality readily gives $b_{2,T} = \frac{a_{1,T}^2}{2}$, which finishes the proof. ■

6. Conclusion. We have established an approximation pricing formula for call/put options in the time dependent Heston models. We prove that the error is of order 3 w.r.t. the volatility of volatility and 2 w.r.t. the maturity. In practice, taking the Fourier method as a benchmark, the accuracy is excellent for a large range of strikes and maturities. In addition, the computational time is about 100 to 1000 times smaller than using an efficient Fourier method.

Following the arguments in [8], our formula extends immediately to other payoffs depending on S_T (note that the identities (2.2) and (5.1) are valid for any payoff of this type). As explained in [8], the smoother the payoff, the higher the error order w.r.t. T; the less smooth the payoff, the lower the error order w.r.t. T. For digital options, the error order w.r.t. T becomes 3/2 instead of 2.

Extensions to exotic options and to the third order expansion formula w.r.t. the volatility of volatility are left for further research.

7. Appendix: Closed formulas in the Heston model. There are few closed representations for the call/put prices written on the asset $S_t = e^{\int_0^t (r_s - q_s)ds} e^{X_t}$ in the Heston model (defined in (1.1) and (1.2)). We focus on the Heston formula [22] and on the Lewis formula [27]. Both of them rely on the knowledge of the characteristic function of the log-asset price $(X_t)_t$ and on Fourier transform-based approaches.

(i) In [22], Heston obtains a representation in a *Black–Scholes* form

$$Call_{Heston}(t, S_t, v_t; T, K) = S_t e^{-\int_t^T q_s ds} P_1 - K e^{-\int_t^T r_s ds} P_2,$$

where both probabilities P_1 and P_2 are equal to a one-dimensional integral of characteristic functions.

(ii) In [27], Lewis takes advantage of the generalized Fourier transform, by using an integration along a straight line in the complex plane parallel to the real axis. It is important to detect the strip where the integration is safe. Lewis suggests the use of complex numbers z such that $\mathcal{I}\mathrm{m}(z) = \frac{1}{2}$. His formula writes

$$Call_{Heston}(t, S_t, v_t; T, K) = S_t e^{-\int_t^T q_s ds} - \frac{K e^{-\int_t^T r_s ds}}{2\pi} \int_{\frac{i}{2}-\infty}^{\frac{i}{2}+\infty} e^{-izX} \phi_T(-z) \frac{dz}{z^2 - iz},$$

where $X = \log(\frac{S_t e^{-\int_t^T q_s ds}}{K e^{-\int_t^T r_s ds}})$ and $\phi_T(z) = \mathbb{E}(e^{z(X_T - X_t)} | \mathcal{F}_t)$. Then, the above integral is evaluated by numerical integration.

Using PDE arguments in combination with affine models, we can obtain an explicit formula for $\phi_T(z)$ in the case of constant Heston parameters. In addition, it can be computed without discontinuities in z, following the arguments in [23]. For piecewise constant parameters, the characteristic function $\phi_T(z)$ can be computed recursively using nested Riccati equations with constant coefficients: we refer the reader to the work by Mikhailov and Nogel [31].

In our numerical tests, we prefer the Lewis formula, which gives better numerical results, in particular for very small or very large strikes, compared to the Heston formula.

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