CS177 Homework 1

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Exercise 1.1

1. If $F = 1$, we obviously have $V = E$ so $V - E + F = 1$ is satisfied. For a more complicated polygonal disk such as FIG. 1, we want to show that F can be reduced one by one without changing the topology.

2. There must be some simple polygons whose edges coincide with the boundary of the disk. We call such edges **boundary edges** and the others **internal edges**, and define **boundary vertices** similarly. Furthermore, there must be a simple polygon with at least one boundary edge and adjacent boundary vertices. Removing such simple polygon does not change the topology.

3. To show the claim, suppose there is a simple polygon with separated boundary vertices such as the blue triangle $[2, 3, 8]$ in FIG. 1. Because edges cannot cross each other, the boundary vertices of a simple polygon is a subset of either $[8, 9, 1, 2]$ or $[3, 4, 5, 6, 7, 8]$, both of which contain adjacent but less boundary vertices. Repeating the same logic, we must be able to find a desired simple polygon when or before each subset contains three boundary vertices since themselves form a triangle.

4. Now a desired simple polygon, say, the red hexagon in FIG. 1 is found. Suppose the simple polygon has n boundary edges. Because the internal edges are adjacent, removing the simple polygon from the polygonal disk causes $F \to F - 1$, $E \to E - n$ and $V \to V - (n - 1)$. The combination $V - E + F$ is invariant. By induction to the $F = 1$ case we prove that $V - E + F = 1$ for any polygonal disk.

5. A polygonal disk is formed when we remove a simple polygon from a polyhedron. Such procedure causes $F \to F - 1$, $E \to E$ and $V \to V$ so we know $V - E + F = 2$ for any polyhedron.

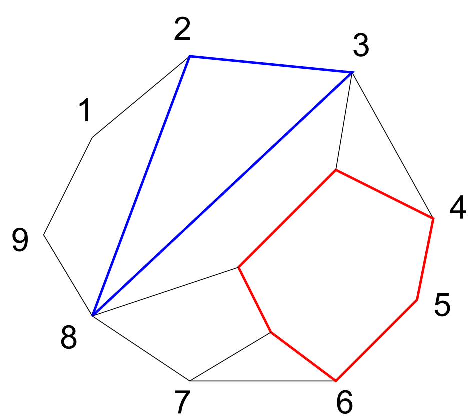

FIG. 1: A polygonal disk.

Exercise 1.2

The valance of each vertex is 6 so each triangle gives one face, three $\frac{1}{2}$-edges and three $\frac{1}{6}$-vertices. Hence

$$g = 1 - \frac{1}{2}(V - E + F) = 1 - \frac{1}{2}\left(\frac{1}{2}F - \frac{3}{2}F + F\right) = 1. \tag{1}$$

Exercise 1.3

For simplicity, assume we have m regular valance vertices and n irregular valance vertices with the same valance $v \neq 6$. Each regular valance vertices one vertex, six $\frac{1}{2}$-edges and six $\frac{1}{3}$ faces. Each irregular valance vertices gives one vertex, v $\frac{1}{2}$-edges and v $\frac{1}{3}$ faces. Using the Euler-Poincaré formula, we have

$$2 - 2g = m\,(1 - 3 + 2) + n\left(1 - \frac{v}{2} + \frac{v}{3}\right) = n\left(1 - \frac{v}{6}\right). \tag{2}$$

- $g = 0 : v \geq 3 \Rightarrow n \geq 4$.
- $g = 1$: We can always take $n = 0$.
- $g \geq 2$: We can fix $n = 1$ and take $v = 12g - 6$.

Exercise 1.4

Let n_v be the number of vertices with valance v and $N = \sum_v n_v$. The Euler-Poincaré formula then gives

$$\begin{aligned} 2 - 2g = \sum_v n_v \left(1 - \frac{v}{6}\right) = N - \frac{1}{6}\sum_v n_v \cdot v \\ \Rightarrow \;\bar{v} = 6 - \frac{12}{N}(1 - g) = 6\left(1 - \frac{\chi}{N}\right). \end{aligned} \tag{3}$$

Take the limit $N \to \infty$ and then we have $\bar{v} \to 6$ and $V : E : F = 1 : \bar{v}/2 : \bar{v}/3 \to 1 : 3 : 2$ (c.f. Eq. (2)).

Exercise 1.5

From the hint FIG. 2, kindly provided by Peter, we can read out $A_i = 4\alpha_i$ and $A_1 + A_2 + A_3 = 4\pi + 2A$. Hence

$$A = \alpha_1 + \alpha_2 + \alpha_3 - \pi. \tag{4}$$

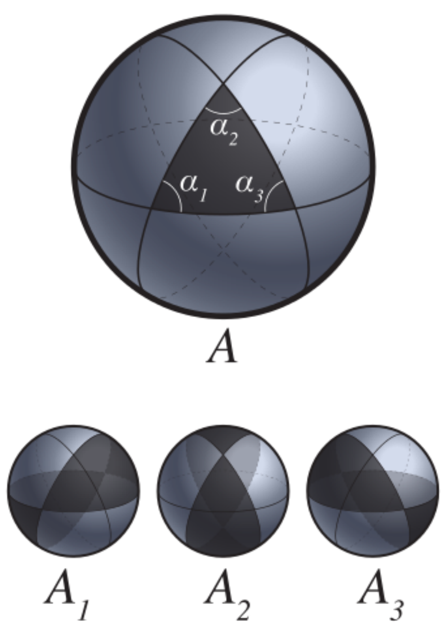

FIG. 2: Area of a spherical triangle.

Exercise 1.6

We can divide a spherical n-gon into $n-2$ spherical triangles while preserving the total internal angle. Hence

$$A = \sum_{i=1}^{n-2} (\alpha_{i,1} + \alpha_{i,2} + \alpha_{i,3} - \pi) = (2-n)\,\pi + \sum_{i=1}^{n} \beta_i. \tag{5}$$

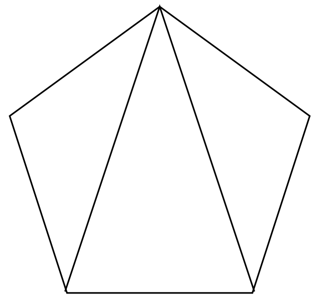

FIG. 3: A pentagon divided into three triangles.

Exercise 1.7

Each plane spanned by two consecutive normal vectors corresponds to a great circle on the sphere. Let us call such planes **great planes**. The intersections of normals with the unit sphere are vertices of a spherical polygon, and thus great planes are projected to edges. A great plane is spanned by two normals, so it is perpendicular to the edge between the two corresponding faces. Since the intersection of two consecutive great planes is just the normal vector, the intersection angle is equal to the internal angle of the spherical polygon. From FIG. 4 we know the internal angle is given by $\beta_i = \pi - \angle_i$. Then we can use Eq. (5) to obtain

$$A = (2-n)\,\pi + \sum_{i=1}^{n} \beta_i = 2\pi - \sum_{i=1}^{n} \angle_i. \tag{6}$$

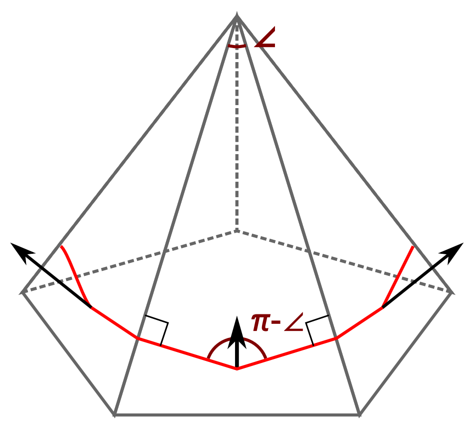

FIG. 4: Computing angle defect.

Exercise 1.8

Since the surface is simplicial, we have

$$\sum_{v\in V} d(v) = \sum_{v\in V}\left(2\pi - \sum_{f\in F}\angle_f(v)\right) = 2\pi\,|V| - \sum_{f\in F}\left(\sum_{v\in V}\angle_f(v)\right) = 2\pi\,|V| - \pi\,|F|\,. \tag{7}$$

Then we can use the result of Exercise 1.4 to get

$$\sum_{v\in V} d(v) = 2\pi\,|V|\left(1-\frac{\bar{v}}{6}\right) = 2\pi\sum_{v} n_v\left(1-\frac{v}{6}\right) = 2\pi\chi. \tag{8}$$