Exponentiated Weibull Family for Analyzing Bathtub Failure-Rate Data

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Reader Aids —
General purpose: Widen the state of art
Special math needed for explanations: Probability and elementary statistics
Special math needed to use results: Same
Results useful to: Reliability analysts and theoreticians

Abstract — **A simple generalization of the Weibull distribution offered here is well suited for modeling bathtub failure rate lifetime data and for testing goodness-of-fit of the Weibull and negative exponential models as subhypotheses.**

1. INTRODUCTION

The negative exponential model (though often criticized for lifetime data) provides simple, elegant, closed-form solutions to many problems in reliability analysis. Its classic generalization introduced by Weibull [15], now known as the Weibull family, is commonly used for modeling systems with monotone failure rates. However, data in reliability analysis, especially over the life-cycle of the product, can involve high initial failure rates (infant mortality), and eventual high failure rates due to aging and wearout, indicating a bathtub failure rate. Models which allow only monotone failure rates might not be appropriate or adequate for modeling the populations giving rise to such data. There have been several attempts to answer the need for a family of distributions which allow flexibility in modeling. For example Mann, Schafer, Singpurwalla [12] proposed mixtures of Weibull distributions, and Hjorth [8] studied the 3-parameter family obtained by generalizing the Rayleigh distribution which itself is a generalization of the exponential distribution. The Hjorth family [8] includes distributions with increasing, decreasing, and bathtub failure rates. Gaver & Acar [7] consider a similar family with 4 parameters.

This note presents a simple generalization of the Weibull family called the exponentiated-Weibull family; it not only includes distributions with bathtub and unimodal failure rates but provides a broader class of monotone failure rates. Neither the Hjorth nor the Gaver & Acar model includes the Weibull family; the exponentiated Weibull model is parsimonious in parameters as compared with the mixture model in [12]. A graphical method based on the total time on test (TTT) transform introduced by Barlow & Campo [2] and further extended by Bergman & Klefsjö [4] illustrates the variety of hazard-rate shapes available in the exponentiated-Weibull family. Aarset [1] proposes and illustrates the use of empirical TTT-transform for identifying bathtub failure rates and offers a goodness of fit test of exponentiality. We illustrate the usefulness of the family by modeling the bathtub failure rate behavior of the data in Aarset [1] and show that the family can be used to test not only exponentiality but also goodness of fit of the Weibull model.

Notation

T_i	$i=1, \ldots, N;$ a random sample from a life distribution
$T_{N:i}$	$i=1, \ldots, N;$ the ordered sample
$F, \bar{F}$	absolutely continuous [Cdf, Sf] of a life distribution
$F^{-1}(u)$	lub{t: F(t) $\geq u$}; also know as quantile function and conventionally denoted by $Q(u)$
f, h	pdf, hazard rate
μ	mean time to failure, $H_F^{-1}(1)$
N	number of items on test
r	number of failed items, $r \leq N$
Σ_i	$\Sigma_{i=1}^{N}$
TTT	total time on test
$H_F^{-1}(u)$	TTT-Transform: $\int_0^{F?-1?(u)} \bar{F}(t)\, dt,\ 0 \leq u \leq 1$
$\phi_F(u)$	scaled TTT-Transform: $H_F^{-1}(u)/H_F^{-1}(1)$
$\phi_N(r/N)$	scaled empirical TTT-transform: $H_N^{-1}(u)/H_N^{-1}(1) = [\Sigma_{i=1}^{r} T_{N:i} + (N-r)T_{N:r}]/\Sigma_i T_i$
L, l	likelihood, log likelihood
Λ	likelihood ratio statistic
ˆ	implies an estimate
e, w	(subscript) implies the [exponential, Weibull] model
ML	maximum likelihood.

Other, standard notation is given in "Information for Readers & Authors" at the rear of each issue.

2. EXPONENTIATED-WEIBULL FAMILY

The exponentiated-Weibull family Cdf, pdf, quantile-function, and hazard (failure) rate are:
(The α, θ, σ are positive parameters; σ is a true scale parameter.)

$$F(t) = [1 - \exp(-(t/\sigma)^{\alpha}]^{\theta},\ 0 \leq t \leq \infty, \quad (1)$$

$$f(t) = (\alpha\theta/\sigma)[1 - \exp(-(t/\sigma)^{\alpha}]^{\theta-1} \cdot \exp(-(t/\sigma)^{\alpha}) \cdot (t/\sigma)^{\alpha-1}, \quad (2)$$

$$Q(u) = F^{-1}(u) = \sigma[-\log(1 - u^{1/\theta})]^{1/\alpha},\ 0 \leq u \leq 1, \quad (3)$$

$$h(t) = f(t)/[1-F(t)]. \tag{4}$$

The exponentiated-Weibull family accommodates unimodal, bathtub, and a broad variety of monotone failure rates; examples are shown in the table.

α	θ	failure-rate behavior
1	1	constant (exponential)
	1	monotonic (Weibull)
< 1	< 1	decreasing
> 1	> 1	increasing
> 1	< 1	bathtub or increasing
< 1	> 1	unimodal or decreasing.

Figure 1a illustrates the four types of hazard shapes.

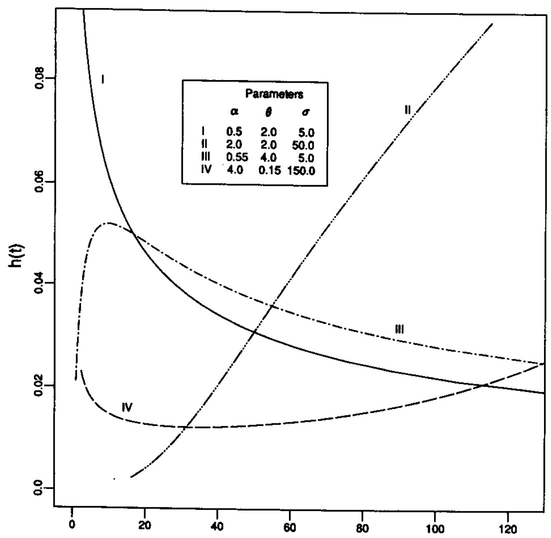

Figure 1a. 4 Types of Hazard (Failure) Rate Functions

3. SOME EXPONENTIATED-WEIBULL TTT-TRANSFORMS

An interesting device for exploring the hazard shape appropriate for a data-set is the empirical version of the TTT-transform [2]. It shows that the hazard function of F is increasing (decreasing) iff the TTT-transform, $H_F^{-1}(u)$, is concave (convex). Bergman & Klefsjö [4] consider the TTT-transform graphs in the context of monotone as well as non-monotone failure rate models and extend application of the TTT concept to determination of replacement policies; see also [3, 5, 10]. Aarset [1] introduces $\phi_N(r/N)$, the scaled empirical TTT-transform, for identifying a bathtub failure rate. Specifically, for a distribution with bathtub (unimodal) failure rate the TTT-transform is first convex (concave) and then concave (convex).

The TTT-transform illustrates the variety of hazard-rate functions included in the exponentiated-Weibull family of distributions. In order to obtain $\phi_F(u)$ — the scaled TTT-transform — the definite integrals (see Notation) are evaluated using the IMSL [9] routine DQDAG. Figure 1b gives the scaled TTT-transforms of the 4 distributions in figure 1a, and clarifies the use of the TTT-transform for identifying the hazard rate shape and illustrates the model flexibility of exponentiated-Weibull family.

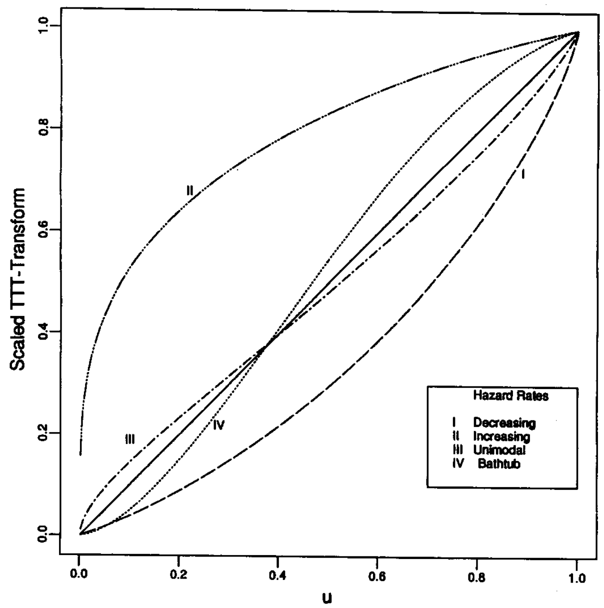

Figure 1b. Scaled TTT-Transforms of the Distributions in Figure 1a

4. MAXIMUM LIKELIHOOD ESTIMATION

Given a sample of size N from an exponentiated-Weibull distribution, its parameters can be estimated by the ML method [6, 11, 12].

$$l = N\cdot\log(\alpha\theta/\sigma) + (\theta-1)\cdot\Sigma_i \log(g(T_i)) - \Sigma_i (T_i/\sigma)^\alpha + (\alpha-1)\cdot\Sigma_i \log(T_i/\sigma), \tag{5}$$

$g(T_i) = g(T_i; \alpha, \sigma) = 1 - \exp(-(T_i/\sigma)^\alpha)$. The three equations obtained by differentiation from (5) are:

$$0 = \partial l/\partial\alpha = N/\alpha + (\theta-1)\cdot\Sigma_i g_\alpha(T_i)/g(T_i) - \Sigma_i (T_i/\sigma)^\alpha\cdot\log(T_i/\sigma) + \Sigma_i \log(T_i/\sigma), \tag{6}$$

$$0 = \partial l/\partial\theta = N/\theta + \Sigma_i \log(g(T_i)), \tag{7}$$

$$0 = \partial l/\partial\sigma = -(N\alpha/\sigma) + (\theta-1)\cdot\Sigma_i\ g_\sigma(T_i)/g(T_i)$$

$$+ (\alpha/\sigma)\cdot\Sigma_i\ (T_i/\sigma)^\alpha, \quad (8)$$

$$g_\alpha(T_i) \equiv \exp(-(T_i/\sigma)^\alpha)\cdot(T_i/\sigma)^\alpha\cdot\log(T_i/\sigma),$$

$$g_\sigma(T_i) \equiv -[\alpha\cdot\exp(-(T_i/\sigma)^\alpha)\cdot(T_i/\sigma)^\alpha]/\sigma.$$

In practice, (6) - (8) can be solved by using NAG [13] routines CO5NCF & CO5PBF, or IMSL [9] routine DNEQNF. The sample information matrix, obtained by differentiating l twice, can be used for estimating the standard errors (standard deviations) of the estimates [14].

5. GOODNESS OF FIT

The problem of testing goodness-of-fit of a Weibull model against the unrestricted class of alternatives is complex. However, by restricting the alternatives to the exponentiated-Weibull family, we can use the usual likelihood ratio statistics [14] to test the adequacy of a Weibull submodel. The null hypothesis, H_{01}: $\theta=1$, corresponds to the Weibull submodel, and H_{02}: $\{\theta=1,\ \alpha=1\}$ corresponds to the exponential submodel (of the exponentiated-Weibull model). If one or both of these hypotheses are rejected in the class of exponentiated-Weibull family then they are of questionable merit in the class of all distributions. The likelihood ratio statistics for H_{0i} $(i=1,2)$ [14] are:

$$\Lambda_i = \sup_{R_{0i}}\{L(\alpha,\ \theta,\ \sigma)\}/\sup_R\{L(\alpha,\ \theta,\ \sigma)\},\ i=1,2;$$

R_{0i} is the parametric space corresponding to H_{0i}, $i=1,2$;

R is the unrestricted parametric space.

In terms of the ML estimates, the likelihood ratio statistics reduce to:

$$\Lambda_1 = L(\alpha_w,\ \theta=1,\ \sigma_w)/L(\alpha,\ \theta,\ \sigma), \quad (9a)$$

$$\Lambda_2 = L(\alpha=1,\ \theta=1,\ \sigma_e)/L(\alpha,\ \theta,\ \sigma). \quad (9b)$$

Under the null hypothesis, $-2\cdot\log(\Lambda_1)$ follows a χ^2 distribution with 2 degrees of freedom (df), and $-2\cdot\log(\Lambda_2)$ follows a χ^2 distribution with 1 df.

6. EXAMPLE

The use of the exponentiated-Weibull family for modeling and testing goodness-of- fit hypotheses is illustrated using the data in table 1.

TABLE 1
Lifetimes of 50 Devices [1]

.1	.2	1	1	1	1	1	2	3	6	7	11	12	18	18	18	18	18	21
32	36	40	45	46	47	50	55	60	63	63	67	67	67	67	72	75	79	82
82	83	84	84	84	85	85	85	85	85	86	86							

An exponentiated-Weibull fit for the data in table 1 is obtained by solving the ML equations (6) - (8) for α, θ, σ. The solutions used the IMSL routine DNEQNF. The standard errors (SE) of the estimates based on the sample information matrix involves 6 second order partial derivatives computed at the ML estimates. The numbers in [] are the standard errors of the estimates.

$$\alpha = 4.69 \quad [1.6],$$

$$\theta = 0.146 \quad [0.053],$$

$$\sigma = 91.023 \quad [0.056] \text{ (scale parameter).} \quad (10)$$

The plots of the fitted scaled TTT-transform with the estimates in (10) and the empirical scaled TTT-transform,

$$\phi_N(r/N) = [\Sigma_i^r\ T_{N:i} + (N-r)\cdot T_{N:r}]/\Sigma_i\ T_i,$$

are shown in figure 2a. The corresponding hazard plot of the exponentiated Weibull distribution with the parameters in (10) is given in figure 2b.

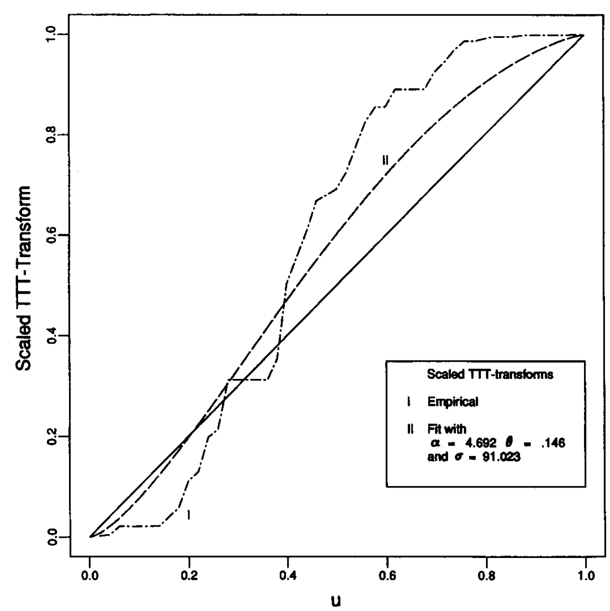

Figure 2a. Empirical & Fitted Scaled TTT-Transforms for the Aarset Data

The values of the likelihood ratio statistics for testing the Weibull and exponentiality goodness-of-fit hypotheses are:

$$-2\cdot\log(\Lambda_1) = 165.1,\ -2\cdot\log(\Lambda_2) = 161.8.$$

The s-significance levels corresponding to these values are so small that the Weibull and exponential models are untenable.

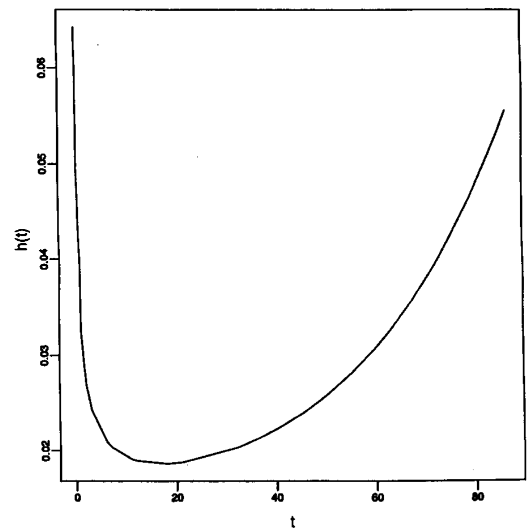

Figure 2b. Hazard Plot of Exponentiated-Weibull Fit to the Aarset Data

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