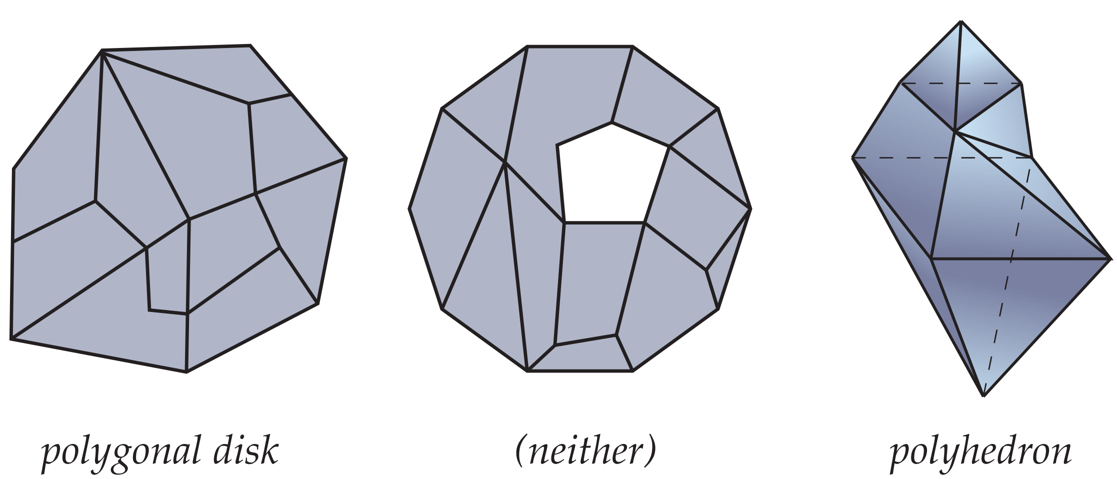
Homework 1: Topological Invariants of Discrete Surfaces

October 9, 2013 Homework

This homework is due 10/17 at 5pm. Submissions should be sent to me. I have a very strong preference for all online submissions. If that doesn't work for you for some reason come talk to me so we can see whether we can address the underlying issue. Homework will be graded not only on the basis of correctness, but also on ***presentation!*** Remember that proofs are meant for human beings, not for machines. Complete sentences, an outline of your approach, and illustrations (where appropriate) are all strongly recommended if you wish to receive a good score.

Euler Characteristic

Exercise 1.1: Polyhedral Formula A topological *disk* is, roughly speaking, any shape you can get by deforming the region bounded by a circle without tearing it, puncturing it, or gluing its edges together. Some examples of shapes that are disks include a flag, a leaf, and a glove. Some examples of shapes that are *not* disks include a circle (why?), a ball, a sphere, a DVD, a donut, and a teapot. A *polygonal disk* is any disk constructed out of simple polygons. Similarly, a topological *sphere* is any shape resembling the standard sphere, and a *polyhedron* is a sphere made of polygons. More generally, a *piecewise linear* surface is any surface made by gluing together polygons along their edges; a *simplicial surface* is a special case of a piecewise linear surface where all the faces are triangles. The *boundary* of a piecewise linear surface is the set of edges that are contained in only a single face (all other edges are shared by exactly two faces). For example, a disk has a boundary whereas a polyhedron does not. You may assume that surfaces have no boundary unless otherwise stated.



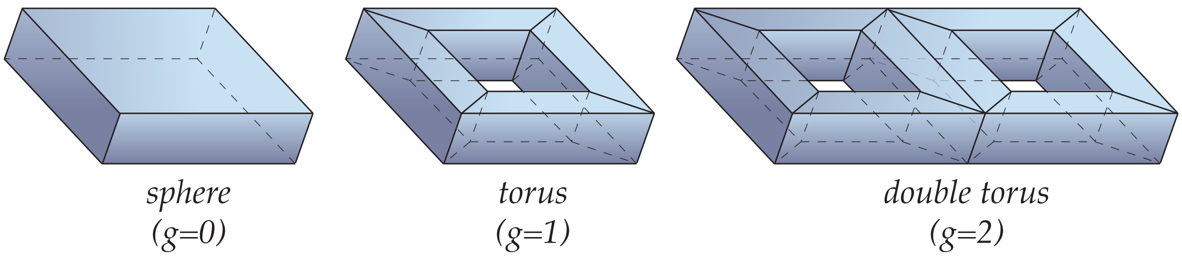
Show that for any polygonal disk with V vertices, E edges, and F faces, the following relationship holds:

$$V - E + F = 1$$

and conclude that for any polyhedron $V - E + F = 2$.

*Hint: use induction. Note that induction is generally easier if you start with a given object and decompose it into **smaller** pieces rather than trying to make it **larger**, because there are fewer cases to think about.*

Euler-Poincaré Formula



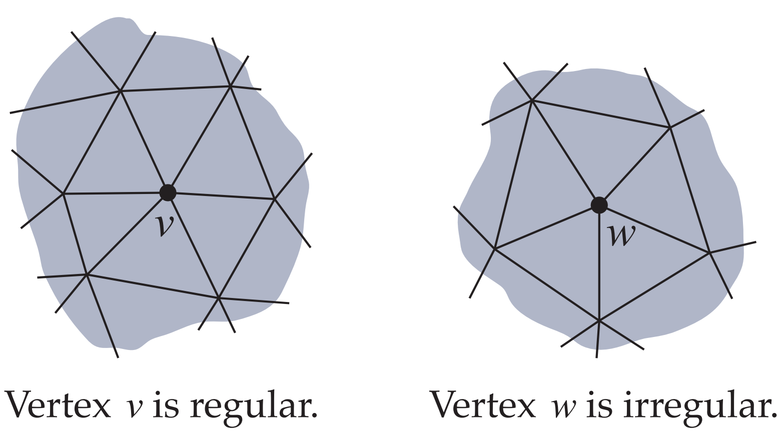
Clearly not all surfaces look like disks or spheres. Some surfaces have additional *handles* that distinguish them topologically; the number of handles g is known as the *genus* of the surface (see illustration above for examples). In fact, among all surfaces that have no boundary and are connected (meaning a single piece), compact (meaning closed and contained in a ball of finite size), and orientable (having two distinct sides), the genus is the *only* thing that distinguishes two surfaces. A more general formula applies to such surfaces, namely

$$V - E + F = 2 - 2g,$$

which is known as the *Euler-Poincaré formula.* (You do not have to prove anything about this statement, but it will be useful in later calculations.)

Tessellation

Exercise 1.2: Regular Valence



The *valence* of a vertex in a piecewise linear surface is the number of faces that contain that vertex. A vertex of a *simplicial* surface is said to be *regular* when its valence equals six. Show that the only (connected, orientable) simplicial surface for which every vertex has regular valence is a torus ($g = 1$). You may assume that the surface has finitely many faces.

Hint: apply the Euler-Poincaré formula.

Exercise 1.3: Minimally Irregular Valence Show that the minimum possible number of irregular valence vertices in a (connected, orientable) simplicial surface K of genus g is given by

$$m(K) = \begin{cases} 4, & g = 0 \\ 0, & g = 1 \\ 1, & g \geq 2, \end{cases}$$

assuming that all vertices have valence at least three and that there are finitely many faces.

Exercise 1.4: Mean Valence Show that the mean valence approaches six as the number of vertices in a (connected, orientable) simplicial surface goes to infinity, and that the ratio of vertices to edges to faces hence approaches

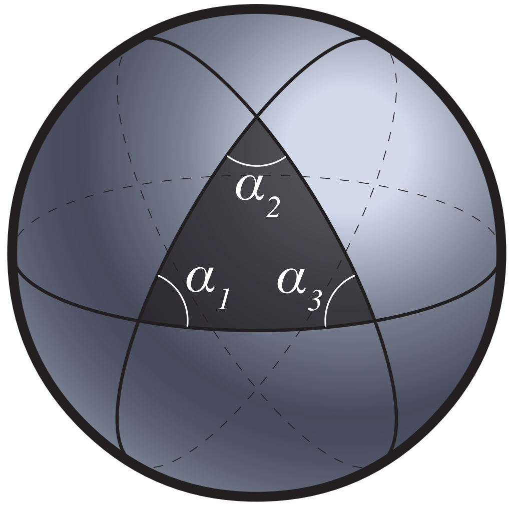
$$V : E : F = 1 : 3 : 2.$$

Discrete Gaussian Curvature

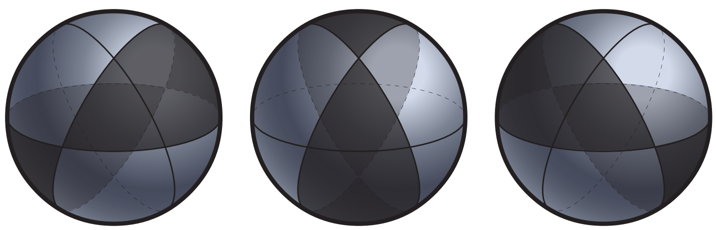
Exercise 1.5: Area of a Spherical Triangle Show that the area of a spherical triangle on the unit sphere with interior angles $\alpha_1, \alpha_2, \alpha_3$ is

$$A = \alpha_1 + \alpha_2 + \alpha_3 - \pi.$$

Hint: consider the areas A_1, A_2, A_3 of the three shaded regions (called "diangles") pictured below.



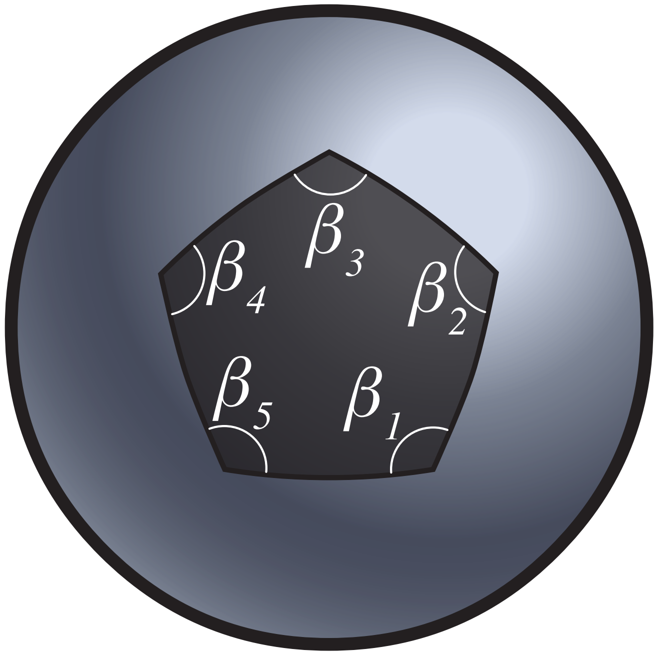
A



A_1 A_2 A_3

Exercise 1.6: Area of a Spherical Polygon Show that the area of a spherical polygon with consecutive interior angles $\beta_1, \ldots, \beta_n$ is

$$A = (2-n)\pi + \sum_{i=1}^{n} \beta_i.$$



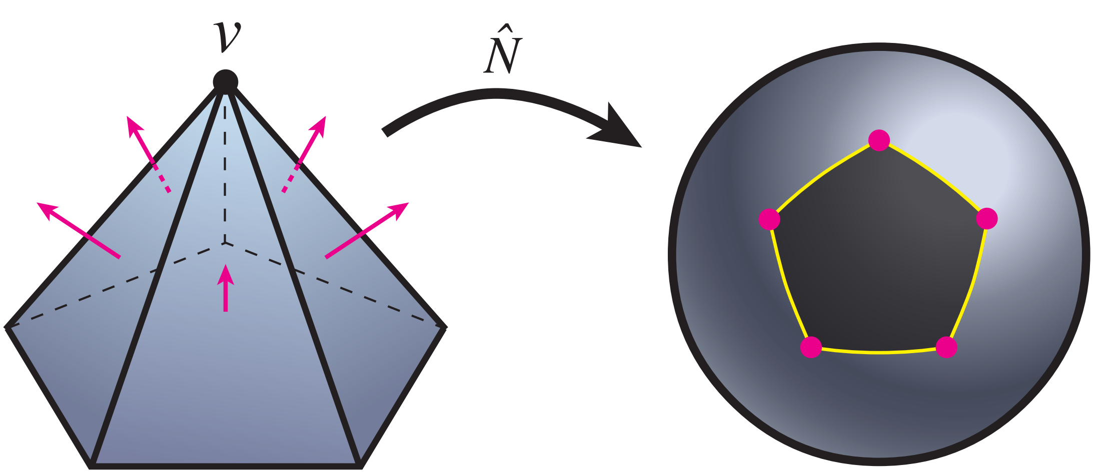
Hint: use the expression for the area of a spherical triangle you just derived!

Exercise 1.7: Angle Defect Recall that for a discrete planar curve we can define the curvature at a vertex as the distance on the unit circle between the two adjacent normals. For a discrete *surface* we can define (Gaussian) curvature at a vertex v as the *area* on the unit sphere bounded by a spherical polygon whose vertices are the unit normals of the faces around v. Show that this area is equal to the *angle defect*

$$d(v) = 2\pi - \sum_{f \in F_v} \angle_f(v)$$

where F_v is the set of faces containing v and $\angle_f(v)$ is the interior angle of the face f at vertex v.

Hint: consider planes that contain two consecutive normals and their intersection with the unit sphere.



Exercise 1.8: Discrete Gauss-Bonnet Theorem Consider a (connected, orientable) simplicial surface K

with finitely many vertices V, edges E and faces F. Show that a discrete analog of the Gauss-Bonnet theorem holds for simplicial surfaces, namely

$$\sum_{v \in V} d(v) = 2\pi\chi$$

where $\chi = |V| - |E| + |F|$ is the *Euler characteristic* of the surface.